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Engineering Mathematics - 1

1) Find non-singular matrices P and Q such that PAQ is in the normal form. Hence find the rank of A and A^{-1} if exists where $A = \begin{bmatrix} 1 & 2 & -2 \\ -1 & 3 & 0 \\ 0 & -2 & 1 \end{bmatrix}$

→ Let $A = I_3 A I_3$

$$\therefore \begin{bmatrix} 1 & 2 & -2 \\ -1 & 3 & 0 \\ 0 & -2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$R_2 + R_1$

$$\sim \begin{bmatrix} 1 & 2 & -2 \\ 0 & 5 & -2 \\ 0 & -2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$C_2 - 2C_1$

$$\sim \begin{bmatrix} 1 & 0 & -2 \\ 0 & 5 & -2 \\ 0 & -2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & -2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$C_3 + 2C_1$,

$$\Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 5 & -2 \\ 0 & -2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & -2 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$C_2 + 2C_3$

$$\Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 2 & 2 \\ 0 & 1 & 0 \\ 0 & 2 & 1 \end{bmatrix}$$

$R_2 + 2R_3$

$$\Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 2 & 2 \\ 0 & 1 & 0 \\ 0 & 2 & 1 \end{bmatrix}$$ ~~wh~~ which

is in normal form i.e. $[I_3] = PAQ$.

$\therefore$ Rank of $A = 3$

$$\therefore P = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}, \quad Q = \begin{bmatrix} 1 & 2 & 2 \\ 0 & 1 & 0 \\ 0 & 2 & 1 \end{bmatrix}.$$

For A^{-1}, we know that

$$A^{-1} = QP = \begin{bmatrix} 1 & 2 & 2 \\ 0 & 1 & 0 \\ 0 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\therefore \quad A^{-1} = \begin{bmatrix} 3 & 2 & 6 \\ 1 & 1 & 2 \\ 2 & 2 & 5 \end{bmatrix}$$

2) Prove that

$$\frac{1+\cos 9A}{1+\cos A} = \left[16\cos^4 A - 8\cos^3 A - 12\cos^2 A + 4\cos A + 1\right]^2$$

$\rightarrow$ We know that,

$$\frac{1+\cos 9A}{1+\cos A} = \frac{2\cos^2\left(\frac{9A}{2}\right)}{2\cos^2\left(A/2\right)}$$

$$= \frac{2\cos^2\left(\frac{9A}{2}\right)}{2\cos^2\left(A/2\right)} \times \frac{2\sin^2\left(A/2\right)}{2\sin^2\left(A/2\right)}$$

$$= \left[\frac{2\cos\left(\frac{9A}{2}\right)\sin\left(A/2\right)}{2\cos\left(A/2\right)\sin\left(A/2\right)}\right]^2$$

$$= \left[\frac{\sin\left(\frac{9A}{2}+\frac{A}{2}\right) - \sin\left(\frac{9A}{2}-\frac{x}{2}\right)}{\sin A}\right]^2$$

$$\therefore \quad \frac{1+\cos 9A}{1+\cos A} = \left[\frac{\sin 5A - \sin 4A}{\sin A}\right]^2 \quad \text{—— (i)}$$

Now,

$$\sin 5A = 5\cos^4 A \sin A - 10\cos^2 A \sin^3 A + \sin^5 A$$
$$= \sin A\left[5\cos^4 A - 10\cos^2 A \sin^2 A + \sin^4 A\right]$$

$$\sin 4A = 4\cos^3 A \sin A - 4\cos A \sin^3 A$$
$$= \sin A\left[4\cos^3 A - 4\cos A \sin^2 A\right]$$

Now,

$$LHS = \frac{1+\cos 9A}{1+\cos A}$$

$$= \left[\frac{\sin 5A - \sin 4A}{\sin A}\right]^2 \quad \cdots \left[\text{from equation (i)}\right]$$

$$= \left(\frac{\sin A}{\sin A}\right)^2 \left[5\cos^4 A - 10\cos^2 A \sin^2 A + \sin^4 A - 4\cos^3 A + 4\cos A \sin^2 A\right]^2$$

$$= \left[5\cos^4 A - 10\cos^2 A \sin^2 A + \sin^4 A - 4\cos^3 A + 4\cos A \sin^2 A\right]^2$$

$$= \left[5\cos^4 A - 10\cos^2 A(1-\cos^2 A) + (1-\cos^2 A)^2 - 4\cos^3 A + 4\cos A(1-\cos^2 A)^2\right]^2$$

$$= \cancel{+16\cos^4 x - 8\cos^3 x - 12\cos^2 x + 4\cos x}$$

$$= \left[16\cos^4 A - 8\cos^3 A - 12\cos^2 A + 4\cos A + 1\right]^2$$
$$= RHS$$

$\therefore$ LHS = RHS, Hence Proved.

