

## Review:

### • Probability Formalism

↳ Goal: use set theory to rigorize our discussion of probability

↳ Random Experiment: any scenario w/ uncertainty, e.g.

- Coin toss
- Sequence of dice rolls
- Outcome of an election where every voter votes randomly.

↳ Any RE can be defined with the following

- A sample space  $\Omega$ , which is a set of all possible outcomes.
- A function  $P: \Omega \rightarrow \mathbb{R}$  satisfying
  - $P[\omega] \in [0, 1] \quad \forall \omega \in \Omega$ , and
  - $\sum_{\omega \in \Omega} P[\omega] = 1$ .

↳ An event in an RE is a subset  $E \subseteq \Omega$

- Can typically be thought of "all outcomes with a certain property".
- e.g., if the RE is two coin flips,  
 $E = \{ (H, H), (T, T) \}$  is "the event that the two flips are the same".
- Can define  $P[E] = \sum_{e \in E} P[e]$ .

- Principle of Inclusion - Exclusion (PIE)

↳ Idea: If you have bunch of sets  $A_i$  that overlap with each other, how do you figure out how many elements total across all of them?

↳ Solution: iteratively correct for over/under counting in the following way:

$$\left| \bigcup_i A_i \right| = \sum_i |A_i| - \sum_{i_1 \neq i_2} |A_{i_1} \cap A_{i_2}| + \sum_{i_1 \neq i_2 \neq i_3} |A_{i_1} \cap A_{i_2} \cap A_{i_3}| - \dots + (-1)^{n-1} \sum_{i_1 \neq i_2 \neq \dots \neq i_n} |A_{i_1} \cap \dots \cap A_{i_n}|$$

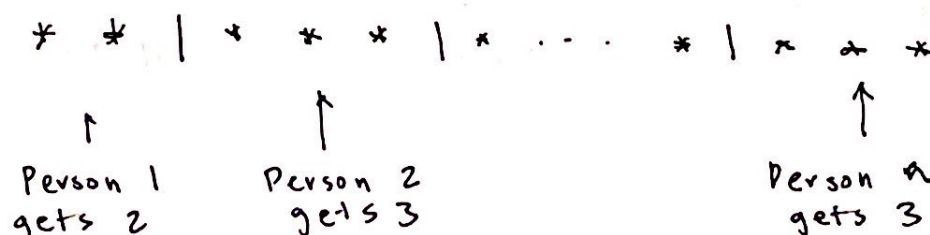
↳ Each term  $\sum_{i_1 \neq \dots \neq i_k} |A_{i_1} \cap \dots \cap A_{i_k}|$  corrects for under/overcounting.

- If  $x$  is in  $k$  sets, it will be overcounted by the first sum, undercounted by the second, etc, until it hits the  $k^{\text{th}}$  term.

- Stars + Bars (Balls + Bins, Balls + Urns, ...)

↳ Problem: How many ways can I split  $K$  indistinguishable objects among  $n$  people?

↳ Solution: Consider the following sequence of "stars" and "bars", where we have  $K$  stars and  $n-1$  bars:



There are  $k$  stars,  $n-1$  bars, so  $n+k-1$  symbols total  $\Rightarrow$  total # of sequences is  $\binom{n+k-1}{n-1}$ .

$\hookrightarrow$  Can be rephrased " $k$  from  $n$  w/ replacement where order doesn't matter."

### • Derangements:

$\hookrightarrow$  A derangement of length  $n$  is a permutation  $\pi$  of  $[1, 2, \dots, n]$  with no fixed points, e.g.  $\forall i, \pi(i) \neq i$ .

$\hookrightarrow$  Can be counted using complement + PIE!

$\hookrightarrow$  Lets count all the permutations w/ at least one fixed point

- Let  $A_i = \{ \pi \mid \pi(i) = i \}$ , i.e. the permutations that fix  $i$ .
- We want  $|\bigcup_i A_i|$ ; all permutations with at least one fixed point. We can use PIE!

$$\hookrightarrow \left| \bigcup_i A_i \right| = \sum |A_i| - \sum |A_{i_1} \cap A_{i_2}| + \dots$$

$\hookrightarrow$  What does  $|A_{i_1} \cap A_{i_2} \dots \cap A_{i_k}|$  look like?

- The set of all permutations that fix  $i_1, i_2, \dots, i_k$  — so there are  $(n-k)!$  of them!

$$\hookrightarrow \left| \bigcup_i A_i \right| = \sum_i n! - \sum_{i_1 \neq i_2} (n-1)! + \sum_{i_1 \neq i_2 \neq i_3} (n-2)! - \dots$$

$$\begin{aligned}
 \hookrightarrow \left| \bigcup_i A_i \right| &= \sum_i (n-1)! - \sum_{i_1 \neq i_2} (n-2)! + \sum_{i_1 \neq i_2 \neq i_3} (n-3)! - \dots \\
 &= n! - \frac{n!}{2!} + \frac{n!}{3!} - \frac{n!}{4!} + \dots \\
 &= n! \sum_{i=1}^n (-1)^{i-1} \cdot \frac{1}{i!} = \overline{D}_n
 \end{aligned}$$

Thus, the # of derangements is

$$D_n = n! - \overline{D}_n = n! \sum_{i=0}^n (-1)^i \cdot \frac{1}{i!}.$$