Mini-Project

Stability of Parametrically Excited Coplanar Pendulum

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Stability of 2^{nd} order ODE

$$\ddot{x} + a(t)x = 0$$

$a(t) \longrightarrow$ periodic with period T.

Let $x_1 = x$ & $x_2 = x_1'$

$$\therefore \boxed{\begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -a(t) & \cancel{t}0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}} \Rightarrow \boxed{\ddot{x}(t) = A(t)\,x(t)}$$

Choosing initial Condition,

$$\begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

we obtain a solution of the form $\begin{bmatrix} x_1^{(1)}(t) \\ x_1'^{(1)}(t) \end{bmatrix}$

Choosing initial condition,

$$\begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

we obtain a solution of the form $\begin{bmatrix} x_1^{(2)}(t) \\ x_1'^{(2)}(t) \end{bmatrix}$

$$\therefore X(0) = I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$B = X(T) = \begin{bmatrix} x_1^{(1)}(T) & x_1^{(2)}(T) \\ x_1'^{(1)}(T) & x_1'^{(2)}(T) \end{bmatrix}$$

$$\boxed{\ddot{X}(t) = +A(t)\,X(t)}$$

Now from Floquet's Theorem,

$$\rho_1 \rho_2 = \exp\left(\int_0^T \mathrm{tr}(A(s))\,ds\right)$$

$$\therefore \boxed{\rho_1 \rho_2 = 1}$$

$$\rho_1 + \rho_2 = \mathrm{tr}(B) = x_1^{(1)}(T) + x_1'^{(2)}(T)$$

$$\boxed{\phi = \frac{\mathrm{tr}(B)}{2}}$$

$$\therefore \boxed{\rho_1\rho_2 = 1} \ \& \ \boxed{\rho_1+\rho_2 = 2\Phi}$$

$$\therefore \rho = \Phi \pm \sqrt{\Phi^2 - 1}$$

$$\rho_i = e^{\mu_i T} \Rightarrow \boxed{\mu_1+\mu_2 = 0 \ ; \ e^{\mu_1 T} + e^{\mu_2 T} = 2\Phi}$$

$$\therefore \boxed{\cosh(\mu_1 T) = \Phi}$$

1] For $\Phi > 1 \Rightarrow$ we have an unstable sol^n of the form,

$$\boxed{x(t) = c_1 e^{\mu_1 t} p_1(t) + c_2 e^{-\mu_1 t} p_2(t)}$$

2] For ~~$-1<\Phi<1$~~ $\Phi = 1 \Rightarrow$ we have an unstable sol^n

$$\boxed{x(t) = (c_1 + t c_2)\, p_1(t) + c_2 p_2(t)}$$

3] For ~~$\Phi=1$~~ $(-1<\Phi<1) \Rightarrow$ we have a stable periodic (pseudo) sol^n

$$\boxed{x(t) = c_1 \operatorname{Re}\left(e^{i\sigma t} p(t)\right) + c_2 \operatorname{Im}\left(e^{i\sigma t} p(t)\right)}$$

4] For $\Phi = -1 \Rightarrow$ we have an unstable sol^n

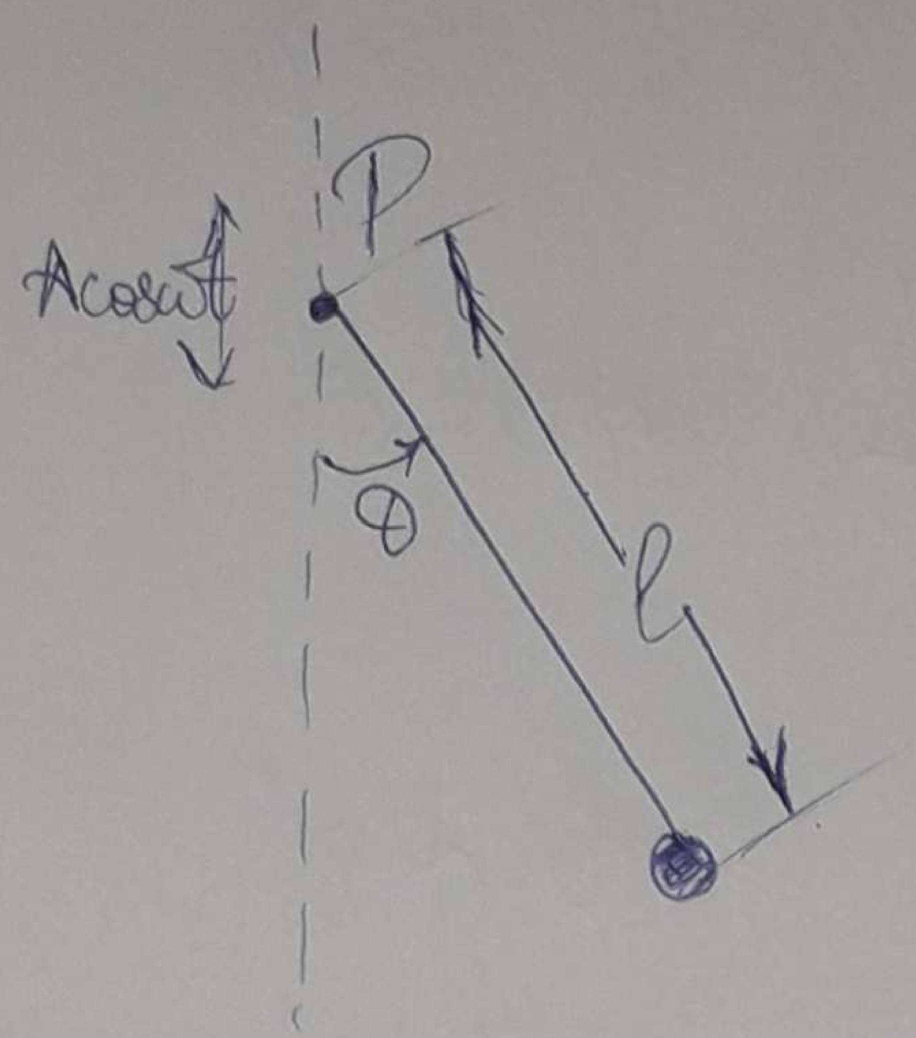
$$\boxed{x(t) = (c_1 + t c_2)\, q_1(t) + c_2 q_2(t)}$$