5)

a) Show that the minimum value of $u = xy + a^3\left(\frac{1}{x} + \frac{1}{y}\right)$ is $3a^2$

→ We have $f(x,y) = xy + a^3\left(\frac{1}{x} + \frac{1}{y}\right)$

Step 1: $f_x = y - \frac{a^3}{x^2}$, $f_{xx} = \frac{2a^3}{x^3}$

$f_y = x - \frac{a^3}{y^2}$, $f_{yy} = \frac{2a^3}{y^3}$, $f_{xy} = 1$

Step 2 Solving, $f_x = 0$ and $f_y = 0$

$\therefore y - \frac{a^3}{x^2} = 0$, $x - \frac{a^3}{y^2} = 0$

$\therefore x^2 y = a^3$ and $x = \frac{a^3}{y^2}$

$\therefore \frac{a^6}{y^3} = a^3$

$\therefore y = a$

$\therefore x^2 = a^2$

$\therefore x = a$ or $-a$

Step 3 (i) When $x = -a$, $y = a$

$r = f_{xx} = -2$, $t = f_{yy} = 2$

$s = f_{xy} = 1$

$\therefore rt - s^2 = -4 - 1 = -5 < 0$

$\therefore f(x,y)$ is neither maximum nor minimum.

(ii) when $x = a,\ y = a$

$\therefore r = f_{xx} = 2,\quad s = f_{xy} = 1,\quad t = f_{yy} = 2$

$\therefore rt - s^2 = 4 - 1 = 3 > 0$

And $r = f_{xx} = 2 > 0$

$\therefore f(x,y)$ is ~~maximum~~ minimum at (a,a)

$\therefore$ At (a,a) : $u = a^2 + a^3\left(\frac{1}{a} + \frac{1}{a}\right) = 3a^2$

Hence,

$\therefore$ Minimum value of $u = xy + a^3\left(\frac{1}{x} + \frac{1}{y}\right)$ is $3a^2$

5) b) If $x = uv,\ y = \frac{u}{v}$ prove that $JJ' = 1$

$\rightarrow$ $\frac{\partial x}{\partial u} = v,\quad \frac{\partial x}{\partial v} = u$

$\frac{\partial y}{\partial u} = \frac{1}{v},\quad \frac{\partial y}{\partial v} = \frac{-u}{v^2}$

$$\therefore J = \begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix} = \begin{vmatrix} v & u \\ 1/v & -u/v^2 \end{vmatrix} = \frac{-2u}{v}$$

But,

$u^2 = xy$

$\therefore 2u \frac{\partial u}{\partial x} = y$ and $2u \frac{\partial u}{\partial y} = x$

And, $v^2 = \frac{x}{y}$

$\therefore 2v \frac{\partial v}{\partial x} = \frac{1}{y}$ and $2v \frac{\partial v}{\partial y} = -\frac{x}{y^2}$

$$\therefore J' = \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix} = \begin{vmatrix} y/2u & x/2u \\ 1/2vy & -x/2vy^2 \end{vmatrix} = \frac{-x}{4uvy} - \frac{x}{4uvy}$$

$$\therefore J' = \frac{-1}{2uv} \cdot \frac{x}{y} = \frac{-1}{2uv} \cdot v^2 = \frac{-v}{2u}$$

$$\therefore JJ' = \left(-\frac{2u}{v}\right)\left(-\frac{v}{2u}\right) = 1$$

- Hence Proved.

6) Solve the following equations by Gauss-Seidel method.

$$27x + 6y - z = 85$$
$$6x + 15y + 2z = 72$$
$$x + y + 54z = 110$$

$\rightarrow$ $x = \frac{1}{27}(85 - 6y + z)$ — (1)

$y = \frac{1}{15}(72 - 6x - 2z)$ — (2)

$z = \frac{1}{54}(110 - x - y)$ — (3)

First iteration: $y=0, z=0$ from (i)

$\therefore x_1 = \frac{85}{27} = 3.15$

we put $x = 3.15, z = 0$ in (2)

$y_1 = \frac{1}{15}[72 - 6(3.15)] = 3.54$

We use these values of x_1 and y_1 to find z_1
$x = 3.15, y_1 = 3.54$ in (3)

$z_1 = \frac{1}{54}(110 - 3.15 - 3.54) = 1.91$

Second Iteration: we use latest value of y and z to find x i.e. $y_1 = 3.54$, and $z_1 = 1.91$

$$x_2 = \frac{1}{27}\left[85 - 6(3.54) + 1.91\right] = 2.43$$

Put $x_2 = 2.43$, $z_1 = 1.91$ to find y_2 from (2)

$$\therefore y_2 = \frac{1}{15}\left[72 - 6(2.43) - 2(1.91)\right] = 3.57$$

Put $x_2 = 2.43$, $y_2 = 3.57$ in (3)

$$\therefore \cancel{z_2 = \frac{1}{27}\left[85 - 6(3.57) + 1.93\right] = 1.93}$$

$$z_2 = \frac{1}{54}\left(110 - 2.43 - 3.57\right) = 1.93$$

Third Iteration

Put $y_2 = 3.57$, $z_2 = 1.93$ in (i),

$$x_3 = \frac{1}{27}\left[85 - 6(3.57) + 1.93\right] = 2.43$$

Put $x_3 = 2.43$, $z_2 = 1.93$ in (2),

$$y_3 = \frac{1}{15}\left[72 - 6(2.43) - 2(1.93)\right] = 3.57$$

Put $x_3 = 2.43$, $y_3 = 3.57$ in (3),

$$z_3 = \frac{1}{54}\left(110 - 2.43 - 3.57\right) = 1.93$$

Since, second and third iterations give the same value.

$$\therefore \boxed{\begin{aligned} x &= 2.43 \\ y &= 3.57 \\ z &= 1.93 \end{aligned}}$$