Induction

1 The Basics

Let $P(n)$ be a statement that depends on a positive integer n. The **Principle of Mathematical Induction** (usually just called "induction") states that if $P(1)$ is true, and if $P(k)$ implies $P(k+1)$ for any positive integer k, then $P(n)$ is true for all positive integers n.

Induction is often likened to toppling over a row of dominoes: If the first domino is tipped over, and if the dominoes are placed close enough so that if one domino falls then the next domino falls, then all the dominoes will eventually fall.

A typical induction argument consists of two parts:

- Verifying the base case (which is usually $n = 0$ or $n = 1$), and
- Assuming that $P(k)$ is true to prove $P(k+1)$. (This sometimes called the "inductive step," and the assumption that $P(k)$ is true is called the "inductive hypothesis.")

Thus, if you are considering using induction, you should first check if there is a reasonable way of linking $P(k)$ to $P(k+1)$.

Here is the classic example of proof by induction:

Problem 1.1. Prove that, for all positive integers n,

$$1 + 2 + \cdots + n = \frac{n(n+1)}{2}.$$

Solution: We prove the result using induction. When $n = 1$, the left-hand side is 1, and the right-hand side is

$$\frac{1 \cdot 2}{2} = 1,$$

so the result is true for $n = 1$.

Now, assume that the result holds for a given positive integer k, so that

$$1 + 2 + \cdots + k = \frac{k(k+1)}{2}.$$

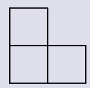
Then

$$1 + 2 + \cdots + k + (k+1) = \frac{k(k+1)}{2} + k + 1 = (k+1)\left(\frac{k}{2} + 1\right) = \frac{(k+1)(k+2)}{2}.$$

Hence, the result is true for $n = k+1$. By induction, we conclude that the result holds for all positive integers n. $\square$

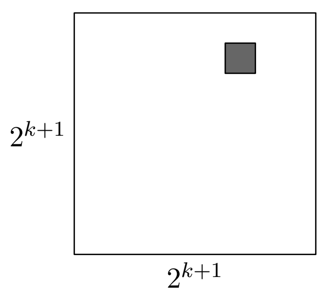
In the previous problem, linking the sum for k terms and the sum for $k+1$ terms was easy, because all we had to do was add a single term. Typically this linking will not be as obvious, as in the next problem:

Problem 1.2. Let n be a positive integer. One square of a $2^n \times 2^n$ chessboard is removed. Prove that the remaining chessboard can be tiled with L-triominoes, as shown below.

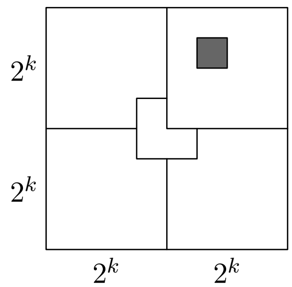


Solution: We prove the result using induction. For $n = 1$, the result is trivial, since when we remove any square from a 2×2 chessboard, the remaining chessboard is itself an L-triomino.

Assume that the result holds for a given positive integer k, meaning that a $2^k \times 2^k$ chessboard with any square deleted can be tiled with L-triominoes. Consider a $2^{k+1} \times 2^{k+1}$ chessboard, with one square deleted.



We want to use the induction hypothesis: we should think about how to use the fact that every $2^k \times 2^k$ chessboard with a deleted square can be tiled with L-triominoes. First, it makes sense to carve out the $2^k \times 2^k$ quadrant that contains the deleted square. We then divide the remaining chessboard into three $2^k \times 2^k$ chessboards, each with a deleted square, such that the three deleted squares are adjacent to each other.



We then have four $2^k \times 2^k$ chessboards with a deleted square, and by the induction hypothesis, each can be tiled with L-triominoes. Also, the three deleted squares form another L-triomino. Therefore, the whole $2^{k+1} \times 2^{k+1}$ chessboard with a deleted square can be tiled with L-triominoes. Hence, the result is true for $n = k + 1$, and by induction, the result is true for all positive integers n. □

2 Strong Induction

Normally, when using induction, we assume that $P(k)$ is true to prove $P(k + 1)$. In **strong induction**, we assume that all of $P(1), P(2), \ldots, P(k)$ are true to prove $P(k + 1)$.

