AIP-Assignment 1

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1 Question 1

We reffered to following papper,
Compressive hyperspectral imaging by random separable projections in both the spatial and the spectral domains

Summarizing the architecture describe in the paper.

1. This paper presents a new method for HS image acquisition using CS separable encoding both in the spacial and spectral domains.

2. The proposed CHISSS architecture implements an optical CS system using separable operators.
   Separable Compressive Sensing Separable CS is used to overcome the practical limitations in compressive imaging implementations involving large data. Separable sensing operator ϕ, can be represented in the form of $\phi_y \bigotimes \phi_x$,where $\bigotimes$ denotes kronecker product.
   The CHSISS system uses two separable random encoding codes, one for the spatial domain and the other for the spectral domain.

3. Details of CHISSS architecture

   chisss.png

   Figure 1: CHISSS Architecture

   (a) **Spacial encoding** is done using scheme of single pixel CS camera.
   - The lens L1 is used to image the object on the digital micro-mirror device (DMD) D1.
   - A random code of size $N_x * N_y$ is displayed by L1.
   - The encoded light reflected from D1 is then focused on the central point of the G1 grating using the lens L2.
   - The spot on the G1 plane contains the same mixed spatial information for the entire spectrum.

   (b) **Spectral encoding**
   - For spectral encoding the output of single pixel CS is taken as input.
   - A **dispersive element** is present at S3 which works as spectral to spatial convertor.
   - Grating G1 splits and diffracts the beam S3 into N λ spectral spots, which are spread along parallel rays on the coding device C1 by means of the cylindrical lens L3.

- These spectral line is spatially encoded using the coded aperture mask C1($M_\lambda * N_\lambda$). Therefore, each horizontal spectral geometrical line is encoded by a different random pattern.
- To focus and collect the different spectral components, regular converging lenses L4 is used.
- spectral encoding is done in parallel with spacial encoding.

(c) The encoding process with the CHISSS system is separable in the x–y and λ domains.

(d) **Sensing matrix** : Is assumed to be a random matrix, since no priory information about the spatial or spectral features of the imaged scene is assumed to be available. Since, We need to make M measurements for CS, each measurement is the result of different encoding of the datacube.

How it is different from the architecture studied in class that was most similar to it.
Answer) We studied in class a method for acquisition of compressed hyper spectral cube using CASSI. In CASSI the spatial information is first randomly encoded and then the spectral information is mixed by a shearing operation.That is, we don't encode in spectral domain. In CHISSS, encoding is done both in spectral encoding as well as spacial domain.

Objective function and some key equation presented in the paper for compressive reconstruction.

- By defination of compressed Sensing,
  g=ϕf where,
  $\phi \in R^{M \times N}$ is a sensing matrix
  $g \in R^k$
  $f = \psi\alpha, a \in R^N$ and $k \ll N$
- F and G are matrix representation of f and g.
- Objective function,
  vec(G) = $\phi_{yx} \times vec(F) = (\phi_y^T \bigotimes \phi_x) \times vec(F)$,
  By properties of Kronecker product, we can write
  G=$\phi_y F \phi_x$
  F=$\psi \widehat{A} \psi^T$ subject to
  $min_a\{\parallel vec(G) - vec(\phi_y \psi A \psi_T \phi_x) \parallel_2 + \gamma \parallel vec(A) \parallel_1\}$
  a=vec(A)
- used the 3D Haar wavelets as the sparsifying operators, ψ^T , together with l_1 regularization
- **Mutual coherence** of the separable sensing system is given by
  $\mu(\phi_{yx}, \psi yx) = \mu(\phi_y \bigotimes \phi_x, \psi_y \bigotimes \psi_x)$
  =$\mu(\phi_y, \psi_y)\mu(\phi_x, \psi_x)$

- $\frac{\mu(\phi_y \bigotimes \psi_y, \psi)}{\mu(\phi, \psi)} \approx \frac{2log_{10}(\sqrt{N}}{\sqrt{2log_{10}(N)}} = \sqrt{\frac{1}{2}log_{10}(N)}$
- It means $\sqrt{\frac{1}{2}log_{10}(N)}$ more measurements are required to accurately reconstruct the signal using a separable sensing operator than with a non separable random operator.This is a reasonable cost for gaining the computational simplification.
- Compression ratio is given as $\frac{M_x \times M_y \times M_\lambda}{N_x \times N_y \times N_\lambda}$

2 Question 2

Lower bound on the number of compressive measurements(m) for exact reconstruction of a signal in R^n that is s in some orthonormal basis ψ if

a) For BP (P1)
From theorm 1 we know,
$m \geq Clog(n/\delta) \parallel \theta \parallel_0 \mu^2(\psi, \phi)$
For lower bound we assume μ=1(high incohernace between ψ and ϕ)
This Gives,$m \geq Clog(n/\delta) \parallel \theta \parallel_0$
If signal obeys the RIP of order S,then
$m \geq CSlog(n/S)$
Hence this gives the Lower bound on the number of compressive measurements.

b)For P0
For a s sparse signal it is necessary and sufficient that any 2s columns of ϕ are linearly independent. This will be only possible when minimum 2s measurements are taken.So lower bound on the number of compressive measurements for exact reconstruction of a signal in R^n that is s in some orthonormal basis ψ is 2s.

Lower bound on the compressive measurements changes if signal was s sparse in different orthonormal basis ?
a)For P0
It will not change provided ϕ and ψ remains incoherent(A satisfies RIP). Suppose we have random matrix phi then this holds for any orthonormal basis ψ with very high probability.

b)For BP
Since m(number of compressive measurements) depends upon $\mu^2(\psi, \phi)$, lower bound on the compressive measurements may change if incoherance between ϕ and ψ reduces. If on changing ψ , there is still high incohernace between ψ and ϕ (which has high probability if ϕ is a random matrix), Lower bounds will not change.

