Assignment-4

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***Abstract*—This document contains solution of Problem Geolin**(1.12)

Download latex-tikz codes from

https://github.com/ayushkesh/Matrix−Theory−EE5609/tree/master/A4

1 Question

Show that $\sin 30° = \frac{1}{2}$ and $\cos 30° = \frac{\sqrt{3}}{2}$.

2 Solution

Consider an equilateral $\triangle ABC$ as shown in figure:1. Let the angle bisector of $\angle$A intersect side **BC** at a point D between B and C. In equilateral triangle all sides are equal.Hence,

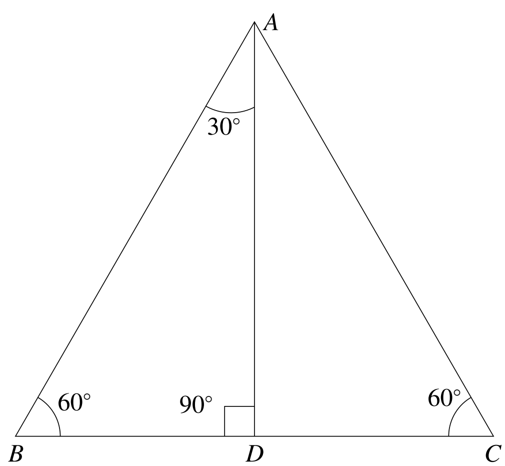

Fig. 1: Equilateral $\triangle ABC$

$$\|\mathbf{A}-\mathbf{B}\| = \|\mathbf{B}-\mathbf{C}\| = \|\mathbf{A}-\mathbf{C}\| = 2\|\mathbf{B}-\mathbf{D}\| \tag{2.0.1}$$

Putting **B**= 0 in (2.0.1) we have,

$$\|\mathbf{A}\| = \|\mathbf{C}\| \tag{2.0.2}$$

$$\|\mathbf{A}\| = \|\mathbf{A}-\mathbf{C}\| \tag{2.0.3}$$

Squaring equation (2.0.2)

$$\|\mathbf{A}\|^2 = \|\mathbf{C}\|^2 \tag{2.0.4}$$

Squaring equation (2.0.3)

$$\|\mathbf{A}\|^2 = \|\mathbf{A}\|^2 - 2(\mathbf{A}^T)(\mathbf{C}) + \|\mathbf{C}\|^2$$
$$\implies \|\mathbf{A}\|^2 = 2(\mathbf{A}^T)(\mathbf{C}) \tag{2.0.5}$$

Taking the inner product of sides AB,BC we have:

$$(\mathbf{A}-\mathbf{B})^T(\mathbf{B}-\mathbf{C}) = \|\mathbf{A}-\mathbf{B}\|\,\|\mathbf{B}-\mathbf{C}\|\cos ABC \tag{2.0.6}$$

The angle ABC from the above equation is:

$$\cos ABC = \frac{(\mathbf{A}-\mathbf{B})^T(\mathbf{B}-\mathbf{C})}{\|\mathbf{A}-\mathbf{B}\|\,\|\mathbf{B}-\mathbf{C}\|} \tag{2.0.7}$$

Substituting value in (2.0.8) and putting we have:

$$\cos ABC = \frac{(\mathbf{A})^T(\mathbf{C})}{\|\mathbf{A}\|^2} \tag{2.0.8}$$

From (2.0.5) we have:

$$\cos ABC = \frac{(\mathbf{A})^T(\mathbf{C})}{2(\mathbf{A})^T(\mathbf{C})}$$
$$\implies \cos ABC = 1/2$$
$$\implies \angle ABC = 60° \tag{2.0.9}$$

Taking the inner product of sides AB,AC we have:

$$(\mathbf{B}-\mathbf{A})^T(\mathbf{A}-\mathbf{C}) = \|\mathbf{B}-\mathbf{A}\|\,\|\mathbf{A}-\mathbf{C}\|\cos BAC \tag{2.0.10}$$

The angle BAC from the above equation is:

$$\cos BAC = \frac{(\mathbf{B}-\mathbf{A})^T(\mathbf{A}-\mathbf{C})}{\|\mathbf{B}-\mathbf{A}\|\,\|\mathbf{A}-\mathbf{C}\|} \tag{2.0.11}$$

Substituting value in (2.0.11) and putting we have:

$$\cos BAC = \frac{(\mathbf{A})^T(\mathbf{A}-\mathbf{C})}{\|\mathbf{A}\|^2} \tag{2.0.12}$$

$$\implies \frac{(\mathbf{A})^T(\mathbf{A}) - (\mathbf{A})^T(\mathbf{C})}{\|\mathbf{A}\|^2} \tag{2.0.13}$$

We know $(\mathbf{A})^T(\mathbf{A}) = \|\mathbf{A}\|^2$
From equation (2.0.5) we have: $(\mathbf{A})^T(\mathbf{C}) = \frac{1}{2}\|\mathbf{A}\|^2$

