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Q4.

$$1_\wedge d = d_\wedge$$
$$-1_\wedge d = -1 * d_\wedge$$
} for simplication.

(i) To prove: $\phi(-1_{\wedge^c} d) = \phi(1_\wedge d)$

→ as d is Nullspace of ϕ i.e
$d \in N(\phi) \setminus \{0\}$

$$\therefore \phi d = 0$$

$$\phi[d_\wedge + d_{\wedge^c}] = 0$$

$$\phi d_\wedge = -\phi d_{\wedge^c}$$

$$\phi(1_\wedge d) = \phi(-1_{\wedge^c} d)$$ hence proved.

(ii) To prove: $\|1_\wedge d\|_1 < \|1_{\wedge^c} d\|_1$

→ From (i) we know

$$\phi(-1_{\wedge^c} d) = \phi(1_\wedge d)$$

$$\sum_{i \in \wedge^c} -\phi_i d_i = \sum_{j \in \wedge} \phi_j d_j$$

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As by Lemma 1.1 (i), we assumed it to true, so $(1_\Lambda d)$ is the unique "minimizer" of $\|c\|_0$, means it is the solution to minization problem (seps) such that its l_1 norm is minimum.

$\therefore \|1_\Lambda d\|_1 = \|c\|_1$

$\therefore$ So $\|-1_{\Lambda^c} d\|$ will not be the ~~minizer~~ minimize of the l_1 problem

$$\|-1_{\Lambda^c} d\|_1 \geq \|1_\Lambda d\|_1.$$

So we get

$$\boxed{\begin{gathered}\|1_\Lambda d\|_1 < \|1_{\Lambda^c} d\|_1 \\ \text{or} \\ \|d_\Lambda\|_1 < \|d_{\Lambda^c}\|_1\end{gathered}} \quad (2).$$

Hence proved.

(ii) To prove $\|1_\Lambda d\|_1 < \frac{1}{2}\|d\|_1$ by using (2)

$$\|d_\Lambda\|_1 < \|d_{\Lambda^c}\|_1$$

$$\frac{1}{2}\|d_\Lambda\|_1 < \frac{1}{2}\|d_{\Lambda^c}\|_1$$

$$\|d_\Lambda\|_1 - \frac{1}{2}\|d_\Lambda\|_1 < \frac{1}{2}\|d_{\Lambda^c}\|_1$$

$$\|d_\Lambda\|_1 < \frac{1}{2}\|d_{\Lambda^c}\|_1 + \|d_\Lambda\|_1$$

$$\boxed{\|d_\Lambda\|_1 < \frac{1}{2}\|d_{\Lambda^c} + d_\Lambda\|_1}$$

as d_{Λ^c} & d_Λ are disjoint

$\therefore$ we get

$$\boxed{\begin{array}{c} \|d\wedge\|_1 < \frac{1}{2}\|d\|_1 \\ \text{or} \\ \|1\wedge d\|_1 < \frac{1}{2}\|d\|_1 \end{array}}$$

Hence proved

(iv) Too prove $\|C_2\|_1 - \|C_1\| = \|1\wedge^c C_2\|_1 + \|1\wedge C_2\|_1 - \|1\wedge C_1\|_1$

We know

$C_2 = C_2\wedge + C_2\wedge^c$ (a)

$C_1 = C_1\wedge + C_1\wedge^c$ (b)

Also $\|C_1\|_0 \le k$ $\therefore \|C_1\wedge^c\|_1 = 0$. (c)

LHS

$\|C_2\|_1 - \|C_1\|$

$\rightarrow$ using (a) & (b)

$\|C_2\wedge\|_1 + / \|e_2$

$= \|C_2\wedge + C_2\wedge^c\|_1 + \|C_1\wedge + C_1\wedge^c\|_1$

= as $\wedge$ & $\wedge^c$ are disjoint, we get

$\|C_2\wedge\| + \|C_2\wedge^c\| \bar{+} \|C_1\wedge\| \bar{+} \|C_1\wedge^c\|_1$

= by (c) we get

$\|C_2\wedge\| + \|C_2\wedge^c\| \mp \|C_1\wedge\|$

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$= \|1 \wedge^c c_2\|_1 + \|1 \wedge c_2\|_1 - \|1 \wedge c_1\|_1$

Hence proved.

To prove,

$\|c_2 \wedge^c\|_1 + \|c_2 \wedge\|_1 - \|c_1 \wedge\|_1 \geq \|1 \wedge^c d\|_1 - \|d \wedge\|_1$.

→ we know $d = c_2 - c_1$

$c_2 = d + c_1$

$c_{2\wedge} = d_\wedge + c_{1\wedge}$ (a)

$c_2 \wedge^c = d_{\wedge^c} + c_{1\wedge^c}$ (b)

→ using (a) & (b) in LHS

$\|d_{\wedge^c} + c_{1\wedge^c}\|_1 + \|d_\wedge + c_{1\wedge}\|_1 - \|c_{1\wedge}\|_1$

$= \|d_{\wedge c} - (-1)c_{1\wedge^c}\|_1 + \|d_\wedge + c_{1\wedge}\|_1 - \|c_{1\wedge}\|_1$

using $\|a - b\| \geq \|a\| - \|b\|$

$\geq \|d_{\wedge^c}\| - \underbrace{\|c_{1\wedge^c}\|}_{0 \text{ from (c)}} + \cancel{\|c_{1\wedge}\|_1} - \|d_\wedge\|_1 \cancel{- \|c_{1\wedge}\|_1}$

$\geq \|d_{\wedge^c}\| - \|d_\wedge\|_1$

$\geq \|1 \wedge^c d\| - \|1 \wedge d_\wedge\|_1$

Hence proved.

