Page No.:
Date: / /

Q2. Theorem 3 Proof:

① Justify both inequalities:

$$\|\phi(x^*-x)\|_2 \leq \|\phi x^* - y\|_2 + \|y - \phi x\|_2 \leq 2\epsilon$$

$\Rightarrow$ by theorem 3 constraint $\|y - \phi x\|_2 \leq \epsilon$ (i)

$= \|\phi x^* - y\|_2 + \|y - \phi x\|_2$

= by using cauchy-schwartz inequality,
$\|v + w\| \leq \|v\| + \|w\|$.

$\therefore$ here $v = \phi x^* - y$ &
$w = y - \phi x$.

$\therefore$ we get,

$$\|\phi x^* - \cancel{y} + \cancel{y} - \phi x\|_2 \leq \|\phi x^* - y\|_2 + \|y - \phi x\|_2$$

$$\therefore \boxed{\|\phi x^* - \phi x\|_2 \leq \|\phi x^* - y\|_2 + \|y - \phi x\|_2.}$$ (a)

Prooving upper bound:

By theorem 3 constraint:

$\|y - \phi x\| \leq \epsilon$

$\therefore \|y - \phi x\|_2 \leq \epsilon$ & $\|\phi x^* - y\| \leq 3$

adding both equation, we get,

$$\boxed{\begin{aligned}\|\phi x^* - y\|_2 + \|y - \phi x\|_2 &\leq \epsilon + \epsilon \\ &\leq 2\epsilon\end{aligned}}$$

hence,

$$\|\phi(x^* - x)\|_2 \le \|\phi x^* - y\|_2 + \|y - \phi x\|_2 \le 2\epsilon$$

(2) Justify:

for j ≥ 2 $\quad \|h_{T_j}\|_2 \le \sqrt{s}\,\|h_{T_j}\|_\infty \le \frac{1}{\sqrt{s}}\|h_{T_{j-1}}\|_1$

Proof:

⇒ ~~Given~~: h : $\boxed{x^* = x + h}$.

Let h decompose into sum of vectors $h_{T_0}, h_{T_1}, \ldots$ each of sparsity "s".

T_0 ≡ location 's' largest cofficients of x

T_1 ≡ location 's' largest ~~cof~~ coeff. of $h_{T_0^c}$

so on...

(a) ~~j≥2~~ $\|h_{T_j}\|_2 = \left(\sum_{i \in T_j} (h_i)^2\right)^{1/2}$ (by def.)

→ Let 'p' be the max. value of "h_{T_j}"

i.e. $p = \max_{i \in T_j} (h_i)$

→ ∴ each value of $h_{T_j} \le p$.

$$\Rightarrow \sum_{i \in T_j} (h_i)^2 \le \sum_{i \in T_j} p^2.$$

Gurukul
Page No.:
Date: / /

but we know by definition J_i that every "h_{T_j} is 's' sparse".

$$\therefore \sum_{i \in T_j} h_i^2 \le s \cdot \sum_{\substack{\text{upto} \\ \text{'s' time}}} p^2$$

$$\sum_{i \in T_j} (h_i)^2 \le s \cdot p^2$$

$\Rightarrow$ as ~~bo~~ both sq. are +ve. take sq. root both side.

$$\underbrace{\left(\sum_{i \in T_j} (h_i)^2\right)^{1/2}}_{\downarrow \text{ by def}} \le \sqrt{s} \cdot p$$

$$\| h_{T_j} \|_2 \le \sqrt{s}\ \underbrace{\max_{i \in T_j} (h_i)}_{\text{by def. of } \|\cdot\|_\infty}$$

we get:

$$\boxed{\| h_{T_j} \|_2 \le \sqrt{s}\, \| h_j \|_\infty}$$

Scanned by CamScanner

Gurukul
Page No.:
Date:

(b) To prove : $\sqrt{s}\,\|h_{T_j}\|_\infty \leq \frac{1}{\sqrt{2}}\|h_{T_{j-1}}\|_1$

→ Let p be max value in h_{T_j}
i.e $p = \max_{i \in T_j} (h_i)$.

→ by def. of h_{T_j} that

~~w.r.t~~ $h_{T_{j-1}}$: it will have s longest value $h^c_{T_{j-2}}$

& h_{T_j} : it will have s largest value. $h^c_{T_{j-1}}$

which means, "highest value in h_{T_j} will less than every value of $h_{T_{j-1}}$.

