SAP ID - 60004200132
Name - Ayush Jain

02/08/2021

Engineering Maths

Tutorial 6 : Higher order differential Equations

1) Solve $(D^2+5D+6)y = e^{-2x}\sec^2 x\,(1+2\tan x)$

2) Solve $(D^2-4D+3)y = 2xe^{3x}+3e^{x}\cos 2x$

3) Solve $(D^2-1)y = x\sin 3x + \cos x$

4) Solve by the method of variation of parameters

$$(D^2-4D+4)y = e^{2x}\sec^2 x$$

5) Solve $x^2\dfrac{d^3y}{dx^3} + 3x\dfrac{d^2y}{dx^2} + \dfrac{dy}{dx} = x^2\log x$

Solutions

1.) $(D^2+5D+6)y = e^{-2x}\sec^2 x(1+2\tan x)$

Auxillary equation is,

$D^2+5D+6=0$

$D=-3,-2$

C.F. $= y_c = c_1 e^{-3x} + c_2 e^{-2x}$

$y_p = \frac{1}{(D^2+5D+6)} e^{-2x}\sec^2 x(1+2\tan x)$

$= \frac{1}{(D+2)(D+3)} e^{-2x}\sec^2 x(1+2\tan x)$

$= \frac{e^{-2x}}{D+3}\int \sec^2 x(1+2\tan x)\,dx$

Put $\tan x = t$

$\sec^2 x\,dx = dt$

$\therefore \frac{e^{-2x}}{D+3}\int(1+2t)\,dt = \frac{e^{-2x}}{(D+3)}(t+t^2)$

$= \frac{1}{D+3}\left[e^{-2x}(\tan x+\tan^2 x)\right]$

$= e^{-3x}\int e^{3x}e^{-2x}(\tan x+\tan^2 x)\,dx$

$= e^{-3x}\int e^{x}(\tan x+\sec^2 x-1)\,dx$

$= e^{-3x}\left[e^{x}\tan x - e^{x}\right]$

$= e^{-2x}(\tan x-1)$

$\therefore$ Solution : $\boxed{\begin{aligned} y &= y_c + y_p \\ &= c_1e^{-3x} + c_2e^{-2x} + e^{-2x}(\tan x-1)\end{aligned}}$

2) $(D^2-4D+3)y = 2xe^{3x} + 3e^{x}\cos 2x$

Auxillary equation is $D^2-4D+3=0$

$\therefore D = 3, 1$

$C.F = y_c = c_1 e^{3x} + c_2 e^{x}$

$$y_p = \frac{1}{(D^2-4D+3)} (2xe^{3x} + 3e^{x}\cos 2x)$$

$$= \frac{1}{(D-3)(D-1)}(2xe^{3x}) + \frac{1}{(D-3)(D-1)} 3e^{x}\cos 2x$$

$$= 2e^{3x} \cdot \frac{1}{D(D+2)} \cdot x + 3e^{x} \cdot \frac{1}{(D-2)D} \cos 2x$$

$$= 2e^{3x} \cdot \frac{1}{2D(1+D/2)} \cdot x + 3e^{x} \cdot \frac{1}{-4-2d} \cos 2x$$

$$= e^{3x} \cdot \frac{1}{D}\left(1+\frac{D}{2}\right)^{-1} x - \frac{3e^{x}}{2} \cdot \frac{1}{D+2} \cdot \cos 2x$$

$$= e^{3x} \cdot \frac{1}{D}\left(1-\frac{D}{2}+\frac{D^2}{4} \cdots\right)x - \frac{3e^{x}}{2} \cdot \frac{2-D}{4-D^2} \cos 2x$$

$$= e^{3x} \frac{1}{D}\left(1-\frac{D}{2}+\frac{D^2}{4} \cdots\right)x - \frac{3e^{x}}{2} \cdot \frac{2-D}{8} \cos 2x$$

$$= e^{3x}\left(\frac{x^2}{2} - \frac{x}{2} + \frac{1}{4}\right) - \frac{3e^{x}}{16}(2\cos 2x + 2\sin 2x)$$

$\therefore$ Solution: $y = y_c + y_p$

$$= c_1 e^{3x} + c_2 e^{x} + e^{3x}\left(\frac{x^2}{2} - \frac{x}{2} + \frac{1}{4}\right) - \frac{3e^{x}}{8}(\cos 2x + \sin 2x)$$

$$\boxed{y = c_1 e^{3x} + c_2 e^{x} + e^{3x}\left(\frac{x^2}{2} - \frac{x}{2}\right) - \frac{3e^{x}}{8}(\cos 2x + \sin 2x)}$$

