Mathieu Equation (Parametric Oscillator)

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1. Introduction

Mathieu Equations are encountered in various areas of physics and engineering. Certain problems in theoretical physics lead to Mathieu equation, particularly the problem of the propagation of electromagnetic waves in a medium with a periodic structure, the problem of motion of electrons in a crystal lattice in the quantum theory of metals. This equation is also encountered in the investigation of stability in the nonlinear oscillations, namely inverted pendulum, parametric oscillator etc. The Mathieu equation occurs in the study of parametric excitation and parametric resonance in the theory of nonlinear resonances.

2. Mathieu Equation

The Mathieu equation is a second-order homogenous differential equation, given as

$$\frac{d^2y}{dz^2} + \left(a - 2q\cos(2z)\right)y = 0 \qquad (1)$$

Where a, q are constants and are called as characteristic numbers. Mathieu equation is a differential equation with periodic coefficients. This behavior makes the difference in its study from the other differential equations. The solution of the Mathieu equation may be stable, unstable or periodic bounded, non periodic unbounded depending on the parameters a, q. Therefore what is of interest is to find the behaviour of the differential equation according to the values of a and q. We are strictly interested in knowing under what values of a, q the solution of Eq (1) i.e. y is stable. These solutions are shown in the form of charts called stability charts. A stability chart is a plot of q vs. a values, and it has regions of stability and instability. A typical stability chart of Eq (1), within the bounds of 0 to 5 for a, q is shown in Fig.1. In the figure 1 the solution is stable for a, q values which lie in the white region, and unstable for the values which lie inside the colorful region, and show beating phenomenon on the lines which separates stable and unstable regions.

We can include the damping term in the Mathieu equation to get damped Mathieu equation as follows:

$$\frac{d^2y}{dz^2} + \gamma\frac{dy}{dz} + \left(a - 2q\cos(2z)\right)y = 0 \qquad (2)$$

Where γ is the damping term.

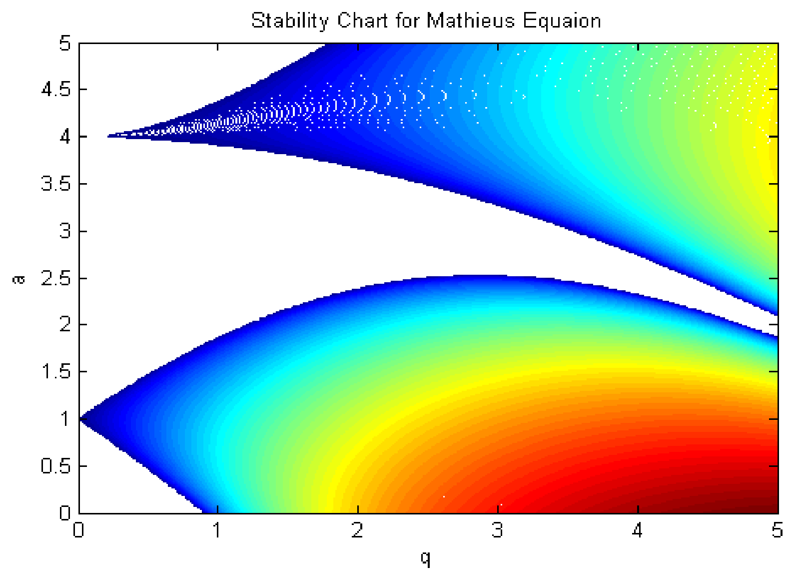

Figure 1: Stability Chart of Mathieu Equation

3. Parametric Oscillator

When a simple pendulum is excited at its pivot periodically in the vertical direction, it is called a parametric oscillator and it gives rise to Mathieu equation. A parametric oscillator is shown in figure 2.

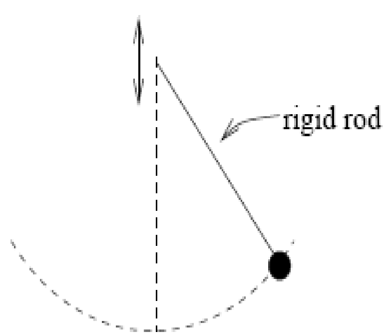

Figure 2: Parametric Oscillator

In parametric oscillator the external excitation enters into the system and gives a homogenous second order differential equation with periodic coefficients which explains the behavior the system. Let the simple pendulum be excited periodically in the vertical direction as follows

