AI1103-Assignment 8

Name : Ayush Jha
Roll Number: CS20BTECH11006

Download all latex-tikz codes from

https://github.com/ayushjha2612/AI11003/tree/main/Assignment8

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Let $X_1, \cdots, X_n$ be independent and identically distributed random variables with probability density function

$$f(x) = \frac{1}{2}\lambda^3 x^2 e^{-\lambda x}; x > 0; \lambda > 0 \tag{0.0.1}$$

Then which of the following statements are true?

1) $\frac{2}{n}\sum_{i=1}^{n}\frac{1}{X_i}$ is an unbiased estimator of λ
2) $\frac{3n}{\sum_{i=1}^{n} X_i}$ is an unbiased estimator of λ
3) $\frac{2}{n}\sum_{i=1}^{n}\frac{1}{X_i}$ is a consistent estimator of λ
4) $\frac{3n}{\sum_{i=1}^{n} X_i}$ is a consistent estimator of λ

Solution

Definition 0.1. An **estimator** is a statistic that estimates some fact about the population. The quantity that is being estimated is called the **estimand.**

Definition 0.2. Let $\Theta = h(X_1, X_2, \cdots, X_n)$ be a point estimator for θ. The **bias** of the estimator Θ is defined by

$$B(\Theta) = E[\Theta] - \theta \tag{0.0.2}$$

where $E[\Theta]$ is the expectation value of the estimator Θ and θ is the estimand.

Definition 0.3. Let $\Theta = h(X_1, X_2, \cdots, X_n)$ be a point estimator for a parameter θ. We say that Θ is an **unbiased estimator** of θ if

$$B(\Theta) = 0, \text{ for all possible values of } \theta. \tag{0.0.3}$$

Definition 0.4. Let $\Theta_1, \Theta_2, \cdots, \Theta_n, \cdots$, be a sequence of point estimators of θ. We say that Θ_n is a **consistent** estimator of θ, if

$$\lim_{n\to\infty} \Pr\left(|\Theta_n - \theta| \geq \epsilon\right) = 0 \text{ ,for all } \epsilon > 0. \tag{0.0.4}$$

Definition 0.5. The **mean squared error (MSE)** of a point estimator Θ, shown by $MSE(\Theta)$, is defined as

$$MSE(\Theta) = E[(\Theta - \theta)^2] \tag{0.0.5}$$
$$= Var(\Theta) + B(\Theta)^2 \tag{0.0.6}$$

where $B(\Theta)$ is the bias of Θ.

Theorem 0.1. Let $\Theta_1, \Theta_2, \cdots$ be a sequence of point estimators of θ. If

$$\lim_{n\to\infty} MSE(\Theta_n) = 0, \tag{0.0.7}$$

then Θ_n is a consistent estimator of θ.

Definition 0.6. The **moment generating function (MGF)** of a random variable X is a function $M_X(s)$ defined as

$$M_X(s) = E[e^{sX}]. \tag{0.0.8}$$

Lemma 0.2. A well known result of Integratin by Parts is

$$\int_0^\infty x^a e^{-bx}\, dx = \frac{\Gamma(a+1)}{b^{a+1}} \tag{0.0.9}$$

where $\Gamma(a) = (a-1)!$ for $a \in Z$

Theorem 0.3. The Moment generating function of a gamma distribution with PDF as,

$$f_X(x) = \begin{cases} \frac{\lambda^\alpha x^{\alpha-1} e^{-\lambda x}}{\Gamma(\alpha)} & x > 0 \\ 0 & otherwise \end{cases} \tag{0.0.10}$$

where $\Gamma(\alpha) = \frac{1}{(\alpha-1)!}$ is given by,

$$M_X(s) = \left(\frac{\lambda}{\lambda - s}\right)^\alpha \tag{0.0.11}$$

Proof.

$$M_X(s) = E[e^{sX}] \quad (0.0.12)$$
$$= \int_{-\infty}^{\infty} e^{sx} f(x)\, dx \quad (0.0.13)$$
$$= \int_{0}^{\infty} e^{sx} \frac{1}{\Gamma(\alpha)} \lambda^{\alpha} x^{\alpha-1} e^{-\lambda x}\, dx \quad (0.0.14)$$
$$= \frac{\lambda^{\alpha}}{\Gamma(\alpha)} \int_{0}^{\infty} x^{\alpha-1} e^{-(\lambda-s)x}\, dx \quad (0.0.15)$$

From lemma 0.2 we have,

$$\int_{0}^{\infty} x^{a} e^{-bx}\, dx = \frac{\Gamma(a+1)}{b^{a+1}} \quad (0.0.16)$$

Making this substitution we get,

$$\int_{0}^{\infty} x^{\alpha-1} e^{-(\lambda-s)x}\, dx = \frac{\Gamma(\alpha)}{(\lambda-s)^{\alpha}} \quad (0.0.17)$$

Therefore,

$$M_X(s) = \frac{\lambda^{\alpha}}{\Gamma(\alpha)} \times \frac{\Gamma(\alpha)}{(\lambda-s)^{\alpha}} \quad (0.0.18)$$
$$= \left(\frac{\lambda}{\lambda-s}\right)^{\alpha} \quad (0.0.19)$$

□

Theorem 0.4. The MGF (if it exists) uniquely determines the distribution. That is, if two random variables have the same MGF, then they must have the same distribution.

