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Maths Tutorial 5

Q. 2) Solution :

$$f(-x) = \begin{cases} x + \pi/2 & , -\pi < -x < 0 \\ \pi/2 + x & , 0 < -x < \pi \end{cases}$$

$$f(-x) = \begin{cases} \pi/2 - x & , \pi > x > 0 \\ \pi/2 + x & , 0 < x < -\pi \end{cases} = f(x)$$

$\therefore f(x) = f(-x)$

Hence, it is an even function.

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + b_n \sin nx$$

$b_n = 0$ $\because f(x)$ is an even function.

$$\therefore f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx \quad \text{——} (i)$$

where $a_0 = \frac{1}{\pi}\int_0^{\pi} f(x)\,dx = \frac{1}{\pi}\int_0^{\pi}\left(\frac{\pi}{2} - x\right)dx = \frac{1}{\pi}\left(\frac{\pi x}{2} - \frac{x^2}{2}\right)_0^{\pi}$

$\therefore a_0 = 0$

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx = \frac{2}{\pi}\int_0^{\pi}\left(\frac{\pi}{2} - x\right)\cos nx\,dx$$

$$= \frac{2}{\pi}\left[\left(\frac{\pi}{2} - x\right)\left(\frac{\sin nx}{n}\right) - \left(\frac{\cos nx}{n^2}\right)\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[-\frac{(-1)^n}{n^2} + \frac{1}{n^2}\right]$$

$$a_n = \frac{2}{\pi}\left[\frac{1}{n^2} - \frac{(-1)^n}{n^2}\right]$$

$$\therefore f(x) = \frac{4}{\pi}\left[\frac{1}{1^2}\cos x + \frac{1}{3^2}\cos 3x + \frac{1}{5^2}\cos 5x + \cdots\right] \quad \text{--- (2)}$$

(i) Now $f(x)$ is discontinuous at $x=0$,

But at a point of discontinuity $x = c$

$$\therefore f(x) = \frac{1}{2}\left[\lim_{x\to c^-} f(x) + \lim_{x \to c^+} f(x)\right]$$

$$\therefore f(x) = \frac{1}{2}\left[\frac{\pi}{2} + \frac{\pi}{2}\right] = \frac{\pi}{2}$$

Putting $x=0$, in (2)

$$\frac{\pi}{2} = \frac{4}{\pi}\left[\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots\right]$$

$$\therefore \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots \frac{\pi^2}{8}$$

(ii) Using Perseval's identity in $(-\pi, \pi)$

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}[f(x)]^2 dx = a_0^2 + \frac{1}{2}\sum_{n=1}^{\infty}(a_n^2 + b_n^2) \quad \text{--- (3)}$$

Here, $a_0 = 0$ and $b_n = 0$

$$\therefore \frac{1}{2\pi}\int_{-\pi}^{\pi}[f(x)]^2 dx = \frac{1}{2\pi}\left[\int_{-\pi}^{0}\left(x+\frac{\pi}{2}\right)^2 dx + \int_{0}^{\pi}\left(\frac{\pi}{2}-x\right)^2 dx\right]$$

$$= \frac{1}{2\pi}\left[\int_{-\pi}^{0}\left(x^2 + \pi + \frac{\pi^2}{4}\right)dx + \int_{0}^{\pi}\left(\frac{\pi^2}{4} - \pi x + x^2\right)\right.$$

$$= \frac{1}{2\pi}\left[\left(\frac{x^3}{3} + \frac{\pi x^2}{2} + \frac{\pi^2 x}{4}\right)_{-\pi}^{0} + \left(\frac{\pi^2 x}{4} - \frac{\pi^2 x}{2} + \frac{x^3}{3}\right)\right]$$

$$= \frac{2\pi^3}{2\pi}\left[\frac{1}{4} - \frac{1}{2} + \frac{1}{3}\right]$$

$$\therefore \frac{1}{2\pi}\int_{-\pi}^{\pi}[f(x)]^2 dx = \frac{\pi^2}{12}$$

Substituting the values of an in (3)

$$\frac{\pi^2}{12} = \frac{1}{2} \cdot \frac{16}{\pi^2}\left[\frac{1}{1^4} + \frac{1}{3^4} + \frac{1}{5^4} \cdots\right]$$

$$\therefore \frac{\pi^4}{96} = \frac{1}{1^4} + \frac{1}{3^4} + \frac{1}{5^4} + \cdots$$

$$\frac{1}{1^4} + \frac{1}{3^4} + \frac{1}{5^4} + \cdots = \frac{\pi^4}{96}$$

Q.3) Solution:

