

15/03/2021

Engineering MathematicsTutorial 3: Hyperbolic and logarithmic of complex number.

1) If $\tan\left(\frac{x}{2}\right) = \tanh\left(\frac{y}{2}\right)$, then show that

i) $y = \log\left(\tan\left(\frac{\pi}{4} + \frac{x}{2}\right)\right)$

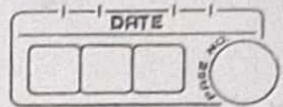
ii) $\cosh y \cdot \cos x = 1$

2) Separate into real and imaginary parts.
 $\cos^{-1}(e^{i\theta})$

3) If $\tan \log(x+iy) = a+ib$ and $a^2+b^2 \neq 1$ then prove that
 $\tan \log(x^2+y^2) = \frac{2a}{1-a^2-b^2}$

4) Separate into real and imaginary parts.
 $(-i)^{-(1-i)}$

5) Prove that real part of the principal value of
 $i^{\log(1+i)}$ is $e^{-\pi/8} \cos\left(\frac{\pi}{4} \log 2\right)$

Solutions

$$\Rightarrow \text{Given, } \tan\left(\frac{x}{2}\right) = \tanh\left(\frac{y}{2}\right)$$

$$\tan\left(\frac{x}{2}\right) = \frac{\sinh(y/2)}{\cosh(y/2)}$$

$$\tan\left(\frac{x}{2}\right) = \frac{e^{y/2} - e^{-y/2}}{e^{y/2} + e^{-y/2}}$$

$$\tan\left(\frac{x}{2}\right) = \frac{e^y - 1}{e^y + 1}$$

$$(e^y + 1) \tan\left(\frac{x}{2}\right) = e^y - 1$$

$$e^y \tan\left(\frac{x}{2}\right) + \tan\frac{x}{2} = e^y - 1$$

$$\therefore 1 + \tan\frac{x}{2} = e^y (1 - \tan\frac{x}{2})$$

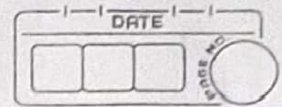
$$e^y = \frac{1 + \tan x/2}{1 - \tan x/2}$$

$$e^y = \frac{\tan(\pi/4) + \tan x/2}{1 - \tan(\pi/4) \tan x/2}$$

$$e^y = \log$$

$$e^y = \tan\left(\frac{\pi}{4} + \frac{x}{2}\right)$$

$$\therefore y = \log \tan\left(\frac{\pi}{4} + \frac{x}{2}\right)$$



ii) To prove: $\cosh y \cdot \cos x = 1$

$$\therefore \cos x = \frac{1 - \tan^2(x/2)}{1 + \tan^2(x/2)} \quad \dots \dots (1)$$

$$\cosh y = \frac{1 + \tanh^2(y/2)}{1 - \tanh^2(y/2)}$$

$$\text{As } \tan\left(\frac{x}{2}\right) = \tanh\left(\frac{y}{2}\right)$$

$$\therefore \cosh y = \frac{1 + \tan^2(x/2)}{1 - \tan^2(x/2)} \quad \dots \dots (2)$$

Multiplying equation (1) and (2)

$$\cosh y \cdot \cos x = \frac{1 + \tan^2(x/2)}{1 - \tan^2(x/2)} \times \frac{1 - \tan^2(x/2)}{1 + \tan^2(x/2)}$$

$$\therefore \boxed{\cosh y \cdot \cos x = 1}$$

2) Let $\cos^{-1}(e^{i\theta}) = x + iy$

$$e^{i\theta} = \cos(x + iy)$$

$$\cos \theta + i \sin \theta = \cos(x + iy)$$

$$\therefore \cos \theta + i \sin \theta = \cos x \cosh y - \sin x \sinh y$$

$$\therefore \cos \theta + i \sin \theta = \cos x \cosh y - i \sin x \sinh y$$

\therefore On comparing real and imaginary parts,

$$\cos \theta = \cos x \cdot \cosh y \quad \dots (1)$$

$$\sin \theta = -\sin x \sinh y \quad \dots (2)$$



$$\therefore \cosh y = \frac{\cos \theta}{\cos x}$$

$$\therefore \sinh y = \frac{-\sin \theta}{\sin x}$$

NOW,

$$\cosh^2 y - \sinh^2 y = 1$$

$$\left(\frac{\cos \theta}{\cos x}\right)^2 - \left(\frac{-\sin \theta}{\sin x}\right)^2 = 1$$

$$\frac{\cos^2 \theta}{\cos^2 x} - \frac{\sin^2 \theta}{\sin^2 x} = 1$$

$$\therefore \frac{\cos^2 \theta \cdot \sin^2 x - \sin^2 \theta \cdot \cos^2 x}{\cos^2 x \cdot \sin^2 x} = 1$$

