Hyperbola

***Abstract*—This document contains solution of Problem Loney**(314, 7)

Download latex-tikz codes from

https://github.com/ayushkesh/Matrix−Theory−EE5609/tree/master/A7

1 Question

Find the asymptotes of the hyperbola given below and also the equations to their conjugate hyperbolas. $8x^2 + 10xy - 3y^2 - 2x + 4y - 2 = 0$

2 Solution

The above equation can be expressed in the form

$$\mathbf{x}^T\mathbf{V}\mathbf{x} + 2\mathbf{u}^T\mathbf{x} + f = 0 \tag{2.0.1}$$

Comparing equation we get

$$\mathbf{V} = \mathbf{V}^T = \begin{pmatrix} 8 & 5 \\ 5 & -3 \end{pmatrix} \tag{2.0.2}$$

$$\mathbf{u} = \begin{pmatrix} -1 \\ 2 \end{pmatrix} \tag{2.0.3}$$

$$f = -2 \tag{2.0.4}$$

Expanding the Determinant of V.

$$\Delta_V = \begin{vmatrix} 8 & 5 \\ 5 & -3 \end{vmatrix} < 0 \tag{2.0.5}$$

Hence from (2.0.5) given equation represents the hyperbola The characteristic equation of $\mathbf{V}$ is obtained by evaluating the determinant

$$\left| V - \lambda\mathbf{I} \right| = 0 \tag{2.0.6}$$

$$\begin{vmatrix} 8-\lambda & 5 \\ 5 & -3-\lambda \end{vmatrix} = 0 \tag{2.0.7}$$

$$(8-\lambda)(-3-\lambda) - 25 = 0 \tag{2.0.8}$$

$$\lambda_1 = \frac{5+\sqrt{221}}{2} \tag{2.0.9}$$

$$\lambda_2 = \frac{5-\sqrt{221}}{2} \tag{2.0.10}$$

The eigenvector $\mathbf{p}$ is defined as

$$\mathbf{V}\mathbf{p} = \lambda\mathbf{p} \tag{2.0.11}$$

$$\implies (\mathbf{V} - \lambda\mathbf{I})\mathbf{p} = 0 \tag{2.0.12}$$

For $\lambda_1 = \frac{5+\sqrt{221}}{2}$,

$$(\mathbf{V} - \lambda_1\mathbf{I}) = \begin{pmatrix} \frac{11-\sqrt{221}}{2} & 5 \\ 5 & \frac{-11-\sqrt{221}}{2} \end{pmatrix} \tag{2.0.13}$$

By row reduction ,

$$\begin{pmatrix} \frac{11-\sqrt{221}}{2} & 5 \\ 5 & \frac{-11-\sqrt{221}}{2} \end{pmatrix} \tag{2.0.14}$$

$$\xleftrightarrow{R_1 \leftarrow R_2} \begin{pmatrix} \frac{-11-\sqrt{221}}{2} & 5 \\ \frac{11-\sqrt{221}}{2} & 5 \end{pmatrix} \tag{2.0.15}$$

$$\xleftrightarrow{R_2 \leftarrow R_2 - \frac{11-\sqrt{221}}{10}R_1} \begin{pmatrix} 5 & \frac{-11-\sqrt{221}}{2} \\ 0 & 0 \end{pmatrix} \tag{2.0.16}$$

$$\xleftrightarrow{R_1 \leftarrow R_1/5} \begin{pmatrix} 1 & \frac{-11-\sqrt{221}}{10} \\ 0 & 0 \end{pmatrix} \tag{2.0.17}$$

Subsituting equation 2.0.17 in equation 2.0.12 we get

$$\begin{pmatrix} 1 & \frac{-11-\sqrt{221}}{10} \\ 0 & 0 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{2.0.18}$$

Where, $\mathbf{p} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$ Let $v_2 = t$

$$v_1 = \frac{t(11+\sqrt{221})}{10} \tag{2.0.19}$$

Eigen vector $\mathbf{p_1}$ is given by

$$\mathbf{p_1} = \begin{pmatrix} \frac{t(11+\sqrt{221})}{10} \\ t \end{pmatrix} \tag{2.0.20}$$

