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Name - Ayush Jain

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Engineering Mathematics

Tutorial 4 - Partial Differentiation

1) If $u = \log(\tan x + \tan y + \tan z)$, prove that

$$\sin(2x)\frac{\partial u}{\partial x} + \sin(2y)\frac{\partial u}{\partial y} + \sin(2z)\frac{\partial u}{\partial z} = 2$$

2) If $u = f(r^2)$ where $r^2 = x^2 + y^2 + z^2$, prove that

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 4r^2 f''(r^2) + 6f'(r^2)$$

3) If $\frac{x^2}{a^2+4} + \frac{y^2}{b^2+4} + \frac{z^2}{c^2+4} = 1$

where u is a homogeneous function of degree n in x, y, z prove that

$$u_x^2 + u_y^2 + u_z^2 = 2nu$$

4) If $u = f\left(\frac{x-y}{xy}, \frac{z-x}{zx}\right)$, prove that

$$x^2\frac{\partial u}{\partial x} + y^2\frac{\partial u}{\partial y} + z^2\frac{\partial u}{\partial z} = 0$$

5) If $z = \tan(y+ax) + (y-ax)^{3/2}$

Show that $\frac{\partial^2 z}{\partial x^2} = a^2\frac{\partial^2 z}{\partial y^2}$

Solutions:

1) $u = \log(\tan x + \tan y + \tan z)$

Differentiate u partially with x, y, z respectively

$$\therefore \frac{\partial u}{\partial x} = \left(\frac{1}{\tan x + \tan y + \tan z}\right) \cdot \sec^2 x \quad -(i)$$

$$\therefore \frac{\partial u}{\partial y} = \left(\frac{1}{\tan x + \tan y + \tan z}\right) \cdot \sec^2 y \quad -(2)$$

$$\therefore \frac{\partial u}{\partial z} = \left(\frac{1}{\tan x + \tan y + \tan z}\right) \cdot \sec^2 z \quad -(3)$$

Now,

$\sin(2x) \times (i) + \sin(2y) \times (2) + \sin(2z) \times (3)$

$$\therefore \sin(2x)\frac{\partial u}{\partial x} + \sin(2y)\frac{\partial u}{\partial y} + \sin(2z)\frac{\partial u}{\partial z}$$

$$= \frac{2\sin x \cos x}{(\tan x + \tan y + \tan z)} \cdot \sec^2 x + \frac{2\sin y \cos y}{(\tan x + \tan y + \tan z)} \cdot \sec^2 y + \frac{2\sin z \cos z}{(\tan x + \tan y + \tan z)} \cdot \sec^2 z$$

$$= 2\left[\frac{\tan x}{\tan x + \tan y + \tan z} + \frac{\tan y}{\tan x + \tan y + \tan z} + \frac{\tan z}{\tan x + \tan y + \tan z}\right]$$

$$= 2\left[\frac{\tan x + \tan y + \tan z}{\tan x + \tan y + \tan z}\right] = 2$$

$$\therefore \sin(2x)\frac{\partial u}{\partial x} + \sin(2y)\frac{\partial u}{\partial y} + \sin(2z)\frac{\partial u}{\partial z} = 2$$

2) $u = f(r^2)$, $r^2 = x^2+y^2+z^2$

$$u \xrightarrow{D} r^2 \xrightarrow{P} x, y, z$$

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$$\therefore \frac{\partial u}{\partial x} = \frac{du}{dr} \cdot \frac{\partial r}{\partial x} = 2r f'(r^2) \frac{\partial r}{\partial x}$$

$$\therefore \frac{\partial u}{\partial y} = \frac{du}{dr} \cdot \frac{\partial r}{\partial y} = 2r f'(r^2) \cdot \frac{\partial r}{\partial y}$$

$$\therefore \frac{\partial u}{\partial z} = \frac{du}{dr} \cdot \frac{\partial r}{\partial z} = 2r f'(r^2) \cdot \frac{\partial r}{\partial z}$$