∴ we can say:

$$\min_{i \in T_{j-1}} (h_i) \geq p$$

⇒ let $k = \min_{i \in T_{j-1}} (h_i)$

$\therefore k \geq p$

$= \cancel{k^2 \geq p^2}$

$= \cancel{\sum_{i \in T_j} k^2 \geq p^2/\sum_i}$

so, $\left(\sum_{i \in T_{j-1}} h_i\right) \geq p \left(\sum_{i \in T_{j-1}} p\right)$.

as T_{j-1} is 's' sparse, we get

$$\underbrace{\left(\sum_{i \in T_{j-1}} h_i\right)}_{\text{by def } \|\cdot\|_1} \geq s \cdot p$$

$$\|h_{T_{j-1}}\|_1 \geq \underbrace{s \cdot p}_{\text{by def } \|\cdot\|_\infty}$$

$$\frac{\|h_{T_{j-1}}\|_1}{\sqrt{s}} \geq \sqrt{s} \max_{i \in T_j} (h_i)$$

$$\therefore \frac{1}{\sqrt{s}} \|h_{T_{j-1}}\| \geq \sqrt{s} \|h_{T_j}\|_\infty$$

$$\Rightarrow \boxed{\sqrt{s} \|h_{T_j}\|_\infty \leq \frac{1}{\sqrt{s}} \|h_{T_{j-1}}\|_1}$$

by combining (a) & (b) result

$$\boxed{\|h_{T_j}\|_2 \leq (s)^{1/2} \|h_{T_j}\|_\infty \leq (s)^{-1/2} \|h_{T_{j-1}}\|_1}$$

hence proved.

Gurukul
Page No.:
Date:

③ Justify:

$$\sum_{j\geq 2} \|h_{T_j}\|_2 \leq (s)^{-1/2} (\|h_{T_1}\|_1 + \|h_{T_2}\|_1 + \cdots) \leq (s^{-1/2}) \|h_{T_0^c}\|_1$$

Proof

⇒ ④ $$\sum_{j\geq 2} \|h_{T_j}\|_2 \leq s^{-1/2} (\|h_{T_1}\| + \|h_{T_2}\| \cdots)$$

→ In prev. part ② we proved that

$$\|h_{T_j}\|_2 \leq (s)^{-1/2} \|h_{T_{j-1}}\|$$

So by using this property we get

$$\|h_{T_2}\|_2 \leq (s^{-1/2}) \|h_{T_1}\|_1$$

$$\|h_{T_3}\|_2 \leq (s)^{-1/2} \|h_{T_2}\|_1$$

all are ≥ zero

so on.

⇒ left & right ~~both~~ part are true, we can add them.

$$\therefore \sum_{j\geq 2} \|h_{T_j}\|_2 \leq (s)^{-1/2} \left(\sum_{j\geq 1} \|h_{T_j}\|_1\right)$$

i.e $$\boxed{\sum_{j\geq 2} \|h_{T_j}\|_2 \leq (s^{-1/2}) \left(\|h_{T_1}\| + \|h_{T_2}\|_1 \cdots\right)}$$

Hen

Gurukul
Page No.:
Date: \ \

(b) $(S)^{-1/2} (\|h_1\|_1 + \|h_2\|_1 \cdots) \leq (S^{1/2}) \|h_{T_0^c}\|_1$

$\Rightarrow$ by def. $h_{T_0^c} = h - h_{T_0}$

$\Rightarrow$ $T_0, T_1, T_2 \cdots$ all are "disjoint support"

$\therefore \|h\|_1 = \|h_{T_0} + h_{T_1} + h_{T_2} \cdots\|_1$ (i)

$\therefore \|h_{T_0^c}\|_1 = \|h - h_{T_0}\|_1$.

by triangle inverse in-equality
$\|V - W\| \geq \|V\| - \|W\|$.

$\therefore \|h - h_{T_0}\|_1 \geq \|h\|_1 - \|h_{T_0}\|_1$

by using (i) property of $\|h\|_1$

we get

$\|h - h_{T_0}\|_1 \geq \underbrace{\|h_{T_0} + h_{T_1} \cdots\|_1}_{*} - \|h_{T_0}\|$

by traiangle inequality ~~equal~~

$\|h_{T_0} + h_{T_1} \cdots\|_1 - \|h_{T_0}\| \leq (\cancel{\|h_{T_0}\|_1} + \|h_{T_1}\| \cdots) - \cancel{\|h_{T_0}\|}$

So we get,

$\|h - h_{T_0}\|_1 \leq (\|h_{T_1}\|_1 + \|h_{T_2}\| \cdots)$.

i.e. $(S)^{-1/2} \underbrace{\|h - h_{T_0}\|_1}_{= \|h_{T_0^c}\|_1} \leq (S)^{1/2} \cdot (\|h_{T_1}\|_1 + \|h_{T_2}\| \cdots$

by combining (a) & (b) result we got

$$\sum_{j\geq 2} \|h_{T_j}\|_2 \leq (S)^{-1/2} (\|h_{T_1}\|_1 + \|h_{T_2}\|_1 \cdots) \leq (S)^{-1/2} \|(S)^{-1/2} \|h_{T_0^c}\|_1$$

hence, proved.

(a) Justify inequality

$$\|h_{(T_0 \cup T_1)^c}\|_2 = \|\sum_{j\geq 2} h_{T_j}\|_2 \leq \sum_{j\geq 2} \|h_{T_j}\|_2 \leq s^{-1/2} \|h_{T_0^c}\|_1$$

Proof.

