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Maths - III

Tutorial - 9 : Z - Transform

1) Find $Z\{c^k \sin \alpha k\}$ from $Z\{\sin \alpha k\}$

2) Find z - transform of $\{k e^{-ak}\}$, $k \geq 0$

3) Find $Z\{{}^nC_k\}$, $0 \leq k \leq n$

4) Find the inverse z - transform of

$$F(z) = \frac{2z^2 - 10z + 13}{(z-3)^2 (z-2)}, \quad 2 < |z| < 3$$

Solution :

→ 1) Assuming for $k < 0$, $f(k) = 0$

By defination

$$Z\{f(k)\} = \sum_{k=-\infty}^{\infty} f(k) z^{-k}$$

$$= \sum_{k=-\infty}^{-1} 0 \cdot z^{-k} + \sum_{k=0}^{\infty} f(k) z^{-k}$$

$$= \sum_{k=0}^{\infty} f(k) z^{-k}$$

$$= \sum_{k=0}^{\infty} \sin \alpha k \, z^{-k}$$

$$= \frac{1}{2i} \sum_{k=0}^{\infty} \left(\frac{e^{i\alpha k}}{z^k} - \frac{e^{-i\alpha k}}{z^k} \right)$$

$$= \frac{1}{2i}\sum_{k=0}^{\infty}\left(\frac{e^{i\alpha}}{z}\right)^{k} - \frac{1}{2i}\sum_{k=0}^{\infty}\left(\frac{e^{-i\alpha}}{z}\right)^{k}$$

$$= \frac{1}{2i}\left(\frac{1}{1-\frac{e^{i\alpha}}{z}} - \frac{1}{1-\frac{e^{-i\alpha}}{z}}\right), \text{ For } |z|>1$$

$$= \frac{1}{2i}\left(\frac{z}{z-e^{i\alpha}} - \frac{z}{z-e^{-i\alpha}}\right)$$

$$= \frac{1}{2i}\left(\frac{z(z-e^{-i\alpha}) - z(z-e^{i\alpha})}{(z-e^{i\alpha})(z-e^{-i\alpha})}\right)$$

$$= \frac{1}{2i}\left(\frac{z(e^{i\alpha}-e^{-i\alpha})}{z^2 - z(e^{i\alpha}+e^{-i\alpha})+1}\right)$$

$$= \frac{1}{2i}\left(\frac{z \cdot 2i \cdot \sin\alpha}{z^2 - 2z\cos\alpha + 1}\right)$$

$$\therefore z\{f(k)\} = z\{\sin\alpha k\} = \frac{z\sin\alpha}{z^2-2z\cos\alpha+1}, \quad |z|>1$$

By change of scalar property,

$$z\{(c^k \sin\alpha k)\} = \frac{(z/c)\sin\alpha}{(z/c)^2 - 2(z/c)\cos\alpha + 1}$$

$$\therefore z\{(c^k \sin\alpha k)\} = \frac{cz\sin\alpha}{z^2-2cz\cos\alpha+c^2}, \quad |z|>c$$

ROC

→ 2) let $f(k) = e^{-ak}$

$$Z\{f(k)\} = \sum_{k=-\infty}^{\infty} f(k) z^{-k} = \sum_{k=0}^{\infty} e^{-ak} z^{-k}$$

$$= 1 + \frac{e^{-a}}{z} + \frac{e^{-2a}}{z^2} + \frac{e^{-3a}}{z^3} + \cdots$$

$$= \frac{1}{1 - \frac{e^{-a}}{z}}, \quad |z| > e^{-a}$$

$$\therefore Z\{e^{-ak}\} = \frac{z}{z - e^{-a}}, \quad |z| > e^{-a}$$

$$\therefore Z\{ke^{-ak}\} = -z \frac{d}{dz}\left(\frac{z}{z-e^{-a}}\right) \quad \ldots \left[\because Z\{kf(k)\} = -z\frac{d}{dz} Z f(k)\right]$$

