Name - Ayush Jain
SAP ID - 60004200132

DATE:

Engineering Mathematics - II

Tutorial - 1 : Beta and Gamma Functions, DUIS and Rectification of plane curves.

1.) Show that $\int_0^\infty x e^{-x^8} dx \cdot \int_0^\infty x^2 e^{-x^4} dx = \frac{\pi}{16\sqrt{2}}$

2) Evaluate $\int_0^1 \frac{(1-x^4)^{3/4}}{(1+x^4)^2} dx$

3) Prove that $\int_0^\pi x \sin^5 x \cos^4 x \, dx = \frac{8\pi}{315}$

4) Assuming the validity of differentiation under the integral sign, prove that

$$\int_0^\infty \frac{\tan^{-1}(ax) - \tan^{-1}(bx)}{x} dx = \frac{\pi}{2} \log\left(\frac{a}{b}\right)$$

5) Find the length of the parabola $x^2 = 4y$ which lies inside the circle $x^2 + y^2 = 6y$.

Solutions:

1.) Let $I = \int_0^\infty x e^{-x^8} dx \cdot \int_0^\infty x^2 e^{-x^4} dx$

$= I_1 \cdot I_2$

$\therefore I_1 = \int_0^\infty x e^{-x^8} dx$

Put $x^8 = t$

$x = t^{1/8}$

$\therefore dx = \frac{1}{8} t^{-7/8} dt$

when $x = 0,\ t = 0$

$x = \infty,\ t = \infty$

$\therefore I_1 = \int_0^\infty t^{1/8} e^{-t} \frac{1}{8} t^{-7/8} dt$

$= \frac{1}{8} \int_0^\infty e^{-t} t^{-6/8} dt$

$= \frac{1}{8} \int_0^\infty e^{-t} t^{-3/4} dt$

$= \frac{1}{8} \int_0^\infty e^{-t} t^{1/4 - 1} dt$

$I_1 = \frac{1}{8} \Gamma\frac{1}{4}$

Now,

$I_2 = \int_0^\infty x^2 e^{-x^4} dx$

Put $x^4 = t$

$x = t^{1/4}$

$dx = \frac{1}{4} t^{-3/4} dt$

DATE:

when $x = 0$, $t = 0$

$x = \infty$, $t = \infty$

$$\therefore I_2 = \int_0^\infty t^{1/2} \cdot e^{-t} \cdot \frac{1}{4} t^{-3/4} \, dt$$

$$= \frac{1}{4} \int_0^\infty e^{-t} \, t^{-1/4} \, dt$$

$$= \frac{1}{4} \int_0^\infty e^{-t} \cdot t^{3/4-1} \, dt$$

$$I_2 = \frac{1}{4} \Gamma\left(\frac{3}{4}\right)$$

$$\therefore I = I_1 \cdot I_2$$

$$= \frac{1}{8} \Gamma\left(\frac{1}{4}\right) \cdot \frac{1}{4} \Gamma\left(\frac{3}{4}\right)$$

$$= \frac{1}{32} \Gamma\left(\frac{1}{4}\right) \Gamma\left(\frac{3}{4}\right)$$

$$= \frac{1}{32} \sqrt{2}\, \pi$$ .... [By Duplication formula of Gamma functions. $\Gamma\left(\frac{1}{4}\right) \Gamma\left(\frac{3}{4}\right) = \sqrt{2}\,\pi$]

$$I = \frac{\pi}{16\sqrt{2}}$$

$$\therefore \int_0^\infty x e^{-x^8} dx \cdot \int_0^\infty x^2 e^{-x^4} dx = \frac{\pi}{16\sqrt{2}}$$

DATE:

2) let $I = \int_0^1 \frac{(1-x^4)^{3/4}}{(1+x^4)^2} dx$

Put $x^4 = t$

$x = t^{1/4}$

$dx = \frac{1}{4} t^{-3/4} dt$

when $x = 0, \ t = 0$

$x = 1, \ t = 1$

$\therefore I = \int_0^1 \frac{(1-t)^{3/4}}{(1+t)^2} \cdot \frac{1}{4} t^{-3/4} dt$

