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Maths - III

Tutorial 4

1.) Solve using Laplace transform $(D^2 - D - 2)y = 20 \sin 2t$, with $y(0) = 1$, $y'(0) = 2$

2) Solve using laplace transform the following pair of simultaneous differential equation.

$$2x' + y' = 5e^t$$
$$y' - 3x' = 5$$

given that when $t = 0$, $x = 0$ and $y = 0$

3) Find the laplace Transform of $f(t) = |\sin pt|$, $t \geq 0$

4) Using Laplace Transform evaluate

$$\int_0^\infty e^{-t}(1 + 2t - t^2 + t^3)\, H(t-1)\, dt$$

5) Find $L\left[\cos t \left(H\left(t - \frac{\pi}{2}\right) - H\left(t - \frac{3\pi}{2}\right)\right)\right]$

6) Find $L\left[t^2 H(t-2) - \cosh t\, \delta(t-4)\right]$

Solutions

1.) $(D^2 - D - 2)y = 20\sin 2t$ $\quad y(0) = 1, y'(0) = 2$

Taking Laplace Transform on both sides,

$\therefore L\left[\frac{d^2y}{dt^2} - \frac{dy}{dt} - 2y\right] = 20\, L[\sin 2t]$

$\therefore \left[s^2 L[y(t)] - s\,y(0) - y'(0)\right] - \left[s L[y(t)] - y(0)\right] - 2L[y(t)] = 20 \times \frac{2}{s^2+4}$

$\therefore$ Now, putting $y(0) = 1$, $y'(0) = 2$

$\therefore s^2 L[y(t)] - s - 2 - sL[y(t)] + 1 - 2L[y(t)] = \frac{40}{s^2+4}$

$\therefore L[y(t)](s^2 - s - 2) - s - 1 = \frac{40}{s^2+4}$

$\therefore L[y(t)](s^2 - s - 2) = \frac{40}{s^2+4} + s + 1$

$\therefore L[y(t)](s+1)(s-2) = \frac{s^3 + s^2 + 4s + 44}{s^2+4}$

$\therefore L[y(t)] = \frac{s^3 + s^2 + 4s + 44}{(s^2+4)(s+1)(s-2)}$

$\therefore \frac{s^3 + s^2 + 4s + 44}{(s+1)(s-2)(s^2+4)} = \frac{A}{s+1} + \frac{B}{s-2} + \frac{Cs+D}{s^2+4}$

$\therefore s^3 + s^2 + 4s + 44 = A(s-2)(s^2+4) + B(s+1)(s^2+4) + (Cs+D)(s+1)(s-2)$

$= s^3(A+B+C) + s^2(-2A+B-C+D) + s(4A+4B-2C-D) - (8A-4B+2D)$

comparing both sides,

$A+B+C=1$, $-2A+B-C+D=1$, $4A+4B-2C-D=4$, $8A-4B+2D=-44$

solving equations we get,

$A=-\frac{8}{3}$, $B=\frac{8}{3}$, $C=1$, $D=-6$

$\therefore L[y(t)] = -\frac{8}{3}\cdot\frac{1}{S+1} + \frac{8}{3}\cdot\frac{1}{S-2} + \frac{S-6}{S^2+4}$

Taking inverse laplace Transform,

$y(t) = -\frac{8}{3}L^{-1}\left[\frac{1}{S+1}\right] + \frac{8}{3}L^{-1}\left[\frac{1}{S-2}\right] + L^{-1}\left[\frac{S}{S^2+4}\right] - \frac{6}{2}L^{-1}\left[\frac{2}{S^2+4}\right]$

$\therefore y(t) = -\frac{8}{3}e^{-t} + \frac{8}{3}e^{2t} + \cos 2t - 3\sin 2t$

$\therefore y = -\frac{8}{3}e^{-t} + \frac{8}{3}e^{2t} + \cos 2t - 3\sin 2t$

2) $2x' + y' = 5e^t$

$y' - 3x' = 5$

When $t=0$, $x=0$, $y=0$

Taking Laplace Transform on both sides,

$\therefore L[2x'+y'] = 5L[e^t]$

$L[y'-3x'] = 5L[1]$

$\therefore 2[sL[x(t)] - x(0)] + sL[y(t)] - y(0) = \frac{5}{s-1}$

$sL[y(t)] - y(0) - 3[sL[x(t)] - x(0)] = \frac{5}{s}$

Now, Put $x(0)=0$, $y(0)=0$

$\therefore 2sL[x(t)] + sL[y(t)] = \frac{5}{s-1}$

$sL[y(t)] - 3sL[x(t)] = \frac{5}{s}$

$\therefore 2sL[x(t)] + sL[y(t)] = \frac{5}{s-1}$ —— (1)