5] For $\Phi < -1 \Rightarrow$ we have an unstable sol^n

$$\boxed{x(t) = c_1 e^{\delta t} q_1(t) + c_2 e^{-\delta t} q_2(t)}$$

$p_i(t) \longrightarrow$ period T ; $q_i(t) \longrightarrow$ period $2T$

Pendulum with Pivot oscillating vertically



Point P is oscillating vertically with $\boxed{y = A\cos\omega t}$

$$x = l\sin\theta \quad ; \quad \dot{x} = (l\cos\theta)\,\dot{\theta}$$

$$y = A\cos\omega t - l\cos\theta$$

$$\dot{y} = -A\omega\sin\omega t + (l\sin\theta)\,\dot{\theta}$$

V = Potential Energy = $mg\left[A\cos\omega t - l\cos\theta\right]$

T = Kinetic Energy = $\frac{1}{2}m\left[\dot{x}^2 + \dot{y}^2\right]$

$$= \frac{1}{2}m\left[l^2\cos^2\theta\,(\dot{\theta})^2 + l^2\sin^2\theta\,(\dot{\theta})^2 + A^2\omega^2\sin^2\omega t - 2Al\omega(\sin\omega t)(\sin\theta)\dot{\theta}\right]$$

$$T = \frac{1}{2}m\left[l^2(\dot{\theta})^2 + A^2\omega^2\sin^2\omega t - 2Al\omega(\sin\omega t)(\sin\theta)\dot{\theta}\right]$$

$$L = T - V$$

$$\boxed{L = \frac{1}{2}m\left[l^2(\dot{\theta})^2 + A^2\omega^2\sin^2\omega t - 2Al\omega(\sin\omega t)(\sin\theta)\dot{\theta}\right] - mg\left[A\cos\omega t - l\cos\theta\right]}$$

Now $\boxed{\dfrac{\partial L}{\partial \theta} = \dfrac{d}{dt}\left(\dfrac{\partial L}{\partial \dot{\theta}}\right)}$

$$\frac{\partial L}{\partial \theta} = -mAl\omega(\sin\omega t)\cdot(\cos\theta)\dot{\theta} - mgl\sin\theta$$

$$\frac{\partial L}{\partial \dot{\theta}} = ml^2\dot{\theta} - mAl\omega(\sin\omega t)(\sin\theta)$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) = ml^2\ddot{\theta} - mAl\omega(\sin\omega t)(\cos\theta)\dot{\theta} - mAl\omega^2(\cos\omega t)(\sin\theta)$$

∴ we have,

$$-\cancel{m}gl\sin\theta = \cancel{m}l^2\ddot{\theta} - \cancel{m}Al\omega^2(\cos\omega t)\sin\theta$$

$$\therefore \quad \ddot{\theta} + \left(\frac{g - A\omega^2\cos\omega t}{l}\right)\sin\theta = 0$$

Now for small θ, $\sin\theta \sim \theta$

$$\therefore \quad \boxed{\ddot{\theta} + \left(\frac{g - A\omega^2\cos\omega t}{l}\right)\theta = 0}$$

Undamped Motion $\longrightarrow$

$$\ddot{\theta} + \left(\frac{g - A\omega^2 \cos\omega t}{l}\right)\theta = 0$$

Let $x = \theta - \pi$, so $|x| \ll 1$; $\theta_{max} \approx \pi$ (near the top)

$$\therefore \ddot{x} + \left(\frac{-g + A\omega^2 \cos\omega t}{l}\right) x = 0$$

Let $2\tau = \omega t$; $\delta = \frac{-4g}{\omega^2 l}$; $\epsilon = \frac{4A}{l}$

$$\therefore \boxed{\ddot{x} + (\delta + \epsilon\cos 2\tau)x = 0} \quad ; \text{ small } x$$

$\hookrightarrow$ Mathieu's Eq^n.

Stability Boundary of Mathieu's Eq^n

At the edge of region of stability,

$\Phi = 1$ or $\Phi = -1$

$\Phi = 1$ $\rightarrow$ periodic sol^s with T

$\Phi = -1$ $\rightarrow$ periodic sol^s with $2T$.

$$\ddot{x} + [\delta + \epsilon \cos 2t]x = 0$$

$\hookrightarrow$ To determine Region of Stability in δ-ϵ plane.

We need to have sol^n periodic with period either π or 2π.

F_n^{s} of Period π

$$x = \sum_{n=0}^{\infty} a_n \cos(2nt) + \sum_{n=1}^{\infty} b_n \sin(2nt)$$

$$0 = \ddot{x} + [\delta + \epsilon \cos 2t]x$$

$$0 = \sum_{n=0}^{\infty} (\delta - 4n^2) a_n \cos 2nt + \sum_{n=1}^{\infty} (\delta - 4n^2) b_n \sin 2nt$$

$$+ \epsilon \sum_{n=0}^{\infty} a_n \cos(2nt) \cdot \cos 2t + \epsilon \sum_{n=1}^{\infty} b_n \sin(2nt) \cdot \cos 2t$$

Using Trigonometric Identities,

$$\therefore 0 = \sum_{n=0}^{\infty} (\delta - 4n^2) a_n \cos(2nt) + \sum_{n=1}^{\infty} (\delta - 4n^2) b_n \sin(2nt)$$

$$+ \frac{\epsilon}{2} \sum_{n=0}^{\infty} a_n \left[\cos(2(n+1)t) + \cos(2(n-1)t)\right]$$

$$+ \frac{\epsilon}{2} \sum_{n=1}^{\infty} b_n \left[\sin(2(n+1)t) + \sin(2(n-1)t)\right]$$

