

08/03/2021

Engineering MathematicsTutorial 2 : Complex numbers.

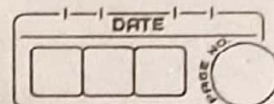
1) If z_1 and z_2 are two complex numbers such that $|z_1| = |z_2|$, show that $\frac{z_1 + z_2}{z_1 - z_2}$ is purely imaginary.

2) Find the roots common to $x^4 + 1 = 0$ and $x^6 - i = 0$

3) Show that

$$\frac{\sin 7\theta}{\sin \theta} = 7 - 56 \sin^2 \theta + 112 \sin^4 \theta - 64 \sin^6 \theta$$

4) Express $\cos^4 \theta \cdot \sin^3 \theta$ in terms of sine multiples of θ .



Solutions.

1) Let $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$

$$\therefore |z_1| = \sqrt{x_1^2 + y_1^2}, \quad |z_2| = \sqrt{x_2^2 + y_2^2}$$

Given that, $|z_1| = |z_2|$

$$\therefore \sqrt{x_1^2 + y_1^2} = \sqrt{x_2^2 + y_2^2}$$

$$\therefore x_1^2 + y_1^2 = x_2^2 + y_2^2 \quad \text{--- (1)}$$

Now,

$$\frac{z_1 + z_2}{z_1 - z_2} = \frac{x_1 + iy_1 + x_2 + iy_2}{x_1 + iy_1 - x_2 - iy_2}$$

$$= \frac{(x_1 + x_2) + i(y_1 + y_2)}{(x_1 - x_2) + i(y_1 - y_2)}$$

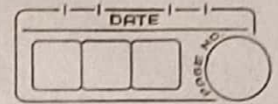
$$= \frac{[(x_1 + x_2) + i(y_1 + y_2)] \times [(x_1 - x_2) - i(y_1 - y_2)]}{[(x_1 - x_2) + i(y_1 - y_2)] \times [(x_1 - x_2) - i(y_1 - y_2)]}$$

$$= \frac{(x_1^2 - x_2^2) + i^2(y_1^2 - y_2^2)}{(x_1 - x_2)^2 + (y_1 - y_2)^2}$$

$$= \frac{(x_1^2 - x_2^2) - i(x_1 + x_2)(y_1 - y_2) + i(y_1 + y_2)(x_1 - x_2) - i^2(y_1 + y_2)(y_1 - y_2)}{(x_1 - x_2)^2 + (y_1 - y_2)^2}$$

$$= \frac{x_1^2 - x_2^2 + y_1^2 - y_2^2 + i[(x_1 - x_2)(y_1 + y_2) - (x_1 + x_2)(y_1 - y_2)]}{(x_1 - x_2)^2 + (y_1 - y_2)^2}$$

$$= \frac{0 + i[(x_1 - x_2)(y_1 + y_2) - (x_1 + x_2)(y_1 - y_2)]}{(x_1 - x_2)^2 + (y_1 - y_2)^2} \quad \dots [\text{from i}]$$



$$\therefore \frac{z_1 + z_2}{z_1 - z_2} = \frac{i[(x_1 - x_2)(y_1 + y_2) - (x_1 + x_2)(y_1 - y_2)]}{(x_1 - x_2)^2 + (y_1 - y_2)^2}$$

\therefore Since, the real part of $\frac{z_1 + z_2}{z_1 - z_2}$ is 0,

$\frac{z_1 + z_2}{z_1 - z_2}$ is purely imaginary.

2> Consider $x^4 + 1 = 0$,

$$\therefore x^4 = -1$$

$$x^4 = (\cos \pi + i \sin \pi)$$

$$x^4 = [\cos(2n\pi + \pi) + i \sin(2n\pi + \pi)]$$

$$\therefore x = \cos \pi \quad x = [\cos(2n\pi + \pi) + i \sin(2n\pi + \pi)]^{1/4}$$

$$\therefore x = \left[\frac{\cos(2n\pi + \pi)}{4} + i \frac{\sin(2n\pi + \pi)}{4} \right] \dots \text{[By De Moivre's theorem]}$$

