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Engineering Mathematics

Tutorial (1) : Matrices

1.) Show that every square matrix can be expressed uniquely as the sum of a Hermition matrix and a skew Hermition matrix.

2) Prove that matrix A is unitary

$$A = \begin{bmatrix} i/2 & \sqrt{3}/2 \\ \sqrt{3}/2 & i/2 \end{bmatrix}$$

3) Find the rank of the matrix by reducing it to Echelon form.

$$A = \begin{bmatrix} 3 & -2 & 0 & -1 \\ 0 & 2 & 2 & 1 \\ 1 & -2 & -3 & 2 \\ 0 & 1 & 2 & 1 \end{bmatrix}$$

4) Find non-singular matrix P and Q such that PAQ is in normal form. Hence find the rank of A.

$$A = \begin{bmatrix} 1 & 1 & 1 & 2 \\ 3 & -3 & 1 & 2 \\ 2 & 1 & -3 & -6 \end{bmatrix}$$

5) Examine the consistency and solve

$$x_1 + 2x_2 - x_3 = 1$$
$$3x_1 - 2x_2 + 2x_3 = 2$$
$$7x_1 - 2x_2 + 3x_3 = 5$$

Solutions:

1.) We can write,

$A = \frac{1}{2}(A + A^\theta) + \frac{1}{2}(A - A^\theta) = P + Q$

where $P = \frac{1}{2}(A + A^\theta)$, $Q = \frac{1}{2}(A - A^\theta)$

Now,

$P^\theta = \frac{1}{2}(A + A^\theta)^\theta = \frac{1}{2}(A^\theta + A^{\theta\theta}) = \frac{1}{2}(A^\theta + A) = P$

$\therefore$ P is Hermition.

Also,

$Q^\theta = \frac{1}{2}(A - A^\theta)^\theta = \frac{1}{2}(A^\theta - A^{\theta\theta}) = \frac{1}{2}(A^\theta - A) = \frac{-1}{2}(A - A^\theta) = Q$

$\therefore$ Q is skew-Hermition.

Thus, A is sum of Hermition and skew-Hermition matrix.

To prove uniqueness,

let $A = R + S$,

where R is Hermition and S is skew-Hermition

$\therefore A^\theta = (R+S)^\theta = R^\theta + S^\theta = R - S \quad [\because R^\theta = R, S^\theta = -S]$

$\therefore P = \frac{1}{2}(A + A^\theta) = \frac{1}{2}(R + S + R - S) = R$

$\therefore R = P$

and, $Q = \frac{1}{2}(A - A^\theta) = \frac{1}{2}(R + S - R + S) = S$

$\therefore Q = S$

Hence, the representation $A = P + Q$ is unique.

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2.) $A = \begin{bmatrix} i/2 & \sqrt{3}/2 \\ \sqrt{3}/2 & i/2 \end{bmatrix}$

To prove: $AA^{\theta} = A^{\theta}A = I$

Sol:

$$A^{\theta} = \bar{A}' = \begin{bmatrix} -i/2 & \sqrt{3}/2 \\ \sqrt{3}/2 & -i/2 \end{bmatrix}$$

$$\therefore AA^{\theta} = \begin{bmatrix} i/2 & \sqrt{3}/2 \\ \sqrt{3}/2 & i/2 \end{bmatrix} \begin{bmatrix} -i/2 & \sqrt{3}/2 \\ \sqrt{3}/2 & -i/2 \end{bmatrix}$$

$$= \begin{bmatrix} -i^2/4 + 3/4 & \sqrt{3}i/4 - \frac{\sqrt{3}i}{4} \\ -\frac{\sqrt{3}i}{4} + \frac{\sqrt{3}i}{4} & \frac{3}{4} - \frac{i^2}{4} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

$\therefore AA^{\theta} = I$

Now,

$$A^{\theta}A = \begin{bmatrix} -i/2 & \sqrt{3}/2 \\ \sqrt{3}/2 & -i/2 \end{bmatrix} \begin{bmatrix} i/2 & \sqrt{3}/2 \\ \sqrt{3}/2 & i/2 \end{bmatrix}$$

$$= \begin{bmatrix} -\frac{i^2}{4} + \frac{3}{4} & -\frac{\sqrt{3}i}{4} + \frac{\sqrt{3}i}{4} \\ \frac{\sqrt{3}i}{4} - \frac{\sqrt{3}i}{4} & \frac{3}{4} - \frac{i^2}{4} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

$\therefore AA^{\theta} = A^{\theta}A = I$

$\therefore$ A is unitary matrix.

3) $A = \begin{bmatrix} 3 & -2 & 0 & -1 \\ 0 & 2 & 2 & 1 \\ 1 & -2 & -3 & 2 \\ 0 & 1 & 2 & 1 \end{bmatrix}$

Apply R_{13}

$\sim \begin{bmatrix} 1 & -2 & -3 & 2 \\ 0 & 2 & 2 & 1 \\ 3 & -2 & 0 & -1 \\ 0 & 1 & 2 & 1 \end{bmatrix}$

Apply $R_3 - 3R_1$, R_{24}

$\sim \begin{bmatrix} 1 & -2 & -3 & 2 \\ 0 & 1 & 2 & 1 \\ 0 & 4 & 9 & -7 \\ 0 & 2 & 2 & 1 \end{bmatrix}$

Apply $R_3 - 4R_2$, $R_4 - 2R_2$

$\sim \begin{bmatrix} 1 & -2 & -3 & 2 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & -11 \\ 0 & 0 & -2 & -1 \end{bmatrix}$

Apply $R_4 + 2R_3$

$$\sim \begin{bmatrix} 1 & -2 & -3 & 2 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & -11 \\ 0 & 0 & 0 & -23 \end{bmatrix}$$

Appy $R_4\left(\frac{-1}{23}\right)$

$$\sim \begin{bmatrix} 1 & -2 & -3 & 2 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & -11 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

The above matrix is in Echelon form.