$\Rightarrow$ by def. $h_{(T_0 \cup T_1)^c} = h - h_{T_0} - h_{T_1}$
$= h_{T_2} + h_{T_3} \cdots\cdots$

$\therefore \|h_{(T_0 \cup T_1)^c}\|_2 = \|\sum_{j\geq 2} h_{T_j}\|_2$

(a) By triangle inequality
$\|V + W\| \leq \|V\| + \|W\|$.

$\therefore \|h_{T_2} + h_{T_3} \cdots\|_2 \leq \|h_{T_2}\|_2 + \|h_{T_3}\|_2 \cdots$

$$\boxed{\|h_{T_2} + h_{T_3} \cdots\|_2 \leq \sum_{j\geq 2} \|h_{T_j}\|_2}$$

(b) In previous we proved that

$$\sum_{j\geq 2} \|h_{T_j}\|_2 \leq \cancel{\|h\|} (S)^{-1/2} \|h_{T_0^c}\|_1$$

Gurukul
Page No.:
Date: / /

$\therefore$ by combining ⓐ & ⓑ we get

$$\underbrace{\Big\| \sum_{j \geq 2} h_{T_j} \Big\|_2}_{\text{by def. } \| h_{(T_0 \cup T_1)^c} \|_2} \leq \sum_{j \geq 2} \| h_{T_j} \|_2 \leq s^{-1/2} \| h_{T_0^c} \|_1$$

so we get,

$$\| h_{(T_0 \cup T_1)^c} \|_2 \leq \sum_{j \geq 2} \| h_{T_j} \|_2 \leq s^{-1/2} \| h_{T_0^c} \|_2 .$$

Hence proved.

⑤ Justify "inequality."

$$\|x+h\| = \sum_{i \in T_0} |x_i + h_i| + \sum_{i \in T_0^c} |x_i + h_i| \geq \|x_{T_0}\|_1 - \|h_{T_0}\|_1 + \|h_{T_0^c}\|_1 - \|x_{T_0^c}\|_1$$

Proof.

$$\Rightarrow \sum_{i \in T_0} |x_i + h_i| = \| x_{T_0} + h_{T_0} \|_1$$

&

$$\sum_{i \in T_0^c} |x_i + h_i| = \| x_{T_0^c} + h_{T_0^c} \|_1 .$$

$\therefore$ ~~To proof~~ To prove:

$$\| x_{T_0} + h_{T_0} \|_1 + \| x_{T_0^c} + h_{T_0^c} \|_1 \geq \| x_{T_0} \|_1 - \| h_{T_0} \|_1 + \| h_{T_0^c} \| - \| x_{T_0} \|$$

Gurukul
Page No.:
Date: \ \

$\Rightarrow \underbrace{\| x_{T_0} - (-1) * h_{T_0} \|_1}_{\Downarrow} = \| x_{T_0} + h_{T_0} \|_1$

by inverse triangle inequality

$\Rightarrow \| x_{T_0} - (-1) h_{T_0} \|_1 \geq \| x_{T_0} \|_1 - \| (-1) \cdot h_{T_0} \|_1$

$\Rightarrow \| x_{T_0} + h_{T_0} \|_1 \geq \| x_{T_0} \| - |-1| \, \| h_{T_0} \|_1$

$\therefore \boxed{\| x_{T_0} + h_{T_0} \|_1 \geq \| x_{T_0} - \| h_{T_0} \|_1}$ (i)

$\Rightarrow \| x_{T_0^c} + h_{T_0^c} \|_1 = \| h_{T_0^c} + x_{T_0^c} \|_1 = \| h_{T_0^c} - (-1) x_{T_0^c} \|_1$

by using inverse traingle inequality,

$\| h_{T_0^c} - (-1) x_{T_0^c} \|_1 \geq \| h_{T_0^c} \|_1 - |-1| \, \| x_{T_0^c} \|_1$

$\therefore \boxed{\| h_{T_0^c} + x_{T_0^c} \|_1 \geq \| h_{T_0^c} \| - \| x_{T_0^c} \|_1}$ (ii)

by adding (i) & (ii)

$\| x_{T_0} + h_{T_0} \| + \| x_{T_0^c} + h_{T_0^c} \| \geq \| x_{T_0} \| - \| h_{T_0} \|_1 + \| h_{T_0^c} \|_1 - \| x_{T_0^c} \|_1$

$\therefore$ Hence proved

Scanned by CamScanner

(6) Justify inequality

$$\|h_{T_0^c}\|_1 \leq \|h_{T_0}\|_1 + 2\|x_{T_0^c}\|_1$$

Proof:

$\Rightarrow$ By def: $\|x_{T_0^c}\|_1 = \|x - x_s\|_1$
where x_s has top 's' element of x.

