Q-6)

(a) To prove $y^T P y \geq 0$.

$\because P = A^T A$

$\Rightarrow y^T A^T A y$

$\Rightarrow (Ay)^T A y$

$\Rightarrow \|Ay\|^2$, is positive or zero

Hence, we can say that $y^T P y \geq 0$

To prove $z^T Q z \geq 0$

$\because Q = A A^T$

$\Rightarrow z^T A A^T z$

$\Rightarrow z^T A (z^T A)^T$

$\Rightarrow \|z^T A\|^2$, is positive or zero

Hence, we can say that $z^T Q z \geq 0$

Eigen values of P and Q are non-negative, as

$PY = \lambda Y$ [for any eigen values λ & eigen vector Y]

Multiplying Both sides by Y^T

$Y^T P Y = \lambda$

And we have proved above that $y^T P y \geq 0$. Hence eigen values of P & Q are non-negative

Q-6).

(b) To prove: $Q(Au) = \lambda(Au)$

We know,

$Pu = \lambda u$, where λ is eigen value of P & u is eigen vector of P.

$APu = \lambda Au$, Multiplying both sides by A

$AA^TAu = \lambda Au$, $\because P = A^TA$

$Q(Au) = \lambda(Au)$, $\because Q = AA^T$

Hence proved, $Q(Au) = \lambda(Au)$ i.e Au is an eigen vector of Q with eigen value λ

To prove : $P(A^Tv) = \mu(A^Tv)$

We know,

$Qv = \mu v$, where μ is eigen value of Q & v is eigen vector of Q.

$A^TQv = \mu A^Tv$, Multiplying both sides by A^T

$A^TAA^Tv = \mu A^Tv$, $\because Q = AA^T$

$P(A^Tv) = \mu(A^Tv)$, $\because P = A^TA$

Hence proved, $P(A^Tv) = \mu(A^Tv)$ i.e. A^Tv is an eigen vector of P with eigen value μ

Number of elements in u is n

Number of elements in v is m

(c) To prove $\gamma_i \geq 0$ such that $A u_i = \gamma_i v_i$

Given $u_i \triangleq \frac{A^T v_i}{\|A^T v_i\|_2}$

$$A \frac{A^T v_i}{\|A^T v_i\|_2} = \gamma_i v_i$$

$v_i^t A A^T v_i = \gamma_i \cdot \|A^T v_i\|_2$, multiplying both sides by v_i^t

We have proved in part a that $v_i^t Q v_i \geq 0$

Hence,

$$\gamma_i \|A^T v_i\|_2 \geq 0$$

$$\gamma_i \geq 0 \quad \because \|A^T v_i\| \geq 0$$

Hence Proved.

(d) Given :→ $U = [v_1 | v_2 \cdots | v_m]$, $V = [u_1 | u_2 \cdots | u_n]$

To show, $A = U\Gamma V^T$

We know from part a,

$P = VSV^T$, where V contains eigen vector, S is a diagnol matrix contain eigen values of P.

$P_{n\times n} = V_{n\times n} S_{n\times n} V^T_{n\times n}$

Now S is a diagnol matrix,

$\because S_{n\times n} = T^T_{n\times m} \times T_{m\times n}$, where $T_{m\times n}$ contains all sq. root values of $S_{n\times n}$

$\because$ S contains all non-negative values, T will contain all non-negative values.

Putting value of $S_{n\times n}$ in eq ①.

$P = V(\Gamma^T \Gamma)V^T$

$P = V(\Gamma^T I \Gamma)V^T$

$P = V(\Gamma^T U^T U \Gamma)V^T$ where $U^T U = I$

$P = (U\Gamma V^T)^T (U\Gamma V^T)$

$A^T A = (U\Gamma V^T)^T (U\Gamma V^T)$

Hence, $A = U\Gamma V^T$

<u>Proved</u>.