60004200132_Maths
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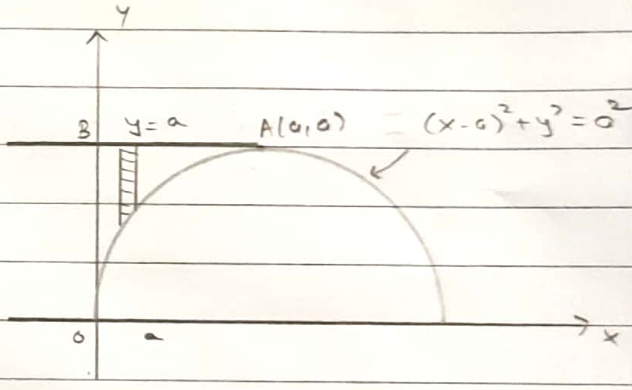
ESE_Maths

2) Change the order of integration and evaluate

$$\int_0^a dy \int_0^{a-\sqrt{a^2-y^2}} \frac{xy\log(x+a)}{(x-a)^2}\,dx$$

$\rightarrow$ To change the order of integration consider a strip parallel to y-axis. On this strip y varies from $y = \sqrt{a^2-(x-a)^2} = \sqrt{2ax-x^2}$ to $y=a$. Then the strip moves from $x=0$ to $x=a$



$$\therefore I = \int_0^a dx \int_{\sqrt{2ax-x^2}}^{a} \frac{xy\log(x+a)\,dy}{(x-a)^2}$$

$$= \int_0^a \frac{x\log(x+a)}{(x-a)^2}\left[\frac{y^2}{2}\right]_{\sqrt{2ax-x^2}}^{a} dx$$

$$= \int_0^a \frac{x\log(x+a)}{2(x-a)^2}\left[a^2-2ax+x^2\right]dx$$

$$= \frac{1}{2}\int_0^a x\log(x+a)\,dx$$

$$= \frac{1}{2}\left[\log(x+a)\cdot\frac{x^2}{2} - \int\frac{x^2}{2(x+a)}\,dx\right]_0^a \quad \ldots \text{[By parts]}$$

$$\cancel{= \frac{1}{2}\left[\frac{x^2}{2}\log(x+a) - \frac{1}{2}\int(x-a) - \frac{a^2}{2}\int\frac{dx}{x+a}\right]_0^a}$$

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$$= \frac{1}{2}\left[\frac{x^2}{2}\log(x+a) - \frac{1}{2}\int \frac{(x^2-a^2)+a^2}{(x+a)}\,dx\right]_0^a$$

$$= \frac{1}{2}\left[\frac{x^2}{2}\log(x+a) - \frac{1}{2}\int(x-a) - \frac{a^2}{2}\int\frac{dx}{x+a}\right]_0^a$$

$$= \frac{1}{2}\left[\frac{x^2}{2}\log(x+a) - \frac{1}{2}\left(\frac{x^2}{2} - ax\right) - \frac{a^2}{2}\log(x+a)\right]_0^a$$

$$= \frac{1}{2}\left[\frac{a^2}{2}\log(a+a) - \frac{1}{2}\left(\frac{a^2}{2} - a^2\right) - \frac{a^2}{2}\log(a+a) + \frac{a^2}{2}\log a\right]$$

$$I = \frac{a^2}{8}\left[1 + 2\log a\right]$$

Q. 5) $(D^3+2D^2+D)y = x^2 e^{3x} + \sin^2 x + 2^x$

Aux. equation: $D^3+2D^2+D=0$

$D(D^2+2D+1)=0$

$D = 0, -1, -1$

$y_c = C_1 + (C_2 + C_3 x) e^{-x}$

$y_P = \frac{1}{(D^3+2D^2+D)} x^2 e^{3x} + \sin^2 x + 2^x$

Now,

$\frac{1}{(D^3+2D^2+D)} x^2 e^{3x} = e^{3x} \cdot \frac{1}{(D+3)^3 + 2(D+3)^2 + (D+3)} x^2$

$= e^{3x} \cdot \frac{1}{D^3+27+27D+9D^2+2(D^2+9+6D)+D+3} x^2$

$= e^{3x} \cdot \frac{1}{D^3+11D^2+40D+48} \cdot x^2$

$= \frac{e^{3x}}{48}\left[1 + \frac{5D}{6} + \frac{11D^2}{48} + \frac{D^3}{48}\right]^{-1} \cdot x^2$