Let $t = 1$, we get

$$\mathbf{p_1} = \begin{pmatrix} \frac{11+\sqrt{221}}{10} \\ 1 \end{pmatrix} \tag{2.0.21}$$

For $\lambda_2 = \frac{5-\sqrt{221}}{2}$,

$$(\mathbf{V}-\lambda_2\mathbf{I}) = \begin{pmatrix} \frac{11+\sqrt{221}}{2} & 5 \\ 5 & \frac{-11+\sqrt{221}}{2} \end{pmatrix} \tag{2.0.22}$$

By row reduction ,

$$\begin{pmatrix} \frac{11+\sqrt{221}}{2} & 5 \\ 5 & \frac{-11+\sqrt{221}}{2} \end{pmatrix} \xleftrightarrow{R_1 \leftarrow R_2 + \frac{11-\sqrt{221}}{10} R_1} \begin{pmatrix} \frac{11+\sqrt{221}}{2} & 5 \\ 0 & 0 \end{pmatrix} \tag{2.0.23}$$

$$\xleftrightarrow{R_1 \leftarrow \frac{R_1}{\frac{11+\sqrt{221}}{10}}} \begin{pmatrix} 1 & \frac{10}{11+\sqrt{221}} \\ 0 & 0 \end{pmatrix} \tag{2.0.24}$$

Substiuting equation 2.0.24 in equation 2.0.12 we get

$$\begin{pmatrix} 1 & \frac{10}{11+\sqrt{221}} \\ 0 & 0 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{2.0.25}$$

Where, $\mathbf{p} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$ Let $v_2 = t$

$$v_1 = \frac{-t\,(10)}{11+\sqrt{221}} \tag{2.0.26}$$

Eigen vector $\mathbf{p_2}$ is given by

$$\mathbf{p_2} = \begin{pmatrix} \frac{-t(10)}{11+\sqrt{221}} \\ t \end{pmatrix} \tag{2.0.27}$$

Let $t = 1$, we get

$$\mathbf{p_2} = \begin{pmatrix} \frac{(-10)}{11+\sqrt{221}} \\ 1 \end{pmatrix} \tag{2.0.28}$$

By eigen decompostion $\mathbf{V}$ can be represented by

$$\mathbf{V} = \mathbf{PDP}^T \tag{2.0.29}$$

where

$$\mathbf{P} = \begin{pmatrix} \mathbf{p_1} & \mathbf{p_2} \end{pmatrix} \tag{2.0.30}$$

$$\mathbf{D} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \tag{2.0.31}$$

Substituting equations 2.0.21, 2.0.28 in equation 2.0.30 we get

$$\mathbf{P} = \begin{pmatrix} \frac{11+\sqrt{221}}{10} & \frac{-10}{11+\sqrt{221}} \\ 1 & 1 \end{pmatrix} \tag{2.0.32}$$

Substituting equations 2.0.9, 2.0.10 in 2.0.31 we get

$$\mathbf{D} = \begin{pmatrix} \frac{5+\sqrt{221}}{2} & 0 \\ 0 & \frac{5-\sqrt{221}}{2} \end{pmatrix} \tag{2.0.33}$$

Centre of the hyperbola is given by

$$\mathbf{c} = -\mathbf{V}^{-1}\mathbf{u} \tag{2.0.34}$$

$$\implies \mathbf{c} = -\begin{pmatrix} \frac{3}{49} & \frac{5}{49} \\ \frac{5}{49} & \frac{-8}{49} \end{pmatrix} \begin{pmatrix} -1 \\ 2 \end{pmatrix} \tag{2.0.35}$$

$$\implies \mathbf{c} = \begin{pmatrix} \frac{-3}{49} & \frac{-5}{49} \\ \frac{-5}{49} & \frac{8}{49} \end{pmatrix} \begin{pmatrix} -1 \\ 2 \end{pmatrix} \tag{2.0.36}$$

$$\implies \mathbf{c} = \begin{pmatrix} \frac{-1}{7} \\ \frac{3}{7} \end{pmatrix} \tag{2.0.37}$$