Here $2L = 2$, $L = 1$

$$f(x) = a_0 + \sum_{n=1}^{\infty}\left[a_n \cos\left(\frac{n x \pi}{l}\right) + b_n \sin\left(\frac{n x \pi}{l}\right)\right]$$

$$\therefore f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos(n\pi x) + b_n \sin(\pi n x) \quad -(1)$$

where $a_0 = \frac{1}{2l}\int_0^{2L} f(x)\,dx = \frac{1}{2}\int_0^{2}(4 - x^2)\,dx = \frac{1}{2}\left(4x - \frac{x^3}{3}\right]_0^2$

$$\therefore a_0 = \frac{8}{3}$$

$$a_n = \frac{1}{L}\int_0^{2L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx = \frac{1}{1}\int_0^{2}(4 - x^2)(\cos n\pi x)\,dx$$

$$= \left[(4 - x^2)\left(\frac{\sin n\pi x}{n\pi}\right) - (2x)\left(\frac{\cos n\pi x}{n^2\pi^2}\right) + 2\left(\frac{\sin n\pi x}{n^3\pi^3}\right)\right]_0^2$$

$$= \left[0 - \frac{4}{n^2\pi^2} + 0\right) - (0)\right]$$

$$a_n = \frac{-4}{n^2\pi^2}$$

$$b_n = \frac{1}{l}\int_0^{2l} f(x)\sin\frac{n\pi x}{l}\,dx = \int_0^2 (4-x^2)\sin n\pi x\,dx$$

$$= \left[(4-x^2)\left(\frac{-\cos n\pi x}{n\pi}\right) - (2x)\left(\frac{\sin n\pi x}{n^2\pi^2}\right) - 2\left(\frac{\cos n\pi x}{n^3\pi^3}\right)\right]_0^2$$

$$= \left[\left\{0 - \frac{2}{n^3\pi^3}\right\} - \left\{-\frac{4}{n\pi} - \frac{2}{n^3\pi^3}\right\}\right]$$

$$b_n = \frac{4}{n\pi}$$

$$f(x) = \frac{8}{3} - \frac{4}{\pi^2}\left[\frac{1}{1^2}\cos\pi x + \frac{1}{2^2}\cos 2x + \cdots\right] + \frac{4}{\pi}\left[\sin x + \frac{1}{2}\sin 2x + \cdots\right]$$

$L(2)$

Since $f(x)$ is discontinuous at $x = 0, 2, 4, 6\ldots$ we find its value as

$$f(c) = \frac{1}{2}\left[\lim_{x\to c^-} f(x) + \lim_{x\to c^+} f(x)\right]$$

when $c = 0$,

$$f(0) = \frac{1}{2}\left[\lim_{x\to 0^-} f(x) + \lim_{x\to 0^+} f(x)\right] = \frac{0+4}{2} = 2$$

$$f(2) = \frac{1}{2}\left[\lim_{x\to 2^-} f(x) + \lim_{x\to 2^+} f(x)\right] = \frac{4+0}{2} = 2$$

Similarly $f(4) = f(6) = f(8) = f(10) = 2$ since $f(x)$ is periodic with period 2

Now, at $x = 1$, the function is continuous. $\therefore f(1) = 4 - 1 = 3$

Also at $x = 1$, the function is continuous, $\therefore f(11) = 3$

$\therefore f(1) = 3$, $f(2) = 2$, $f(10) = 2$ and $f(11) = 3$

Put $x=0$ and $x=2$ in eq (2)

$$4-0 = \frac{8}{3} - \frac{4}{\pi^2}\left[\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\cdots\right]$$

$$\therefore \frac{1}{3} = \frac{-1}{\pi^2}\left[\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\cdots\right] \quad -(3)$$

$$4-4 = \frac{8}{3} - \frac{4}{\pi^2}\left[\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\cdots\right]$$

$$\therefore -\frac{2}{3} = \frac{-1}{\pi^2}\left[\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\cdots\right] \quad -(4)$$

Adding (3) and (4), we get

$$-\frac{1}{3} = \frac{-2}{\pi^2}\left[\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\cdots\right]$$

$$\therefore \frac{\pi^2}{6} = \frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\cdots$$

$\therefore$ ~~$\frac{1}{1^2}+$~~

Q. 4) Solution:

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx \qquad (l=\pi)$$

where $b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\,dx = \frac{2}{\pi}\left[\int_0^{\pi/3} \frac{2x}{3}\sin(nx)\,dx + \int_{\pi/3}^{\pi}\left(\frac{\pi-x}{3}\right)\sin(nx)\,dx\right]$

$$= \frac{2}{3\pi}\left[\left\{(2x)\left(\frac{-\cos nx}{n}\right)+\left(\frac{2\sin nx}{n^2}\right)\right\}_0^{\pi/3} + \left\{(\pi-x)\cdot\left(\frac{-\cos nx}{n}\right)-\left(\frac{\sin nx}{n^2}\right)\right\}_{\pi/3}^{\pi}\right]$$

$$\therefore b_n = \frac{2}{3\pi}\cdot\frac{3}{n^2}\sin\frac{n\pi}{3} = \frac{2}{\pi n^2}\sin\frac{n\pi}{3}$$

$$\therefore f(x) = \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{1}{n^2} \sin\frac{n\pi}{3} \cdot \sin nx$$

$$\therefore f(x) = \frac{2}{\pi}\left[\frac{1}{1^2} \frac{\sqrt{3}}{2} \sin x + \frac{1}{1^2} \cdot \frac{\sqrt{3}}{2} \sin 2x - \frac{1}{4^2} \frac{\sqrt{3}}{2} \sin 4x - \frac{1}{5^2} \cdot \frac{\sqrt{3}}{2} \sin 5x \cdots\right]$$

$$\therefore f(x) = \frac{\sqrt{3}}{\pi}\left[\frac{1}{1^2} \sin x + \frac{1}{2^2} \sin 2x - \frac{1}{4^2} \sin 4x - \frac{1}{5^2} \sin 5x \cdots\right]$$