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Maths - III

Tutorial 1 - Laplace Transform.

1.) Find the Laplace Transform of $f(t) = \begin{cases} \cos t & , 0 < t < \pi \\ \sin t & , t > \pi \end{cases}$

2) Find $L\left[\frac{\cos\sqrt{t}}{\sqrt{t}}\right]$

3) Evaluate by Transform,

$$\int_0^\infty e^{-4t} \sin^3 t \, dt$$

4) If $L[\sin\sqrt{t}] = \frac{\pi}{2s\sqrt{s}} e^{-1/4s}$, find $L[\sin 2\sqrt{t}]$

5) Find $L\left[\frac{\cos 2t \sin t}{e^t}\right]$

Sundaram

FOR EDUCATIONAL USE

Solutions

1.> $f(t) = \begin{cases} \cos t, & 0 < t < \pi \\ \sin t, & t > \pi \end{cases}$

$$L[f(t)] = \int_0^\infty e^{-st} f(t)\, dt$$

$$= \int_0^\pi e^{-st} \cos t\, dt + \int_\pi^\infty e^{-st} \sin t\, dt$$

But, $\int e^{ax} \cos bx\, dx = \frac{1}{(a^2+b^2)} \cdot e^{ax}(a\cos bx + b\sin bx)$

$\int e^{ax} \sin bx\, dx = \frac{1}{(a^2+b^2)} \cdot e^{ax}(a\sin bx - b\cos bx)$

$$\therefore L[f(t)] = \frac{1}{s^2+1}\left[e^{-st}(-s\cos t + \sin t)\right]_0^\pi + \frac{1}{s^2+1}\left[e^{-st}(-s\cdot\sin t - \cos t\right]_\pi^\infty$$

$$= \frac{1}{s^2+1}\left[e^{-s\pi}(s) - (-s)\right] + \frac{1}{s^2+1}\left[-e^{-s\pi}\right]$$

$$= \frac{1}{s^2+1}\left[e^{-s\pi}\cdot s + s - e^{-s\pi}\right]$$

$$= \frac{1}{s^2+1}\left[s + (s-1)e^{-s\pi}\right]$$

2) $L\left[\frac{\cos\sqrt{t}}{\sqrt{t}}\right]$

$\rightarrow$ We know, $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$

$$\cos\sqrt{t} = 1 - \frac{t}{2!} + \frac{t^2}{4!} - \frac{t^3}{6!} \cdots$$

$$\therefore \frac{\cos\sqrt{t}}{\sqrt{t}} = t^{-1/2} - \frac{t^{1/2}}{2!} + \frac{t^{3/2}}{4!} - \frac{t^{5/2}}{6!} + \cdots$$

$$\therefore L\left[\frac{\cos\sqrt{t}}{\sqrt{t}}\right] = L\left[t^{-1/2} - \frac{t^{1/2}}{2!} + \frac{t^{3/2}}{4!} - \frac{t^{5/2}}{6!} + \cdots\right]$$

$$= \frac{\Gamma 1/2}{s^{1/2}} - \frac{1}{2!}\frac{\Gamma 3/2}{s^{3/2}} + \frac{1}{4!}\frac{\Gamma 5/2}{s^{5/2}} - \frac{1}{6!}\frac{\Gamma 7/2}{s^{7/2}} + \cdots$$

$$= \frac{\Gamma 1/2}{s^{1/2}} - \frac{1}{2!}\frac{(1/2)\,\Gamma 1/2}{s^{3/2}} + \frac{1}{4!}\frac{(3/2)(1/2)\,\Gamma 1/2}{s^{5/2}} - \frac{1}{6!}\frac{(5/2)(3/2)(1/2)\,\Gamma 1/2}{s^{7/2}} \cdots$$

$$= \frac{\sqrt{\pi}}{\sqrt{s}}\left[1 - \frac{1}{4s} + \frac{1}{2!(4s)^2} - \frac{1}{3!.(4s)^3} + \cdots\right]$$

$$= \sqrt{\frac{\pi}{s}}\left[e^{-1/4s}\right]$$

3) $\int_0^\infty e^{-4t} \sin^3 t \, dt$

$\longrightarrow$ $\therefore \int_0^\infty e^{-4t} \sin^3 t \, dt = L[\sin^3 t]$, where $s = 4$

$$L[\sin^3 t] = L\left[\frac{3}{4}\sin t - \frac{1}{4}\sin 3t\right]$$

$$= \frac{3}{4} L[\sin t] - \frac{1}{4} L[\sin 3t]$$

$$= \frac{3}{4}\left[\frac{1}{s^2+1}\right] - \frac{1}{4}\left[\frac{3}{s^2+9}\right]$$