(6) To prove $d_i = -\sum_{j \neq i} \langle \phi_i, \phi_j \rangle d_j$

→ we know

→ $\phi^T \phi d = 0$ → ϕ is col. are normalized

→ $$A_{n\times n} = \phi^T \phi = \begin{bmatrix} 1 & \langle \phi_1, \phi_2 \rangle & \langle \phi_1, \phi_3 \rangle & \cdots & \langle \phi_1, \phi_n \rangle \\ \langle \phi_2, \phi_1 \rangle & 1 & \cdots & & \langle \phi_2, \phi_n \rangle \\ \vdots & & & & \\ \langle \phi_n, \phi_1 \rangle & \langle \phi_n, \phi_2 \rangle & \cdots & & 1 \end{bmatrix}_{n\times n}$$

→ $$A \cdot d = 0 = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$$

A^i = ith row of A

→ we get

$\langle A^1, d \rangle = 0$

$\langle A^2, d \rangle = 0$

$\vdots$

$\langle A^i, d \rangle = 0.$

$\vdots$

$\langle A^n, d \rangle = 0$

→ $\langle A^i, d \rangle = 0$ = :

$$= [1 \ \langle \phi_1, \phi_2 \rangle \ \langle \phi_1, \phi_3 \rangle \cdots \langle \phi_1, \phi_n \rangle] \begin{bmatrix} d_1 \\ \vdots \\ d_n \end{bmatrix} = 0$$

$= d_1 + d_2 \langle \phi_1, \phi_2 \rangle + d_3 \langle \phi_1, \phi_3 \rangle \cdots = 0$

$= d_1 + \sum_{j \neq 1} d_j \langle \phi_1, \phi_j \rangle = 0$

$= d_1 = - \sum_{j \neq 1} d_j \langle \phi_1, \phi_j \rangle = 0$

similarly for d_i

$$\boxed{d_i = - \sum_{j \neq i} d_j \langle \phi_i, \phi_j \rangle}$$

Hence proved.

(7). To prove.

$\|1_\Lambda d\|_1 \leq |\Lambda| \cdot \left(1 + \frac{1}{\mu(\Phi)}\right)^{-1} \|d\|_1 = \|C\|_0 \cdot \left(1 + \frac{1}{\mu(\Phi)}\right)^{-1}$
$\times \|d_1\| < \frac{1}{2} \|d_1\|$

→ we know that

$|d_i| \leq \left(1 + \frac{1}{\mu(\Phi)}\right)^{-1} \|d\|_1$ ⓐ

&

$\|1_\Lambda d\|_1 \leq \frac{1}{2} \|d\|_1$ ⓑ

already proved.

$\|\gamma_{x}d\|_1 = cH$

$\rightarrow$ by using (a)

$$|d_i| \leq \left(1+\frac{1}{\mu(\Phi)}\right)^{-1}\|d\|_1$$

i.e

$$|d_1| \leq \left(1+\frac{1}{\mu(\Phi)}\right)^{-1}\|d\|_1$$

$$|d_2| \leq \left(1+\frac{1}{\mu(\Phi)}\right)^{-1}\|d\|_1$$

$$\vdots$$

$$|d_n| \leq \left(1+\frac{1}{\mu(\Phi)}\right)^{-1}\|d\|_1$$

$$\Rightarrow \|1_\Lambda d\|_1 = \sum_{i\in\Lambda}|d_i|$$

$$= \sum_{i\in\Lambda}|d_i| \leq \sum_{i\in\Lambda}\left(1+\frac{1}{\mu(\Phi)}\right)^{-1}\|d\|_1$$

$$\Rightarrow \sum_{i\in\Lambda}|d_i| \leq |\Lambda|\left(1+\frac{1}{\mu(\Phi)}\right)^{-1}\|d\|_1$$

$\therefore$ we get $\|1_\Lambda d\|_1 \leq |\Lambda|\left(1+\frac{1}{\mu(\Phi)}\right)^{-1}\|d\|_1$

$$\leq \|c_0\|\left(1+\frac{1}{\mu(\Phi)}\right)^{-1}\|d_1\|$$

& by using (b) we know $\|1_\Lambda d\|_1 \leq \frac{1}{2}\|d\|_1$

$\therefore$ we get

$$\boxed{\|1_\Lambda d\|_1 \leq \|c_0\|\left(1+\frac{1}{\mu(\Phi)}\right)^{-1}\|d_1\| \leq \frac{1}{2}\|d\|_1}$$

Hence proved.

* Adv. of Theorem 1.1 over RIP

→ Theorem 1.1 say, if

$$\|C_0\|_0 < \frac{1}{2}\left(1 + \frac{1}{\mu([\Phi_1|\Phi_2])}\right) \text{ then}$$

l_1 minimization solution coincides with l_0 problem.

(i) μ i.e mutual coherance μ can easily be computed w.r.t δ_{2S} of Φ in RIP

(ii) ~~Time~~ Time complexity ~~of~~ an computing δ_{2S} is exponential w.r.t for computing Φ.

(iii) RIP depends on the underlying signal i.e Θ

$\forall$ ~~Φ~~ Θ

$$(1-\delta_S)\|\Theta\|^2 \leq \|\Phi\Theta\|^2 \leq (1+\delta_S)\|\Theta\|^2.$$

for $\forall$ Θ the RIP should be satisfied. This is not the case with Theorem 1.1.

Dis adv.

(i) → Big dis. adv of Theorem 1.1 over RIP is "sparsity". i.e we know the $\|C\|_0$ sparity before applying Theorem 1.1.
But in real world knowing sparsity before hand is not possible.

(ii) This is must ~~re~~ restrictive property than ~~the~~ RIP.