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Maths - III

Tutorial 3 - Inverse Laplace Transform, Convolution Theorem

1) Find $L^{-1}\left[\frac{3s+5}{9s^2-25}\right]$

2) Find $L^{-1}\left[\frac{s+2}{s^2+4s+7}\right]$

3) Find $L^{-1}\left[\frac{5s^2+8s-1}{(s+3)(s^2+1)}\right]$

4) Find $L^{-1}\left[\cot^{-1}(s+1)\right]$

5) Find using convolution theorem

$$L^{-1}\left[\frac{s}{(s^2+4)(s^2+1)}\right]$$

6) Find using convolution theorem

$$L^{-1}\left[\frac{1}{(s-2)^4(s+3)}\right]$$

Solutions :

1) $L^{-1}\left[\dfrac{3s+5}{9s^2-25}\right]$

$= L^{-1}\left[\dfrac{3s+5}{(3s)^2-(5)^2}\right]$

$= L^{-1}\left[\dfrac{\cancel{3s+5}}{\cancel{(3s+5)}(3s-5)}\right]$

$= L^{-1}\left[\dfrac{1}{3s-5}\right]$

$= \dfrac{1}{3} L^{-1}\left[\dfrac{1}{s-5/3}\right]$

$\therefore$ By First shifting theorem,

$= \dfrac{1}{3} e^{(5/3)t} L^{-1}\left[\dfrac{1}{s}\right]$

$= \dfrac{1}{3} e^{(5/3)t} (1)$

$= \dfrac{1}{3} e^{\frac{5}{3}t}$

$$\therefore \boxed{L^{-1}\left[\dfrac{3s+5}{9s^2-25}\right] = \dfrac{1}{3} e^{\frac{5}{3}t}}$$

2) $L^{-1}\left[\frac{s+2}{s^2+4s+7}\right]$

$= L^{-1}\left[\frac{s+2}{s^2+4s+4+3}\right]$

$= L^{-1}\left[\frac{s+2}{(s+2)^2+3}\right]$

$\therefore$ By using First shifting Theorem,

$= e^{-2t} L^{-1}\left[\frac{s}{s^2+3}\right]$

$= e^{-2t} L^{-1}\left[\frac{s}{s^2+(\sqrt{3})^2}\right]$

$= e^{-2t}\left(\cos\sqrt{3}t\right)$

$\therefore \boxed{L^{-1}\left[\frac{s+2}{s^2+4s+7}\right] = e^{-2t}\cdot\cos\sqrt{3}\,t}$

3) $L^{-1}\left[\dfrac{5s^2+8s-1}{(s+3)(s^2+1)}\right]$

Let $\dfrac{5s^2+8s-1}{(s+3)(s^2+1)} = \dfrac{A}{s+3} + \dfrac{Bs+C}{s^2+1}$

$\therefore\ 5s^2+8s-1 = A(s^2+1) + (Bs+C)(s+3)$

$= (A+B)s^2 + (3B+C)s + (A+3C)$

Equating the coefficients, we get

$A+B=5,\quad 3B+C=8,\quad A+3C=-1$

$\therefore\ A=2,\quad B=3,\quad C=-1$

$\therefore\ L^{-1}\left[\dfrac{5s^2+8s-1}{(s+3)(s^2+1)}\right] = L^{-1}\left[\dfrac{2}{s+3} + \dfrac{3s-1}{s^2+1}\right]$

$= 2L^{-1}\left[\dfrac{1}{s+3}\right] + 3L^{-1}\left[\dfrac{s}{s^2+1}\right] - L^{-1}\left[\dfrac{1}{s^2+1}\right]$

$= 2e^{-3t} + 3\cos t - \sin t$

$\therefore\ \boxed{L^{-1}\left[\dfrac{5s^2+8s-1}{(s+3)(s^2+1)}\right] = 2e^{-3t} + 3\cos t - \sin t}$

4) $L^{-1}[\cot^{-1}(s+1)]$

$= \frac{-1}{t} L^{-1}\left[\frac{d}{ds}\cot^{-1}(s+1)\right]$ ... Using multiplication of power of t

$= \frac{-1}{t} L^{-1}\left[\frac{-1}{1+(s+1)^2}\right]$

$= \frac{1}{t} L^{-1}\left[\frac{1}{1+(s+1)^2}\right]$

$= \frac{1}{t} L^{-1}\left[\frac{1}{(s+1)^2+1}\right]$

$\therefore$ By using First shifting theorem,

$= \frac{1}{t} \cdot e^{-t} L^{-1}\left[\frac{1}{s^2+1}\right]$

$= \frac{1}{t} \cdot e^{-t} \cdot \sin t$

$\therefore \boxed{L^{-1}[\cot^{-1}(s+1)] = \frac{1}{t} \cdot e^{-t} \cdot \sin t}$

