

Assignment 9

Abstract—This document contains Solution of Problem using SVD.

Download latex-tikz codes from

<https://github.com/ayushkesh/Matrix-Theory-EE5609/tree/master/A9>

Download Python code from

<https://github.com/ayushkesh/Matrix-Theory-EE5609/tree/master/A9.pynb>

1 PROBLEM

Find the foot of the perpendicular from,

$$\mathbf{B} = \begin{pmatrix} 3 \\ -2 \\ 2 \end{pmatrix} \quad (1.0.1)$$

to the plane,

$$(2 \ -3 \ 1)\mathbf{x} = 0 \quad (1.0.2)$$

2 SOLUTION

The equation of plane is give

$$\mathbf{n}^T \mathbf{x} = c \quad (2.0.1)$$

Hence the normal vector \mathbf{n} is,

$$\mathbf{n} = \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix} \quad (2.0.2)$$

Let, the normal vectors \mathbf{m}_1 and \mathbf{m}_2 to the normal vector \mathbf{n} be,

$$\mathbf{m} = \begin{pmatrix} a \\ b \\ c \end{pmatrix} \quad (2.0.3)$$

$$\text{then, } \mathbf{m}^T \mathbf{n} = 0 \quad (2.0.4)$$

$$\Rightarrow (a \ b \ c) \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix} = 0 \quad (2.0.5)$$

Let, $a=0$ and $b=1$ we get,

$$\mathbf{m}_1 = \begin{pmatrix} 1 \\ 0 \\ -2 \end{pmatrix} \quad (2.0.6)$$

Let, $a=1$ and $b=0$,

$$\mathbf{m}_2 = \begin{pmatrix} 0 \\ 1 \\ 3 \end{pmatrix} \quad (2.0.7)$$

Now solving the equation,

$$\mathbf{M}\mathbf{x} = \mathbf{b} \quad (2.0.8)$$

Where,

$$\mathbf{M} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ -2 & 3 \end{pmatrix}, \mathbf{b} = \begin{pmatrix} 3 \\ -2 \\ 2 \end{pmatrix} \quad (2.0.9)$$

To solve (2.0.8) we perform singular value decomposition on \mathbf{M} given by,

$$\mathbf{M} = \mathbf{U}\mathbf{S}\mathbf{V}^T \quad (2.0.10)$$

substituting the value of \mathbf{M} from equation (2.0.10) to (2.0.8),

$$\Rightarrow \mathbf{U}\mathbf{S}\mathbf{V}^T \mathbf{x} = \mathbf{b} \quad (2.0.11)$$

$$\Rightarrow \mathbf{x} = \mathbf{V}\mathbf{S}_+ \mathbf{U}^T \mathbf{b} \quad (2.0.12)$$

where, \mathbf{S}_+ is Moore-Pen-rose Pseudo-Inverse of \mathbf{S} . Columns of \mathbf{U} are eigenvectors of $\mathbf{M}\mathbf{M}^T$, columns of \mathbf{V} are eigenvectors of $\mathbf{M}^T\mathbf{M}$ and \mathbf{S} is diagonal matrix of singular value of eigenvalues of $\mathbf{M}^T\mathbf{M}$. First calculating the eigenvectors corresponding to $\mathbf{M}^T\mathbf{M}$.

$$\mathbf{M}^T\mathbf{M} = \begin{pmatrix} 1 & 0 & -2 \\ 0 & 1 & 3 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ -2 & 3 \end{pmatrix} = \begin{pmatrix} 5 & -6 \\ -6 & 10 \end{pmatrix} \quad (2.0.13)$$

Eigenvalues corresponding to $\mathbf{M}^T\mathbf{M}$ is,

$$|\mathbf{M}^T\mathbf{M} - \lambda\mathbf{I}| = 0 \quad (2.0.14)$$

$$\Rightarrow \begin{pmatrix} 5 - \lambda & -6 \\ -6 & 10 - \lambda \end{pmatrix} \quad (2.0.15)$$

$$\Rightarrow (\lambda - 14)(\lambda - 1) = 0 \quad (2.0.16)$$

$$\therefore \lambda_1 = 14, \lambda_2 = 1, \quad (2.0.17)$$

Hence the eigenvectors corresponding to λ_1 and λ_2 respectively is,

$$\mathbf{v}_1 = \begin{pmatrix} -\frac{2}{3} \\ 1 \end{pmatrix}, \mathbf{v}_2 = \begin{pmatrix} \frac{3}{2} \\ 1 \end{pmatrix} \quad (2.0.18)$$