Problem 2.1. The Fibonacci sequence is defined by $F_1 = F_2 = 1$, and $F_n = F_{n-1} + F_{n-2}$ for all $n \geq 3$. Prove that every positive integer N can be represented in the form

$$N = F_{a_1} + F_{a_2} + \cdots + F_{a_m}$$

for some integers $a_1, \ldots, a_m$ satisfying $2 \leq a_1 < a_2 < \cdots < a_m$.

Solution: We're asked to prove a property for all positive integers, so induction is a natural tactic to consider. The base case $1 = F_2$ is trivial. However, it's not very clear how to build the sum for $K + 1$ from the sum for K.

To get a feel for the problem, we start by looking at a specific example. Suppose we want to write 77 as the sum of distinct Fibonacci numbers. First, we can choose the greatest Fibonacci number that is less than or equal to 77, which is $F_{10} = 55$. Subtracting, we get $77 - 55 = 22$. We can then repeat this procedure of subtracting the largest Fibonacci number possible. This gives us $F_8 = 21$, and $F_2 = 1$. Hence, we can write

$$77 = 1 + 21 + 55 = F_2 + F_8 + F_{10}.$$

In the example above, we started with 77, and subtracted 55 to get 22. Thus, we reduced the problem to finding a representation for 22. In general, if we apply our tactic of subtracting the largest Fibonacci number from $K+1$, we won't get K, but we will get some number that is smaller than $K+1$. This suggests that we can use a strong induction argument: if F_a is the largest Fibonacci number less than or equal to $K+1$, then $K+1 = F_a + ((K+1) - F_a)$, and we use the strong inductive hypothesis to write $K+1-F_a$ as a sum of distinct Fibonacci numbers. All we have left to worry about is whether a Fibonacci number will be repeated. The only way that this can happen is if the largest Fibonacci number that appears in the sum of $K+1-F_a$ is itself F_a. But this cannot be, as

$$(K+1) - F_a < F_{a+1} - F_a = F_{a-1},$$

so $(K+1) - F_a$ is smaller than the next-smallest Fibonacci number F_{a-1}. Thus our strong induction argument works. □

Here is how we could write our argument as a formal solution:

Solution: We prove the result by strong induction.

For the base case $N = 1$, we have $1 = F_2$.

For the strong inductive step, assume there exists a positive integer K such that all the integers from 1 to K can be represented in the given form. We will show that $K+1$ can also be represented in the given form.

Let a be the greatest positive integer such that $F_a \le K+1$. If $K+1 = F_a$, then we are done. Otherwise, $F_a < K+1 < F_{a+1}$, and we have

$$0 < (K+1) - F_a < F_{a+1} - F_a = F_{a-1}.$$

By the strong induction hypothesis, there exist positive integers $a_1, a_2, \ldots, a_m$ such that

$$(K+1) - F_a = F_{a_1} + F_{a_2} + \cdots + F_{a_m},$$

where $2 \le a_1 < a_2 < \cdots < a_m$. Furthermore, since $(K+1) - F_a < F_{a-1}$, all of the terms $F_{a_1}, F_{a_2}, \ldots, F_{a_m}$ are less than F_a. Hence, the sum

$$K+1 = F_{a_1} + F_{a_2} + \cdots + F_{a_m} + F_a$$

satisfies the given conditions.

Therefore, by strong induction, all positive integers can be represented in the given form.

3 Strengthening the Hypothesis

In some problems, the result must be modified, or more assumptions must be imposed, to make the induction step possible. This technique is known as **strengthening the hypothesis**. To continue the domino analogy, the dominoes may be placed too far apart for one to knock down the next. The idea is to extend the length of the dominoes, so that they will now be within reach of each other.

