Name - Ayush Jain
SAP ID - 60004200132
Div - J1

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Engineering Mathematics

Tutorial 3: Triple Integration

1) Evaluate $\int_{-1}^{1} \int_{0}^{z} \int_{x-z}^{x+z} (x+y+z)\, dy\, dx\, dz$

2) Evaluate $\iiint x^2 y z \, dx\, dy\, dz$ throughout the volume bounded by the planes $x=0$, $y=0$, $z=0$, $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$

3) Use spherical coordinates to evaluate $\iiint_V \frac{dx\, dy\, dz}{(x^2+y^2+z^2)^{3/2}}$ where V is the volume bounded by the spheres $x^2+y^2+z^2=a^2$ and $x^2+y^2+z^2=b^2$, $(b>a)$

4) Evaluate $\iiint z^2 \, dx\, dy\, dz$ over the volume bounded by the cylinder $x^2+y^2=a^2$ and the paraboloid $x^2+y^2=z$ and the plane $z=0$.

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Solutions:

1) let $$I = \int_{z=-1}^{1} \int_{x=0}^{z} \int_{y=x-z}^{x+z} (x+y+z)\, dy\, dx\, dz$$

$$I = \int_{z=-1}^{1} \left(\int_{x=0}^{z} \left[xy + \frac{y^2}{2} + yz \right]_{x-z}^{x+z} dx \right) dz$$

$$= \int_{z=-1}^{1} \left(\int_{x=0}^{z} \left[(x+z)^2 + \frac{(x+z)^2}{2} - (x^2 - z^2) - \frac{(x-z)^2}{2} \right] dx \right) dz$$

$$= \int_{z=-1}^{1} \int_{x=0}^{z} \left(\left[\frac{3}{2}(x+z)^2 - x^2 + z^2 - \frac{(x-z)^2}{2} \right] dx \right) dz$$

$$= \int_{z=-1}^{1} \left[\frac{3}{2} \cdot \frac{(x+z)^3}{3} - \frac{x^3}{3} + z^2 x - \frac{(x-z)^3}{6} \right]_0^z dz$$

$$= \int_{z=-1}^{1} \left(4z^3 - \frac{z^3}{3} + z^3 - \frac{z^3}{2} - \frac{z^3}{6} \right) dz$$

$$= \int_{z=-1}^{1} 4z^3\, dz$$

$$= \left[\frac{4z^4}{4} \right]_{-1}^{1} = 0$$

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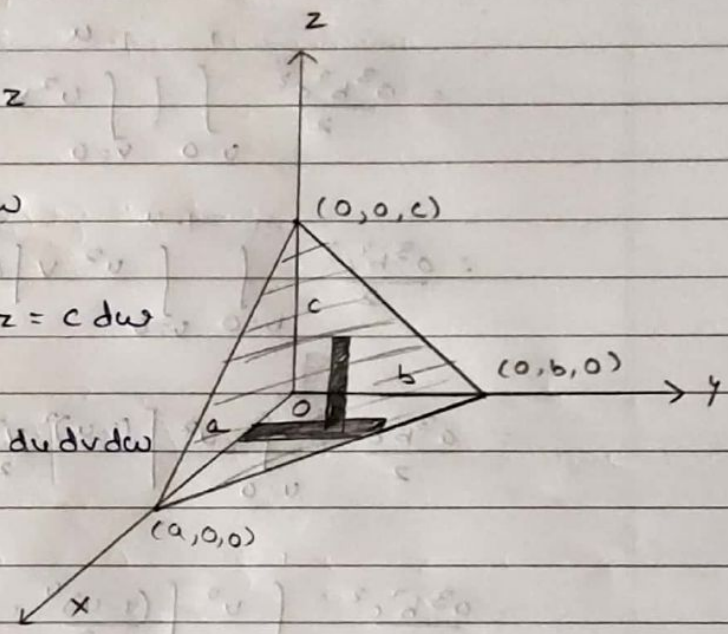
2) Let $I = \iiint x^2 y z \, dx\,dy\,dz$



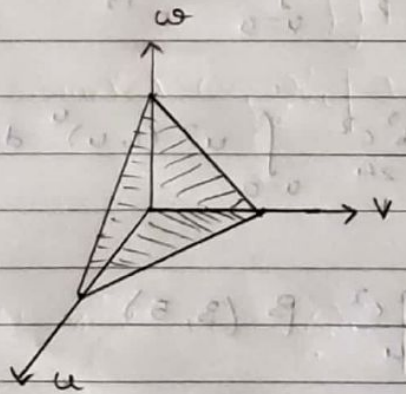
Let $\frac{x}{a} = u$, $\frac{y}{b} = v$, $\frac{z}{c} = w$

