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(5) N identical lotteries

Prize range $1, 2, \dots N$

n tickets are drawn with replacement

$M =$ Maximum prize ticket

$E[M] = ?$

$$E[M] = E[M=k] = \sum_{k=1}^{N} k\, P(M=k) \quad -(1)$$

We need to find probability of maximum prize ticket

$[1, 2, \dots N]$

like $[4]$ $[2]$ we do it n times

so we have total ways $= \underbrace{N \cdot N \cdot N \cdots N}_{n\,time}$

$= N^n$

$[1, 2, \dots (k-1), k, \dots N]$

k be any number

let (max) $M \le k$

so k can be drawn k^n ways

k is max

$k-1$ can be drawn $(k-1)^n$ ways

$$P(M=k) = \frac{(k)^n - (k-1)^n}{N^n} \quad -(2)$$

(k is max)

put in eq (1)

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$$E[M] = \sum_{k=1}^{N} k\left[\frac{k^n - (k-1)^n}{N^n}\right]$$

$$= \sum_{k=1}^{N} \frac{k^{n+1} - k(k-1)^n}{N^n}$$

$$= \sum_{k=1}^{N} \frac{[k^{n+1} - (k+1-1)(k-1)^n]}{N^n}$$

$$= \sum_{k=1}^{N} [k^{n+1} - (k-1)^{n+1} - (k-1)^n] / N^n$$

$$= \underbrace{\sum_{k=1}^{N} \frac{[k^{n+1} - (k-1)^{n+1}}{N^n}}_{\text{sequence after solving it}} - \sum_{k=1}^{N} \left(\frac{k-1}{N}\right)^n$$

$$= [\cancel{(1)^{n+1}} - 0 + \cancel{(2)^{n+1}} - \cancel{(1)^{n+1}} + \cdots N^{n+1} - (N-1)^{n+1}] - \sum_{k=1}^{N} \left(\frac{k-1}{N}\right)^n$$

$$= \frac{N^{n+1}}{N^n} - \sum_{k=1}^{N} \left(\frac{k-1}{N}\right)^n$$

$$\boxed{E[M] = N - \sum_{k=1}^{N} \left(\frac{k-1}{N}\right)^n} \quad //$$

⑥ find the probability when two points are selected on line segment of length d and distance between them ~~them~~ is $d/3$

Solution

consider the line segment of length d

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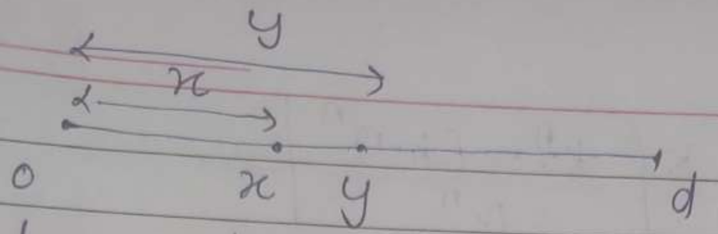
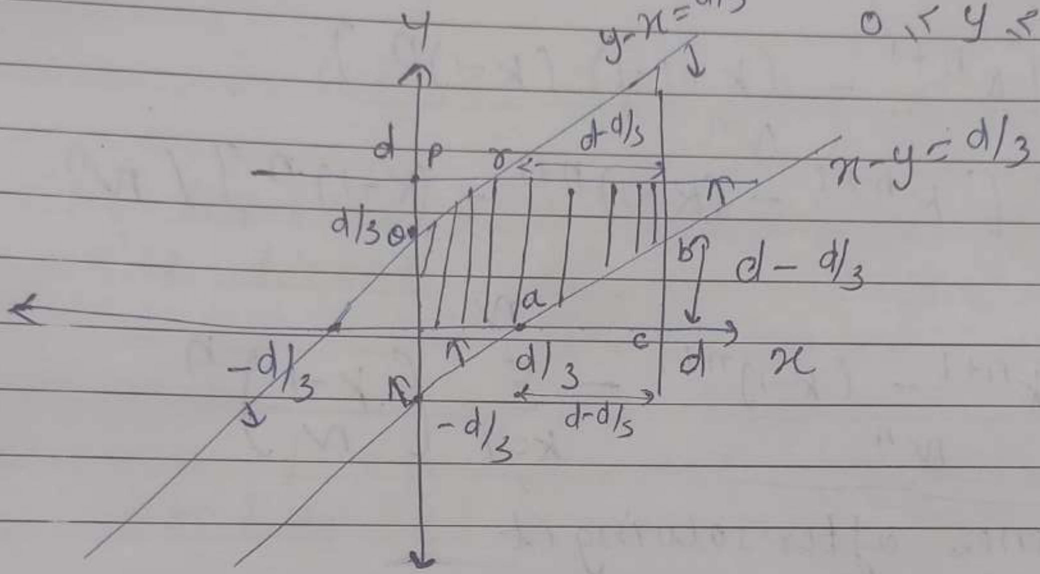


