Assignment-4

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Abstract—**This document contains solution of Problem Geolin**(1.12)

1 QUESTION

Show that $\sin 30° = \frac{1}{2}$ and $\cos 30° = \frac{\sqrt{3}}{2}$.

2 SOLUTION

Consider an equilateral $\triangle ABC$ as shown in figure:1. Let the angle bisector of ∠A intersect side **BC** at a point D between B and C.
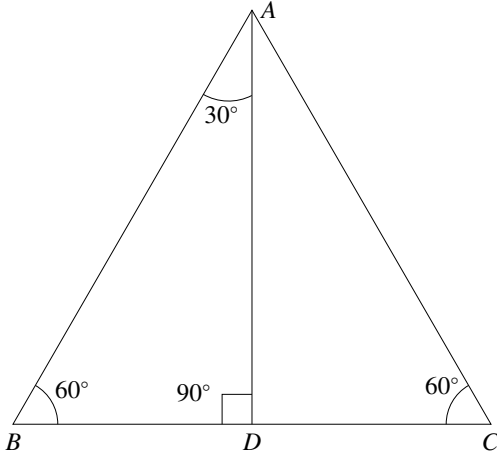

Fig. 1: Equilateral $\triangle ABC$

$$\|\mathbf{A} - \mathbf{B}\| = \|\mathbf{B} - \mathbf{C}\| = \|\mathbf{A} - \mathbf{C}\| = 2\|\mathbf{B} - \mathbf{D}\| \quad (2.0.1)$$

To Find AD.

$$(\mathbf{B} - \mathbf{A})^T (\mathbf{B} - \mathbf{A})$$
$$= (\mathbf{B} - \mathbf{D} + \mathbf{D} - \mathbf{A})^T (\mathbf{B} - \mathbf{D} + \mathbf{D} - \mathbf{A})$$
$$= [(\mathbf{B} - \mathbf{D})^T + (\mathbf{D} - \mathbf{A})^T][(\mathbf{B} - \mathbf{D}) + (\mathbf{D} - \mathbf{A})]$$
$$= (\mathbf{B} - \mathbf{D})^T (\mathbf{B} - \mathbf{D}) + (\mathbf{B} - \mathbf{D})^T (\mathbf{D} - \mathbf{A}) +$$
$$(\mathbf{D} - \mathbf{A})^T (\mathbf{B} - \mathbf{D}) + (\mathbf{D} - \mathbf{A})^T (\mathbf{D} - \mathbf{A}) \quad (2.0.2)$$

Since BD is the perpendicular to AD the inner product is zero

$$(\mathbf{B} - \mathbf{D})^T (\mathbf{D} - \mathbf{A}) = 0 \quad (2.0.3)$$
$$(\mathbf{D} - \mathbf{A})^T (\mathbf{B} - \mathbf{D}) = 0 \quad (2.0.4)$$

which gives

$$(\mathbf{B} - \mathbf{A})^T (\mathbf{B} - \mathbf{A}) =$$
$$(\mathbf{B} - \mathbf{D})^T (\mathbf{B} - \mathbf{D}) + (\mathbf{D} - \mathbf{A})^T (\mathbf{D} - \mathbf{A})$$
$$\implies \|\mathbf{B} - \mathbf{A}\|^2 = \|\mathbf{B} - \mathbf{D}\|^2 + \|\mathbf{D} - \mathbf{A}\|^2 \quad (2.0.5)$$

$$\|\mathbf{D} - \mathbf{A}\|^2 = \|\mathbf{B} - \mathbf{A}\|^2 - \|\mathbf{B} - \mathbf{D}\|^2 \quad (2.0.6)$$

Substitutiing Eq (2.0.1)

$$\|\mathbf{D} - \mathbf{A}\|^2 = \|\mathbf{B} - \mathbf{A}\|^2 - \frac{1}{4}\|\mathbf{B} - \mathbf{A}\|^2 \quad (2.0.7)$$

$$\|\mathbf{D} - \mathbf{A}\| = \frac{\sqrt{3}}{2}\|\mathbf{B} - \mathbf{A}\| \quad (2.0.8)$$

$$\implies \|\mathbf{B} - \mathbf{A}\| = \frac{2}{\sqrt{3}}\|\mathbf{D} - \mathbf{A}\| \quad (2.0.9)$$