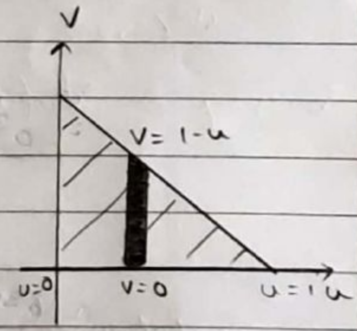
$dx = a\,du$, $dy = b\,dv$, $dz = c\,dw$

$\therefore I = \iiint a^2u^2 \cdot bv \cdot cw \cdot abc \; du\,dv\,dw$

$= \iiint a^3 b^3 c^2 u^2 v w \; du\,dv\,dw$

The planes are $u=0$, $v=0$, $w=0$, $u+v+w=1$





$I = \int_{u=0}^{1} \int_{v=0}^{1-u} \int_{w=0}^{1-u-w} a^3 b^2 c^2 u^2 v w \; du\,dv\,dw$

$\cancel{= a^3b^2c^2 \int_{u=0}^{1} u^2\,du \int_{v=0}^{1-u} v\,dv \int_{w=0}^{1-u-v} w\,dw}$

$\cancel{= a^3b^2c^2} = a^3b^2c^2 \int_{u=0}^{1} \left(\int_{v=0}^{1-u} \left[u^2 v \frac{w^2}{2} \right]_0^{1-u-v} dv \right) du$

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$\therefore$

$$= \frac{a^3b^2c^2}{2} \int_{u=0}^{1} \left(\int_{v=0}^{1-u} u^2 v (1-u-v)^2 \, dv \right) du$$

$$= \frac{a^3b^2c^2}{2} \int_{u=0}^{1} \int_{v=0}^{1-u} u^2 v \left[(1-u)^2 - 2(1-u)v + v^2\right] dv \, du$$

$$= \frac{a^3b^2c^2}{2} \int_{u=0}^{1} u^2 \left[(1-u)^2 \cdot \frac{v^2}{2} - 2(1-u) \cdot \frac{v^3}{3} + \frac{v^4}{4} \right]_0^{1-u} du$$

$$= \frac{a^3b^2c^2}{2} \int_{u=0}^{1} u^2 \left[\frac{(1-u)^4}{2} - 2\frac{(1-u)^4}{3} + \frac{(1-u)^4}{4} \right] du$$

$$= \frac{a^3b^2c^2}{2} \int_{u=0}^{1} \frac{u^2(1-u)^4}{12} \, du$$

$$= \frac{a^3b^2c^2}{24} \int_{u=0}^{1} u^2 (1-u)^4 \, du$$

$$= \frac{a^3b^2c^2}{24} \beta(3,5)$$

$$= \frac{a^3b^2c^2}{24} \frac{\Gamma 3 \, \Gamma 5}{\Gamma 8}$$

$$= \frac{a^3b^2c^2}{\cancel{24}} \times \frac{\cancel{2} \times \cancel{24}}{7 \times 6 \times 5 \times 4 \times 3 \times \cancel{2}}$$

$$I = \frac{a^3b^2c^2}{2520}$$

$$\therefore I = \frac{a^3b^2c^2}{2520}$$

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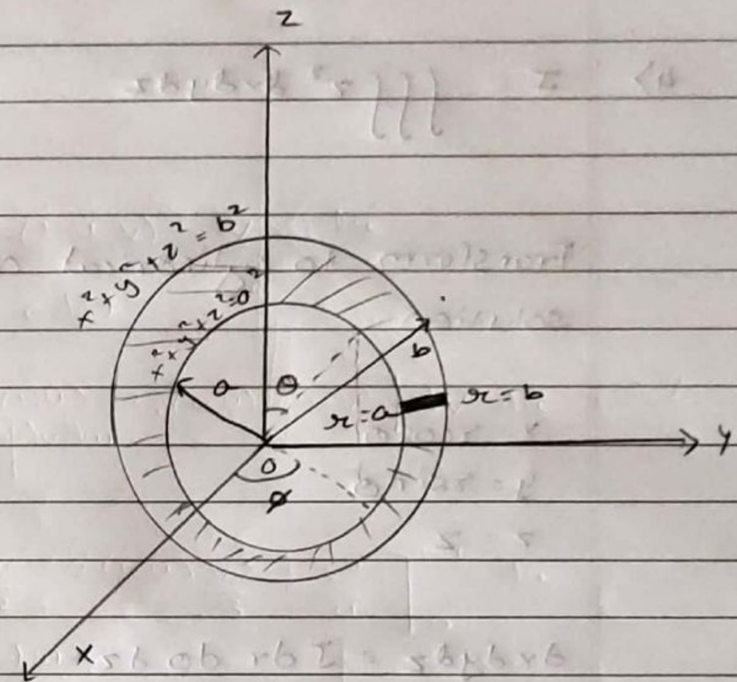
3) $\displaystyle\iiint_V \frac{dx\,dy\,dz}{(x^2+y^2+z^2)^{3/2}}$