Normalizing the eigenvectors we get,

$$\mathbf{v}_1 = \frac{1}{\sqrt{13}} \begin{pmatrix} -2 \\ 3 \end{pmatrix} \quad (2.0.19)$$

$$\mathbf{v}_2 = \frac{1}{\sqrt{13}} \begin{pmatrix} 3 \\ 2 \end{pmatrix} \quad (2.0.20)$$

$$\Rightarrow \mathbf{V} = \frac{1}{\sqrt{13}} \begin{pmatrix} -2 & 3 \\ 3 & 2 \end{pmatrix} \quad (2.0.21)$$

Now calculating the eigenvectors corresponding to $\mathbf{M}\mathbf{M}^T$

$$\mathbf{M}\mathbf{M}^T = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ -2 & 3 \end{pmatrix} \begin{pmatrix} 1 & 0 & -2 \\ 0 & 1 & 3 \end{pmatrix} \quad (2.0.22)$$

$$\Rightarrow \begin{pmatrix} 1 & 0 & -2 \\ 0 & 1 & 3 \\ -2 & 3 & 13 \end{pmatrix} \quad (2.0.23)$$

Eigenvalues corresponding to $\mathbf{M}\mathbf{M}^T$ is,

$$|\mathbf{M}\mathbf{M}^T - \lambda\mathbf{I}| = 0 \quad (2.0.24)$$

$$\Rightarrow \begin{pmatrix} 1 - \lambda & 0 & -2 \\ 0 & 1 - \lambda & 3 \\ -2 & 3 & 13 - \lambda \end{pmatrix} \quad (2.0.25)$$

$$\Rightarrow -\lambda^3 + 15\lambda^2 - 14\lambda = 0 \quad (2.0.26)$$

$$\Rightarrow -\lambda(\lambda - 1)(\lambda - 14) = 0 \quad (2.0.27)$$

$$\therefore \lambda_3 = 14, \lambda_4 = 1, \lambda_5 = 0 \quad (2.0.28)$$

Hence the eigenvectors corresponding to λ_3, λ_4 and λ_5 respectively is,

$$\mathbf{v}_3 = \begin{pmatrix} -\frac{2}{3} \\ \frac{3}{13} \\ 1 \end{pmatrix}, \mathbf{v}_4 = \begin{pmatrix} \frac{3}{2} \\ 1 \\ 0 \end{pmatrix}, \mathbf{v}_5 = \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix} \quad (2.0.29)$$

Normalizing the eigenvectors we get,

$$\mathbf{v}_3 = \frac{1}{\sqrt{182}} \begin{pmatrix} -2 \\ 3 \\ 13 \end{pmatrix} = \begin{pmatrix} -\sqrt{\frac{2}{91}} \\ \frac{3}{\sqrt{182}} \\ \sqrt{\frac{13}{14}} \end{pmatrix} \quad (2.0.30)$$

$$\mathbf{v}_4 = \frac{1}{\sqrt{13}} \begin{pmatrix} 3 \\ 2 \\ 0 \end{pmatrix} = \begin{pmatrix} \frac{3}{\sqrt{13}} \\ \frac{2}{\sqrt{13}} \\ 0 \end{pmatrix} \quad (2.0.31)$$

$$\mathbf{v}_5 = \frac{1}{\sqrt{14}} \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix} = \begin{pmatrix} \sqrt{\frac{2}{7}} \\ -\frac{3}{\sqrt{14}} \\ \sqrt{\frac{1}{14}} \end{pmatrix} \quad (2.0.32)$$

$$\Rightarrow \mathbf{U} = \begin{pmatrix} -\sqrt{\frac{2}{91}} & \frac{3}{\sqrt{13}} & \sqrt{\frac{2}{7}} \\ \frac{3}{\sqrt{182}} & \frac{2}{\sqrt{13}} & -\frac{3}{\sqrt{14}} \\ \sqrt{\frac{13}{14}} & 0 & \sqrt{\frac{1}{14}} \end{pmatrix} \quad (2.0.33)$$

Now \mathbf{S} corresponding to eigenvalues λ_3, λ_4 and λ_5 is as follows,

$$\mathbf{S} = \begin{pmatrix} \sqrt{14} & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \quad (2.0.34)$$

Now, Moore-Penrose Pseudo inverse of \mathbf{S} is given by,

$$\mathbf{S}_+ = \begin{pmatrix} \frac{1}{\sqrt{14}} & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \quad (2.0.35)$$