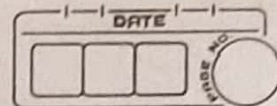
where $n = 0, 1, 2, 3$

$$\therefore x_0 = \cos\left(\frac{\pi}{4}\right) + i \sin\left(\frac{\pi}{4}\right)$$

$$\therefore x_1 = \cos\left(\frac{3\pi}{4}\right) + i \sin\left(\frac{3\pi}{4}\right)$$

$$\therefore x_2 = \cos\left(\frac{5\pi}{4}\right) + i \sin\left(\frac{5\pi}{4}\right)$$

$$\therefore x_3 = \cos\left(\frac{7\pi}{4}\right) + i \sin\left(\frac{7\pi}{4}\right)$$



NOW,

$$\text{consider } x^6 - i = 0,$$

$$\therefore x^6 = i$$

$$x^6 = \cos\left(\frac{\pi}{2}\right) + i \sin\left(\frac{\pi}{2}\right)$$

$$x^6 = \cos\left(2n\pi + \frac{\pi}{2}\right) + i \sin\left(2n\pi + \frac{\pi}{2}\right)$$

$$x^6 = \cos\left(\frac{4n\pi + \pi}{2}\right) + i \sin\left(\frac{4n\pi + \pi}{2}\right)$$

$$x = \left[\cos\left(\frac{4n\pi + \pi}{2}\right) + i \sin\left(\frac{4n\pi + \pi}{2}\right) \right]^{1/6}$$

$$\therefore x = \cos\left(\frac{4n\pi + \pi}{12}\right) + i \sin\left(\frac{4n\pi + \pi}{12}\right) \dots [\text{By De Moivre's theorem}]$$

$$\text{where } n = 0, 1, 2, 3, 4, 5$$

$$\therefore x_0 = \cos\left(\frac{\pi}{12}\right) + i \sin\left(\frac{\pi}{12}\right)$$

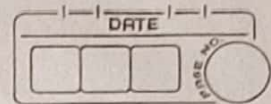
$$\therefore x_1 = \cos\left(\frac{5\pi}{12}\right) + i \sin\left(\frac{5\pi}{12}\right)$$

$$\therefore x_2 = \cos\left(\frac{9\pi}{12}\right) + i \sin\left(\frac{9\pi}{12}\right) = \cos\left(\frac{3\pi}{4}\right) + i \sin\left(\frac{3\pi}{4}\right)$$

$$\therefore x_3 = \cos\left(\frac{13\pi}{12}\right) + i \sin\left(\frac{13\pi}{12}\right)$$

$$\therefore x_4 = \cos\left(\frac{17\pi}{12}\right) + i \sin\left(\frac{17\pi}{12}\right)$$

$$\therefore x_5 = \cos\left(\frac{21\pi}{12}\right) + i \sin\left(\frac{21\pi}{12}\right) = \cos\left(\frac{7\pi}{4}\right) + i \sin\left(\frac{7\pi}{4}\right)$$



\therefore Common roots for $x^4 + 1 = 0$ and $x^6 - i = 0$ are

$$\cos\left(\frac{3\pi}{4}\right) + i\sin\left(\frac{3\pi}{4}\right) \quad \text{and} \quad \cos\left(\frac{7\pi}{4}\right) + i\sin\left(\frac{7\pi}{4}\right)$$

3) To prove that:

$$\frac{\sin 7\theta}{\sin \theta} = 7 - 56 \sin^2 \theta + 112 \sin^4 \theta - 64 \sin^6 \theta$$

Solution: We know that,

$$\cos 7\theta + i \sin 7\theta = (\cos \theta + i \sin \theta)^7 \dots \text{(By De Moivre's theorem)}$$

$$= \cos^7 \theta + 7 \cos^6 \theta \sin \theta + 21 \cos^5 \theta \sin^2 \theta + 35 \cos^4 \theta \sin^3 \theta + 35 \cos^3 \theta \sin^4 \theta + 21 \cos^2 \theta \sin^5 \theta + 7 \cos \theta \sin^6 \theta + \sin^7 \theta$$

$$\dots \text{(By Binomial Expansion)}$$

$$= \cos^7 \theta - 21 \cos^5 \theta \sin^2 \theta + 35 \cos^3 \theta \sin^4 \theta - 7 \cos \theta \sin^6 \theta + i(7 \cos^6 \theta \sin \theta - 35 \cos^4 \theta \sin^3 \theta + 21 \cos^2 \theta \sin^5 \theta - \sin^7 \theta)$$

