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DATE:

12/07/2021

Engineering Mathematics - II

Tutorial 4 : Application of multiple integrals

1) Find by double integration the area common to the circle $x^2+y^2=10$ and the parabola $y^2=9x$

2) Find by double integration the area inside the circle $r=a\sin\theta$ and outside the cardioid $r=a(1-\cos\theta)$

3) Find the volume bounded by $y^2=4ax$ and $x^2=4ay$ and the planes $z=0$ and $z=3$.

4) Find by double integration the mass of a thin plate bounded by $y^2=x$ and $y=x^3$ if the density at any point varies as the square of its distance from the origin.

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Solutions

1.) $x^2+y^2=10 \qquad y^2=9x$

$x^2+9x-10=0$

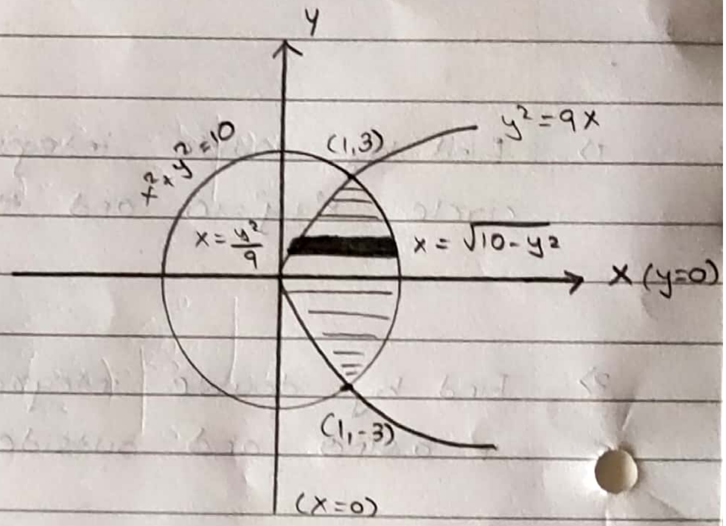
$x^2+10x-x-10=0$

$x(x+10)-1(x+10)=0$

$x=1$

$y^2=9$

$\therefore y=\pm 3$



$\therefore$ (1,3) and (1,-3) are points of intersection.

$\therefore$ ~~A~~ By symmetry,

Required Area = 2 [Area in 1st quadrant]

$$\therefore A = 2\int_{y=0}^{3}\left(\int_{x=\frac{y^2}{9}}^{\sqrt{10-y^2}} dx\right)dy$$

$$= 2\int_{y=0}^{3}\left[x\right]_{y^2/9}^{\sqrt{10-y^2}} dy$$

$$= 2\int_{y=0}^{3}\sqrt{10-y^2}-\frac{y^2}{9}\,dy$$

$$= 2\left[\frac{y}{2}\sqrt{10-y^2}+\frac{10}{2}\sin^{-1}\frac{y}{\sqrt{10}}-\frac{y^3}{27}\right]_0^3$$

$$= 2\left[\frac{3}{2}+5\sin^{-1}\frac{3}{\sqrt{10}}-1\right] = 2\left[\frac{1}{2}+5\sin^{-1}\frac{3}{\sqrt{10}}\right]$$

$$\therefore A = 1+10\sin^{-1}\frac{3}{\sqrt{10}}$$

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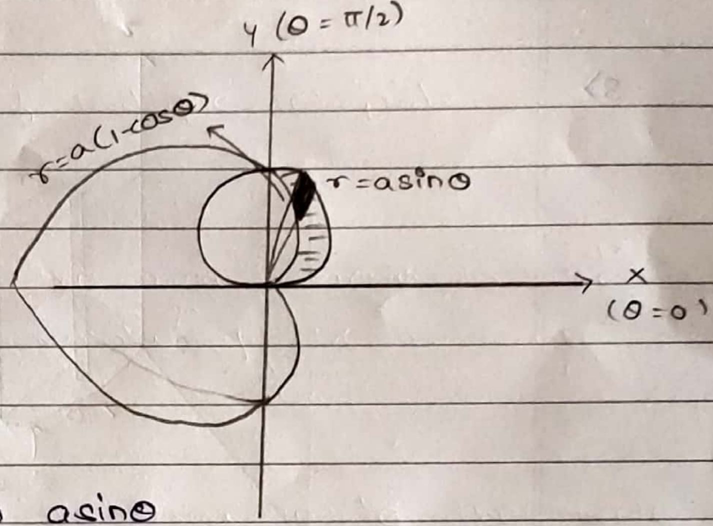
2) $a\sin\theta = a(1-\cos\theta)$

