Q3 $\hat{x}$: oracular knowing the index indeices set S where $K = |S|$

(a) find $\hat{x}$?

we have min the error:

$$\hat{x} = \min_{x} \|Y - \eta - \phi_s x\|_2^2$$

where ϕ_s is submatrix with coloumn ~~ide~~ index is from S set.

Taking derivative

$$\frac{\partial \|Y - \eta - \phi_s x\|_2^2}{\partial x} = 0$$

$$2\phi_s^T (Y - \eta - \phi_s x) = 0$$

$$\phi_s^T (Y - \eta - \phi_s x) = 0$$

$$\phi_s^T (Y - \eta) = \phi_s^T \phi_s x$$

as $\phi_s^T \phi_s$ inverse exists we get

$$\hat{x}_s \equiv x = (\phi_s^T \phi_s)^{-1} \phi_s^T (Y - \eta)$$

$$\therefore \boxed{\hat{x}_s = (\phi_s^T \phi_s)^{-1} \phi_s^T (Y - \eta)}$$

$\therefore \hat{x} \equiv$ will have element from $\hat{x}_s$ at index from set S & remaining as zero.

(b) $\|\hat{x}-x\|_2$

$\rightarrow$ as we know $\hat{x} = \Phi_s^{+}(y-n)$

$\therefore = \|\Phi_s^{+}(y-n)-x\|_2$ where $\Phi_s^{+} \triangleq (\Phi_s^{T}\Phi_s)^{-1}\Phi_s^{T}$

$= \|\underbrace{\Phi_s^{+}y}_{\text{true result i.e } x} - \Phi_s^{+}n - x\|_2$

$= \|-\Phi_s^{+}n\|_2 = \|\Phi_s^{+}n\|_2$

$\therefore$ we get,

$$\boxed{\|\hat{x}-x\|_2 = \|\Phi_s^{+}n\|_2}$$

By using cachy schwarz inequality

$\|\Phi_s^{+}n\|_2 \leq \|\Phi_s^{+}\|_2\|n\|_2$

$$\boxed{\therefore \|\hat{x}-x\|_2 = \|\Phi_s^{+}n\|_2 \leq \|\Phi_s^{+}\|_2\|n\|_2}$$

(c) Using RIP

$(1-\delta_{2k})\|\hat{x}-x\|_2^2 \leq \|\Phi(\hat{x}-x)\|_2^2 \leq (1+\delta_{2k})\|\hat{x}-x\|_2^2$

we know that
$\|\Phi(\hat{x}-x)\|_2^2 = \|\Phi\hat{x}-\Phi x\|_2^2$

$= \|y-n-y\|_2^2$

$= \|n\|_2^2$

$\therefore (1-\delta_{2k}) \|\hat{x}-x\|_2^2 \le \|h\|_2^2 \le (1+\delta_{2k}) \|\hat{x}-x\|_2^2$

→ right side inequality

$\|h\|_2^2 \le (1+\delta_{2k}) \|\hat{x}-x\|_2^2$

$\frac{\|h\|_2^2}{(1+\delta_{2k})} \le \|\hat{x}-x_2\|_2^2$ we know that

$\|\hat{x}-x_2\|_2 \le \|\Phi_s^{+}\|_2 \|h\|_2$

$\therefore \frac{\|\cancel{h}\|_2^2}{(1+\delta_{2k})} \le \|\Phi_s^{+}\|_2^2 \|\cancel{h}\|_2^2$

$\therefore$ we get $\frac{1}{(1+\delta_{2k})} \le \|\Phi_s^{+}\|_2^2$

which is $\boxed{\frac{1}{\sqrt{1+\delta_{2k}}} \le \|\Phi_s^{+}\|_2}$

→ left side.

$(1-\delta_{2k}) \|\hat{x}-x\|_2 \le \|h\|_2^2$

Similarly as above we get,

$\|\Phi_s^{+}\|_2 \ge \frac{1}{\sqrt{1-\delta_{2k}}}$

$\therefore$ $\boxed{\text{$\Phi_s^{+}$ largest singular values ~~line~~ lies between $\frac{1}{\sqrt{1+\delta_{2k}}}$ \& $\frac{1}{\sqrt{1-\delta_{2k}}}$.}}$

i.e $\frac{1}{\sqrt{1+\delta_{2k}}} \le \|\Phi_s^{+}\|_2 \le \frac{1}{\sqrt{1-\delta_{2k}}}$

Page No.:
Date: \ \

(d) By RIP (as previously proved)

$$(1-\delta_{2k})\,\|\hat{x}-x\|_2^2 \le \|n\|_2^2 \le (1+\delta_{2k})\,\|\hat{x}-x\|_2^2$$

$\therefore \|n\|_2^2 \le \epsilon$

so we get

$$\frac{\epsilon}{\sqrt{1+\delta_{2k}}} \le \|\hat{x}-x\|_2 \le \frac{\epsilon}{\sqrt{1-\delta_{2k}}}$$

$$\therefore \frac{\epsilon}{\sqrt{1+\delta_{2k}}} \le \|x-\hat{x}\|_2 \le \frac{\epsilon}{\sqrt{1-\delta_{2k}}}$$

Let x^* be the solution of Theorem 3

i.e $\min_x \|x\|_1$ such that $\|y-\phi x\|_2 \le \epsilon$

$\therefore$ by theorem 3 constraint we get

$$\|y-\phi x^*\|_2 \le \epsilon$$

& we know that

$$\frac{\epsilon}{\sqrt{1+\delta_{2k}}} \le \|x-\hat{x}\|_2$$

$$\epsilon \le (\sqrt{1+\delta_{2k}})\,\|x-\hat{x}\|_2$$

$$\|\phi x-\phi x^*\|_2 \le \epsilon$$

$$\|\phi(x-x^*)\|_2 \le \epsilon$$

we say that $\|x-x^*\|_2$ difference is less than $\|x-\hat{x}\|_2$

$$\therefore \|x-x^*\|_2 \le C\,\|x-\hat{x}\|_2$$