AI1103-Assignment 8

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Download all latex-tikz codes from

https://github.com/ayushjha2612/AI11003/tree/main/Assignment8

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Let $X_1, \cdots, X_n$ be independent and identically distributed random variables with probability density function

$$f(x) = \frac{1}{2}\lambda^3 x^2 e^{-\lambda x}; x > 0; \lambda > 0$$

Then which of the following statements are true?

1) $\frac{2}{n}\sum_{i=1}^{n}\frac{1}{X_i}$ is an unbiased estimator of λ
2) $\frac{3n}{\sum_{i=1}^{n} X_i}$ is an unbiased estimator of λ
3) $\frac{2}{n}\sum_{i=1}^{n}\frac{1}{X_i}$ is a consistent estimator of λ
4) $\frac{3n}{\sum_{i=1}^{n} X_i}$ is a consistent estimator of λ

SOLUTION

Definition 0.1. An **estimator** is a statistic that estimates some fact about the population. The quantity that is being estimated is called the **estimand.**

Definition 0.2. Let $\Theta = h(X_1, X_2, \cdots, X_n)$ be a point estimator for θ. The **bias** of the estimator Θ is defined by

$$B(\Theta) = E[\Theta] - \theta \tag{0.0.1}$$

where $E[\Theta]$ is the expectation value of the estimator Θ and θ is the estimand.

Definition 0.3. Let $\Theta = h(X_1, X_2, \cdots, X_n)$ be a point estimator for a parameter θ. We say that Θ is an **unbiased estimator** of θ if

$$B(\Theta) = 0, \text{ for all possible values of } \theta. \tag{0.0.2}$$

Definition 0.4. Let $\Theta_1, \Theta_2, \cdots, \Theta_n, \cdots,$ be a sequence of point estimators of θ. We say that Θ_n is a **consistent** estimator of θ, if

$$\lim_{n\to\infty} \Pr\left(|\Theta_n - \theta| \geq \epsilon\right) = 0 \text{ ,for all } \epsilon > 0. \tag{0.0.3}$$

Definition 0.5. The **mean squared error (MSE)** of a point estimator Θ, shown by $MSE(\Theta)$, is defined as

$$MSE(\Theta) = E[(\Theta - \theta)^2] \tag{0.0.4}$$
$$= Var(\Theta) + B(\Theta)^2 \tag{0.0.5}$$

where $B(\Theta)$ is the bias of Θ.

Theorem 0.1. Let $\Theta_1, \Theta_2, \cdots$ be a sequence of point estimators of θ. If

$$\lim_{n\to\infty} MSE(\Theta_n) = 0, \tag{0.0.6}$$

then Θ_n is a consistent estimator of θ.

Solving all options :

1) Now here we have our estimator Θ and estimand θ as,

$$\Theta = \frac{2}{n}\sum_{i=1}^{n}\frac{1}{X_i} \text{ and } \theta = \lambda \tag{0.0.7}$$

The expectation value of the estimator is given by,

$$E[\Theta] = E\left[\frac{2}{n}\sum_{i=1}^{n}\frac{1}{X_i}\right] \tag{0.0.8}$$
$$= \frac{2}{n}\sum_{i=1}^{n} E\left[\frac{1}{X_i}\right] \tag{0.0.9}$$
$$= \frac{2}{n}\sum_{i=1}^{n}\int_{-\infty}^{\infty}\frac{1}{x}f(x)\,dx \tag{0.0.10}$$
$$= \frac{2n}{n}\int_{0}^{\infty}\frac{1}{x}\frac{1}{2}\lambda^3 x^2 e^{-\lambda x}\,dx \tag{0.0.11}$$
$$= \lambda^3\int_{0}^{\infty} xe^{-\lambda x}\,dx \tag{0.0.12}$$
$$= \lambda \tag{0.0.13}$$

So the bias of estimator is given by,

$$B(\Theta) = E[\Theta] - \theta \tag{0.0.14}$$
$$= \lambda - \lambda = 0 \tag{0.0.15}$$

Therefore $\frac{2}{n}\sum_{i=1}^{n}\frac{1}{X_i}$ is an unbiased estimator of λ
Option 1 is correct.

