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Maths - III

Tutorial 7

1) Find the Fourier Transform of

$$f(x) = \begin{cases} 1 + \dfrac{x}{a}, & -a < x < 0 \\ 1 - \dfrac{x}{a}, & 0 < x < a \\ 0, & \text{otherwise} \end{cases}$$

2) Find the fourier cosine and sine transform of

$$f(x) = \begin{cases} x, & 0 < x < 1 \\ 2-x, & 1 < x < 2 \\ 0, & x > 2 \end{cases}$$

3) Find the fourier sine and cosine transform of

(i) x^{m-1}, (ii) $\dfrac{1}{\sqrt{x}}$

Solutions:

i) let $F(\alpha)$ be the Fourier transform of $f(x)$

$$\therefore \mathcal{F}[f(x)] = F(\alpha) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{i\alpha x} dx$$

$$\therefore F(\alpha) = \frac{1}{\sqrt{2\pi}} \left[\int_{-a}^{0} \left(1+\frac{x}{a}\right) e^{i\alpha x} dx + \int_{0}^{a} \left(1-\frac{x}{a}\right) e^{i\alpha x} dx \right]$$

$$= \frac{1}{\sqrt{2\pi}} \left\{ \left[\left(1+\frac{x}{a}\right) \frac{e^{i\alpha x}}{i\alpha} \right]_{-a}^{0} - \left[\frac{1}{a} \frac{e^{i\alpha x}}{(i\alpha)^2} \right]_{-a}^{0} + \left[\left(1-\frac{x}{a}\right) \frac{e^{i\alpha x}}{i\alpha} \right]_{0}^{a} + \left[\frac{1}{a} \frac{e^{i\alpha x}}{(i\alpha)^2} \right]_{0}^{a} \right\}$$

$$= \frac{1}{\sqrt{2\pi}} \left[\frac{1}{i\alpha} + \frac{1}{a\alpha^2} - \frac{e^{-i\alpha a}}{a\alpha^2} - \frac{1}{i\alpha} - \frac{e^{i\alpha a}}{a\alpha^2} + \frac{1}{a\alpha^2} \right]$$

$$= \frac{1}{\sqrt{2\pi}} \left[\frac{2 - (e^{i\alpha a} + e^{-i\alpha a})}{a\alpha^2} \right]$$

$$= \frac{1}{\sqrt{2\pi}} \left(\frac{2 - 2\cos a\alpha}{a\alpha^2} \right) \qquad \left[\frac{e^{ix} + e^{-ix}}{2} = \cos x \right]$$

$$\therefore \boxed{F(\alpha) = \sqrt{\frac{2}{\pi}} \left(\frac{1 - \cos a\alpha}{a\alpha^2} \right)}$$

2) Let $F_c(\alpha)$ be the Fourier cosine transform of $F(x)$ ~~is $F_c(x)$~~

i.e. $F_c(\alpha) = \mathfrak{F}_c[f(x)]$

$$\therefore F_c(\alpha) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x)\cos\alpha x \, dx$$

$$= \sqrt{\frac{2}{\pi}} \left[\int_0^1 x\cos\alpha x \, dx + \int_1^2 (2-x)\cos\alpha x \, dx \right]$$

$$= \sqrt{\frac{2}{\pi}} \left[\left(\frac{x\sin\alpha x}{\alpha} + \frac{\cos\alpha x}{\alpha^2} \right)_0^1 + \left(\frac{(2-x)\sin\alpha x}{\alpha} - \frac{\cos\alpha x}{\alpha^2} \right)_1^2 \right]$$

$$= \sqrt{\frac{2}{\pi}} \left[\frac{\sin\alpha}{\alpha} + \frac{\cos\alpha}{\alpha^2} - \frac{1}{\alpha^2} + \left(\frac{-\cos2\alpha}{\alpha^2} - \left(\frac{\sin\alpha}{\alpha} - \frac{\cos\alpha}{\alpha^2} \right) \right) \right]$$

$$= \sqrt{\frac{2}{\pi}} \left[\frac{\sin\alpha}{\alpha} + \frac{\cos\alpha - 1}{\alpha^2} - \frac{\cos2\alpha}{\alpha^2} - \frac{\sin\alpha}{\alpha} + \frac{\cos\alpha}{\alpha^2} \right]$$

$$= \sqrt{\frac{2}{\pi}} \left(\frac{2\cos\alpha - (1+\cos2\alpha)}{\alpha^2} \right)$$

$$\boxed{F_c(\alpha) = \sqrt{\frac{2}{\pi}} \left(\frac{2\cos\alpha(1-\cos\alpha)}{\alpha^2} \right)}$$

