# AI1103-Assignment 9

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Download all python codes from

https://github.com/ayushjha2612/AI11003/tree/main/Assignment9/Codes

and latex-tikz codes from

https://github.com/ayushjha2612/AI11003/tree/main/Assignment9

### CSIR UGC NET EXAM (June 2017) Q. 103

Let $c \in \mathbb{R}$ be a constant. Let $X$, $Y$ be random variables with joint probability density function

$$f(x,y) = \begin{cases} cxy & , \text{if } 0 < x < y < 1 \\ 0 & , \text{otherwise} \end{cases}$$

Which of the following statements are correct ?

1) $c = \frac{1}{8}$
2) $c = 8$
3) $X$ and $Y$ are independent
4) $\Pr(X = Y) = 0$

### Answer

Option (2) $c = 8$ and option (4) $\Pr(X = Y) = 0$.

### Solution

**Solving all options :**

1)
2) X and Y are two random variables with joint pdf

$$f(x,y) = \begin{cases} cxy & , \text{if } 0 < x < y < 1 \\ 0 & , \text{otherwise} \end{cases} \tag{0.0.1}$$

The marginal probability density functions are as follows :

$$f_X(x) = \int_{-\infty}^{\infty} f(x,y)\,dy \tag{0.0.2}$$

$$f_Y(y) = \int_{-\infty}^{\infty} f(x,y)\,dx \tag{0.0.3}$$

Calculating $f_X(x)$

$$f_X(x) = \int_{-\infty}^{\infty} f(x,y)\,dy \tag{0.0.4}$$

$$= \int_{x}^{1} cxy\,dy \tag{0.0.5}$$

$$= cx\left(\frac{y^2}{2}\right)\Bigg|_x^1 \tag{0.0.6}$$

$$= cx\left(\frac{1-x^2}{2}\right) \tag{0.0.7}$$

$$f_X(x) = \begin{cases} cx\left(\frac{1-x^2}{2}\right) & , \text{if } 0 < x < 1 \\ 0 & , \text{otherwise} \end{cases} \tag{0.0.8}$$

Calculating $f_Y(y)$

$$f_y(y) = \int_{-\infty}^{\infty} f(x,y)\,dx \tag{0.0.9}$$

$$= \int_{0}^{y} cxy\,dx \tag{0.0.10}$$

$$= cy\left(\frac{x^2}{2}\right)\Bigg|_0^y \tag{0.0.11}$$

$$= \frac{cy^3}{2} \tag{0.0.12}$$

$$f_Y(y) = \begin{cases} \frac{cy^3}{2} & , \text{if } 0 < y < 1 \\ 0 & , \text{otherwise} \end{cases} \tag{0.0.13}$$

Now by using property of pdf and equation (0.0.13) we have,

$$\int_{-\infty}^{\infty} f_Y(y)\,dy = 1 \tag{0.0.14}$$

$$\int_0^1 c\frac{y^3}{2} = 1 \tag{0.0.15}$$

$$\frac{c}{8} = 1 \tag{0.0.16}$$

$$c = 8 \tag{0.0.17}$$

Therefore option (2) is correct.

3) The pdf of X is

$$f_X(x) = \begin{cases} 4x\left(1 - x^2\right) & \text{, if } 0 < x < 1 \\ 0 & \text{, otherwise} \end{cases} \tag{0.0.18}$$

The pdf of Y is

$$f_Y(y) = \begin{cases} 4y^3 & \text{, if } 0 < y < 1 \\ 0 & \text{, otherwise} \end{cases} \tag{0.0.19}$$

To check whether X and Y are independent, we calculate $f_X(x) \times f_Y(y)$. From (0.0.18) and (0.0.19)

$$f_X(x) \times f_Y(y) = \begin{cases} 16xy^3\left(1 - x^2\right) & \text{, if } 0 < x, y < 1 \\ 0 & \text{, otherwise} \end{cases} \tag{0.0.20}$$

$$\neq f(x, y) \tag{0.0.21}$$

Since $f(x, y)$ and $f_X(x) \times f_Y(y)$ are different, random variables X and Y are not independent. Therefore option (3) is not correct.

4) The marginal PDF of X is given by,

$$f_X(x) = \begin{cases} 4x\left(1 - x^2\right) & \text{, if } 0 < x < 1 \\ 0 & \text{, otherwise} \end{cases} \tag{0.0.22}$$

The marginal CDF of X, $F_X(x)$ is given by,

$$F_X(x) = \int_{-\infty}^{x} f_X(x)\, dx \tag{0.0.23}$$

$$= \int_0^x 4x(1 - x^2)\, dx \tag{0.0.24}$$

$$= \int_0^x 4x - 4x^3\, dx \tag{0.0.25}$$

$$= 2x^2 - 4x^4 \text{ for } 0 < x < 1 \tag{0.0.26}$$

Therefore we have,

$$F_X(x) = \begin{cases} 0 & \text{, if } x \leq 0 \\ 2x^2 - 4x^4 & \text{, if } 0 < x < 1 \\ 1 & \text{, if } x \geq 1 \end{cases} \tag{0.0.27}$$

