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Div - I

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Engineering Mathematics

Tutorial 5: Successive Differentiation

1.) Find the n^{th} derivative of $\frac{x^2}{(x+2)(2x+3)}$

2.) If $y = 2^x \sin^2 x \cos^3 x$, find y_n

3.) If $y = 2x\sqrt{1-x^2}$, prove that

$(1-x^2)y_{n+2} - (2n+1)x y_{n+1} - (n^2-4)y_n = 0$

4.) If $y = e^{ax} \cos^2 x \sin x$, find y_n

5.) If $y = (\sin^{-1} x)^2$, prove that

$(1-x^2)y_{n+2} - (2n+1)x y_{n+1} - n^2 y_n = 0$

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Solutions

1.) $y = \dfrac{x^2}{(x+2)(2x+3)}$

$$\frac{x^2}{(x+2)(2x+3)} = A + \frac{B}{(x+2)} + \frac{C}{(2x+3)}$$

$$x^2 = A(2x+3)(x+2) + B(2x+3) + C(x+2)$$

$$x^2 = x^2(2A) + x(7A+2B+C) + (6A+3B+2C)$$

$\therefore\ A = \frac{1}{2}$, $7A+2B+C = 0$, $6A+3B+2C = 0$

$\therefore$ On solving, $A = \frac{1}{2}$, $B = -4$, $C = \frac{9}{2}$

$$\therefore\ y = \frac{1}{2} - \frac{4}{(x+2)} + \frac{9}{2(2x+3)}$$

Differentiate n times, we have

$$y_n = \frac{-4(-1)^n n!}{(x+2)^{n+1}} + \frac{9}{2}\frac{(-1)^n n!\, 2^n}{(2x+3)^{n+1}}$$

$$y_n = (-1)^n n! \left[\frac{9\, 2^n}{2(2x+3)^{n+1}} - \frac{4}{(x+2)^{n+1}} \right]$$

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2) $y = 2^{x} \sin^2 x \cos^3 x$

$y = e^{x\log 2} \cdot \frac{4\sin^2 x \cos^2 x \cdot \cos x}{4}$

$= e^{x\log 2} \cdot \frac{\sin^2 2x}{4} \cos x$

$= e^{x\log 2} \cdot \frac{(1-\cos 4x)\cdot \cos x}{8}$

$= e^{x\log 2}\left[\frac{1}{8}\cos x - \frac{1}{8}\cos 4x \cos x\right]$

$= e^{x\log 2}\left[\frac{1}{8}\cos x - \frac{2}{16}\cos 4x \cos x\right]$

$= e^{x\log 2}\left[\frac{1}{8}\cos x - \frac{1}{16}(\cos 5x + \cos 3x)\right]$

$y = e^{x\log 2}\left[\frac{1}{8}\cos x - \frac{1}{16}\cos 5x - \frac{1}{16}\cos 3x\right]$

$y = \frac{1}{8}e^{x\log 2}\cos x - \frac{1}{16}e^{x\log 2}\cos 5x - \frac{1}{16}e^{x\log 2}\cos 3x$

Differentiate n times, we get

$y_n = \frac{1}{8} r_1^n e^{\log 2 \cdot x}\cos(x + n\theta_1) - \frac{1}{16} r_2^n e^{\log 2 \cdot x}\cos(5x + n\theta_2)$

$- \frac{1}{16} r_3^n e^{\log 2 \cdot x}\cos(3x + n\theta_3)$

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$\therefore \quad y_n = \frac{1}{8} r_1^n 2^x \cos(x + n\theta_1) - \frac{1}{16} r_2^n 2^x \cos(5x + n\theta_2) - \frac{1}{16} r_3^n 2^x \cos(3x + n\theta_3)$

where, $r_1 = \sqrt{(\log 2)^2 + (1)^2}$, $\theta_1 = \tan^{-1}\left(\frac{1}{\log 2}\right)$, $r_2 = \sqrt{(\log 2)^2 + 5^2}$,

$\theta_2 = \tan^{-1}\left(\frac{5}{\log 2}\right)$, $r_3 = \sqrt{(\log 2)^2 + 3^2}$, $\theta_3 = \tan^{-1}\left(\frac{3}{\log 2}\right)$