$\therefore$ we must have

$$0 = \sum_{n=0}^{\infty} (\delta - 4n^2) a_n \cos(2nt) + \frac{\epsilon}{2} \sum_{n=0}^{\infty} a_n \left[\cos(2(n+1)t) + \cos(2(n-1)t)\right]$$

$$0 = \left(\delta a_0 + \frac{\epsilon}{2} a_1\right) + \left[(\delta - 4) a_1 + \frac{\epsilon}{2}(2a_0 + a_2)\right] \cos 2t$$

$$+ \sum_{n=2}^{\infty} \left[(\delta - 4n^2) a_n + \frac{\epsilon}{2}(a_{n-1} + a_{n+1})\right] \cos 2nt$$

&

$$0 = \sum_{n=1}^{\infty} (\delta - 4n^2) b_n \sin 2nt + \frac{\epsilon}{2} \sum_{n=1}^{\infty} b_n \left[\sin(2(n+1)t) + \sin(2(n-1)t)\right]$$

$$0 = \left[(\delta - 4) b_1 + \frac{\epsilon}{2} b_2\right] \sin 2t + \sum_{n=2}^{\infty} \left[(\delta - 4n^2) b_n + \frac{\epsilon}{2}(b_{n-1} + b_{n+1})\right] \sin(2nt)$$

$\therefore$ By orthogonality of sine & cosine,

$$\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ \vdots \end{bmatrix} = \begin{bmatrix} \delta & \epsilon/2 & & & 0 \\ \epsilon & \delta-4\cdot1^2 & \epsilon/2 & & \\ \vdots & \epsilon/2 & \delta-4\cdot2^2 & \epsilon/2 & \\ 0 & & \epsilon/2 & \delta-4\cdot3^2 & \epsilon/2 \ \ldots \\ & & & \ddots & \ddots \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ \vdots \end{bmatrix}$$

&

$$\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ \vdots \end{bmatrix} = \begin{bmatrix} \delta-4\cdot1^2 & \epsilon/2 & & 0 \\ \epsilon/2 & \delta-4\cdot2^2 & \epsilon/2 & \\ & \epsilon/2 & \delta-4\cdot3^2 & \epsilon/2 \\ \vdots & & \ddots & \ddots \\ 0 & & & \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ \vdots \end{bmatrix}$$

In order to have a non-zero sol^n, the determinant of at least one of the matrices must be 0.

This gives the relation btw ϵ & δ

We can approximate the det by the det of finite $(n\times n)$ matrices of the same form.

Fn^s of Period 2π

$$x = \sum_{\substack{n=1 \\ odd}}^{\infty} a_n \cos(nt) + \sum_{\substack{n=1 \\ odd}}^{\infty} b_n \sin(nt)$$

$$0 = \ddot{x} + [\delta + \epsilon \cos 2t]\, x$$

$$0 = \sum_{\substack{n=1 \\ odd}}^{\infty} (\delta - n^2) a_n \cos nt + \sum_{\substack{n=1 \\ odd}}^{\infty} (\delta - n^2) b_n \sin nt$$

$$+ \frac{\epsilon}{2} \sum_{\substack{n=1 \\ odd}}^{\infty} a_n \left[\cos(n+2)t + \cos(n-2)t\right]$$

$$+ \frac{\epsilon}{2} \sum_{n=1\, odd}^{\infty} b_n \left[\sin(n+2)t + \sin(n-2)t\right]$$

We must then have,

$$0 = \sum_{\substack{n=1 \\ odd}}^{\infty} \left[(\delta-1)a_1 + \frac{\epsilon}{2}(a_1+a_3)\right]\cos t + \sum_{\substack{n=1 \\ odd}}^{\infty} \left[(\delta-n^2)a_n + \frac{\epsilon}{2}\left(a_{n-2}+a_{n+2}\right)\right]\cos nt$$

$$0 = \sum_{\substack{n=1 \\ odd}}^{\infty} \left[(\delta-1)b_1 + \frac{\epsilon}{2}(-b_1+b_3)\right]\sin t + \sum_{\substack{n=1 \\ odd}}^{\infty} \left[(\delta-n^2)b_n + \frac{\epsilon}{2}\left(b_{n-2}+b_{n+2}\right)\right]\sin nt$$

∴ We can write

$$\begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \end{bmatrix} = \begin{bmatrix} \delta-1^2+\epsilon/2 & \epsilon/2 & & & 0 \\ \epsilon/2 & \delta-3^2 & \epsilon/2 & & \\ & \epsilon/2 & \delta-5^2 & \epsilon/2 & \\ \vdots & & & \ddots & \\ 0 & & & & \ddots \end{bmatrix} \begin{bmatrix} b_1 \\ b_3 \\ b_5 \\ \vdots \end{bmatrix}$$

&

$$\begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \end{bmatrix} = \begin{bmatrix} \delta-1^2+\epsilon/2 & \epsilon/2 & & & 0 \\ \epsilon/2 & \delta-3^2 & \epsilon/2 & & \\ & \epsilon/2 & \delta-5^2 & \epsilon/2 & \\ \vdots & & & \ddots & \\ 0 & & & & \ddots \end{bmatrix} \begin{bmatrix} a_1 \\ a_3 \\ a_5 \\ \vdots \end{bmatrix}$$

∴ $\underline{\underline{a_{even}}}$:
$$\begin{vmatrix} \delta & \epsilon/2 & & & 0 \\ \epsilon & \delta-4\cdot1^2 & \epsilon/2 & & \\ & \epsilon/2 & \delta-4\cdot2^2 & \epsilon/2 & \\ & & & \ddots & \\ 0 & & & & \ddots \end{vmatrix} = 0$$

