AI1103-Assignment 9

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Download all python codes from

https://github.com/ayushjha2612/AI11003/tree/main/Assignment9/Codes

and latex-tikz codes from

https://github.com/ayushjha2612/AI11003/tree/main/Assignment9

CSIR UGC NET EXAM (June 2017) Q. 103

Let $c \in \mathbb{R}$ be a constant. Let X, Y be random variables with joint probability density function

$$f(x,y) = \begin{cases} cxy & \text{, if } 0 < x < y < 1 \\ 0 & \text{, otherwise} \end{cases}$$

Which of the following statements are correct ?

1) $c = \frac{1}{8}$
2) $c = 8$
3) X and Y are independent
4) $\Pr(X = Y) = 0$

Answer

Option (2) $c = 8$ and option (4) $\Pr(X = Y) = 0$.

Solution

Solving all options :

1)
2) X and Y are two random variables with joint pdf

$$f(x,y) = \begin{cases} cxy & \text{, if } 0 < x < y < 1 \\ 0 & \text{, otherwise} \end{cases} \tag{0.0.1}$$

The marginal probability density functions are as follows :

$$f_X(x) = \int_{-\infty}^{\infty} f(x,y)\,dy \tag{0.0.2}$$

$$f_Y(y) = \int_{-\infty}^{\infty} f(x,y)\,dx \tag{0.0.3}$$

Calculating $f_X(x)$

$$f_X(x) = \int_{-\infty}^{\infty} f(x,y)\,dy \tag{0.0.4}$$

$$= \int_{x}^{1} cxy\,dy \tag{0.0.5}$$

$$= cx\left(\frac{y^2}{2}\right)\Bigg|_x^1 \tag{0.0.6}$$

$$= cx\left(\frac{1-x^2}{2}\right) \tag{0.0.7}$$

$$f_X(x) = \begin{cases} cx\left(\frac{1-x^2}{2}\right) & \text{, if } 0 < x < 1 \\ 0 & \text{, otherwise} \end{cases} \tag{0.0.8}$$

Calculating $f_Y(y)$

$$f_y(y) = \int_{-\infty}^{\infty} f(x,y)\,dx \tag{0.0.9}$$

$$= \int_{0}^{y} cxy\,dx \tag{0.0.10}$$

$$= cy\left(\frac{x^2}{2}\right)\Bigg|_0^y \tag{0.0.11}$$

$$= \frac{cy^3}{2} \tag{0.0.12}$$

$$f_Y(y) = \begin{cases} \frac{cy^3}{2} & \text{, if } 0 < y < 1 \\ 0 & \text{, otherwise} \end{cases} \tag{0.0.13}$$

Now by using property of pdf and equation (0.0.13) we have,

$$\int_{-\infty}^{\infty} f_Y(y)\,dy = 1 \tag{0.0.14}$$

$$\int_0^1 c\frac{y^3}{2} = 1 \tag{0.0.15}$$

$$\frac{c}{8} = 1 \tag{0.0.16}$$

$$c = 8 \tag{0.0.17}$$

Therefore option (2) is correct.

3) The pdf of X is

$$f_X(x) = \begin{cases} 4x\left(1-x^2\right) & \text{, if } 0 < x < 1 \\ 0 & \text{, otherwise} \end{cases} \tag{0.0.18}$$

The pdf of Y is

$$f_Y(y) = \begin{cases} 4y^3 & \text{, if } 0 < y < 1 \\ 0 & \text{, otherwise} \end{cases} \tag{0.0.19}$$

To check whether X and Y are independent, we calculate $f_X(x) \times f_Y(y)$. From (0.0.18) and (0.0.19)

$$f_X(x) \times f_Y(y) = \begin{cases} 16xy^3\left(1-x^2\right) & \text{, if } 0 < x, y < 1 \\ 0 & \text{, otherwise} \end{cases} \tag{0.0.20}$$

$$\neq f(x,y) \tag{0.0.21}$$

Since $f(x,y)$ and $f_X(x) \times f_Y(y)$ are different, random variables X and Y are not independent. Therefore option (3) is not correct.

