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Engineering Mathematics - II

Term-Test 1 Assignment

1) Evaluate $\int_0^1 \frac{dx}{\sqrt{-\log x}}$

Solution Let $I = \int_0^1 \frac{dx}{\sqrt{-\log x}}$

Put $\log x = -t$

$\therefore x = e^{-t}$

$dx = -e^{-t}\, dt$

when $x = 0$, $t = \infty$

$x = 1$, $t = 0$

$\therefore I = \int_\infty^0 \frac{-e^{-t}}{\sqrt{t}}$

$= \int_0^\infty e^{-t} \cdot t^{-1/2}\, dt$

$= \int_0^\infty e^{-t}\, t^{-1/2+1-1}\, dt$

$= \int_0^\infty e^{-t}\, t^{1/2-1}\, dt$

$= \Gamma\frac{1}{2}$ ... (Using defination of Gamma function)

$= \sqrt{\pi}$ (Property of Gamma function. $\Gamma\frac{1}{2} = \sqrt{\pi}$)

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2) Evaluate $\int_0^1 \sqrt{1-\sqrt{x}}\,dx \cdot \int_0^{1/2} \sqrt{2y-4y^2}\,dy$

Solution Let $I = \int_0^1 \sqrt{1-\sqrt{x}}\,dx \cdot \int_0^{1/2} \sqrt{2y-4y^2}\,dy$

$= I_1 \cdot I_2$

$\therefore I_1 = \int_0^1 \sqrt{1-\sqrt{x}}\,dx$

Put $\sqrt{x} = t$

$x = t^2$

$dx = 2t\,dt$

when $x=0, t=0$ and $x=1, t=1$

$\therefore I_1 = \int_0^1 \sqrt{1-t}\; 2t \cdot dt$

$= 2\int_0^1 t\,(1-t)^{1/2}\,dt$

$= 2\int_0^1 t^{2-1}\,(1-t)^{3/2-1}\,dt$

$I_1 = 2\,\beta(2, \frac{3}{2})$

$\therefore I_2 = \int_0^{1/2} \sqrt{2y-4y^2}\,dy$

Put $2y = t$

$2dy = dt$

when $y=0, t=0$ and $y=1/2, t=1$

$\therefore I_2 = \int_0^1 \sqrt{t-t^2} \cdot \frac{1}{2}\,dt$

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$$\therefore\ I_2 = \frac{1}{2}\int_0^1 t^{1/2}(1-t)^{1/2}\,dt$$

$$= \frac{1}{2}\int_0^1 t^{3/2-1}(1-t)^{3/2-1}\,dt$$

$$\therefore\ I_2 = \frac{1}{2}\beta\left(\frac{3}{2},\frac{3}{2}\right)$$

$$\therefore\ I = I_1 \cdot I_2$$

$$= \cancel{2}\,\beta\left(2,\frac{3}{2}\right)\cdot\frac{1}{\cancel{2}}\beta\left(\frac{3}{2},\frac{3}{2}\right)$$

$$= \frac{\Gamma 2\,\Gamma 3/2}{\Gamma 7/2}\cdot\frac{\Gamma 3/2\,\Gamma 3/2}{\Gamma 3}$$

$$= \frac{1\times\cancel{\Gamma 3/2}}{\frac{5}{\cancel{2}}\cdot\frac{3}{\cancel{2}}\cdot\cancel{\Gamma 3/2}}\cdot\frac{\left(\frac{1}{\cancel{2}}\Gamma\frac{1}{2}\right)^2}{2!}$$

$$= \frac{\left(\Gamma\frac{1}{2}\right)^2}{15\times 2}$$

$$\therefore\ I = \frac{\pi}{30} \quad \ldots \quad \left(\Gamma\tfrac{1}{2} = \sqrt{\pi}\right)$$

$$\therefore\ \int_0^1 \sqrt{1-\sqrt{x}}\,dx \cdot \int_0^{1/2}\sqrt{2y-4y^2}\,dy = \frac{\pi}{30}$$

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3) Prove that $B(x,x) = \frac{1}{2^{2x-1}} B(x, \frac{1}{2})$

Solution $\therefore\ B(x,x) = \frac{\Gamma x\ \Gamma x}{\Gamma 2x}$ — (i)