$\underline{\underline{b_{even}}}$:
$$\begin{vmatrix} \delta-4\cdot1^2 & \epsilon/2 & & & 0 \\ \epsilon/2 & \delta-4\cdot2^2 & \epsilon/2 & & \\ & \epsilon/2 & \delta-4\cdot3^2 & \epsilon/2 & \\ & & \ddots & & \\ 0 & & & \ddots & \end{vmatrix} = 0$$

$\underline{\underline{a_{odd}}}$:

$$\begin{vmatrix} \delta-1^2+\epsilon/2 & \epsilon/2 & & & 0 \\ \epsilon/2 & \delta-3^2 & \epsilon/2 & & \\ & \epsilon/2 & \delta-5^2 & \epsilon/2 & \\ & & \ddots & \ddots & \ddots \\ 0 & & & \ddots & \end{vmatrix} = 0$$

$\underline{\underline{b_{odd}}}$:

$$\begin{vmatrix} \delta-1^2-\epsilon/2 & \epsilon/2 & & & 0 \\ \epsilon/2 & \delta-3^2 & \epsilon/2 & & \\ & \epsilon/2 & \delta-5^2 & \epsilon/2 & \\ & & \ddots & \ddots & \ddots \\ 0 & & & \ddots & \end{vmatrix} = 0$$

These are also called Hill's Determinants.

$$\delta = n^2 + \delta_1\epsilon + \delta_2\epsilon^2 + \dots$$

Expanding a (3×3) truncation of a_{odd} determinant

$$\begin{vmatrix} \delta-1+\epsilon/2 & \epsilon/2 & 0 \\ \epsilon/2 & \delta-9 & \epsilon/2 \\ 0 & \epsilon/2 & \delta-25 \end{vmatrix}$$

$$\Rightarrow \left(\delta-1+\epsilon/2\right)\left[(\delta-9)(\delta-25)-\frac{\epsilon^2}{4}\right] - \frac{\epsilon}{2}\left[\frac{\epsilon}{2}(\delta-25)\right]$$

$$\Rightarrow \frac{-\epsilon^3}{8} - \frac{\delta\epsilon^2}{2} + \frac{13\epsilon^2}{2} + \frac{\delta^2\epsilon}{2} - 17\delta\epsilon + 225\frac{\epsilon}{2} + \delta^3 - 35\delta^2 + 259\delta - 225$$

Substituting for δ with $n=1$

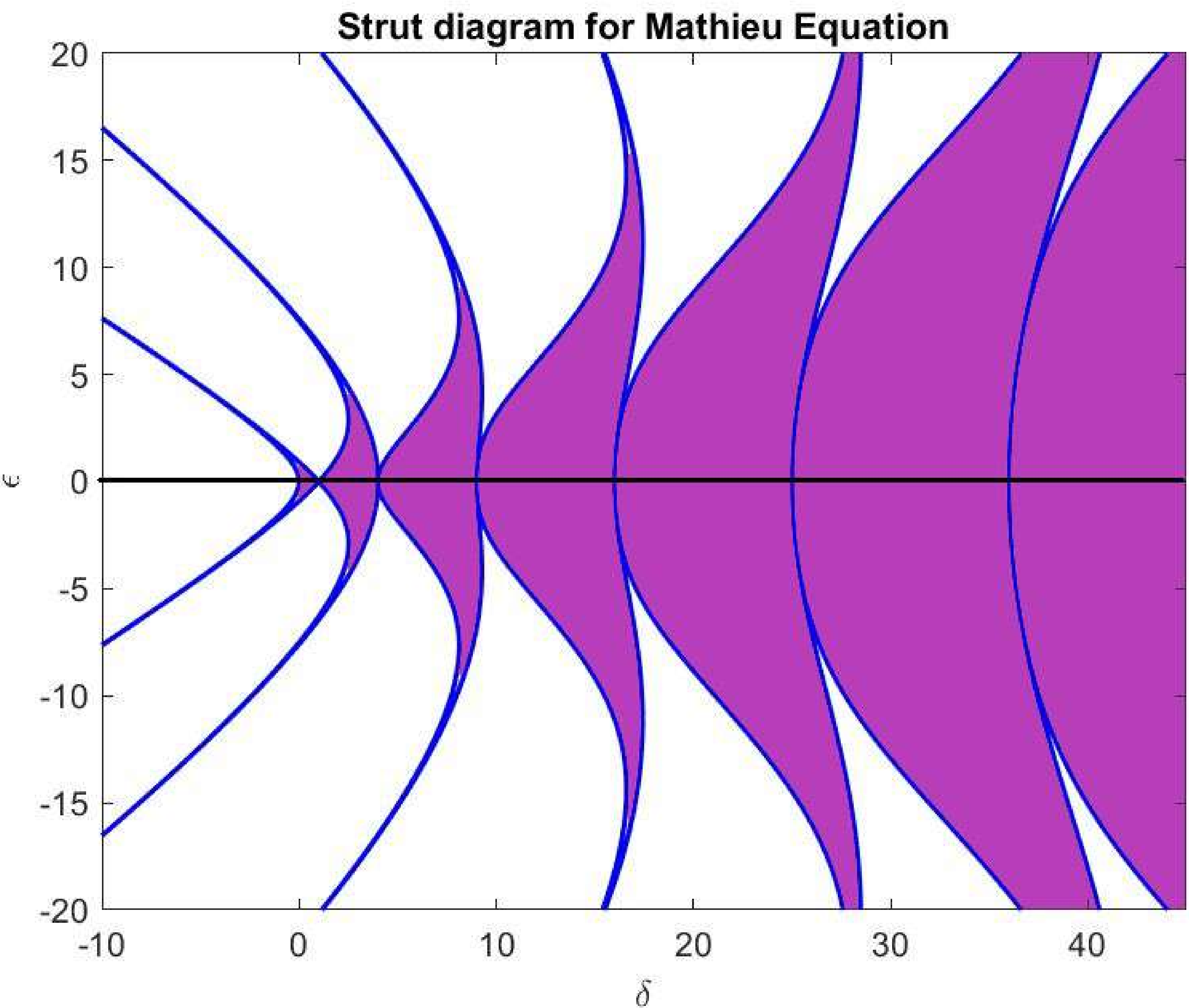
& collecting terms gives

$$(12\delta_1+6)\epsilon + \frac{(24\delta_2-16\delta_1^2-8\delta_1+3)\epsilon^2}{2} + \dots$$

Requiring Coeff's of ϵ & ϵ^2 to vanish,

$$\delta_1 = -\frac{1}{2} \quad ; \quad \delta_2 = -1/8$$

The process can be continued to any order of truncation. Here are first few transition curves.


