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Maths - III

Tutorial - 6

1) Find the complex form of fourier series

$$f(x) = \begin{cases} x^2, & 0 \le x < 1 \\ 1, & 1 < x < 2 \end{cases}$$

2) Show that set of function $Q_n(x) = \sin\left(\frac{n\pi x}{L}\right)$, $n = 1, 2, 3 \dots$ is orthogonal set on interval $0 \le x < L$ and find corresponding orthogonal set.

3) Find the fourier integral representation of the function

$$f(x) = \begin{cases} 0, & x < 0 \\ 1/2, & x = 0 \\ e^{-x}, & x > 0 \end{cases}$$

Solutions :

1) solution:

The complex form of fourier series is given by

$$f(x) = \sum_{n=-\infty}^{\infty} C_n e^{\frac{in\pi x}{L}} \quad \text{where} \quad C_n = \frac{1}{2L}\int_0^{2L} f(x)\, e^{-\frac{in\pi x}{L}}\, dx$$

Here $l = 1$,

$$C_n = \frac{1}{2}\left[\int_0^1 x^2 e^{-in\pi x}\, dx + \int_1^2 e^{-in\pi x}\, dx\right]$$

$$= \frac{1}{2}\left[\frac{e^{-in\pi x}}{-in\pi}\left(x^2 - \frac{2x}{(in\pi)} + \frac{2}{(-in\pi)^2}\right)\right]_0^1 + \frac{1}{2}\left[\frac{e^{-in\pi x}}{-in\pi}\right]_1^2$$

$$= \frac{1}{2}\left[\frac{e^{-in\pi}}{-in\pi}\left(1 + \frac{2}{in\pi} - \frac{2}{i^2 n^2\pi^2}\right) - \frac{2}{in^3\pi^3} + \frac{e^{-in\pi}}{in\pi} - \frac{e^{-2n\pi i}}{in\pi}\right]$$

$$= \frac{1}{2}\left[\frac{-e^{-in\pi}}{in\pi} + \frac{2e^{-in\pi}}{n^2\pi^2} + \frac{2e^{-in\pi}}{in^3\pi^3} - \frac{2}{in^3\pi^3} + \frac{e^{-in\pi}}{in\pi} - \frac{e^{-2n\pi i}}{in\pi}\right]$$

$$= \frac{1}{2}\left[\frac{2e^{-n\pi i}}{n^2\pi^2} + \frac{2e^{n\pi i}}{in^3\pi^3} - \frac{2}{in^3\pi^3} - \frac{e^{-2n\pi i}}{in\pi}\right]$$

$$e^{-in\pi} = \cos n\pi - i\sin n\pi = (-1)^n$$

$$e^{-2n\pi i} = \cos 2n\pi i - i\sin 2n\pi i = 1$$

$$C_n = \frac{1}{2}\left[\frac{2(-1)^n}{n^2\pi^2} - \frac{2i}{n^3\pi^3}\left((-1)^n - 1\right) + \frac{i}{n\pi}\right]$$

when $n=0$,

$$c_0 = \frac{1}{2l}\int_0^{2l} f(x)\, e^{\frac{-i0\pi x}{2}}\, dx = \frac{1}{2}\int_0^2 f(x)\, dx$$

$$c_0 = \frac{1}{2}\left[\int_0^1 x^2\, dx + \int_1^2 dx\right] = \frac{1}{2}\left[\frac{1}{3}+1\right] = \frac{2}{3}$$

$\therefore$ Complex form of fourier series is

$$f(x) = \frac{2}{3} + \frac{1}{2}\sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty}\left[\frac{2(-1)^n}{n^2\pi^2} - \frac{2i}{n^3\pi^3}((-1)^n-1) + \frac{i}{n\pi}\right]e^{in\pi x}$$

