AI1103-Assignment 8

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Download all latex-tikz codes from

https://github.com/ayushjha2612/AI11003/tree/main/Assignment8

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Let $X_1, \cdots, X_n$ be independent and identically distributed random variables with probability density function

$$f(x) = \frac{1}{2}\lambda^3 x^2 e^{-\lambda x}; x > 0; \lambda > 0$$

Then which of the following statements are true?

1) $\frac{2}{n}\sum_{i=1}^{n}\frac{1}{X_i}$ is an unbiased estimator of λ
2) $\frac{3n}{\sum_{i=1}^{n} X_i}$ is an unbiased estimator of λ
3) $\frac{2}{n}\sum_{i=1}^{n}\frac{1}{X_i}$ is a consistent estimator of λ
4) $\frac{3n}{\sum_{i=1}^{n} X_i}$ is a consistent estimator of λ

Solution

Solving all options :

1)

Definition 0.1. Let $\Theta = h(X_1, X_2, \cdots, X_n)$ be a point estimator for a parameter θ. We say that Θ is an **unbiased estimator** of θ if

$$B(\Theta) = 0, \text{ for all possible values of } \theta. \tag{0.0.1}$$

Definition 0.2. The **bias** of the estimator Θ is defined by

$$B(\Theta) = E[\Theta] - \theta \tag{0.0.2}$$

Now here we have our estimator Θ and quantity to be estimated θ as,

$$\Theta = \frac{2}{n}\sum_{i=1}^{n}\frac{1}{X_i} \text{ and } \theta = \lambda \tag{0.0.3}$$

The expectation value of the estimator is given by,

$$E[\Theta] = E\left[\frac{2}{n}\sum_{i=1}^{n}\frac{1}{X_i}\right] \tag{0.0.4}$$
$$= \frac{2}{n}\sum_{i=1}^{n} E\left[\frac{1}{X_i}\right] \tag{0.0.5}$$
$$= \frac{2}{n}\sum_{i=1}^{n}\int_{-\infty}^{\infty}\frac{1}{x}f(x)\,dx \tag{0.0.6}$$
$$= \frac{2n}{n}\int_{0}^{\infty}\frac{1}{x}\frac{1}{2}\lambda^3 x^2 e^{-\lambda x}\,dx \tag{0.0.7}$$
$$= \lambda^3\int_{0}^{\infty} x e^{-\lambda x}\,dx \tag{0.0.8}$$
$$= \lambda^3\left(\frac{-xe^{-\lambda x}}{\lambda} - \frac{xe^{-\lambda x}}{\lambda^2}\right)\Bigg|_0^{\infty} \tag{0.0.9}$$
$$= \lambda^3\left(-\left[0 - \frac{1}{\lambda^2}\right]\right) \tag{0.0.10}$$
$$= \lambda \tag{0.0.11}$$

So the bias of estimator is given by,

$$B(\Theta) = E[\Theta] - \theta \tag{0.0.12}$$
$$= \lambda - \lambda = 0 \tag{0.0.13}$$

Therefore $\frac{2}{n}\sum_{i=1}^{n}\frac{1}{X_i}$ is an unbiased estimator of λ
Option 1 is correct.

2) Now in this option we have our estimator Θ and quantity to be estimated θ as,

$$\Theta = \frac{3n}{\sum_{i=1}^{n} X_i} \text{ and } \theta = \lambda \tag{0.0.14}$$

The expectation value of the estimator is given by,

$$E[\Theta] = E\left[\frac{3n}{\sum_{i=1}^{n} X_i}\right] \tag{0.0.15}$$

$$= \frac{3n}{\sum_{i=1}^{n}} E\left[\frac{1}{X_i}\right] \tag{0.0.16}$$

The value of $E\left[\frac{1}{X_i}\right]$ can be obtained from (0.0.7) as

$$E\left[\frac{1}{X_i}\right] = \frac{\lambda}{2} \tag{0.0.17}$$

So we have,

$$E[\Theta] = \frac{3n}{\sum_{i=1}^{n}} \frac{\lambda}{2} \tag{0.0.18}$$

$$= \frac{3n}{n} \frac{\lambda}{2} \tag{0.0.19}$$

$$= \frac{3\lambda}{2} \tag{0.0.20}$$

So the bias of estimator is given by,

$$B(\Theta) = E[\Theta] - \theta \tag{0.0.21}$$

$$= \frac{3\lambda}{2} - \lambda \tag{0.0.22}$$

$$= \frac{\lambda}{2} \neq 0 \tag{0.0.23}$$

Therefore $\frac{3n}{\sum_{i=1}^{n} X_i}$ is not an unbiased estimator of λ
Option 2 is not correct.

3)

Definition 0.3. Let $\Theta_1, \Theta_2, \cdots, \Theta_n, \cdots$, be a sequence of point estimators of θ. We say that Θ_n is a **consistent** estimator of θ, if

$$\lim_{n\to\infty} \Pr\left(|\Theta_n - \theta| \geq \epsilon\right) = 0 \text{ ,for all } \epsilon > 0. \tag{0.0.24}$$

Theorem 0.1. Let $\Theta_1, \Theta_2, \cdots$ be a sequence of point estimators of θ. If

$$\lim_{n\to\infty} MSE(\Theta_n) = 0, \tag{0.0.25}$$

then Θ_n is a consistent estimator of θ.