Strut diagram for Mathieu Equation
20
15
10
5
0
-5
-10
-15
-20
-10
0
10
20
30
40
ϵ
δ

$$\delta = -\frac{\epsilon^2}{2} + \frac{7\epsilon^4}{32} - \frac{29\epsilon^6}{144} + \frac{68687\epsilon^8}{294912} - \frac{123707\epsilon^{10}}{409600} + \cdots$$

$$\delta = 1 - \frac{\epsilon}{2} - \frac{\epsilon^2}{8} + \frac{\epsilon^3}{32} - \frac{\epsilon^4}{384} - \frac{11\epsilon^5}{4608} + \frac{49\epsilon^6}{36864} + \cdots$$

$$\delta = 1 + \frac{\epsilon}{2} - \frac{\epsilon^2}{8} - \frac{\epsilon^3}{32} - \frac{\epsilon^4}{384} + \frac{11\epsilon^5}{4608} + \frac{49\epsilon^6}{36864} + \cdots$$

$$\delta = 1 - \frac{\epsilon^2}{12} + \frac{5\epsilon^4}{3456} - \frac{289\epsilon^6}{4976640} + \cdots$$

The Ince-Strutt diagram is plotted.
Stable Regions are bounded in the course of time.
Unbounded Regions are unbounded in the course of time.

Hence Shaded region → Stable
Unshaded Region → Unstable

Also $\delta = -\frac{4g}{\omega^2 l}$; $\epsilon = \frac{4A}{l}$

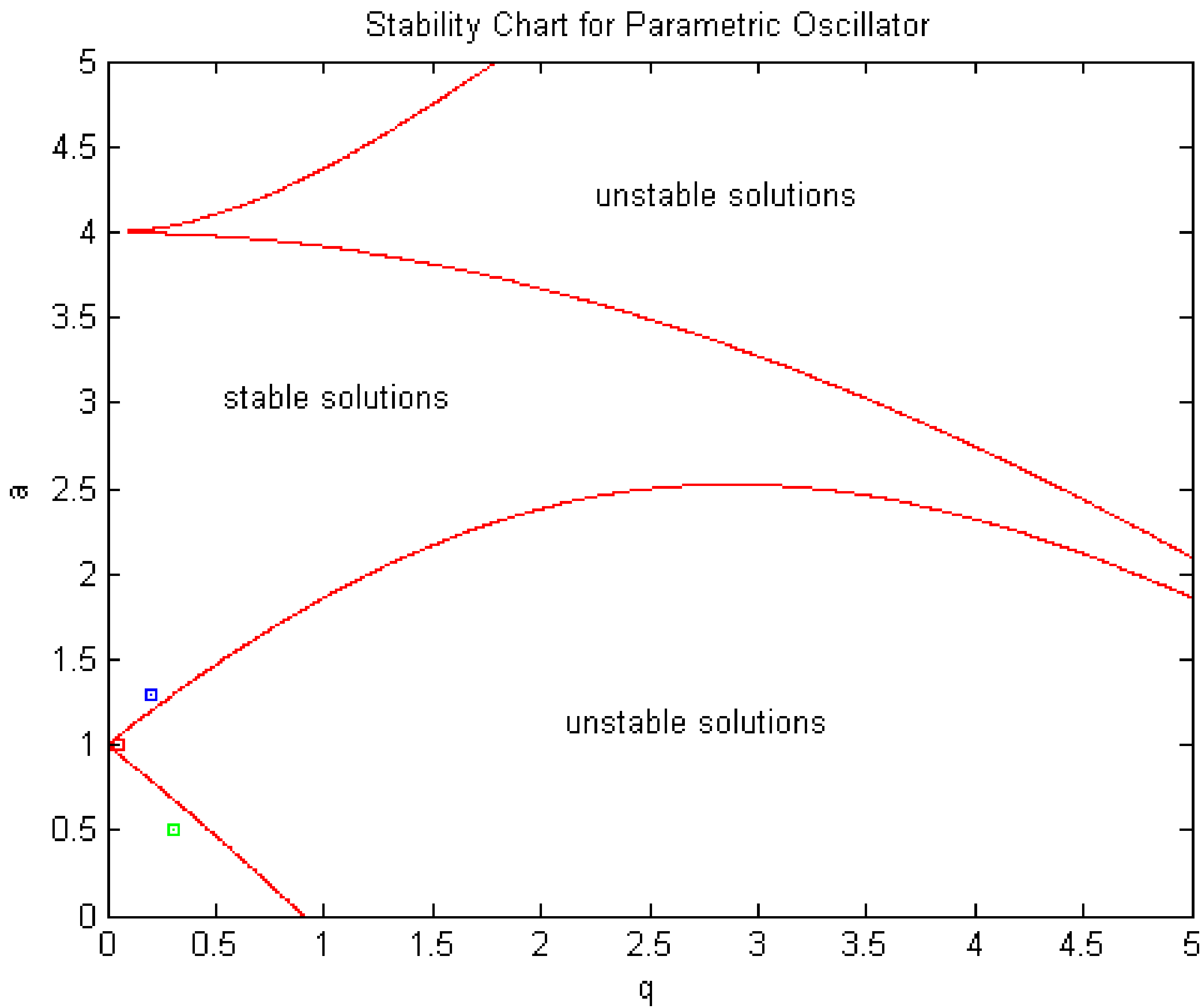
It is clear that $\epsilon > 0$ & $\delta < 0$.

