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Maths - III

Tutorial 8

1) Find the Inverse Fourier Transform of $F(\alpha) = e^{-|\alpha|} \frac{\sin\alpha}{\alpha}$

2) Find the value of $\int_0^\infty \frac{e^{-x^2/2}}{x^2+1} dx$

3) Find finite fourier sine Transform of $\frac{\sinh a(\pi - x)}{\sinh a\pi}$, $0 \le x \le \pi$

Solutions:

1) First find $\mathcal{F}^{-1}[e^{-|\alpha|}]$ and $\mathcal{F}^{-1}\left[\frac{\sin\alpha}{\alpha}\right]$ seperately.

$$\therefore \mathcal{F}^{-1}[e^{-|\alpha|}] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-|\alpha|} e^{-i\alpha x} d\alpha$$

$$= \frac{1}{\sqrt{2\pi}} \left\{ \int_{-\infty}^{0} e^{\alpha} e^{-i\alpha x} d\alpha + \int_0^\infty e^{-\alpha} e^{-i\alpha x} d\alpha \right\}$$

$$= \frac{1}{\sqrt{2\pi}} \left\{ \int_{-\infty}^{0} e^{\alpha(1-ix)} d\alpha + \int_0^\infty e^{-\alpha(1+ix)} d\alpha \right\}$$

$$= \frac{1}{\sqrt{2\pi}} \left\{ \left[\frac{e^{\alpha(1-ix)}}{1-ix} \right]_{\infty}^{0} + \left[\frac{e^{-\alpha(1+ix)}}{-(1+ix)} \right]_0^\infty \right\}$$

$$= \frac{1}{\sqrt{2\pi}} \left[\frac{1}{1-ix} + \frac{1}{1+ix} \right]$$

$$= \frac{1}{\sqrt{2\pi}} \left[\frac{2}{1+x^2} \right]$$

$$\mathcal{F}^{-1}\left[e^{-|\alpha|}\right] = \sqrt{\frac{2}{\pi}}\left(\frac{1}{1+x^2}\right)$$

$$\mathcal{F}^{-1}\left[\frac{\sin\alpha}{\alpha}\right] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\frac{\sin\alpha}{\alpha}\, e^{-i\alpha x}\, d\alpha$$

$$= \frac{1}{\sqrt{2\pi}}\left\{\int_{-\infty}^{\infty}\underbrace{\frac{\sin\alpha}{\alpha}\cos\alpha x\, d\alpha}_{\text{even function}} - i\int_{-\infty}^{\infty}\underbrace{\frac{\sin\alpha}{\alpha}\sin\alpha x\, d\alpha}_{\text{odd function}}\right\}$$

$$= \frac{2}{\sqrt{2\pi}}\int_{0}^{\infty}\frac{\sin\alpha\cos\alpha x}{\alpha}\, d\alpha$$

$$= \sqrt{\frac{2}{\pi}}\, f(x) \quad \text{where } f(x) = \begin{cases}\pi/2, & 0<x<1\\ 0, & \text{otherwise}\end{cases}$$

$\therefore$ By convolution theorem,

$$\mathcal{F}^{-1}\left[H(\alpha)\cdot G(\alpha)\right] = \mathcal{F}^{-1}\left[H(\alpha)\right] * \mathcal{F}^{-1}\left[G(\alpha)\right]$$

$$\therefore \mathcal{F}^{-1}\left[e^{-|\alpha|}\frac{\sin\alpha}{\alpha}\right] = \mathcal{F}^{-1}\left[e^{-|\alpha|}\right] * \mathcal{F}^{-1}\left[\frac{\sin\alpha}{\alpha}\right]$$

$$= \underbrace{\left[\sqrt{\frac{2}{\pi}}\left(\frac{1}{1+x^2}\right)\right]}_{\hookrightarrow h(x)} * \underbrace{\left(\sqrt{\frac{2}{\pi}}\, f(x)\right)}_{\hookrightarrow k(x)}$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} k(t)\cdot h(x-t)\, dt$$

$$= \frac{1}{\sqrt{2\pi}}\cdot\frac{2}{\pi}\int_{-\infty}^{\infty} f(t)\cdot\frac{1}{(1+(x-t)^2)}\, dt$$

$$= \frac{\sqrt{2}}{\pi\sqrt{\pi}}\cdot\frac{\pi}{2}\int_{0}^{1}\frac{1}{1+(x-t)^2}\, dt$$

$$= \frac{1}{\sqrt{2\pi}} \int_0^1 \frac{1}{1+(x-t)^2}\, dt$$

Put $x - t = u$, $-dt = du$

$$\therefore \mathcal{F}^{-1}\left[e^{-|\alpha|} \frac{\sin\alpha}{\alpha}\right] = \frac{-1}{\sqrt{2\pi}} \int_x^{x-1} \frac{1}{1+u^2}\, du$$

$$= \frac{-1}{\sqrt{2\pi}} \left[\tan^{-1} u\right]_x^{x-1}$$

$$\therefore \mathcal{F}^{-1}\left[e^{-|\alpha|} \frac{\sin\alpha}{\alpha}\right] = \frac{-1}{\sqrt{2\pi}} \left[\tan^{-1} x - \tan^{-1}(x-1)\right]$$

2) Let $f(x) = e^{-x^2/2}$

$g(x) = e^{-x}$

$$\therefore \mathcal{F}_c\left[e^{-x^2/2}\right] = e^{-\alpha^2/2} = F_c(\alpha)$$

$$\mathcal{F}_c\left[e^{-x}\right] = \sqrt{\frac{2}{\pi}} \frac{1}{1+\alpha^2} = G_c(\alpha)$$