By duplication formula of Gamma functions,

$$2^{2m-1}\, \Gamma m\ \Gamma m+1/2 = \sqrt{\pi}\ \Gamma 2m$$

$$\therefore\ \frac{\Gamma m}{\Gamma 2m} = \frac{\sqrt{\pi}}{2^{2m-1} \cdot \Gamma m+1/2}$$

$$\therefore\ \frac{\Gamma x}{\Gamma 2x} = \frac{\sqrt{\pi}}{2^{2x-1}\ \Gamma x+1/2} \quad \text{— (ii)}$$

$\therefore$ from (i) and (ii)

$$B(x,x) = \frac{\Gamma x \cdot \sqrt{\pi}}{2^{2x-1}\ \Gamma x+1/2}$$

$$= \frac{1}{2^{2x-1}} \cdot \frac{\Gamma x\ \Gamma 1/2}{\Gamma x+1/2} \quad \ldots\ \left[\because\ \Gamma \tfrac{1}{2} = \sqrt{\pi}\right]$$

$$= \frac{1}{2^{2x-1}} \cdot B(x, \tfrac{1}{2}) \quad \ldots\ \left[\because\ B(x,\tfrac{1}{2}) = \frac{\Gamma x\ \Gamma 1/2}{\Gamma x+1/2}\right]$$

$$\therefore\ B(x,x) = \frac{1}{2^{2x-1}}\ B(x, \tfrac{1}{2})$$

-Hence Proved.

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4) Prove that $\int_0^\infty \frac{dx}{(e^x+e^{-x})^n} = \frac{1}{4}\beta\left(\frac{n}{2},\frac{n}{2}\right)$ and hence evaluate

$\int_0^\infty \operatorname{sech}^8 x \, dx$

Solution Let $I = \int_0^\infty \frac{dx}{(e^x+e^{-x})^n}$

$$= \frac{1}{2}\int_{-\infty}^{\infty} \frac{dx}{(e^x+e^{-x})^n}$$

Put $e^x = \tan\theta$

$\therefore e^x dx = \sec^2\theta \, d\theta$

$\therefore dx = \frac{\sec^2\theta \, d\theta}{e^x} = \frac{\sec^2\theta \, d\theta}{\tan\theta}$

when $x = \infty$, $\theta = \pi/2$

$x = -\infty$, $\theta = 0$

$$\therefore I = \frac{1}{2}\int_0^{\pi/2} \frac{1}{(\tan\theta+\cot\theta)^n} \cdot \frac{\sec^2\theta}{\tan\theta} \, d\theta$$

$$= \frac{1}{2}\int_0^{\pi/2} \frac{1}{\left(\frac{\sin\theta}{\cos\theta}+\frac{\cos\theta}{\sin\theta}\right)^n} \cdot \frac{1}{\cos^2\theta} \cdot \frac{\cos\theta}{\sin\theta} \, d\theta$$

$$= \frac{1}{2}\int_0^{\pi/2} \frac{\sin^n\theta \, \cos^n\theta}{\sin\theta \cdot \cos\theta} \, d\theta$$

$$= \frac{1}{2}\int_0^{\pi/2} \sin^{n-1}\theta \cdot \cos^{n-1}\theta \, d\theta$$

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$$= \frac{1}{2} \cdot \frac{1}{2} \beta\left(\frac{n}{2}, \frac{n}{2}\right) \quad \because \left[\int_0^{\frac{\pi}{2}} \sin^p \theta \cdot \cos^q \theta = \frac{1}{2} \beta\left(\frac{p+1}{2}, \frac{q+1}{2}\right) \right.$$

$$I = \frac{1}{4} \cdot \beta\left(\frac{n}{2}, \frac{n}{2}\right)$$

$$\therefore \int_0^{\infty} \frac{dx}{(e^x + e^{-x})^n} = \frac{1}{4} \beta\left(\frac{n}{2}, \frac{n}{2}\right)$$

- Hence Proved.

