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Engineering Mathematics

Tutorial 6 : Application of Partial Differentiation and Expansion of functions

1) Find the extreme values of $u = xy(a-x-y)$

2) Examine the maximum and minimum on the surface

$$u = xy + a^2\left(\frac{1}{x} + \frac{1}{y}\right)$$

3) Expand the function in an ascending power of x
$\sin(e^x - 1)$

4) Using Taylor's series, find the expansion of
$f(x) = x^4 - 3x^3 + 2x^2 - x + 4$ in the power of $(x-1)$

5) Expand the function in an ascending power of x to a minimum of 3 non-zero terms.

$$y = \frac{xe^x}{e^x - 1}$$

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Solutions

1) $u(x,y) = xy(a-x-y)$
$= axy - x^2y - xy^2$

Extreme values of $u(x,y)$ are given by
$\frac{\partial u}{\partial x} = 0$

$\therefore ay - 2xy - y^2 = 0$ — (i)

$\frac{\partial u}{\partial y} = 0$

$\therefore ax - x^2 - 2xy = 0$ — (2)

Solving the equation (i) and (2) simultaneously,

$ay - 2x - y^2 = 0 \Rightarrow y(a-2x-y) = 0$
$ax - 2xy - x^2 = 0 \Rightarrow x(a-2y-x) = 0$

$y = 0$ $\qquad$ $a - 2x - y = 0$
$x = 0$ $\qquad$ $a - 2y - x = 0$

$\therefore y = 0,\ x = 0 \Rightarrow (0,0)$
$y = 0,\ a - x - 2y \Rightarrow (a,0)$
$a - 2x - y = 0,\ x = 0 \Rightarrow (0,a)$
$a - 2x - y = 0,\ a - 2y - x = 0 \Rightarrow \left(\frac{a}{3}, \frac{a}{3}\right)$

$\therefore$ The pair of values of x and y which makes the function stationary are
$(0,0)$, $(a,0)$, $(0,a)$, $\left(\frac{a}{3}, \frac{a}{3}\right)$

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$\therefore r = \frac{\partial^2 u}{\partial x^2} = -2y$

$\therefore s = \frac{\partial^2 u}{\partial x \partial y} = a - 2x - 2y$

$\therefore t = \frac{\partial^2 u}{\partial y^2} = -2x$

Now,

at $(0,0)$

$rt - s^2 = -a^2 < 0$

Hence, $(0,0)$ is not an extreme value of $u(x,y)$

at $(a,0)$

$rt - s^2 = -a^2 < 0$

Hence $(a,0)$ is not an extreme value of $u(x,y)$

at $(0,a)$

$rt - s^2 = 0 - (a-2a)^2 = -a^2 < 0$

Hence $(0,a)$ is not an extreme value of $u(x,y)$

at $\left(\frac{a}{3}, \frac{a}{3}\right)$

$rt - s^2 = \cancel{a} \left(-\frac{2a}{3}\right)\left(-\frac{2a}{3}\right) - \left(\frac{3a}{3} - \frac{2a}{3} - \frac{2a}{3}\right)^2 = \frac{a^2}{3} > 0$

$\therefore r = -\frac{2a}{3}$

Hence, $\left(\frac{a}{3}, \frac{a}{3}\right)$ is an extreme value and will be maximum or minimum according as r is -ve or +ve i.e. according as a is positive or negative.

$\therefore$ The extreme value is $u\left(\frac{a}{3}, \frac{b}{3}\right) = \frac{a^3}{27}$

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2) $u(x,y) = xy + a^2\left(\frac{1}{x} + \frac{1}{y}\right)$

Extreme values of $u(x,y)$ are given by

$\frac{\partial u}{\partial x} = 0$

$\therefore y - \frac{a^2}{x^2} = 0$ — (i)

$\frac{\partial u}{\partial y} = 0$

$\therefore x - \frac{a^2}{y^2} = 0$ — (2)

Solving equation (i) and (2) simultaneously,

$y - \frac{a^2}{x^2} = 0$, $x - \frac{a^2}{y^2} = 0$

$y = \frac{a^2}{x^2}$ $\therefore x - \frac{a^2}{a^4} x^4 = 0$

$x - \frac{x^4}{a^2} = 0$

$x\left[1 - \frac{x^3}{a^2}\right] = 0$

$\therefore x = 0,\ y = \infty$

$\therefore x = a^{2/3},\ y = a^{2/3}$

Hence, the stationary values of x and y are $(0, \infty)$ and $(a^{2/3}, a^{2/3})$

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$$r = \frac{\partial^2 u}{\partial x^2} = \frac{-a^2(-2)}{x^3} = \frac{2a^2}{x^3}$$

$$s = \frac{\partial^2 u}{\partial x \partial y} = 1$$

$$t = \frac{\partial^2 u}{\partial y^2} = \frac{-a^2(-2)}{y^3} = \frac{2a^2}{y^3}$$

Now,

At $(0, \infty)$

$rt - s^2 =$ not defined.

$\therefore$ The extreme values are not defined.