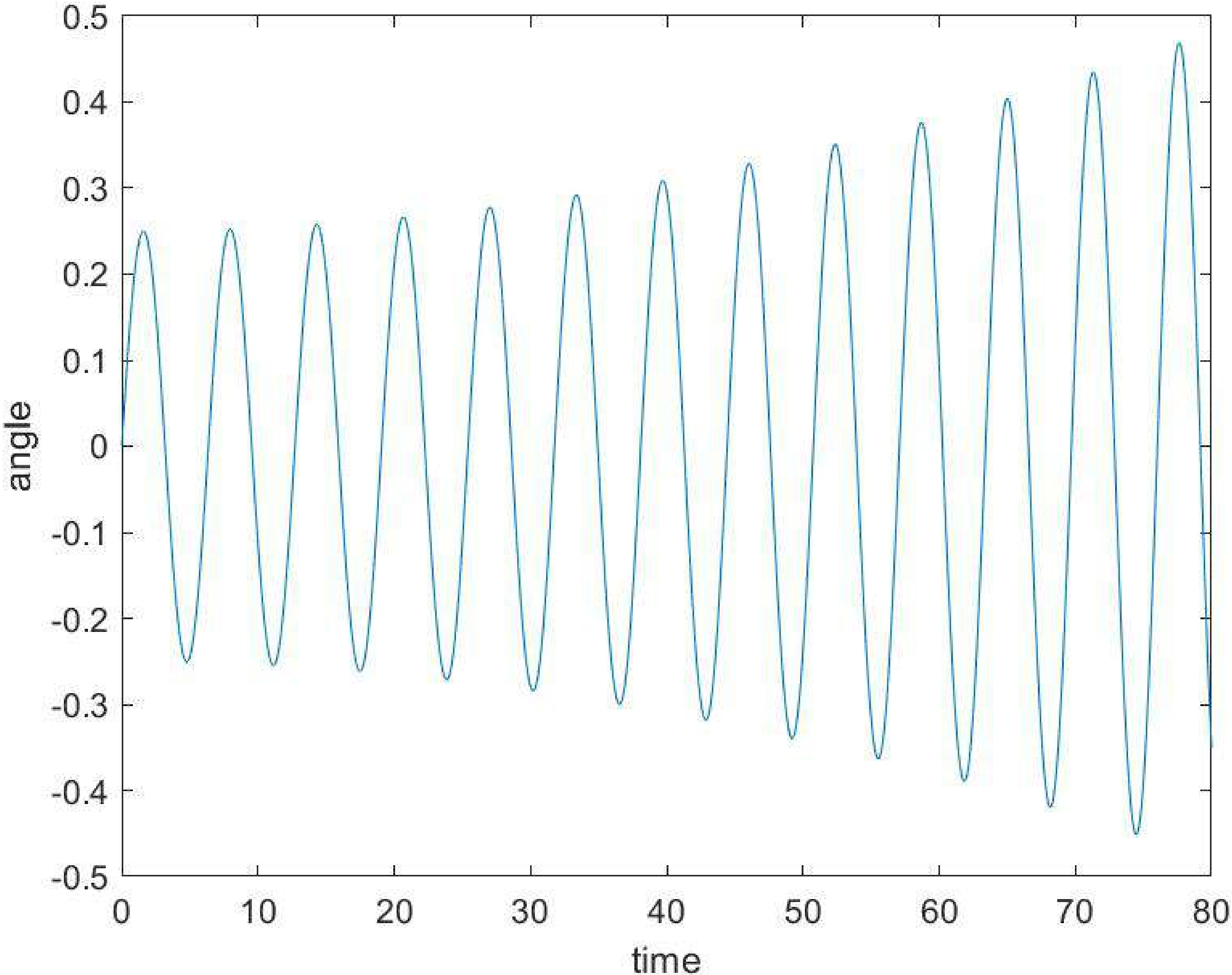
A code is written for the Parametric Oscillator to get the sol^n. The results obtained is also shown. I have taken →