$= \frac{e^{3x}}{48}\left[1 - \left(\frac{5D}{6}\right) - \left(\frac{11D^2}{48}\right) + \left(\frac{25D^2}{36}\right)\right] x^2$

$= \frac{e^{3x}}{48}\left[x^2 - \frac{5x}{3} - \frac{11}{24} + \frac{25}{18}\right]$

$= \frac{e^{3x}}{48}\left[x^2 - \frac{5x}{3} + \frac{67}{72}\right]$

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$$\frac{1}{D^3+2D^2+D} \cdot \sin^2 x = \frac{1}{D^3+2D^2+D}\left(\frac{1-\cos 2x}{2}\right)$$

$$= \frac{1}{D^3+2D^2+D} \cdot \frac{1}{2} - \frac{1}{2} \cdot \frac{1}{D^3+2D^2+D} \cos 2x$$

$$= \frac{1}{2} \cdot \frac{1}{D(D+1)} e^{-x} \int e^{x} dx - \frac{1}{2}\left(\frac{1}{D(-4)-8+D} \cos 2x\right)$$

$$= \frac{1}{2} \cdot \frac{1}{D} \cdot 1 - \frac{1}{2}\left(\frac{1}{-3D-8} \cos 2x\right)$$

$$= \frac{x}{2} - \frac{1}{2}\left(\frac{-3D+8}{9D^2-64}\right) \cos 2x$$

$$= \frac{x}{2} - \frac{1}{2}\left(\frac{6\sin 2x + 8\cos 2x}{-100}\right)$$

$$= \frac{x}{2} + \frac{1}{100}(3\sin 2x + 4\cos 2x)$$

$$\frac{1}{D^3+2D^2+D} 2^x = \frac{1}{(\log 2)^3 + 2(\log 2)^2 + \log 2} \cdot 2^x$$

$$\therefore y = y_c + y_p$$

$$= C_1 + (C_2 + C_3 x)e^{-x} + \frac{e^{3x}}{48}\left[x^2 - \frac{5x}{3} + \frac{67}{72}\right] + \frac{x}{2} + \frac{1}{100}(3\sin 2x + 4\cos 2x)$$

$$+ \frac{1 \cdot 2^x}{(\log 2)^3 + 2(\log 2)^2 + \log 2}$$

→ 6) We divide the interval into six equal sub-intervals by taking each sub-interval equal to $\frac{1.4 - 0.2}{6} = 0.2$

x :	0.2	0.4	0.6	0.8	1.0	1.2	1.4
y :	3.02950	2.79753	2.89759	3.16604	3.55975	4.06984	4.70418
Ordinate :	y_0	y_1	y_2	y_3	y_4	y_5	y_6

(i) By Trapezoidal rule,

$$I = \frac{h}{2}[X + 2R]$$

Here, $h = 0.2$

X = Sum of the extremes $= 3.02950 + 4.70418 = 7.73368$

R = Sum of the remaining

$= 2.79753 + 2.89759 + 3.16604 + 3.55975 + 4.06984$

$= 16.49075$

$\therefore I = \frac{0.2}{2}[7.73368 + 2(16.49075)] = 4.071518$

$\therefore I = 4.071518$

ii) By Simpson's $\left(\frac{1}{3}\right)$rd Rule,

$$I = \frac{h}{3}[X + 2E + 4O]$$

Here,

$h = 0.2$

X = Sum of the extremes $= 3.0295 + 4.70418 = 7.73368$

E = Sum of the even ordinates

$= 2.89759 + 3.55975 = 6.45734$

O = Sum of the odd ordinates

$= 2.79753 + 3.16604 + 4.06984 = 10.03341$

$\therefore I = \frac{0.2}{3}\left[7.73368 + 2(6.45734) + 4(10.03341)\right]$

$\therefore I = 4.05213$

iii) By Simpson's $\left(\frac{3}{8}\right)$th Rule,

$I = \frac{3h}{8}\left[X + 2T + 3R\right]$

here, $h = 0.2$

X = Sum of extremes $= 3.0295 + 4.70418 = 7.73368$

T = Sum of multiples of three $= 3.16604$

R = Sum of remaining

$= 2.79753 + 2.89759 + 3.55975 + 4.06984 = 13.32471$

$\therefore I = \frac{3(0.2)}{8}\left[7.73368 + 2(3.16604) + 3(13.32471)\right]$

$\therefore I = 4.05299$

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1.) a Prove that $\int_0^\infty \frac{e^{-x^3}}{\sqrt{x}} dx \cdot \int_0^\infty y^4 \cdot e^{-y^6} dy = \frac{\pi}{9}$