3) $(D^2-1)\,y = x\sin 3x + \cos x$

Auxillary equation is $D^2-1=0$

$\therefore D = \pm 1$

$C.F = y_c = c_1e^{x} + c_2e^{-x}$

$$y_p = \frac{1}{D^2-1}(x\sin 3x + \cos x)$$

$$= \frac{x}{D^2-1}\sin 3x - \frac{2D}{(D^2-1)^2}\sin 3x + \frac{1}{D^2-1}\cos x$$

$$= \frac{-x\sin 3x}{10} - \frac{1}{50}D\sin 3x - \frac{1}{2}\cos x$$

$$= \frac{-x}{10}\sin 3x - \frac{3}{50}\cos 3x - \frac{1}{2}\cos x$$

Solution: $y = y_c + y_p$

$$\boxed{y = c_1e^{x} + c_2e^{-x} - \frac{x}{10}\sin 3x - \frac{\cos x}{2} - \frac{3}{50}\cos 3x}$$

4) $(D^2-4D+4)y = e^{2x}\sec^2 x$

Auxillary equation:

$D^2-4D+4 = 0$

$m = 2, 2$

Complementary equation :

$y_c = (C_1x+C_2)e^{2x}$

$= C_1xe^{2x} + C_2e^{2x}$

$= C_1y_1 + C_2y_2$

$y_1 = xe^{2x}, \quad y_2 = e^{2x}$

Assume particular integral as,

$y_p = uy_1 + vy_2, \quad X = e^{2x}\sec^2 x$

$u = \int -\frac{y_2 X}{\omega}dx, \quad v = \int \frac{y_1 X}{\omega}dx$

$\omega = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} xe^{2x} & e^{2x} \\ e^{2x}(1+2x) & 2e^{2x} \end{vmatrix}$

$= -e^{4x}$

$\therefore u = \int -\frac{y_2 X}{\omega}dx = -\int \frac{e^{2x}.e^{2x}\sec^2 x\, dx}{-e^{4x}}$

$= \int \sec^2 x\, dx$

$= \tan x$

$$V = \int \frac{y_1 x \, dx}{\omega} = \int \frac{x e^{2x} e^{2x} \sec^2 x \, dx}{e^{-4x}}$$

$$= -\int x \sec^2 x \, dx$$

$$= -\left[x \tan x - \int \tan x \right]$$

$$= -\left[x \tan x - \log \sec x \right]$$

$$\therefore y_P = u y_1 + v y_2$$

$$= (\tan x) x e^{2x} - \left[x \tan x - \log \sec x \right] e^{2x}$$

$$= e^{2x} \log \sec x$$

$\therefore$ The Complete solution is :

$$y = y_C + y_P$$

$$\therefore \boxed{y = c_1 x e^{2x} + c_2 e^{2x} + e^{2x} \log \sec x}$$

5) $x^2 \cdot \frac{d^3y}{dx^3} + 3x\frac{d^2y}{dx^2} + \frac{dy}{dx} = x^2 \log x$.

Multiplying throughout by x,

$\therefore x^3\frac{d^3y}{dx^3} + 3x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} = x^3 \log x$ — (i)

It is cauchy's linear differential equation.

Put $x = e^z$ $\therefore z = \log x$

$x\frac{dy}{dx} = Dy$ $\left(D = \frac{d}{dz}\right)$

$x^2\frac{d^2y}{dx^2} = D(D-1)y$

$x^3\frac{d^3y}{dx^3} = D(D-1)(D-2)y$

On substituting in equation (i)

$D(D-1)(D-2)y + 3D(D-1)y + Dy = e^{3z} \cdot z$

$\therefore D^3y = e^{3z} \cdot z$ — (2)

Equation (2) is a linear differential equation with constant coefficients.

Auxillary equation : $D^3 = 0$

$m = 0, 0, 0$

$\therefore$ Complementery function $y_c = c_1 z^2 + c_2 z + c_3$

By defination $\frac{1}{\phi(D)} f(x) = y_P$

$$y_P = \frac{1}{D^3} e^{3z} \cdot z$$

$$= e^{3z} \cdot \frac{1}{(D+3)^3} z$$

$$= e^{3z} \cdot \frac{1}{D^3 + 3D^2 + 9D + 27} \cdot z$$

$$= \frac{e^{3z}}{27} \cdot \frac{1}{\left(1 + D + \frac{D^2}{3} + \frac{D^3}{27}\right)} \cdot z$$

$$= \frac{e^{3z}}{27}\left(1 - D - \frac{1}{3}D^2 \dots\right) z = \frac{e^{3z}}{27}(z-1) = \frac{e^{3z} z}{27} - \frac{e^{3z}}{27}$$

Complete equation, $y = y_c + y_P$

$$y = c_1 z^2 + c_2 z + c_3 + \frac{e^{3z} \cdot z}{27} - \frac{e^{3z}}{27}$$

Resubstitute $z = \log x$

$$\boxed{y = c_1 (\log x)^2 + c_2 (\log x) + c_3 + \frac{x^3}{27} \log x - \frac{x^3}{27}}$$