4) For a small positive epislon, ϵ, we have conditional PDF as,

$$f_{Y|X}(y|a-\epsilon \leq x \leq a) = \frac{f_{XY}(x,y)}{f_X(x)} \tag{0.0.22}$$

$$= \frac{8xy}{4x(1-x^2)} \tag{0.0.23}$$

$$= \frac{2y}{1-x^2} \tag{0.0.24}$$

The conditional CDF is defined as

$$F_{Y|X}(Y \leq y|a-\epsilon \leq X \leq a) \tag{0.0.25}$$

$$= \Pr\left(Y \leq y|a-\epsilon \leq X \leq a\right) \tag{0.0.26}$$

$$= \int_{-\infty}^{y} f_{Y|X}(y|a-\epsilon \leq x \leq a)\, dy \tag{0.0.27}$$

Therefore,

$$\Pr\left(a-\epsilon \leq Y \leq a|a-\epsilon \leq X \leq a\right) \tag{0.0.28}$$

$$= \int_{a-\epsilon}^{a} f_{Y|X}(y|a-\epsilon \leq x \leq a)\, dy \tag{0.0.29}$$

$$= \int_{a-\epsilon}^{a} \frac{2y}{1-x^2}\, dy \tag{0.0.30}$$

$$= \frac{a^2-(a-\epsilon)^2}{1-x^2} \tag{0.0.31}$$

Now, we have that

$$\Pr\left(a-\epsilon \leq Y \leq a|a-\epsilon \leq X \leq a\right) \tag{0.0.32}$$

$$= \frac{\Pr\left(a-\epsilon \leq Y \leq a, a-\epsilon \leq X \leq a\right)}{\Pr\left(a-\epsilon \leq X \leq a\right)} \tag{0.0.33}$$

As $\epsilon \to 0$ we have,

$$\lim_{\epsilon\to 0} \Pr\left(a-\epsilon \leq Y \leq a|a-\epsilon \leq X \leq a\right) \tag{0.0.34}$$

$$= \lim_{\epsilon\to 0} \frac{a^2-(a-\epsilon)^2}{1-x^2} \tag{0.0.35}$$

$$= \lim_{\epsilon\to 0} \frac{2a\epsilon-\epsilon^2}{1-x^2} \tag{0.0.36}$$

$$= 0 \tag{0.0.37}$$

So by (0.0.33) and (0.0.37) we have,

$$\lim_{\epsilon\to 0} \Pr\left(a-\epsilon \leq Y \leq a, a-\epsilon \leq X \leq a\right) \tag{0.0.38}$$

$$= 0 \tag{0.0.39}$$

But as $\epsilon \to 0$,

$$\lim_{\epsilon\to 0} \Pr\left(a-\epsilon \leq Y \leq a, a-\epsilon \leq X \leq a\right) \tag{0.0.40}$$

$$= \Pr\left(X = Y\right) = 0 \tag{0.0.41}$$

Therefore option (2) and (4) are correct.

The marginal PDF of X and Y are shown at figure 4 and figure 4.

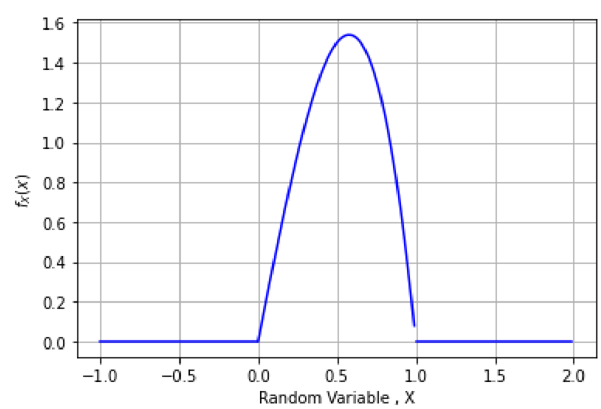

Fig. 4: The marginal PDF of X

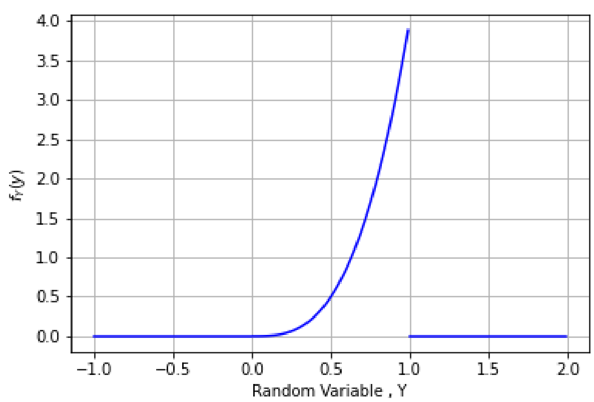

Fig. 4: The marginal PDF of Y