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08/03/2021

Engineering Mathematics

Tutorial 2 : Complex numbers.

1.) If z_1 and z_2 are two complex numbers such that $|z_1| = |z_2|$, show that $\frac{z_1+z_2}{z_1-z_2}$ is purely imaginary.

2.) Find the roots common to $x^4+1=0$ and $x^6-i=0$

3) Show that

$$\frac{\sin 7\theta}{\sin\theta} = 7 - 56\sin^2\theta + 112\sin^4\theta - 64\sin^6\theta$$

4) Express $\cos^4\theta \cdot \sin^3\theta$ in terms of sine multiples of θ.

FOR EDUCATIONAL USE

Solutions.

1) Let $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$

$\therefore |z_1| = \sqrt{x_1^2 + y_1^2}$, $|z_2| = \sqrt{x_2^2 + y_2^2}$

Given that, $|z_1| = |z_2|$

$$\therefore \sqrt{x_1^2 + y_1^2} = \sqrt{x_2^2 + y_2^2}$$

$$\therefore x_1^2 + y_1^2 = x_2^2 + y_2^2 \quad \text{— (i)}$$

Now,

$$\frac{z_1 + z_2}{z_1 - z_2} = \frac{x_1 + iy_1 + x_2 + iy_2}{x_1 + iy_1 - x_2 - iy_2}$$

$$= \frac{(x_1 + x_2) + i(y_1 + y_2)}{(x_1 - x_2) + i(y_1 - y_2)}$$

$$= \frac{[(x_1 + x_2) + i(y_1 + y_2)]}{[(x_1 - x_2) + i(y_1 - y_2)]} \times \frac{[(x_1 - x_2) - i(y_1 - y_2)]}{[(x_1 - x_2) - i(y_1 - y_2)]}$$

$$= \cancel{(x_1^2 - x_2^2) + i}$$

$$= \frac{(x_1^2 - x_2^2) - i(x_1 + x_2)(y_1 - y_2) + i(y_1 + y_2)(x_1 - x_2) - i^2(y_1 + y_2)(y_1 - y_2)}{(x_1 - x_2)^2 + (y_1 - y_2)^2}$$

$$= \frac{x_1^2 - x_2^2 + y_1^2 - y_2^2 + i\left((x_1 - x_2)(y_1 + y_2) - (x_1 + x_2)(y_1 - y_2)\right)}{(x_1 - x_2)^2 + (y_1 - y_2)^2}$$

$$= \frac{0 + i\left[(x_1 - x_2)(y_1 + y_2) - (x_1 + x_2)(y_1 - y_2)\right]}{(x_1 - x_2)^2 + (y_1 - y_2)^2} \quad \ldots \boxed{\text{from i}}$$

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$$\therefore \frac{z_1+z_2}{z_1-z_2} = \frac{i\left[(x_1-x_2)(y_1+y_2)-(x_1+x_2)(y_1-y_2)\right]}{(x_1-x_2)^2+(y_1-y_2)^2}$$

$\therefore$ Since, the real part of $\frac{z_1+z_2}{z_1-z_2}$ is 0,

$\frac{z_1+z_2}{z_1-z_2}$ is purely imaginary.

2.) Consider $x^4+1=0$,

$$\therefore x^4 = -1$$

$$x^4 = (\cos\pi + i\sin\pi)$$

$$x^4 = [\cos(2n\pi+\pi) + i\sin(2n\pi+\pi)]$$

$\therefore \cancel{x = \cos\pi}$ $\quad x = [\cos(2n\pi+\pi) + i\sin(2n\pi+\pi)]^{1/4}$

$$\therefore x = \left[\cos\left(\frac{2n\pi+\pi}{4}\right) + i\sin\left(\frac{2n\pi+\pi}{4}\right)\right] \quad \ldots \text{[By DeMoivre's theorem]}$$

where $n = 0, 1, 2, 3$

$$\therefore x_0 = \cos\left(\frac{\pi}{4}\right) + i\sin\left(\frac{\pi}{4}\right)$$

$$\therefore x_1 = \cos\left(\frac{3\pi}{4}\right) + i\sin\left(\frac{3\pi}{4}\right)$$

$$\therefore x_2 = \cos\left(\frac{5\pi}{4}\right) + i\sin\left(\frac{5\pi}{4}\right)$$

$$\therefore x_3 = \cos\left(\frac{7\pi}{4}\right) + i\sin\left(\frac{7\pi}{4}\right)$$