$\Rightarrow$ From previous proof:

$$\|x\|_1 = \|x+h\|_1 \geq \|x_{T_0}\|_1 - \|h_{T_0}\|_1 + \|h_{T_0^c}\|_1 - \|x_{T_0^c}\|_1.$$

$\Rightarrow$ by def: $x_{T_0} = x - x_{T_0^c}$

$$\|x\|_1 \geq \underbrace{\|x_{T_0}\|_1}_{\text{by def}} - \|h_{T_0}\|_1 + \|h_{T_0^c}\|_1 - \|x_{T_0^c}\|_1$$

$$\Rightarrow \|x\|_1 \geq \|x - x_{T_0^c}\|_1 - \|h_{T_0}\|_1 + \|h_{T_0^c}\|_1 - \|x_{T_0^c}\|_1$$

$\Rightarrow$ by ~~tracing~~ inverse triangle inequality

$$\|x - x_{T_0^c}\| \geq \|x\| - \|x_{T_0^c}\|.$$

$\therefore$ we get,

$$\cancel{\|x\|_1} \geq \cancel{\|x\|} - \|x_{T_0^c}\|_1 - \|h_{T_0}\|_1 + \|h_{T_0^c}\|_1 - \|x_{T_0^c}\|_1$$

$$\therefore 2\|x_{T_0^c}\|_1 \geq -\|h_{T_0}\|_1 + \|h_{T_0^c}\|_1$$

Gurukul
Page No.:
Date: \ \

∴ we get

$$2\|x_{T_0^c}\|_1 + \|h_{T_0}\|_1 \geq \|h_{T_0^c}\|_1.$$

hence,

$$\|h_{T_0^c}\|_1 \leq \|h_{T_0}\|_1 + 2\|x_{T_0^c}\|_1$$

Hence proved.

(7) Justify inequality

$$\|h_{(T_0 \cup T_1)^c}\|_2 \leq \|h_{T_0}\|_2 + 2e_0$$

$$e_0 = s^{-1/2}\|x - x_s\|_1$$

Proof

⇒ From previous proof, we know.

$$\|h_{(T_0 \cup T_1^c)}\|_2 \leq s^{-1/2}\|h_{T_0^c}\|_2 \quad (i)$$

&

$$\|h_{T_0^c}\|_1 \leq \|h_{T_0}\|_1 + 2\|x_{T_0^c}\| \quad (ii)$$

by norm property

$$\|v\|_2 \leq \|v\|_1 \leq \sqrt{n}\|v\|_2 \quad (iii)$$

⇒ using (i) & (iii) we get.

$$\|h_{(T_0 \cup T_1^c)}\|_2 \leq (s)^{-\frac{1}{2}}\|h_{T_0^c}\|_2 \leq \boxed{(s)^{-1/2}\|h_{T_0^c}\|_1}$$

∴ we get,

$$\|h_{(T_0 \cup T_1)^c}\|_2 \leq s^{-1/2}\|h_{T_0}\|_1 + 2s^{-1/2}\|x_{T_0^c}\|_1. \quad (iv)$$

Gurukul
Page No.:
Date: / /

$\Rightarrow$ by using (iv) & (iii)

by norm property $\|h_{T_0}\|_1 \le (S)^{1/2}\|h_{T_0}\|_2$

$\therefore \dfrac{\|h_{T_0}\|_1}{(S)^{1/2}} \le \|h_{T_0}\|_2$

i.e $(S)^{-1/2}\|h_{T_0}\|_1 + 2S^{-1/2}\|x_{T_0^c}\|_1 \le \|h_{T_0}\|_2 + 2S^{-1/2}\|x_{T_0^c}\|_1$

So we get

$\|h_{(T_0 \cup T_1)^c}\|_2 \le \|h_{T_0}\|_2 + 2S^{-1/2}\underbrace{\|x_{T_0^c}\|_1}_{\text{by def}}$

$\|h_{(T_0 \cup T_1)^c}\|_2 \le \|h_{T_0}\|_2 + 2S^{-1/2}\|x - x_s\|_1$

let $e_0 = S^{-1/2}\|x - x_s\|_1$

hence,

$$\boxed{\|h_{(T_0 \cup T_1)^c}\|_2 \le \|h_{T_0}\|_2 + 2e_0}$$

Hence proved.

8 Justify:

$|\langle \Phi h_{T_0 \cup T_1}, \Phi h\rangle| \le \|\Phi h_{T_0 \cup T_1}\|_2 \|\Phi h\|_2 \le 2\sqrt{1+\delta_{2S}}\,\epsilon\,\|h_{T_0 \cup T_1}\|_2$

Gurukul
Page No.:
Date: \ \

(a) By cachy-schwarz inequality

$$|\langle \Phi h_{T_0 \cup T_1}, \Phi h \rangle| \le \|\Phi h_{T_0 \cup T_1}\|_2 \|\Phi h\|_2.$$

(b) By RIP.