$\therefore$ By Parseval's identity,

$$\int_0^\infty f(x)\, g(x)\, dx = \int_0^\infty F_c(\alpha)\, G_c(\alpha)\, d\alpha$$

$$\therefore \int_0^\infty e^{-x^2/2} e^{-x}\, dx = \int_0^\infty e^{-\alpha^2/2} \sqrt{\frac{2}{\pi}} \cdot \frac{1}{\alpha^2+1}\, d\alpha$$

$$\therefore \int_0^\infty \frac{e^{-\alpha^2/2}}{\alpha^2+1} = \sqrt{\frac{\pi}{2}} \int_0^\infty e^{-x^2/2} \cdot e^{-x}\, dx$$

$$= \sqrt{\frac{\pi}{2}} \int_0^\infty e^{-\left(\frac{1+x}{\sqrt{2}}\right)^2 + 1/2}\, dx$$

$$= \sqrt{\frac{\pi}{2}}\, \sqrt{e} \int_0^{\infty} e^{-\left(\frac{1+x}{\sqrt{2}}\right)^2} dx$$

Put $\left(\frac{1+x}{\sqrt{2}}\right)^2 = u$, $(1+x) = \sqrt{2u}$

$$dx = \frac{du}{\sqrt{2}\sqrt{u}}$$

$$\therefore \int_0^{\infty} \frac{e^{-\alpha^2/2}}{\alpha^2+1} = \sqrt{\frac{\pi e}{2}} \left\{ \int_0^{\infty} \frac{1}{\sqrt{2}} \cdot \frac{1}{\sqrt{u}} e^{-u}\, du \right) - \int_0^{1/2} \frac{1}{\sqrt{2}} \cdot \frac{1}{\sqrt{u}} e^{-u}\, du \right\}$$

$$= \sqrt{\frac{\pi e}{2}} \left\{ \frac{1}{\sqrt{2}} \Gamma\frac{1}{2} - \int_0^{1/2} \frac{e^{-u}}{\sqrt{2u}}\, du \right\}$$

$$\therefore \int_0^{\infty} \frac{e^{-\alpha^2/2}}{\alpha^2+1} = \frac{\pi}{2}\sqrt{e} - \frac{\sqrt{\pi e}}{2} \int_0^{1/2} \frac{e^{-u}}{\sqrt{u}}\, du$$

3) The finite fourier sine Transform of $f(x)$ is given by,

$$\mathcal{F}_s[f(x)] = F_s(n) = \int_0^{\pi} f(x) \sin nx\, dx$$

$$= \int_0^{\pi} \frac{\sinh a(\pi - x)}{\sinh a\pi} \sin nx\, dx$$

Here, $\frac{\sinh a(\pi-x)}{\sinh a\pi} = \frac{\sinh a\pi \cosh ax - \cosh a\pi \sinh ax}{\sinh a\pi}$

$$= \cosh ax - \coth a\pi \cdot \sinh ax$$

$$\therefore \frac{\sinh a(\pi - x)}{\sinh a\pi} = \frac{e^{ax} + e^{-ax}}{2} - \coth a\pi \left[\frac{e^{ax} - e^{-ax}}{2} \right]$$

$$\therefore\ F_s(n) = \int_0^{\pi} \left(\frac{e^{ax}+e^{-ax}}{2}\right)\sin nx\,dx - \coth a\pi \int_0^{\pi}\left(\frac{e^{ax}-e^{-ax}}{2}\right)\sin nx\,dx$$

$$= \frac{1}{2}\left\{\left[\frac{e^{ax}}{a^2+n^2}\left[a\sin nx - n\cos nx\right]\right]_0^{\pi} + \left[\frac{e^{-ax}}{a^2+n^2}(-a\sin nx - n\cos nx)\right]_0^{\pi}\right\}$$

$$-\frac{\coth a\pi}{2}\left[\frac{e^{ax}}{a^2+n^2}(a\sin nx - n\cos nx) - \frac{e^{-ax}}{a^2+n^2}(-a\sin nx - n\cos nx)\right]_0^{\pi}$$

$$= \frac{1}{2}\left[\frac{e^{a\pi}}{a^2+n^2}(-n\cos n\pi) + \frac{n}{a^2+n^2} + \frac{e^{-a\pi}}{a^2+n^2}\left[-n\cos n\pi\right] + \frac{n}{a^2+n^2}\right]$$

$$-\frac{\coth a\pi}{2}\left[\frac{e^{a\pi}}{a^2+n^2}(-n\cos n\pi) + \frac{n}{a^2+n^2} - \frac{e^{-a\pi}}{a^2+n^2}(-n\cos n\pi) - \frac{n}{a^2+n^2}\right]$$

$$= \frac{1}{2}\left[(e^{a\pi}+e^{-a\pi})\left(\frac{-n\cos n\pi}{a^2+n^2}\right) + \frac{2n}{a^2+n^2}\right] - \frac{\coth a\pi}{2}\left[(e^{a\pi}-e^{-a\pi})\left(\frac{-n\cos n\pi}{a^2+n^2}\right)\right]$$

$$= \frac{1}{2}\left[(e^{a\pi}+e^{-a\pi})\left(\frac{-n\cos n\pi}{a^2+n^2}\right) + \frac{2n}{a^2+n^2}\right] - \frac{1}{2}\left[(e^{a\pi}+e^{-a\pi})\left(\frac{-n\cos n\pi}{a^2+n^2}\right)\right]$$

where ~~coth aπ~~ $\coth a\pi = \frac{e^{a\pi}+e^{-a\pi}}{e^{a\pi}-e^{-a\pi}}$

$$\therefore\ F_s(n) = \frac{n}{a^2+n^2}$$