Name - Ayush Jain
SAP ID - 60004200132
Div - B | B1

01-12-2021 Maths - III

1.) Find the laplace transform of $t^2 e^t \sin(4t)$

→ $L[\sin 4t] = \frac{4}{s^2+16}$ $\quad \because \left(L[\sin at] = \frac{a}{s^2+a^2} \right)$

$\therefore L[t^2 \sin 4t] = (-1)^2 \frac{d^2}{ds^2}\left[\cancel{\sin 4} \frac{4}{s^2+16}\right]$ $\quad \because \left(L[t^n F(t)] = (-1)^n \frac{d^n}{ds^2} F(s) \right)$

$= \frac{d^2}{ds^2}\left(\frac{4}{s^2+16}\right)$

$= 4\frac{d}{ds}\left(\frac{-2s}{(s^2+16)^2}\right)$

$= -8\frac{d}{ds}\left(\frac{s}{(s^2+16)^2}\right)$

$= -8\left(\frac{s^2+16-4s^2}{(s^2+16)^3}\right)$

$= -8\left(\frac{16-3s^2}{(s^2+16)^3}\right)$

$L[t^2 \sin 4t] = \frac{8(3s^2-16)}{(s^2+16)^3}$

$\therefore$ By First Shifting Theorem,

$L[e^{-t} t^2 \sin 4t] = \frac{8(3(s+1)^2-16)}{((s+1)^2+16)^3}$

$= \frac{8(3s^2+6s+3-16)}{(s^2+2s+1+16)^3}$

60004200132

$$= \frac{8(3s^2+6s-13)}{(s^2+2s+17)^3}$$

$$\therefore \boxed{L[t^2e^{-t}\sin 4t] = \frac{8(3s^2+6s-13)}{(s^2+2s+17)^3}}$$

2) Evaluate $\int_0^\infty \frac{t^2 \sin 3t \, dt}{e^{2t}}$

$\rightarrow$ $\therefore \int_0^\infty e^{-2t} t^2 \sin 3t \, dt = L[t^2 \sin 3t]$, where $s = 2$

$\therefore L[\sin 3t] = \frac{3}{s^2+9}$ $\quad \because \left(L[\sin at] = \frac{a}{s^2+a^2} \right)$

$\therefore L[t^2 \sin 3t] = (-1)^2 \frac{d^2}{ds^2}\left(\frac{3}{s^2+9}\right)$ $\quad \because \left(L[t^n f(t)] = (-1)^n \frac{d^n}{ds^n} F(s) \right)$

$$= \frac{d^2}{ds^2}\left(\frac{3}{s^2+9}\right)$$

$$= -6 \frac{d}{ds}\left[\frac{s}{(s^2+9)^2}\right]$$

$$= -6\left[\frac{s^2+9-4s^2}{(s^2+9)^3}\right]$$

$$L[t^2 \sin 3t] = 6\left[\frac{3s^2-9}{(s^2+9)^3}\right]$$

Now,

Put $s = 2$

$$\therefore L[t^2 \sin 3t] = 6\left[\frac{3(2)^2-9}{(2^2+9)^3}\right]$$

$$= 6\left[\frac{12 - 9}{13^3}\right]$$

$$= 6\left[\frac{3}{13^3}\right]$$

$$= \frac{18}{2197}$$

$$\therefore \cancel{L\left[t^2 \sin 3t\right] = 18}$$

$$\therefore \boxed{\int_0^\infty \frac{t^2 \sin 3t}{e^{2t}}\, dt = \frac{18}{2197}}$$