Since,

$$\mathbf{u}^T\mathbf{V}^{-1}\mathbf{u} - f = 1 > 0 \tag{2.0.38}$$

there isn't a need to swap axes In hyperbola,

$$axes = \begin{cases} \sqrt{\frac{\mathbf{u}^T\mathbf{V}^{-1}\mathbf{u}-f}{\lambda_1}} \\ \sqrt{\frac{f-\mathbf{u}^T\mathbf{V}^{-1}\mathbf{u}}{\lambda_2}} \end{cases} \tag{2.0.39}$$

From above equations we can say that,

$$\sqrt{\frac{\mathbf{u}^T\mathbf{V}^{-1}\mathbf{u}-f}{\lambda_1}} = \sqrt{\frac{2}{5+\sqrt{221}}} \tag{2.0.40}$$

$$\sqrt{\frac{f-\mathbf{u}^T\mathbf{V}^{-1}\mathbf{u}}{\lambda_2}} = \sqrt{\frac{2}{5-\sqrt{221}}} \tag{2.0.41}$$

Now we have,

$$\mathbf{y}^T\mathbf{D}\mathbf{y} = \mathbf{u}^T\mathbf{V}^{-1}\mathbf{u} - f \tag{2.0.42}$$

where ,

$$\mathbf{y} = \mathbf{P}^T(\mathbf{x}-\mathbf{c}) \tag{2.0.43}$$

To get $\mathbf{y}$,

$$\mathbf{y} = \mathbf{P}^T\mathbf{x} - \mathbf{P}^T\mathbf{c} \tag{2.0.44}$$

$$\mathbf{y} = \begin{pmatrix} \frac{11+\sqrt{221}}{10} & 1 \\ \frac{-10}{11+\sqrt{221}} & 1 \end{pmatrix}\mathbf{x} - \begin{pmatrix} \frac{11+\sqrt{221}}{10} & 1 \\ \frac{-10}{11+\sqrt{221}} & 1 \end{pmatrix} \begin{pmatrix} \frac{-1}{7} \\ \frac{3}{7} \end{pmatrix} \tag{2.0.45}$$

$$\mathbf{y} = \begin{pmatrix} \frac{11+\sqrt{221}}{10} & 1 \\ \frac{-10}{11+\sqrt{221}} & 1 \end{pmatrix}\mathbf{x} - \begin{pmatrix} \frac{-11-\sqrt{221}}{70} + \frac{3}{7} \\ \frac{10}{(7)11+(7)\sqrt{221}} + \frac{3}{7} \end{pmatrix} \tag{2.0.46}$$

Subsituting the eqauations (2.0.38), (2.0.33) in equation (2.0.42)

$$\implies \mathbf{y}^T \begin{pmatrix} \frac{5+\sqrt{221}}{2} & 0 \\ 0 & \frac{5-\sqrt{221}}{2} \end{pmatrix} \mathbf{y} + 2 = 0 \tag{2.0.47}$$

