Quick Review

Let G be a graph:

- G is connected, planar $\Rightarrow v - e + f = 2$
  - If $v \geq 3$, then $e \leq 3v - 6$
  - If $v \geq 3$ and G is bipartite, then $e \leq 2v - 4$
- (Kuratowski) G is nonplanar if and only if it contains something equivalent to K_5 or $K_{3,3}$.
- The chromatic number $\chi(G)$ is # of colors needed to color the vertices of G so that no adjacent vertices have the same color.
  - If G is planar, $\chi(G) \leq 5$
  - If H is a subgraph of G, $\chi(G) \geq \chi(H)$

General Notes:

- If you show a subgraph of G is nonplanar, then G is nonplanar.
- When inducting on graphs, be wary of build-up error and removal error.
- Don't be afraid to work with connected components individually instead of the whole graph.

Short Answer

2C #1

(a) Bob removed a degree 3 node from an n vertex tree. How many connected components does the resulting graph have?

(b) Starting with an n vertex tree, Bob adds 10 edges. Then, Alice removes 5 edges (not necessarily the ones Bob added). If the resulting graph has 3 connected components, how many more edges does Alice have to remove to eliminate all cycles?

Planarity

Let G be a graph with the property that for any 3 distinct vertices v_1, v_2, v_3, there at least two edges between them. Using contradiction, prove that if G has 7 or more vertices, then G is not planar. (Hint: try to show that G contains K_5)

Graph Coloring

2C #3

Prove that a graph with maximum degree K is $(K+1)$ colorable.

Hypercubes:

2C #4

Recall that an n-dimensional hypercube is a graph generated by n-bit strings such that

- Each vertex corresponds to a unique string of n 0s and 1s,
- Two vertices are connected if their bit strings differ at exactly one place.

Prove that for all n, the n-dimensional hypercube is bipartite. (Hint: draw out small hypercubes and look for a pattern)