$$\therefore \cos^2 \theta \cdot \sin^2 x - \sin^2 \theta \cdot \cos^2 x = \cos^2 x \cdot \sin^2 x$$

$$\therefore \cos^2 \theta \cdot \sin^2 x - \sin^2 \theta (1 - \sin^2 x) = \sin^2 x (1 - \sin^2 x)$$

$$\therefore \cos^2 \theta \cdot \sin^2 x - \sin^2 \theta + \sin^2 \theta \cdot \sin^2 x = \sin^2 x - \sin^4 x$$

$$\therefore \sin^2 x (\cos^2 \theta + \sin^2 \theta) - \sin^2 \theta = \sin^2 x - \sin^4 x$$

$$\therefore \cancel{\sin^2 x} - \sin^2 \theta = \cancel{\sin^2 x} - \sin^4 x$$

$$\therefore \sin^2 \theta = \sin^4 x$$

$$\therefore \sin^4 x = \sin^2 \theta$$

$$\therefore \sin^2 x = \pm \sin \theta$$

Since, $\sin^2 x$ cannot be negative, $-\sin \theta$ is ignored.

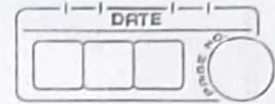
$$\therefore \sin^2 x = \sin \theta$$

$$\therefore \sin x = \sqrt{\sin \theta}$$

$$x = \sin^{-1} \sqrt{\sin \theta}$$

NOW,

Substitute $\sin x = \sqrt{\sin \theta}$ in equation (2)



$$\therefore \sin \theta = -\sin x \sinh y$$

$$\therefore \sin \theta = -\sqrt{\sin \theta} \cdot \sinh y$$

$$\therefore \sinh y = -\sqrt{\sin \theta}$$

$$\therefore y = \sinh^{-1}(-\sqrt{\sin \theta})$$

$$\therefore y = \log(-\sqrt{\sin \theta} + \sqrt{1 + \sin \theta})$$

$$\therefore y = \log(\sqrt{1 + \sin \theta} - \sqrt{\sin \theta})$$

$$\therefore \cos^{-1}(e^{i\theta}) = x + iy$$

$$= \sin^{-1} \sqrt{\sin \theta} + i \log(\sqrt{1 + \sin \theta} - \sqrt{\sin \theta})$$

\therefore

$$\therefore \boxed{\cos^{-1}(e^{i\theta}) = \sin^{-1}(\sqrt{\sin \theta}) + i \log(\sqrt{1 + \sin \theta} - \sqrt{\sin \theta})}$$

3) We have,

$$\tan \log(x + iy) = a + ib$$

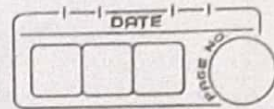
$$\therefore \tan \log(x - iy) = a - ib$$

$$\text{To prove: } \tan \log(x^2 + y^2) = \frac{2a}{1 - a^2 - b^2}$$

$$\therefore \text{LHS} = \tan \log(x^2 + y^2)$$

$$= \tan[\log(x + iy)(x - iy)]$$

$$= \tan[\log(x + iy) + \log(x - iy)]$$



$$= \frac{\tan \log(x+iy) + \tan \log(x-iy)}{1 - \tan \log(x+iy) \tan \log(x-iy)}$$

$$= \frac{a+ib + a-ib}{1 - (a+ib)(a-ib)}$$

$$= \frac{2a}{1 - (a^2 + b^2)}$$

$$= \frac{2a}{1 - a^2 - b^2}$$

$$= \text{RHS}$$

$$\therefore \tan \log(x^2 + y^2) = \frac{2a}{1 - a^2 - b^2}$$

$$4) (-i)^{-(1-i)} = e^{-(1-i) \log(-i)}$$

$$= e^{-(1-i) \log(-i)}$$

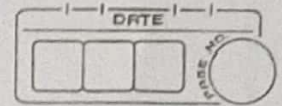
$$\therefore \log(-i) = \log e^{i(2n\pi - \pi/2)}$$

$$\log(-i) = i(2n\pi - \pi/2)$$

$$\therefore (-i)^{-(1-i)} = e^{-(1-i)(2n\pi - \pi/2)}$$

$$= e^{-(2n\pi - \pi/2)} \cdot e^{-i(2n\pi - \pi/2)}$$

$$= e^{-(2n\pi - \pi/2)} \cdot (\cos(2n\pi - \pi/2) - i \sin(2n\pi - \pi/2))$$



$$= e^{-(2n\pi - \pi/2)} \cdot [0 - i(-1)]$$

$$= e^{-(2n\pi - \pi/2)} \cdot (i)$$

$$= i e^{-(2n\pi - \pi/2)}$$

\therefore Real part = 0

Imaginary part = $e^{-(2n\pi - \pi/2)}$

$$\therefore (-i)^{-(1-i)} = i e^{-(2n\pi - \pi/2)}$$