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NOW,

consider $x^6 - i = 0$,

$\therefore x^6 = i$

$x^6 = \cos\left(\frac{\pi}{2}\right) + i\sin\left(\frac{\pi}{2}\right)$

$x^6 = \cos\left(2n\pi + \frac{\pi}{2}\right) + i\sin\left(2n\pi + \frac{\pi}{2}\right)$

$x^6 = \cos\left(\frac{4n\pi + \pi}{2}\right) + i\sin\left(\frac{4n\pi + \pi}{2}\right)$

$x = \left[\cos\left(\frac{4n\pi+\pi}{2}\right) + i\sin\left(\frac{4n\pi+\pi}{2}\right)\right]^{1/6}$

$\therefore x = \cos\left(\frac{4n\pi+\pi}{12}\right) + i\sin\left(\frac{4n\pi+\pi}{12}\right)$ ... [By Demoivre's theorem]

where $n = 0, 1, 2, 3, 4, 5$

$\therefore x_0 = \cos\left(\frac{\pi}{12}\right) + i\sin\left(\frac{\pi}{12}\right)$

$\therefore x_1 = \cos\left(\frac{5\pi}{12}\right) + i\sin\left(\frac{5\pi}{12}\right)$

$\therefore x_2 = \cos\left(\frac{9\pi}{12}\right) + i\sin\left(\frac{9\pi}{12}\right) = \cos\left(\frac{3\pi}{4}\right) + i\sin\left(\frac{3\pi}{4}\right)$

$\therefore x_3 = \cos\left(\frac{13\pi}{12}\right) + i\sin\left(\frac{13\pi}{12}\right)$

$\therefore x_4 = \cos\left(\frac{17\pi}{12}\right) + i\sin\left(\frac{17\pi}{12}\right)$

$\therefore x_5 = \cos\left(\frac{21\pi}{12}\right) + i\sin\left(\frac{21\pi}{12}\right) = \cos\left(\frac{7\pi}{4}\right) + i\sin\left(\frac{7\pi}{4}\right)$

$\therefore$ Common roots for $x^4+1=0$ and $x^6-i=0$ are

$\cos\left(\frac{3\pi}{4}\right)+i\sin\left(\frac{3\pi}{4}\right)$ and $\cos\left(\frac{7\pi}{4}\right)+i\sin\left(\frac{7\pi}{4}\right)$

3) To prove that:

$$\frac{\sin 7\theta}{\sin\theta} = 7-56\sin^2\theta+112\sin^4\theta-64\sin^6\theta.$$

Solution: We know that,

$\cos 7\theta + i\sin 7\theta = (\cos\theta + i\sin\theta)^7$ ... (By Demoivre's theorem)

~~$= \cos^7\theta + 7\cos^6\theta\sin\theta + 21$~~

$= \cos^7\theta + 7\cos^6\theta(i\sin\theta) + 21\cos^5\theta(i\sin\theta)^2 + 35\cos^4\theta\cdot(i\sin\theta)^3$
$+35\cos^3\theta(i\sin\theta)^4 + 21\cos^2\theta(i\sin\theta)^5 + 7\cos\theta(i\sin\theta)^6 + (i\sin\theta)^7$ ... (By Binomial Expansion)