Lemma 0.5. If X_i for $i = 1, 2, \dots, n$ are independent and identically distributed random variables (i.i.ds), then we have the following property,

$$Var\left(a \sum_{i=1}^{n} g(X_i)\right) = a^2 \sum_{i=1}^{n} Var(g(X_i)) \quad (0.0.20)$$

Lemma 0.6. One of the property of gamma distributions states that,

$$\int_{0}^{\infty} \lambda^{\alpha} t^{\alpha-1} e^{-\lambda t}\, dt \quad (0.0.21)$$
$$= \frac{1}{\Gamma(\alpha)} \quad (0.0.22)$$

Lemma 0.7. The expected value of $\frac{1}{\bar{X}}$ with distribution of X from equation (0.0.1) is

$$E\left[\frac{1}{\bar{X}}\right] = \frac{n\lambda}{3n-1} \quad (0.0.23)$$

Proof. Let r.v. T be,

$$T = \sum_{i=1}^{n} X_i \sim \Gamma(3n, \lambda) \quad (0.0.24)$$

with pdf,

$$f_T(t) = \frac{\lambda^{3n} t^{3n-1} e^{-\lambda t}}{\Gamma(3n)}, t > 0 \quad (0.0.25)$$

Using, $\frac{1}{\bar{X}} = \frac{n}{T}$

$$E\left[\frac{1}{\bar{X}}\right] = \int_{0}^{\infty} \frac{n}{t} \frac{1}{\Gamma(3n)} \lambda^{3n} t^{3n-1} e^{-\lambda t}\, dt \quad (0.0.26)$$
$$= \frac{n\lambda}{(3n-1)} \int_{0}^{\infty} \frac{1}{\Gamma(3n-1)} \lambda^{3n-1} t^{3n-2} e^{-\lambda t}\, dt \quad (0.0.27)$$

Using lemma 0.6 we have,

$$\int_{0}^{\infty} \frac{1}{\Gamma(3n-1)} \lambda^{3n-1} t^{3n-2} e^{-\lambda t}\, dt = 1 \quad (0.0.28)$$

Using equations (0.0.28) and (0.0.27) we have,

$$E\left[\frac{1}{\bar{X}}\right] = \frac{n\lambda}{3n-1} \quad (0.0.29)$$

□

Lemma 0.8. The variance of $\frac{1}{\bar{X}}$ with distribution of X from equation (0.0.1) is

$$Var\left(\frac{1}{\bar{X}}\right) = \frac{n^2\lambda^2}{(3n-1)^2(3n-2)} \quad (0.0.30)$$

Proof. Similar to lemma 0.7 we have, rv T

$$Var\left(\frac{1}{\bar{X}}\right) = \left(E\left[\frac{1}{\bar{X}}^{2}\right] - E\left[\frac{1}{\bar{X}}\right]^{2}\right) \quad (0.0.31)$$

To calculate, $E\left[\frac{1}{\bar{X}}^{2}\right]$ we use,

$$\frac{1}{\bar{X}}^{2} = \frac{n^2}{t^2} \quad (0.0.32)$$

$$E\left[\frac{1}{\bar{X}}^{2}\right] = \int_{0}^{\infty} \frac{n^2}{t^2} \frac{1}{\Gamma(3n)} \lambda^{3n} t^{3n-1} e^{-\lambda t}\, dt \quad (0.0.33)$$
$$= \frac{n^2\lambda^2}{(3n-1)(3n-2)} \times (1) \quad (0.0.34)$$

As from lemma 0.6 we have,

$$\int_0^\infty \frac{1}{\Gamma(3n-2)}\lambda^{3n-2}t^{3n-3}e^{-\lambda t}\,dt = 1 \tag{0.0.35}$$