NOW,

$$\because \frac{e^x + e^{-x}}{2} = \cosh x$$

$$\therefore e^x + e^{-x} = 2\cosh x$$

Putting $n = 8$ in the integral,

$$\therefore \int_0^{\infty} \frac{dx}{(e^x + e^{-x})^8} = \frac{1}{4} \beta(4,4)$$

$$\therefore \int_0^{\infty} \frac{dx}{2^8 \cosh^8 x} = \frac{1}{4} \beta(4,4)$$

$$\therefore \int_0^{\infty} \operatorname{sech}^8 x \, dx = \frac{2^8}{4} \cdot \beta(4,4)$$

$$= \frac{2^8}{4} \cdot \frac{\Gamma 4 \, \Gamma 4}{\Gamma 8} = 2^6 \frac{3! \cdot 3!}{7!} = \frac{16}{35}$$

$$\therefore \int_0^{\infty} \operatorname{sech}^8 x \, dx = \frac{16}{35}$$

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5) Assuming the validity of differentiation under the integral sign, prove that

$$\int_0^{\frac{\pi}{2}} \log\left(\frac{a+b\sin\theta}{a-b\sin\theta}\right)\operatorname{cosec}\theta\, d\theta = \pi \sin^{-1}\left(\frac{b}{a}\right), \quad a > b$$

Solution Let $I(b) = \int_0^{\frac{\pi}{2}} \log\left(\frac{a+b\sin\theta}{a-b\sin\theta}\right)\operatorname{cosec}\theta\, d\theta$ — (i)

$$= \int_0^{\frac{\pi}{2}} \left(\log\left[(a+b\sin\theta)\right] - \log\left[a-b\sin\theta\right]\right) \cdot \frac{d\theta}{\sin\theta}$$

By the rule of differentiation under the integral sign,

$$\therefore \frac{dI}{db} = \int_0^{\pi/2} \left[\frac{\sin\theta}{a+b\sin\theta} + \frac{\sin\theta}{a-b\sin\theta}\right] \cdot \frac{d\theta}{\sin\theta}$$

$$= \int_0^{\frac{\pi}{2}} \left[\frac{1}{a+b\sin\theta} + \frac{1}{a-b\sin\theta}\right] d\theta$$

$$= \int_0^{\frac{\pi}{2}} \frac{2a}{a^2 - b^2\sin^2\theta}\, d\theta$$

$$= \int_0^{\frac{\pi}{2}} \frac{2a\operatorname{cosec}^2\theta}{a^2\operatorname{cosec}^2\theta - b^2}\, d\theta$$

Put $\cot\theta = t$

$\therefore -\operatorname{cosec}^2\theta\, d\theta = dt$

and $\operatorname{cosec}^2\theta = 1+\cot^2\theta = 1+t^2$

when $\theta = 0$, $t = \infty$

$\theta = \frac{\pi}{2}$, $t = 0$

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$$\therefore \frac{dI}{db} = \int_{\infty}^{0} \frac{-2a\,dt}{(1+t^2)a^2-b^2}$$

$$= \int_{0}^{\infty} \frac{2a\,dt}{(1+t^2)a^2-b^2}$$

$$= \int_{0}^{\infty} \frac{2a\,dt}{a^2t^2+(a^2-b^2)}$$

$$= \frac{2}{a}\int_{0}^{\infty} \frac{dt}{t^2+(a^2-b^2)/a^2}$$

$$= \frac{2}{a}\cdot\frac{a}{\sqrt{a^2-b^2}}\left[\tan^{-1}\frac{ta}{\sqrt{a^2-b^2}}\right]_0^{\infty}$$

$$= \frac{2}{\sqrt{a^2-b^2}}\left[\frac{\pi}{2}-0\right]$$

$$\therefore \frac{dI}{db} = \frac{2}{\sqrt{a^2-b^2}}\cdot\frac{\pi}{2} = \frac{\pi}{\sqrt{a^2-b^2}}$$

Integrating both sides wrt b,

$$\therefore \int dI = \int \frac{\pi}{\sqrt{a^2-b^2}}\,db + c$$

$$\therefore I(b) = \pi\int\frac{db}{\sqrt{a^2-b^2}} + c$$

$$= \pi\sin^{-1}\left(\frac{b}{a}\right) + c \quad \text{—— (2)}$$

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To find c, we put $b=0$ in (2), $\therefore I(0)=C$

Now, putting $b=0$ in $I(b)$ i.e. in (i) we get,

$$I(0)=\int_0^{\frac{\pi}{2}}[\log b-\log a]\frac{d\theta}{\sin\theta} = \int_0^{\frac{\pi}{2}} 0\,dx = 0$$

