AI1103-Assignment 3

Name : Ayush Jha
Roll Number: CS20BTECH11006

Download all python codes from

https://github.com/ayushjha2612/AI11003/tree/main/Assignment3/Codes

and latex-tikz codes from

https://github.com/ayushjha2612/AI11003/tree/main/Assignment3

GATE Problem 34

Let X and Y be two statistically independent random variables uniformly distributed in the range (-1, 1) and (-2, 1) respectively. Let $Z = X + Y$, then the probability that $[Z \leq -2]$ is

(A) zero (B) $\frac{1}{6}$ (C)$\frac{1}{3}$ (D)$\frac{1}{12}$

Answer

Option (D) $\frac{1}{12}$

Solution

X and Y are two independent random variables.
The range of X is $-1 \leq X \leq 1$ and the range of Y is $-2 \leq X \leq 1$.
Let $p_X(x) = \Pr(X = x)$, $p_Y(y) = \Pr(Y = y)$ and $p_Z(z) = \Pr(Z = z)$ be the probability densities of random variables X ,Y and Z.
X lies in range(-1,1), therefore,

$$\int_{-1}^{1} p_X(x)\, dx = 1 \tag{0.0.1}$$

$$2 \times p_X(x) = 1 \tag{0.0.2}$$

$$p_X(x) = 1/2 \tag{0.0.3}$$

Similarly for Y we have,

$$\int_{-2}^{1} p_Y(y)\, dy = 1 \tag{0.0.4}$$

$$3 \times p_Y(y) = 1 \tag{0.0.5}$$

$$p_Y(y) = 1/3 \tag{0.0.6}$$

The density for X is

$$p_X(x) = \begin{cases} \frac{1}{2} & -1 \leq x \leq 1 \\ 0 & otherwise \end{cases} \tag{0.0.7}$$

As $Z = X + Y$ we have $z = x + y$ and $x = z - y$, The density of X can also be represented as,

$$p_X(z-y) = \begin{cases} \frac{1}{2} & -1 \leq z - y \leq 1 \\ 0 & otherwise \end{cases} \tag{0.0.8}$$

and the density of Y is,

$$p_Y(y) = \begin{cases} \frac{1}{3} & -2 \leq y \leq 1 \\ 0 & otherwise \end{cases} \tag{0.0.9}$$

The density of Z i.e. $Z = X + Y$ is given by the convolution of the densities of X and Y

$$p_Z(z) = \int_{-\infty}^{\infty} p_X(z-y) p_Y(y)\, dy \tag{0.0.10}$$

From 0.0.8 and 0.0.9 we have,
The integrand is $\frac{1}{6}$ when $2 \leq y \leq 1$ and $-1 \leq z - y \leq 1$ i.e. $z - 1 \leq y \leq z + 1$ and zero, otherwise Now when $-3 \leq z \leq -2$ them we have,

$$p_Z(z) = \int_{-2}^{z+1} \frac{1}{6}\, dy \tag{0.0.11}$$

$$= \frac{1}{6} \times (z + 1 - (-2)) \tag{0.0.12}$$

$$= \frac{1}{6}(z + 3) \tag{0.0.13}$$

For $-2 < z \leq -1$,

$$p_Z(z) = \int_{-2}^{z+1} \frac{1}{6}\, dy \tag{0.0.14}$$

$$= \frac{1}{6} \times (z + 1 - (-2)) \tag{0.0.15}$$

$$= \frac{1}{6}(z + 3) \tag{0.0.16}$$

For $-1 < z \leq 0$,

$$p_Z(z) = \int_{z-1}^{z+1} \frac{1}{6}\, dy \tag{0.0.17}$$

$$= \frac{1}{6} \times (z + 1 - (z - 1)) \tag{0.0.18}$$

$$= \frac{1}{3} \tag{0.0.19}$$

For $0 < z \leq 2$,

$$p_Z(z) = \int_{z-1}^{1} \frac{1}{6}\, dy \tag{0.0.20}$$

$$= \frac{1}{6} \times (1 - (z - 1)) \tag{0.0.21}$$

$$= \frac{1}{6}(2 - z) \tag{0.0.22}$$

Therefore the density of Z is given by

$$p_Z(z) = \begin{cases} \frac{1}{6}(z+3) & -3 \leq y \leq -2 \\ \frac{1}{6}(z+3) & -2 < y \leq -1 \\ \frac{1}{3} & -1 < y \leq 0 \\ \frac{1}{6}(z+3) & 0 < y \leq 2 \\ 0 & otherwise \end{cases} \tag{0.0.23}$$

Now,

$$\Pr(Z \leq -2) = \int_{-\infty}^{-2} p_Z(z)\, dz \tag{0.0.24}$$

$$= \int_{-3}^{-2} \frac{1}{6}(z+3)\, dz \tag{0.0.25}$$

$$= \frac{1}{6}\left(\frac{z^2}{2} + 3z\right)\Bigg|_{-3}^{-2} \tag{0.0.26}$$

$$= \frac{1}{6} \times \left((2 - 6) - \left(\frac{9}{2} - 9\right)\right) \tag{0.0.27}$$

$$= \frac{1}{6} \times \frac{1}{2} = \frac{1}{12} \tag{0.0.28}$$

So $\Pr(Z \leq -2) = \frac{1}{12}$ i.e. option (D).
The theory vs simulation plot can be viewed at figure 0.

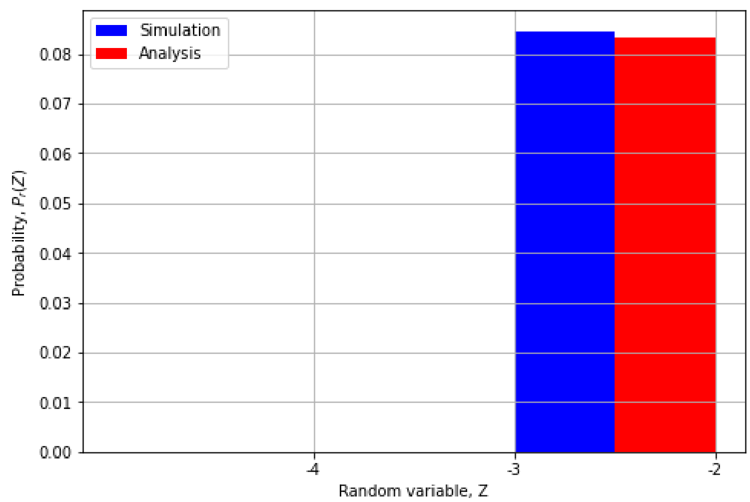

Fig. 0: Theory VS Simulation plot