AI1103-Assignment 9

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Download all python codes from

```
https://github.com/ayushjha2612/AI11003/tree/main
/Assignment9/Codes
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and latex-tikz codes from

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https://github.com/ayushjha2612/AI11003/tree/main
/Assignment9
```

CSIR UGC NET EXAM (June 2017) Q. 103

Let $c \in \mathbb{R}$ be a constant. Let X, Y be random variables with joint probability density function

$$f(x,y) = \begin{cases} cxy & \text{, if } 0 < x < y < 1 \\ 0 & \text{, otherwise} \end{cases}$$

Which of the following statements are correct ?

1) $c = \frac{1}{8}$
2) $c = 8$
3) X and Y are independent
4) $\Pr(X = Y) = 0$

Answer

Option (2) $c = 8$ and option (4) $\Pr(X = Y) = 0$.

Solution

Solving all options :

1)
2) X and Y are two random variables with joint pdf

$$f(x,y) = \begin{cases} cxy & \text{, if } 0 < x < y < 1 \\ 0 & \text{, otherwise} \end{cases} \tag{0.0.1}$$

The marginal probability density functions are as follows :

$$f_X(x) = \int_{-\infty}^{\infty} f(x,y)\,dy \tag{0.0.2}$$

$$f_Y(y) = \int_{-\infty}^{\infty} f(x,y)\,dx \tag{0.0.3}$$

Calculating $f_X(x)$

$$f_X(x) = \int_{-\infty}^{\infty} f(x,y)\,dy \tag{0.0.4}$$

$$= \int_{x}^{1} cxy\,dy \tag{0.0.5}$$

$$= cx\left(\frac{y^2}{2}\right)\Bigg|_x^1 \tag{0.0.6}$$

$$= cx\left(\frac{1-x^2}{2}\right) \tag{0.0.7}$$

$$f_X(x) = \begin{cases} cx\left(\frac{1-x^2}{2}\right) & \text{, if } 0 < x < 1 \\ 0 & \text{, otherwise} \end{cases} \tag{0.0.8}$$

Calculating $f_Y(y)$

$$f_y(y) = \int_{-\infty}^{\infty} f(x,y)\,dx \tag{0.0.9}$$

$$= \int_{0}^{y} cxy\,dx \tag{0.0.10}$$

$$= cy\left(\frac{x^2}{2}\right)\Bigg|_0^y \tag{0.0.11}$$

$$= \frac{cy^3}{2} \tag{0.0.12}$$

$$f_Y(y) = \begin{cases} \frac{cy^3}{2} & \text{, if } 0 < y < 1 \\ 0 & \text{, otherwise} \end{cases} \tag{0.0.13}$$

Now by using property of pdf and equation 0.0.13 we have,

$$\int_{-\infty}^{\infty} f_Y(y)\,dy = 1 \tag{0.0.14}$$

$$\int_0^1 c\frac{y^3}{2} = 1 \tag{0.0.15}$$

$$\frac{c}{8} = 1 \tag{0.0.16}$$

$$c = 8 \tag{0.0.17}$$

Therefore option (2) is correct.

3) The pdf of X is

$$f_X(x) = \begin{cases} 4x\left(1-x^2\right) & \text{, if } 0 < x < 1 \\ 0 & \text{, otherwise} \end{cases} \quad (0.0.18)$$

The pdf of Y is

$$f_Y(y) = \begin{cases} 4y^3 & \text{, if } 0 < y < 1 \\ 0 & \text{, otherwise} \end{cases} \quad (0.0.19)$$

To check whether X and Y are independent, we calculate $f_X(x) \times f_Y(y)$. From 0.0.18 and 0.0.19

$$f_X(x) \times f_Y(y) = \begin{cases} 16xy^3\left(1-x^2\right) & \text{, if } 0 < x, y < 1 \\ 0 & \text{, otherwise} \end{cases} \quad (0.0.20)$$

$$\neq f(x, y) \quad (0.0.21)$$

Since $f(x, y)$ and $f_X(x) \times f_Y(y)$ are different, random variables X and Y are not independent. Therefore option (3) is not correct.

4) The conditional PDF of Y given $X = x$ is defined as

$$f_{Y|X}(x|y) = \frac{f_{XY}(x, y)}{f_X(x)} \quad (0.0.22)$$

The conditional CDF of Y given $X = x$ is defined as

$$F_{Y|X}(y|x) = \Pr(Y \leq y | X = x) \quad (0.0.23)$$

$$= \int_{-\infty}^{y} f_{Y|X}(x|y)\, dy \quad (0.0.24)$$

$$= \int_{x}^{y} \frac{f_{XY}(x, y)}{f_X(x)}\, dy \quad (0.0.25)$$

$$= \int_{x}^{y} \frac{8xy}{4x(1-x^2)}\, dy \quad (0.0.26)$$

Now Random variable,

$$X = Y \iff x = y \quad (0.0.27)$$

Therefore,

$$F_{Y|X}(y|x) = \Pr(Y \leq y | X = y) \quad (0.0.28)$$

$$= \int_{y}^{y} \frac{8xy}{4x(1-x^2)}\, dy \quad (0.0.29)$$

$$= 0 \quad (0.0.30)$$

This implies,

$$\Pr(Y \leq x | X = x) = 0 \quad (0.0.31)$$

$$\iff \Pr(Y \leq X) = 0 \quad (0.0.32)$$

$$\iff \Pr(X = Y) = 0 \quad (0.0.33)$$

Therefore option (2) and (4) are correct.

The marginal PDF of X and Y are shown at figure 4 and figure 4.

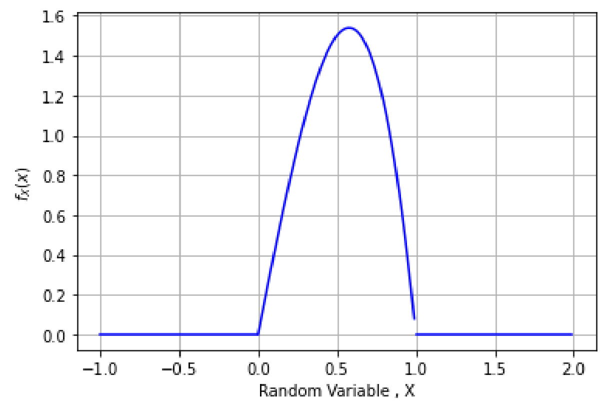

Fig. 4: The marginal PDF of X

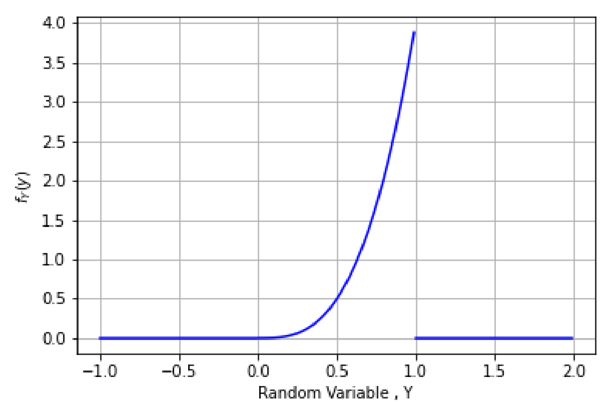

Fig. 4: The marginal PDF of Y