2) Now in this option we have our estimator Θ and quantity to be estimated θ as,

$$\Theta = \frac{3n}{\sum_{i=1}^{n} X_i} \text{ and } \theta = \lambda \tag{0.0.16}$$

We have that sample mean, $\bar{X}$,

$$\bar{X} = \frac{X_1 + X_2 + \cdots + X_n}{n} \tag{0.0.17}$$
$$= \frac{\sum_{i=1}^{n} X_i}{n} \tag{0.0.18}$$

Therefore estimator,

$$\Theta = \frac{3n}{n\bar{X}} = \frac{3}{\bar{X}} \tag{0.0.19}$$

The distribution is gamma distribution, i.e. $X \sim \Gamma(\alpha, \lambda)$ with pdf,

$$f_X(x) = \begin{cases} \frac{\lambda^\alpha x^{\alpha-1}e^{-\lambda x}}{(\alpha-1)!} & x > 0 \\ 0 & otherwise \end{cases} \tag{0.0.20}$$

where $\alpha = 3$.
Let r.v. T be,

$$T = \sum_{i=1}^{n} X_i \sim \Gamma(3n, \lambda) \tag{0.0.21}$$

with pdf,

$$f_T(t) = \frac{\lambda^{3n} t^{3n-1} e^{-\lambda t}}{(3n-1)!}, t > 0 \tag{0.0.22}$$

Using, $\frac{1}{\bar{X}} = \frac{n}{T}$

$$E\left[\frac{1}{\bar{X}}\right] = \int_0^\infty \frac{n}{t}\frac{1}{(3n-1)!}\lambda^{3n}t^{3n-1}e^{-\lambda t}\,dt \tag{0.0.23}$$

$$= \frac{n\lambda}{(3n-1)}\int_0^\infty \frac{1}{(3n-2)!}\lambda^{3n-1}t^{3n-2}e^{-\lambda t}\,dt \tag{0.0.24}$$

Using property of gamma distributions that

$$\int_0^\infty \lambda^\alpha t^{\alpha-1} e^{-\lambda t}\,dt \tag{0.0.25}$$
$$= \frac{1}{(\alpha-1)!} \tag{0.0.26}$$

So we have,

$$\int_0^\infty \frac{1}{(3n-2)!}\lambda^{3n-1}t^{3n-2}e^{-\lambda t}\,dt = 1 \tag{0.0.27}$$

$$E[\Theta] = \frac{3n\lambda}{3n-1} \tag{0.0.28}$$

So we calculate bias as follows,

$$B(\Theta) = E[\Theta] - \lambda \tag{0.0.29}$$
$$= \frac{3n\lambda}{3n-1} - \lambda \tag{0.0.30}$$
$$= \frac{\lambda}{3n-1} \neq 0 \tag{0.0.31}$$

Therefore $\frac{3n}{\sum_{i=1}^{n} X_i}$ is not an unbiased estimator of λ
Option 2 is not correct.

3) Now here we have our estimator Θ and quantity to be estimated θ as,

$$\Theta = \frac{2}{n}\sum_{i=1}^{n}\frac{1}{X_i} \text{ and } \theta = \lambda \tag{0.0.32}$$

Now the variance of Θ is calculated as

$$Var(\Theta) = Var\left(\frac{2}{n}\sum_{i=1}^{n}\frac{1}{X_i}\right) \tag{0.0.33}$$
$$= \frac{4}{n^2}\sum_{i=1}^{n} Var\left(\frac{1}{X_i}\right) \tag{0.0.34}$$
$$= \frac{4n}{n^2}\left(E\left[\frac{1}{X_i}^2\right] - E\left[\frac{1}{X_i}\right]^2\right) \tag{0.0.35}$$
$$= \frac{4}{n}\left(\int_{-\infty}^{\infty}\frac{1}{x^2}f(x)\,dx - \left(\frac{\lambda}{2}\right)^2\right) \tag{0.0.36}$$
$$= \frac{4}{n}\left(\int_0^\infty \frac{1}{x^2}\frac{1}{2}\lambda^3 x^2 e^{-\lambda x}\,dx - \frac{\lambda^2}{4}\right) \tag{0.0.37}$$