5) $L^{-1}\left[\frac{s}{(s^2+4)(s^2+1)}\right]$

$\therefore L^{-1}\left[\frac{s}{(s^2+4)(s^2+1)}\right] = L^{-1}\left[\frac{s}{s^2+4}\cdot\frac{1}{s^2+1}\right] = L^{-1}\left[F(s)\cdot G(s)\right]$

where $F(s) = \frac{s}{s^2+4}$, $G(s) = \frac{1}{s^2+1}$

$\therefore f(t) = L^{-1}[F(s)] = L^{-1}\left[\frac{s}{s^2+4}\right] = \cos 2t$

$g(t) = L^{-1}[G(s)] = L^{-1}\left[\frac{1}{s^2+1}\right] = \sin t$

$\therefore$ By convolution theorem,

$$L^{-1}\left[\frac{s}{(s^2+4)(s^2+1)}\right] = f(t) * g(t)$$

$$= \int_0^t f(u)\cdot g(t-u)\,du$$

$$= \int_0^t \cos 2u \cdot \sin(t-u)\,du$$

$$= \frac{1}{2}\int_0^t 2\cos 2u \cdot \sin(t-u)\,du$$

$$= \frac{1}{2}\int_0^t \left(\sin(2u+t-u) - \sin(2u-t+u)\right)du$$

$$= \frac{1}{2}\int_0^t \left(\sin(u+t) - \sin(3u-t)\right)du$$

$$= \frac{1}{2}\left[-\cos(u+t) + \frac{\cos(3u-t)}{3} \right]_0^t$$

$$= \frac{1}{2}\left[-\cos 2t + \frac{1}{3}\cos 2t + \cos t - \frac{1}{3}\cos t \right]$$

$$= \frac{1}{2}\left[-\frac{2}{3}\cos 2t + \frac{2}{3}\cos t \right]$$

$$= \frac{1}{3}\cos t - \frac{1}{3}\cos 2t$$

$$= \frac{1}{3}\left[\cos t - \cos 2t \right]$$

$$\therefore \boxed{L^{-1}\left[\frac{s}{(s^2+4)(s^2+1)} \right] = \frac{1}{3}(\cos t - \cos 2t)}$$

6> $L^{-1}\left[\frac{1}{(s-2)^4(s+3)} \right] = L^{-1}\left[\frac{1}{(s+3)} \cdot \frac{1}{(s-2)^4} \right]$

$$= L^{-1}[F(s) \cdot G(s)]$$

where $F(s) = \frac{1}{s+3}$, $G(s) = \frac{1}{(s-2)^4}$

$$\therefore f(t) = L^{-1}[F(s)] = L^{-1}\left[\frac{1}{s+3} \right] = e^{-3t}$$

$$\therefore g(t) = L^{-1}\left[\frac{1}{(s-2)^4} \right] = e^{2t} L^{-1}\left[\frac{1}{s^4} \right] = e^{2t} \cdot \frac{t^3}{3!} = e^{2t} \cdot \frac{t^3}{6}$$

$\therefore$ By convolution theorem,

$$L^{-1}\left[\frac{1}{(s+3)(s-2)^4}\right] = f(t) * g(t)$$

$$= \int_0^t f(u) \cdot g(t-u)\, du$$

$$= \int_0^t e^{-3u} \cdot e^{2(t-u)} \cdot \frac{(t-u)^3}{6}\, du$$

$$= \int_0^t e^{(2t-5u)} \cdot \frac{(t-u)^3}{6}\, du$$

$$= e^{2t} \int_0^t e^{-5u} \frac{(t-u)^3}{6}\, du$$

$$= e^{2t}\left[\frac{(t-u)^3}{6}\left(\frac{-e^{-5u}}{5}\right) - \frac{(t-u)^2}{2}(-1)\left(\frac{e^{-5u}}{25}\right) + (t-u)\left(\frac{e^{-5u}}{125}\right) + (1)\left(\frac{e^{-5u}}{625}\right)\right]_0^t$$

$$= e^{2t}\left[\frac{e^{-5t}}{625} - \left\{-\frac{t^3}{30} + \frac{t^2}{50} - \frac{t}{125} + \frac{1}{625}\right]\right.$$

$$= \frac{e^{-3t}}{625} - e^{2t}\left[\frac{1}{625} - \frac{t}{125} + \frac{t^2}{50} - \frac{t^3}{30}\right]$$

$$\therefore \boxed{L^{-1}\left[\frac{1}{(s+3)(s-2)^4}\right] = \frac{e^{-3t}}{625} - e^{2t}\left[\frac{1}{625} - \frac{t}{125} + \frac{t^2}{50} - \frac{t^3}{30}\right]}$$