Now, $\cancel{\frac{\partial r}{\partial x}} =$

$$r^2 = x^2+y^2+z^2$$

$$\therefore 2r \frac{\partial r}{\partial x} = 2x$$

$$\therefore \frac{\partial r}{\partial x} = \frac{x}{r}$$

Similarly, $\frac{\partial r}{\partial y} = \frac{y}{r}$, $\frac{\partial r}{\partial z} = \frac{z}{r}$

$$\therefore \frac{\partial u}{\partial x} = 2r f'(r^2) \cdot \frac{x}{r} = 2 f'(r^2) \cdot x$$

$$\therefore \frac{\partial u}{\partial y} = 2r \cdot f'(r^2) \cdot \frac{y}{r} = 2 f'(r^2) \cdot y$$

$$\therefore \frac{\partial u}{\partial z} = 2r \cdot f'(r^2) \cdot \frac{z}{r} = 2 f'(r^2) \cdot z$$

$$\frac{\partial^2 u}{\partial x^2} = 2\left[f'(r^2) + x \cdot f''(r^2) \cdot 2r \frac{\partial r}{\partial x}\right]$$

$$= 2\left[f'(r^2) + x \cdot f''(r^2) \cdot 2 \cdot r \cdot \frac{x}{r}\right]$$

$$\therefore \frac{\partial^2 u}{\partial x^2} = 2f'(r^2) + 4f''(r^2) \cdot x^2 \quad \text{— (i)}$$

Similarly,

$$\frac{\partial^2 u}{\partial y^2} = 2f'(r^2) + 4f''(r^2) \cdot y^2 \quad \text{— (2)}$$

$$\frac{\partial^2 u}{\partial z^2} = 2f'(r^2) + 4f''(r^2) \cdot z^2 \quad \text{— (3)}$$

Adding (i), (ii) and (3)

$$\therefore \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 6f'(r^2) + 4f''(r^2)\left[x^2+y^2+z^2\right]$$

$$= 6f'(r^2) + 4f''(r^2) \cdot (r^2)$$

$$= 6f'(r^2) + 4r^2 f''(r^2)$$

$$\therefore \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 4r^2 f''(r^2) + 6f'(r^2)$$

3) $\because$ u is homogeneous function of degree n in x, y, z

$\therefore$ By Euler's theorem,

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = nu \quad -(i)$$

$$\frac{x^2}{a^2+u} + \frac{y^2}{b^2+u} + \frac{z^2}{c^2+u} = 1$$

Differentiate w.r.t x,

$$x^2\left(\frac{-1}{(a^2+u)^2}\right)u_x + \frac{2x}{a^2+u} + y^2\left(\frac{-1}{(b^2+u)^2}\right)u_x + z^2\left(\frac{-1}{(c^2+u)^2}\right)u_x = 0$$

$$\therefore u_x\left[\frac{x^2}{(a^2+u)^2} + \frac{y^2}{(b^2+u)^2} + \frac{z^2}{(c^2+u)^2}\right] = \frac{2x}{a^2+u}$$

$$\therefore u_x = \frac{2x}{L(a^2+u)}$$

where $L = \frac{x^2}{(a^2+u)^2} + \frac{y^2}{(b^2+u)^2} + \frac{z^2}{(c^2+u)^2}$

Similarly,

$$u_y = \frac{2y}{L(b^2+u)}, \quad u_z = \frac{2z}{L(c^2+u)}$$

$$\therefore u_x^2 + u_y^2 + u_z^2 = \frac{4}{L^2}\left[\frac{x^2}{(a^2+u)^2} + \frac{y^2}{(b^2+u)^2} + \frac{z^2}{(c^2+u)^2}\right]$$

$$= \frac{4}{L^2}[L] = \frac{4}{L} \quad - (2)$$

from (i)

$$x u_x + y u_y + z u_z = nu$$

$$x\left(\frac{2x}{L(a^2+u)}\right) + y\left(\frac{2y}{L(b^2+u)}\right) + z\left(\frac{2z}{L(c^2+u)}\right) = nu$$

$$\frac{2}{L}\left[\frac{x^2}{a^2+u} + \frac{y^2}{b^2+u} + \frac{z^2}{c^2+u}\right] = nu$$

$$\frac{2}{L}(1) = nu$$

$$\therefore L = \frac{2}{nu}$$

Substitute L in eq (2)

$$\therefore u_x^2 + u_y^2 + u_z^2 = \frac{4}{L} = \frac{4nu}{2} = 2nu$$

$$\therefore u_x^2 + u_y^2 + u_z^2 = 2nu$$

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4) $u = f\left(\frac{x-y}{xy}, \frac{z-x}{zx}\right)$