$= \cos^7\theta - 21\cos^5\theta\sin^2\theta + 35\cos^3\theta\cdot\sin^4\theta - 7\cos\theta\sin^6\theta$
$+ i(7\cos^6\theta\cdot\sin\theta - 35\cos^4\theta\sin^3\theta + 21\cos^2\theta\sin^5\theta - \sin^7\theta$

$\therefore$ Comparing Imaginary parts on both sides,

$\therefore \sin 7\theta = 7\cos^6\theta\cdot\sin\theta - 35\cos^4\theta\sin^3\theta + 21\cos^2\theta\sin^5\theta - \sin^7\theta$

Now,

$$\frac{\sin 7\theta}{\sin\theta} = \frac{7\cos^6\theta\cdot\sin\theta - 35\cos^4\theta\cdot\sin^3\theta + 21\cos^2\theta\cdot\sin^5\theta - \sin^7\theta}{\sin\theta}$$

$$= 7\cos^6\theta - 35\cos^4\theta\cdot\sin^2\theta + 21\cos^2\theta\cdot\sin^4\theta - \sin^6\theta$$

$$= 7(1-\sin^2\theta)^3 - 35\sin^2\theta(1-\sin^2\theta)^2 + 21\sin^4\theta(1-\sin^2\theta) - \sin^6\theta$$

$$= 7(1-3\sin^2\theta+3\sin^4\theta-\sin^6\theta) - 35\sin^2\theta(1-2\sin^2\theta+\sin^4\theta)$$
$$+21\sin^4\theta - 21\sin^6\theta - \sin^6\theta$$

$$= 7 - 21\sin^2\theta + 21\sin^4\theta - 7\sin^6\theta - 35\sin^2\theta + 70\sin^4\theta - 35\sin^6\theta$$
$$+21\sin^4\theta - 21\sin^6\theta - \sin^6\theta$$

$$= 7 - 56\sin^2\theta + 112\sin^4\theta - 64\sin^6\theta$$

$$\therefore \frac{\sin 7\theta}{\sin\theta} = 7 - 56\sin^2\theta + 112\sin^4\theta - 64\sin^6\theta$$

$\therefore$ Hence proved

4) Let $x = \cos\theta + i\sin\theta$

$\therefore x + \frac{1}{x} = 2\cos\theta$ and $x - \frac{1}{x} = 2i\sin\theta$

Now,

$$(2\cos\theta)^4(2i\sin\theta)^3 = \left(x+\frac{1}{x}\right)^4\left(x-\frac{1}{x}\right)^3$$

$$2^7(-i)\cos^4\theta\cdot\sin^3\theta = \left(x+\frac{1}{x}\right)\left(x+\frac{1}{x}\right)^3\left(x-\frac{1}{x}\right)^3$$

$$= \left(x+\frac{1}{x}\right)\left(x^2-\frac{1}{x^2}\right)^3$$

$$= \left(x+\frac{1}{x}\right)\left(x^6 - 3x^2 + \frac{3}{x^2} - \frac{1}{x^6}\right)$$

$$= x^7 - 3x^3 + \frac{3}{x} - \frac{1}{x^5} + x^5 - 3x + \frac{3}{x^3} - \frac{1}{x^7}$$

$$= \left(x^7 - \frac{1}{x^7}\right) - 3\left(x^3 - \frac{1}{x^3}\right) + 1\left(x^5 - \frac{1}{x^5}\right) - 3\left(x - \frac{1}{x}\right)$$

$$= 2i\sin 7\theta - 3(2i\sin 3\theta) + 2i\sin 5\theta - 3(2i\sin\theta)$$

$$-2^7 i \cos^4\theta \cdot \sin^3\theta = 2i\sin 7\theta - 6i\sin 3\theta + 2i\sin 5\theta - 6i\sin\theta$$

$$\therefore \boxed{\cos^4\theta \cdot \sin^3\theta = \frac{-1}{64}\left[\sin 7\theta + \sin 5\theta - 3\sin 3\theta - 3\sin\theta\right]}$$