Therefore,

$$Var\left(\frac{1}{\bar{X}}\right) = \left(\frac{n^2\lambda^2}{(3n-1)(3n-2)} - \frac{n^2\lambda^2}{(3n-1)^2}\right) \tag{0.0.36}$$

$$= \frac{n^2\lambda^2}{3n-1}\left(\frac{1}{3n-2} - \frac{1}{3n-1}\right) \tag{0.0.37}$$

$$= \frac{n^2\lambda^2}{(3n-1)^2(3n-2)} \tag{0.0.38}$$

□

Lemma 0.9. The expected value of $\frac{1}{X_i}$ with distribution of X from equation (0.0.1) is

$$E\left[\frac{1}{X_i}\right] = \frac{\lambda}{2} \tag{0.0.39}$$

Proof.

$$E\left[\frac{1}{X_i}\right] = \int_{-\infty}^\infty \frac{1}{x}f(x)\,dx \tag{0.0.40}$$

$$= \int_0^\infty \frac{1}{x}\frac{1}{2}\lambda^3x^2e^{-\lambda x}\,dx \tag{0.0.41}$$

$$= \lambda^3\int_0^\infty xe^{-\lambda x}\,dx \tag{0.0.42}$$

$$= \frac{\lambda}{2} \tag{0.0.43}$$

□

Lemma 0.10. The variance of $\frac{1}{\bar{X}}$ with distribution of X from equation (0.0.1) is

$$Var\left(\frac{1}{X_i}\right) = \frac{\lambda^2}{4} \tag{0.0.44}$$

Proof.

$$Var\left(\frac{1}{X_i}\right) = \left(E\left[\frac{1}{X_i}^2\right] - E\left[\frac{1}{X_i}\right]^2\right) \tag{0.0.45}$$

$$= \left(\int_{-\infty}^\infty \frac{1}{x^2}f(x)\,dx - \left(\frac{\lambda}{2}\right)^2\right) \tag{0.0.46}$$

$$= \left(\int_0^\infty \frac{1}{x^2}\frac{1}{2}\lambda^3x^2e^{-\lambda x}\,dx - \frac{\lambda^2}{4}\right) \tag{0.0.47}$$

$$= \left(\frac{\lambda^3}{2}\int_0^\infty e^{-\lambda x}\,dx - \frac{\lambda^2}{4}\right) \tag{0.0.48}$$

$$= \left(\frac{\lambda^2}{2} - \frac{\lambda^2}{4}\right) \tag{0.0.49}$$

$$= \frac{\lambda^2}{4} \tag{0.0.50}$$

□

Solving all options :

1) Now here we have our estimator Θ and estimand θ as,

$$\Theta = \frac{2}{n}\sum_{i=1}^n \frac{1}{X_i} \text{ and } \theta = \lambda \tag{0.0.51}$$

The expectation value of the estimator using lemma 0.9 is given by,

$$E[\Theta] = E\left[\frac{2}{n}\sum_{i=1}^n \frac{1}{X_i}\right] \tag{0.0.52}$$

$$= \frac{2}{n}\sum_{i=1}^n E\left[\frac{1}{X_i}\right] \tag{0.0.53}$$

$$= \frac{2n}{n}\times\frac{\lambda}{2} \tag{0.0.54}$$

$$= \lambda \tag{0.0.55}$$

So the bias of estimator is given by,

$$B(\Theta) = E[\Theta] - \theta \tag{0.0.56}$$

$$= \lambda - \lambda = 0 \tag{0.0.57}$$

Therefore $\frac{2}{n}\sum_{i=1}^n \frac{1}{X_i}$ is an unbiased estimator of λ
Option 1 is correct.

2) Now in this option we have our estimator Θ and quantity to be estimated θ as,

$$\Theta = \frac{3n}{\sum_{i=1}^n X_i} \text{ and } \theta = \lambda \tag{0.0.58}$$

We have that sample mean, $\bar{X}$,

$$\bar{X} = \frac{X_1 + X_2 + \cdots + X_n}{n} \tag{0.0.59}$$

$$= \frac{\sum_{i=1}^n X_i}{n} \tag{0.0.60}$$

Therefore estimator,

$$\Theta = \frac{3n}{n\bar{X}} = \frac{3}{\bar{X}} \tag{0.0.61}$$

To check the distribution is gamma or not we calculate, MGF , i.e., $M_X(s)$ for random variable, X

$$M_X(s) = E[e^{sX}] \tag{0.0.62}$$
$$= \int_{-\infty}^{\infty} e^{sx} f(x)\, dx \tag{0.0.63}$$

Using theorem 0.3 and lemma 0.2 we have,

$$M_X(s) = \frac{\lambda^3}{2} \times \frac{\Gamma(3)}{(\lambda - s)^3} \tag{0.0.64}$$
$$= \left(\frac{\lambda}{\lambda - s}\right)^3, \text{ as } \Gamma(3) = 2! \tag{0.0.65}$$
$$= \left(\frac{\lambda}{\lambda - s}\right)^\alpha, \text{ for } \alpha = 3 \tag{0.0.66}$$