As $h_{T_0 \cup T_1}$ is 2S sparse

$$(1-\delta_{2S}) \|h_{T_0 \cup T_1}\|_2^2 \le \|\Phi h_{T_0 \cup T_1}\|_2^2 \le (1+\delta_{2S}) \|h_{T_0 \cup T_1}\|_2^2$$

∴ we say (i) $\|\Phi h_{T_0 \cup T_1}\|_2 \le \sqrt{1+\delta_{2S}} \|h_{T_0 \cup T_1}\|_2$

(c) $h = x^* - x$ as $x^* = x + h$.
& we have proved previously that

$$\|\Phi h\|_2 = \|\Phi (x^* - x)\|_2 \le 2\epsilon \quad \text{(ii)}$$

(d) As (i) & (ii) both are true,
we got

$$\|\Phi h_{T_0 \cup T_1}\|_2 \|\Phi h\|_2 \le \sqrt{1+\delta_{2S}} \|h_{T_0 \cup T_1}\|_2 \, 2\epsilon$$

∴ by using (a) & (d) we get.

$$|\langle \Phi h_{T_0 \cup T_1}, \Phi h \rangle| \le \|\Phi h_{T_0 \cup T_1}\|_2 \|\Phi h\|_2 \le 2\epsilon \sqrt{1+\delta_{2S}} \|h_{T_0 \cup T_1}\|_2$$

Hence proved.

Gurukul
Page No.:
Date: \ \

(a) Justify
$|\langle \Phi h_{T_0}, \Phi h_{T_j} \rangle| \le \delta_{2s} \|h_{T_0}\|_2 \|h_{T_j}\|_2$

Proof
⇒ by Lemma 1:

$|\langle \Phi x, \Phi x' \rangle| \le \delta_{s+s'} \|x\|_2 \|x'\|_2$

for x & x' having disjoint support such that $|x| \le s$ & $|x'| \le s'$

→ So here, by def every $|h_{T_j}| \le s$ i.e every h_{T_j} is 's' sparse.

→ Also, by def h_{T_0}, h_{T_j} are have disjoint support for $j \ge 1$.

∴ we can use lemma 1.

∴ $|\langle \Phi h_{T_0}, \Phi h_{T_j} \rangle| \le \delta_{s+s} \|h_{T_0}\|_2 \|h_{T_j}\|_2$

i.e

$\boxed{|\langle \Phi h_{T_0}, \Phi h_{T_j} \rangle| \le \delta_{2s} \|h_{T_0}\|_2 \|h_{T_j}\|_2}$

hence proved.

Scanned by CamScanner

Gurukul
Page No.:
Date: / /

(10) Justify inequality

$\|h_{T_0}\|_2 + \|h_{T_1}\|_2 \leq \sqrt{2}\,\|h_{T_0 \cup T_1}\|_2$.

for T_0 & T_1 are disjoint.

Proof

$\Rightarrow$ let $\sum_{i \in T_0} (h_i)^2 = P$, by def. $Q \leq P$ (i)

$\sum_{i \in T_1} (h_i)^2 = Q$

$\Rightarrow$ Also, T_0 & T_1 are disjoint

$\therefore h_{T_0 \cup T_1} = h_{T_0} + h_{T_1}$. (ii)

$\|h_{T_0 \cup T_1}\|_2^2 = \sum_{i \in T_0 \cup T_1} (h_i)^2$

by usin (ii) & T_0 & T_1 are disjoint $= \sum_{i \in T_0} (h_i)^2 + \sum_{i \in T_1} (h_i)^2$

$\therefore$ To proove : $\cancel{(P)^{1/2}}$

$\|h_{T_0}\|_2 + \|h_{T_1}\|_2 \leq \sqrt{2}\,\|h_{T_0 \cup T_1}\|_2$

$\|_2$ $\therefore$ in terms of P & Q

$(P)^{1/2} + (Q)^{1/2} \leq \sqrt{2}\,\sqrt{P+Q}$

$\Rightarrow$ Taking LHS.

$(\sqrt{P} + \sqrt{Q})^2 = P + q + 2\sqrt{PQ}$

we know by definition $Q \leq P$ (using (i))

$\therefore P = Q + \delta \implies \boxed{Q = P - \delta}$ (iii)

δ = some ~~constant~~ scalar

Gurukul
Page No.:
Date: / /

So we get

$$\left((P)^{1/2}+(Q)^{1/2}\right)^2 = p+q+2\sqrt{P(P-\delta)}$$ ← using (iii)

$$= p+q+2\sqrt{p^2\left(1-\frac{\delta}{p}\right)}$$

$$= p+q+2P\left(1-\frac{\delta}{p}\right)^{1/2} \quad \text{(iv)}$$

$\frac{\delta}{p} < 1$ as $\delta < p$

Using binomeal expansion

$$(1+x)^n = 1+nx+\frac{n(n-1)}{2}x^2 \ldots$$

$$\left(1\cancel{+}\frac{\delta}{p}\right)^{1/2} = 1\cancel{+}\frac{1}{2}\frac{\delta}{p}$$

$$\left(1-\frac{\delta}{p}\right)^{2} = 1-\frac{1}{2}\frac{\delta}{p}-\frac{1}{8}\frac{\delta^2}{p^2}\ldots\ldots \quad \text{(v)}$$