3) $L^{-1}\left[\frac{3s+1}{(s+1)(s^2+2)}\right]$

Let $\frac{3s+1}{(s+1)(s^2+2)} = \frac{A}{s+1} + \frac{Bs+C}{s^2+2}$

$\therefore 3s+1 = A(s^2+2) + (Bs+C)(s+1)$

$\therefore 3s+1 = (A+B)s^2 + (B+C)s + 2A + C$

Equating the coefficients, we get

$\therefore A+B=0$, $B+C=3$, $2A+C=1$

Solving this equations, we get

$A = -\frac{2}{3}$, $B = \frac{2}{3}$, $C = \frac{7}{3}$

$$\therefore L^{-1}\left[\frac{3s+1}{(s+1)(s^2+2)}\right] = -\frac{2}{3}L^{-1}\left[\frac{1}{s+1}\right] + \frac{2}{3}L^{-1}\left[\frac{s}{s^2+2}\right] + \frac{7}{3}L^{-1}\left[\frac{1}{s^2+2}\right]$$

$$= \frac{-2}{3}\left[e^{-t}\right] + \frac{2}{3}L^{-1}\left[\frac{s}{s^2+(\sqrt{2})^2}\right] + \frac{7}{3}L^{-1}\left[\frac{1}{s^2+(\sqrt{2})^2}\right]$$

$$= \frac{-2}{3}(e^{-t}) + \frac{2}{3}\cos\sqrt{2}t + \frac{7}{3\sqrt{2}}\sin\sqrt{2}t$$

$$\boxed{\therefore L^{-1}\left[\frac{3s+1}{(s+1)(s^2+2)}\right] = \frac{-2e^{-t}}{3} + \frac{2}{3}\cos\sqrt{2}t + \frac{7}{3\sqrt{2}}\sin\sqrt{2}t}$$

4) $L^{-1}\left[\frac{1}{s^2(s+1)^2}\right] = L^{-1}\left[\frac{1}{(s+1)^2}\cdot\frac{1}{s^2}\right] = L^{-1}\left[F(s)\cdot G(s)\right]$

where $F(s) = \frac{1}{(s+1)^2}$, $G(s) = \frac{1}{s^2}$

$$\therefore f(t) = L^{-1}[F(s)] = L^{-1}\left[\frac{1}{(s+1)^2}\right]$$

$$= e^{-t}L^{-1}\left[\frac{1}{s^2}\right] \quad \ldots \text{(By first shifting theorem)}$$

$$\therefore f(t) = e^{-t}\cdot t$$

$$g(t) = L^{-1}[G(s)] = L^{-1}\left[\frac{1}{s^2}\right] = t$$

$$\therefore g(t) = t$$

$\therefore$ By Convolution theorem,

$$L^{-1}\left[\frac{1}{s^2(s+1)^2}\right] = f(t) * g(t)$$

$$= \int_0^t f(u)\cdot g(t-u)\,du$$

$$= \int_0^t e^{-u}\cdot u\cdot(t-u)\,du$$

$$= t\int_0^t e^{-u} u\,du - \int_0^t e^{-u}\cdot u^2\,du$$

$$= t\left[\frac{u\cdot e^{-u}}{-1} - \frac{e^{-u}}{1}\right]_0^t - \left[\frac{u^2\cdot e^{-u}}{-1} - \frac{e^{-u}}{1}\cdot 2u + \frac{e^{-u}}{-1}\cdot 2\right]_0^t$$

$$= t\left[-(te^{-t}) - e^{-t} + 1\right] - \left[-(t^2e^{-t}) - e^{-t}\cdot 2t - e^{-t}\cdot 2 + 2\right]$$

$$= \cancel{-t^2e^{-t}} - e^{-t}\cdot t + t + \cancel{t^2e^{-t}} + e^{-t}\cdot 2t + e^{-t}\cdot 2 - 2$$

$$= te^{-t} + 2e^{-t} + t - 2$$

$$\therefore \boxed{L^{-1}\left[\frac{1}{s^2(s+1)^2}\right] = te^{-t} + 2e^{-t} + t - 2}$$

5) $(D^2-3D+2)y = 4e^{2t}$, $y(0)=-3$, $y'(0)=5$

Taking Laplace Transform on both sides, we get

$$\therefore L\left[\frac{d^2y}{dt^2} - 3\frac{dy}{dt} + 2y\right] = 4L\left[e^{2t}\right]$$

$$\therefore \left[s^2 L[y(t)] - s\,y(0) - y'(0)\right] - 3\left[sL[y(t)] - y(0)\right] + 2L[y(t)] = \frac{4}{s-2}$$

