WOOT Week 1: Induction

1. For any positive integer n, let $S(n)$ be the sum of digits in the decimal representation of n. Any positive integer obtained by removing one or more digits from the right end of the decimal representation of n is called a stump of n. Let $T(n)$ be the sum of all stumps of n. Prove that $n = S(n) + 9T(n)$. (For example, if $n = 238$, we have $S(n) = 2 + 3 + 8 = 13$, and stumps 2 and 23, so $T(n) = 2 + 23 = 25$. We verify that $238 = 13 + 9(25)$.)

2. Prove that
$$2(\sqrt{n+1} - 1) < 1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \cdots + \frac{1}{\sqrt{n}} < 2\sqrt{n}$$
for all positive integers n.

3. Given $0 < a < b < c < d < e < 1$, prove that $abcde > a + b + c + d + e - 4$.

4. Let p be a prime, and let a, k be positive integers such that $p^k \mid (a - 1)$. Show that $p^{n+k} \mid (a^{p^n} - 1)$ for all positive integers n.

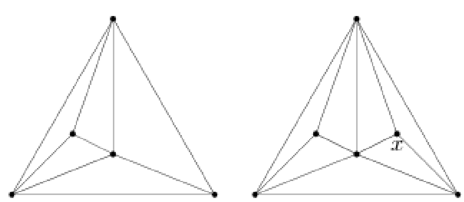
5. Find, in terms of n, the sum of the digits of
$$9 \times 99 \times 9999 \times \cdots \times \left(10^{2^n} - 1\right),$$
where each factor has twice as many nines as the previous factor.

6. Let $P(z)$ be a polynomial with complex coefficients of degree 1992 with 1992 distinct zeros. Prove that there exist complex numbers $a_1, a_2, \ldots, a_{1992}$ such that $P(z)$ divides the polynomial
$$(\cdots((z - a_1)^2 - a_2)^2 \cdots - a_{1991})^2 - a_{1992}.$$

7. For any positive integer $n \geq 2$, let S_n be the set of all fractions of the form $\frac{1}{pq}$, where p and q are relatively prime, $0 < p < q \leq n$, and $p + q > n$. Show that the sum of the elements of S_n is $\frac{1}{2}$.

8. Here is a problem and a proposed solution.

   Problem. Let n be a nonnegative integer. Suppose we are given a triangle and n points inside it, with no three of the given $n + 3$ points collinear. We divide the triangle into smaller triangles, using the $n + 3$ points as vertices. Show that we always end up with $2n + 1$ triangles.

   $k = 2$ $k = 3$

   Solution. For the base case $n = 0$, there is clearly $2n + 1 = 1$ triangle. For the inductive step, assume that k points inside the triangle define $2k + 1$ triangles. If we add a point x, as shown, then we lose one triangle but create three more triangles, for a net addition of two triangles. Hence, there are a total of $2k + 1 + 2 = 2k + 3 = 2(k + 1) + 1$ triangles, which completes the induction. ■

   This proposed solution has a major conceptual flaw. Identify the flaw, and fix the induction argument.

9. Prove that every positive integer can be written as a finite sum of distinct integral powers of the golden ratio. (Recall that the golden ratio is $\tau = \frac{1}{2}(1 + \sqrt{5})$.)

10. Let S be a finite nonempty set of points in three-dimensional space. Let S_x, S_y, S_z be the sets consisting of the orthogonal projections of the points of S onto the yz-plane, zx-plane, xy-plane respectively. Prove that
$$|S|^2 \leq |S_x| \cdot |S_y| \cdot |S_z|,$$
where $|A|$ denotes the number of elements in the finite set A.