Assignment-4

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***Abstract*—This document contains solution of Problem Geolin**(1.12)

Download latex-tikz codes from

https://github.com/ayushkesh/Matrix−Theory−EE5609/tree/master/A4

1 Question

Show that $\sin 30° = \frac{1}{2}$ and $\cos 30° = \frac{\sqrt{3}}{2}$.

2 Solution

Consider an equilateral $\triangle ABC$ as shown in figure:1. Draw a Perpendicular $\mathbf{AD}$ from A on the $\mathbf{BC}$, such that $\mathbf{AD}$ bisect $\angle A$ and $\mathbf{BC}$.

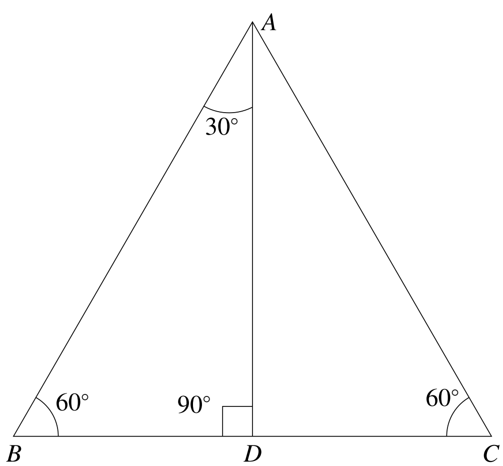

Fig. 1: Equilateral $\triangle ABC$

$$\|\mathbf{A}-\mathbf{B}\| = \|\mathbf{B}-\mathbf{C}\| = \|\mathbf{A}-\mathbf{C}\| = 2\,\|\mathbf{B}-\mathbf{D}\| \tag{2.0.1}$$

To Find AD.

$$\begin{aligned}(\mathbf{B}-\mathbf{A})^T(\mathbf{B}-\mathbf{A}) \\ &= (\mathbf{B}-\mathbf{D}+\mathbf{D}-\mathbf{A})^T(\mathbf{B}-\mathbf{D}+\mathbf{D}-\mathbf{A}) \\ &= [(\mathbf{B}-\mathbf{D})^T + (\mathbf{D}-\mathbf{A})^T][(\mathbf{B}-\mathbf{D}) + (\mathbf{D}-\mathbf{A})] \\ &= (\mathbf{B}-\mathbf{D})^T(\mathbf{B}-\mathbf{D}) + (\mathbf{B}-\mathbf{D})^T(\mathbf{D}-\mathbf{A}) + \\ &\quad (\mathbf{D}-\mathbf{A})^T(\mathbf{B}-\mathbf{D}) + (\mathbf{D}-\mathbf{A})^T(\mathbf{D}-\mathbf{A})\end{aligned} \tag{2.0.2}$$

Since BD is the perpendicular to AD the inner product is zero

$$(\mathbf{B}-\mathbf{D})^T(\mathbf{D}-\mathbf{A}) = 0 \tag{2.0.3}$$

$$(\mathbf{D}-\mathbf{A})^T(\mathbf{B}-\mathbf{D}) = 0 \tag{2.0.4}$$

which gives

$$\begin{aligned}(\mathbf{B}-\mathbf{A})^T(\mathbf{B}-\mathbf{A}) &= \\ (\mathbf{B}-\mathbf{D})^T(\mathbf{B}-\mathbf{D}) &+ (\mathbf{D}-\mathbf{A})^T(\mathbf{D}-\mathbf{A}) \\ \implies \|\mathbf{B}-\mathbf{A}\|^2 &= \|\mathbf{B}-\mathbf{D}\|^2 + \|\mathbf{D}-\mathbf{A}\|^2\end{aligned} \tag{2.0.5}$$

$$\|\mathbf{D}-\mathbf{A}\|^2 = \|\mathbf{B}-\mathbf{A}\|^2 - \|\mathbf{B}-\mathbf{D}\|^2 \tag{2.0.6}$$