$$= \frac{4}{n}\left(\frac{\lambda^3}{2}\int_0^\infty e^{-\lambda x}\,dx - \frac{\lambda^2}{4}\right) \tag{0.0.38}$$

$$= \frac{4}{n}\left(\frac{\lambda^2}{2} - \frac{\lambda^2}{4}\right) \tag{0.0.39}$$

$$= \frac{\lambda^2}{n} \tag{0.0.40}$$

The bias of Θ from option 1 is given as

$$B(\Theta) = 0 \tag{0.0.41}$$

So we have,

$$MSE(\Theta_n) = Var(\Theta) + B(\Theta)^2 \tag{0.0.42}$$

$$= \frac{\lambda^2}{n} \tag{0.0.43}$$

Now,

$$\lim_{n\to\infty} MSE(\Theta_n) = \lim_{n\to\infty}\frac{\lambda^2}{n} \tag{0.0.44}$$

$$= 0 \tag{0.0.45}$$

Therefore, $\frac{2}{n}\sum_{i=1}^{n}\frac{1}{X_i}$ is a consistent estimator of λ. Option 3 is correct.

4) Now in this option we have our estimator Θ and quantity to be estimated θ as,

$$\Theta = \frac{3n}{\sum_{i=1}^{n} X_i} \text{ and } \theta = \lambda \tag{0.0.46}$$

Similar to option 2 we have, rv T

$$Var(\Theta) = Var(\frac{3}{\bar{X}}) \tag{0.0.47}$$

$$= 9\left(E\left[\frac{1}{\bar{X}}^2\right] - E\left[\frac{1}{\bar{X}}\right]^2\right) \tag{0.0.48}$$

To calculate, $E\left[\frac{1}{\bar{X}}^2\right]$ we use,

$$\frac{1}{\bar{X}}^2 = \frac{n^2}{t^2} \tag{0.0.49}$$

$$E\left[\frac{1}{\bar{X}}^2\right] = \int_0^\infty \frac{n^2}{t^2}\frac{1}{(3n-1)!}\lambda^{3n}t^{3n-1}e^{-\lambda t}\,dt \tag{0.0.50}$$

$$= \frac{n^2\lambda^2}{(3n-1)(3n-2)} \times (1) \tag{0.0.51}$$

As from property of gamma distribution we have,

$$\int_0^\infty \frac{1}{(3n-3)!}\lambda^{3n-2}t^{3n-3}e^{-\lambda t}\,dt = 1 \tag{0.0.52}$$

Therefore,

$$Var(\Theta) = 9\left(\frac{n^2\lambda^2}{(3n-1)(3n-2)} - \frac{n^2\lambda^2}{(3n-1)^2}\right) \tag{0.0.53}$$

$$= \frac{9n^2\lambda^2}{3n-1}\left(\frac{1}{3n-2} - \frac{1}{3n-1}\right) \tag{0.0.54}$$

$$= \frac{9n^2\lambda^2}{(3n-1)^2(3n-2)} \tag{0.0.55}$$

The bias calculated from option 2 is

$$B(\Theta) = \frac{\lambda}{3n-1} \tag{0.0.56}$$

So we have,

$$MSE(\Theta) = Var(\Theta) + B(\Theta)^2 \tag{0.0.57}$$

$$= \frac{9n^2\lambda^2}{(3n-1)^2(3n-2)} + \frac{\lambda^2}{(3n-1)^2} \tag{0.0.58}$$

Finally,

$$\lim_{n\to\infty} MSE(\Theta_n) \tag{0.0.59}$$

$$= \lim_{n\to\infty}\frac{9n^2\lambda^2}{(3n-1)^2(3n-2)} + \frac{\lambda^2}{(3n-1)^2} \tag{0.0.60}$$

$$\tag{0.0.61}$$

Now in first limit multiply and divide by n^2 and $n \to \infty$ we get,

$$\lim_{n\to\infty} MSE(\Theta_n) = 0 \tag{0.0.62}$$

Therefore, $\frac{3n}{\sum_{i=1}^{n} X_i}$ is a consistent estimator of λ.
Option 4 is correct.

Therefore option 1, option 3 and option 4 are correct.