Python Code

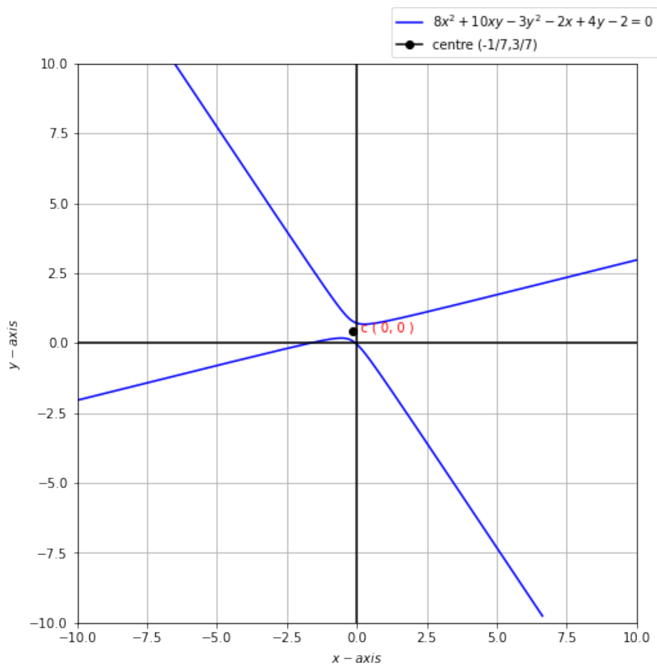

Fig. 1: Hyperbola $8x^2 + 10xy - 3y^2 - 2x + 4y - 2 = 0$

https://github.com/ayushkesh/Matrix−Theory−EE5609/blob/master/A7/codes/A7_1.ipynb

2.1 Asymptotes of hyperbola

Equation of a hyperbola and the combined equation of the Asymptotes differ only in the constant term.

$$8x^2 + 10xy - 3y^2 2x + 4y + K = 0 \tag{2.1.1}$$

The above equation can be expressed in the form

$$\mathbf{x}^T \mathbf{V} \mathbf{x} + 2\mathbf{u}^T \mathbf{x} + f = 0 \tag{2.1.2}$$

Comparing equation we get

$$\mathbf{V} = \mathbf{V}^T = \begin{pmatrix} 8 & 5 \\ 5 & -3 \end{pmatrix} \tag{2.1.3}$$

$$\mathbf{u} = \begin{pmatrix} -1 \\ 2 \end{pmatrix} \tag{2.1.4}$$

$$f = K \tag{2.1.5}$$

$$\Delta = \begin{vmatrix} 8 & 5 & -1 \\ 5 & -3 & 2 \\ -1 & 2 & K \end{vmatrix} \tag{2.1.6}$$

$$\implies K = -1 \tag{2.1.7}$$

Similar way expanding the Determinant of V.

$$\Delta_V = \begin{vmatrix} 8 & 5 \\ 5 & -3 \end{vmatrix} < 0 \tag{2.1.8}$$

From (2.1.8) we could say that the given equation represents two straight lines Let the equations of lines be,

$$\left(\mathbf{n_1}^T \mathbf{x} - c_1\right)\left(\mathbf{n_1}^T \mathbf{x} - c_1\right) = \mathbf{x}^T \mathbf{V} \mathbf{x} + 2\mathbf{u}^T \mathbf{x} + f = 0 \tag{2.1.9}$$

$$\left(\mathbf{n_1}^T \mathbf{x} - c_1\right)\left(\mathbf{n_2}^T \mathbf{x} - c_2\right) = \mathbf{x}^T \begin{pmatrix} 8 & 5 \\ 5 & -3 \end{pmatrix} \mathbf{x}$$

$$+2\begin{pmatrix} -1 & 2 \end{pmatrix} \mathbf{x} - 1 \tag{2.1.10}$$

$$\mathbf{n_1} * \mathbf{n_2} = \begin{pmatrix} a \\ 2b \\ c \end{pmatrix} = \begin{pmatrix} 8 \\ 10 \\ -3 \end{pmatrix} \tag{2.1.11}$$

$$c_2 \mathbf{n_1} + c_1 \mathbf{n_2} = -2\begin{pmatrix} -1 \\ 2 \end{pmatrix} \tag{2.1.12}$$

$$c_1 c_2 = -1 \tag{2.1.13}$$

The slopes of the lines are given by the roots of the polynomial

$$cm^2 + 2bm + a = 0 \tag{2.1.14}$$

$$\implies m_i = \frac{-b \pm \sqrt{-\Delta_V}}{c} \tag{2.1.15}$$

$$\mathbf{n_i} = k\begin{pmatrix} -m_i \\ 1 \end{pmatrix} \tag{2.1.16}$$

Substituting the given data in above equations (2.1.14) we get,

$$-3m^2 + 10m + 8 = 0 \tag{2.1.17}$$

$$m_1 = 4, m_2 = \frac{-2}{3} \tag{2.1.18}$$

$$= \mathbf{n_1} = \begin{pmatrix} -4 \\ 1 \end{pmatrix}, \mathbf{n_2} = \begin{pmatrix} -2 \\ -3 \end{pmatrix} \tag{2.1.19}$$