Hence, $C=0$

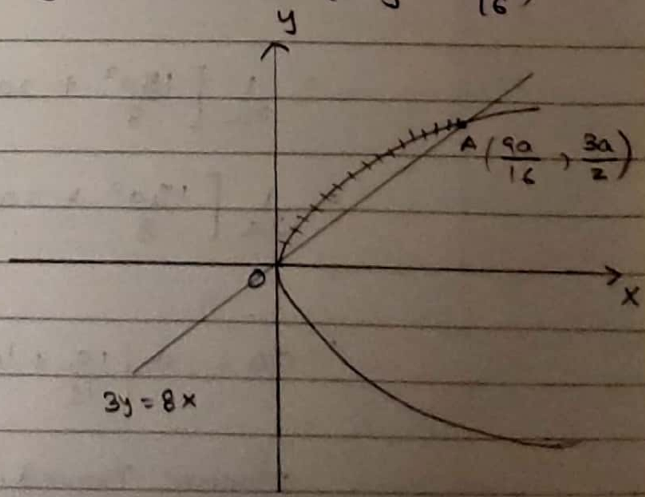
$\therefore$ From (2), we get

$$I(b)=\pi\sin^{-1}\left(\frac{b}{a}\right)$$

$$\therefore \int_0^{\frac{\pi}{2}} \log\left(\frac{a+b\sin\theta}{a-b\sin\theta}\right)\operatorname{cosec}\theta\,d\theta = \pi\sin^{-1}\left(\frac{b}{a}\right)$$

-Hence Proved.

6) Show that the length of the arc of the parabola $y^2=4ax$ cut-off by the line $3y=8x$ is $a\left(\log 2+\frac{15}{16}\right)$



Solution: Solving $y^2=4ax$ and $3y=8x$ for point of intersection A

$\therefore \left(\frac{8x}{3}\right)^2 = 4ax$

$\therefore 64x^2 = 4ax(9)$

$\therefore 64x^2-36ax=0$

$\therefore x=0, \quad x=\frac{9a}{16}$

$\therefore y=0; \quad y=\frac{3a}{2}$

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$\therefore$ Point A is $\left(\frac{9a}{16}, \frac{3a}{2}\right)$

Now, $x = \frac{y^2}{4a}$ $\therefore \frac{dx}{dy} = \frac{2y}{4a} = \frac{y}{2a}$

$\therefore$ Required arc length $= OA = \int_{y_1}^{y_2} \sqrt{1+\left(\frac{dx}{dy}\right)^2}\, dy$

$$\therefore OA = \int_0^{\frac{3a}{2}} \sqrt{1+\frac{y^2}{4a^2}}\, dy = \frac{1}{2a}\int_0^{\frac{3a}{2}} \sqrt{y^2+4a^2}\, dy$$

$$= \frac{1}{2a}\left[\frac{y}{2}\sqrt{y^2+4a^2} + \frac{4a^2}{2}\log\left(y+\sqrt{y^2+4a^2}\right)\right]_0^{\frac{3a}{2}}$$

$$= \frac{1}{2a}\left[\frac{3a}{4}\sqrt{\frac{25a^2}{4}} + \frac{4a^2}{2}\log\left(\frac{3a}{2}+\sqrt{\frac{25a^2}{4}}\right) - \frac{4a^2}{2}\log(2a)\right]$$

$$= \frac{1}{2a}\left[\frac{15a^2}{8} + 2a^2\log\left(\frac{3a}{2}+\frac{5a}{2}\right) - 2a^2\log 2a\right]$$

$$= \frac{1}{2a}\left[\frac{15a^2}{8} + 2a^2\left(\log 4a - \log 2a\right)\right]$$

$$= \frac{1}{2a}\left[\frac{15a^2}{8} + 2a^2\log 2\right] = a\left[\frac{15}{16}+\log 2\right]$$

$$\therefore OA = a\left[\frac{15}{16}+\log 2\right]$$

- Hence Proved.