Take two points x and y

Total possibility $= d^2$ $\quad 0 \leq x \leq d$

$\qquad\qquad 0 \leq y \leq d$



Area of square $= d^2$

favourable condition $|x-y| < d/3$

$x-y < d/3$

$y-x < d/3$

($\triangle abc$ and $\triangle pqr$)

favorable area = Area of square − Area of both triangle

$= d^2 - \frac{\cancel{2}}{\cancel{2}}(d-d/3)^2$

$= \cancel{d^2} - \cancel{d^2} - \frac{d^2}{9} + \frac{2d^2}{3}$

$= \frac{2d^2}{3} - \frac{d^2}{9} = \frac{5d^2}{9}$

Probability $= \frac{\text{favourable}}{\text{Total}} = \frac{\frac{5\cancel{d^2}}{9}}{\cancel{d^2}} = \frac{5}{9}$

Probability of two points ($|x-y| < d/3$) $= \frac{5}{9}$ //

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(1) We need to find probability that at least one letter is in correct envelop.
We use principle of inclusion - exclusion.

E_i is the event that (i^{th} letter is in i^{th} envelop)

$$P(\text{at least one } E_i \text{ occur}) = P\left(\bigcup_{i=1}^{N} E_i\right)$$

$$P\left(\bigcup_{i=1}^{N} E_i\right) = \sum_{i=1}^{N} P(E_i) - \sum_{\substack{i_1 \neq i_2 \\ i_1 < i_2}} P(E_{i_1} \cap E_{i_2}) + \sum_{i_1 \neq i_2 \neq i_3} P(E_{i_1} \cap E_{i_2} \cap E_{i_3})$$

$$+ \cdots (-1)^{N-1} \sum (E_{i_1} \cap E_{i_2} \cap \cdots E_{i_N})$$

$$= N \times \frac{1}{N} - {}^NC_2 \frac{(N-2)!}{N!} + {}^NC_3 \frac{(N-3)!}{N!}$$

$$+ \cdots \cdots {}^NC_N \frac{(-1)^{N-1}}{N!}$$

$$\Rightarrow 1 - \frac{\cancel{N!}}{\cancel{(N-2)!}\, 2!} \frac{\cancel{(N-2)!}}{\cancel{N!}} + \frac{\cancel{N!}}{\cancel{(N-3)!}\, 3!} \frac{\cancel{(N-3)!}}{\cancel{N!}} + \cdots \frac{(-1)^{N-1}}{N!}$$

$$= 1 - \frac{1}{2!} + \frac{1}{3!} + \cdots \frac{(-1)^{N-1}}{N!}$$

$$P\left(\bigcup_{i=1}^{N} E_i\right) = \sum_{K=1}^{N} \frac{(-1)^{K-1}}{K!} \quad //$$

Approximation of this probability for $N = 50$

$$P\left(\bigcup_{i=1}^{N} E_i\right) = \sum_{K=1}^{N} \frac{(-1)^{K-1}}{K!} = 1 - \frac{1}{2!} + \frac{1}{3!} + \cdots \frac{(-1)^{N-1}}{N!}$$

Add ~~and~~ and subtract 1

$$\Rightarrow 1 - 1 + 1 - \frac{1}{2!} + \frac{1}{3!} + \cdots \frac{(-1)^{N-1}}{N!} = 1 - e^{-1}$$

$$P\left(\bigcup_{i=1}^{50} E_i\right) = 1 - e^{-1} = 1 - \frac{1}{e} \approx 0.6321$$

(2) we have three presents

let

C_1 = \$ 1000 is in present (1)

C_2 = \$ 1000 is in present (2)

C_3 = \$ 1000 is in present (3)

Conditions.

