

# AI1103-Assignment 8

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Download all latex-tikz codes from

<https://github.com/ayushjha2612/AI11003/tree/main/Assignment8>

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Let  $X_1, \dots, X_n$  be independent and identically distributed random variables with probability density function

$$f(x) = \frac{1}{2} \lambda^3 x^2 e^{-\lambda x}; x > 0; \lambda > 0$$

Then which of the following statements are true?

- 1)  $\frac{2}{n} \sum_{i=1}^n \frac{1}{X_i}$  is an unbiased estimator of  $\lambda$
- 2)  $\frac{3n}{\sum_{i=1}^n X_i}$  is an unbiased estimator of  $\lambda$
- 3)  $\frac{2}{n} \sum_{i=1}^n \frac{1}{X_i}$  is a consistent estimator of  $\lambda$
- 4)  $\frac{3n}{\sum_{i=1}^n X_i}$  is a consistent estimator of  $\lambda$

SOLUTION

**Solving all options :**

1) **Definition Unbiased Estimator :**

Let  $\Theta = h(X_1, X_2, \dots, X_n)$  be a point estimator for a parameter  $\theta$ . We say that  $\Theta$  is an unbiased estimator of  $\theta$  if

$$B(\Theta) = 0, \text{ for all possible values of } \theta. \quad (0.0.1)$$

Where the bias of the estimator  $\Theta$  is defined by

$$B(\Theta) = E[\Theta] - \theta \quad (0.0.2)$$

Now here we have our estimator  $\Theta$  and quantity to be estimated  $\theta$  as,

$$\Theta = \frac{2}{n} \sum_{i=1}^n \frac{1}{X_i} \text{ and } \theta = \lambda \quad (0.0.3)$$

The expectation value of the estimator is given by,

$$E[\Theta] = E\left[\frac{2}{n} \sum_{i=1}^n \frac{1}{X_i}\right] \quad (0.0.4)$$

$$= \frac{2}{n} \sum_{i=1}^n E\left[\frac{1}{X_i}\right] \quad (0.0.5)$$

$$= \frac{2}{n} \sum_{i=1}^n \int_{-\infty}^{\infty} \frac{1}{x} f(x) dx \quad (0.0.6)$$

$$= \frac{2n}{n} \int_0^{\infty} \frac{1}{x} \frac{1}{2} \lambda^3 x^2 e^{-\lambda x} dx \quad (0.0.7)$$

$$= \lambda^3 \int_0^{\infty} x e^{-\lambda x} dx \quad (0.0.8)$$

$$= \lambda^3 \left( \frac{-x e^{-\lambda x}}{\lambda} - \frac{x e^{-\lambda x}}{\lambda^2} \right) \Big|_0^{\infty} \quad (0.0.9)$$

$$= \lambda^3 \left( - \left[ 0 - \frac{1}{\lambda^2} \right] \right) \quad (0.0.10)$$

$$= \lambda \quad (0.0.11)$$

So the bias of estimator is given by,

$$B(\Theta) = E[\Theta] - \theta \quad (0.0.12)$$

$$= \lambda - \lambda = 0 \quad (0.0.13)$$

Therefore  $\frac{2}{n} \sum_{i=1}^n \frac{1}{X_i}$  is an unbiased estimator of  $\lambda$

Option 1 is correct.

2) Now in this option we have our estimator  $\Theta$  and quantity to be estimated  $\theta$  as,

$$\Theta = \frac{3n}{\sum_{i=1}^n X_i} \text{ and } \theta = \lambda \quad (0.0.14)$$

The expectation value of the estimator is given

by,

$$E[\Theta] = E\left[\frac{3n}{\sum_{i=1}^n X_i}\right] \quad (0.0.15)$$

$$= \frac{3n}{\sum_{i=1}^n} E\left[\frac{1}{X_i}\right] \quad (0.0.16)$$

The value of  $E\left[\frac{1}{X_i}\right]$  can be obtained from (0.0.7) as

$$E\left[\frac{1}{X_i}\right] = \frac{\lambda}{2} \quad (0.0.17)$$

So we have,

$$E[\Theta] = \frac{3n}{\sum_{i=1}^n} \frac{\lambda}{2} \quad (0.0.18)$$

$$= \frac{3n}{n} \frac{\lambda}{2} \quad (0.0.19)$$

$$= \frac{3\lambda}{2} \quad (0.0.20)$$

So the bias of estimator is given by,

$$B(\Theta) = E[\Theta] - \theta \quad (0.0.21)$$

$$= \frac{3\lambda}{2} - \lambda \quad (0.0.22)$$

$$= \frac{\lambda}{2} \neq 0 \quad (0.0.23)$$

Therefore  $\frac{3n}{\sum_{i=1}^n X_i}$  is not an unbiased estimator of  $\lambda$   
Option 2 is not correct.

### 3) Definition Consistent Estimator :

Let  $\Theta_1, \Theta_2, \dots, \Theta_n, \dots$ , be a sequence of point estimators of  $\theta$ . We say that  $\Theta_n$  is a **consistent** estimator of  $\theta$ , if

$$\lim_{n \rightarrow \infty} \Pr(|\Theta_n - \theta| \geq \epsilon) = 0, \text{ for all } \epsilon > 0. \quad (0.0.24)$$

We can equivalently say that

$$\lim_{n \rightarrow \infty} MSE(\Theta_n) = 0, \quad (0.0.25)$$

then  $\Theta_n$  is a consistent estimator of  $\theta$ .

