AI1103-Assignment 3

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Download all python codes from

```
https://github.com/ayushjha2612/AI11003/tree/main
    /Assignment3/Codes
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and latex-tikz codes from

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https://github.com/ayushjha2612/AI11003/tree/main
    /Assignment3
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GATE Problem 34

Let X and Y be two statistically independent random variables uniformly distributed in the range (-1, 1) and (-2, 1) respectively. Let $Z = X + Y$, then the probability that $[Z \leq -2]$ is

(A) zero (B) $\frac{1}{6}$ (C)$\frac{1}{3}$ (D)$\frac{1}{12}$

Answer

Option (D) $\frac{1}{12}$

Solution

X and Y are two independent random variables. Let

$$p_X(x) = \Pr(X = x) \tag{0.0.1}$$
$$p_Y(y) = \Pr(Y = y) \tag{0.0.2}$$
$$p_Z(z) = \Pr(Z = z) \tag{0.0.3}$$

be the probability densities of random variables X ,Y and Z.
X lies in range(-1,1), therefore,

$$\int_{-1}^{1} p_X(x)\ dx = 1 \tag{0.0.4}$$
$$2 \times p_X(x) = 1 \tag{0.0.5}$$
$$p_X(x) = 1/2 \tag{0.0.6}$$

Similarly for Y we have,

$$\int_{-2}^{1} p_Y(y)\ dy = 1 \tag{0.0.7}$$
$$3 \times p_Y(y) = 1 \tag{0.0.8}$$
$$p_Y(y) = 1/3 \tag{0.0.9}$$

The density for X is

$$p_X(x) = \begin{cases} \frac{1}{2} & -1 \leq x \leq 1 \\ 0 & otherwise \end{cases} \tag{0.0.10}$$

We have ,

$$Z = X + Y \iff z = x + y \iff x = z - y \tag{0.0.11}$$

The density of X can also be represented as,

$$p_X(z-y) = \begin{cases} \frac{1}{2} & -1 \leq z-y \leq 1 \\ 0 & otherwise \end{cases} \tag{0.0.12}$$

and the density of Y is,

$$p_Y(y) = \begin{cases} \frac{1}{3} & -2 \leq y \leq 1 \\ 0 & otherwise \end{cases} \tag{0.0.13}$$

The density of Z i.e. $Z = X + Y$ is given by the convolution of the densities of X and Y

$$p_Z(z) = \int_{-\infty}^{\infty} p_X(z-y) p_Y(y)\, dy \tag{0.0.14}$$

From 0.0.12 and 0.0.13 we have,
The integrand is $\frac{1}{6}$ when,

$$2 \leq y \leq 1 \tag{0.0.15}$$
$$-1 \leq z - y \leq 1 \tag{0.0.16}$$
$$z - 1 \leq y \leq z + 1 \tag{0.0.17}$$

and zero, otherwise.