$-3sL[x(t)] + sL[y(t)] = \frac{5}{s}$ —— (2)

Subtracting (1) and (ii)

$\therefore L[x(t)] = \frac{1}{s^2(s-1)}$ —— (3)

$$\therefore \quad 2sL[x(t)] + sL[y(t)] = \frac{5}{s-1} \quad \times(3s)$$

$$-3sL[x(t)] + sL[y(t)] = \frac{5}{s} \quad \times(2s)$$

Subtracting the equations,

$$L[y(t)] = \frac{5s-2}{s^2(s-1)} \quad -(4)$$

$$\therefore \quad L[x(t)] = \frac{1}{s^2(s-1)}$$

$$\therefore \quad x(t) = L^{-1}\left[\frac{1}{s^2(s-1)}\right]$$

$$\therefore \quad x(t) = L^{-1}\left[\frac{1}{s-1}\right] + L^{-1}\left[-\frac{1}{s}\right] + L^{-1}\left[\frac{-1}{s^2}\right]$$

$$= e^t - 1 - t$$

$$\therefore \quad x(t) = e^t - t - 1$$

Now, $L[y(t)] = \frac{5s-2}{s^2(s-1)}$

$$\therefore \quad y(t) = L^{-1}\left[\frac{5s-2}{s^2(s-1)}\right]$$

$$\frac{5s-2}{(s-1)(s^2)} = \frac{A}{s-1} + \frac{B}{s} + \frac{C}{s^2}$$

$$5s-2 = A(s^2) + B(s)(s-1) + C(s-1)$$

$$\therefore \quad A = 3, \quad B = -3, \quad C = 2$$

60004200132

$\therefore y(t) = 3L^{-1}\left[\frac{1}{s-1}\right] - 3L^{-1}\left[\frac{1}{s}\right] + 2L^{-1}\left[\frac{1}{s^2}\right]$

$\therefore y(t) = 3e^{t} - 3 + 2t$

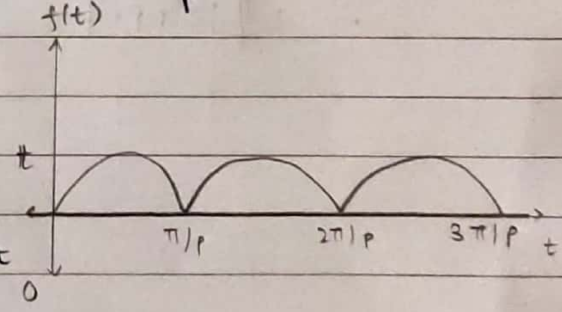
$\therefore y(t) = 3e^{t} + 2t - 3$

$\therefore x = e^{t} - t - 1, \quad y = 3e^{t} + 2t - 3$

3) $f(t) = |\sin pt|, \ t \geq 0$

$\rightarrow$ $f\left(t+\frac{\pi}{p}\right) = \sin p\left(t+\frac{\pi}{p}\right) = |\sin(pt+\pi)| = |\sin pt|$

$\therefore f(t)$ is a periodic function with period $\frac{\pi}{p}$



$\therefore L[f(t)] = \frac{1}{1-e^{-sT}} \int_0^T e^{-st} f(t)\, dt$

$= \frac{1}{1-e^{-\pi s/p}} \int_0^{\pi/p} e^{-st} |\sin pt|\, dt$

$= \frac{1}{1-e^{-\pi s/p}} \int_0^{\pi/p} e^{-st} \sin pt\, dt \qquad \left[\because \sin pt > 0, \text{ for } 0 \leq t \leq \frac{\pi}{p}\right]$

$= \frac{1}{1-e^{-\pi s/p}} \left[\frac{e^{-st}}{s^2+p^2}(-s\sin pt - p\cos pt)\right]_0^{\pi/p}$

$\left[\because \int e^{ax}\sin bx = \frac{e^{ax}}{a^2+b^2}(a\sin bx - b\cos bx)\right]$

$$\therefore L[f(t)] = \frac{1}{1-e^{-\pi s/p}} \cdot \frac{1}{s^2+p^2} \left[e^{-\pi s/p}(p) - (-p) \right]$$

$$= \frac{1}{1-e^{-\pi s/p}} \cdot \frac{1}{(s^2+p^2)} \cdot p(1+e^{-\pi s/p})$$

$$= \frac{p}{s^2+p^2} \left(\frac{1+e^{-\pi s p}}{1-e^{-\pi s p}} \right)$$

$$= \frac{p}{s^2+p^2} \left(\frac{e^{\pi s/2p} + e^{-\pi s/2p}}{e^{\pi s/2p} - e^{-\pi s/2p}} \right)$$

$$= \frac{p}{s^2+p^2} \coth\left(\frac{\pi s}{2p}\right) \quad \cdots \left[\because \cosh t = \frac{e^t+e^{-t}}{e^t-e^{-t}} \right]$$

$$\therefore L[|\sin pt|] = \frac{p}{s^2+p^2} \coth\left(\frac{\pi s}{2p}\right)$$