Now, Put $y(0)=-3$, $y'(0)=5$

$$\therefore \left[s^2 L[y(t)] + 3s - 5\right] - 3\left[sL[y(t)] + 3\right] + 2L[y(t)] = \frac{4}{s-2}$$

$$\therefore L[y(t)](s^2-3s+2) + 3s - 5 - 9 = \frac{4}{s-2}$$

$$L[y(t)](s-2)(s-1) = \frac{4}{s-2} + 14 - 3s$$

$$\therefore L[y(t)](s-2)(s-1) = \frac{4 + (14-3s)(s-2)}{(s-2)}$$

$$\therefore L[y(t)] = \frac{-3s^2+20s-24}{(s-1)(s-2)^2}$$

$$\frac{-3s^2+20s-24}{(s-1)(s-2)^2} = \frac{A}{(s-1)} + \frac{B}{(s-2)} + \frac{C}{(s-2)^2}$$

$$-3s^2+20s-24 = A(s-2)^2 + B(s-1)(s-2) + C(s-1)$$

Put $s=1 \implies -3+20-24 = A$

$\therefore A = -7$

Put $s=2 \implies -12+40-24=C$

$\therefore C=4$

Put $s=0 \implies -24 = 4(-7)+2B-4$

$\therefore B=4$

$\therefore L[y(t)] = \cancel{-7} \; \frac{-7}{s-1} + \frac{4}{s-2} + \frac{4}{(s-2)^2}$

Taking inverse laplace transform,

$y(t) = -7L^{-1}\left[\frac{1}{s-1}\right] + 4L^{-1}\left[\frac{1}{s-2}\right] + 4L^{-1}\left[\frac{1}{(s-2)^2}\right]$

$\therefore y(t) = -7e^{t} + 4e^{2t} + 4e^{2t}L^{-1}\left[\frac{1}{s^2}\right]$ ... (By first shifting)

$\therefore y(t) = -7e^{t} + 4e^{2t} + 4e^{2t}.t$

$\therefore \boxed{y(t) = -7e^{t} + 4e^{2t} + 4te^{2t}}$

6> (i) $L\left[\sin t H(t-\pi) + t^2 \delta(t-2)\right]$

$\rightarrow$ $= L\left[\sin t \; H(t-\pi) + t^2 \delta(t-2)\right]$

$= L\left[\sin t \; H(t-\pi)\right] + L\left[t^2 \delta(t-2)\right]$

We know that,

$L\left[f(t) \cdot H(t-a)\right] = e^{-as} L\left[f(t+a)\right]$ and

$L\left[f(t) \cdot \delta(t-a)\right] = e^{-as} f(a)$

Now,

$L\left[\sin t H(t-\pi)\right] + L\left[t^2 \delta(t-2)\right]$

$= e^{-\pi s} L\left[\sin(t+\pi)\right] + e^{-2s} (2)^2$

$= -e^{-\pi s} L\left[\sin t\right] + e^{-2s} \cdot 4$

$= -e^{-\pi s} \cdot \left(\frac{1}{s^2+1}\right) + 4e^{-2s}$ .... $\left[L\left[\sin at\right] = \frac{a}{s^2+a^2}\right]$

$= 4e^{-2s} - \frac{e^{-\pi s}}{s^2+1}$

$\therefore$ $\boxed{L\left[\sin t \; H(t-\pi) + t^2 \delta(t-2)\right] = 4e^{-2s} - \frac{e^{-\pi s}}{s^2+1}}$