$$= \frac{-z(-e^{-a})}{(z-e^{-a})^2}$$

$$= \frac{ze^{-a}}{(z-e^{-a})^2}, \quad |z| > e^{-a}$$

$$\therefore Z\{ke^{-ak}\} = \frac{ze^{a}}{(ze^{a}-1)^2}, \quad e^{a}|z| > 1$$

→ 3) Find $Z\{^nC_k\}$, $0 \le k \le n$

let $f(k) = {^nC_k}$

By defination

$$Z\{f(k)\} = \sum_{k=-\infty}^{\infty} {^nC_k}\, z^{-k}$$

$$= \sum_{k=0}^{n} {^nC_k}\, z^{-k} \quad \ldots \{\text{Here } k=0 \text{ to } n, \text{ since } {^nC_k} \text{ is defined between } 0 \text{ to } n\}$$

$= {}^nC_0 + {}^nC_1 \frac{1}{z} + {}^nC_2 \frac{1}{z^2} + \ldots\ldots {}^nC_n \frac{1}{z^n}$

$= \left(1 + \frac{1}{z}\right)^n$

The series being finite is obviously convergent if $z \neq 0$

$\therefore$ Region of Convergence (ROC) is whole z-plane except the origin.

→ 4) $F(z) = \frac{2z^2 - 10z + 13}{(z-3)^2(z-2)}$

Consider,

$\frac{2z^2-10z+13}{(z-3)^2(z-2)} = \frac{A}{(z-2)} + \frac{Bz+C}{(z-3)^2}$

$\therefore 2z^2 - 10z + 13 = A(z-3)^2 + (Bz+C)(z-2)$

Put $z = 2$

$\therefore 8 - 20 + 13 = A$

$\therefore A = 1$

Put $z = 0$

$13 = 9A - 2C$

$\therefore C = -2$

Put $z = 3$

$3B + C = 1$

$\therefore B = 1$

$\therefore \frac{2z^2-10z+13}{(z-3)^2(z-2)} = \frac{1}{z-2} + \frac{z-2-1+1}{(z-3)^2}$

$$\therefore F(z) = \frac{1}{z-2} + \frac{1}{z-3} + \frac{1}{(z-3)^2}$$

NOW, $\left|\frac{z}{2}\right| < 1$, $\left|\frac{z}{3}\right| < 1$

$$\therefore F(z) = \frac{1}{2}\frac{1}{(1-z/2)} + \frac{1}{3}\frac{1}{(z/3-1)} + \frac{1}{9}\frac{1}{(z/3-1)^2}$$

$$= \frac{1}{2}\left(1-\frac{z}{2}\right)^{-1} - \frac{1}{3}\left(1-\frac{z}{3}\right)^{-1} + \frac{1}{9}\left(1-\frac{z}{3}\right)^{-2}$$

$$= \frac{1}{2}\left(1 + \frac{z}{2} + \frac{z^2}{2^2} + \frac{z^3}{2^3} + \cdots \frac{z^{k-1}}{2^{k-1}} + \cdots\right)$$

$$-\frac{1}{3}\left(1 + \frac{z}{3} + \frac{z^2}{3^2} + \frac{z^3}{3^3} + \cdots \frac{z^k}{3^k} + \cdots\right)$$

$$+\frac{1}{9}\left(1 + 2\cdot\frac{z}{3} + 3\cdot\frac{z^2}{3^2} + \cdots (k+1)\frac{z^k}{3^k} + \cdots\right)$$

From 1^{st} series,

coefficient of $z^{-k} = 2^{k-1}$, $k \geq 1$

From 2^{nd} series,

Coefficient of $z^k = -\frac{1}{3^{k+1}}$

From 3^{rd} series,

$$\text{Coefficient of } z^k = \frac{k+1}{3^{k+2}} - \frac{1}{3^{k+1}}$$

$$= \frac{k-2}{3^{k+2}}, \quad k > 0$$

$$\therefore \text{Coefficient of } z^{-k} = \frac{(-k)-2}{3^{-k+2}}, \quad k \leq 0$$

Hence,

$$z^{-k}\left[F(z)\right] = 2^{k-1} \quad , k \geq 1$$
$$= \frac{-k-2}{3^{-k+2}} \quad , k \leq 0$$