We know that,

$$\mathbf{n_1} * \mathbf{n_2} = \begin{pmatrix} a \\ 2b \\ c \end{pmatrix} \tag{2.1.20}$$

Verification using Toeplitz matrix, From equation (2.1.19)

$$\mathbf{n_1} = \begin{pmatrix} -4 & 0 \\ 1 & -4 \\ 0 & -1 \end{pmatrix} \mathbf{n_2} = \begin{pmatrix} -2 \\ -3 \end{pmatrix} \tag{2.1.21}$$

$$\implies \begin{pmatrix} -4 & 0 \\ 1 & -4 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ -1 \end{pmatrix} = \begin{pmatrix} 8 \\ 10 \\ -3 \end{pmatrix} = \begin{pmatrix} a \\ 2b \\ c \end{pmatrix} \tag{2.1.22}$$

$\implies$ Equation (2.1.19) satisfies (2.1.20)
c_1 and c_2 can be obtained as,

$$\begin{pmatrix} \mathbf{n_1} & \mathbf{n_2} \end{pmatrix} \begin{pmatrix} c_2 \\ c_1 \end{pmatrix} = -2\mathbf{u} \tag{2.1.23}$$

Substituting (2.1.19) in (2.1.23), the augmented matrix is,

$$\begin{pmatrix} -4 & -2 & -2 \\ 1 & -3 & 4 \end{pmatrix} \xleftrightarrow[R_2 \leftarrow R_2 - R_1]{R_1 \leftarrow -R_1/4} \begin{pmatrix} 1 & \frac{1}{2} & \frac{1}{2} \\ 0 & -\frac{7}{2} & \frac{7}{2} \end{pmatrix} \tag{2.1.24}$$

$$\xleftrightarrow[R_1 \leftarrow R_1 - \frac{1}{2}R_2]{R_2 \leftarrow -\frac{2}{7}R_2} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \end{pmatrix} \tag{2.1.25}$$

$$\implies c_1 = 1, c_2 = -1 \tag{2.1.26}$$

Equations (2.1.9), can be modified as,from (2.1.19) and (2.1.26) in we get,

$$\begin{pmatrix} -4 & 1 \end{pmatrix} \mathbf{x} = 1 \tag{2.1.27}$$

$$\begin{pmatrix} -2 & -3 \end{pmatrix} \mathbf{x} = -1 \tag{2.1.28}$$

$$\implies (-4x + y - 1)(-2x - 3y + 1) = 0$$

$$\implies \boxed{(4x - y + 1)(2x + 3y - 1) = 0} \tag{2.1.29}$$

The angle between the lines can be expressed as,

$$\mathbf{n_1} = \begin{pmatrix} -4 \\ 1 \end{pmatrix}, \quad \mathbf{n_2} = \begin{pmatrix} -2 \\ -3 \end{pmatrix} \tag{2.1.30}$$

$$\cos\theta = \frac{\mathbf{n_1}^T \mathbf{n_2}}{\|\mathbf{n_1}\| \|\mathbf{n_2}\|} \tag{2.1.31}$$

$$\implies \quad \theta = \cos^{-1}\left(\frac{0}{\sqrt{221}}\right) = 90°. \tag{2.1.32}$$

Python Code

https://github.com/ayushkesh/Matrix−Theory−EE5609/blob/master/A7/codes/A7_2.ipynb

