

1/03/2021

DATE		PAGE NO.

Engineering MathematicsTutorial (1) : Matrices

1.) Show that every square matrix can be expressed uniquely as the sum of a Hermitian matrix and a skew Hermitian matrix.

2.) Prove that matrix A is unitary

$$A = \begin{bmatrix} i/2 & \sqrt{3}/2 \\ \sqrt{3}/2 & i/2 \end{bmatrix}$$

3.) Find the rank of the matrix by reducing it to Echelon form.

$$A = \begin{bmatrix} 3 & -2 & 0 & -1 \\ 0 & 2 & 2 & 1 \\ 1 & -2 & -3 & 2 \\ 0 & 1 & 2 & 1 \end{bmatrix}$$

4.) Find non-singular matrix P and Q such that PAQ is in normal form. Hence find the rank of A.

$$A = \begin{bmatrix} 1 & 1 & 1 & 2 \\ 3 & -3 & 1 & 2 \\ 2 & 1 & -3 & -6 \end{bmatrix}$$

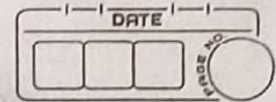
5.) Examine the consistency and solve

$$x_1 + 2x_2 - x_3 = 1$$

$$3x_1 - 2x_2 + 2x_3 = 2$$

$$7x_1 - 2x_2 + 3x_3 = 5$$





### Solutions:

1) We can write,

$$A = \frac{1}{2} (A + A^0) + \frac{1}{2} (A - A^0) = P + Q$$

$$\text{where } P = \frac{1}{2} (A + A^0) \quad , \quad Q = \frac{1}{2} (A - A^0)$$

Now,

$$P^0 = \frac{1}{2} (A + A^0)^0 = \frac{1}{2} (A^0 + A^{00}) = \frac{1}{2} (A^0 + A) = P$$

$\therefore P$  is Hermitian.

Also,

$$Q^0 = \frac{1}{2} (A - A^0)^0 = \frac{1}{2} (A^0 - A^{00}) = \frac{1}{2} (A^0 - A) = -\frac{1}{2} (A - A^0) = -Q$$

$\therefore Q$  is skew-Hermitian.

Thus,  $A$  is sum of Hermitian and skew-Hermitian matrix.

To prove uniqueness,

$$\text{let } A = R + S,$$

where  $R$  is Hermitian and  $S$  is skew-Hermitian

$$\therefore A^0 = (R + S)^0 = R^0 + S^0 = R - S \quad [\because R^0 = R, S^0 = -S]$$

$$\therefore P = \frac{1}{2} (A + A^0) = \frac{1}{2} (R + S + R - S) = R$$

$$\therefore R = P$$

$$\text{and, } Q = \frac{1}{2} (A - A^0) = \frac{1}{2} (R + S - R + S) = S$$

$$\therefore Q = S$$

Hence, the representation  $A = P + Q$  is unique.





$$2.) \quad A = \begin{bmatrix} i/2 & \sqrt{3}/2 \\ \sqrt{3}/2 & i/2 \end{bmatrix}$$

To prove:  $AA^{\theta} = A^{\theta}A = I$

Sol:

$$A^{\theta} = \overline{A^T} = \begin{bmatrix} -i/2 & \sqrt{3}/2 \\ \sqrt{3}/2 & -i/2 \end{bmatrix}$$

$$\therefore AA^{\theta} = \begin{bmatrix} i/2 & \sqrt{3}/2 \\ \sqrt{3}/2 & i/2 \end{bmatrix} \begin{bmatrix} -i/2 & \sqrt{3}/2 \\ \sqrt{3}/2 & -i/2 \end{bmatrix}$$

$$= \begin{bmatrix} -i^2/4 + 3/4 & \sqrt{3}i/4 - \sqrt{3}i/4 \\ -\sqrt{3}i/4 + \sqrt{3}i/4 & \frac{3}{4} - \frac{i^2}{4} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