$\therefore$ LHS = $I_1 \times I_2$

$\therefore I_1 = \int_0^\infty e^{-x^3} x^{-1/2} dx$

let $x^3 = t$

$x = t^{1/3}$

$dx = \frac{1}{3} t^{1/3 - 1}$

At x : 0 ∞

t 0 ∞

$\therefore I_1 = \int_0^\infty e^{-t} \cdot t^{-1/6} \cdot \frac{1}{3} t^{1/3-1} dt$

$= \int_0^\infty e^{-t} \cdot t^{1/6 - 1} dt$

$I_1 = \frac{1}{3} \Gamma \frac{1}{6}$

$I_2 = \int_0^\infty y^4 \cdot e^{-y^6} dy$

let $y^6 = t$

$y = t^{1/6}$

$dy = \frac{1}{6} \cdot t^{1/6 - 1}$

At $y = 0$, $t = 0$

$y = \infty$, $t = \infty$

$I_2 = \int_0^\infty t^{4/6} e^{-t} \cdot \frac{1}{6} t^{1/6-1} dt$

$= \frac{1}{6} \int_0^\infty e^{-t} \cdot t^{5/6-1} dt$

$I_2 = \frac{1}{6} \Gamma \frac{5}{6}$

$\therefore$ LHS : $I_1 \times I_2$

$= \frac{1}{3} \Gamma \frac{1}{6} \times \frac{1}{6} \Gamma \frac{5}{6}$

$= \frac{1}{18} \Gamma \frac{1}{6} \Gamma \frac{5}{6}$

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If $p = \frac{1}{6}$, $1-p = \frac{5}{6}$ $\qquad \Gamma p\, \Gamma 1-p = \frac{\pi}{\sin p\pi}$

$$\therefore \frac{1}{18} \frac{\pi}{\sin(\pi/6)} = \frac{1}{18} \cdot \frac{\pi}{(1/2)}$$

$$= \frac{2\pi}{18}$$

$$= \frac{\pi}{9}$$

Hence, $\int_0^\infty \frac{e^{-x^3}}{\sqrt{x}}\,dx \cdot \int_0^\infty y^4 \cdot e^{-y^6}\,dy = \frac{\pi}{9}$

- Hence Proved.

1.) b $J = \int_0^{\frac{\pi}{2}} \frac{dx}{1+a\cos^2 x}$

Dividing Num and Den by $\cos^2 x$,

$$\therefore J = \int_0^{\pi/2} \frac{\sec^2 x}{a + \sec^2 x}\,dx$$

$$= \int_0^{\frac{\pi}{2}} \frac{\sec^2 x}{(1+a) + \tan^2 x}\,dx$$

Now, put $t = \tan x$

$\therefore \sec^2 x\,dx = dt$

when $x = 0$, $t = 0$

$x = \pi/2$, $t = \infty$

$$\therefore \quad I = \int_0^{\infty} \frac{dt}{(1+a)+t^2}$$

$$= \cancel{\frac{1}{\sqrt{1+a}}\left[\frac{\tan^{-1} t}{\sqrt{1+a}}\right.} = \frac{1}{\sqrt{1+a}}\left[\tan^{-1}\frac{t}{\sqrt{1+a}}\right]_0^{\infty}$$

$$= \frac{\pi}{2\sqrt{1+a}}$$

$$\therefore \quad I = \frac{\pi}{2\sqrt{1+a}}$$

$$\therefore \quad \frac{dI}{da} = \frac{\pi}{2}\left[-\frac{1}{2}\right](1+a)^{-3/2}$$