The plot for marginal CDF is at figure 4.
Now, marginal PDF of Y is

$$f_Y(y) = \begin{cases} 4y^3 & \text{, if } 0 < y < 1 \\ 0 & \text{, otherwise} \end{cases} \tag{0.0.28}$$

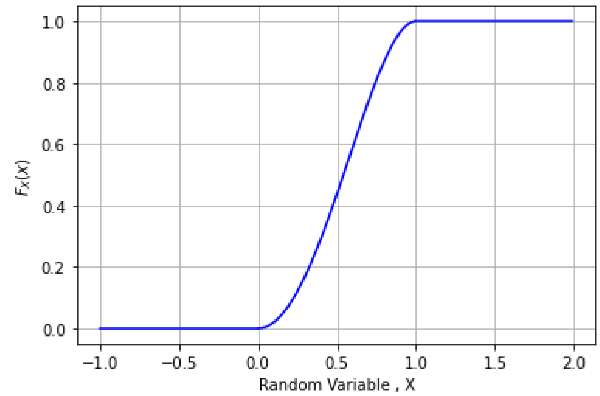


Fig. 4: The marginal CDF of X

Using the marginal CDF from (0.0.27) we have,

$$\Pr(Y - \epsilon < X < Y + \epsilon) = F_X(Y + \epsilon) - F_X(Y - \epsilon) \tag{0.0.29}$$

$$= (2(y + \epsilon)^2 - (y + \epsilon)^4) - (2(y - \epsilon)^2 - (y - \epsilon)^4) \tag{0.0.30}$$

$$= 2(4\epsilon y) - (4\epsilon y)((y - \epsilon)^2 + (y + \epsilon)^2) \tag{0.0.31}$$

$$= 8\epsilon y(1 - y^2 - \epsilon^2) \tag{0.0.32}$$

Taking expectation of RHS w.r.t. Y we have,

$$\text{Let RHS } = g(Y) = 8\epsilon y(1 - y^2 - \epsilon^2) \tag{0.0.33}$$

$$E[g(Y)] = \int_{-\infty}^{\infty} g(y) f_Y(y)\, dy \tag{0.0.34}$$

From equation (0.0.28) we have,

$$E[g(Y)] = \int_0^1 (4y^3)(8\epsilon y)(1 - y^2 - \epsilon^2)\, dy \tag{0.0.35}$$

$$= 32\epsilon \int_0^1 (y^4 - y^6 - y^4\epsilon^2)\, dy \tag{0.0.36}$$

$$= 32\epsilon \left( \frac{y^5}{5} - \frac{y^7}{7} - \frac{y^5\epsilon^2}{5} \right)\Bigg|_0^1 \tag{0.0.37}$$

$$= 32\epsilon \left( \frac{2 - 7\epsilon^2}{35} \right) \tag{0.0.38}$$

Now substituting $\epsilon = 0$ we have,

$$E[g(Y)] = 0 \tag{0.0.39}$$

Therefore,

$$\Pr(X = Y) = 0 \tag{0.0.40}$$

**Therefore option (2) and (4) are correct.**

The marginal PDF of X and Y are shown at figure 4 and figure 4.

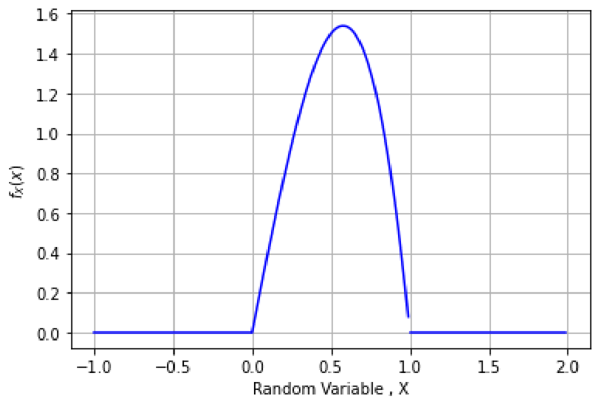


Fig. 4: The marginal PDF of X

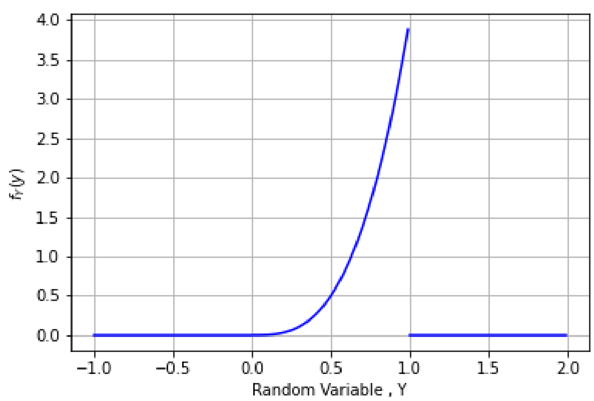


Fig. 4: The marginal PDF of Y