SAP ID - 60004200132
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Engineering Mathematics

Tutorial 3: Hyperbolic and logarithmic of complex number.

1.) If $\tan\left(\frac{x}{2}\right) = \tanh\left(\frac{y}{2}\right)$, then show that

i) $y = \log\left(\tan\left(\frac{\pi}{4} + \frac{x}{2}\right)\right)$

ii) $\cosh y \cdot \cos x = 1$

2) Seperate into real and imaginary parts.
$\cos^{-1}(e^{i\theta})$

3) If $\tan\log(x+iy) = a+ib$ and $a^2+b^2 \neq 1$ then prove that
$\tan\log(x^2+y^2) = \frac{2a}{1-a^2-b^2}$

4) Seperate into real and imaginary parts.
$(-i)^{-(1-i)}$

5) Prove that real part of the principal value of
$i^{\log(1+i)}$ is $e^{-\pi^2/8}\cos\left(\frac{\pi}{4}\log 2\right)$

DATE PAGE NO

Solutions

1.) i) Given, $\tan\left(\frac{x}{2}\right) = \tanh\left(\frac{y}{2}\right)$

$$\tan\left(\frac{x}{2}\right) = \frac{\sinh(y/2)}{\cosh(y/2)}$$

$$\tan\left(\frac{x}{2}\right) = \frac{e^{y/2} - e^{-y/2}}{e^{y/2} + e^{-y/2}}$$

$$\tan\left(\frac{x}{2}\right) = \frac{e^{y} - 1}{e^{y} + 1}$$

$$(e^{y}+1)\tan\left(\frac{x}{2}\right) = e^{y} - 1$$

$$e^{y}\tan\left(\frac{x}{2}\right) + \tan\frac{x}{2} = e^{y} - 1$$

$$\therefore 1 + \tan\frac{x}{2} = e^{y}\left(1 - \tan\frac{x}{2}\right)$$

$$e^{y} = \frac{1 + \tan x/2}{1 - \tan x/2}$$

$$e^{y} = \frac{\tan(\pi/4) + \tan x/2}{1 - \tan(\pi/4)\tan x/2}$$

$$\cancel{e^{y}} = \log$$

$$e^{y} = \tan\left(\frac{\pi}{4} + \frac{x}{2}\right)$$

$$\therefore \boxed{y = \log\tan\left(\frac{\pi}{4} + \frac{x}{2}\right)}$$

DATE

ii) To prove: $\cosh y \cdot \cos x = 1$

$$\therefore \cos x = \frac{1 - \tan^2 (x/2)}{1 + \tan^2 (x/2)} \quad \ldots\ldots (i)$$

$$\cosh y = \frac{1 + \tanh^2 (y/2)}{1 - \tanh^2 (y/2)}$$

As $\tan\left(\frac{x}{2}\right) = \tanh\left(\frac{y}{2}\right)$

$$\therefore \cosh y = \frac{1 + \tan^2 (x/2)}{1 - \tan^2 (x/2)} \quad \ldots\ldots (2)$$

Multiplying equation (i) and (ii)

$$\cosh y \cdot \cos x = \frac{1 + \tan^2 (x/2)}{1 - \tan^2 (x/2)} \times \frac{1 - \tan^2 (x/2)}{1 + \tan^2 (x/2)}$$

$$\therefore \boxed{\cosh y \cdot \cos x = 1}$$

2) Let $\cos^{-1}(e^{i\theta}) = x + iy$

$$e^{i\theta} = \cos(x + iy)$$

$$\cos\theta + i\sin\theta = \cos(x + iy)$$

$$\therefore \cos\theta + i\sin\theta = \cos x \cos iy - \sin x \sin iy$$

$$\therefore \cos\theta + i\sin\theta = \cos x \cosh y - i\sin x \sinh y$$

$\therefore$ On comparing real and imaginary parts,

$$\cos\theta = \cos x \cdot \cosh y \quad — (1)$$

$$\sin\theta = -\sin x \sinh y \quad — (2)$$

DATE

$\therefore \cosh y = \dfrac{\cos\theta}{\cos x}$

$\therefore \sinh y = \dfrac{-\sin\theta}{\sin x}$

Now,

$$\cosh^2 y - \sinh^2 y = 1$$

$$\left(\frac{\cos\theta}{\cos x}\right)^2 - \left(\frac{-\sin\theta}{\sin x}\right)^2 = 1$$

$$\frac{\cos^2\theta}{\cos^2 x} - \frac{\sin^2\theta}{\sin^2 x} = 1$$

$$\therefore \frac{\cos^2\theta\cdot\sin^2 x - \sin^2\theta\cdot\cos^2 x}{\cos^2 x\cdot\sin^2 x} = 1$$