Let $p = \frac{x-y}{xy}$, $q = \frac{z-x}{zx}$

$u \xrightarrow{P} p, q \xrightarrow{P} x, y, z$

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial p} \cdot \frac{\partial p}{\partial x} + \frac{\partial u}{\partial q} \cdot \frac{\partial q}{\partial x} \quad \text{— (i)}$$

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial p} \cdot \frac{\partial p}{\partial y} + \frac{\partial u}{\partial q} \cdot \frac{\partial q}{\partial y} \quad \text{— (2)}$$

$$\frac{\partial u}{\partial z} = \frac{\partial u}{\partial p} \cdot \frac{\partial p}{\partial z} + \frac{\partial u}{\partial q} \cdot \frac{\partial q}{\partial z} \quad \text{— (3)}$$

Now,

$$p = \frac{x-y}{xy} = \frac{1}{y} - \frac{1}{x}$$

$$\therefore \frac{\partial p}{\partial x} = \frac{1}{x^2} , \quad \frac{\partial p}{\partial y} = \frac{-1}{y^2} , \quad \frac{\partial p}{\partial z} = 0$$

$$q = \frac{z-x}{zx} = \frac{1}{x} - \frac{1}{z}$$

$$\frac{\partial q}{\partial x} = \frac{-1}{x^2} , \quad \frac{\partial q}{\partial y} = 0 , \quad \frac{\partial q}{\partial z} = \frac{1}{z^2}$$

Substitute all the values in eq (i), (2) and (3)

$$\therefore \frac{\partial u}{\partial x} = \frac{\partial u}{\partial p}\left(\frac{1}{x^2}\right) + \frac{\partial u}{\partial q}\left(\frac{-1}{x^2}\right)$$

$$\therefore \cancel{\frac{\partial y}{\partial x} = \partial u}$$

$$\therefore \frac{\partial u}{\partial y} = \frac{\partial u}{\partial p}\left(\frac{-1}{y^2}\right) + \frac{\partial u}{\partial q}(0)$$

$$\therefore \frac{\partial u}{\partial z} = \frac{\partial u}{\partial p}(0) + \frac{\partial u}{\partial q}\left(\frac{1}{z^2}\right)$$

$$\therefore \cancel{x^2 \partial^2 u}$$

$$\therefore x^2\frac{\partial u}{\partial x} + y^2\frac{\partial u}{\partial y} + z^2\frac{\partial u}{\partial z}$$

$$= x^2 \cdot \frac{\partial u}{\partial p} \cdot \left(\frac{1}{x^2}\right) + x^2\frac{\partial u}{\partial q}\left(\frac{-1}{x^2}\right) + y^2 \cdot \frac{\partial u}{\partial p}\left(\frac{-1}{y^2}\right) + z^2 \cdot \frac{\partial u}{\partial q} \cdot \frac{1}{z^2}$$

$$= \frac{\partial u}{\partial p} - \frac{\partial u}{\partial q} - \frac{\partial u}{\partial p} + \frac{\partial u}{\partial q} = 0$$

$$\therefore x^2\frac{\partial u}{\partial x} + y^2\frac{\partial u}{\partial y} + z^2\frac{\partial u}{\partial z} = 0$$

5) $z = \tan(y+ax) + (y-ax)^{3/2}$

$$\frac{\partial z}{\partial x} = a \cdot \sec^2(y+ax) - a \cdot \frac{3}{2}(y-ax)^{1/2}$$

$$\therefore \frac{\partial^2 z}{\partial x^2} = a^2 \cdot 2\sec^2(y+ax) \cdot \tan(y+ax) + a^2 \cdot \frac{3}{4}(y-ax)^{-1/2} \quad \text{--- (i)}$$

Now,

$$\frac{\partial z}{\partial y} = \sec^2(y+ax) + \frac{3}{2}(y-ax)^{1/2}$$

$$\therefore \frac{\partial^2 z}{\partial y^2} = 2\sec^2(y+ax) \cdot \tan(y+ax) + \frac{3}{4}(y-ax)^{-1/2} \quad \text{--- (2)}$$

$\therefore$ From equation (i) and (2)

$$\therefore \frac{\partial^2 z}{\partial x^2} = a^2 \cdot \frac{\partial^2 z}{\partial y^2}$$