In the following problem we find that straightforward induction won't cut it. Instead we prove a more elaborate result which follows easily from induction and use that to prove what we want.

Problem 3.1. Show that for all $n \geq 1$,

$$1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{n^2} < 2.$$

Solution: It is not obvious how to prove the result using induction, as is. For example, the result for $n = 1$ is

$$1 < 2,$$

and the result for $n = 2$ is

$$1 + \frac{1}{4} < 2.$$

The only way to get from the first inequality to the second is to add the inequality

$$\frac{1}{4} < 0,$$

which is not true. We need to modify the inequality, to give more "elbow-room" on the right-hand side.

So instead, let's try to prove an inequality of the form

$$1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{n^2} \leq 2 - f(n),$$

where f is some function. To determine what an appropriate function f would be, we start by listing the criteria we want f to satisfy.

First, since we want our original inequality to follow, we should have $f(n) \geq 0$ for all positive integers n.

Second, our goal is to prove this inequality using induction. For the base case $n = 1$, the inequality becomes

$$1 \leq 2 - f(1),$$

so $f(1) \leq 1$. To make the induction step possible, we need to go from the inequality

$$1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{k^2} \leq 2 - f(k)$$

to the inequality

$$1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{k^2} + \frac{1}{(k+1)^2} \leq 2 - f(k+1).$$

We can do so by adding the inequality

$$\frac{1}{(k+1)^2} \le f(k) - f(k+1)$$

to both sides. In other words, we would like f to satisfy

$$f(k) - f(k+1) \ge \frac{1}{(k+1)^2}$$

for all positive integers k.

So to summarize, we have the following conditions on f:

- $f(n) \ge 0$ for all positive integers n,
- $f(1) \le 1$,
- $f(k) - f(k+1) \ge \dfrac{1}{(k+1)^2}$ for all positive integers k.

After some experimentation, we find that the function $f(n) = \frac{1}{n}$ satisfies all these conditions. Hence, by induction, we can conclude that

$$1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{n^2} \le 2 - \frac{1}{n} < 2$$

for all positive integers n. □

Here is the formal solution:

Solution: We prove, by induction, the stronger result that

$$1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{n^2} \le 2 - \frac{1}{n},$$

from which the desired result follows.

For the base case $n = 1$, we observe that $1 = 2 - \frac{1}{1}$.

For the inductive step, assume that for a given positive integer k,

$$1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{k^2} \le 2 - \frac{1}{k}.$$

Adding the inequality $\frac{1}{(k+1)^2} \le \frac{1}{k(k+1)}$ to both sides gives

$$\begin{aligned}1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{k^2} + \frac{1}{(k+1)^2} &\le 2 - \frac{1}{k} + \frac{1}{k(k+1)} \\ &= 2 - \frac{(k+1)-1}{k(k+1)} \\ &= 2 - \frac{k}{k(k+1)} \\ &= 2 - \frac{1}{k+1},\end{aligned}$$

completing the inductive step.

If a problem looks like it can be proven using induction, but you can't quite make the induction step work, try modifying or generalizing the problem to make the induction step possible. Wishful thinking is a powerful tool in finding a successful modification. We used this tool in the previous problem when we hoped we could find a suitable function f.

Problem 3.2. An $n \times n$ matrix (square array) whose entries come from the set $S = \{1, 2, \ldots, 2n-1\}$ is called a *silver* matrix if, for each $i = 1, 2, \ldots, n$, the i^{th} row and the i^{th} column together contain all elements of S. Show that silver matrices exist for infinitely many values of n. (IMO, 1997)

Solution: We start with small cases. For $n = 2$, we have the 2×2 silver matrix

$$\begin{bmatrix} 1 & 3 \\ 2 & 1 \end{bmatrix}.$$

To generate larger silver matrices, we should think about how to use the silver matrices that we already know. For example, let's try to construct a 4×4 silver matrix by stacking two copies of the above 2×2 silver matrix:

$$\left[\begin{array}{cc|cc} 1 & 3 & & \\ 2 & 1 & & \\ \hline & & 1 & 3 \\ & & 2 & 1 \end{array}\right].$$