By using Eq (2.0.1) and (2.0.6)

$$\|\mathbf{D}-\mathbf{A}\| \frac{\sqrt{3}}{2} \|\mathbf{B}-\mathbf{A}\| \tag{2.0.7}$$

$$\implies \|\mathbf{B}-\mathbf{A}\| = \frac{2}{\sqrt{3}} \|\mathbf{D}-\mathbf{A}\| \tag{2.0.8}$$

Let $\mathbf{A} = 0$. Then substituting in (2.0.1) and (2.0.8)

$$\|\mathbf{B}\| = 2\,\|\mathbf{B}-\mathbf{D}\| \tag{2.0.9}$$

$$\|\mathbf{B}\| = \frac{2}{\sqrt{3}} \|\mathbf{D}\| \tag{2.0.10}$$

Square on both sides in (2.0.9).

$$\|\mathbf{B}\|^2 = 4\,\|\mathbf{B}-\mathbf{D}\|^2 \tag{2.0.11}$$

$$\frac{1}{4}\|\mathbf{B}\|^2 = \|\mathbf{B}\|^2 + \|\mathbf{D}\|^2 - 2\mathbf{B}^T\mathbf{D} \tag{2.0.12}$$

Square on both sides in (2.0.10).

$$\|\mathbf{B}\|^2 = \frac{4}{3}\|\mathbf{D}\|^2 \tag{2.0.13}$$

Using (2.0.12) and (2.0.13)

$$\frac{1}{3}\|\mathbf{D}\|^2 = \frac{4}{3}\|\mathbf{D}\|^2 + \|\mathbf{D}\|^2 - 2\mathbf{D}^T\mathbf{D} \tag{2.0.14}$$

$$\implies 0 = \|\mathbf{D}\|^2 - 2\mathbf{B}^T\mathbf{D} \tag{2.0.15}$$

$$\implies \mathbf{B}^T\mathbf{D} = \|\mathbf{D}\|^2 \tag{2.0.16}$$

Let $\theta = \angle BAD$. and Taking the inner product of sides BA and AD.

$$(\mathbf{B}-\mathbf{A})^T(\mathbf{A}-\mathbf{D}) = \|\mathbf{B}-\mathbf{A}\|\,\|\mathbf{A}-\mathbf{D}\|\cos\theta \tag{2.0.17}$$

$$\cos\theta = \frac{(\mathbf{B}-\mathbf{A})^T(\mathbf{A}-\mathbf{D})}{\|\mathbf{B}-\mathbf{A}\|\,\|\mathbf{A}-\mathbf{D}\|} \tag{2.0.18}$$

Substitute $\mathbf{A} = 0$ in (2.0.18)

$$\implies \cos\theta = \frac{\mathbf{B}^T\mathbf{D}}{\|\mathbf{B}\|\,\|\mathbf{D}\|} \tag{2.0.19}$$

Using (2.0.10)

$$\implies \cos\theta = \frac{\mathbf{B}^T\mathbf{D}}{\frac{2}{\sqrt{3}}\|\mathbf{D}\|\,\|\mathbf{D}\|} \tag{2.0.20}$$

$$\implies \cos\theta = \frac{\mathbf{B}^T\mathbf{D}}{\frac{2}{\sqrt{3}}\|\mathbf{D}\|^2} \tag{2.0.21}$$

Substitute (2.0.16) in (2.0.21)

$$\cos\theta = \frac{\frac{\sqrt{3}}{2}\|\mathbf{D}\|^2}{\|\mathbf{D}\|^2} \tag{2.0.22}$$

$$\implies \cos\theta = \frac{\sqrt{3}}{2} \tag{2.0.23}$$

$$\because cos30^\circ = \frac{\sqrt{3}}{2} \tag{2.0.24}$$

$$\implies \theta = 30^\circ \tag{2.0.25}$$

$$\because \cos^2\theta + \sin^2\theta = 1 \tag{2.0.26}$$

$$\sin 30^\circ = \sqrt{1-\cos^2 30^\circ} \tag{2.0.27}$$

$$\implies \sin 30^\circ = \frac{1}{2}. \tag{2.0.28}$$