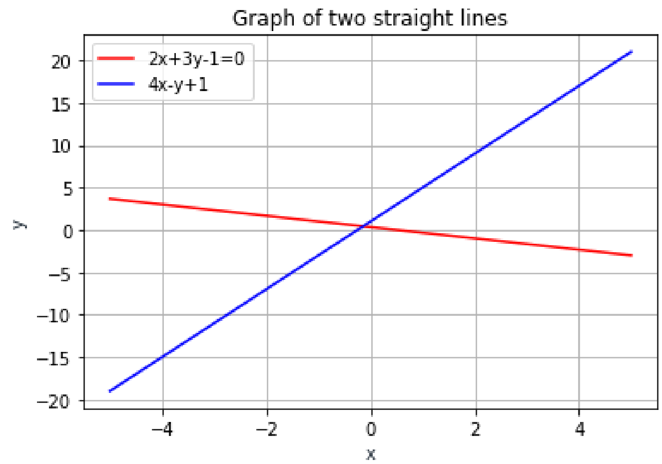

Fig. 1: Pair of straight lines

2.2 *Equation of Asymptotes*

The characteristic equation of $\mathbf{V}$ is obtained by evaluating the determinant (2.1.3)

$$\left| V - \lambda\mathbf{I} \right| = 0 \tag{2.2.1}$$

$$\begin{vmatrix} 8 - \lambda & 5 \\ 5 & -3 - \lambda \end{vmatrix} = 0 \tag{2.2.2}$$

$$(8 - \lambda)(-3 - \lambda) - 25 = 0 \tag{2.2.3}$$

$$\lambda_1 = \frac{5 + \sqrt{221}}{2} \tag{2.2.4}$$

$$\lambda_2 = \frac{5 - \sqrt{221}}{2} \tag{2.2.5}$$

The eigenvector $\mathbf{p}$ is defined as

$$\mathbf{Vp} = \lambda\mathbf{p} \tag{2.2.6}$$

$$\implies (\mathbf{V} - \lambda\mathbf{I})\mathbf{p} = 0 \tag{2.2.7}$$

For $\lambda_1 = \frac{5+\sqrt{221}}{2}$,

$$(\mathbf{V} - \lambda_1\mathbf{I}) = \begin{pmatrix} \frac{11-\sqrt{221}}{2} & 5 \\ 5 & \frac{-11-\sqrt{221}}{2} \end{pmatrix} \tag{2.2.8}$$

By row reduction ,

$$\begin{pmatrix} \frac{11-\sqrt{221}}{2} & 5 \\ 5 & \frac{-11-\sqrt{221}}{2} \end{pmatrix} \tag{2.2.9}$$

$$\xleftrightarrow{R_1 \leftarrow R_2} \begin{pmatrix} \frac{-11-\sqrt{221}}{2} & 5 \\ \frac{11-\sqrt{221}}{2} & 5 \end{pmatrix} \tag{2.2.10}$$

$$\xleftrightarrow{R_2 \leftarrow R_2 - \frac{11-\sqrt{221}}{10} R_1} \begin{pmatrix} 5 & \frac{-11-\sqrt{221}}{2} \\ 0 & 0 \end{pmatrix} \tag{2.2.11}$$

$$\xleftrightarrow{R_1 \leftarrow R_1/5} \begin{pmatrix} 1 & \frac{-11-\sqrt{221}}{10} \\ 0 & 0 \end{pmatrix} \tag{2.2.12}$$

Subsituting equation 2.2.12 in equation 2.2.7 we get

$$\begin{pmatrix} 1 & \frac{-11-\sqrt{221}}{10} \\ 0 & 0 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{2.2.13}$$

Where, $\mathbf{p} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$ Let $v_2 = t$

$$v_1 = \frac{t(11 + \sqrt{221})}{10} \tag{2.2.14}$$

Eigen vector $\mathbf{p_1}$ is given by

$$\mathbf{p_1} = \begin{pmatrix} \frac{t(11+\sqrt{221})}{10} \\ t \end{pmatrix} \tag{2.2.15}$$