We want the numbers $1, 2, \ldots, 7$ to appear in the i^{th} row and the i^{th} column combined. Can we use the 2×2 silver matrix again to achieve this? We can, at least partially, by adding 4 to each entry, and then placing a copy in the upper-right and lower-left quadrants:

$$\left[\begin{array}{cc|cc} 1 & 3 & 5 & 7 \\ 2 & 1 & 6 & 5 \\ \hline 5 & 7 & 1 & 3 \\ 6 & 5 & 2 & 1 \end{array}\right].$$

This matrix almost satisfies the condition: For each i, in the i^{th} row and the i^{th} column combined, there are two 5s and no 4s. The easiest way to remedy this is by replacing every 5 in the lower-left quadrant with a 4:

$$\left[\begin{array}{cc|cc} 1 & 3 & 5 & 7 \\ 2 & 1 & 6 & 5 \\ \hline 4 & 7 & 1 & 3 \\ 6 & 4 & 2 & 1 \end{array}\right].$$

Thus, we have successfully constructed a silver matrix for $n = 4$.

We can generalize this argument, to construct a $2^{k+1} \times 2^{k+1}$ silver matrix from a $2^k \times 2^k$ silver matrix. In the construction above, we had to replace every 5 in the lower-left quadrant with a 4. In the general case, we will have to replace every instance of $2^{k+1} + 1$ in the lower-left quadrant with 2^{k+1}.

Furthermore, every instance of $2^{k+1} + 1$ was generated by adding 2^{k+1} to a 1 on the main diagonal of the smaller $2^k \times 2^k$ silver matrix. This means that the induction step will rely on the fact that every entry on the main diagonal is a 1, so this is an additional condition that we must impose. Thus, we have another example of an induction problem that requires strengthening the hypothesis.

The claim is now as follows: For every positive integer m, there is a $2^m \times 2^m$ silver matrix, such that every entry on the main diagonal is a 1. We now begin the proof, which we give by induction on m.

We have already seen an example for $m = 1$, namely

$$\begin{bmatrix} 1 & 3 \\ 2 & 1 \end{bmatrix},$$

which takes care of the base case.

Now, assume that the result is true for some positive integer $m = k$, i.e. there exists a $2^k \times 2^k$ silver matrix such that every entry on the main diagonal is a 1. Let M_k denote such a matrix.

Let A_k be the matrix obtained by adding 2^{k+1} to each entry of M_k, and let B_k be the matrix obtained by replacing every instance of $2^{k+1} + 1$ in A_k to 2^{k+1}. Finally, let

$$M_{k+1} = \begin{bmatrix} M_k & A_k \\ B_k & M_k \end{bmatrix}.$$

We claim that M_{k+1} is a silver matrix. Let G_i be the set of numbers appearing in the i^{th} row and the i^{th} column combined. We must prove that G_i contains all the numbers $1, 2, \dots, 2^{k+2}-1$, for all i.

First, consider the case $1 \le i \le 2^k$. By the induction hypothesis, the portion of G_i contained in the upper-left quadrant, namely in M_k, contains the numbers $\{1, 2, \dots, 2^{k+1}-1\}$. Now, the portion of G_i contained in the upper-right quadrant is the i^{th} row of A_k, and the portion of G_i contained in the lower-left quadrant is the i^{th} column of B_k. Let's take a closer look at what this means.

Both A_k and B_k are generated by adding 2^{k+1} to each entry of M_k, and ignoring the 1s on the main diagonal, the i^{th} row and the i^{th} column of M_k combined contain the numbers $2, 3, \dots, 2^{k+1}-1$. Hence, ignoring the main diagonals of A_k and B_k, the i^{th} row of A_k and the i^{th} column of B_k combined contain the numbers $2^{k+1}+2, 2^{k+1}+3, \dots, 2^{k+2}-1$. Finally, by construction, every entry on the main diagonal of A_k is $2^{k+1}+1$, and every entry on the main diagonal of B_k is 2^{k+1}.