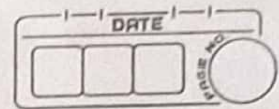
$$\therefore AA^{\theta} = I$$

NOW,

$$A^{\theta}A = \begin{bmatrix} -i/2 & \sqrt{3}/2 \\ \sqrt{3}/2 & -i/2 \end{bmatrix} \begin{bmatrix} i/2 & \sqrt{3}/2 \\ \sqrt{3}/2 & i/2 \end{bmatrix}$$

$$= \begin{bmatrix} -\frac{i^2}{4} + \frac{3}{4} & -\frac{\sqrt{3}i}{4} + \frac{\sqrt{3}i}{4} \\ \frac{\sqrt{3}i}{4} - \frac{\sqrt{3}i}{4} & \frac{3}{4} - \frac{i^2}{4} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$



$$\therefore AA^{\circ} = A^{\circ}A = I$$

$\therefore A$  is unitary matrix.

$$3) A = \begin{bmatrix} 3 & -2 & 0 & -1 \\ 0 & 2 & 2 & 1 \\ 1 & -2 & -3 & 2 \\ 0 & 1 & 2 & 1 \end{bmatrix}$$

Apply  $R_{13}$

$$\hookrightarrow \begin{bmatrix} 1 & -2 & -3 & 2 \\ 0 & 2 & 2 & 1 \\ 3 & -2 & 0 & -1 \\ 0 & 1 & 2 & 1 \end{bmatrix}$$

Apply  $R_3 - 3R_1, R_{24}$

$$\hookrightarrow \begin{bmatrix} 1 & -2 & -3 & 2 \\ 0 & 1 & 2 & 1 \\ 0 & 4 & 9 & -7 \\ 0 & 2 & 2 & 1 \end{bmatrix}$$

Apply  $R_3 - 4R_2, R_4 - 2R_2$

$$\hookrightarrow \begin{bmatrix} 1 & -2 & -3 & 2 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & -11 \\ 0 & 0 & -2 & -1 \end{bmatrix}$$



Apply  $R_4 + 2R_3$ 

$$\sim \begin{bmatrix} 1 & -2 & -3 & 2 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & -11 \\ 0 & 0 & 0 & -23 \end{bmatrix}$$

Apply  $R_4 \left( -\frac{1}{23} \right)$ 

$$\sim \begin{bmatrix} 1 & -2 & -3 & 2 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & -11 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

The above matrix is in Echelon form.

 $\therefore$  no. of non-zero rows = Rank of matrix.

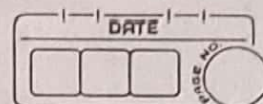
$$\therefore \text{Rank}(A) = 4$$

$$4) \quad A = \begin{bmatrix} 1 & 1 & 1 & 2 \\ 3 & -3 & 1 & 2 \\ 2 & 1 & -3 & -6 \end{bmatrix}$$

$$\text{Let } A = \begin{matrix} I & A & I \\ 3 \times 3 & 3 \times 4 & 4 \times 4 \end{matrix}$$

$$\begin{bmatrix} 1 & 1 & 1 & 2 \\ 3 & -3 & 1 & 2 \\ 2 & 1 & -3 & -6 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$





Apply,  $R_2 - 3R_1$ ,  $R_3 - 2R_1$

∴

$$\begin{bmatrix} 1 & 1 & 1 & 2 \\ 0 & -6 & -2 & -4 \\ 0 & -1 & -5 & -10 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Apply  $C_2 - C_1$ ,  $C_3 - C_1$ ,  $C_4 - 2C_1$

$$\therefore \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -6 & -2 & -4 \\ 0 & -1 & -5 & -10 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & -1 & -1 & -2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Apply  $C_4 - 2C_3$

$$\therefore \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -6 & -2 & 0 \\ 0 & -1 & -5 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & -1 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Apply  $C_2 - 2C_3$

$$\therefore \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -2 & -2 & 0 \\ 0 & 9 & -5 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 1 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -2 & 1 & -2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$R_2 (-1/2)$