$\therefore \int_0^\infty e^{-4t} \sin^3 t \, dt = \frac{3}{4}\left(\frac{1}{s^2+1}\right) - \frac{1}{4}\left(\frac{3}{s^2+9}\right)$, where $s = 4$

$$= \frac{3}{4}\left(\frac{1}{16+1}\right) - \frac{1}{4}\left(\frac{3}{16+9}\right)$$

$$= \frac{3}{4}\left(\frac{1}{17}\right) - \frac{1}{4}\left(\frac{3}{25}\right)$$

$$= \frac{3}{4}\left(\frac{1}{17} - \frac{1}{25}\right)$$

$$= \frac{3}{4}\left(\frac{8}{25 \times 17}\right)$$

$$\int_0^\infty e^{-4t} \sin^3 t \, dt = \frac{6}{425}$$

4) If $L[\sin\sqrt{t}] = \frac{\sqrt{\pi}}{2s\sqrt{s}} e^{-1/4s}$, find $L[\sin 2\sqrt{t}]$

$\longrightarrow$ $L[\sin\sqrt{t}] = \frac{\sqrt{\pi}}{2s\sqrt{s}} e^{-1/4s} = F(s)$

$\therefore$ By change of scalar theorem,

$L[\sin 2\sqrt{t}] = L[\sin(\sqrt{4t})]$

$= \frac{1}{4} F\left(\frac{s}{4}\right)$

$\cancel{= \frac{1}{4} \cdot \frac{\sqrt{\pi}}{2s\sqrt{s}} e}$

$= \frac{1}{4} \frac{\sqrt{\pi}}{2\left(\frac{s}{4}\right)\left(\sqrt{\frac{s}{4}}\right)} \cdot e^{-1/4\left(\frac{s}{4}\right)}$

$= \frac{1}{4} \cdot \frac{2\sqrt{\pi}}{s\sqrt{s}} \cdot 2 \; e^{-1/s}$

$= \frac{\sqrt{\pi}}{s\sqrt{s}} e^{-1/s}$

$\therefore L[\sin 2\sqrt{t}] = \frac{\sqrt{\pi}}{s\sqrt{s}} e^{-1/s}$

5> $L\left[\dfrac{\cos 2t \cdot \sin t}{e^{t}}\right]$

$= L\left[e^{-t} \cos 2t \cdot \sin t\right]$

Now, $\cos 2t \cdot \sin t = \frac{1}{2} \cdot 2\sin t \cdot \cos 2t = \frac{1}{2}\left[\sin 3t - \sin t\right]$

$\therefore L\left[\cos 2t \sin t\right] = \frac{1}{2} L\left[\sin 3t - \sin t\right]$

$L\left[\cos 2t \sin t\right] = \frac{1}{2}\left(\frac{3}{s^2+9} - \frac{1}{s^2+1}\right)$

$\therefore$ By First shifting Theorem,

$L\left[e^{-t} \cos 2t \cdot \sin t\right] = \frac{1}{2}\left(\frac{3}{(s+1)^2+9} - \frac{1}{(s+1)^2+1}\right)$

$= \frac{1}{2}\left[\frac{3}{s^2+2s+1+9} - \frac{1}{s^2+2s+1+1}\right]$

$= \frac{1}{2}\left[\frac{3(s^2+2s+2) - s^2-2s-10}{(s^2+2s+10)(s^2+2s+2)}\right]$

$= \frac{1}{2}\left[\frac{2s^2+4s-4}{(s^2+2s+10)(s^2+2s+2)}\right]$

$\therefore L\left[e^{-t} \cos 2t \sin 2t\right] = \dfrac{s^2+2s-2}{(s^2+2s+10)(s^2+2s+2)}$