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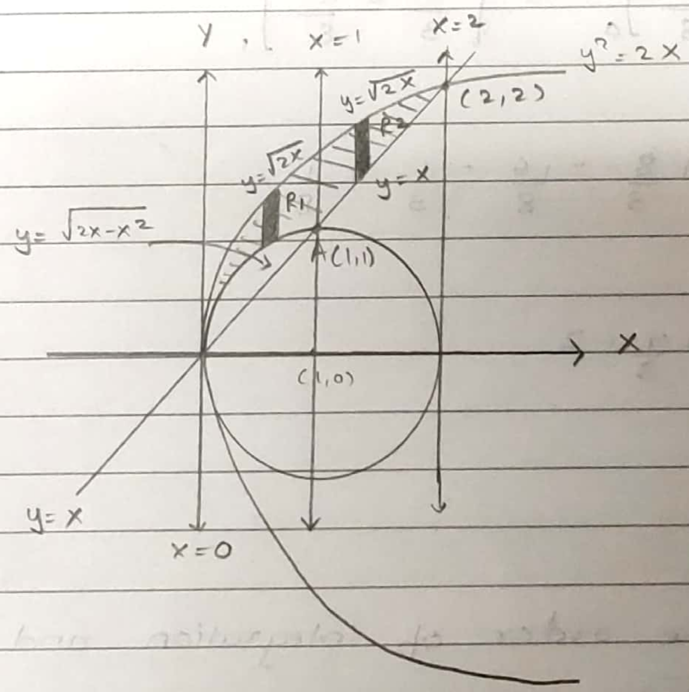
7) Evaluate $\iint_R xy\,dx\,dy$ over the region R given by

$x^2+y^2-2x=0$, $y^2=2x$ and $y=x$

Solution



$$\therefore \iint_R xy\,dx\,dy = \iint_{R1} xy\,dx\,dy + \iint_{R2} xy\,dx\,dy$$

$$\therefore I = \int_{x=0}^{1}\left(\int_{y=\sqrt{2x-x^2}}^{\sqrt{2x}} xy\,dy\right)dx + \int_{x=1}^{2}\left(\int_{y=x}^{\sqrt{2x}} xy\,dy\right)dx$$

$$= \int_{x=0}^{1} x\left[\frac{y^2}{2}\right]_{\sqrt{2x-x^2}}^{\sqrt{2x}} dx + \int_{x=1}^{2} x\left[\frac{y^2}{2}\right]_{x}^{\sqrt{2x}} dx$$

$$= \int_{x=0}^{1}\left(\frac{2x-2x+x^2}{2}\right)x\,dx + \int_{x=1}^{2}\left(\frac{2x-x^2}{2}\right)x\,dx$$

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$$\therefore I = \int_{x=0}^{1} \frac{x^3}{2}\,dx + \int_{x=1}^{2} \left(x^2 - \frac{x^3}{2}\right) dx$$

$$= \left[\frac{x^4}{8}\right]_0^1 + \left[\frac{x^3}{3} - \frac{x^4}{8}\right]_1^2$$

$$= \frac{1}{8} + \frac{8}{3} - \frac{16}{8} - \frac{1}{3} + \frac{1}{8}$$

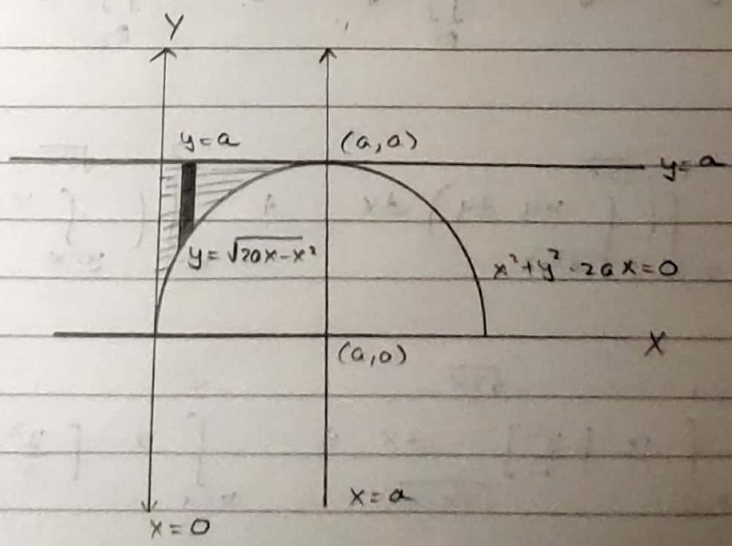
$$= \frac{1}{4} + \frac{7}{3} - 2$$

$$\therefore \boxed{I = \frac{7}{12}}$$

8) Change the order of integration and evaluate

$$\int_{y=0}^{a} dy \int_{x=0}^{a-\sqrt{a^2-y^2}} \frac{xy \log(x+a)}{(x-a)^2}\,dx$$

