AI1103-Assignment 9

Name : Ayush Jha
Roll Number: CS20BTECH11006

CSIR UGC NET EXAM (June 2017) Q. 103

Let $c \in \mathbb{R}$ be a constant. Let X, Y be random variables with joint probability density function

$$f(x,y) = \begin{cases} cxy & \text{, if } 0 < x < y < 1 \\ 0 & \text{, otherwise} \end{cases}$$

Which of the following statements are correct ?
  1) $c = \frac{1}{8}$
  2) $c = 8$
  3) X and Y are independent
  4) $\Pr(X = Y) = 0$

ANSWER

Option (2) $c = 8$ and option (4) $\Pr(X = Y) = 0$.

SOLUTION

Solving all options :
  1)
  2) X and Y are two random variables with joint pdf

$$f(x,y) = \begin{cases} cxy & \text{, if } 0 < x < y < 1 \\ 0 & \text{, otherwise} \end{cases} \tag{0.0.1}$$

The marginal probability density functions are as follows :

$$f_X(x) = \int_{-\infty}^{\infty} f(x,y)\,dy \tag{0.0.2}$$

$$f_Y(y) = \int_{-\infty}^{\infty} f(x,y)\,dx \tag{0.0.3}$$

Calculating $f_X(x)$

$$f_X(x) = \int_{-\infty}^{\infty} f(x,y)\,dy \tag{0.0.4}$$

$$= \int_{x}^{1} cxy\,dy \tag{0.0.5}$$

$$= cx \left(\frac{y^2}{2}\right)\bigg|_{x}^{1} \tag{0.0.6}$$

$$= cx \left(\frac{1 - x^2}{2}\right) \tag{0.0.7}$$

$$f_X(x) = \begin{cases} cx\left(\frac{1-x^2}{2}\right) & \text{, if } 0 < x < 1 \\ 0 & \text{, otherwise} \end{cases} \tag{0.0.8}$$

Calculating $f_Y(y)$

$$f_y(y) = \int_{-\infty}^{\infty} f(x,y)\,dx \tag{0.0.9}$$

$$= \int_{0}^{y} cxy\,dx \tag{0.0.10}$$

$$= cy \left(\frac{x^2}{2}\right)\bigg|_{0}^{y} \tag{0.0.11}$$

$$= \frac{cy^3}{2} \tag{0.0.12}$$

$$f_Y(y) = \begin{cases} \frac{cy^3}{2} & \text{, if } 0 < y < 1 \\ 0 & \text{, otherwise} \end{cases} \tag{0.0.13}$$

Now by using property of pdf and equation (0.0.13) we have,

$$\int_{-\infty}^{\infty} f_Y(y)\,dy = 1 \tag{0.0.14}$$

$$\int_{0}^{1} c\frac{y^3}{2} = 1 \tag{0.0.15}$$

$$\frac{c}{8} = 1 \tag{0.0.16}$$

$$c = 8 \tag{0.0.17}$$

Therefore option (2) is correct.

3) The pdf of X is

$$f_X(x) = \begin{cases} 4x\left(1 - x^2\right) & \text{, if } 0 < x < 1 \\ 0 & \text{, otherwise} \end{cases}$$

$$(0.0.18)$$

The pdf of Y is

$$f_Y(y) = \begin{cases} 4y^3 & \text{, if } 0 < y < 1 \\ 0 & \text{, otherwise} \end{cases} \quad (0.0.19)$$

To check whether X and Y are independent, we calculate $f_X(x) \times f_Y(y)$. From (0.0.18) and (0.0.19)

$$f_X(x) \times f_Y(y) = \begin{cases} 16xy^3\left(1 - x^2\right) & \text{, if } 0 < x, y < 1 \\ 0 & \text{, otherwise} \end{cases}$$

$$(0.0.20)$$

$$\neq f(x, y) \qquad (0.0.21)$$

Since $f(x, y)$ and $f_X(x) \times f_Y(y)$ are different, random variables X and Y are not independent. Therefore option (3) is not correct.