Therefore, G_i contains every number in $\{1, 2, \dots, 2^{k+2}-1\}$. The proof for $2^k+1 \le i \le 2^{k+1}$ is similar. Finally, by construction, every entry on the main diagonal of M_{k+1} is clearly 1.

Hence, the result is true for $m = k+1$, and by induction, it is true for all positive integers m. It follows that there are silver matrices for infinitely many values of n. □

Remarks: The problem, as stated above, was actually part (b) of problem 4 of the 1997 IMO. Part (a) asked to show that there is no silver matrix for $n = 1997$. It can be shown that silver matrices exist for every even n, not just powers of 2.

The adjective "silver" was chosen as a double pun: The 1997 IMO was held in Mar del Plata, Argentina. The name "Argentina" comes from "argentum," the Latin word for silver. And in Spanish, the official language of Argentina, "plata" means silver.

Problem 3.3. Define a sequence by $a_0 = 1$, together with the rules $a_{2n+1} = a_n$ and $a_{2n+2} = a_n + a_{n+1}$ for each integer $n \ge 0$. Prove that every positive rational number appears in the set

$$\left\{\frac{a_{n-1}}{a_n} : n \ge 1\right\} = \left\{\frac{1}{1}, \frac{1}{2}, \frac{2}{1}, \frac{1}{3}, \frac{3}{2}, \dots\right\}.$$

(Putnam, 2002)

Solution: There's a lot here to swallow. First, we want to show that we can get every positive rational, but it's not even clear how we expect that rational number to appear. For example will we be able to find 3 and 4 consecutively in the sequence to make $\frac{3}{4}$, or will we have to wait until we reach a fraction like $\frac{51}{68}$, which reduces to $\frac{3}{4}$? Second, the recursive nature of the problem screams for us to use induction, but it looks like we're going to have to induct on the positive rational numbers, which is more than a little intimidating. After all, we can only induct on the positive integers! Or can we?

Let's begin by looking at a few more terms of the sequence a_n:

$$1, 1, 2, 1, 3, 2, 3, 1, 4, 3, 5, 2, 5, 3, 4, 1, 5, 4, 7, 3, 8, 5, 7, \dots.$$

Ok, that wasn't as helpful as it could have been. The only thing that we see is that the terms seem to be growing, but very slowly. This might be enough of a hint, though. Since we care for the terms in neighboring pairs, and we are trying to construct quotients out of these neighbors, then we should expect quotients of small numbers to come first and quotients of larger numbers to come later. This suggests thinking of $\frac{p}{q}$ as having size $p+q$, since this is a nice integer "measure" for the size of the rational number. This is the integer variable that we are going to induct on.

However, as we've noted, there are many ways to write a rational number. This looks like a good point for invoking a classic problem-solving strategy: make a wild assumption and hope for the best. Let's try to build rational numbers with a relatively prime numerator and denominator. This may or may not work but is as good a place as any to start. (It's actually encouraging to notice that every consecutive pair in our list is relatively prime!)

The base case is $n = 2$, and the only positive rational number with numerator and denominator that sum to 2 is

$$\frac{1}{1} = \frac{a_0}{a_1}.$$

Thus the base case is proven. Let's clearly state the inductive hypothesis, and we expect that we'll probably need strong induction. We let N be given, and assume that for every pair of relatively prime positive integers p, q such that $p + q \le N$, there is some n with $a_{n-1} = p$ and $a_n = q$, and in this way

$$\frac{p}{q} = \frac{a_{n-1}}{a_n}.$$

Next let's explore what the inductive step may be. Since we have the sequence defined in terms of a recursion, we check what this recursion says for the ratios. There are two cases, depending on whether the index in the numerator is even or odd:

$$\frac{a_{2n+1}}{a_{2n+2}} = \frac{a_n}{a_n + a_{n+1}},$$

and

$$\frac{a_{2n+2}}{a_{2n+3}} = \frac{a_n + a_{n+1}}{a_{n+1}}.$$

We notice that these cases are different. The first fraction is always less than one and the second fraction is always greater than one, so we consider two cases. (Note that $p = q = 1$ is covered in the base case.)