$$y(t) = hcos(2\omega t) \tag{3}$$

Where h is the amplitude and ω is the frequency of oscillation. Since the entire pendulum accelerates in the vertical direction, the net acceleration is given by

$$g - \ddot{y}(t) = g - 4\omega^2 hcos(2\omega t) \tag{4}$$

Therefore the equation of motion of pendulum is given by

$$ml^2\ddot{\theta} + m(g - \ddot{y})lsin\theta = 0 \tag{5}$$

On substituting for the vertical excitation from Eq (4), we get

$$ml^2\ddot{\theta} + m(g - 4\omega^2 hcos(2\omega t))lsin\theta = 0$$

$$\ddot{\theta} + \left(\frac{g}{l} - \frac{4\omega^2 h}{l}\cos(2\omega t)\right)\sin\theta = 0 \tag{6}$$

Let z = ωt, on substituting this in Eq (6), we get

$$\frac{d^2\theta}{dz^2} + \left(a - 2q\cos(2z)\right)\sin\theta = 0 \tag{7}$$

Where $a = \frac{g}{l\omega^2} = \frac{\omega_0^2}{\omega^2}$ and $q = \frac{2h}{l}$. Now a, q have become dimension less parameters. Equation (7) is Mathieu equation.

Adding a dimensionless damping term γ into Eq (7), we get damped Mathieu equation, as follows

$$\frac{d^2\theta}{dz^2} + \gamma\frac{d\theta}{dz} + \left(a - 2q\cos(2z)\right)\sin\theta = 0 \tag{8}$$

The stability chart for the equation (7) is shown in figure 3. It is same as shown in figure 1, but here only the boundaries which separate stable and unstable regions are shown.

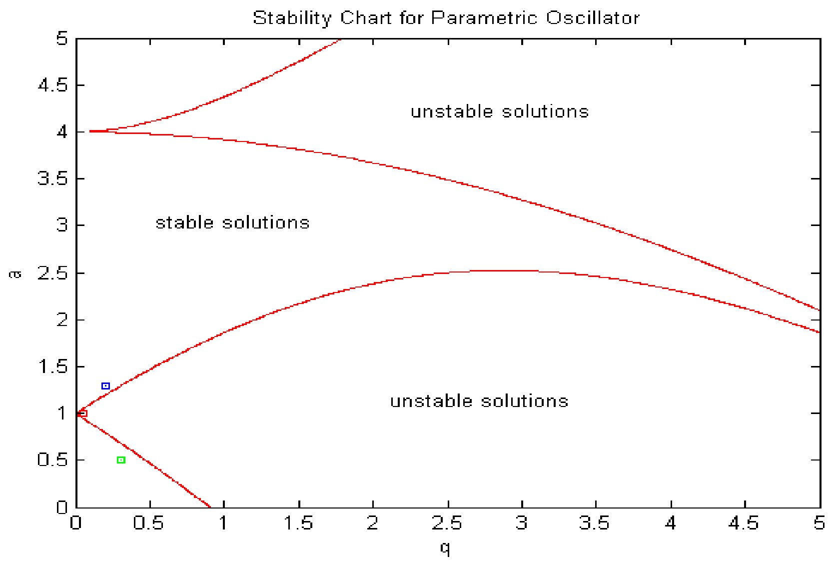

Figure 3 : Stability Chart for the Parametric Oscillator

4. Numerical Analysis of Mathieu Equation

In this section Mathieu equation i.e. the parametric oscillator equation is solved numerically for time history. It is initial boundary value problem. The equation can be time integrated using methods like Euler method, Runge-Kutta method using initial boundary values. Runge-Method is the most accurate method. To use this method the equation is transferred into a system of two differential equations of first order as follows:

$$\frac{d\theta}{dz} = y$$

$$\frac{dy}{dz} = -\gamma \frac{d\theta}{dz} - \big(a - 2q\cos(2z)\big)\sin\theta \qquad (9)$$

To solve the above system of equation initial boundary values are needed.

5. Code for the Mathieu Equation

A code is written for the Mathieu equation/ parametric oscillator to get the solution. By plotting the phase planes of the equation the stability and instability of the equation can be understood. The results obtained using the code for the Mathieu equation is shown here for three different cases. These cases are shown in square markers in figure 3.

Figure 4 shows the behavior of the oscillator for the conditions a = 1.3, q = 0.2 without damping and initial conditions as y=0, y’=0.3. The position is shown in blue color square in figure 3. This point lies in the stable region. The initial condition is shown in red circle in the figure.