4) To find $\Pr(X = Y)$, we use the conditional pdf,

$$f_{X|Y}(x|y) = \frac{f_{XY}(x, y)}{f_Y(y)} \qquad (0.0.22)$$

$$= \lim_{\Delta x, \Delta y \to 0} \Pr(x < X \le x + \Delta x | y < Y \le y + \Delta y)$$

$$(0.0.23)$$

$$\approx \frac{\Pr(X = x, Y = y)}{\Pr(X = x)} \qquad (0.0.24)$$

Similarly we have,

$$f_{Y|X}(y|x) = \frac{f_{XY}(x, y)}{f_X(x)} \qquad (0.0.25)$$

$$\approx \frac{\Pr(X = x, Y = y)}{\Pr(Y = y)} \qquad (0.0.26)$$

$$f_{XY}(x, y) = \begin{cases} 8xy & \text{, if } 0 < x < y < 1 \\ 0 & \text{, otherwise} \end{cases}$$

$$(0.0.27)$$

Calculation of conditional PDF and using

(0.0.27),

$$f_{X|Y}(x|y) = \frac{f_{XY}(x, y)}{f_Y(y)} \qquad (0.0.28)$$

$$= \frac{0}{f_Y(y)} \text{ for } x = y \qquad (0.0.29)$$

$$= 0 \text{ for } x = y \qquad (0.0.30)$$

Similarly we have,

$$f_{Y|X}(y|x) = 0 \text{ for } x = y \qquad (0.0.31)$$

$$(0.0.32)$$

Therefore for $x = y$ we have,

$$f_{X|Y}(x|y) = 0 \qquad (0.0.33)$$

$$f_{Y|X}(y|x) = 0 \qquad (0.0.34)$$

$$\implies \frac{\Pr(X = x, Y = y)}{\Pr(Y = y)} = 0 \qquad (0.0.35)$$

$$\Pr(X = x, Y = y) = 0 \text{ for } x = y \qquad (0.0.36)$$

$$\implies \Pr(X = Y) = 0 \qquad (0.0.37)$$

Therefore option (2) and (4) are correct.

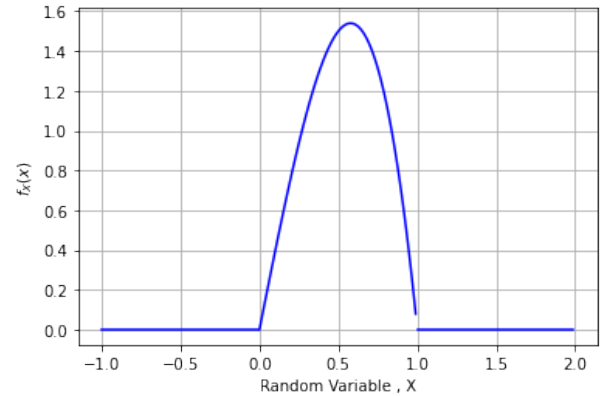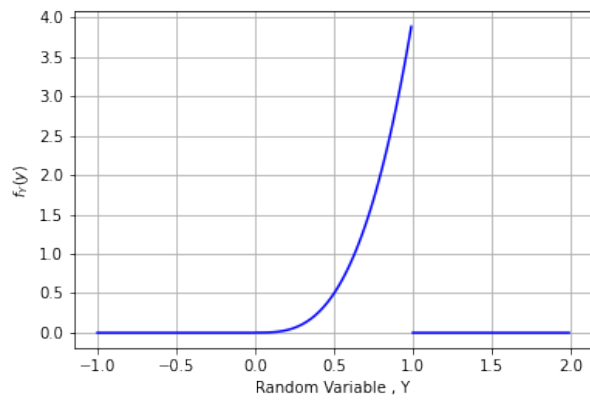The marginal PDF of X and Y are shown at figure 4 and figure 4.

Fig. 4: The marginal PDF of X

Fig. 4: The marginal PDF of Y