Name: Ayush Jain
SAP ID: 60004200132
Div: B1

15/02/22

Engineering Maths III

Solutions:

→ 3) By defination,

$$F_s(s) = \sqrt{\frac{2}{\pi}} \cdot \int_0^\infty \frac{e^{-ax}}{x} \sin sx \, dx \qquad \text{—— (1)}$$

Differentiating both sides w.r.t s,

$$\frac{d}{ds}[F_s(s)] = \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{e^{-ax}}{x} \cos sx \cdot x \, dx$$

$$= \sqrt{\frac{2}{\pi}} \cdot \int_0^\infty e^{-ax} \cos sx \, dx$$

$$= \sqrt{\frac{2}{\pi}} \left[\frac{1}{a^2+s^2} e^{-ax} (-a \cos sx + s \cdot \sin ax) \right]_0^\infty$$

$$= \sqrt{\frac{2}{\pi}} \cdot \frac{a}{a^2+s^2}$$

By integration, $F_s(s) = \sqrt{\frac{2}{\pi}} \cdot \tan^{-1} \frac{s}{a} + C$

But when $s=0$, from (1) $F_s(s) = 0$ $\therefore C = 0$

$$\therefore F_s(s) = \sqrt{\frac{2}{\pi}} \cdot \tan^{-1}\left(\frac{s}{a}\right) \qquad \text{—— (2)}$$

Now, we use inverse Fourier sine transform.

By defination, $f(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty F_s(s) \sin sx \, ds$

$$\therefore f(x) = \sqrt{\frac{2}{\pi}} \cdot \int_0^\infty \sqrt{\frac{2}{\pi}} \cdot \tan^{-1} \frac{s}{a} \cdot \sin sx \, dx$$

Putting $x = 1$, we get

$$\int_0^\infty \frac{\pi}{2} e^{-a} = \int_0^\infty \tan^{-1}\frac{s}{a} \cdot \sin s \, ds$$

Changing s to x, we get

$$\boxed{\int_0^\infty \tan^{-1}\frac{x}{a} \cdot \sin x \, dx = \frac{\pi}{2} e^{-a}}$$

Cor. : If we put $a = 0$, then from (1)

$$F_s(s) = \sqrt{\frac{2}{\pi}} \cdot \int_0^\infty \frac{\sin sx}{x} dx \quad \text{and}$$

from (2), $F_s(s) = \sqrt{\frac{2}{\pi}} \tan^{-1}\infty = \sqrt{\frac{2}{\pi}} \cdot \frac{\pi}{2} = \sqrt{\frac{\pi}{2}}$

$$\therefore \int_0^\infty \frac{\sin sx}{x} dx = \frac{\pi}{2}$$

60004200132

→ 5) $\frac{d^2y}{dt^2} + 2\frac{dy}{dt} + 5y = e^{-t}\sin t$, $y(0) = 0$, $y'(0) = 1$

Let $L(y) = \bar{y}$ Then taking Laplace Transform on both sides,

$L(y'') + 2L(y') + 5L(y) = L(e^{-t}\sin t)$

But $L(y') = s\bar{y} - y(0) = s\bar{y}$

and $L(y'') = s^2\bar{y} - s\,y(0) - y'(0) = s^2\bar{y} - 1$

and $L(e^{-t}\sin t) = \frac{1}{(s+1)^2+1}$

∴ The equation becomes

$$(s^2\bar{y} - 1) + 2s\bar{y} + 5\bar{y} = \frac{1}{(s+1)^2+1}$$

$$\therefore (s^2+2s+5)\bar{y} = 1 + \frac{1}{s^2+2s+2} = \frac{s^2+2s+3}{s^2+2s+2}$$

$$\therefore \bar{y} = \frac{s^2+2s+3}{(s^2+2s+5)(s^2+2s+2)}$$

Let $\frac{s^2+2s+3}{(s^2+2s+5)(s^2+2s+2)} = \frac{as+b}{(s^2+2s+5)} + \frac{cs+d}{(s^2+2s+2)}$

After simplification we get

$$s^2+2s+s = (a+c)s^3 + (2a+b+2c+d)s^2 + (2a+2b+5c+2d)s + (2b+5d)$$

Equating the coefficients of like powers of s, we get

$a+c=0$, $2a+b+2c+d=1$,

$2a+2b+5c+2d=2$, $2b+5d=3$.