$\sin\frac{\theta}{2}\left[\sin\frac{\theta}{2} - \cos\frac{\theta}{2}\right] = 0$

when $\sin\frac{\theta}{2} = 0$, $\theta = 0$

$\sin\frac{\theta}{2} = \cos\frac{\theta}{2}$, $\theta = \frac{\pi}{2}$



r varies from $a(1-\cos\theta)$ to $a\sin\theta$

$$\therefore A = \int_{\theta=0}^{\frac{\pi}{2}} \int_{r=a(1-\cos\theta)}^{a\sin\theta} r\,dr\,d\theta$$

$$= \int_{\theta=0}^{\frac{\pi}{2}} \left[\frac{r^2}{2}\right]_{a(1-\cos\theta)}^{a\sin\theta} d\theta$$

$$= \int_{\theta=0}^{\frac{\pi}{2}} \frac{a^2\sin^2\theta}{2} - \frac{a^2}{2} - \frac{a^2\cos^2\theta}{2} + \frac{2\cos\theta}{2}\,d\theta$$

$$= \frac{a^2}{2}\int_{\theta=0}^{\pi/2} \sin^2\theta - 1 + 2\cos\theta - \cos^2\theta\,d\theta$$

$$= \frac{a^2}{2}\left[-\theta + 2\sin\theta - \frac{\sin 2\theta}{2}\right]_0^{\pi/2}$$

$$= \frac{a^2}{2}\left[-\frac{\pi}{2} + 2\right] = \frac{a^2}{2}\left(\frac{-\pi+4}{2}\right)$$

$$\therefore A = \frac{a^2}{4}(4-\pi)$$

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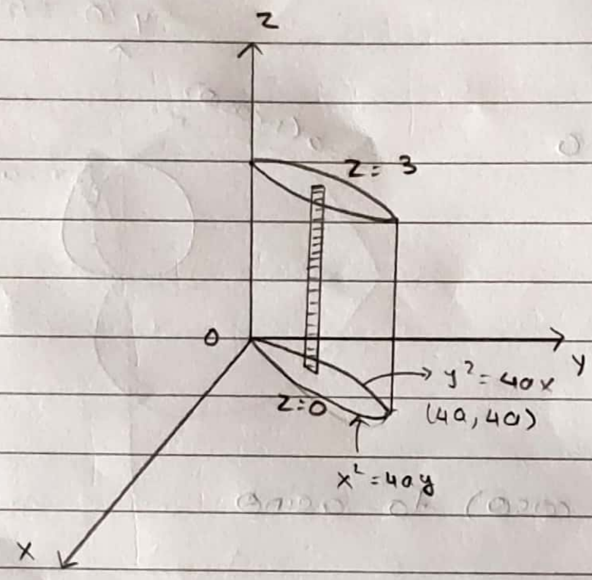