$x^2+y^2+z^2=a^2$ $\qquad x^2+y^2+z^2=b^2$

Transform to spherical coordinate system.

$x = r\sin\theta\cos\phi$

$y = r\sin\theta\sin\phi$

$z = r\cos\theta$

$\phi = \tan^{-1}\left(\frac{y}{x}\right), \qquad \theta = \tan^{-1}\left(\frac{\sqrt{x^2+y^2}}{z}\right)$

$dx\,dy\,dz = J\,dr\,d\theta\,d\phi$

$= r^2\sin\theta\;dr\,d\theta\,d\phi$

$\theta = 0$ to π $\qquad \phi = 0$ to 2π $\qquad r = a$ to b

$\therefore\ x^2+y^2+z^2 = r^2$

$$\therefore I = \int_{\theta=0}^{\pi}\int_{\phi=0}^{2\pi}\int_{r=a}^{b} \frac{r^2\sin\theta\,dr\,d\theta\,d\phi}{(r^2)^{3/2}}$$

$$= \int_{\phi=0}^{2\pi} d\phi \cdot \int_{\theta=0}^{\pi}\sin\theta\,d\theta \int_{r=a}^{b}\frac{dr}{r}$$

$$= [2\pi][2]\left[\log b - \log a\right]$$

$$\therefore I = 4\pi\log\left(\frac{b}{a}\right)$$

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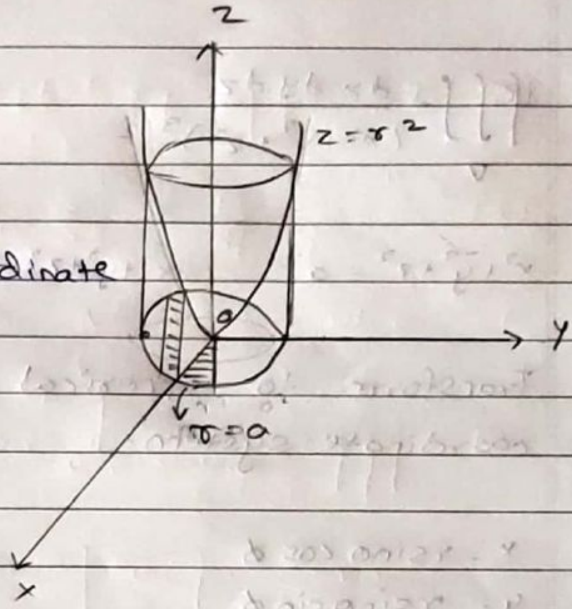
4) $I = \iiint z^2 \, dx\,dy\,dz$



Transform to cylindrical coordinate solution.

$x = r\cos\theta$
$y = r\sin\theta$
$z = z$

$dx\,dy\,dz = J\,dr\,d\theta\,dz$
$= r\,dr\,d\theta\,dz$

$x^2 + y^2 = a^2$
$r^2 = a^2$
$x^2 + y^2 = z$
$r^2 = z$

$\theta = 0$ to 2π, $r = 0$ to a, $z = 0$ to r^2

$\therefore I = \int_{\theta=0}^{2\pi} \int_{r=0}^{a} \int_{z=0}^{z} z^2 r\,dr\,d\theta\,dz$

$= \int_{\theta=0}^{2\pi} \int_{r=0}^{a} \left[\frac{z^3}{3}\right]_0^{r^2} r\,dr\,d\theta$

$= \int_{\theta=0}^{2\pi} \int_{r=0}^{a} \frac{r^7}{3}\,dr\,d\theta$

$= \int_{\theta=0}^{2\pi} \left[\frac{r^8}{24}\right]_0^a d\theta = \frac{a^8}{24}\left[\theta\right]_0^{2\pi} = \frac{\pi a^8}{12}$