60004200182

$\therefore a = 0, \quad b = \frac{2}{3}, \quad c = 0, \quad d = \frac{1}{3}$

$$\therefore \bar{y} = \frac{2}{3} \cdot \frac{1}{s^2+2s+5} + \frac{1}{3} \cdot \frac{1}{s^2+2s+2}$$

$$= \frac{2}{3} \cdot \frac{1}{(s+1)^2+2^2} + \frac{1}{3} \cdot \frac{1}{(s+1)^2+1^2}$$

Taking inverse Laplace Transform,

$$y = \frac{2}{3} e^{-t} \cdot L^{-1}\left[\frac{1}{s^2+2^2}\right] + \frac{1}{3} e^{-t} L^{-1}\left[\frac{1}{s^2+1^2}\right]$$

$$= \frac{2}{3} e^{-t} \cdot \frac{1}{2} \sin 2t + \frac{1}{3} e^{-t} \sin t = \frac{e^{-t}}{3}(\sin 2t + \sin t)$$

$$\cancel{\therefore y = \frac{2e^{-t}}{3} + \frac{1}{2}\sin 2t + \frac{1}{3} e^{-t} \sin t}$$

$$\therefore \boxed{y = \frac{e^{-t}}{3}(\sin 2t + \sin t)}$$

6000420013 2

→ 2) $f(x) = \frac{(\pi - x)^2}{4}$, $(0, 2\pi)$

Let $f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$ — (i)

$\therefore a_0 = \frac{1}{2\pi} \int_0^{2\pi} f(x)\,dx = \frac{1}{2\pi} \int_0^{2x} \frac{(\pi - x)^2}{4}\,dx$

$\therefore a_0 = \frac{1}{8\pi} \left[\frac{(\pi - x)^3}{-3} \right]_0^{2x} = \frac{-1}{24\pi} \left[-\pi^3 - \pi^3 \right] = \frac{\pi^2}{12}$ — (A)

$a_n = \frac{1}{\pi} \int_0^{2\pi} f(x) \cos nx\,dx = \frac{1}{\pi} \int_0^{2\pi} \frac{(\pi - x)^2}{4} \cos nx\,dx$

$= \frac{1}{4\pi} \left[(\pi - x)^2 \left(\frac{\sin nx}{n} \right) - 2(\pi - x)(-1)\left(\frac{-\cos nx}{n^2} \right) + 2(-1)(-1)\left(\frac{-\sin nx}{n^3} \right) \right]_0^{2\pi}$

(By generalised rule of integration by parts)

$= \frac{1}{4\pi} \left[\left(0 + \frac{2\pi \cos 2n\pi}{n^2} - 0 \right) - \left(0 - \frac{2\pi}{n^2} - 0 \right) \right]$

$\therefore a_n = \frac{1}{4\pi} \left[\frac{2\pi}{n^2} + \frac{2\pi}{n^2} \right] = \frac{1}{n^2}$ $[\because \cos 2n\pi = 1]$ .... (B)

$b_n = \frac{1}{\pi} \int_0^{2\pi} f(x) \sin nx\,dx = \frac{1}{\pi} \int_0^{2x} \frac{(\pi - x)^2}{4} . \sin nx\,dx$

$= \frac{1}{4\pi} \left[(\pi - x)^2 \left(-\frac{\cos nx}{n} \right) - 2(\pi - x)(-1)\left(-\frac{\sin nx}{n^2} \right) + 2(-1)(-1)\left(\frac{\cos nx}{n^3} \right) \right]_0^{2\pi}$