6) ii) $L\left[(1+3t-4t^2+2t^3)H(t-3)\right]$

$$= e^{-3s} L\left[1+3(t+3)-4(t+3)^2+2(t+3)^3\right]$$

$$\because L\left[f(t)H(t-a)\right] = e^{-as} L\left[F(t+a)\right]$$

$$= e^{-3s} L\left[1+3t+9-4t^2-24t-36+2t^3+18t^2+54t+54\right]$$

$$= e^{-3s} L\left[2t^3+14t^2+33t+28\right]$$

$$= e^{-3s}\left[2L[t^3]+14L[t^2]+33L[t]+28L[1]\right]$$

$$= e^{-3s}\left[2\cdot\frac{3!}{s^4}+14\cdot\frac{2!}{s^3}+33\cdot\frac{1}{s^2}+\frac{28}{s}\right]$$

$$= e^{-3s}\left[\frac{12}{s^4}+\frac{28}{s^3}+\frac{33}{s^2}+\frac{28}{s}\right]$$

$$\boxed{\therefore L\left[(1+3t-4t^2+2t^3)H(t-3)\right] = e^{-3s}\left[\frac{12}{s^4}+\frac{28}{s^3}+\frac{33}{s^2}+\frac{28}{s}\right]}$$

60004200132

7) Find the Fourier expansion of $f(x) = \sqrt{1-\cos x}$ in $(0, 2\pi)$

→ $f(x) = \sqrt{1-\cos x}$

$= \sqrt{2\sin^2 x/2}$

$= \sqrt{2}\left|\sin \frac{x}{2}\right|$

But $\sin \frac{x}{2} > 0$ in interval $(0, 2\pi)$

$\therefore f(x) = \sqrt{2} \sin \frac{x}{2}$

Fourier series expansion is given by,

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx \quad \text{— (i)}$$

where $a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)\,dx$

$= \frac{1}{\pi}\int_0^{2\pi} \sqrt{2}\sin \frac{x}{2}\,dx$

$= \frac{\sqrt{2}}{\pi}\left[-2\cos \frac{x}{2}\right]_0^{2\pi}$

$= \frac{\sqrt{2}}{\pi}\left[2-(-2)\right] = \frac{\sqrt{2}}{\pi}(4) = \frac{4\sqrt{2}}{\pi}$

$\therefore a_0 = \frac{4\sqrt{2}}{\pi}$

$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,dx$

$= \frac{1}{\pi}\int_0^{2\pi} \sqrt{2}\sin \frac{x}{2} \cos nx\,dx$

$$= \frac{\sqrt{2}}{2\pi}\int_0^{2\pi}\left[\sin\left(n+\frac{1}{2}\right)x - \sin\left(n-\frac{1}{2}\right)x\right]dx$$

$$= \frac{\sqrt{2}}{2\pi}\left[\frac{-\cos(n+1/2)x}{n+1/2} + \frac{\cos(n-1/2)x}{n-1/2}\right]_0^{2\pi}$$

$$= \frac{\sqrt{2}}{2\pi}\left[\frac{-[(-1)-1]}{n+1/2} + \frac{(-1)-(1)}{n-1/2}\right]$$

$$= \frac{\sqrt{2}}{2\pi}\left[\frac{2}{n+1/2} - \frac{2}{n-1/2}\right] \quad \left[\because \cos(2n+1)\pi = -1 \text{ and } \cos(2n-1)\pi = -1\right]$$

$$= \frac{2\sqrt{2}}{\pi}\left[\frac{1}{2n+1} - \frac{1}{2n-1}\right]$$

$$= \frac{-4\sqrt{2}}{\pi}\left[\frac{1}{4n^2-1}\right]$$

$$= \frac{-4\sqrt{2}}{\pi(4n^2-1)}$$

$$\therefore a_n = \frac{-4\sqrt{2}}{\pi(4n^2-1)}$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,dx$$

$$= \frac{1}{\pi}\int_0^{2\pi}\sqrt{2}\sin\frac{x}{2}\sin nx\,dx$$

$$= \frac{\sqrt{2}}{2\pi}\int_0^{2\pi}\left[\cos\left(n-\frac{1}{2}\right)x - \cos\left(n+\frac{1}{2}\right)x\right]dx$$

$$= \frac{\sqrt{2}}{2\pi}\left[\frac{\sin(n-1/2)x}{n-1/2} - \frac{\sin(n+1/2)x}{n+1/2}\right]_0^{2\pi}$$