Solution



After changing the order of integration,

$$R = \left\{ (x,y) \in IR^2 \;\middle|\; \begin{array}{c} 0 < x < a \\ \sqrt{2ax-x^2} < y < a \end{array} \right\}$$

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$$\therefore I = \int_{x=0}^{a} \left(\int_{y=\sqrt{2ax-x^2}}^{a} \frac{x \log(x+a)}{(x-a)^2} \, y \, dy \right) dx$$

$$= \int_{x=0}^{a} \frac{x \log(x+a)}{(x-a)^2} \left[\frac{y^2}{2} \right]_{\sqrt{2ax-x^2}}^{a} dx$$

$$= \int_{x=0}^{a} \frac{x \log(x+a)}{(x-a)^2} \frac{(-2ax+x^2+a^2)}{2} \, dx$$

$$= \int_{x=0}^{a} \frac{x \log(x+a)}{2} \, dx$$

$$= \frac{1}{2} \left[\frac{x^2}{2} \log(x+a) - \frac{1}{2} \int \frac{x^2}{(x+a)} dx \right]_0^a$$

$$= \frac{1}{2} \left[\frac{x^2}{2} \log(x+a) - \frac{1}{2} \int \frac{(x^2-a^2)+a^2}{(x+a)} dx \right]_0^a$$

$$= \frac{1}{2} \left[\frac{x^2}{2} \log(x+a) - \frac{1}{2} \int (x-a) dx - \frac{a^2}{2} \int \frac{1}{(x+a)} dx \right]_0^a$$

$$= \frac{1}{2} \left[\frac{x^2}{2} \log(x+a) - \frac{1}{2} \left(\frac{x^2}{2} - ax \right) - \frac{a^2}{2} \log(x+a) \right]_0^a$$

$$= \frac{1}{2} \left[\frac{a^2}{2} \log 2a - \frac{1}{2} \left(\frac{a^2}{2} - a^2 \right) - \frac{a^2}{2} \log 2a + \frac{a^2}{2} \log a \right]$$

$$= \frac{1}{2} \left[\frac{a^2}{4} + \frac{a^2}{2} \log a \right] = \frac{a^2}{8} \left[1 + 2 \log a \right]$$

$$\therefore \boxed{I = \frac{a^2}{8} (1 + 2 \log a]}$$

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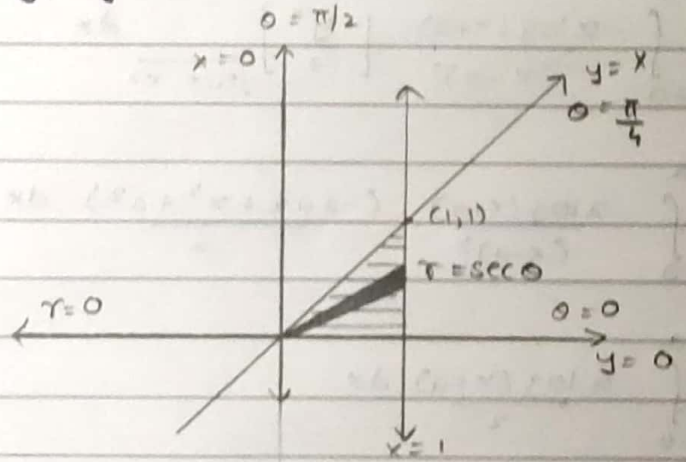
9) Change to polar coordinates and evaluate

$$\int_{x=0}^{1}\int_{y=0}^{x}(x+y)\,dy\,dx$$

Solution



Changing to polar coordinates,

$x = r\cos\theta$, $y = r\sin\theta$

$x^2+y^2=r^2$, $\theta = \tan^{-1}(y/x)$

$$J = \frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \partial x/\partial r & \partial x/\partial\theta \\ \partial y/\partial r & \partial y/\partial\theta \end{vmatrix} = \begin{vmatrix} \cos\theta & r\sin\theta \\ \sin\theta & -r\cos\theta \end{vmatrix} = r$$

