Name - Ayush Jain
SAP ID - 60004200132
Div - A1

19/11/2021

Maths - III

Tutorial 2 : Laplace Transform

1) Evaluate $\int_0^\infty e^{-t} \sin\left(\frac{t}{2}\right) \sinh\left(\frac{\sqrt{3}}{2} t\right) dt$

2) Find $L\left[t e^{-4t} \sin 3t\right]$

3) Find $L\left[t^5 \cosh t\right]$

4) Prove that $\int_0^\infty e^{-t} \frac{\sin^2 t}{t} dt = \frac{1}{4} \log 5$

5) Find $L\left[\int_0^t \frac{1-e^{au}}{u} du\right]$

Solutions :

1) $$\int_0^\infty e^{-t} \sin\left(\frac{t}{2}\right) \sinh\left(\frac{\sqrt{3}}{2}t\right) dt$$

$$= \int_0^\infty e^{-t} \left(\frac{e^{(\sqrt{3}/2)t} - e^{-(\sqrt{3}/2)t}}{2} \right) \sin\left(\frac{t}{2}\right) dt$$

$$\cancel{= \frac{1}{2}\int_0^\infty \left[e^{-1+\sqrt{3}/2} \right.} \quad \cancel{\frac{1}{2}\int_0^\infty \left[e^{-(t-\sqrt{3}/2)t} \right.}$$

$$= \frac{1}{2}\int_0^\infty \left[e^{-(1-\sqrt{3}/2)t} - e^{-(1+\sqrt{3}/2)t} \right] \sin\left(\frac{t}{2}\right) dt$$

$$= \frac{1}{2}\left[\int_0^\infty e^{-(1-\sqrt{3}/2)t} \cdot \sin\frac{t}{2}\, dt - \int_0^\infty e^{-(1+\sqrt{3}/2)t} \cdot \sin\frac{t}{2}\, dt \right]$$

$$= \frac{1}{2}\left[L\left(\sin\frac{t}{2}\right) - L\left(\sin\frac{t}{2}\right) \right]$$

where $s = 1 - \frac{\sqrt{3}}{2}$ where $s = 1 + \sqrt{3}/2$

Consider,

$$L\left[\sin\frac{t}{2}\right] = \frac{1/2}{s^2 + 1/4} = \frac{2}{4s^2+1}$$

$$\therefore \frac{1}{2}\left[\frac{2}{4s^2+1} - \frac{2}{4s^2+1} \right]$$

wher $s = 1 - \frac{\sqrt{3}}{2}$ $s = 1 + \sqrt{3}/2$

$$\therefore \frac{1}{2}\left[\frac{2}{4(1-\sqrt{3}/2)^2+1} - \frac{2}{4(1+\sqrt{3}/2)^2+1}\right]$$

$$\therefore \frac{1}{2}\left[\frac{2}{4(1-\sqrt{3}+3/4)+1} - \frac{2}{4(1+\sqrt{3}+3/4)+1}\right]$$

$$\therefore \frac{1}{2}\left[\frac{2}{4-4\sqrt{3}+4} - \frac{2}{4+4\sqrt{3}+4}\right]$$

$$\therefore \frac{1}{\cancel{2}}\left[\frac{\cancel{2}}{8-4\sqrt{3}} - \frac{\cancel{2}}{8+4\sqrt{3}}\right]$$

$$\therefore \frac{1}{2}\left[\frac{1}{4-2\sqrt{3}} - \frac{1}{4+2\sqrt{3}}\right]$$

$$\therefore \frac{1}{2}\left[\frac{4+2\sqrt{3}-4+2\sqrt{3}}{16-12}\right] = \frac{1}{2}\left[\frac{4\sqrt{3}}{4}\right] = \frac{\sqrt{3}}{2}$$

$$\boxed{\therefore \int_0^\infty e^{-t}\sin\left(\frac{t}{2}\right)\sinh\left(\frac{\sqrt{3}}{2}\right)t\ dt = \frac{\sqrt{3}}{2}}$$

2) $L\left[t e^{-4t} \sin 3t\right]$

Consider, $L[\sin 3t] = \frac{3}{s^2+9}$

$\therefore L\left[e^{-4t} \sin 3t\right] = \frac{3}{(s+4)^2+9}$ ... By First shifting theorem.

NOW,

$$\therefore L\left[t e^{-4t} \sin 3t\right] = (-1)\frac{d}{ds} L\left[e^{-4t} \sin 3t\right]$$

$$= (-1)\frac{d}{ds}\left[\frac{3}{(s+4)^2+9}\right]$$

$$= -\frac{d}{ds}\left[\frac{3}{s^2+8s+25}\right]$$

$$= \frac{6(s+4)}{(s^2+8s+25)^2}$$

$$\therefore \boxed{L\left[t e^{-4t} \sin 3t\right] = \frac{6(s+4)}{(s^2+8s+25)^2}}$$

3) $L[t^5 \cosh t]$

$= L\left[t^5 \left(\frac{e^t + e^{-t}}{2}\right)\right]$

$\cancel{= L[} \quad = \frac{1}{2}L\left[e^t \cdot t^5 + e^{-t} \, t^5\right]$

Consider, $L[t^5] = \frac{5!}{s^6}$

$\therefore$ By ~~F~~ First Shifing Theorem,

$\cancel{\frac{1}{2}L[} \quad = \frac{1}{2}\left[\frac{5!}{(s-1)^6} + \frac{5!}{(s+1)^6}\right]$