$$= \frac{\sqrt{2}}{2\pi}(0-0) = 0$$

$\therefore b_n = 0$

Substitute a_0, a_n, b_n in (i),

Fourier expansion of $f(x)$ is:

$$f(x) = \frac{4\sqrt{2}}{2\pi} + \sum_{n=1}^{\infty}\left[\frac{-4\sqrt{2}}{\pi(4n^2-1)}\right]\cos nx + \sum_{n=1}^{\infty} 0 \cdot \sin nx$$

$$\therefore f(x) = \frac{2\sqrt{2}}{\pi} + \sum_{n=1}^{\infty}\left(\frac{-4\sqrt{2}}{\pi(4n^2-1)}\right)\cos nx$$

$$\therefore \boxed{f(x) = \frac{2\sqrt{2}}{\pi} - \frac{4\sqrt{2}}{\pi}\sum_{n=1}^{\infty} \cdot \frac{1}{4n^2-1} \cdot \cos nx}$$

8) Find the Fourier expansion of $f(x)$ whose period is 3 where $f(x) = 2x - x^2$, $0 \le x \le 3$

→ $f(x) = 2x - x^2$, $x \in (0,3)$

$2L = 3$

$\therefore L = \frac{3}{2}$

$\therefore$ Fourier series expansion is given by:

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right) \right]$$

$$\therefore f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos\left(\frac{2n\pi x}{3}\right) + b_n \sin\left(\frac{2n\pi x}{3}\right) \right] \quad \text{— (i)}$$

where $a_0 = \frac{1}{L} \int_0^{2L} f(x)\, dx$

$$= \frac{2}{3} \int_0^3 (2x - x^2)\, dx$$

$$= \frac{2}{3} \left[x^2 - \frac{x^3}{3} \right]_0^3 = 0$$

$\therefore a_0 = 0$

$$a_n = \frac{1}{L} \int_0^{2L} f(x) \cdot \cos\left(\frac{n\pi x}{L}\right) dx$$

$$= \frac{2}{3} \int_0^3 (2x - x^2) \cdot \cos\left(\frac{2n\pi x}{3}\right) dx$$

6000 4200132

$$= \frac{2}{3}\left[(2x-x^2)\left(\frac{3}{2n\pi}\sin\frac{2\pi nx}{3}\right) - (2-2x)\left[\frac{-9}{4n^2\pi^2}\cos\frac{2n\pi x}{3}\right]\right.$$

$$\left.+(-2)\left(\frac{-27}{8\pi^3 n^3}\sin\left(\frac{2\pi nx}{3}\right)\right)\right]_0^3$$

$$= \frac{2}{3}\left[(2-2x)\cos\frac{2n\pi x}{3}\right]_0^3 \left(\frac{9}{4n^2\pi^2}\right)$$

$$= \frac{3}{2n^2\pi^2}\left[(2-6)\cos 2n\pi - 2\cos 0\right]$$

$$= \frac{3}{2n^2\pi^2}\left[-4(1)-2\right]$$

$$= \frac{-9}{n^2\pi^2}$$

$$\therefore a_n = \frac{-9}{n^2\pi^2}$$

$$\therefore b_n = \frac{1}{L}\int_0^{2L} f(x)\sin\left(\frac{n\pi x}{L}\right)dx$$

$$= \frac{2}{3}\int_0^3 (2x-x^2)\sin\left(\frac{2n\pi x}{3}\right)dx$$

$$= \frac{2}{3}\left[(2x-x^2)\left(\frac{-3}{2n\pi}\cdot\cos\frac{2n\pi x}{3}\right) - (2-2x)\left(\frac{-9}{4n^2\pi^2}\cdot\sin\frac{2n\pi x}{3}\right]\right.$$

$$\left.+(-2)\left[\frac{27}{8n^3\pi^3}\cdot\cos\frac{2\pi nx}{3}\right)\right]_0^3$$

$$= \frac{2}{3}\left[\left\{\frac{9}{2n\pi} - 0 - \frac{27}{4n^3\pi^3}\right\} - \left\{0 - 0 - \frac{27}{4n^3\pi^3}\right\}\right]$$