Sobsituting (v) in eq (iv)

$$= p+q+2p\left(1-\frac{1}{2}\frac{\delta}{p}-\frac{1}{8}\frac{\delta^2}{p^2}\ldots\right)$$

$$= p+q+\left(\underbrace{2P-\frac{2P\cdot\delta}{2\cdot P}}_{\text{taking 1st two terms}}-\frac{2P}{8}\frac{\delta^2}{P^2}\ldots\right)$$

i.e

$$\boxed{\leq P+q+2P-\frac{2P\delta}{2\cdot P}}$$

$$(\sqrt{P}+\sqrt{q})^2 \leq P+q+2P-\delta$$ by def $Q = P-\delta$

$$(\sqrt{P}+\sqrt{q})^2 \leq P+q+P+(P-\delta)$$

$$(\sqrt{P}+\sqrt{q})^2 \leq P+q+P+q$$

$$(\sqrt{P}+\sqrt{q})^2 \leq 2(P+q).$$

$$\therefore (\sqrt{P}+\sqrt{q}) \leq \sqrt{2}\,(P+q)^{1/2}.$$

Scanned by CamScanner

Gurukul
Page No.:
Date: / /

Subsituting value of p & q.

$$\underbrace{\left(\sum_{i\in T_0}(h_i)^2\right)^{1/2}}_{\|h_{T_0}\|_2} + \underbrace{\left(\sum_{i\in T_1}(h_i)^2\right)^{1/2}}_{\|h_{T_1}\|_2} \leq \sqrt{2}\underbrace{\left(\sum_{i\in T_0}(h_i^2)+\sum_{i\in T_1}(h_i)^2\right)^{1/2}}_{\|h_{T_0\cup T_1}\|_2}$$

$\therefore$ we get

$$\|h_{T_0}\|_2 + \|h_{T_1}\|_2 \leq \sqrt{2}\,\|h_{T_0\cup T_1}\|_2$$

Hence proved.

(ii) Justify inequality:

$$(1-\delta_{2s})\|h_{T_0\cup T_1}\|_2^2 \leq \|\Phi h_{T_0\cup T_1}\|_2^2 \leq \|h_{T_0\cup T_1}\|_2 \left(2\epsilon\sqrt{1+\delta_{2s}} + \sqrt{2}\,\delta_{2s}\sum_{j\geq 2}\|h_{T_j}\|_2\right)$$

Proof:

$\Rightarrow$ By RIP we get

$$(1-\delta_{2s})\|h_{T_0\cup T_1}\|_2^2 \leq \|\Phi h_{T_0\cup T_1}\|^2 \leq (1+\delta_{2s})\|h_{T_0\cup T_1}\|_2$$

as $h_{T_0\cup T_1}$ is 2s sparse by def.

$\rightarrow$ * inner product property: (we will use)

$\langle u+v, w\rangle = \langle u,w\rangle + \langle v,w\rangle$ (i)

$\langle u,v\rangle = \langle v,u\rangle$ (ii)

$\Rightarrow$ from prev proofs we know:

$$\|\Phi h_{T_0\cup T_1}\|_2^2 = \underbrace{\langle \Phi h_{T_0\cup T_1}, \Phi h\rangle}_{A} - \underbrace{\langle \Phi h_{T_0\cup T_1}, \sum_{j\geq 2}\Phi T_j\rangle}_{B}$$

let $A = \langle \Phi h_{T_0\cup T_1}, \Phi h\rangle$

$B = \langle \Phi h_{T_0\cup T_1}, \sum_{j\geq 2}\Phi T_j\rangle$

Page No.:
Date: \ \

(a) from previous proof we know that

$$A = \langle \phi h_{T_{OUT_1}}, \phi h \rangle \leq 2G\sqrt{1+\delta_{2S}}\, \|h_{T_{OUT_1}}\|_2$$

i.e

$$A \leq 2G\sqrt{1+\delta_{2S}}\, \|h_{T_{OUT_1}}\|_2 \quad (iii)$$

(b) $B = \langle \phi h_{T_{OUT_1}}, \sum_{j\geq 2} \phi h_{T_j} \rangle$

by using (i) property we get.

$$\langle \phi h_{T_{OUT_1}}, \sum_{j\geq 1} \phi h_{T_j} \rangle = \langle \phi(\overbrace{h_{T_0} + h_{T_1}}^{\text{disjoint}}), \sum_{j\geq 2} \phi h_{T_j} \rangle$$

$$= \langle \phi h_{T_0}, \sum_{j\geq 2} \phi h_{T_j} \rangle + \langle \phi h_{T_1}, \sum_{j\geq 2} \phi h_{T_j} \rangle \quad (iv)$$

(c) $\langle \phi h_{T_0}, \sum_{j\geq 2} \phi h_{T_j} \rangle = ?$

using again (i) property of inner product we get

$$\langle \phi h_{T_0}, \sum_{j\geq 2} \phi h_{T_j} \rangle = \langle \phi h_{T_0}, \phi h_{T_2} \rangle + \langle \phi h_{T_0}, \phi h_{T_3} \rangle \cdots$$

$$= \sum_{j\geq 2} \langle \phi h_{T_0}, \phi h_{T_j} \rangle$$

→ from lemma 2.1 (of paper)
as ~~h_0~~, h_{T_0}, h_{T_j} has disjoint support and $|h_{T_0}| \leq S$ & $|h_{T_1}| \leq S$ by def.
we get,

$$|\langle \phi h_{T_0}, \phi h_{T_j} \rangle| \leq \delta_{2S} \|h_{T_0}\|_2 \|h_{T_j}\|_2$$