In both cases, what is the maximum bound allowed on the RIC of the matrix? $\phi\psi$
Note the requirement that δ_{2s} should be less than $2^{0.5} - 1$.

3 Question 3

To prove : the relationship between the restricted isometry constant of order s of a matrix A (denoted as δ_s) and the mutual coherence μ of A: $\delta_s \leq$ (s - 1)μ.

1. By definition of RIC :
   $\delta_s = max\{1 - \lambda_{min}, \lambda_{max} - 1\}$, where
   $\lambda_m ax = max_{x \in R^S, |\gamma| \leq S} \frac{\|A_\gamma x\|^2}{\|x\|^2}$
   This therefore requires us to enumerate different subsets γ of maximum size S, and compute the maximum and minimum eigenvalue of matrices $(A_\gamma)^T(A\gamma)$.

2. **We will prove this using Greshgorin's disc/circle theorem on** $A^T A$
   Now by Greshgorin's disc/circle theorem, we have

   ∃i $\in \{1, ..., n\}$ such that $B_{ii} - r_i \leq \lambda \leq B_{ii} + r_i$
   where,
   B is a square matrix
   r_i is the radius of the i-th disc and is defined as the sum of the absolute values of the off-diagonal elements of the i-th row.

3. We will also use the following fact that X is S sparse and A_s corresponds to only those indices of X which are non-zero. Hence A_s will have maximum of s elements in a row.

4. Now, Since A is unit normalized $A_s^T A_s$ will have 1 in its digonal (So B_{ii} = 1).

5. Now by Defination of mutual coherance :
   $\mu(A) = max_{i,j,i \neq j} |A_i \dot{A}_j|$
   So, let μ be the mutual coherence
   We want max/min range for the eigen values of the $B = A_s^T A_s$

   By **Greshgorin's theorem** the eigen value will lie in the union of all Greshgorin's disc. All disc can be represented as:

   $|\lambda - b_{ii}| <= \sum_{j=1j}^{s} b_{ij}$

   But $b_{ij} \leq \mu$

   Therefore, $|\lambda - b_{ii}| \leq \sum_{j=1j}^{s} \mu$

We get, $|\lambda - 1| \leq (s-1)\mu$ as $b_{ii} = 1$

$-1 * (s-1)\mu \leq \lambda - 1 \leq (s-1)\mu$

$-(s-1)\mu + 1 \leq \lambda \leq (s-1)\mu + 1$

6. λ_{max} is the largest and λ_{min} smallest eigen value of $A^T A$

7. Using Greshgorin's theorem we get following equation,
   $\lambda_{max} \leq 1 + \mu(s$ - $1)$
   $\lambda_{min} \geq 1 - \mu(s$ - $1)$

8. By RIC theorem we have,
   $\delta_s \leq max(\lambda_{max} - 1, 1 - \lambda_{min})$

9. Hence,
   $\delta_s \leq (s$ - $1)\mu$.

4 Question 4

Upper Bound Proof

$\mu(\phi, \psi) = \sqrt{n}\ max_{i,j \in \{0,1,....,n-1\}}\ < \phi^i, \psi_j >$

If $\phi = \psi^t$ then max value of inner product will be 1 as ith row of ϕ and jth col of ψ are same when $i = j$.

Therefore $\mu(\phi, \psi) = \sqrt{n} * 1 = \sqrt{n}$

Lower Bound Proof

Let : g=$\sum_{k=1}^{n} \alpha_k \psi_k$

Taking square on both sides, we get

$||g||^2$=$\sum_{k=1}^{n} \alpha_k {\psi_k}^t \sum_{j=1}^{n} \alpha_j \psi_j$

$||g||^2$=$\sum_{k=1}^{n} \alpha_k \alpha_j {\psi_k}^t \psi_j$

$||g||^2$=$\sum_{k=1}^{n} \alpha_k \alpha_j$ {Since ψ is an orthonormal matrix}

$\sum_{k=1}^{n} \alpha_k^2$=1 {Since g is a unit vector }

Now,

$\mu(g, \psi) = \sqrt{n}\ max_{j \in \{1,....,n\}}\ < g, \psi_j >$

$\mu(g,\psi) = \sqrt{n}\ max_{j\in\{1,....,n\}}\ g^t * \psi_j$

$\mu(g,\psi) = \sqrt{n}\ max_{j\in\{1,....,n\}}\ (\sum_{k=1}^{n}\alpha_k\psi_k)^t * \psi_j$

$\mu(g,\psi)$=$\sqrt{n}\ max_{j\in\{1,....,n\}}|\alpha_i|$ {Since $\sum_{k=1}^{n}\alpha_k^2$=1 }

As $||\alpha|| = \sum_{k=1}^{n}\alpha_k^2 = 1$
Let say lowest bound on the value of a_k be **t** i.e $|a_k| \geq t$ s.t $\sum_{k=1}^{n}\alpha_k^2 = 1$
Therefore we get, $\sum_{k=1}^{n}t^2 = 1$
t = $\frac{1}{\sqrt{n}}$

For the lowest case, values of all α_k will be i.e α_i=$\sqrt{1/n}$
So, $\mu(g,\psi)$=$\sqrt{n}\ max_{j\in\{1,....,n\}}|\alpha_i|$
$\mu(g,\psi) = \sqrt{n} * \frac{1}{\sqrt{n}} = 1$

Hence, the lower bound values is 1

5 Question 5

Separate Report file