

AI1103-Assignment 9

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Download all python codes from

<https://github.com/ayushjha2612/AI11003/tree/main/Assignment9/Codes>

and latex-tikz codes from

<https://github.com/ayushjha2612/AI11003/tree/main/Assignment9>

CSIR UGC NET EXAM (JUNE 2017) Q. 103

Let $c \in \mathbb{R}$ be a constant. Let X, Y be random variables with joint probability density function

$$f(x, y) = \begin{cases} cxy & , \text{ if } 0 < x < y < 1 \\ 0 & , \text{ otherwise} \end{cases}$$

Which of the following statements are correct ?

- 1) $c = \frac{1}{8}$
- 2) $c = 8$
- 3) X and Y are independent
- 4) $\Pr(X = Y) = 0$

ANSWER

Option (2) $c = 8$ and option (4) $\Pr(X = Y) = 0$.

SOLUTION

Solving all options :

- 1)
- 2) X and Y are two random variables with joint pdf

$$f(x, y) = \begin{cases} cxy & , \text{ if } 0 < x < y < 1 \\ 0 & , \text{ otherwise} \end{cases} \quad (0.0.1)$$

The marginal probability density functions are as follows :

$$f_X(x) = \int_{-\infty}^{\infty} f(x, y) dy \quad (0.0.2)$$

$$f_Y(y) = \int_{-\infty}^{\infty} f(x, y) dx \quad (0.0.3)$$

Calculating $f_X(x)$

$$f_X(x) = \int_{-\infty}^{\infty} f(x, y) dy \quad (0.0.4)$$

$$= \int_x^1 cxy dy \quad (0.0.5)$$

$$= cx \left(\frac{y^2}{2} \right) \Big|_x^1 \quad (0.0.6)$$

$$= cx \left(\frac{1 - x^2}{2} \right) \quad (0.0.7)$$

$$f_X(x) = \begin{cases} cx \left(\frac{1 - x^2}{2} \right) & , \text{ if } 0 < x < 1 \\ 0 & , \text{ otherwise} \end{cases} \quad (0.0.8)$$

Calculating $f_Y(y)$

$$f_Y(y) = \int_{-\infty}^{\infty} f(x, y) dx \quad (0.0.9)$$

$$= \int_0^y cxy dx \quad (0.0.10)$$

$$= cy \left(\frac{x^2}{2} \right) \Big|_0^y \quad (0.0.11)$$

$$= \frac{cy^3}{2} \quad (0.0.12)$$

$$f_Y(y) = \begin{cases} \frac{cy^3}{2} & , \text{ if } 0 < y < 1 \\ 0 & , \text{ otherwise} \end{cases} \quad (0.0.13)$$

Now by using property of pdf and equation 0.0.13 we have,

$$\int_{-\infty}^{\infty} f_Y(y) dy = 1 \quad (0.0.14)$$

$$\int_0^1 c \frac{y^3}{2} dy = 1 \quad (0.0.15)$$

$$\frac{c}{8} = 1 \quad (0.0.16)$$

$$c = 8 \quad (0.0.17)$$

Therefore option (2) is correct.

3) The pdf of X is

$$f_X(x) = \begin{cases} 4x(1-x^2) & , \text{ if } 0 < x < 1 \\ 0 & , \text{ otherwise} \end{cases} \quad (0.0.18)$$

The pdf of Y is

$$f_Y(y) = \begin{cases} 4y^3 & , \text{ if } 0 < y < 1 \\ 0 & , \text{ otherwise} \end{cases} \quad (0.0.19)$$

To check whether X and Y are independent, we calculate $f_X(x) \times f_Y(y)$. From 0.0.18 and 0.0.19

$$f_X(x) \times f_Y(y) = \begin{cases} 16xy^3(1-x^2) & , \text{ if } 0 < x, y < 1 \\ 0 & , \text{ otherwise} \end{cases} \quad (0.0.20)$$

$$\neq f(x, y) \quad (0.0.21)$$

Since $f(x, y)$ and $f_X(x) \times f_Y(y)$ are different, random variables X and Y are not independent. Therefore option (3) is not correct.

4) The joint CDF of X and y is defined as

$$F_{XY}(x, y) = \Pr(X \leq x, Y \leq y) \quad (0.0.22)$$

$$= \int_{-\infty}^y \int_{-\infty}^x f_{XY}(u, v) du dv \quad (0.0.23)$$

$$= \int_x^y \int_0^x 4uv du dv \quad (0.0.24)$$

$$= \int_x^y 2x^2v dv \quad (0.0.25)$$

$$= x^2v^2 \Big|_x^y \quad (0.0.26)$$

$$= x^2(y^2 - x^2) \quad (0.0.27)$$

Now Random variable,

$$X = Y \iff x = y \quad (0.0.28)$$

Therefore,

$$F_{XY}(x, x) = \Pr(X \leq x, Y \leq x) = x^2(x^2 - x^2) \quad (0.0.29)$$

$$= 0 \quad (0.0.30)$$

This implies,

$$\Pr(X \leq x, Y \leq x) = 0 \quad (0.0.31)$$

$$\iff \Pr(Y \leq X) = 0 \quad (0.0.32)$$

$$\iff \Pr(X = Y) = 0 \quad (0.0.33)$$

Therefore option (2) and (4) are correct.

The marginal PDF of X and Y are shown at figure 4 and figure 4.

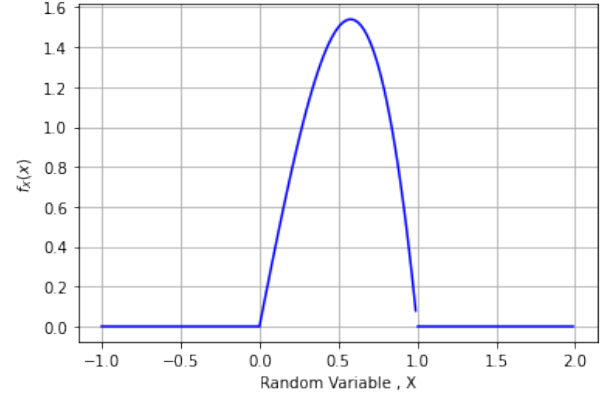


Fig. 4: The marginal PDF of X

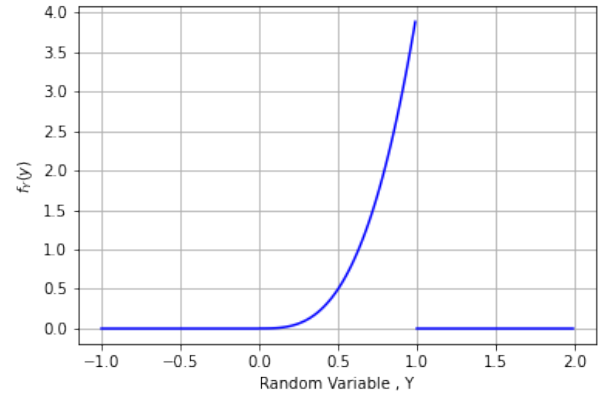


Fig. 4: The marginal PDF of Y