Substituting values in (2.0.12) we have:

$$\cos BAC = \frac{\frac{1}{2}\left\|\mathbf{A}\right\|^2}{\left\|\mathbf{A}\right\|^2}$$
$$\implies \cos BAC = 1/2$$
$$\implies \angle BAC = 60^\circ \tag{2.0.14}$$

Taking the inner product of sides AC,BC we have:

$$(\mathbf{C}-\mathbf{A})^T(\mathbf{C}-\mathbf{B}) = \left\|\mathbf{C}-\mathbf{A}\right\|\left\|\mathbf{C}-\mathbf{B}\right\|\cos ACB \tag{2.0.15}$$

The angle ACB from the above equation is:

$$\cos ACB = \frac{(\mathbf{C}-\mathbf{A})^T(\mathbf{C}-\mathbf{B})}{\left\|\mathbf{C}-\mathbf{A}\right\|\left\|\mathbf{C}-\mathbf{B}\right\|} \tag{2.0.16}$$

Substituting value in (2.0.16) and putting we have:

$$\cos ACB = \frac{(\mathbf{C}-\mathbf{A})^T(\mathbf{C})}{\left\|\mathbf{A}\right\|^2}$$
$$\implies \frac{(\mathbf{C})^T(\mathbf{C})-(\mathbf{A})^T(\mathbf{C})}{\left\|\mathbf{A}\right\|^2} \tag{2.0.17}$$

We know $(\mathbf{C})^T(\mathbf{C}) = \left\|\mathbf{C}\right\|^2$
From (2.0.4) and (2.0.5) we have:
$(\mathbf{A})^T(\mathbf{C}) = \frac{1}{2}\left\|\mathbf{A}\right\|^2 = \frac{1}{2}\left\|\mathbf{C}\right\|^2$
Substituting values in (2.0.17) we have:

$$\cos ACB = \frac{\frac{1}{2}\left\|\mathbf{C}\right\|^2}{\left\|\mathbf{C}\right\|^2}$$
$$\implies \cos ACB = 1/2$$
$$\implies \angle ACB = 60^\circ \tag{2.0.18}$$

Hence from equation (2.0.9),(2.0.14) and(2.0.16)

$$\angle ABC = \angle BAC = \angle ACB = 60^\circ \tag{2.0.19}$$

Now To Find AD.

$$\begin{aligned}(\mathbf{B}-\mathbf{A})^T(\mathbf{B}-\mathbf{A})&\\ = (\mathbf{B}-\mathbf{D}+\mathbf{D}-\mathbf{A})^T(\mathbf{B}-\mathbf{D}+\mathbf{D}-\mathbf{A})&\\ = [(\mathbf{B}-\mathbf{D})^T+(\mathbf{D}-\mathbf{A})^T][(\mathbf{B}-\mathbf{D})+(\mathbf{D}-\mathbf{A})]&\\ = (\mathbf{B}-\mathbf{D})^T(\mathbf{B}-\mathbf{D})+(\mathbf{B}-\mathbf{D})^T(\mathbf{D}-\mathbf{A})+&\\ (\mathbf{D}-\mathbf{A})^T(\mathbf{B}-\mathbf{D})+(\mathbf{D}-\mathbf{A})^T(\mathbf{D}-\mathbf{A})&\end{aligned} \tag{2.0.20}$$

$$\because \triangle ABD = \angle BAD + \angle ADB + \angle DBA \tag{2.0.21}$$
$$180^\circ = 30^\circ + \angle ADB + 60^\circ. \tag{2.0.22}$$
$$\angle ADB = 180^\circ - (60^\circ + 30^\circ) = 90^\circ. \tag{2.0.23}$$
$$\implies (\mathbf{B}-\mathbf{D})^T(\mathbf{D}-\mathbf{A}) = 0 \tag{2.0.24}$$
$$\implies (\mathbf{D}-\mathbf{A})^T(\mathbf{B}-\mathbf{D}) = 0 \tag{2.0.25}$$

which gives

$$\begin{aligned}(\mathbf{B}-\mathbf{A})^T(\mathbf{B}-\mathbf{A}) =&\\ (\mathbf{B}-\mathbf{D})^T(\mathbf{B}-\mathbf{D})+(\mathbf{D}-\mathbf{A})^T(\mathbf{D}-\mathbf{A})&\\ \implies \left\|\mathbf{B}-\mathbf{A}\right\|^2 = \left\|\mathbf{B}-\mathbf{D}\right\|^2+\left\|\mathbf{D}-\mathbf{A}\right\|^2&\end{aligned} \tag{2.0.26}$$

$$\left\|\mathbf{D}-\mathbf{A}\right\|^2 = \left\|\mathbf{B}-\mathbf{A}\right\|^2 - \left\|\mathbf{B}-\mathbf{D}\right\|^2 \tag{2.0.27}$$