$I = \frac{1}{4} \int_0^1 \frac{t^{-3/4} (1-t)^{3/4}}{(1+t)^2} dt$

let $t = \frac{a}{2-a}$

$\therefore 1-t = 1 - \frac{a}{2-a} = \frac{2(1-a)}{2-a}, \quad 1+t = 1 + \frac{a}{2-a} = \frac{2}{2-a}$

$\therefore dt = \frac{2}{(2-a)^2} da$

when $t = 0, \ a = 0$

$\therefore \ t = 1, \ a = 1$

$\therefore I = \frac{1}{4} \int_0^1 \frac{a^{-3/4}}{(2-a)^{-3/4}} \cdot \frac{2^{3/4}(1-a)^{3/4}}{(2-a)^{3/4}} \cdot \frac{(2-a)^2}{2^2} \cdot \frac{2}{(2-a)^2} da$

$\therefore I = \frac{1 \times 2^{3/4}}{4 \cdot 2} \int_0^1 a^{-3/4} (1-a)^{3/4} da$

DATE:

$\therefore I = \dfrac{2^{3/4}}{8} \int_0^1 a^{1/4-1} (1-a)^{7/4-1} \, da$

$\therefore I = \dfrac{2^{3/4}}{8} \cdot \beta\left(\dfrac{1}{4}, \dfrac{7}{4}\right)$

$= \dfrac{2^{3/4}}{8} \dfrac{\Gamma 1/4 \, \Gamma 7/4}{\Gamma 2}$

$= \dfrac{2^{3/4}}{8} \Gamma\dfrac{1}{4} \left(\dfrac{3}{4}\right) \Gamma\dfrac{3}{4}$

$= \dfrac{2^{3/4}}{8} \cdot \dfrac{3}{4} \sqrt{2}\,\pi$ $\quad \because$ [By duplication formula of Gamma functions. $\Gamma\frac{1}{4}\Gamma\frac{3}{4} = \sqrt{2}\,\pi$]

$I = \dfrac{3 \cdot 2^{5/4}}{32} \pi$

$\therefore \int_0^1 \dfrac{(1-x^4)^{3/4}}{(1+x^4)^2} \, dx = \dfrac{3}{32} \cdot 2^{5/4} \pi$

3) Let $I = \int_0^{\pi} x \sin^5 x \cos^4 x \, dx$ — (1)

$\therefore I = \int_0^{\pi} (\pi - x) \sin^5 x \cos^4 x \, dx$ — (2) $\quad \ldots \left[\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx \right]$

Adding (1) and (2)

$\therefore 2I = \pi \int_0^{\pi} \sin^5 x \cos^4 x \, dx$

$\therefore 2I = \pi \int_0^{\pi/2} \sin^5 x \cdot \cos^4 dx + \pi \int_0^{\pi/2} \sin^5(\pi - x) \cos^4(\pi - x) \, dx$

$\ldots \left[\int_0^{2a} f(x)\,dx = \int_0^a f(x)\,dx + \int_0^a f(2a-x)\,dx \right]$

DATE:

$$\therefore \cancel{2} I = \cancel{2}\pi \int_0^{\pi/2} \sin^5 x \cos^4 x \, dx$$

$$\therefore I = \frac{\pi}{2} \beta\left(3, \frac{5}{2}\right)$$

$$= \frac{\pi}{2} \frac{\Gamma 3 \, \Gamma 5/2}{\Gamma 11/2}$$

$$= \frac{\pi}{2} \times \frac{(2\times 1)\left(\frac{\cancel{3}}{2}\times\frac{1}{2}\,\cancel{\Gamma\frac{1}{2}}\right)}{\left(\frac{9}{2}\times\frac{7}{2}\times\frac{5}{2}\times\cancel{\frac{3}{2}}\times\cancel{\Gamma\frac{1}{2}}\right)}$$

$$I = \frac{8\pi}{315}$$

$$\therefore \int_0^{\pi} x \sin^5 x \cos^4 x \, dx = \frac{8\pi}{315}$$

4) let $I(a) = \int_0^{\infty} \frac{\tan^{-1}(ax) - \tan^{-1}(bx)}{x} \, dx$ — (i)