$\therefore \sum_{j\geq 2} |\langle \Phi h_{T_0}, \Phi h_{T_j}\rangle| \leq \delta_{2S} \|h_{T_0}\|_2 \left(\sum_{j\geq 2} \|h_{T_j}\|_2\right)$

$\therefore \langle \Phi h_{T_0}, \sum_{j\geq 2} \Phi h_{T_j}\rangle = \sum_{j\geq 2} \langle \Phi h_{T_0}, \Phi h_{T_j}\rangle \leq \delta_{2S} \|h_{T_0}\|_2 \left(\sum_{j\geq 2} \|h_{T_j}\|_2\right)$ (v)

(d) $\langle \Phi h_{T_1}, \sum_{j\geq 2} \Phi h_{T_j}\rangle = ?$

Similarly as (c) part, we get

$\langle \Phi h_{T_1}, \sum_{j\geq 2} \Phi h_{T_j}\rangle = \sum_{j\geq 2} \langle \Phi h_{T_1}, \Phi h_{T_j}\rangle \leq \delta_2 \|h_{T_1}\|_2 \left(\sum_{j\geq 2} \|h_{T_j}\|_2\right)$ (vi)

(e) adding both (v) & (vi) we get

$\overbrace{\langle \Phi h_{T_0 \cup T_1}, \sum_{j\geq 2} \Phi h_{T_j}\rangle}^{B} = \overbrace{\langle \Phi h_0, \sum_{j\geq 2} \Phi h_{T_j}\rangle}^{c} + \overbrace{\langle \Phi h_1, \sum_{j\geq 2} \Phi h_{T_j}\rangle}^{d}$

$\leq \delta_{2S} \left(\|h_{T_0}\|_2 + \|h_{T_1}\|_2\right)\left(\sum_{j\geq 2} \|h_{T_j}\|_2\right)$

Prev. we proved that

$\|h_{T_0}\|_2 + \|h_{T_1}\|_2 \leq \sqrt{2} \|h_{T_0 \cup T_1}\|_2$

$\therefore B = \langle \Phi h_{T_0 \cup T_1}, \sum_{j\geq 2} \Phi h_{T_j}\rangle \leq \sqrt{2}\delta_{2S} \|h_{T_0 \cup T_1}\|_2 \left(\sum_{j\geq 2} \|h_{T_j}\|_2\right)$ (vii)

Gurukul
Page No.:
Date: / /

(8) $\|\phi h_{T_0 \cup T_1}\|_2^2 = \underbrace{\langle \phi_{T_0 \cup T_1}, \phi h \rangle}_{A} - \underbrace{\langle \phi_{T_0 \cup T_1}, \sum_{j \geq 2} \phi h_{T_j} \rangle}_{B}$

as we know that $A \geq 0$ & $B \geq 0$

$\therefore \|\phi h_{T_0 \cup T_1}\|_2^2 = A - B \leq A + B$

$\therefore \|\phi h_{T_0 \cup T_1}\|_2^2 \leq A + B$

by subsituting value of A & B from eq. (iii) & (vii) we get.

$\|\phi h_{T_0 \cup T_1}\|_2^2 \leq 2\epsilon \sqrt{1+\delta_{2s}} \|h_{T_0 \cup T_1}\|_2 + \sqrt{2}\delta_{2s} \|h_{T_0 \cup T_1}\|_2 \left(\sum_{j \geq 2} \|\phi h_{T_j}\|_2 \right)$

$\therefore \|\phi h_{T_0 \cup T_1}\|_2^2 \leq \|h_{T_0 \cup T_1}\|_2 \left(2\epsilon \sqrt{1+\delta_{2s}} + \sqrt{2}\delta_{2s} \left(\sum_{j \geq 2} \|h_{T_j}\|_2 \right) \right)$

&

$(1-\delta_{2s}) \|h_{T_0 \cup T_1}\|_2^2 \leq \|\phi h_{T_0 \cup T_1}\|_2^2$

Hence proved.

(12) Justify inequality

$\|h_{T_0 \cup T_1}\|_2 \leq \alpha \epsilon + \rho s^{-1/2} \|h_{T_0^c}\|_1$

where $\alpha = \dfrac{2\sqrt{1+\delta_{2s}}}{1-\delta_{2s}}$, $\rho = \dfrac{\sqrt{2}\,\delta_{2s}}{1-\delta_{2s}}$

Proof:

$\Rightarrow$ by eq. (10) of paper, we know that

$\sum_{j \geq 2} \|h_{T_j}\|_2 \leq s^{-1/2} \|h_{T_0^c}\|_1$ (i)