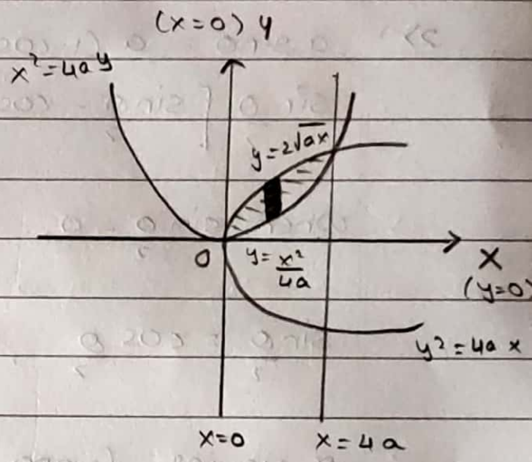

$y^2 = 4ax$ and $x^2 = 4ay$

$\therefore y^2 = 4a \cdot 2\sqrt{ay}$

$\therefore y^4 = 64a^3 y \Rightarrow y^3 = 64a^3$

$\therefore y = 4a, \quad x = 4a$

$\therefore$ Volume $= \iiint_V dx\,dy\,dz$

Limits : z varies from 0 to 3

y varies from $\frac{x^2}{4a}$ to $2\sqrt{ax}$

Z varies from 0 to $4a$

$$\therefore \text{Volume} = \int_{x=0}^{4a} \int_{y=\frac{x^2}{4a}}^{2\sqrt{ax}} \left(\int_{z=0}^{3} dz \right) dy\,dx$$

$$= \int_{x=0}^{4a} \left(\int_{y=\frac{x^2}{4a}}^{2\sqrt{ax}} 3\,dy \right) dx$$

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$$= \int_{x=0}^{4a} 3\,(y)_{x^2/4a}^{2\sqrt{ax}}\,dx$$

$$= \int_{x=0}^{4a} 3\left(2\sqrt{ax} - \frac{x^2}{4a}\right)dx$$

$$= 3\left[\frac{4\sqrt{a}\,(x^{3/2})}{3} - \frac{1}{4a}\left(\frac{x^3}{3}\right)\right]_0^{4a}$$

$$= 3\left[\frac{4}{3}\times 8a^2 - \frac{64\,a^3}{4a\times 3}\right]$$

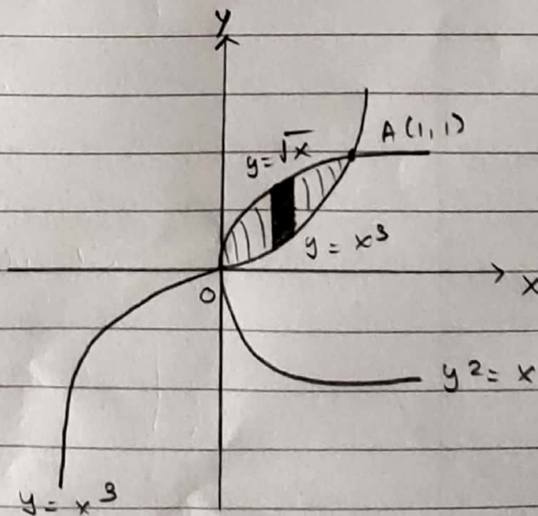
$$= 32a^2 - 16a^2$$

$$= 16a^2$$

$\therefore$ Volume of required region is $16a^2$.

4) The surface density is given by

$\rho = kr^2 = k(x^2+y^2)$



Point of intersection: $A(1,1)$

Limits: $x^3 \le y \le \sqrt{x}$; $0 \le x \le 1$

$$\therefore \text{Mass} = \iint \rho\,dx\,dy$$

$$= \int_{x=0}^{1}\int_{y=x^3}^{\sqrt{x}} k(x^2+y^2)\,dx\,dy$$

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$$= k \int_{x=0}^{1} \left[x^2 y + \frac{y^3}{3} \right]_{x^3}^{\sqrt{x}} dx$$

$$= k \int_{x=0}^{1} \left[x^2 \sqrt{x} + \frac{x^{3/2}}{3} - x^2 \cdot x^3 - \frac{x^9}{3} \right] dx$$

$$= k \left[\frac{2x^{7/2}}{7} + \frac{2x^{5/2}}{3x^5} - \frac{x^6}{6} - \frac{x^{10}}{30} \right]_0^1$$

$$= k \left[\frac{2}{7} + \frac{2}{15} - \frac{1}{6} - \frac{1}{30} \right]$$

$$= \frac{23k}{105}$$

$\therefore$ mass of thin plate is $\frac{23k}{105}$