4) $\int_0^\infty e^{-t}(1+2t-t^2+t^3)\,H(t-1)\,dt$

→ Let

Here,

$L[(1+2t-t^2+t^3)\,H(t-1)]$

$= e^{-s} L[1+2(t+1)-(t+1)^2+(t+1)^3]$ ... $L[f(t)\cdot H(t-a)] = e^{-as} L[f(t+a)]$

$= e^{-s} L[1+2t+2-t^2-2t-1+t^3+1+3t^2+3t]$

$= e^{-s} L[t^3+2t^2+3t+3]$

$= e^{-s}\left[\frac{3!}{s^4}+2\cdot\frac{2!}{s^3}+\frac{3}{s^2}+\frac{3}{s}\right]$

$= e^{-s}\left[\frac{6}{s^4}+\frac{4}{s^3}+\frac{3}{s^2}+\frac{3}{s}\right]$ —— (i)

Now,

$\int_0^\infty e^{-t}(1+2t-t^2+t^3)\,H(t-1)\,dt = L[(1+2t-t^2+t^3)H(t-1)]$, at $s=1$

$= e^{-s}\left[\frac{6}{s^4}+\frac{4}{s^3}+\frac{3}{s^2}+\frac{3}{s}\right]$,

Put $s=1$,

$\therefore \int_0^\infty e^{-t}(1+2t-t^2+t^3)\,H(t-1)\,dt = e^{-1}\left[\frac{6}{1}+\frac{4}{1}+\frac{3}{1}+\frac{3}{1}\right] = \frac{16}{e}$

5) $L\left[\cos t\left(H(t-\frac{\pi}{2})-H(t-\frac{3\pi}{2})\right)\right]$

$= L\left[\cos t\ H(t-\frac{\pi}{2})\right] - L\left[\cos t\ H(t-\frac{3\pi}{2})\right]$

$= e^{-\frac{\pi s}{2}} L\left[\cos(t+\frac{\pi}{2})\right] - e^{-\frac{3\pi s}{2}} L\left[\cos(t+\frac{3\pi}{2})\right]$

$\left[\because L[f(t)\cdot H(t-a)] = e^{-as} L[f(t+a)]\right]$

$= e^{-\frac{\pi s}{2}} L[-\sin t] - e^{-\frac{3\pi s}{2}} L[\sin t]$

$= e^{-\frac{\pi s}{2}}\left(\frac{-1}{s^2+1}\right) - e^{-\frac{3\pi s}{2}}\left(\frac{1}{s^2+1}\right)$

$= \frac{-1}{s^2+1}\left(e^{-\frac{\pi s}{2}} + e^{-\frac{3\pi s}{2}}\right)$

$\therefore L\left[\cos t\left[H(t-\frac{\pi}{2}) - H(t-\frac{3\pi}{2})\right]\right] = -\left(e^{-\frac{\pi s}{2}} + e^{-\frac{3\pi s}{2}}\right)\cdot\frac{1}{s^2+1}$

6) $L[t^2 H(t-2) - \cosh t\, \delta(t-4)]$

→ we know that,

$L[f(t) H(t-a)] = e^{-as} L[f(t+a)]$

$L[f(t) \cdot \delta(t-a)] = e^{-as} f(a)$

Now,

$L[t^2 H(t-2) - \cosh t\, \delta(t-4)] = L[t^2 H(t-2)] - L[\cosh t\, \delta(t-4)]$

$= e^{-2s} L[(t+2)^2] - e^{-4s} \cosh 4$

$= e^{-2s} \{ L[t^2 + 4t + 4] - e^{-4s} \cosh 4$

$= e^{-2s} \left[\frac{2}{s^3} + \frac{4}{s^2} + \frac{4}{s} \right] - e^{-4s} \cosh 4$

$\therefore L[t^2 H(t-2) - \cosh t\, \delta(t-4)] = e^{-2s} \left(\frac{2}{s^3} + \frac{4}{s^2} + \frac{4}{s} \right) - e^{-4s} \cosh 4$