$= \frac{1}{4\pi} \left[\left(-\frac{\pi^2 \cos 2n\pi}{n} + 0 + \frac{2\cos 2n\pi}{n^3} \right) - \left(-\frac{\pi^2}{n} + 0 + \frac{2}{n^3} \right) \right]$

$\therefore b_n = \frac{1}{4\pi} \left[-\frac{\pi^2}{n} + \frac{2}{n^3} + \frac{\pi^2}{n} - \frac{2}{n^3} \right] = 0$ — (C)

Putting these values in (1) we get,

$$\left(\frac{\pi - x}{2}\right)^2 = \frac{\pi^2}{12} + \sum_{n=1}^{\infty} \frac{1}{n^2} \cos nx$$

$$= \frac{\pi^2}{12} + \frac{1}{1^2} \cos x + \frac{1}{2^2} \cos 2x + \frac{1}{3^2} \cos 3x + \cdots \qquad (2)$$

$$\boxed{\therefore f(x) = \frac{(\pi - x)^2}{4} = \frac{\pi^2}{12} + \frac{1}{1^2} \cos x + \frac{1}{2^2} \cos 2x + \frac{1}{3^2} \cos 3x + \cdots}$$

Now, put $x = 0$ in (2)

$$\therefore \frac{\pi^2}{4} = \frac{\pi^2}{12} + \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots$$

$$\therefore \frac{\pi^2}{6} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots$$

60004200132

→ 1a) $\int_0^\infty \frac{\cos 6t - \cos 4t}{t} dt$

we know from defination,

$F(s) = L[f(t)] = \int_0^\infty e^{-st} f(t)\, dt$

Here withing case $s = 0$, $f(t) = \frac{\cos 6t - \cos 4t}{t}$

we first take, $L[\cos 6t - \cos 4t] = \frac{s}{s^2+6^2} - \frac{s}{s^2+4^2}$

$\therefore L[\cos 6t - \cos 4t] = \frac{s}{s^2+6^2} - \frac{s}{s^2+4^2}$

Now, by using division 't' property,

$\therefore L\left[\frac{\cos 6t - \cos 4t}{t}\right] = \int_s^\infty \frac{s}{s^2+36} - \frac{s}{s^2+16}\, ds$

$= \left[\frac{1}{2}\log(s^2+36) - \frac{1}{2}\log(s^2+16)\right]_s^\infty$

$= -\frac{1}{2}\log(s^2+36) + \frac{1}{2}\log(s^2+16)$

$= \frac{1}{2}\log\left(\frac{s^2+16}{s^2+36}\right)$

Now, substitoting $s = 0$,

$\therefore \int_0^\infty \frac{\cos 6t - \cos 4t}{t} dt = \frac{1}{2}\log\left(\frac{16}{36}\right) = \log\sqrt{\frac{16}{36}} = \log\frac{2}{3}$

$\therefore \boxed{\int_0^\infty \frac{\cos 6t - \cos 4t}{t} dt = \log\frac{2}{3}}$

→1 b) $L^{-1}\left[\frac{1}{s^2(s+1)^2}\right]$

Let $\phi_1(s) = \frac{1}{(s+1)^2}$, $\phi_2(s) = \frac{1}{s^2}$

$\therefore L^{-1}\phi_1(s) = L^{-1}\left[\frac{1}{(s+1)^2}\right] = e^{-t} L^{-1}\frac{1}{s^2} = e^{-t}.t$

$\therefore L^{-1}\phi_2(s) = L^{-1}\frac{1}{s^2} = t$

$\therefore L^{-1}\phi(s) = L^{-1}\left[\frac{1}{(s+1)^2}\cdot\frac{1}{s^2}\right] = \int_0^t e^{-u}u.(t-u)\,du$

$= t\int_0^t e^{-u}.u\,du - \int_0^t e^{-u}.u^2\,du$

$= t\left[u(-e^{-u}) - (e^{-u})(1)\right]_0^t - \left[u^2(-e^{-u}) - (e^{-u})(2u) + (-e^{-u})(2)\right]_0^t$