$= \frac{120}{2}\left[\frac{1}{(s-1)^6} + \frac{1}{(s+1)^6}\right]$

$= 60\left[\frac{1}{(s-1)^6} + \frac{1}{(s+1)^6}\right]$

$\therefore \boxed{L[t^5 \cosh t] = 60\left[\frac{1}{(s-1)^6} + \frac{1}{(s+1)^6}\right]}$

4) To prove: $\int_0^\infty e^{-t} \frac{\sin^2 t}{t} \, dt = \frac{1}{4} \log 5$

LHS: $\int_0^\infty e^{-t} \frac{\sin^2 t}{t} \, dt = L\left[\frac{\sin^2 t}{t}\right]$, wher $s = 1$

Consider,

$$L\left[\frac{\sin^2 t}{t}\right] = \int_s^\infty L[\sin^2 t] \, ds$$

$$= \int_s^\infty L\left[\frac{1-\cos 2t}{2}\right] ds$$

$$= \int_s^\infty \frac{1}{2} L[1-\cos 2t] \, ds$$

$$= \int_s^\infty \frac{1}{2}\left[L[1] - L[\cos 2t]\right] ds$$

$$= \int_s^\infty \frac{1}{2}\left[\frac{1}{s} - \frac{s}{s^2+4}\right] ds$$

$$= \int_s^\infty \frac{1}{2} \cdot \frac{1}{s} \, ds - \int_0^\infty \frac{1}{2} \cdot \frac{s}{s^2+4} \, ds$$

$$= \left[\frac{1}{2} \log s - \frac{1}{4} \log(s^2+4)\right]_s^\infty$$

$$= \cancel{\frac{1}{2}\left[\log s - \frac{1}{2}\log(s^2+4)\right]_s^\infty} \qquad = \cancel{\frac{1}{4}\left[\log\left(\frac{s^2}{s^2+4}\right)\right]_s^\infty}$$

$$= \cancel{\frac{1}{2}\left[\log\left(\frac{s}{\sqrt{s^2+4}}\right)\right]_s^\infty} \qquad =$$

$$= \frac{1}{2}\left[\log s - \frac{1}{2}\log(s^2+4)\right]_s^\infty$$

$$= \frac{1}{2}\left[\log\left(\frac{s}{\sqrt{s^2+4}}\right)\right]_s^\infty$$

$$= \frac{1}{2}\lim_{s\to\infty}\log\left(\frac{s}{\sqrt{s^2+4}}\right) - \frac{1}{2}\log\left(\frac{s}{\sqrt{s^2+4}}\right)$$

$$= \frac{1}{2}\lim_{s\to\infty}\log\left(\frac{1}{\sqrt{1+4/s^2}}\right) - \frac{1}{2}\log\left(\frac{s}{\sqrt{s^2+4}}\right)$$

$$= \frac{1}{2}\log(1) - \frac{1}{2}\log\left(\frac{s}{\sqrt{s^2+4}}\right)$$

$$= \frac{1}{2}\log\left(\frac{\sqrt{s^2+4}}{s}\right)$$

$$\therefore \int_0^\infty e^{-t}\frac{\sin^2 t}{t}\,dt = L\left[\frac{\sin^2 t}{t}\right], s=1$$

$$= \frac{1}{2}\log\left(\frac{\sqrt{s^2+4}}{s}\right), s=1$$

$$= \frac{1}{2}\log(\sqrt{5})$$

$$= \frac{1}{2}\cdot\frac{1}{2}\log 5$$

$$= \frac{1}{4}\log 5 = \text{RHS}$$

$$\therefore \boxed{\int_0^\infty e^{-t}\frac{\sin^2 t}{t}\,dt = \frac{1}{4}\log 5}$$

5) ~~Find~~ $L\left[\int_0^t \frac{1-e^{au}}{u}\,du\right]$

Consider, $L[1-e^{au}] = L[1] - L[e^{au}]$

$$= \frac{1}{s} - \frac{1}{s-a}$$

$$\therefore L\left[\frac{1-e^{au}}{u}\right] = \int_s^\infty \left[\frac{1}{s} - \frac{1}{s-a}\right] ds$$

$$= \left[\log s - \log(s-a)\right]_s^\infty$$

$$= \left[\log\left(\frac{s}{s-a}\right)\right]_s^\infty$$

$$= \lim_{s\to\infty} \log\left(\frac{s}{s-a}\right) - \log\left(\frac{s}{s-a}\right)$$

$$= \lim_{s\to\infty} \log\left(\frac{1}{1-a/s}\right) - \log\left(\frac{s}{s-a}\right)$$

$$= -\log\left(\frac{s}{s-a}\right) = \log\left(\frac{s-a}{s}\right) = F(s)$$

$$\therefore \boxed{L\left[\int_0^t \frac{1-e^{au}}{u}\,du\right] = \frac{1}{s}F(s) = \frac{1}{s}\log\left(\frac{s-a}{s}\right)}$$