I) $\delta_0 = 1$; $\epsilon_0 = 0.05$

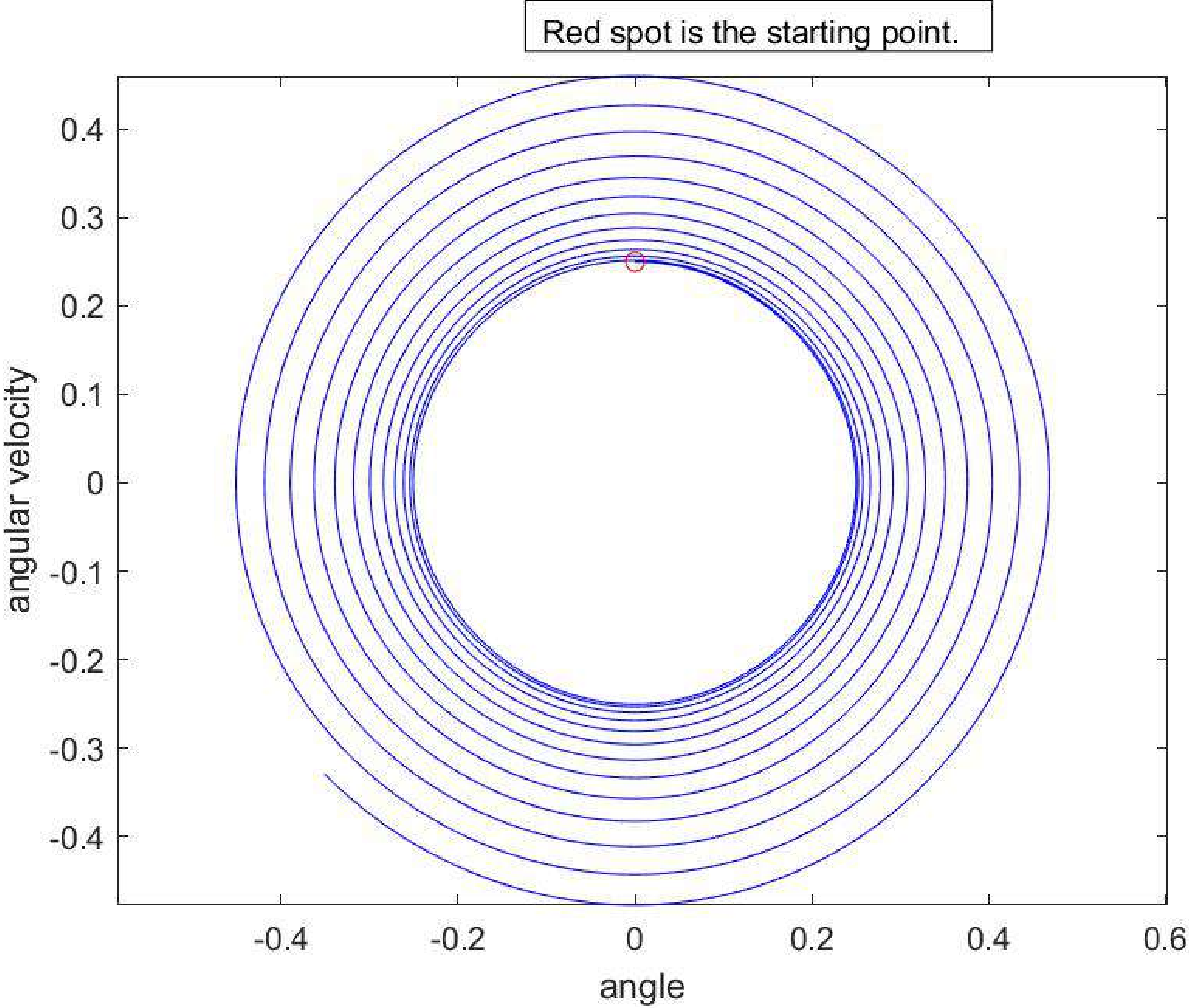
Initial angle $= \theta_0 = 0$; $\omega_0 = 0.25$ (rad/s) = Initial Angular velocity.

∴ <u>Unstable sol^n</u>


Stability Chart for Parametric Oscillator
unstable solutions
stable solutions
unstable solutions
a
q
0
0.5
1
1.5
2
2.5
3
3.5
4
4.5
5



0.5
0.4
0.3
0.2
0.1
0
-0.1
-0.2
-0.3
-0.4
-0.5
angle
0
10
20
30
40
50
60
70
80
time



Red spot is the starting point.
0.4
0.3
0.2
0.1
0
-0.1
-0.2
-0.3
-0.4
angular velocity
-0.4
-0.2
0
0.2
0.4
0.6
angle

It is clear that the amplitude of oscillation keeps on increasing exponentially, showing the instability. Phase plane plot shows that the sol^n moves away from the stable point. This instability is due to a phenomenon called Parametric Resonance. Parametric Resonance takes place when the external excitation frequency approaches integral multiples of the system's natural frequency. This is non-linear Resonance phenomenon. In most cases, Parametric Resonance is catastrophic. This Instability will not be affected by damping as well. Damping may only reduce the rate of increase of amplitude. For higher values of δ_0 & ϵ_0, chaotic behaviour is seen.

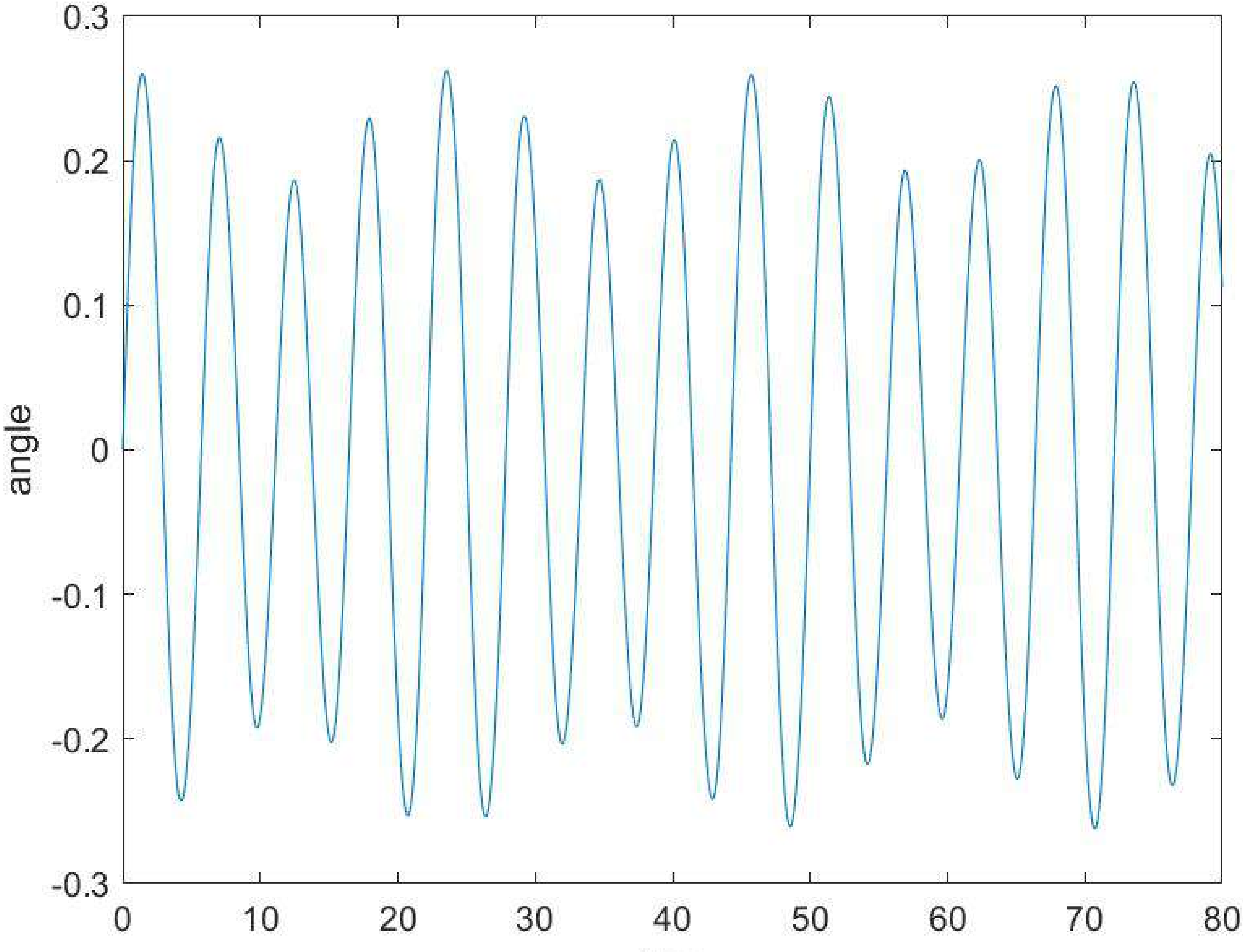
II) $\delta_0 = 0.2$; $\epsilon_0 = 1.3$