Definition 0.4. The **mean squared error (MSE)** of a point estimator Θ, shown by $MSE(\Theta)$, is defined as

$$MSE(\Theta) = E[(\Theta - \theta)^2] \tag{0.0.26}$$

$$= Var(\Theta) + B(\Theta)^2 \tag{0.0.27}$$

where $B(\Theta)$ is the bias of Θ.
Now here we have our estimator Θ and quantity to be estimated θ as,

$$\Theta = \frac{2}{n} \sum_{i=1}^{n} \frac{1}{X_i} \text{ and } \theta = \lambda \tag{0.0.28}$$

Now the variance of Θ is calculated as

$$Var(\Theta) = Var\left(\frac{2}{n} \sum_{i=1}^{n} \frac{1}{X_i}\right) \tag{0.0.29}$$

$$= \frac{4}{n^2} \sum_{i=1}^{n} Var\left(\frac{1}{X_i}\right) \tag{0.0.30}$$

$$= \frac{4n}{n^2} \left(E\left[\frac{1}{X_i}^2\right] - E\left[\frac{1}{X_i}\right]^2\right) \tag{0.0.31}$$

$$= \frac{4}{n} \left(\int_{-\infty}^{\infty} \frac{1}{x^2} f(x)\, dx - \left(\frac{\lambda}{2}\right)^2\right) \tag{0.0.32}$$

$$= \frac{4}{n} \left(\int_{0}^{\infty} \frac{1}{x^2} \frac{1}{2} \lambda^3 x^2 e^{-\lambda x}\, dx - \frac{\lambda^2}{4}\right) \tag{0.0.33}$$

$$= \frac{4}{n} \left(\frac{\lambda^3}{2} \int_{0}^{\infty} e^{-\lambda x}\, dx - \frac{\lambda^2}{4}\right) \tag{0.0.34}$$

$$= \frac{4}{n} \left(\frac{\lambda^2}{2} - \frac{\lambda^2}{4}\right) \tag{0.0.35}$$

$$= \frac{\lambda^2}{n} \tag{0.0.36}$$

The bias of Θ from option 1 is given as

$$B(\Theta) = 0 \tag{0.0.37}$$

So we have,

$$MSE(\Theta_n) = Var(\Theta) + B(\Theta)^2 \tag{0.0.38}$$

$$= \frac{\lambda^2}{n} \tag{0.0.39}$$

Now,

$$\lim_{n\to\infty} MSE(\Theta_n) = \lim_{n\to\infty} \frac{\lambda^2}{n} \tag{0.0.40}$$

$$= 0 \tag{0.0.41}$$

Therefore, $\frac{2}{n} \sum_{i=1}^{n} \frac{1}{X_i}$ is a consistent estimator of λ. Option 3 is correct.

4) Now in this option we have our estimator Θ and quantity to be estimated θ as,

$$\Theta = \frac{3n}{\sum_{i=1}^{n} X_i} \text{ and } \theta = \lambda \tag{0.0.42}$$

Now the variance of Θ is calculated as

$$Var(\Theta) = Var\left(\frac{3n}{\sum_{i=1}^{n} X_i}\right) \tag{0.0.43}$$
$$= \frac{9n^2}{\sum_{i=1}^{n}} Var\left(\frac{1}{X_i}\right) \tag{0.0.44}$$
$$\tag{0.0.45}$$

Now the value of $Var\left(\frac{1}{X_i}\right)$ from (0.0.31) is substituted, we have

$$Var(\Theta) = \frac{9n^2}{\sum_{i=1}^{n}} \frac{\lambda^2}{4} \tag{0.0.46}$$
$$= \frac{9n^2\lambda^2}{4n} \qquad = \frac{9n\lambda^2}{4} \tag{0.0.47}$$

The bias of Θ from option 2 is given as

$$B(\Theta) = \frac{\lambda}{2} \tag{0.0.48}$$

So we have,

$$MSE(\Theta_n) = Var(\Theta) + B(\Theta)^2 \tag{0.0.49}$$
$$= \frac{9n\lambda^2}{4} + \left(\frac{\lambda}{2}\right)^2 \tag{0.0.50}$$
$$= \frac{\lambda^2}{4}(9n + 1) \tag{0.0.51}$$

Now,

$$\lim_{n\to\infty} MSE(\Theta_n) = \lim_{n\to\infty} \frac{\lambda^2}{4}(9n + 1) \tag{0.0.52}$$
$$\tag{0.0.53}$$

Clearly as n grows larger $9n + 1$ also grows larger, so

$$\lim_{n\to\infty} MSE(\Theta_n) \neq 0 \tag{0.0.54}$$

Therefore, $\frac{3n}{\sum_{i=1}^{n} X_i}$ is not a consistent estimator of λ.
Option 4 is not correct.

Therefore option 1 and option 3 are correct.