If $p < q$ and $p + q = N$, then we would like to solve

$$\frac{p}{q} = \frac{a_n}{a_n + a_{n+1}}.$$

We could solve this if we knew that there was some n such that $a_n = p$ and $a_{n+1} = q - p$. Fortunately, we know that such a pair *does* exist, since p and $q - p$ are relatively prime positive integers, and

$$p + (q - p) = q < N + 1.$$

Therefore, by the inductive hypothesis, there is some n that solves this pair of equations. As a consequence, we get the result

$$\frac{a_{2n+1}}{a_{2n+2}} = \frac{a_n}{a_n + a_{n+1}} = \frac{p}{p + (q - p)} = \frac{p}{q}.$$

Let's give a cleaner proof for the other case. Let p and q be relatively prime positive integers with $p > q$ and $p + q = N + 1$. Since q and $p - q$ are relatively prime, and $q + (p - q) \le N$, we know by the inductive hypothesis that there is some n such that $a_n = p - q$ and $a_{n+1} = q$. Then

$$\begin{aligned} a_{2n+2} &= a_n + a_{n+1} = (p - q) + q = p, \\ a_{2n+3} &= a_{n+1} = q. \end{aligned}$$

This proves the desired result. □

Here is how a formally written solution might look:

Solution: We prove the following stronger result: For any two relatively prime positive integers p and q, there exists some nonnegative integer n such that $a_n = p$ and $a_{n+1} = q$, and hence the number $\frac{a_n}{a_{n+1}} = \frac{p}{q}$ is in the given set. We prove this by induction on $p + q$.

For the base case, if $p + q = 2$, then $p = a_0 = 1$ and $q = a_1 = 1$.

For the inductive step, let relatively prime positive integers p, q be given with $p + q = N > 2$, and assume that all relatively prime positive integers p', q' with $p' + q' < N$ satisfy the desired result.

If $p < q$, then by the inductive hypothesis choose n such that $a_n = p$ and $a_{n+1} = q - p$. Then

$$\begin{aligned} a_{2n+1} &= a_n = p, \\ a_{2n+2} &= a_n + a_{n+1} = p + (q - p) = q, \end{aligned}$$

so $(p, q) = (a_{2n+1}, a_{2n+2})$, establishing the result.

If $p > q$, then by the inductive hypothesis choose n such that $a_n = p - q$ and $a_{n+1} = q$. Then

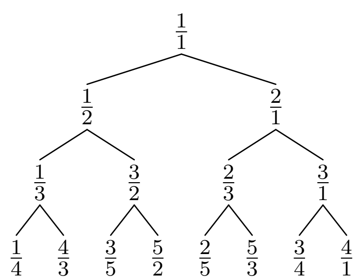
$$\begin{aligned} a_{2n+2} &= a_n + a_{n+1} = (p - q) + q = p, \\ a_{2n+3} &= a_{n+1} = q, \end{aligned}$$

so $(p, q) = (a_{2n+2}, a_{2n+3})$, establishing the result.

Thus, by induction, for any given pair (p, q) of relatively prime positive integers, we have $(p, q) = (a_n, a_{n+1})$ for some nonnegative integer n, and hence any positive rational number $\frac{p}{q}$ is in the given set.

Remarks: Notice that we proved something much stronger than what we were asked to prove. We proved by induction that for any pair of relatively prime positive integers (p, q), there is an n such that $(a_n, a_{n+1}) = (p, q)$. From this very strict statement our result followed. Notice that since we used this strict statement in the inductive step, we were required to prove it for every step; not just that we can construct $\frac{p}{q}$ from an earlier pair. The point here is that the claim that every positive rational number appears in the set is not strong enough for an induction argument; we also require the condition of a relatively prime numerator and denominator.