Therefore the MGF of X is same as the MGF of gamma distribution.
So from the theorem 0.4, the distribution of X is gamma distribution, i.e. $X \sim \Gamma(\alpha, \lambda)$ with PDF,

$$f_X(x) = \begin{cases} \dfrac{\lambda^\alpha x^{\alpha-1} e^{-\lambda x}}{\Gamma(\alpha)} & x > 0 \\ 0 & otherwise \end{cases} \tag{0.0.67}$$

where $\alpha = 3$.
So we have from lemma 0.7 and equation (0.0.61),

$$E[\Theta] = E\left[\frac{3}{\bar{X}}\right] \tag{0.0.68}$$
$$= 3 \times E\left[\frac{1}{\bar{X}}\right] \tag{0.0.69}$$
$$= \frac{3n\lambda}{3n-1} \tag{0.0.70}$$

So we calculate bias as follows,

$$B(\Theta) = E[\Theta] - \lambda \tag{0.0.71}$$
$$= \frac{3n\lambda}{3n-1} - \lambda \tag{0.0.72}$$
$$= \frac{\lambda}{3n-1} \neq 0 \tag{0.0.73}$$

Therefore $\dfrac{3n}{\sum_{i=1}^{n} X_i}$ is not an unbiased estimator of λ
Option 2 is not correct.

3) Now here we have our estimator Θ and quantity to be estimated θ as,

$$\Theta = \frac{2}{n} \sum_{i=1}^{n} \frac{1}{X_i} \text{ and } \theta = \lambda \tag{0.0.74}$$

Now the variance of Θ is calculated using lemma 0.8 as,

$$Var(\Theta) = Var\left(\frac{2}{n} \sum_{i=1}^{n} \frac{1}{X_i}\right) \tag{0.0.75}$$
$$= \frac{4}{n^2} \sum_{i=1}^{n} Var\left(\frac{1}{X_i}\right) \tag{0.0.76}$$

Now using lemma 0.10we have,

$$Var(\Theta) = \frac{4n}{n^2} \times \frac{\lambda^2}{4} \tag{0.0.77}$$
$$= \frac{\lambda^2}{n} \tag{0.0.78}$$

The bias of Θ from option 1 is given as

$$B(\Theta) = 0 \tag{0.0.79}$$

So we have,

$$MSE(\Theta_n) = Var(\Theta) + B(\Theta)^2 \tag{0.0.80}$$
$$= \frac{\lambda^2}{n} \tag{0.0.81}$$

Now,

$$\lim_{n\to\infty} MSE(\Theta_n) = \lim_{n\to\infty} \frac{\lambda^2}{n} \tag{0.0.82}$$
$$= 0 \tag{0.0.83}$$

Therefore, $\dfrac{2}{n} \sum_{i=1}^{n} \dfrac{1}{X_i}$ is a consistent estimator of λ. Option 3 is correct.

4) Now in this option we have our estimator Θ and quantity to be estimated θ as,

$$\Theta = \frac{3n}{\sum_{i=1}^{n} X_i} \text{ and } \theta = \lambda \tag{0.0.84}$$

The variance of estimator similar to option 2 and using lemma 0.8 is,

$$Var(\Theta) = Var\left(\frac{3}{\bar{X}}\right) \tag{0.0.85}$$
$$= 9 \times Var\left(\frac{1}{\bar{X}}\right) \tag{0.0.86}$$
$$= \frac{9n^2\lambda^2}{(3n-1)^2(3n-2)} \tag{0.0.87}$$

The bias calculated from option 2 is

$$B(\Theta) = \frac{\lambda}{3n-1} \tag{0.0.88}$$

So we have,

$$MSE(\Theta) = Var(\Theta) + B(\Theta)^2 \tag{0.0.89}$$

$$= \frac{9n^2\lambda^2}{(3n-1)^2(3n-2)} + \frac{\lambda^2}{(3n-1)^2} \tag{0.0.90}$$

Finally,

$$\lim_{n\to\infty} MSE(\Theta_n) \tag{0.0.91}$$

$$= \lim_{n\to\infty} \frac{9n^2\lambda^2}{(3n-1)^2(3n-2)} + \frac{\lambda^2}{(3n-1)^2} \tag{0.0.92}$$

$$\tag{0.0.93}$$

Now in first limit multiply and divide by n^2 and $n \to \infty$ we get,

$$\lim_{n\to\infty} MSE(\Theta_n) = 0 \tag{0.0.94}$$

Therefore, $\frac{3n}{\sum_{i=1}^{n} X_i}$ is a consistent estimator of λ.
Option 4 is correct.

Therefore option 1, option 3 and option 4 are correct.