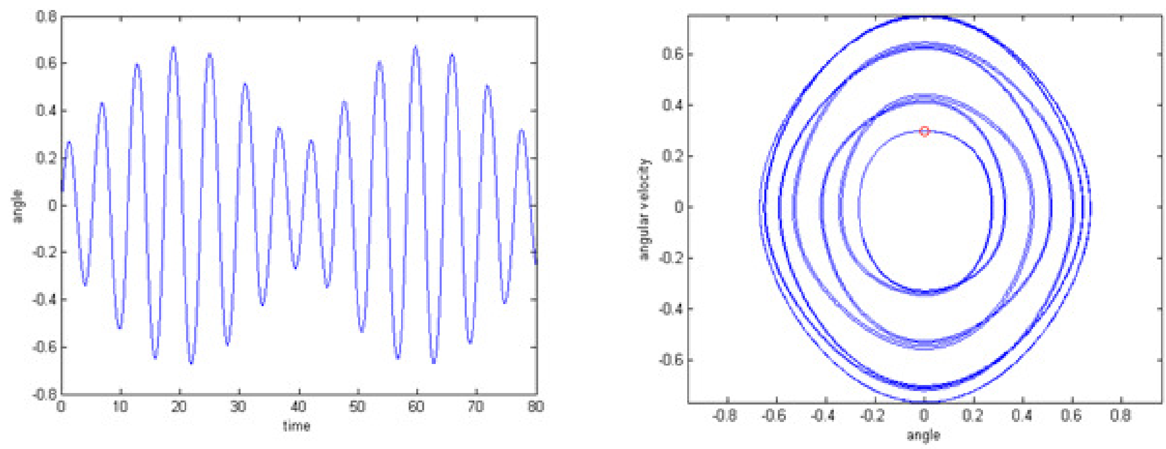

Figure 4 : Beating behaviour of solution

In this case of a, q the oscillator oscillates indefinitely, with the large amplitude of oscillations showing beating like behavior between the systems frequency ω_0 and the external excitation frequency ω. Typically the solution are aperiodic.

Figure 5 shows the behavor of the oscillator for the values, a = 1, q = 0.05 without damping and initial conditions as y = 0, y’ = 0.25. The position of this point in the stability chart figure 3 is shown in red color square. This point lies in the instable region.

From figure 5, it is clear that the amplitude of the oscillation keeps on increasing exponential, showing the instability. Phase plane plot shows that the solution moves away from the stable point. This instability is due to a phenomenon called parametric resonance. Parametric resonance takes place when the external excitation frequency approaches integral multiple of the systems natural frequency. This is a nonlinear resonance phenonmenon. This way different from the normal resonance which takes place when the excitation frequency and fundamental frequency are equal. In most of the cases this parametric resonance is catastrophic. This instability will not be effected by damping as well. Damping only may reduce the rate of increase of amplitude.

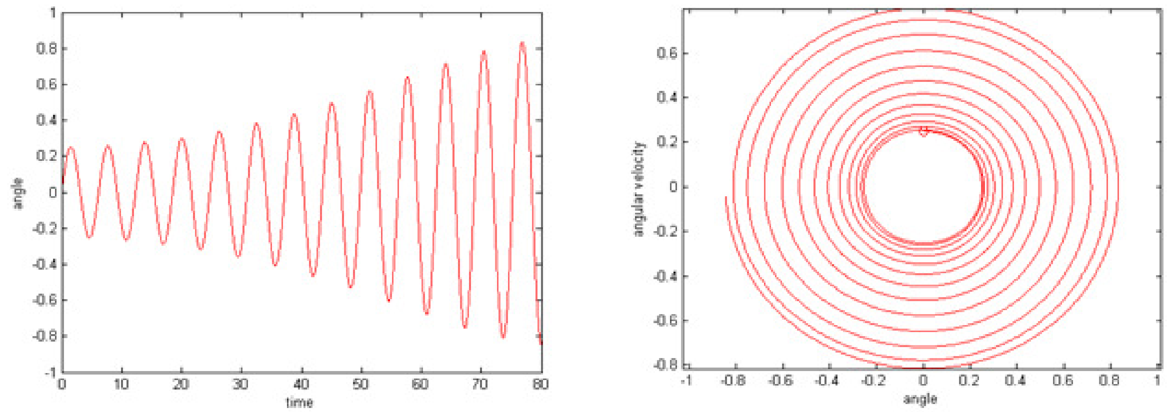

Figure 5: Instability behavior of the solution

Figure 6 shows the behavior of the solution for the same values as above but with damping, the damping value being γ = 0.05. It is clear that the solution in the instable region in presence of damping