Hence we get singular value decomposition of \mathbf{M} as,

$$\mathbf{M} = \frac{1}{\sqrt{13}} \begin{pmatrix} -\sqrt{\frac{2}{91}} & \frac{3}{\sqrt{13}} & \sqrt{\frac{2}{7}} \\ \frac{3}{\sqrt{182}} & \frac{2}{\sqrt{13}} & -\frac{3}{\sqrt{14}} \\ \sqrt{\frac{13}{14}} & 0 & \sqrt{\frac{1}{14}} \end{pmatrix} \begin{pmatrix} \sqrt{14} & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} -2 & 3 \\ 3 & 2 \end{pmatrix}^T \quad (2.0.36)$$

Now substituting the values of (2.0.21), (2.0.35), (2.0.33) and (2.0.9) in (2.0.12),

$$\mathbf{U}^T \mathbf{b} = \begin{pmatrix} -\sqrt{\frac{2}{91}} & \frac{3}{\sqrt{13}} & \sqrt{\frac{2}{7}} \\ \frac{3}{\sqrt{182}} & \frac{2}{\sqrt{13}} & -\frac{3}{\sqrt{14}} \\ \sqrt{\frac{13}{14}} & 0 & \sqrt{\frac{1}{14}} \end{pmatrix}^T \begin{pmatrix} 3 \\ -2 \\ 2 \end{pmatrix} \quad (2.0.37)$$

$$\Rightarrow \mathbf{U}^T \mathbf{b} = \begin{pmatrix} \frac{\sqrt{182}}{13} \\ \frac{5}{\sqrt{13}} \\ \sqrt{14} \end{pmatrix} \quad (2.0.38)$$

$$\mathbf{VS}_+ = \frac{1}{\sqrt{13}} \begin{pmatrix} -2 & 3 \\ 3 & 2 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{14}} & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \quad (2.0.39)$$

$$\Rightarrow \mathbf{VS}_+ = \frac{1}{\sqrt{13}\sqrt{14}} \begin{pmatrix} -2 & 3 & \sqrt{14} \\ 3 & 2 & \sqrt{14} \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad (2.0.40)$$

\therefore from equation (2.0.12),

$$\mathbf{x} = \frac{1}{\sqrt{13}\sqrt{14}} \begin{pmatrix} -2 & 3 & \sqrt{14} \\ 3 & 2 & \sqrt{14} \end{pmatrix} \begin{pmatrix} \frac{\sqrt{182}}{13} \\ \frac{1}{\sqrt{13}} \\ \frac{1}{\sqrt{14}} \end{pmatrix} \quad (2.0.41)$$

$$\Rightarrow \mathbf{x} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad (2.0.42)$$

Verifying the solution using,

$$\mathbf{M}^T \mathbf{M} \mathbf{x} = \mathbf{M}^T \mathbf{b} \quad (2.0.43)$$

$$\Rightarrow \begin{pmatrix} 1 & 0 & -2 \\ 0 & 1 & 3 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ -2 & 3 \end{pmatrix} \mathbf{x} = \begin{pmatrix} 1 & 0 & -2 \\ 0 & 1 & 3 \end{pmatrix} \begin{pmatrix} 3 \\ -2 \\ 2 \end{pmatrix} \quad (2.0.44)$$

$$\Rightarrow \begin{pmatrix} 5 & -6 \\ -6 & 10 \end{pmatrix} \mathbf{x} = \begin{pmatrix} -1 \\ 4 \end{pmatrix} \quad (2.0.45)$$

Solving the augmented matrix we get,

$$\begin{pmatrix} 5 & -6 & -1 \\ -6 & 10 & 4 \end{pmatrix} \xrightarrow{R_1 \leftarrow \frac{R_1}{5}} \begin{pmatrix} 1 & -\frac{6}{5} & -\frac{1}{5} \\ -6 & 10 & 4 \end{pmatrix} \quad (2.0.46)$$

$$\xrightarrow{R_2 \leftarrow R_2 + 6R_1} \begin{pmatrix} 1 & -\frac{6}{5} & -\frac{1}{5} \\ 0 & \frac{14}{5} & \frac{14}{5} \end{pmatrix} \quad (2.0.47)$$

$$\xrightarrow{R_2 \leftarrow \frac{5}{14} R_2} \begin{pmatrix} 1 & -\frac{6}{5} & -\frac{1}{5} \\ 0 & 1 & 1 \end{pmatrix} \quad (2.0.48)$$

$$\xrightarrow{R_1 \leftarrow R_1 + \frac{6}{5} R_2} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix} \quad (2.0.49)$$

$$\Rightarrow \mathbf{x} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad (2.0.50)$$

Hence from equations (2.0.42) and (2.0.50) we conclude that the solution is verified.