2) Solution:

$$\phi(x) = \sin\left(\frac{\pi n x}{L}\right), \quad n = 1, 2, 3 \ldots$$

$$\therefore \int_0^L \phi_m(x)\, \phi_n(x)\, dx = \int_0^L \sin\left(\frac{m\pi x}{L}\right)\sin\left(\frac{n\pi x}{L}\right) dx$$

$$= \frac{1}{2}\int_0^L \left[\cos\left(\frac{m\pi x}{L} - \frac{n\pi x}{L}\right) - \cos\left(\frac{m\pi x}{L} + \frac{n\pi x}{L}\right)\right] dx$$

$$= \frac{1}{2}\left[\frac{\sin\left(\frac{(m-n)\pi x}{L}\right)}{\frac{(m-n)\pi}{L}} - \frac{\sin\left(\frac{(m+n)\pi x}{L}\right)}{\frac{(m+n)\pi}{L}}\right]_0^L$$

$$= \frac{1}{2}\left[\frac{L\sin[(m-n)]\pi}{(m-n)\pi} - \frac{L\sin(m+n)\pi}{(m+n)\pi}\right]$$

Case 1 : $m \neq n$, then

$\because m, n$ are integers $\sin(m-n)\pi = 0$ and $\sin(m+n)\pi = 0$

$$\therefore \int_0^L \phi_m(x)\, \phi_n(x)\, dx = 0 \quad \text{—— (i)}$$

Case 2 : $m = n$, then

$$\int_0^L \phi_m(x)\, \phi_n(x)\, dx = \int_0^L (\phi_n x)^2\, dx = \int_0^L \sin^2\left(\frac{n\pi x}{L}\right) dx$$

$$= \frac{1}{2}\int_0^L \left[1 - \cos\left(\frac{2n\pi x}{l}\right)\right] dx$$

$$= \frac{1}{2}\left[[L] - \left[\frac{\sin\left(\frac{2n\pi x}{l}\right)}{\left(\frac{2n\pi x}{l}\right)}\right]_0^l \right] = \frac{L}{2} \neq 0 \quad -(2)$$

$\therefore$ Given set of function is orthogonal on $[0, L]$

$\therefore$ If the set is orthogonal then we should have
$\int_0^L [\phi_n(x)]^2 dx = 1$

For this we divide polynomial $f_n(x)$ by $\sqrt{\frac{L}{2}}$

$\therefore$ orthogonal set

$$f_n(x) = \sqrt{\frac{2}{L}} \sin\left(\frac{n\pi x}{L}\right), \quad n = 1, 2, 3 \ldots$$

Q. 3 Solution :

The fourier representation of $f(x)$ is given by

$$f(x) = \frac{1}{\pi}\int_0^\infty \left(\int_{-\infty}^{\infty} f(t) \cos \alpha(t-x)\, dt \right) d\alpha$$

$$= \frac{1}{\pi}\int_0^\infty \left(\int_0^\infty e^{-t} \cos \alpha (t-x) dt \right) d\alpha + 0$$

$$= \frac{1}{\pi}\int_0^\infty \left\{ \cos \alpha x \int_0^\infty e^{-t} \cos \alpha t\, dt + \sin \alpha x \int_0^\infty e^{-t} \sin \alpha t\, dt \right\} d\alpha$$

$$= \frac{1}{\pi}\int_0^\infty \left\{ \cos \alpha x \left[\frac{e^{-t}}{1+\alpha^2} \left\{ -\cos \alpha t + \alpha \sin \alpha t \right\} \right]_0^\infty + \right.$$

$$\left. \sin \alpha x \left[\frac{e^{-t}}{1+\alpha^2} \left[\sin \alpha t - \alpha \cos \alpha t \right] \right] \right\}_0^\infty d\alpha$$

As $\lim_{x \to \infty} e^{-x}$, $1/x \to 0$ i.e. $e^{-x} \to 0$

$$\therefore f(x) = \frac{1}{\pi}\int_0^\infty \left[\frac{\cos \alpha x + \alpha \sin \alpha x}{1+\alpha^2} \right] d\alpha \quad \ldots (e^{-\infty} = 0)$$

Hence, Fourier representation of $f(x)$ is

$$\therefore f(x) = \frac{1}{\pi}\int_0^\infty \left(\frac{\cos \alpha x + \alpha \sin \alpha x}{1+\alpha^2} \right) dx$$