The sequence (a_n) is related to the *Calkin-Wilf tree*, which is generated by writing the fraction $\frac{1}{1}$ at the top, and then for each fraction $\frac{a}{b}$ in the tree, writing the fractions $\frac{a}{a+b}$ and $\frac{a+b}{b}$ below it:



Do you see how the fractions $\frac{a_{n-1}}{a_n}$ appear in this tree?

4 Using Induction

- All induction proofs have two major parts: the base case and the inductive step.
- Don't make your reader guess that you're using induction. Start with a sentence like "We will prove the result using induction." Be clear what base case(s) you're proving, where the inductive step begins, and what is the inductive hypothesis. Finish with something like "Therefore, by induction, the result holds for all positive integers n."
- The base case is not necessarily $n = 1$. In general, the base case could be any integer, and there could even be more than one base case, so make sure you have the correct base case(s).
- Induction only works well in problems where there is a reasonable way of linking $P(k)$ to $P(k+1)$. If you don't see how to link $P(k)$ to $P(k+1)$, try looking at a specific case. Once you see how a specific case works, the general case should follow more easily.

- There are actually many variations on induction. For example, one proof of the AM-GM inequality on n variables uses the following induction scheme:

  (1) The inequality holds for $n = 2$.
  (2) If the inequality holds for $n = k$ variables, then it holds for $n = 2k$ variables.
  (3) If the inequality holds for $n = k$ variables, then it holds for $n = k - 1$ variables.

  Statements (1) and (2) show that the inequality holds when n is any power of 2, and since every positive integer is less than some power of 2, it follows from (3) that the inequality holds for all positive integers.

  If you use a non-standard induction scheme, make sure that it covers all cases.

- You can induct on objects other than integers, but the underlying induction parameter must be an integer. For example, you could prove a result about polynomials by inducting on the degree, or a result about matrices by inducting on the size of the matrix.

- You may have noticed that several of the problems above involved powers of 2. If a problem involves powers of 2, try induction.

Review Problems

1. Show that

$$1^2 + 2^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$$

for all positive integers n.

2. Show that for all positive integers n, there exists a polynomial P_n with real coefficients such that for any nonzero real number a,

$$P_n\left(a + \frac{1}{a}\right) = a^n + \frac{1}{a^n}.$$

3. Show that

$$F_n = \binom{n-1}{0} + \binom{n-2}{1} + \binom{n-3}{2} + \cdots$$

for all positive integers n.

4. In the plane, n lines are drawn so that no two lines are parallel and no three lines pass through the same point. How many regions do these n lines determine?

5. (Bernoulli's Inequality) Let n be a positive integer, and let $x \geq -1$. Show that

$$(1 + x)^n \geq 1 + nx.$$

Induction

6. On a large, flat field, n people are positioned so that for each person the distances to all the other people are different. Each person holds a water pistol and at a given signal fires and hits the person who is closest. When n is odd, show that there is at least one person left dry. Is this always true when n is even? (Canada, 1987)

7. (Fermat's Little Theorem) Let p be a prime number. Prove that $n^p - n$ is divisible by p for all positive integers n.

8. Prove that for every positive integer n, the integers $1, 2, \ldots, 2^{n+1}$ can be partitioned into two sets A and B, such that for all $0 \le i \le n$,

$$\sum_{x \in A} x^i = \sum_{x \in B} x^i.$$

For example, for $n = 2$, we can take $A = \{1, 4, 6, 7\}$ and $B = \{2, 3, 5, 8\}$, since

$$\begin{aligned} 1^0 + 4^0 + 6^0 + 7^0 &= 2^0 + 3^0 + 5^0 + 8^0, \\ 1^1 + 4^1 + 6^1 + 7^1 &= 2^1 + 3^1 + 5^1 + 8^1, \\ 1^2 + 4^2 + 6^2 + 7^2 &= 2^2 + 3^2 + 5^2 + 8^2. \end{aligned}$$