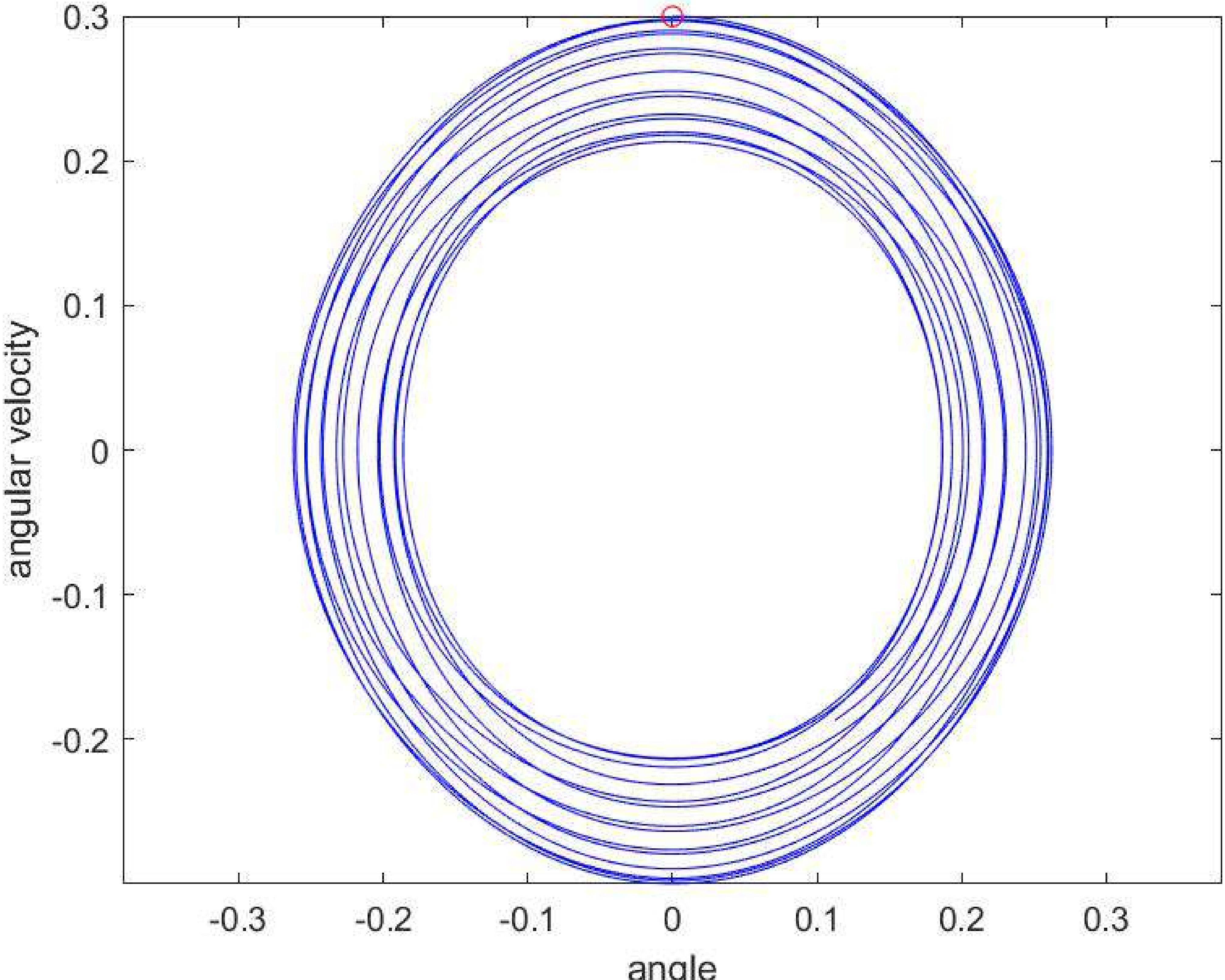
Initial angle $= \theta_0 = 0$; $\dot{\theta}_0 = 0.3$ rad/s = Initial Angular Velocity.

Stable Sol^n

In this case, the oscillator oscillates indefinitely, with the large amplitude of oscillations showing beating like behaviour between the system's frequency ω_0 & the external excitation frequency ω.

Typically the sol^n is aperiodic.


0.3
0.2
0.1
0
-0.1
-0.2
-0.3
angle
0
10
20
30
40
50
60
70
80
time



0.3
0.2
0.1
0
-0.1
-0.2
angular velocity
-0.3
-0.2
-0.1
0
0.1
0.2
0.3
angle