CSIR-UGC NET-June 2013-Problem(68)

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What is estimator?

An estimator is a statistic that estimates some fact about the population. For example, the sample mean $(\bar{X})$ is an estimator for the population mean, μ. The quantity that is being estimated is called the **estimand.**

Bias of estimator

Let $\Theta = h(X_!, X_2, \cdots, X_n)$ be a point estimator for θ. The **bias** of the estimator Θ is defined by

$$B(\Theta) = E[\Theta] - \theta \tag{1}$$

where $E[\Theta]$ is the expectation value of the estimator Θ and θ is the estimand.

Unbiased estimator

Let $\Theta = h(X_1, X_2, \cdots, X_n)$ be a point estimator for a parameter θ. We say that Θ is an **unbiased estimator** of θ if

$$B(\Theta) = 0, \text{ for all possible values of } \theta. \tag{2}$$

Consistent estimator

Let $\Theta_1, \Theta_2, \cdots, \Theta_n, \cdots,$ be a sequence of point estimators of θ. We say that Θ_n is a **consistent** estimator of θ, if

$$\lim_{n \to \infty} \Pr\left(|\Theta_n - \theta| \geq \epsilon\right) = 0 \text{ ,for all } \epsilon > 0. \tag{3}$$

Mean Squared Error (MSE)

The **mean squared error (MSE)** of a point estimator Θ, shown by $MSE(\Theta)$, is defined as

$$MSE(\Theta) = E[(\Theta - \theta)^2] \tag{4}$$

$$= Var(\Theta) + B(\Theta)^2 \tag{5}$$

where $B(\Theta)$ is the bias of Θ.

Theorem

Let $\Theta_1, \Theta_2, \cdots$ be a sequence of point estimators of θ. If

$$\lim_{n \to \infty} MSE(\Theta_n) = 0, \tag{6}$$

then Θ_n is a consistent estimator of θ.

Question

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Let $X_1, \cdots, X_n$ be independent and identically distributed random variables with probability density function

$$f(x) = \frac{1}{2}\lambda^3 x^2 e^{-\lambda x}; x > 0; \lambda > 0$$

Then which of the following statements are true?

1. $\frac{2}{n}\sum_{i=1}^{n}\frac{1}{X_i}$ is an unbiased estimator of λ
2. $\frac{3n}{\sum_{i=1}^{n} X_i}$ is an unbiased estimator of λ
3. $\frac{2}{n}\sum_{i=1}^{n}\frac{1}{X_i}$ is a consistent estimator of λ
4. $\frac{3n}{\sum_{i=1}^{n} X_i}$ is a consistent estimator of λ

Solution: Option 1

Now here we have our estimator Θ and estimand θ as,

$$\Theta = \frac{2}{n}\sum_{i=1}^{n}\frac{1}{X_i} \text{ and } \theta = \lambda \tag{7}$$

To check if this is an unbiased estimator or not the bias needs to be calculated.

$$B(\Theta) = E[\Theta] - \theta \tag{8}$$

Option 1 Contd.

The expectation value of the estimator is given by,

$$E[\Theta] = E\left[\frac{2}{n}\sum_{i=1}^{n}\frac{1}{X_i}\right] \tag{9}$$

$$= \frac{2}{n}\sum_{i=1}^{n} E\left[\frac{1}{X_i}\right] \tag{10}$$

$$= \frac{2}{n}\sum_{i=1}^{n}\int_{-\infty}^{\infty}\frac{1}{x}f(x)\,dx \tag{11}$$

$$= \frac{2n}{n}\int_{0}^{\infty}\frac{1}{x}\frac{1}{2}\lambda^3x^2e^{-\lambda x}\,dx \tag{12}$$

$$= \lambda^3\int_{0}^{\infty}xe^{-\lambda x}\,dx \tag{13}$$

$$= \lambda \tag{14}$$

Option 1 Contd.

From above calculations we can say,

$$E\left[\frac{1}{X}\right] = \frac{\lambda}{2} \tag{15}$$

So the bias of estimator is given by,

$$B(\Theta) = E[\Theta] - \theta \tag{16}$$
$$= \lambda - \lambda = 0 \tag{17}$$

Therefore $\frac{2}{n}\sum_{i=1}^{n}\frac{1}{X_i}$ is an unbiased estimator of λ .
Option 1 is correct.

Option 2

Here in this option, we have our estimator Θ and quantity to be estimated θ as,

$$\Theta = \frac{3n}{\sum_{i=1}^{n} X_i} \text{ and } \theta = \lambda \tag{18}$$

The expectation value of the estimator is given by,

$$E[\Theta] = E\left[\frac{3n}{\sum_{i=1}^{n} X_i}\right] \tag{19}$$

$$= \frac{3n}{\sum_{i=1}^{n}} E\left[\frac{1}{X_i}\right] \tag{20}$$

Option 2 Contd.

