

Assignment-4

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Abstract—This document contains solution of Problem Geolin(1.12)

Download latex-tikz codes from

<https://github.com/ayushkesh/Matrix-Theory-EE5609/tree/master/A4>

1 QUESTION

Show that $\sin 30^\circ = \frac{1}{2}$ and $\cos 30^\circ = \frac{\sqrt{3}}{2}$.

2 SOLUTION

Consider an equilateral $\triangle ABC$ as shown in figure:1. Draw a Perpendicular \mathbf{AD} from A on the \mathbf{BC} , such that \mathbf{AD} bisect $\angle A$ and \mathbf{BC} .

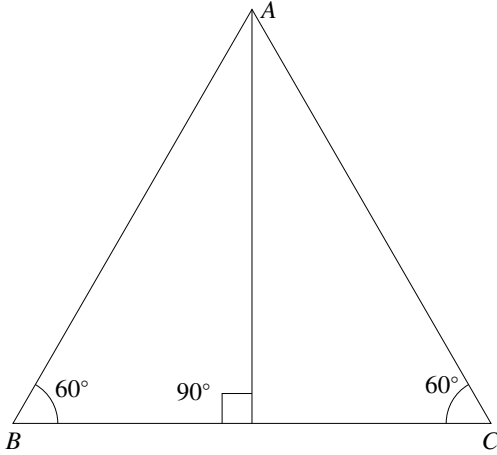


Fig. 1: Equilateral $\triangle ABC$

$$\|\mathbf{A} - \mathbf{B}\| = \|\mathbf{B} - \mathbf{C}\| = \|\mathbf{A} - \mathbf{C}\| = 2 \|\mathbf{B} - \mathbf{D}\| \quad (2.0.1)$$

To Find AD.

$$\begin{aligned} (\mathbf{B} - \mathbf{A})^T (\mathbf{B} - \mathbf{A}) &= (\mathbf{B} - \mathbf{D} + \mathbf{D} - \mathbf{A})^T (\mathbf{B} - \mathbf{D} + \mathbf{D} - \mathbf{A}) \\ &= [(\mathbf{B} - \mathbf{D})^T + (\mathbf{D} - \mathbf{A})^T][(\mathbf{B} - \mathbf{D}) + (\mathbf{D} - \mathbf{A})] \\ &= (\mathbf{B} - \mathbf{D})^T (\mathbf{B} - \mathbf{D}) + (\mathbf{B} - \mathbf{D})^T (\mathbf{D} - \mathbf{A}) + \\ &\quad (\mathbf{D} - \mathbf{A})^T (\mathbf{B} - \mathbf{D}) + (\mathbf{D} - \mathbf{A})^T (\mathbf{D} - \mathbf{A}) \quad (2.0.2) \end{aligned}$$

Since BD is the perpendicular to AD the inner product is zero

$$(\mathbf{B} - \mathbf{D})^T (\mathbf{D} - \mathbf{A}) = 0 \quad (2.0.3)$$

$$(\mathbf{D} - \mathbf{A})^T (\mathbf{B} - \mathbf{D}) = 0 \quad (2.0.4)$$

which gives

$$\begin{aligned} (\mathbf{B} - \mathbf{A})^T (\mathbf{B} - \mathbf{A}) &= (\mathbf{B} - \mathbf{D})^T (\mathbf{B} - \mathbf{D}) + (\mathbf{D} - \mathbf{A})^T (\mathbf{D} - \mathbf{A}) \\ \Rightarrow \|\mathbf{B} - \mathbf{A}\|^2 &= \|\mathbf{B} - \mathbf{D}\|^2 + \|\mathbf{D} - \mathbf{A}\|^2 \quad (2.0.5) \end{aligned}$$

$$\|\mathbf{D} - \mathbf{A}\|^2 = \|\mathbf{B} - \mathbf{A}\|^2 - \|\mathbf{B} - \mathbf{D}\|^2 \quad (2.0.6)$$