(1) I select present (1)

(2) Host select present (2)

Initially $P(C_1) = P(C_2) = P(C_3) = 1/3$

we have to calculate probability when host select (2) and I switch =

$P(C_3 | B_2) = ?$

where B_2 = present (2) is open by Host

B_3 = Host choose (3) present

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Using Bayes theorem we know the formula.

$$P(C_3/B_2) = \frac{P(C_3)\,P(B_2/C_3)}{P(C_1)\,P(B_2/C_1) + P(C_2)\,P(B_2/C_2) + P(C_3)\,P(B_2/C_3)} \quad - (1)$$

$$P(C_1) = P(C_2) = P(C_3) = 1/3.$$

let case (1) occurs

\$1000 present is in (1) = C_1

Host choose B_2 or B_3

$$P(B_2/C_1) = 1/2 \text{ and } P(B_3/C_1) = 1/2$$

Case II (\$1000 present is in (2) = C_2)

Host choose confirm B_3

$$P(B_2/C_2) = 0$$

Case III (\$1000 present is in (3) = C_3)

Host choose confirmly B_2

$$P(B_2/C_3) = 1$$

put in eq (1)

$$P(C_3/B_2) = \frac{\frac{1}{3}\times 1}{\frac{1}{3}\times\frac{1}{2} + \frac{1}{3}\times 0 + \frac{1}{3}\times 1} = \frac{1}{\frac{1}{2}+1} = \frac{1}{\frac{3}{2}} = \frac{2}{3}$$

$$\boxed{P(C_3/B_2) = 2/3}$$

expected wining if I switch

$$= \frac{2}{3} \times \overset{333.3}{1000} + \frac{1}{3}\times 0 = \$666.67 \quad //$$

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(3) let A, B, C, D be four events such that $P(B \cap C) > 0$

(a) $P(A \cap B / C) = P(A / B \cap C)\, P(B / C)$

we know that $P(A/B) = \frac{P(A \cap B)}{P(B)}$ $P(B) > 0$

$LHS = P(A \cap B / C) = \frac{P(A \cap B \cap C)}{P(C)}$ - (1)

$RHS = P(A / B \cap C)\, P(B / C) = \frac{P(A \cap B \cap C)}{P(B \cap C)} \frac{P(B \cap C)}{P(C)}$

$= \frac{P(A \cap B \cap C)}{P(C)}$ - (2)

from (1) and (2)

$P(A \cap B / C) = P(A / B \cap C)\, P(B / C)$ is [TRUE] //

(b) $P(A \cap B / C) = P(A/C)\, P(B/C)$ for independent events A and B

A and B are independent so

$P(A \cap B) = P(A) \cdot P(B)$

→ It is false.

Counter example. let $\{1, 2, 3, 4, 5, 6\}$

$A = \{1, 2, 4\}$ $B = \{1, 3\}$ $C = \{1, 2, 3\}$

$P(A) = 3/6$ $P(B) = 2/6$ $P(C) = 3/6$

$P(A \cap B) = 1/6$

here satisfy $P(A\cap B) = P(A)\cdot P(B)$

$$\frac{1}{6} = \frac{1}{2}\cdot\frac{1}{3} = 1/6$$

But not necessarily hold condition (b) we prove it.

$$P(A|C) = \frac{P(A\cap C)}{P(C)} = \frac{2/6}{3/6} = 2/3$$

$$P(B|C) = \frac{P(B\cap C)}{P(C)} = \frac{2/6}{3/6} = 2/3$$

$$P(A\cap B|C) = \frac{P(A\cap B\cap C)}{P(C)} = \frac{1/6}{3/6} = 1/3$$

$$\frac{1}{3} \neq 4/9$$

So, condition (b) is $\boxed{\text{FALSE}}$ //

(c) Given $P(A|D\cap B^c) > P(A|D\cap B)$

$P(A|D^c\cap B^c) > P(A|D^c\cap B)$

Check $P(A/B) > P(A/B^c)$

→ Using conditional probabilities we can write

$P(A|B) = P(A\cap D|B) + P(A\cap D^c|B)$

$P(A|B) = P(A|B\cap D) + P(D|B) + P(A|B\cap D^c) \cdot P(D^c/B)$ — (1)

Similarly

$P(A|B^c) = P(A|B^c\cap D)\, P(D|B^c) + P(A|B^c\cap D^c) \cdot P(D^c/B^c)$ — (2)