The value of $E\left[\frac{1}{X_i}\right]$ can be obtained from (15) as so we have,

$$E[\Theta] = \frac{3n}{\sum_{i=1}^{n}} \frac{\lambda}{2} \tag{21}$$

$$= \frac{3n}{n} \frac{\lambda}{2} = \frac{3\lambda}{2} \tag{22}$$

So the bias of estimator is given by,

$$B(\Theta) = E[\Theta] - \theta \tag{23}$$

$$= \frac{3\lambda}{2} - \lambda \neq 0 \tag{24}$$

Therefore $\frac{3n}{\sum_{i=1}^{n} X_i}$ is not an unbiased estimator of λ

Option 2 is not correct.

Option 3

Now here we have our estimator Θ and quantity to be estimated θ as,

$$\Theta = \frac{2}{n}\sum_{i=1}^{n}\frac{1}{X_i} \text{ and } \theta = \lambda \tag{25}$$

Now the variance of Θ is calculated as

$$Var(\Theta) = Var\left(\frac{2}{n}\sum_{i=1}^{n}\frac{1}{X_i}\right) \tag{26}$$

$$= \frac{4}{n^2}\sum_{i=1}^{n} Var\left(\frac{1}{X_i}\right) \tag{27}$$

$$= \frac{4n}{n^2}\left(E\left[\frac{1}{X_i}^2\right] - E\left[\frac{1}{X_i}\right]^2\right) \tag{28}$$

$$= \frac{4}{n}\left(\int_{-\infty}^{\infty}\frac{1}{x^2}f(x)\,dx - \left(\frac{\lambda}{2}\right)^2\right) \tag{29}$$

Option 3 Contd.

$$Var(\Theta) = \frac{4}{n}\left(\int_0^\infty \frac{1}{x^2}\frac{1}{2}\lambda^3 x^2 e^{-\lambda x}\, dx - \frac{\lambda^2}{4}\right) \tag{30}$$

$$= \frac{\lambda^2}{n} \tag{31}$$

The bias of Θ from option 1 is given as $B(\Theta) = 0$ So we have,

$$MSE(\Theta_n) = Var(\Theta) + B(\Theta)^2 \tag{32}$$

$$= \frac{\lambda^2}{n} \tag{33}$$

$$\lim_{n\to\infty} MSE(\Theta_n) = \lim_{n\to\infty} \frac{\lambda^2}{n} \tag{34}$$

$$= 0 \tag{35}$$

Therefore, $\frac{2}{n}\sum_{i=1}^{n}\frac{1}{X_i}$ is a consistent estimator of λ.

Option 3 is correct.

Option 4

Now in this option we have our estimator Θ and quantity to be estimated θ as,

$$\Theta = \frac{3n}{\sum_{i=1}^{n} X_i} \text{ and } \theta = \lambda \tag{36}$$

Now the variance of Θ is calculated as

$$Var(\Theta) = Var\left(\frac{3n}{\sum_{i=1}^{n} X_i}\right) \tag{37}$$

$$= \frac{9n^2}{\sum_{i=1}^{n}} Var\left(\frac{1}{X_i}\right) \tag{38}$$

$$\tag{39}$$

Now the value of $Var\left(\frac{1}{X_i}\right)$ from (28) is substituted, we have

Option 4 Contd.

$$Var(\Theta) = \frac{9n^2}{\sum_{i=1}^{n}} \frac{\lambda^2}{4} \tag{40}$$

$$= \frac{9n^2\lambda^2}{4n} \tag{41}$$

$$= \frac{9n\lambda^2}{4} \tag{42}$$

The bias of Θ from option 2 is given as $B(\Theta) = \frac{\lambda}{2}$ So we have,

$$MSE(\Theta_n) = Var(\Theta) + B(\Theta)^2 \tag{43}$$

$$= \frac{9n\lambda^2}{4} + \left(\frac{\lambda}{2}\right)^2 \tag{44}$$

$$= \frac{\lambda^2}{4}(9n+1) \tag{45}$$

Option 4 Contd.

Now,

$$\lim_{n\to\infty} MSE(\Theta_n) = \lim_{n\to\infty} \frac{\lambda^2}{4}(9n+1) \tag{46}$$

$$\tag{47}$$

Clearly as n grows larger $9n+1$ also grows larger, so

$$\lim_{n\to\infty} MSE(\Theta_n) \neq 0 \tag{48}$$

Therefore, $\frac{3n}{\sum_{i=1}^{n} X_i}$ is not a consistent estimator of λ.
Option 4 is not correct.
Therefore option 1 and option 3 are correct.