$$= -\frac{\pi}{4}(1+a)^{-3/2} \quad \text{— (i)}$$

But $\quad I = \int_0^{\pi/2} \frac{dx}{1+a\cos^2 x}$

By the rule of differentiation under the integral sign

$$\frac{dI}{da} = \int_0^{\frac{\pi}{2}} \frac{\partial f}{\partial a}\, dx = \int_0^{\pi/2} -\frac{1}{(1+a\cos^2 x)} \cdot \cos^2 x \, dx$$

$$\therefore \quad \frac{dI}{da} = -\int_0^{\frac{\pi}{2}} \frac{\cos^2 x}{(1+a\cos^2 x)^2}\, dx \quad \text{— (2)}$$

From (i) and (ii), we get

$$\int_0^{\frac{\pi}{2}} \frac{\cos^2 x}{(1+a\cos^2 x)^2}\, dx = \frac{\pi}{4}(1+a)^{-3/2}$$

Put $a = \frac{1}{3}$

$$\therefore \int_0^{\frac{\pi}{2}} \frac{\cos^2 x}{[1+\cos^2 x/3]^2}\, dx = \frac{\pi}{4}\left(1+\frac{1}{3}\right)^{-3/2}$$

$$\therefore \int_0^{\frac{\pi}{2}} \frac{9\cos^2 x}{(3+\cos^2 x)^2}\, dx = \frac{\pi}{4}\left(\frac{3}{4}\right)^{3/2}$$

$$\therefore \int_0^{\pi/2} \frac{\cos^2 x}{(3+\cos^2 x)^2}\, dx = \frac{\pi\sqrt{3}}{96}$$

Q.4) a Solve $\frac{dy}{dx} = \frac{y^3}{e^{2x} + y^2}$

$\rightarrow$ The equation can be written as $e^{2x} + y^2 = y^3 \frac{dx}{dy}$

$\therefore \frac{dx}{dy} - \frac{1}{y} = e^{2x} \cdot \frac{1}{y^3}$

Dividing by ~~e^{2x}~~, e^{2x},

$$e^{-2x} \frac{dx}{dy} - \frac{1}{y} e^{-2x} = \frac{1}{y^3}$$

Putting $e^{-2x} = v$, $-2e^{-2x} \frac{dx}{dy} = \frac{dv}{dy}$, we get

$-\frac{1}{2} \cdot \frac{dv}{dy} - \frac{1}{y} \cdot v = \frac{1}{y^3}$ i.e. $\frac{dv}{dy} + \frac{2}{y} \cdot v = -\frac{2}{y^3}$

This is a linear differential equation.

NOW, $I.F = e^{\int P\,dy} = e^{\int\left(\frac{2}{y}\right)dy} = e^{2\log y} = e^{\log y^2} = y^2$

$\therefore$ The solution is $v \cdot y^2 = \int y^2 \left(-\frac{2}{y^3}\right) dy + c$

$\therefore vy^2 = \int -\frac{2}{y} dy + c = -2\log y + c$

$\therefore e^{-2x} y^2 + 2\log y = c$

Q.4) b The equation of L-R circuit is given by

$$L\frac{di}{dt} + Ri = E$$

$$\therefore \frac{di}{dt} + \frac{Ri}{L} = \frac{E}{L}$$

Compare with

$$\frac{di}{dt} + Pi = Q$$

$$\therefore I.F = e^{\int P\,dt} = e^{\int \frac{R}{L}dt} = e^{Rt/L}$$

$$\therefore i e^{Rt/L} = \int e^{Rt/L} \cdot \frac{E}{L}\, dt + c$$

$$i e^{Rt/L} = \frac{E}{L} \frac{e^{Rt/L}}{R} \times L + c = \frac{E}{R} e^{Rt/L} + c$$

At $t=0, i=0$. Put $i=0$ and $t=0$

$$\therefore i e^{Rt/L} = \frac{E}{R} e^{Rt/L} + c$$

$$\therefore (0) e^{(0)} = \frac{E}{R} e^{0} + c$$

$$\therefore \boxed{c = -\frac{E}{R}}$$

$$\therefore i = \frac{E}{R} - \frac{E}{R} e^{-Rt/L}$$

$$\therefore i = \frac{E}{R}\left(1 - e^{-Rt/L}\right)$$