$$\therefore \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 9 & -5 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 3/2 & -1/2 & 0 \\ -2 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 1 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -2 & 1 & -2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$C_3 - C_2$$

$$\Rightarrow \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 9 & -14 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 3/2 & -1/2 & 0 \\ -2 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 1 & -2 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & -2 & 3 & -2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$R_3 - 9R_2,$$

$$\Rightarrow \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -14 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 3/2 & -1/2 & 0 \\ -2 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 1 & -2 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & -2 & 3 & -2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$R_3 (-1/4)$$

$$\Rightarrow \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 3/2 & -1/2 & 0 \\ 3/28 & -9/28 & -1/4 \end{bmatrix} A \begin{bmatrix} 1 & 1 & -2 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & -2 & 3 & -2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

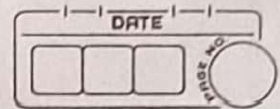
$$\therefore \begin{bmatrix} I_3 & 0 \\ 0 & 0 \end{bmatrix} = P A Q$$

$$\text{where } P = \begin{bmatrix} 1 & 0 & 0 \\ 3/2 & -1/2 & 0 \\ 3/28 & -9/28 & -1/4 \end{bmatrix}, Q = \begin{bmatrix} 1 & 1 & -2 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & -2 & 3 & -2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$\therefore$  Now, it is in normal form.

$$\therefore \text{Rank}(A) = r = 3$$





$$5) \quad x_1 + 2x_2 - x_3 = 1$$

$$3x_1 - 2x_2 + 2x_3 = 2$$

$$7x_1 - 2x_2 + 3x_3 = 5$$

In matrix form,

$$\begin{bmatrix} 1 & 2 & -1 \\ 3 & -2 & 2 \\ 7 & -2 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 5 \end{bmatrix}$$

$$A \cdot x = B$$

$\therefore$  In augmented matrix,

$$C = \left[ \begin{array}{ccc|c} 1 & 2 & -1 & 1 \\ 3 & -2 & 2 & 2 \\ 7 & -2 & 3 & 5 \end{array} \right]$$

Apply  $R_2 - 3R_1$ ,  $R_3 - 7R_1$

$$\hookrightarrow \left[ \begin{array}{ccc|c} 1 & 2 & -1 & 1 \\ 0 & -8 & 5 & -1 \\ 0 & -16 & 10 & -2 \end{array} \right]$$

Apply  $R_3 - 2R_2$

$$\hookrightarrow \left[ \begin{array}{ccc|c} 1 & 2 & -1 & 1 \\ 0 & -8 & 5 & -1 \\ 0 & 0 & 0 & 0 \end{array} \right]$$

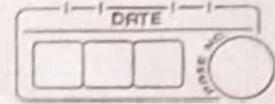
$$\therefore \rho(A) = \rho(C) = 2 < 3 \text{ (number of unknowns)}$$

$\therefore$  It is consistent with infinite solutions.

$$\therefore x_1 + 2x_2 - x_3 = 1 \quad \text{--- (i)}$$

$$-8x_2 + 5x_3 = -1 \quad \text{--- (2)}$$





$$\therefore \text{Let } x_3 = t$$

$$\therefore -8x_2 + 5t = -1$$

$$\therefore -8x_2 = -1 - 5t$$

$$x_2 = \frac{-1 - 5t}{-8} = \frac{1 + 5t}{8} \quad (3)$$

$$\therefore x_1 + 2\left(\frac{1 + 5t}{8}\right) - t = 1$$

$$x_1 + \frac{1 + 5t}{4} - t = 1$$

$$x_1 = 1 + t - \left(\frac{1 + 5t}{4}\right)$$

$$x_1 = \frac{4 + 4t - 1 - 5t}{4}$$

$$x_1 = \frac{3 - t}{4} \quad (4)$$

$$\therefore \text{The value of } x_1 = \frac{3 - t}{4}, \quad x_2 = \frac{1 + 5t}{8}, \quad x_3 = t.$$