$= t\left[-te^{-t} - e^{-t} + 1\right] - \left[-t^2e^{-t} - 2te^{-t} - 2e^{-t} + 2\right]$

$= te^{-t} + 2e^{-t} + t - 2$

$\therefore \boxed{L^{-1}\left[\frac{1}{s^2(s+1)^2}\right] = te^{-t} + 2e^{-t} + t - 2}$

FOR EDUCATIONAL USE

→ 4 a) $z^{-1}\left[\dfrac{z^2}{(z-1/2)(z-1/3)}\right]$, if $\frac{1}{3}<|z|<\frac{1}{2}$

Since, degree of numerator is equal to degree of denominator,

$$\therefore \frac{F(z)}{z} = \frac{z^2}{(z-1/2)(z-1/3)}$$

$$\therefore \frac{F(z)}{z} = \frac{z}{(z-1/2)(z-1/3)} = \frac{3}{(z-1/2)} - \frac{2}{(z-1/3)}$$

$\frac{1}{3}<|z|$ $\therefore \frac{1}{|3z|}<1$, $|z|<\frac{1}{2}$ $\therefore |2z|<1$

$$\therefore \frac{F(z)}{z} = \frac{2\times 3}{2z-1} - \frac{2}{z[1-(1/3z)]}$$

$$= \frac{6}{2z-1} - \frac{2}{z[1-(1/3z)]}$$

$$= -6(1-2z)^{-1} - \frac{2}{z}\left(1-\frac{1}{3z}\right)^{-1}$$

$$= -6(1+2z+(2z)^2+\dots) - \frac{2}{z}\left(1+\frac{3}{3z}+\frac{1}{(3z)^2}+\dots\right)$$

$$\therefore \frac{F(z)}{z} = -3(2+2^2z+\dots 2^k.2^{k-1}+\dots)$$
$$- \frac{2}{z}\left(1+\frac{1}{3z}+\frac{1}{(3z)^2}+\dots\frac{1}{3^k.2^k}+\dots\right)$$

$$\therefore F(z) = -3(2z+2^2z^2+\dots 2^k.z^k+\dots) - 2\left(1+\frac{1}{3z}+\dots\frac{1}{3^k.2^k}\right)$$

$$\therefore F(z) = -3\sum_{k=1} 2^k z^k - 2\sum_{k=0}^{\infty}\frac{1}{3}\left(\frac{1}{2^k}\right)+\dots$$

60004200132

Since, we want coefficient of z^{-k}, we change k to -k in first term,

$$\therefore F(z) = -3 \sum_{k=-1}^{\infty} \left(\frac{1}{2}\right)^{k} \cdot z^{-k} - 2 \sum_{k=0}^{\infty} \left(\frac{1}{3}\right)^{k} \cdot z^{-k}$$

$$\therefore z^{-1}[F(z)] = -3\left(\frac{1}{2}\right)^{k} - 2\left(\frac{1}{3}\right)^{k}$$

$$\therefore z^{-1}[F(z)] = \underset{k<0}{-3\left(\frac{1}{2}\right)^{k}} - \underset{k \geq 0}{2\left(\frac{1}{3}\right)^{k}}$$

→ 4 b) $f(k) = \frac{a^k}{k!}, \quad k \geq 0$

$$z\{f(k)\} = \sum_{k=0}^{\infty} \frac{a^k}{k!} \cdot z^{-k}$$

$$= \sum_{k=0}^{\infty} \frac{(a/z)^k}{k!}$$

$$= 1 + \frac{a}{z} + \frac{1}{2!}\left(\frac{a}{z}\right)^2 + \frac{1}{3!}\left(\frac{a}{z}\right)^3 + \cdots$$