Let $t = 1$, we get

$$\mathbf{p_1} = \begin{pmatrix} \frac{11+\sqrt{221}}{10} \\ 1 \end{pmatrix} \tag{2.2.16}$$

For $\lambda_2 = \frac{5-\sqrt{221}}{2}$,

$$(\mathbf{V} - \lambda_2 \mathbf{I}) = \begin{pmatrix} \frac{11+\sqrt{221}}{2} & 5 \\ 5 & \frac{-11+\sqrt{221}}{2} \end{pmatrix} \tag{2.2.17}$$

By row reduction ,

$$\begin{pmatrix} \frac{11+\sqrt{221}}{2} & 5 \\ 5 & \frac{-11+\sqrt{221}}{2} \end{pmatrix} \xleftrightarrow{R_1 \leftarrow R_2 + \frac{11-\sqrt{221}}{10} R_1} \begin{pmatrix} \frac{11+\sqrt{221}}{2} & 5 \\ 0 & 0 \end{pmatrix} \tag{2.2.18}$$

$$\xleftrightarrow{R_1 \leftarrow \frac{R_1}{\frac{11+\sqrt{221}}{10}}} \begin{pmatrix} 1 & \frac{10}{11+\sqrt{221}} \\ 0 & 0 \end{pmatrix} \tag{2.2.19}$$

Subsituting equation 2.2.19 in equation 2.2.7 we get

$$\begin{pmatrix} 1 & \frac{10}{11+\sqrt{221}} \\ 0 & 0 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{2.2.20}$$

Where, $\mathbf{p} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$ Let $v_2 = t$

$$v_1 = \frac{-t\,(10)}{11 + \sqrt{221}} \tag{2.2.21}$$

Eigen vector $\mathbf{p_2}$ is given by

$$\mathbf{p_2} = \begin{pmatrix} \frac{-t(10)}{11+\sqrt{221}} \\ t \end{pmatrix} \tag{2.2.22}$$

Let $t = 1$, we get

$$\mathbf{p_2} = \begin{pmatrix} \frac{(-10)}{11+\sqrt{221}} \\ 1 \end{pmatrix} \tag{2.2.23}$$

By eigen decompostion $\mathbf{V}$ can be represented by

$$\mathbf{V} = \mathbf{PDP}^T \tag{2.2.24}$$

where

$$\mathbf{P} = \begin{pmatrix} \mathbf{p_1} & \mathbf{p_2} \end{pmatrix} \tag{2.2.25}$$

$$\mathbf{D} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \tag{2.2.26}$$

Substituting equations 2.2.16, 2.2.23 in equation 2.2.25 we get

$$\mathbf{P} = \begin{pmatrix} \frac{11+\sqrt{221}}{10} & \frac{-10}{11+\sqrt{221}} \\ 1 & 1 \end{pmatrix} \tag{2.2.27}$$

$$\mathbf{D} = \begin{pmatrix} \frac{5+\sqrt{221}}{2} & 0 \\ 0 & \frac{5-\sqrt{221}}{2} \end{pmatrix} \tag{2.2.28}$$

Centre of the hyperbola is given by

$$\mathbf{c} = -\mathbf{V}^{-1}\mathbf{u} \tag{2.2.29}$$

$$\implies \mathbf{c} = -\begin{pmatrix} \frac{3}{49} & \frac{5}{49} \\ \frac{5}{49} & \frac{-8}{49} \end{pmatrix} \begin{pmatrix} -1 \\ 2 \end{pmatrix} \tag{2.2.30}$$

$$\implies \mathbf{c} = \begin{pmatrix} \frac{-3}{49} & \frac{-5}{49} \\ \frac{-5}{49} & \frac{8}{49} \end{pmatrix} \begin{pmatrix} -1 \\ 2 \end{pmatrix} \tag{2.2.31}$$

$$\implies \mathbf{c} = \begin{pmatrix} \frac{-1}{7} \\ \frac{3}{7} \end{pmatrix} \tag{2.2.32}$$