The mean squared error (MSE) of a point estimator  $\Theta$ , shown by  $MSE(\Theta)$ , is defined as

$$MSE(\Theta) = E[(\Theta - \theta)^2] \quad (0.0.26)$$

$$= Var(\Theta) + B(\Theta)^2 \quad (0.0.27)$$

where  $B(\Theta)$  is the bias of  $\Theta$ .

Now here we have our estimator  $\Theta$  and quantity to be estimated  $\theta$  as,

$$\Theta = \frac{2}{n} \sum_{i=1}^n \frac{1}{X_i} \text{ and } \theta = \lambda \quad (0.0.28)$$

Now the variance of  $\Theta$  is calculated as

$$Var(\Theta) = Var\left(\frac{2}{n} \sum_{i=1}^n \frac{1}{X_i}\right) \quad (0.0.29)$$

$$= \frac{4}{n^2} \sum_{i=1}^n Var\left(\frac{1}{X_i}\right) \quad (0.0.30)$$

$$= \frac{4n}{n^2} \left( E\left[\frac{1}{X_i}^2\right] - E\left[\frac{1}{X_i}\right]^2 \right) \quad (0.0.31)$$

$$= \frac{4}{n} \left( \int_{-\infty}^{\infty} \frac{1}{x^2} f(x) dx - \left(\frac{\lambda}{2}\right)^2 \right) \quad (0.0.32)$$

$$= \frac{4}{n} \left( \int_0^{\infty} \frac{1}{x^2} \frac{1}{2} \lambda^3 x^2 e^{-\lambda x} dx - \frac{\lambda^2}{4} \right) \quad (0.0.33)$$

$$= \frac{4}{n} \left( \frac{\lambda^3}{2} \int_0^{\infty} e^{-\lambda x} dx - \frac{\lambda^2}{4} \right) \quad (0.0.34)$$

$$= \frac{4}{n} \left( \frac{\lambda^2}{2} - \frac{\lambda^2}{4} \right) \quad (0.0.35)$$

$$= \frac{\lambda^2}{n} \quad (0.0.36)$$

The bias of  $\Theta$  from option 1 is given as

$$B(\Theta) = 0 \quad (0.0.37)$$

So we have,

$$MSE(\Theta_n) = Var(\Theta) + B(\Theta)^2 \quad (0.0.38)$$

$$= \frac{\lambda^2}{n} \quad (0.0.39)$$

Now,

$$\lim_{n \rightarrow \infty} MSE(\Theta_n) = \lim_{n \rightarrow \infty} \frac{\lambda^2}{n} \quad (0.0.40)$$

$$= 0 \quad (0.0.41)$$

Therefore,  $\frac{2}{n} \sum_{i=1}^n \frac{1}{X_i}$  is a consistent estimator of  $\lambda$ . Option 3 is correct.

4) Now in this option we have our estimator  $\Theta$

and quantity to be estimated  $\theta$  as,

$$\Theta = \frac{3n}{\sum_{i=1}^n X_i} \text{ and } \theta = \lambda \quad (0.0.42)$$

Now the variance of  $\Theta$  is calculated as

$$Var(\Theta) = Var\left(\frac{3n}{\sum_{i=1}^n X_i}\right) \quad (0.0.43)$$

$$= \frac{9n^2}{\sum_{i=1}^n} Var\left(\frac{1}{X_i}\right) \quad (0.0.44)$$

$$(0.0.45)$$

Now the value of  $Var\left(\frac{1}{X_i}\right)$  from (0.0.31) is substituted, we have

$$Var(\Theta) = \frac{9n^2}{\sum_{i=1}^n} \frac{\lambda^2}{4} \quad (0.0.46)$$

$$= \frac{9n^2 \lambda^2}{4n} = \frac{9n \lambda^2}{4} \quad (0.0.47)$$

The bias of  $\Theta$  from option 2 is given as

$$B(\Theta) = \frac{\lambda}{2} \quad (0.0.48)$$

So we have,

$$MSE(\Theta_n) = Var(\Theta) + B(\Theta)^2 \quad (0.0.49)$$

$$= \frac{9n \lambda^2}{4} + \left(\frac{\lambda}{2}\right)^2 \quad (0.0.50)$$

$$= \frac{\lambda^2}{4}(9n + 1) \quad (0.0.51)$$

Now,

$$\lim_{n \rightarrow \infty} MSE(\Theta_n) = \lim_{n \rightarrow \infty} \frac{\lambda^2}{4}(9n + 1) \quad (0.0.52)$$

$$(0.0.53)$$

Clearly as  $n$  grows larger  $9n + 1$  also grows larger, so

$$\lim_{n \rightarrow \infty} MSE(\Theta_n) \neq 0 \quad (0.0.54)$$

Therefore,  $\frac{3n}{\sum_{i=1}^n X_i}$  is not a consistent estimator of  $\lambda$ .

Option 4 is not correct.

**Therefore option 1 and option 3 are correct.**