$$\therefore \cos^2\theta\cdot\sin^2 x - \sin^2\theta\cdot\cos^2 x = \cos^2 x\cdot\sin^2 x$$

$$\therefore \cos^2\theta\cdot\sin^2 x - \sin^2\theta\,(1-\sin^2 x) = \sin^2 x\,(1-\sin^2 x)$$

$$\therefore \cos^2\theta\cdot\sin^2 x - \sin^2\theta + \sin^2\theta\cdot\sin^2 x = \sin^2 x - \sin^4 x$$

$$\therefore \sin^2 x\,(\cos^2\theta + \sin^2\theta) - \sin^2\theta = \sin^2 x - \sin^4 x$$

$$\therefore \cancel{\sin^2 x} - \sin^2\theta = \cancel{\sin^2 x} - \sin^4 x$$

$$\therefore \sin^2\theta = \sin^4 x$$

$$\therefore \sin^4 x = \sin^2\theta$$

$$\therefore \sin^2 x = \pm\sin\theta$$

Since, $\sin^2 x$ cannot be negative, $-\sin\theta$ is ignored.

$$\therefore \sin^2 x = \sin\theta$$

$$\therefore \sin x = \sqrt{\sin\theta}$$

$$\boxed{x = \sin^{-1}\sqrt{\sin\theta}.}$$

Now,

substitute $\sin x = \sqrt{\sin\theta}$ in equation (2)

DATE

$\therefore \sin\theta = -\sin x \sinh y$

$\therefore \sin\theta = -\sqrt{\sin\theta} \cdot \sinh y$

$\therefore \sinh y = -\sqrt{\sin\theta}$

$\therefore y = \sinh^{-1}(-\sqrt{\sin\theta})$

$\therefore y = \log(-\sqrt{\sin\theta} + \sqrt{1+\sin\theta})$

$\therefore y = \log(\sqrt{1+\sin\theta} - \sqrt{\sin\theta})$

$\therefore \cos^{-1}(e^{i\theta}) = x + iy$

$= \sin^{-1}\sqrt{\sin\theta} + i\log(\sqrt{1+\sin\theta} - \sqrt{\sin\theta})$

$\therefore$

$\therefore \boxed{\cos^{-1}(e^{i\theta}) = \sin^{-1}(\sqrt{\sin\theta}) + i\log(\sqrt{1+\sin\theta} - \sqrt{\sin\theta})}$

3) We have,

$\tan\log(x+iy) = a+ib$

$\therefore \tan\log(x-iy) = a-ib$

To prove: $\tan\log(x^2+y^2) = \dfrac{2a}{1-a^2-b^2}$

$\therefore$ LHS $= \tan\log(x^2+y^2)$

$= \tan[\log(x+iy)(x-iy)]$

$= \tan[\log(x+iy) + \log(x-iy)]$

DATE

PAGE NO

$$= \frac{\tan\log(x+iy) + \tan\log(x-iy)}{1 - \tan\log(x+iy)\tan\log(x-iy)}$$

$$= \frac{a+i\cancel{b} + a - i\cancel{b}}{1 - (a+ib)(a-ib)}$$

$$= \frac{2a}{1-(a^2+b^2)}$$

$$= \frac{2a}{1-a^2-b^2}$$

$= RHS$

$$\therefore \boxed{\tan\log(x^2+y^2) = \frac{2a}{1-a^2-b^2}}$$

4) $(-i)^{-(1-i)} = e^{\log(-i)^{-(1-i)}}$

$$= e^{-(1-i)\log(-i)}$$

$$\therefore \log(-i) = \log e^{i(2n\pi - \pi/2)}$$

$$\log(-i) = i\left(2n\pi - \frac{\pi}{2}\right)$$

$$\therefore (-i)^{-(1-i)} = e^{-(1-i)i(2n\pi - \pi/2)}$$

$$= e^{-(2n\pi - \pi/2)} \cdot e^{-i(2n\pi - \pi/2)}$$

$$= e^{-(2n\pi - \pi/2)} \cdot \left(\cos\left(2n\pi - \frac{\pi}{2}\right) - i\sin\left(2n\pi - \frac{\pi}{2}\right)\right)$$

$$= e^{-(2n\pi - \pi/2)} \cdot [0 - i(-1)]$$

$$= e^{-(2n\pi - \pi/2)} \cdot (i)$$

$$= i e^{-(2n\pi - \pi/2)}$$

$\therefore$ Real part $= 0$

Imaginary part $= e^{-(2n\pi - \pi/2)}$

$$\therefore \boxed{(-i)^{-(1-i)} = i e^{-(2n\pi - \pi/2)}}$$