By using Eq (2.0.1) and (2.0.6)

$$\|\mathbf{D} - \mathbf{A}\| \frac{\sqrt{3}}{2} \|\mathbf{B} - \mathbf{A}\| \quad (2.0.7)$$

$$\Rightarrow \|\mathbf{B} - \mathbf{A}\| = \frac{2}{\sqrt{3}} \|\mathbf{D} - \mathbf{A}\| \quad (2.0.8)$$

Let $\mathbf{A} = 0$. Then substituting in (2.0.1) and (2.0.8)

$$\|\mathbf{B}\| = 2 \|\mathbf{B} - \mathbf{D}\| \quad (2.0.9)$$

$$\|\mathbf{B}\| = \frac{2}{\sqrt{3}} \|\mathbf{D}\| \quad (2.0.10)$$

Square on both sides in (2.0.9).

$$\|\mathbf{B}\|^2 = 4 \|\mathbf{B} - \mathbf{D}\|^2 \quad (2.0.11)$$

$$\frac{1}{4} \|\mathbf{B}\|^2 = \|\mathbf{B}\|^2 + \|\mathbf{D}\|^2 - 2\mathbf{B}^T \mathbf{D} \quad (2.0.12)$$

Square on both sides in (2.0.10).

$$\|\mathbf{B}\|^2 = \frac{4}{3} \|\mathbf{D}\|^2 \quad (2.0.13)$$

Using (2.0.12) and (2.0.13)

$$\frac{1}{3} \|\mathbf{D}\|^2 = \frac{4}{3} \|\mathbf{D}\|^2 + \|\mathbf{D}\|^2 - 2\mathbf{D}^T \mathbf{D} \quad (2.0.14)$$

$$\Rightarrow 0 = \|\mathbf{D}\|^2 - 2\mathbf{D}^T \mathbf{D} \quad (2.0.15)$$

$$\Rightarrow \mathbf{B}^T \mathbf{D} = \|\mathbf{D}\|^2 \quad (2.0.16)$$

Let $\theta = \angle BAD$. and Taking the inner product of sides BA and AD.

$$(\mathbf{B} - \mathbf{A})^T (\mathbf{A} - \mathbf{D}) = \|\mathbf{B} - \mathbf{A}\| \|\mathbf{A} - \mathbf{D}\| \cos \theta \quad (2.0.17)$$

$$\cos \theta = \frac{(\mathbf{B} - \mathbf{A})^T (\mathbf{A} - \mathbf{D})}{\|\mathbf{B} - \mathbf{A}\| \|\mathbf{A} - \mathbf{D}\|} \quad (2.0.18)$$

Substitute $\mathbf{A} = 0$ in (2.0.18)

$$\Rightarrow \cos \theta = \frac{\mathbf{B}^T \mathbf{D}}{\|\mathbf{B}\| \|\mathbf{D}\|} \quad (2.0.19)$$

Using (2.0.10)

$$\Rightarrow \cos \theta = \frac{\mathbf{B}^T \mathbf{D}}{\frac{2}{\sqrt{3}} \|\mathbf{D}\| \|\mathbf{D}\|} \quad (2.0.20)$$

$$\because \mathbf{B}^T \mathbf{D} = \|\mathbf{D}\|^2 \quad (2.0.21)$$

$$\Rightarrow \cos \theta = \frac{\mathbf{B}^T \mathbf{D}}{\frac{2}{\sqrt{3}} \|\mathbf{D}\|^2} \quad (2.0.22)$$

Substitute (2.0.16) in (2.0.22)

$$\cos \theta = \frac{\frac{\sqrt{3}}{2} \|\mathbf{D}\|^2}{\|\mathbf{D}\|^2} \quad (2.0.23)$$

$$\Rightarrow \cos \theta = \frac{\sqrt{3}}{2} \quad (2.0.24)$$

$$\because \cos 30^\circ = \frac{\sqrt{3}}{2} \quad (2.0.25)$$

$$\Rightarrow \theta = 30^\circ \quad (2.0.26)$$

$$\because \cos^2 \theta + \sin^2 \theta = 1 \quad (2.0.27)$$

$$\sin 30^\circ = \sqrt{1 - \cos^2 30^\circ} \quad (2.0.28)$$

$$\Rightarrow \sin 30^\circ = \frac{1}{2}. \quad (2.0.29)$$