Let $F_s(\alpha)$ be the Fourier sine transform of $F(x)$ ~~i.e. $F_s(x)$~~

i.e. $F_s(\alpha) = \mathfrak{F}_s[f(x)]$

$$\therefore F_s(\alpha) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x)\sin\alpha x \, dx$$

$$= \sqrt{\frac{2}{\pi}} \left[\int_0^1 x\sin\alpha x \, dx + \int_1^2 (2-x)\sin\alpha x \, dx \right]$$

$$= \sqrt{\frac{2}{\pi}} \left[\left(-\frac{x\cos\alpha x}{\alpha} + \frac{\sin\alpha x}{\alpha^2} \right)_0^1 + \left(-\frac{(2-x)\cos\alpha x}{\alpha} - \frac{\sin\alpha x}{\alpha^2} \right)_1^2 \right]$$

$$= \sqrt{\frac{2}{\pi}} \left[\left(-\frac{\cos\alpha}{\alpha} + \frac{\sin\alpha}{\alpha^2} \right) + \left(-\frac{\sin2\alpha}{\alpha^2} - \left(-\frac{\cos\alpha}{\alpha} - \frac{\sin\alpha}{\alpha^2} \right) \right) \right]$$

$$= \sqrt{\frac{2}{\pi}}\left[\frac{-\cos\alpha}{\alpha}+\frac{\sin\alpha}{\alpha^2}-\frac{\sin 2\alpha}{\alpha^2}+\frac{\cos\alpha}{\alpha}+\frac{\sin\alpha}{\alpha^2}\right]$$

$$= \sqrt{\frac{2}{\pi}}\left(\frac{2\sin\alpha-\sin 2\alpha}{\alpha^2}\right)$$

$$= \sqrt{\frac{2}{\pi}}\left(\frac{2\sin\alpha-2\sin\alpha\cos\alpha}{\alpha^2}\right)$$

$$\boxed{\therefore F_s(\alpha) = \sqrt{\frac{2}{\pi}}\left[\frac{2\sin\alpha(1-\cos\alpha)}{\alpha^2}\right]}$$

3) Let $F_c(\alpha)$ be the fourier cosine transform and $F_s(\alpha)$ be the fourier sine transform of the function $f(x)$.

i.e. $F_c(\alpha) = \mathcal{F}_c[f(x)]$ and $F_s(\alpha) = \mathcal{F}_s[f(x)]$

a) i) $\therefore \mathcal{F}_c[x^{m-1}] = \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\cos\alpha x\, dx$

$$= \sqrt{\frac{2}{\pi}}\int_0^\infty x^{m-1}\cos\alpha x\, dx$$

Put $y = i\alpha x$, $dy = i\alpha\, dx$

$$\therefore \int_0^\infty e^{-i\alpha x}x^{m-1}\,dx = \frac{\Gamma m}{(i\alpha)^m}$$

$$\therefore \int_0^\infty \cos\alpha x \cdot x^{m-1}\,dx - i\int_0^\infty \sin\alpha x\cdot x^{m-1}\,dx = \frac{\Gamma m}{(\alpha)^m}(i)^{-m}$$

$$= \frac{\Gamma m}{\alpha^m}\left[\cos\frac{\pi}{2}+i\sin\frac{\pi}{2}\right]^{-m}$$

$$\therefore \int_0^\infty x^{m-1}\cos\alpha x\,dx - i\int_0^\infty x^{m-1}\sin\alpha x\,dx = \frac{\Gamma m}{\alpha^m}\left[\cos\frac{m\pi}{2} - i\sin\frac{m\pi}{2}\right] \quad (1)$$

$\therefore$ Comparing the real part,

$$\int_0^\infty \cos\alpha x \cdot x^{m-1}\,dx = \frac{\Gamma m}{\alpha^m}\cos\left(\frac{m\pi}{2}\right)$$

$$\therefore \boxed{F_c\left[x^{m-1}\right] = \sqrt{\frac{2}{\pi}}\left[\frac{\Gamma m}{\alpha^m}\cos\left(\frac{m\pi}{2}\right)\right]}$$

ii) Put $m = \frac{1}{2}$

$$F_c\left[\frac{1}{\sqrt{x}}\right] = \sqrt{\frac{2}{\pi}}\,\frac{\Gamma 1/2}{\sqrt{\alpha}}\cos\left(\frac{\pi}{4}\right)$$

$$\therefore \boxed{F_c\left[\frac{1}{\sqrt{x}}\right] = \frac{1}{\sqrt{\alpha}}}$$

(b) i) $F_s\left[x^{m-1}\right] = \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\sin\alpha x\,dx$

$$= \sqrt{\frac{2}{\pi}}\int_0^\infty x^{m-1}\sin\alpha x\,dx$$

Comparing the imaginary part from eq (1)

$$\int_0^\infty \sin\alpha x\; x^{m-1}\,dx = \frac{\Gamma m}{\alpha^m}\sin\left(\frac{m\pi}{2}\right)$$

$$\therefore \boxed{F_s\left[x^{m-1}\right] = \sqrt{\frac{2}{\pi}}\left[\sin\left(\frac{m\pi}{2}\right)\frac{\Gamma m}{\alpha^m}\right]}$$

ii) Put $m = 1/2$

$$F_s\left[\frac{1}{\sqrt{x}}\right] = \sqrt{\frac{2}{\pi}}\,\frac{\Gamma(1/2)}{\alpha^{1/2}}\,\sin \pi/4$$

$$\therefore \boxed{F_s\left[\frac{1}{\sqrt{x}}\right] = \frac{1}{\sqrt{\alpha}}}$$