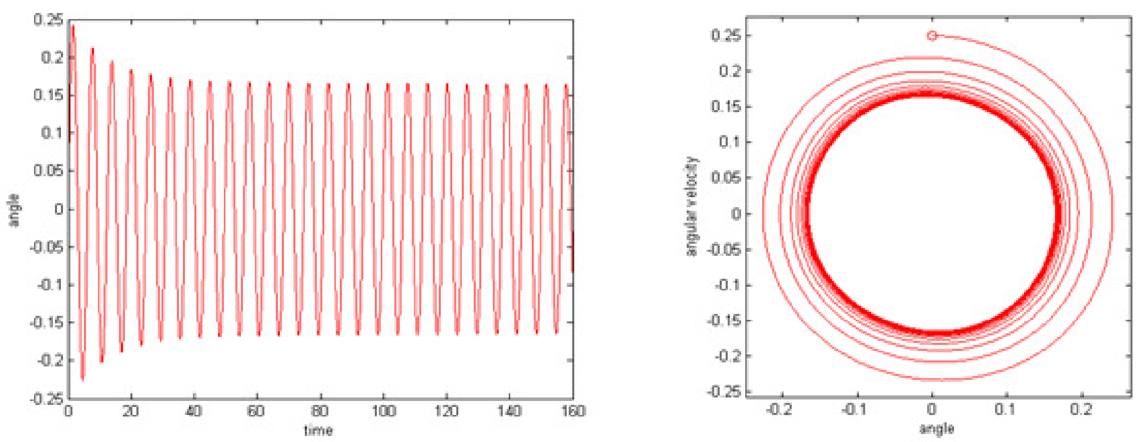

Figure 6: Behavior of solution in instability region with damping

the amplitude of oscillation is decaying very slowly. In the phase plane the solution is coming to stable focus very slowly. In case of high values of a, q the oscillations may not decay even in presence of damping. Damping has no effect on the instability apart from only reducing the rate of increase of amplitude of oscillation.

Figure 7 shows the behavior of the solution for the values a = 0.5, q = 0.3 without damping and initial conditions as y= 0, y' = 0.2. The position of this point in the stability chart figure 3 is shown in green color square. This point lies in the stable region.

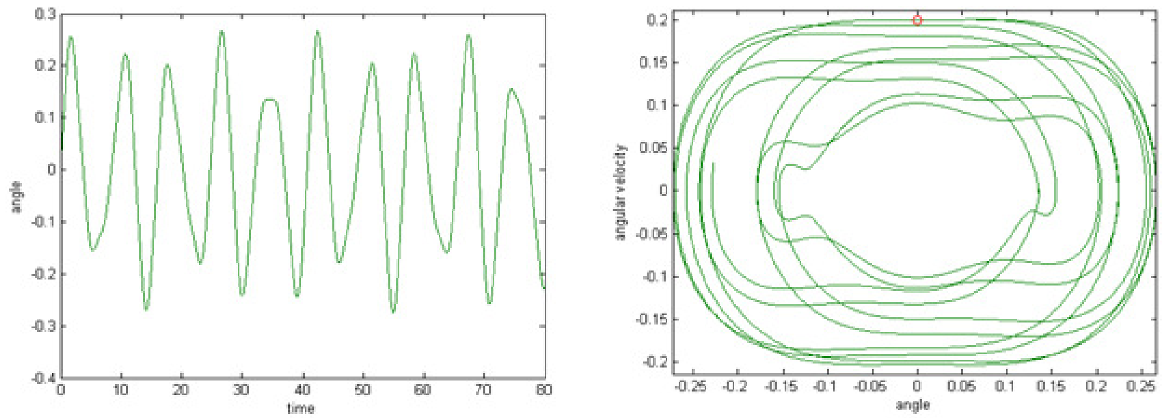

Figure 7: Stable behavior of solution

In this case the values of a, q are such that the point is in stable region, thus the solution is stable, periodic with period 2T and bounded. From phase plane, it is clear that the solution oscillates around the stable focus. If damping is present, the oscillator will come to stable focus.

For the higher values of a, q lying in the unstable region, the solutions or the oscillator shows chaos behavior. The reader can change the values of a, q and study the behavior of the oscillator and the solution.

For any discussions, advice, bugs, developing the code please feel free to mail me. Please share your experience with the code by commenting or rating.