Gurukul
Page No.:
Date: \ \

& from previous proof we know

$$(1-\delta_{2s})\|h_{T_0\cup T_1}\|_2^2 \le \|h_{T_0\cup T_1}\|_2\left(2\epsilon\sqrt{1+\delta_{2s}} + \sqrt{2}\,\delta_{2s}\left(\sum_{j\ge 2}\|h_{T_j}\|_2\right)\right)$$

$$\Rightarrow \|h_{T_0\cup T_1}\|_2 \le \frac{2\epsilon\sqrt{1+\delta_{2s}}}{1+\delta_{2s}} + \frac{\sqrt{2}\,\delta_{2s}}{(1-\delta_{2s})}\left(\sum_{j\ge 2}\|h_{T_j}\|_2\right)$$

let $\alpha = \frac{2\sqrt{1+\delta_{2s}}}{1-\delta_{2s}}$ $\quad \rho = \frac{\sqrt{2}\,\delta_{2s}}{(1-\delta_{2s})}$.

∴ we get

$$\|h_{T_0\cup T_1}\|_2 \le \epsilon\alpha + \rho\sum_{j\ge 2}\|h_{T_j}\|_2$$

as from (i) we know, $\sum_{j\ge 2}\|h_{T_j}\|_2 \le s^{-1/2}\|h_{T_0^c}\|_1$

we get.

$$\boxed{\|h_{T_0\cup T_1}\|_2 \le \epsilon\alpha + \rho\, s^{-1/2}\|h_{T_0^c}\|_1}$$

Hence proved.

13 Justify the inequality

$\|h_{T_0\cup T_1}\|_2 \le \alpha\epsilon + \rho\|h_{T_0\cup T_1}\|_2 + 2\rho\epsilon_0$

Proof.

⇒ from (12) eq of paper, we know

$$\|h_{T_0^c}\|_1 \le \|h_{T_0}\|_1 + 2\|x_{T_0^c}\|_1 \qquad (i)$$

&

from previous proof we know.

$$\|h_{T_0\cup T_1}\|_2 \le \epsilon\alpha + \rho s^{-1/2}\|h_{T_0^c}\|_1 \qquad (ii)$$

→ we get, from (i) & (ii)

$$\|h_{T_0\cup T_1}\|_2 \le \epsilon\alpha + \rho s^{-1/2}\|h_{T_0}\|_1 + 2\rho s^{-1/2}\|x_{T_0^c}\|_1$$

by norm property $\|v\|_1 \leq \sqrt{n}\|v\|_2$, we get
$\|h_{T_0 \cup T_1}\|_2 \leq \epsilon\alpha + \rho s^{-1/2} \cdot \cancel{\sqrt{s}} \|h_{T_0}\|_2 + 2\rho\epsilon_0$
by def. we know $\|h_{T_0}\|_2 \leq \|h_{T_0 \cup T_1}\|$

$\therefore$ we get,
$$\boxed{\|h_{T_0 \cup T_1}\|_2 \leq \epsilon\alpha + \rho\|h_{T_0 \cup T_1}\| + 2\rho\epsilon_0}$$
hence proved.

(14) Justify: $\|h_{T_0 \cup T_1}\|_2 + \|h_{(T_0 \cup T_1)^c}\|_2 \leq 2\|h_{T_0 \cup T_1}\|_2 + 2e_0$
$\leq (1-\rho)^{1/2} 2 \cdot (\alpha\epsilon + (1+\rho)\epsilon_0)$

Proof:
(a) $\|h_{T_0 \cup T_1}\|_2 + \|h_{(T_0 \cup T_1)^c}\|_2$.
from eq (13) of paper: we get
$\|h_{(T_0 \cup T_1)^c}\|_2 \leq \|h_{T_0}\|_2 + 2e_0$.
$\therefore$ we get,
$\|h_{T_0 \cup T_1}\|_2 + \|h_{(T_0 \cup T_1)^c}\|_2 \leq \|h_{T_0 \cup T_1}\|_2 + \|h_{T_0}\|_2 + 2e_0$
also we know $\|h_{T_0}\|_2 \leq \|h_{T_0 \cup T_1}\|_2$ by using property we get
$\therefore \|h_{T_0 \cup T_1}\|_2 + \|h_{(T_0 \cup T_1)^c}\|_2 \leq 2\|h_{T_0 \cup T_1}\|_2 + 2e_0$

(b) using prev. proof we know:
$\|h_{T_0 \cup T_1}\|_2 \leq (1-\rho)^{-1}(\alpha\epsilon + 2\rho\epsilon_0)$.

$$\therefore \|h_{T_0 \cup T_1}\|_2 + 2\epsilon_0 \leq \frac{2(\alpha\epsilon + 2\rho\epsilon_0)}{(1-\rho)} + 2\epsilon_0$$

$$\leq \frac{2\rho\epsilon + 4\rho\epsilon_0 + 2\epsilon_0 - 2\rho\epsilon_0}{(1-\rho)}$$

$\therefore$ we get
$2\|h_{T_0 \cup T_1}\|_2 + 2\epsilon_0 \leq (1-\rho)^{-1} 2 \cdot (\alpha\epsilon + (1+\rho)\epsilon_0)$

Hence proved

X