$\therefore\ dx\,dy = J\,dr\,d\theta = r\,dr\,d\theta$

$x=1 \implies r\cos\theta = 1 \implies r = \sec\theta$

θ varies from 0 to $\frac{\pi}{4}$

$\therefore\ x+y = r\cos\theta + r\sin\theta$

$$\therefore I = \int_{\theta=0}^{\pi/4}\int_{r=0}^{\sec\theta}(r\cos\theta + r\sin\theta)\,r\,dr\,d\theta$$

$$= \int_{\theta=0}^{\pi/4}(\cos\theta+\sin\theta)\left[\frac{r^3}{3}\right]_0^{\sec\theta}d\theta$$

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$$\therefore I = \int_0^{\frac{\pi}{4}} (\cos\theta + \sin\theta)\,\frac{\sec^3\theta}{3}\,d\theta$$

$$= \int_0^{\frac{\pi}{4}} \frac{\sec^2\theta}{3}\,d\theta + \int_0^{\pi/4} \frac{1}{3}\,\frac{\sin\theta}{\cos^3\theta}\,d\theta$$

$$= \frac{1}{3}\left[\tan\theta\right]_0^{\frac{\pi}{4}} + \frac{1}{3}\int_0^{\frac{\pi}{4}} \frac{\sin\theta}{\cos^3\theta}\,d\theta$$

Put $\cos\theta = t$ $\quad \therefore -\sin\theta\,d\theta = dt$

when $\theta = 0$, $t = 1$; when $\theta = \pi/4$, $t = \frac{1}{\sqrt{2}}$

$$\therefore I = \frac{1}{3}(1-0) + \frac{1}{3}\int_1^{1/\sqrt{2}} \frac{-dt}{t^3}$$

$$= \frac{1}{3} + \frac{1}{3}\left[\frac{1}{2t^2}\right]_1^{1/\sqrt{2}}$$

$$= \frac{1}{3} + \frac{1}{3}\left[1 - \frac{1}{2}\right] = \frac{1}{3} + \frac{1}{6} = \frac{1}{2}$$

$$\therefore \boxed{I = \frac{1}{2}}$$

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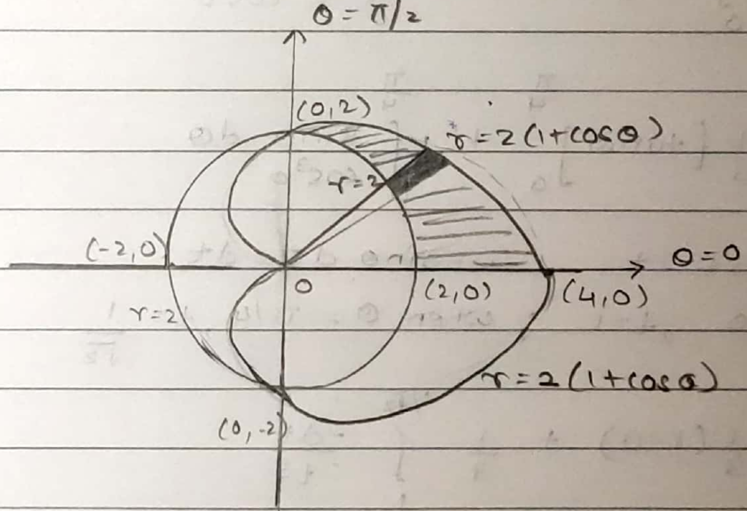
10) Evaluate $\iint_R \sin\theta \, dr \, d\theta$, where R is the region in the first quadrant that is outside the circle $r=2$ and inside the cardiod $r=2(1+\cos\theta)$



In first quadrant, θ varies from 0 to $\frac{\pi}{2}$

r varies from 2 to $2(1+\cos\theta)$.

let $I = \iint_R \sin\theta \, dr \, d\theta$

$$= \int_{\theta=0}^{\frac{\pi}{2}} \left(\int_{r=2}^{2(1+\cos\theta)} \sin\theta \, dr \right) d\theta$$

$$= \int_{\theta=0}^{\frac{\pi}{2}} \sin\theta \, [r]_2^{2(1+\cos\theta)} \, d\theta = \int_{\theta=0}^{\frac{\pi}{2}} \sin\theta \, (2\cos\theta) \, d\theta$$

$$= 2 \int_{\theta=0}^{\frac{\pi}{2}} \sin\theta\cos\theta \, d\theta = 2 \cdot \frac{1}{2} \beta(1,1) = \frac{\Gamma 1 \, \Gamma 1}{\Gamma 2} = 1$$

$$\therefore \boxed{\iint_R \sin\theta \, dr \, d\theta = 1}$$

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