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Q5 Derivation

Given: $Y = \Phi x + \eta$

$\Phi \in \mathbb{R}^{m \times n}$ $y \in \mathbb{R}^m$ $x \in \mathbb{R}^n$

$\eta \sim N(0, \sigma^2 \times I_{m \times m})$

$\Rightarrow$ $\eta \sim N(0, \sigma^2 \times I_{m\times m}) = N(0, \Sigma_{\sigma^2})$

where $\Sigma_{\sigma^2} = \begin{bmatrix} \sigma^2 & 0 & 0 & \cdots \\ 0 & \sigma^2 & 0 & \cdots \\ 0 & 0 & \sigma^2 & \cdots \\ \vdots & & & \\ 0 & \cdots & \cdots & \sigma^2 \end{bmatrix}_{m \times m}$

$\Rightarrow$ we know that 'x' is drawn from zero-mean Gaussian with known Σ_x

$\therefore x \sim N(0, \Sigma_x)$

$$\therefore p(x) = \underbrace{\frac{1}{(2\pi)^{\cancel{n/2}} |\Sigma_x|^{1/2}}}_{c_2} \exp\left(\frac{-1}{2} x^T \Sigma_x^{-1} x\right)$$

$$\therefore p(y|x, \Phi) = \frac{1}{(2\pi)^{1/2}\sigma} \exp\left(\frac{-\|y - \Phi x\|^2}{2\sigma^2}\right)$$

$$= \underbrace{\frac{1}{(2\pi) |\Sigma_{\sigma^2}|^{1/2}}}_{c_1} \exp\left(\frac{-1}{2}(y - \Phi x)^T \Sigma_{\sigma^2} (y - \Phi x)\right)$$

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→ map estimate.

$$\hat{x} = \arg\max_{x} P(x \mid y)$$

$$= \arg\max_{x} \left\{ \frac{P(y \mid x)\, P(x)}{P(y)} \right.$$

$$= \arg\max_{x} P(y \mid x)\, P(x)$$ (as $P(y)$ is independent of x.)

$$= \arg\max_{x} c_1 \exp\left(-\frac{1}{2} (y - \Phi x)^T \Sigma_{\sigma^2}^{-1} (y - \Phi x)\right) * c_2 \exp\left(-\frac{1}{2} x^T \Sigma_x^{-1} x\right).$$

(i) $$= \arg\max_{x} c_1 . c_2 \exp\left(-\frac{1}{2} (y - \Phi x)^T \Sigma_{\sigma^2}^{-1} (y - \Phi x) + -\frac{1}{2} (x^T \Sigma_x^{-1} x) \right)$$

∴ we can write eq. (i) as.

$$= \arg\max_{x} c_1 . c_2 \exp\left(\frac{1}{2} (x - \hat{u})^T \hat{\Sigma}^{-1} (x - \hat{u})\right)$$

for some $\hat{u}$ & $\hat{\Sigma}^{-1}$

or as.

(ii) $$= \arg\max_{x} c_1 . c_2 \exp\left(-\frac{1}{2} (\hat{u} - x)^T \hat{\Sigma}^{-1} (\hat{u} - x)\right)$$

$$\boxed{\hat{x} = \hat{u}}$$

∴ ~~so we have find~~ value of $\hat{u}$

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Finding value of x i.e $\vec{u}$

→ Comparing eq (i) & (ii)

i.e eq (i)

$$\exp\left(-\frac{1}{2}\left((y-\phi x)^T \Sigma_{\sigma^2}^{-1} (y-\phi x) + x^T \Sigma_x^{-1} x\right)\right)$$

$$= \exp\left(-\frac{1}{2}\left((y-\phi x)^T (\Sigma_{\sigma^2}^{-1} y - \Sigma_{\sigma^2}^{-1}\phi x) + x^T \Sigma_x^{-1} x\right)\right)$$

$$= \exp\left(-\frac{1}{2}\left(y^T \Sigma_{\sigma^2}^{-1} y - x^T \phi^T \Sigma_{\sigma^2}^{-1} y - y^T \Sigma_{\sigma^2}^{-1}\phi x + \underbrace{x^T \phi^T \Sigma_{\sigma^2}^{-1} \phi x + x^T \Sigma_x^{-1} x}_{\text{merging}}\right)\right)$$

$$= \exp\left(-\frac{1}{2}\left(y^T \Sigma_{\sigma^2}^{-1} y - \overbrace{x^T \phi^T \Sigma_{\sigma^2}^{-1} y}^{(a)} - y^T \Sigma_{\sigma^2}^{-1}\phi x + \underbrace{x^T (\phi^T \Sigma_{\sigma^2}^{-1} \phi + \Sigma_x^{-1}) x}_{(b)}\right)\right)$$

eq (ii) expanding

$$= \exp\left(-\frac{1}{2}(\hat{u}-x)^T \hat{\Sigma}^{-1} (\hat{u}-x)\right)$$

$$= \exp\left(-\frac{1}{2}(\hat{u}-x)^T (\hat{\Sigma}^{-1}\hat{u} - \hat{\Sigma}^{-1} x)\right)$$

$$= \exp\left(-\frac{1}{2}\left(\hat{u}^T \hat{\Sigma}^{-1} \hat{u} - \underbrace{x^T \hat{\Sigma}^{-1} \hat{u}}_{(c)} - \hat{u} \hat{\Sigma}^{-1} x + \underbrace{x^T \hat{\Sigma}^{-1} x}_{(d)}\right)\right)$$

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As

eq (i) = eq (ii)

which means by similarity of eq.

(a) = (c) & (b) = (d)

⇒ (b) = (d) i.e.

$$x^T(\Phi^T \Sigma_{\sigma^2}^{-1} \Phi + \Sigma_x^{-1}) x = x^T \hat{\Sigma}^{-1} x$$

∴ we get $\boxed{\hat{\Sigma}^{-1} = (\Phi^T \Sigma_{\sigma^2}^{-1} \Phi + \Sigma_x^{-1})}$ (iii)

⇒ as (a) = (c) i.e.

$$x^T \Phi^T \Sigma_{\sigma^2}^{-1} y = x^T \hat{\Sigma}^{-1} \hat{\mu}$$

$$\therefore \Phi^T \Sigma_{\sigma^2}^{-1} y = \hat{\Sigma}^{-1} \hat{\mu}$$

by eq (iii) we get

$$\boxed{\hat{\mu} = (\Phi^T \Sigma_{\sigma^2}^{-1} \Phi + \Sigma_x^{-1})^{-1} \Phi^T \Sigma_{\sigma^2}^{-1} y}$$

$$\therefore \boxed{x = (\Phi^T \Sigma_{\sigma^2}^{-1} \Phi + \Sigma_x^{-1})^{-1} \Phi^T \Sigma_{\sigma^2}^{-1} y}$$ ans

1. or. if we use $\Sigma_{\sigma^2} = \sigma^2 I_{m \times m}$

$$\therefore \Sigma_{\sigma^2}^{-1} = \frac{1}{\sigma^2} I_{m \times m}$$

we get

$$x = \left(\frac{1}{\sigma^2}\Phi^T I \Phi + \Sigma_x^{-1}\right)^{-1} \cdot \frac{1}{\sigma^2} \cdot \Phi^T I \cdot y$$

$$\boxed{x = (\Phi^T \Phi + \sigma^2 \Sigma_x^{-1})^{-1} \cdot \Phi^T y}$$ ans

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