Since,

$$\mathbf{u}^T \mathbf{V}^{-1} \mathbf{u} - f = 0 \tag{2.2.33}$$

Now,

$$\mathbf{y}^T \mathbf{D} \mathbf{y} = \mathbf{u}^T \mathbf{V}^{-1} \mathbf{u} - f \tag{2.2.34}$$

where ,

$$\mathbf{y} = \mathbf{P}^T(\mathbf{x} - \mathbf{c}) \tag{2.2.35}$$

To get $\mathbf{y}$,

$$\mathbf{y} = \mathbf{P}^T\mathbf{x} - \mathbf{P}^T\mathbf{c} \tag{2.2.36}$$

$$\mathbf{y} = \begin{pmatrix} \frac{\frac{11+\sqrt{221}}{10}}{\frac{-10}{11+\sqrt{221}}} & 1 \\ & 1 \end{pmatrix}\mathbf{x} - \begin{pmatrix} \frac{\frac{11+\sqrt{221}}{10}}{\frac{-10}{11+\sqrt{221}}} & 1 \\ & 1 \end{pmatrix}\begin{pmatrix} \frac{-1}{7} \\ \frac{3}{7} \end{pmatrix} \tag{2.2.37}$$

$$\mathbf{y} = \begin{pmatrix} \frac{\frac{11+\sqrt{221}}{10}}{\frac{-10}{11+\sqrt{221}}} & 1 \\ & 1 \end{pmatrix}\mathbf{x} - \begin{pmatrix} \frac{\frac{-11-\sqrt{221}}{70}}{\frac{10}{(7)11+(7)\sqrt{221}}} + \frac{3}{7} \\ + \frac{3}{7} \end{pmatrix} \tag{2.2.38}$$

Subsituting the eqauations (2.2.33), (2.2.28) in equation (2.2.34) Equation of asymptotes is

$$\implies \mathbf{y}^T \begin{pmatrix} \frac{5+\sqrt{221}}{2} & 0 \\ 0 & \frac{5-\sqrt{221}}{2} \end{pmatrix}\mathbf{y} + 1 = 0 \tag{2.2.39}$$

And the Equations of Conjugate hyperbola is 2(Equation of Asymptotes)- Equation of hyperbola.

$$\implies \mathbf{y}^T \begin{pmatrix} \frac{5+\sqrt{221}}{2} & 0 \\ 0 & \frac{5-\sqrt{221}}{2} \end{pmatrix}\mathbf{y} = 0 \tag{2.2.40}$$

Python Code

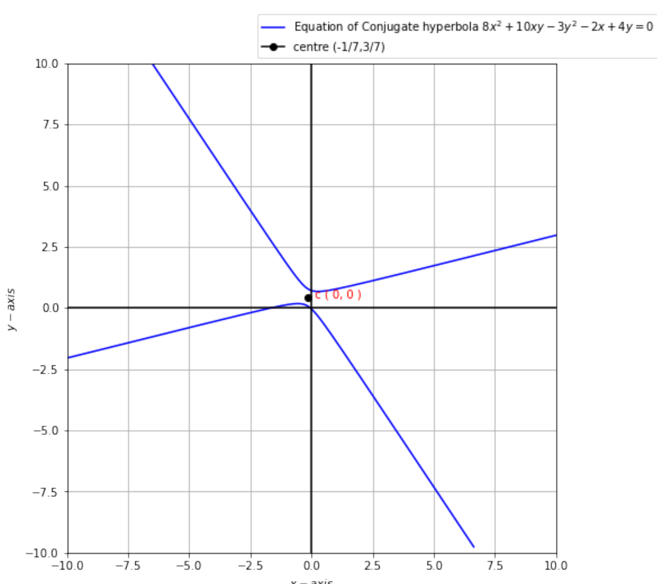

Fig. 1: Conjugate hyperbola

https://github.com/ayushkesh/Matrix−Theory−EE5609/blob/master/A7/codes/A7_3.ipynb