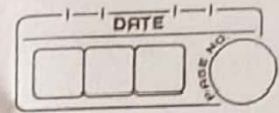
\therefore Comparing Imaginary parts on both sides,

$$\therefore \sin 7\theta = 7 \cos^6 \theta \sin \theta - 35 \cos^4 \theta \sin^3 \theta + 21 \cos^2 \theta \sin^5 \theta - \sin^7 \theta$$

Now,

$$\frac{\sin 7\theta}{\sin \theta} = \frac{7 \cos^6 \theta \sin \theta - 35 \cos^4 \theta \sin^3 \theta + 21 \cos^2 \theta \sin^5 \theta - \sin^7 \theta}{\sin \theta}$$

$$= 7 \cos^6 \theta - 35 \cos^4 \theta \sin^2 \theta + 21 \cos^2 \theta \sin^4 \theta - \sin^6 \theta$$



$$= 7(1 - \sin^2 \theta)^3 - 35 \sin^2 \theta (1 - \sin^2 \theta)^2 + 21 \sin^4 \theta (1 - \sin^2 \theta) - \sin^6 \theta$$

$$= 7(1 - 3\sin^2 \theta + 3\sin^4 \theta - \sin^6 \theta) - 35 \sin^2 \theta (1 - 2\sin^2 \theta + \sin^4 \theta) + 21 \sin^4 \theta - 21 \sin^6 \theta - \sin^6 \theta$$

$$= 7 - 21\sin^2 \theta + 21\sin^4 \theta - 7\sin^6 \theta - 35\sin^2 \theta + 70\sin^4 \theta - 35\sin^6 \theta + 21\sin^4 \theta - 21\sin^6 \theta - \sin^6 \theta$$

$$= 7 - 56\sin^2 \theta + 112\sin^4 \theta - 64\sin^6 \theta$$

$$\therefore \frac{\sin 7\theta}{\sin \theta} = 7 - 56\sin^2 \theta + 112\sin^4 \theta - 64\sin^6 \theta$$

\therefore Hence proved

4) Let $x = \cos \theta + i \sin \theta$

$$\therefore x + \frac{1}{x} = 2\cos \theta \quad \text{and} \quad x - \frac{1}{x} = 2i \sin \theta$$

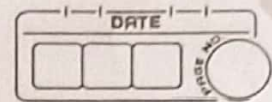
Now,

$$(2\cos \theta)^4 (2i \sin \theta)^3 = \left(x + \frac{1}{x}\right)^4 \left(x - \frac{1}{x}\right)^3$$

$$2^7 (-i) \cos^4 \theta \cdot \sin^3 \theta = \left(x + \frac{1}{x}\right) \left(x + \frac{1}{x}\right)^3 \left(x - \frac{1}{x}\right)^3$$

$$= \left(x + \frac{1}{x}\right) \left(x^2 - \frac{1}{x^2}\right)^3$$

$$= \left(x + \frac{1}{x}\right) \left(x^6 - 3x^2 + \frac{3}{x^2} - \frac{1}{x^4}\right)$$



$$= x^7 - 3x^3 + \frac{3}{x} - \frac{1}{x^5} + x^5 - 3x + \frac{3}{x^3} - \frac{1}{x^7}$$

$$= \left(x^7 - \frac{1}{x^7}\right) - 3\left(x^3 - \frac{1}{x^3}\right) + \left(x^5 - \frac{1}{x^5}\right) - 3\left(x - \frac{1}{x}\right)$$

$$= 2i\sin 7\theta - 3(2i\sin 3\theta) + 2i\sin 5\theta - 3(2i\sin \theta)$$

$$-2^7 i \cos^4 \theta \cdot \sin^3 \theta = 2i\sin 7\theta - 6i\sin 3\theta + 2i\sin 5\theta - 6i\sin \theta$$

$$\therefore \cos^4 \theta \cdot \sin^3 \theta = \frac{-1}{64} [\sin 7\theta + \sin 5\theta - 3\sin 3\theta - 3\sin \theta]$$