Using Differentiation under the integral sign,

$$\frac{dI}{da} = \int_0^{\infty} \frac{x}{1+a^2x^2} \cdot \frac{1}{x} \, dx$$

$$= \int_0^{\infty} \frac{1}{1+a^2x^2} \, dx$$

$$\therefore \frac{dI}{da} = \left[\frac{1}{a}\tan^{-1}(ax)\right]_0^{\infty}$$

DATE:

$$\therefore \frac{dI}{da} = \frac{1}{a} \times \frac{\pi}{2} = \frac{\pi}{2a}$$

Integrating both sides wrt a,

$$\therefore I(a) = \int \frac{\pi}{2a}\, da$$

$$\therefore I(a) = \frac{\pi}{2} \log a + c$$

From eq (1), $a = b$, $I(a) = 0$

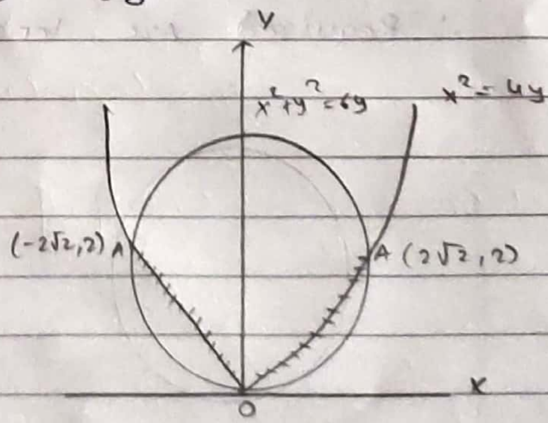
$$\therefore 0 = \frac{\pi}{2} \log b + c$$

$$\therefore c = -\frac{\pi}{2} \log b$$

$$\therefore I(a) = \frac{\pi}{2} \log a - \frac{\pi}{2} \log b$$

$$I(a) = \frac{\pi}{2} \log\left(\frac{a}{b}\right)$$

$$\therefore \int_0^{\infty} \frac{\tan^{-1}(ax) - \tan^{-1}(bx)}{x}\, dx = \frac{\pi}{2} \log\left(\frac{a}{b}\right)$$



5) Equation of parabola:

$$x^2 = 4y$$

$$y = \frac{x^2}{4} \quad -(i)$$

Equation of circle

$$\therefore x^2 + y^2 - 6y + 9 = 9$$

$$\therefore x^2 + (y-3)^2 = 3^2 \quad -(2)$$

$\therefore$ On solving (i) and (2), Points of intersection are $(0,0)$, $(2\sqrt{2}, 2)$, $(-2\sqrt{2}, 2)$

NOW,

$$y = \frac{x^2}{4}$$

$$\frac{dy}{dx} = \frac{x}{2}$$

$\therefore$ Required arc length $= 2 S_{OA}$

$$= 2\int_0^{2\sqrt{2}} \sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx$$

$$= 2\int_0^{2\sqrt{2}} \sqrt{1+\frac{x^2}{4}}\, dx$$

$$= \left[\frac{x}{2}\sqrt{x^2+4} + \frac{4}{2}\log\left(x+\sqrt{x^2+4}\right)\right]_0^{2\sqrt{2}}$$

$$= \sqrt{2}\times\sqrt{12} + 2\log\left(2\sqrt{2}+\sqrt{12}\right) - 2\log 2$$

$$= 2\left[\sqrt{6} + \log\left(\sqrt{2}+\sqrt{3}\right)\right]$$

$\therefore$ Required Arc length is $2\left[\sqrt{6} + \log\left(\sqrt{2}+\sqrt{3}\right)\right]$