$$\boxed{z\{f(k)\} = e^{a/z}}$$

Region of convergence = Entire z plane

→ 6 a) we have to obtain half range sine series for cos x

Let $\cos x = \sum b_n \sin x$ $[\because l = \pi]$

$$b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx \, dx = \frac{2}{\pi}\int_0^{\pi} \cos x \sin nx \, dx \quad \text{--- (1)}$$

$$= \frac{1}{\pi}\int_0^{\pi} \sin(1+n)x - \sin(1-n)x \, dx$$

$$= \frac{1}{\pi}\left[-\frac{\cos(1+n)x}{1+n} + \frac{\cos(1-n)x}{1-n}\right]_0^{\pi}$$

$$= \frac{1}{\pi}\left[-\frac{\cos(n+1)\pi}{n+1} - \frac{\cos(n-1)\pi}{n-1} + \frac{1}{n+1} + \frac{1}{n-1}\right]$$

$$= \frac{1}{\pi}\left[\frac{\cos n\pi}{n+1} + \frac{\cos n\pi}{n-1} + \frac{1}{n+1} + \frac{1}{n-1}\right]$$

$$= \frac{1}{\pi}\left[\frac{1}{n+1} + \frac{1}{n-1}\right](1+\cos n\pi) = \frac{1}{\pi}\cdot\frac{2n}{n^2-1}\left[1+(-1)^n\right]$$

$$= \begin{cases} 0 & , \text{ if } n \text{ is odd and } n \neq 1 \\ \frac{1}{\pi}\cdot\frac{4n}{n^2-1} & , \text{ if } n \text{ is even} \end{cases}$$

when $n=1$, from (1) we get,

$$b_1 = \frac{2}{\pi}\int_0^{\pi}\cos x \sin x \, dx = \frac{1}{\pi}\int_0^{\pi}\sin 2x \, dx$$

$$\therefore b_1 = \frac{1}{\pi}\left[-\frac{\cos 2x}{2}\right]_0^{\pi} = \frac{-1}{2\pi}[1-1] = 0$$

$$\therefore \cos x = \frac{8}{\pi}\left[\frac{1}{2^2-1}\sin 2x + \frac{2}{4^2-1}\sin 4x + \frac{3}{6^2-1}\sin 6x + \cdots\right]$$

$$\boxed{\cos x = \frac{8}{\pi}\sum_{m=1}^{\infty}\frac{m}{4m^2-1}\sin 2mx}$$

G00004200132

$\rightarrow$ 6 b) To prove: $\int_0^\infty \frac{x^2}{(x^2+1)^2} dx = \frac{\pi}{4}$

$f(x) = e^{-x}$

$\therefore \mathcal{F}_s[f(x)] = \sqrt{\frac{2}{\pi}} \frac{\alpha}{1+\alpha^2}$

By Parseval's Identity,

$$\int_0^\infty |f(x)|^2 dx = \int_0^\infty |F_s(\alpha)^2| \, d\alpha$$

$$\therefore \int_0^\infty e^{-2x} dx = \frac{2}{\pi} \int_0^\infty \frac{\alpha^2}{(1+\alpha^2)^2} d\alpha$$

$$\therefore \int_0^\infty e^{-2x} dx = \frac{2}{\pi} \int_0^\infty \frac{\alpha^2}{(1+\alpha^2)^2} d\alpha$$

$$\therefore \frac{1}{(-2)} \left[e^{-2x}\right]_0^\infty = \frac{2}{\pi} \int_0^\infty \frac{\alpha^2}{(1+\alpha^2)^2} d\alpha$$

$$\frac{[0 - 1]}{-2} = \frac{2}{\pi} \int_0^\infty \frac{\alpha^2}{(1+\alpha^2)^2} d\alpha$$

$$\therefore \int_0^\infty \frac{\alpha^2}{(1+\alpha^2)^2} d\alpha = \frac{\pi}{4}$$

Changing α by x,

$$\therefore \boxed{\int_0^\infty \frac{x^2}{(1+x^2)^2} dx = \frac{\pi}{4}}$$