Take an example

$P(D|B) = 0.5$ $\quad$ $P(D^c/B) = 0.5$

$P(D|B^c) = 0.5$ $\quad$ $P(D^c/B^c) = 0.5$

$P(A|D \cap B) = 0.3$ $\quad$ $P(A|D \cap B^c) = 0.4$

$P(A|D^c \cap B^c) = 0.3$ $\quad$ $P(A|D^c \cap B) = 0.2$

put in eq (1) and (2)

$$P(A|B) = 0.3(0.5) + 0.2(0.5)$$
$$= 0.25 \quad -(3)$$
$$P(A|B^c) = 0.4(0.5) + 0.3(0.5)$$
$$= 0.35 \quad -(4)$$

from (3) and (4)

$$P(A/B) < P(A/B^c)$$

So (c) This is $\boxed{\text{FALSE}}$ //

(4) Construct the following or disprove.

(a) A discrete random variable x for which $E(x)$ is finite but $E(x^2)$ is not finite.

x be a random variable

$f(x)$ is probability mass function

we have to find $f(x)$ such that

$\Sigma x f(x)$ converges

$\Sigma x^2 f(x)$ diverges

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let $f(x) = \left\{ \left(c/x^3 \right) \right. \quad x = 1, 2, 3, \dots$

where c is normalising constant

$\because$ we know a result $\sum \frac{1}{x^r}$ converges $r > 1$

diverges $r \leq 1$

$$E[x] = \sum_{x=1}^{\infty} x \cdot \frac{c}{x^3} = c \sum_{x=1}^{\infty} \frac{1}{x^2} < \infty \text{ converges}$$

$$E[x^2] = \sum_{x=1}^{\infty} x^2 \cdot \frac{c}{x^3} = c \sum_{x=1}^{\infty} \frac{1}{x} = \infty \text{ diverges.}$$

It is possible that $E[x]$ finite but $E(x^2]$ may not finite.

(2) A continuous random variable x for which $E(x)$ is finite but $E(x^2)$ is not finite.

x be a random variable with P.d.F.

$$f(x) = \begin{cases} 2/x^3 & \text{if } x \geq 1 \\ 0 & \text{if } x < 1 \end{cases}$$

continuous random variable

$$E(x) = \int_{-\infty}^{\infty} x f(x) dx$$

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$$E(x^2) = \int_{-\infty}^{\infty} x^2 f(x)\, dx$$

$$E(x) = \int_{-\infty}^{\infty} x f(x)\, dx = \int_{-\infty}^{1} 0 \cdot dx + \int_{1}^{\infty} \frac{2}{x^3} \times x\, dx$$

$$= -2\left[\frac{1}{x}\right]_1^{\infty} = -2[0-1]$$
$$= 2$$

$$E(x^2) = \int_{-\infty}^{\infty} x^2 f(x)\, dx = \int_{1}^{\infty} \frac{2}{x^3} \times x^2\, dx$$
$$= 2\int_1^{\infty} \frac{1}{x}\, dx$$
$$= 2[\log x]_1^{\infty} = 2[\infty] = \infty$$

Also $E(x)$ is finite but $E(x^2)$ is not finite.

(c) A random variable x with $E(x)=1$ but $E(e^{-x}) < 1/3$

We cann't construct such example so we disprove it.

let try an example
x be two random variable defined as

$$x = \begin{cases} a & P(a) = P \\ b & P(b) = 1-P \end{cases}$$

$E(x) = 1$

$P(x) = P + 1 - P = 1$

for $E(e^{-x}) < 1/3$

$$E(e^{-x}) = pe^{-a} + (1-p)e^{-b}$$

e^{-a} is negligible

$$E(x) = p.a + (1-p).b = 1$$

$$b = \frac{1-pa}{1-p}$$

$E(e^{-x})$ cann't be $< 1/3$

~~Disprove~~ if we take $a=3$ $p=0.5$

$b=-1$

$$E(e^{-x}) = 0.5 . e^{-3} + 0.5e^{1} \approx 0.5 . 0.0498 + 0.5 \times 2.71)$$

$$\approx 1.38$$

So not possible $E(e^{-x}) < 1/3$

$\boxed{\text{Disprove}}$ It is impossible to construct such a random variable x with $E(x) = 1$

and $E(e^{-x}) < 1/3$

By Jensen's inequality

$$E(e^{-x}) \geq e^{-E(x)} = e^{-1} \approx 0.367$$

$$0.367 > 1/3$$

$$E(e^{-x}) \geq e^{-1} > 1/3$$

such a random variable x cannot exist //