$$\therefore bn = \frac{2}{3}\left[\frac{9}{2n\pi}\right]$$

$$\therefore bn = \frac{3}{n\pi}$$

Substitute ao, an, bn in (i)

$$\therefore f(x) = 0 + \sum_{n=1}^{\infty}\left(\frac{-9}{n^2\pi^2}\right)\cos\frac{2n\pi x}{3} + \sum_{n=1}^{\infty}\left(\frac{3}{n\pi}\right)\cdot\sin\frac{2n\pi x}{3}$$

$$\boxed{\therefore f(x) = \frac{-9}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\cos\left(\frac{2n\pi x}{3}\right) + \frac{3}{\pi}\sum_{n=1}^{\infty}\cdot\frac{1}{n}\cdot\sin\left(\frac{2n\pi x}{3}\right)}$$

9) Find half range sine series for $x\sin x$ in $(0, \pi)$

→ $f(x) = x\sin x$ in interval $(0, \pi)$

Half range sine series in $(0, \pi)$ is given by:

$$f(x) = \sum_{n=1}^{\infty} bn \sin nx \, dx$$

where $bn = \frac{2}{\pi}\int_0^{\pi} f(x)\sin(nx)\,dx$

$$= \frac{2}{\pi}\int_0^{\pi} x\sin x \cdot \sin nx \, dx$$

$$\therefore b_n = \frac{2}{2\pi}\int_0^{\pi} x\left[\cos(n-1)x - \cos(n+1)x\right] dx$$

$$= \frac{1}{\pi}\left[\int_0^{\pi} x\cos(n-1)x\, dx - \int_0^{\pi} x\cos(n+1)x\, dx\right]$$

$$= \frac{1}{\pi}\left[\left(\frac{x\sin(n-1)x}{(n-1)}\right)_0^{\pi} - \int_0^{\pi}\frac{\sin(n-1)x}{n-1}\,dx - \left[\frac{x\sin(n+1)x}{n+1}\right]_0^{\pi} + \int_0^{\pi}\left(\frac{\sin(n+1)x}{n+1}\,dx\right)\right]$$

$$= \frac{1}{\pi}\left[\frac{\pi\sin(n-1)\pi}{n-1} + \frac{\cos(n-1)\pi}{(n-1)^2} - \frac{1}{(n-1)^2} - \frac{\pi\sin(n+1)\pi}{(n+1)} - \frac{\cos(n+1)\pi}{(n+1)^2} + \frac{1}{(n+1)^2}\right]$$

$$= \frac{1}{\pi}\left[\frac{\cos(n-1)\pi - 1}{(n-1)^2} - \frac{\cos(n+1)\pi}{(n+1)^2}\right] \quad \text{if } n \neq 1$$

$$\therefore b_n = \frac{1}{\pi}\left[\frac{(-1)^{n-1}-1}{(n-1)^2} - \frac{(-1)^{n+1}-1}{(n+1)^2}\right] \quad \text{for } n > 1$$

$$\left[\because \cos(n\pm1)\pi = -1 \quad -\cos n\pi = -1\right]$$

To find b_1,

$$b_1 = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\, dx, \quad \text{where } n=1$$

$$b_1 = \frac{2}{\pi}\int_0^{\pi} x\sin x\cdot\sin x\, dx = \frac{2}{\pi}\int_0^{\pi} x\sin^2 x\, dx$$

$$= \frac{2}{2\pi}\int_0^{\pi} x(2\sin^2 x)\, dx = \frac{1}{\pi}\int_0^{\pi} x(1-\cos 2x)\,dx$$

$$= \frac{1}{\pi}\left[\int_0^{\pi} x\, dx - \int_0^{\pi} x\cos 2x\, dx\right]$$

$$= \frac{1}{\pi}\left[\left(\frac{x^2}{2}\right)_0^{\pi} - \left[\left(\frac{x\sin 2x}{2}\right)_0^{\pi} - \int_0^{\pi} \frac{\sin 2x}{2}\, dx \right] \right]$$