$\therefore$ no. of non-zero rows = Rank of matrix.

$\therefore$ Rank (A) = 4

4) $$A = \begin{bmatrix} 1 & 1 & 1 & 2 \\ 3 & -3 & 1 & 2 \\ 2 & 1 & -3 & -6 \end{bmatrix}$$

Let $A = I_{3\times3} \; A_{3\times4} \; I_{4\times4}$

$$\begin{bmatrix} 1 & 1 & 1 & 2 \\ 3 & -3 & 1 & 2 \\ 2 & 1 & -3 & -6 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Apply, $R_2 - 3R_1$, $R_3 - 2R_1$

$$\Rightarrow \begin{bmatrix} 1 & 1 & 1 & 2 \\ 0 & -6 & -2 & -4 \\ 0 & -1 & -5 & -10 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Apply $C_2 - C_1$, $C_3 - C_1$, $C_4 - 2C_1$

$$\Rightarrow \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -6 & -2 & -4 \\ 0 & -1 & -5 & -10 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & -1 & -1 & -2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Apply $C_4 - 2C_3$

$$\Rightarrow \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -6 & -2 & 0 \\ 0 & -1 & -5 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & -1 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Apply $C_2 - 2C_3$

$$\Rightarrow \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -2 & -2 & 0 \\ 0 & 9 & -5 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 1 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -2 & 1 & -2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$R_2(-1/2)$

$$\Rightarrow \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 9 & -5 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 3/2 & -1/2 & 0 \\ -2 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 1 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -2 & 1 & -2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

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$C_3 - C_2$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 9 & -14 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 3/2 & -1/2 & 0 \\ -2 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 1 & -2 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & -2 & 3 & -2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$R_3 - 9R_2$,

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -14 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 3/2 & -1/2 & 0 \\ -2-\frac{27}{2} & 9/2 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 1 & -2 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & -2 & 3 & -2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$R_3(-1/4)$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 3/2 & -1/2 & 0 \\ 31/28 & -9/28 & -1/4 \end{bmatrix} A \begin{bmatrix} 1 & 1 & -2 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & -2 & 3 & -2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\therefore \begin{bmatrix} I_3 & 0 \\ 0 & 0 \end{bmatrix} = PAQ$$

where $P = \begin{bmatrix} 1 & 0 & 0 \\ 3/2 & -1/2 & 0 \\ 31/28 & -9/28 & -1/4 \end{bmatrix}$, $Q = \begin{bmatrix} 1 & 1 & -2 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & -2 & 3 & -2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

$\therefore$ Now, it is in normal form.

$\therefore$ Rank $(A) = r = 3$

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5) $x_1 + 2x_2 - x_3 = 1$

$3x_1 - 2x_2 + 2x_3 = 2$

$7x_1 - 2x_2 + 3x_3 = 5$

In matrix form,

$$\begin{bmatrix} 1 & 2 & -1 \\ 3 & -2 & 2 \\ 7 & -2 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 5 \end{bmatrix}$$

$$A \quad X = B$$

$\therefore$ In augmented matrix,

$$C = \left[\begin{array}{ccc:c} 1 & 2 & -1 & 1 \\ 3 & -2 & 2 & 2 \\ 7 & -2 & 3 & 5 \end{array}\right]$$

Apply $R_2 - 3R_1$, $R_3 - 7R_1$

$$\sim \left[\begin{array}{ccc:c} 1 & 2 & -1 & 1 \\ 0 & -8 & 5 & -1 \\ 0 & -16 & 10 & -2 \end{array}\right]$$

Apply $R_3 - 2R_2$

$$\sim \left[\begin{array}{ccc:c} 1 & 2 & -1 & 1 \\ 0 & -8 & 5 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$\therefore \rho(A) = \rho(C) = 2 < 3$ (number of unknowns)

$\therefore$ It is consistent with infinite solutions.

$\therefore$ $x_1 + 2x_2 - x_3 = 1$ — (i)

$-8x_2 + 5x_3 = -1$ — (2)

$\therefore$ Let $x_3 = t$

$\therefore \quad -8x_2 + 5t = -1$

$\therefore \quad -8x = -1 - 5t$

$$x_2 = \frac{-1-5t}{-8} = \frac{1+5t}{8} \quad \text{— (3)}$$

$$\therefore \quad x_1 + 2\left(\frac{1+5t}{8}\right) - t = 1$$

$$x_1 + \frac{1+5t}{4} - t = 1$$

$$x_1 = 1 + t - \left(\frac{1+5t}{4}\right)$$

$$x_1 = \frac{4+4t-1-5t}{4}$$

$$x_1 = \frac{3-t}{4} \quad -(4)$$

$\therefore$ The value of $x_1 = \frac{3-t}{4}$, $x_2 = \frac{1+5t}{8}$, $x_3 = t$.