Now when $-3 \leq z \leq -2$ them we have,

$$p_Z(z) = \int_{-2}^{z+1} \frac{1}{6}\, dy \tag{0.0.18}$$
$$= \frac{1}{6} \times (z + 1 - (-2)) \tag{0.0.19}$$
$$= \frac{1}{6}(z + 3) \tag{0.0.20}$$

For $-2 < z \le -1$,

$$p_Z(z) = \int_{-2}^{z+1} \frac{1}{6}\,dy \tag{0.0.21}$$
$$= \frac{1}{6} \times (z + 1 - (-2)) \tag{0.0.22}$$
$$= \frac{1}{6}(z + 3) \tag{0.0.23}$$

For $-1 < z \le 0$,

$$p_Z(z) = \int_{z-1}^{z+1} \frac{1}{6}\,dy \tag{0.0.24}$$
$$= \frac{1}{6} \times (z + 1 - (z - 1)) \tag{0.0.25}$$
$$= \frac{1}{3} \tag{0.0.26}$$

For $0 < z \le 2$,

$$p_Z(z) = \int_{z-1}^{1} \frac{1}{6}\,dy \tag{0.0.27}$$
$$= \frac{1}{6} \times (1 - (z - 1)) \tag{0.0.28}$$
$$= \frac{1}{6}(2 - z) \tag{0.0.29}$$

Therefore the density of Z is given by

$$p_Z(z) = \begin{cases} \frac{1}{6}(z+3) & -3 \le z \le -2 \\ \frac{1}{6}(z+3) & -2 < z \le -1 \\ \frac{1}{3} & -1 < z \le 0 \\ \frac{1}{6}(2-z) & 0 < z \le 2 \\ 0 & otherwise \end{cases} \tag{0.0.30}$$

The CDF of Z is defined as,

$$F_Z(z) = \Pr(Z \le z) \tag{0.0.31}$$

Now for $z \le -1$,

$$\Pr(Z \le z) = \int_{-\infty}^{z} p_Z(z)\,dz \tag{0.0.32}$$
$$= \int_{-3}^{z} \frac{1}{6}(z + 3)\,dz \tag{0.0.33}$$
$$= \frac{1}{6}\left(\frac{z^2}{2} + 3z\right)\Bigg|_{-3}^{z} \tag{0.0.34}$$
$$= \frac{1}{6} \times \left(\left(\frac{z^2}{2} + 3z\right) - \left(\frac{9}{2} - 9\right)\right) \tag{0.0.35}$$
$$= \frac{z^2 + 6z + 9}{12} \tag{0.0.36}$$

Similarly for $z \le 0$,

$$\Pr(Z \le z) = \int_{-\infty}^{z} p_Z(z)\,dz \tag{0.0.37}$$
$$= \frac{1}{3} + \int_{-1}^{z} \frac{1}{3}\,dz \tag{0.0.38}$$
$$= \frac{z + 2}{3} \tag{0.0.39}$$

finally for $z \le 2$,

$$\Pr(Z \le z) = \int_{-\infty}^{z} p_Z(z)\,dz \tag{0.0.40}$$
$$= \frac{2}{3} + \int_{0}^{z} \frac{1}{6}(2 - z)\,dz \tag{0.0.41}$$
$$= \frac{2}{3} + \frac{4z - z^2}{12} \tag{0.0.42}$$
$$= \frac{8 + 4z - z^2}{12} \tag{0.0.43}$$

The CDF is as below,

$$F_Z(z) = \begin{cases} 0 & z < 3 \\ \dfrac{z^2 + 6z + 9}{12} & z \le -1 \\ \dfrac{z + 2}{3} & z \le 0 \\ \dfrac{8 + 4z - z^2}{12} & z \le 2 \\ 1 & z > 2 \end{cases} \tag{0.0.44}$$

So

$$\Pr(Z \le -2) = F_Z(2) \tag{0.0.45}$$
$$= \frac{1}{12} \tag{0.0.46}$$

i.e. option (D).
The plot for probability of $Z \le -2$ can be observed at 0.

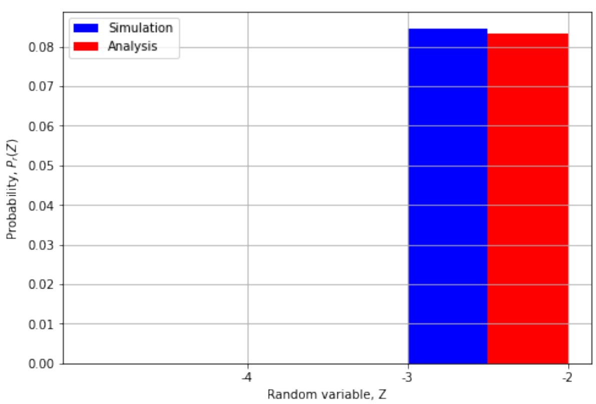

Fig. 0: Theory VS Simulation plot