$$= \frac{1}{\pi}\left[\left(\frac{x^2}{2}\right)_0^{\pi} - \left[\left(\frac{x\sin 2x}{2}\right)_0^{\pi} + \left(\frac{\cos 2x}{4}\right)_0^{\pi} \right] \right]$$

$$= \frac{1}{\pi}\left[\frac{\pi^2}{2} - \left[(0-0) + \left(\frac{1}{4} - \frac{1}{4}\right) \right] \right]$$

$$= \frac{1}{\pi} \times \frac{\pi^2}{2}$$

$$\therefore b_1 = \frac{\pi}{2}$$

Half range sine series of $f(x) = x\sin x$ is,

$$f(x) = b_1 \sin x + \sum_{n=2}^{\infty} \left[\frac{1}{\pi} \left[\frac{(-1)^{n-1} - 1}{(n-1)^2} - \frac{(-1)^{n+1} - 1}{(n+1)^2} \right] \sin nx \right]$$

when n is even

For odd n, $b_n = 0$ $(n>1)$ as $\left[(-1)^{n-1} - 1\right]$ and $\left[(-1)^{n+1} - 1\right] = 0$

$$\therefore f(x) = (\sin x)(x)$$

$$= \frac{\pi}{2}\sin x - \frac{2}{\pi}\left[\left(\frac{1}{1^2} - \frac{1}{3^2}\right)\sin 2x + \left(\frac{1}{3^2} - \frac{1}{5^2}\right)\sin 4x + \left(\frac{1}{5^2} - \frac{1}{7^2}\right)\sin 6x \dots \right]$$

$$\boxed{\therefore f(x) = \frac{\pi}{2}\sin x + \frac{2}{\pi}\left[\left(\frac{1}{3^2} - \frac{1}{1^2}\right)\sin 2x + \left(\frac{1}{5^2} - \frac{1}{3^2}\right)\sin 4x + \left(\frac{1}{7^2} - \frac{1}{5^2}\right)\sin 6x \dots \right]}$$

60004200132

10) Find half range cosine series of $f(x) = x$, $0 < x < 2$

→ $f(x) = x$ in interval $(0,2)$, where $L = 2$

∴ Half range cosine series in $(0,2)$ is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right)$$

where $a_0 = \frac{2}{L}\int_0^L f(x)\,dx = \frac{2}{2}\int_0^2 x\,dx$

$$= \left[\frac{x^2}{2}\right]_0^2 = 2$$

∴ $a_0 = 2$

$$a_n = \frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi x}{L}\right)dx$$

$$= \frac{2}{2}\int_0^2 x\cos\left(\frac{n\pi x}{2}\right)dx$$

$$= \int_0^2 x\cos\left(\frac{n\pi x}{2}\right)dx$$

$$= \left[\frac{2x\sin\left(\frac{n\pi x}{2}\right)}{n\pi}\right]_0^2 - \int_0^2 \frac{2\sin\left(\frac{n\pi x}{2}\right)}{n\pi}dx$$

$$= \left[\frac{2x\sin\left(\frac{n\pi x}{2}\right)}{n\pi}\right]_0^2 + \left[\frac{\cos\left(\frac{n\pi x}{2}\right)}{\left(\frac{n\pi}{2}\right)^2}\right]_0^2$$

$$= \frac{4\sin n\pi}{n\pi} + \frac{4\cos n\pi}{n^2\pi^2} - \frac{4(1)}{n^2\pi^2}$$

$$\therefore \quad a_n = \frac{4}{n^2\pi^2}\left((-1)^n - 1\right)$$

For $n = \text{odd}$, $(-1)^n - 1 = -2$

$n = \text{even}$, $(-1)^n - 1 = 0$

$\therefore$ Half range cosine series of $f(x) = x$ is given by:

$$\boxed{f(x) = 1 - \frac{8}{\pi^2}\left[\frac{1}{1^2}\cos\frac{\pi x}{2} + \frac{1}{3^2}\cos\frac{3\pi x}{2} + \cdots\cdots\right]}$$