At $(a^{2/3}, a^{2/3})$

$rt - s^2 = 2 \times 2 - 1 = 3 > 0$ and $r = 2 > 0$

Hence, $u(a^{2/3}, a^{2/3})$, the function is minimum and $U_{min} = 3a^{4/3}$

3) Let $y = \sin(e^x - 1)$

$$y = \sin\left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots\ldots - 1\right)$$

$$\therefore y = \sin(e^x - 1) = \sin\left(x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} \ldots\right)$$

We know, $\sin\theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} \ldots$

$$\therefore \sin(e^x - 1) = \left[x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} \ldots\right] - \frac{1}{3!}\left[x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots\right]^3 + \ldots$$

$$= x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \ldots\ldots - \frac{x^3}{6} - \frac{1}{4}x^4 + \ldots$$

$$\therefore \sin(e^x - 1) = x + \frac{x^2}{2} - \frac{5}{24}x^4 + \ldots$$

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4) let $f(x) = x^4 - 3x^3 + 2x^2 - x + 4$

let $X = x - 1$, $h = 1$

$f(x) = f(x-1+1) = f(X+h)$

$= f(h) + X \cdot f'(h) + \frac{X^2}{2!} f''(h) + \frac{X^3}{3!} f'''(h) + \cdots$

$= f(1) + X f'(1) + \frac{X^2}{2!} f''(1) + \frac{X^3}{3!} f'''(1) + \cdots$

$f(x) = x^4 - 3x^3 + 2x^2 - x + 4$

$f(1) = 3$

$f'(x) = 4x^3 - 9x^2 + 4x - 1$

$f'(1) = -2$

$f''(x) = 12x^2 - 18x + 4$

$f''(1) = -2$

$f'''(x) = 24x - 18$

$f'''(1) = 6$

$f''''(x) = 24$

$f''''(1) = 1$

$\therefore f(x) = 3 + (x-1)(-2) + \frac{(x-1)^2}{2!}(-2) + \frac{(x-1)^3}{3!}(6) + \frac{(x-1)^4}{4!}(24) + 0$

$f(x) = 3 - 2(x-1) - (x-1)^2 + (x-1)^3 + (x-1)^4$

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5) $y = \dfrac{xe^x}{e^x - 1}$

$y = \dfrac{xe^x + x - x}{e^x - 1}$

$y = \dfrac{x(e^x - 1) + x}{e^x - 1}$

$y = x + \dfrac{x}{e^x - 1}$

let $k = \dfrac{x}{e^x - 1} = \dfrac{x}{\left[1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots\right] - 1}$

$\therefore k = \dfrac{x}{x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots}$

$\therefore k = \dfrac{1}{1 + \frac{x}{2!} + \frac{x^2}{3!} + \frac{x^3}{4!} \cdots}$

$\therefore k = \left[1 + \dfrac{x}{2!} + \dfrac{x^2}{3!} + \dfrac{x^3}{4!} + \cdots\right]^{-1}$

$\therefore k = \left[1 + \left(\dfrac{x}{2!} + \dfrac{x^2}{3!} + \dfrac{x^3}{4!} + \cdots\right)\right]^{-1}$

Now,

$(1+\alpha)^{-1} = 1 - \alpha + \alpha^2 - \alpha^3 + \alpha^4 \cdots$

$\therefore k = 1 - \left(\dfrac{x}{2!} + \dfrac{x^2}{3!} + \dfrac{x^3}{4!} + \cdots\right) + \left(\dfrac{x}{2!} + \dfrac{x^2}{3!} + \dfrac{x^3}{4!} + \cdots\right)^2$

$- \left(\dfrac{x}{2!} + \dfrac{x^2}{3!} + \dfrac{x^3}{4!} + \cdots\right)^3 + \cdots$

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$$\therefore \cancel{K = 1 - \frac{x}{2!} + \frac{x^2}{3!} - \frac{x^4}{4!}}$$

$$\therefore K = 1 - \frac{x}{2!} - \frac{x^2}{3!} + \frac{x^2}{(2!)^2} - \frac{x^3}{4!} + {}^2C_1\left(\frac{x^3}{2!\,3!}\right) - \frac{x^3}{(2!)^3} - \frac{x^4}{5!} + \frac{x^4}{(3!)^2}$$

$$+ 2\left(\frac{x^4}{2!\,4!}\right) - {}^3C_2\,\frac{x^4}{(2!)^2\,3!} + \frac{x^4}{(2!)^4} + \cdots\cdots$$

$$\therefore K = \frac{x}{e^x - 1} = 1 - \frac{x}{2} + \frac{x^2}{12} - \frac{x^4}{720} \cdots\cdots$$

$$\therefore f(x) = x + \frac{x}{e^x - 1}$$

$$= x + \left[1 - \frac{x}{2} + \frac{x^2}{12} - \frac{x^4}{720} \cdots\cdots\right]$$

$$\therefore \frac{xe^x}{e^x - 1} = 1 + \frac{x}{2} + \frac{x^2}{12} - \frac{x^4}{720} \cdots\cdots$$

$$\therefore \frac{xe^x}{e^x - 1} = 1 + \frac{x}{2} + \frac{x^2}{12} - \frac{x^4}{720} \cdots\cdots$$