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Maths - III

Term - Test - 2

Solutions :

3) $f(k) = a^k \quad , \quad a > 0 \quad , k \geq 1$

We have,

$$z\{f(k)\} = \sum_{k=-\infty}^{\infty} f(k) z^{-k}$$

$$= \sum_{k=1}^{\infty} a^k z^{-k} \quad \text{(Assuming for } k<0 \text{ , } f(k)=0\text{)}$$

$$= \frac{a}{z} + \frac{a^2}{z^2} + \frac{a^3}{z^3} + \dots\dots$$

$$= \frac{a}{z}\left[1 + \left(\frac{a}{z}\right) + \left(\frac{a}{z}\right)^2 + \left(\frac{a}{z}\right)^3 + \dots\right]$$

$$= \frac{a}{z}\left[\frac{1}{1-\frac{a}{z}}\right] \quad , \quad \left|\frac{a}{z}\right| < 1$$

$$\therefore z\{f(k)\} = \frac{a}{z-a} \quad , \quad |a| < |z|$$

The z - transform of $f(k)$ is $\frac{a}{z-a}$ and the region of convergence is $|a| < |z|$

1) $f(x) = \begin{cases} 1-x^2, & |x| \leq 1 \\ 0, & |x| > 1 \end{cases}$

The function $f(x)$ is even function.

$\therefore$ The fourier cosine transform of $f(x)$ is given by:

$$\therefore \mathcal{F}_c[f(x)] = F_c(\alpha) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \cos \alpha x \, dx$$

$$= \sqrt{\frac{2}{\pi}} \left[\int_0^1 (1-x^2) \cos \alpha x \, dx + \int_1^\infty 0 \cdot \cos \alpha x \, dx \right]$$

$$= \sqrt{\frac{2}{\pi}} \left\{ \left[\frac{(1-x^2) \sin \alpha x}{\alpha} \right]_0^1 + \left[\frac{(-2x) \cos \alpha x}{\alpha^2} \right]_0^1 - \left[\frac{(-2) \sin \alpha x}{\alpha^3} \right]_0^1 \right\}$$

$$= \sqrt{\frac{2}{\pi}} \left[0 - \frac{2 \cos \alpha}{\alpha^2} + \frac{2 \sin \alpha}{\alpha^3} + 0 \right]$$

$$\therefore F_c(\alpha) = \mathcal{F}_c[f(x)] = \sqrt{\frac{2}{\pi}} \left[2 \left(\frac{\sin \alpha}{\alpha^3} - \frac{\cos \alpha}{\alpha^2} \right) \right]$$

2) Solution :

$f(x) = x e^{-ax}$, $a > 0$

F_s be the fourier sine transform of $f(x)$.

$\therefore \cancel{F_s [x e^{-ax}] = -\frac{d}{d\alpha} F_c(\alpha)}$

$\therefore F_s [x e^{-ax}] = -\frac{d}{d\alpha} \left[\sqrt{\frac{2}{\pi}} \frac{a}{a^2+\alpha^2} \right]$

$\therefore F_s [x e^{-ax}] = \sqrt{\frac{2}{\pi}} \left(\frac{2a\alpha}{(a^2+\alpha^2)^2} \right)$

$$\therefore f(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty F_s(\alpha) \sin \alpha x \, d\alpha$$

$$= \left(\frac{2}{\pi}\right) \int_0^\infty \frac{2a\alpha \sin \alpha x}{(a^2+\alpha^2)^2} d\alpha$$

$$f(x) = \frac{4a}{\pi} \int_0^\infty \frac{\alpha \sin \alpha x}{(a^2+\alpha^2)^2} d\alpha$$

Hence,

$$\therefore \int_0^\infty \frac{\alpha \sin \alpha \, d\alpha}{a^2+\alpha^2} = \frac{\pi}{4a} f(x)$$

$$\therefore \int_0^\infty \frac{\alpha \sin \alpha}{(a^2+\alpha^2)^2} d\alpha = \frac{\pi}{4a} (x e^{-ax})$$

$$\therefore \int_0^\infty \frac{\alpha \sin \alpha}{(a^2+\alpha^2)^2} d\alpha = \frac{\pi}{4a} (e^{-a}) \quad , \text{ at } x = 1$$