Let $\mathbf{A} = 0$. Then substituting in (2.0.1) and (2.0.9)

$$\|\mathbf{B}\| = 2\|\mathbf{B} - \mathbf{D}\| \quad (2.0.10)$$

$$\|\mathbf{B}\| = \frac{2}{\sqrt{3}}\|\mathbf{D}\| \quad (2.0.11)$$

Square on both sides in (2.0.10).

$$\|\mathbf{B}\|^2 = 4\|\mathbf{B} - \mathbf{D}\|^2 \quad (2.0.12)$$

$$\frac{1}{4}\|\mathbf{B}\|^2 = \|\mathbf{B}\|^2 + \|\mathbf{D}\|^2 - 2\mathbf{B}^T\mathbf{D} \quad (2.0.13)$$

Square on both sides in (2.0.11).

$$\|\mathbf{B}\|^2 = \frac{4}{3}\|\mathbf{D}\|^2 \quad (2.0.14)$$

Using (2.0.13) and (2.0.14)

$$\frac{1}{3}\|\mathbf{D}\|^2 = \frac{4}{3}\|\mathbf{D}\|^2 + \|\mathbf{D}\|^2 - 2\mathbf{D}^T\mathbf{D} \quad (2.0.15)$$

$$\implies 0 = \|\mathbf{D}\|^2 - 2\mathbf{B}^T\mathbf{D} \quad (2.0.16)$$

$$\implies \mathbf{B}^T\mathbf{D} = \|\mathbf{D}\|^2 \quad (2.0.17)$$

Let $\theta = \angle BAD$. and Taking the inner product of sides BA and AD.

$$(\mathbf{B} - \mathbf{A})^T (\mathbf{A} - \mathbf{D}) = \|\mathbf{B} - \mathbf{A}\| \, \|\mathbf{A} - \mathbf{D}\| \cos \theta \quad (2.0.18)$$

$$\cos \theta = \frac{(\mathbf{B} - \mathbf{A})^T (\mathbf{A} - \mathbf{D})}{\|\mathbf{B} - \mathbf{A}\| \, \|\mathbf{A} - \mathbf{D}\|} \quad (2.0.19)$$

Substitute $\mathbf{A} = 0$ in (2.0.19)

$$\implies \cos \theta = \frac{\mathbf{B}^T \mathbf{D}}{\|\mathbf{B}\| \, \|\mathbf{D}\|} \quad (2.0.20)$$

Using (2.0.11)

$$\implies \cos \theta = \frac{\mathbf{B}^T \mathbf{D}}{\frac{2}{\sqrt{3}} \|\mathbf{D}\| \, \|\mathbf{D}\|} \quad (2.0.21)$$

$$\implies \cos \theta = \frac{\mathbf{B}^T \mathbf{D}}{\frac{2}{\sqrt{3}} \|\mathbf{D}\|^2} \quad (2.0.22)$$

Substitute (2.0.17) in (2.0.22)

$$\cos \theta = \frac{\frac{\sqrt{3}}{2} \|\mathbf{D}\|^2}{\|\mathbf{D}\|^2} \quad (2.0.23)$$

$$\implies \cos \theta = \frac{\sqrt{3}}{2} \quad (2.0.24)$$

$$\because \cos 30° = \frac{\sqrt{3}}{2} \quad (2.0.25)$$

$$\implies \theta = 30° \quad (2.0.26)$$

$$\because \cos^2 \theta + \sin^2 \theta = 1 \quad (2.0.27)$$

$$\sin 30° = \sqrt{1 - \cos^2 30°} \quad (2.0.28)$$

$$\implies \sin 30° = \frac{1}{2}. \quad (2.0.29)$$