3) $y = 2x\sqrt{1-x^2}$

Differentiate both side wrt x,

$y_1 = 2\left[\frac{x(-\cancel{2}x)}{\cancel{2}\sqrt{1-x^2}} + \sqrt{1-x^2}\right]$

$y_1 = 2\left[\frac{-x^2}{\sqrt{1-x^2}} + \sqrt{1-x^2}\right] = 2\left[\frac{-x^2 + 1 - x^2}{\sqrt{1-x^2}}\right]$

$\therefore y_1\sqrt{1-x^2} = 2[1 - 2x^2]$

$\therefore y_1\sqrt{1-x^2} = 2 - 4x^2$

~~Squaring,~~

$\cancel{y_1^2 (1-x^2) =}$

Differentiate both side wrt x,

$\therefore y_1 \cdot \frac{(-\cancel{2}x)}{\cancel{2}\sqrt{1-x^2}} + \sqrt{1-x^2}\, y_2 = -8x$

$\therefore \frac{-xy_1}{\sqrt{1-x^2}} + y_2\sqrt{1-x^2} = -8x$

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$\therefore \quad -xy_1 + y_2(1-x^2) = -8x\sqrt{1-x^2}$

$\therefore \quad -xy_1 + y_2(1-x^2) + 8x\sqrt{1-x^2} = 0$

$\therefore \quad \underset{u}{y_2}\,\underset{v}{(1-x^2)} - \underset{v}{x}\,\underset{u}{y_1} + 4y = 0$

Applying leibnitz's theorem term by term to find n derivative of y,

$$\therefore \left[y_{n+2}(1-x^2) + n(-2x)y_{n+1} + \frac{n(n-1)}{2!}(-2)y_n\right] - \left[y_{n+1}\,x + n\,y_n\right] + 4y_n = 0$$

$\therefore \quad (1-x^2)\,y_{n+2} - (2n+1)x\,y_{n+1} - (n^2-4)\,y_n = 0$

-Hence Proved.

4) $\quad y = e^{ax}\cos^2 x\sin x$

$y = e^{ax}.\ \cos x\cdot\cos x.\sin x$

$y = e^{ax}\ \cos x\ \dfrac{\sin 2x}{2}$

$y = \dfrac{1}{4}e^{ax}\left[\sin 3x + \sin x\right]$

$y = \dfrac{1}{4}e^{ax}\sin 3x + \dfrac{1}{4}e^{ax}\sin x$

Differentiate n times, we get

$y_n = \dfrac{1}{4}r_1^n e^{ax}\sin(3x+n\theta_1) + \dfrac{1}{4}r_2^n e^{ax}\sin(x+n\theta_2)$

where $r_1 = \sqrt{a^2+9}$, $\theta_1 = \tan^{-1}(3/a)$

$r_2 = \sqrt{a^2+1}$, $\theta_2 = \tan^{-1}(1/a)$

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5) $y = (\sin^{-1}x)^2$

Differentiate wrt x,

$$y_1 = \frac{2\sin^{-1}x}{\sqrt{1-x^2}}$$

$\therefore \sqrt{1-x^2}\, y_1 = 2\sin^{-1}x$

Squaring,

$$(1-x^2)\, y_1^2 = 4(\sin^{-1}x)^2$$

$$(1-x^2)\, y_1^2 = 4y$$

Differentiate wrt x,

$$(1-x^2)2y_1y_2 + y_1^2(-2x) = 4y_1$$

$\therefore \cancel{(1-x^2)y_2 + y_1(-x) - 4y_1 = 0}$ $\quad (1-x^2)y_2 + y_1(-x) = 2$

$\therefore \cancel{(1-x^2)y_2 - xy_1 - 4y_n = 0}$ $\quad (1-x^2)y_2 - xy_1 - 2 = 0$

Applying leibnitz's theorem ~~on~~ term by term to get n derivate of ~~y~~,

$$\left[y_{n+2}(1-x^2) + n\,y_{n+1}(-2x) + \frac{n(n-1)}{2!}y_n(-2)\right] - \left[y_{n+1}(x) + n y_n\right] \cancel{- 4y_{n+1} = 0}$$

$$\therefore (1-x^2)y_{n+2} - (2n+1)x\,y_{n+1} - n^2 y_n = 0$$

-Hence proved.

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