Substitutiing Eq (2.0.1)

$$\left\|\mathbf{D}-\mathbf{A}\right\|^2 = \left\|\mathbf{B}-\mathbf{A}\right\|^2 - \frac{1}{4}\left\|\mathbf{B}-\mathbf{A}\right\|^2 \tag{2.0.28}$$
$$\left\|\mathbf{D}-\mathbf{A}\right\| = \frac{\sqrt{3}}{2}\left\|\mathbf{B}-\mathbf{A}\right\| \tag{2.0.29}$$
$$\implies \left\|\mathbf{B}-\mathbf{A}\right\| = \frac{2}{\sqrt{3}}\left\|\mathbf{D}-\mathbf{A}\right\| \tag{2.0.30}$$

Let $\mathbf{A} = 0$. Then substituting in (2.0.1) and (2.0.30)

$$\left\|\mathbf{B}\right\| = 2\left\|\mathbf{B}-\mathbf{D}\right\| \tag{2.0.31}$$
$$\left\|\mathbf{B}\right\| = \frac{2}{\sqrt{3}}\left\|\mathbf{D}\right\| \tag{2.0.32}$$

Square on both sides in (2.0.31).

$$\left\|\mathbf{B}\right\|^2 = 4\left\|\mathbf{B}-\mathbf{D}\right\|^2 \tag{2.0.33}$$
$$\frac{1}{4}\left\|\mathbf{B}\right\|^2 = \left\|\mathbf{B}\right\|^2 + \left\|\mathbf{D}\right\|^2 - 2\mathbf{B}^T\mathbf{D} \tag{2.0.34}$$

Square on both sides in (2.0.32).

$$\left\|\mathbf{B}\right\|^2 = \frac{4}{3}\left\|\mathbf{D}\right\|^2 \tag{2.0.35}$$

Using (2.0.34) and (2.0.35)

$$\frac{1}{3}\left\|\mathbf{D}\right\|^2 = \frac{4}{3}\left\|\mathbf{D}\right\|^2 + \left\|\mathbf{D}\right\|^2 - 2\mathbf{D}^T\mathbf{D} \tag{2.0.36}$$
$$\implies 0 = \left\|\mathbf{D}\right\|^2 - 2\mathbf{B}^T\mathbf{D} \tag{2.0.37}$$
$$\implies \mathbf{B}^T\mathbf{D} = \left\|\mathbf{D}\right\|^2 \tag{2.0.38}$$

Let $\theta = \angle BAD$. and Taking the inner product of sides BA and AD.

$$(\mathbf{B}-\mathbf{A})^T(\mathbf{A}-\mathbf{D}) = \left\|\mathbf{B}-\mathbf{A}\right\|\left\|\mathbf{A}-\mathbf{D}\right\|\cos\theta \tag{2.0.39}$$

$$\cos\theta = \frac{(\mathbf{B}-\mathbf{A})^T(\mathbf{A}-\mathbf{D})}{\left\|\mathbf{B}-\mathbf{A}\right\|\left\|\mathbf{A}-\mathbf{D}\right\|} \tag{2.0.40}$$

Substitute $\mathbf{A} = 0$ in (2.0.40)

$$\implies \cos\theta = \frac{\mathbf{B}^T\mathbf{D}}{\left\|\mathbf{B}\right\|\left\|\mathbf{D}\right\|} \tag{2.0.41}$$

From (2.0.32) $\|\mathbf{B}\| = \frac{2}{\sqrt{3}}\|\mathbf{D}\|$

$$\implies \cos\theta = \frac{\mathbf{B}^T\mathbf{D}}{\frac{2}{\sqrt{3}}\|\mathbf{D}\|\,\|\mathbf{D}\|} \tag{2.0.42}$$

$$\implies \cos\theta = \frac{\mathbf{B}^T\mathbf{D}}{\frac{2}{\sqrt{3}}\|\mathbf{D}\|^2} \tag{2.0.43}$$

Substitute (2.0.38) in (2.0.43)

$$\cos\theta = \frac{\frac{\sqrt{3}}{2}\|\mathbf{D}\|^2}{\|\mathbf{D}\|^2} \tag{2.0.44}$$

$$\implies \cos\theta = \frac{\sqrt{3}}{2} \tag{2.0.45}$$

$$\because cos 30^\circ = \frac{\sqrt{3}}{2} \tag{2.0.46}$$

$$\implies \theta = 30^\circ \tag{2.0.47}$$

$$\because \cos^2\theta + \sin^2\theta = 1 \tag{2.0.48}$$

$$\sin 30^\circ = \sqrt{1 - \cos^2 30^\circ} \tag{2.0.49}$$

$$\implies \sin 30^\circ = \frac{1}{2}. \tag{2.0.50}$$