Assignment 13

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1 problem

Let R[x] denote the vector space of all real polynomial. Let **D** :**R**[**x**] → **R**[**x**] denote the map $\mathbf{D}f = \frac{df}{dx}, \forall f$ then,

1) **D** is one-one.
2) **D** is onto.
3) There exist $E : \mathbf{R}[\mathbf{x}] \to \mathbf{R}[\mathbf{x}]$ so that $D(E(f)) = f, \forall f$.
4) There exist $E : \mathbf{R}[\mathbf{x}] \to \mathbf{R}[\mathbf{x}]$ so that $E(D(f)) = f, \forall f$.

2 Solution

Let,

$$f = \sum_{i=0}^{n} f_i x^i = f_0 + f_1 x + f_2 x^2 + ...f_n x^n. \tag{2.0.1}$$

f is a real polynomial with degree n. and **V** having dim(**V**)= n.

Now,

$$\tag{2.0.2}$$

$$D(f) = \frac{d}{dx}(f), \forall f \tag{2.0.3}$$

$$= \frac{d}{dx}\left(f_0 + f_1 x + ...f_n x^n\right). \tag{2.0.4}$$

$$= f_1 + 2f_2 x + ...n f_n x^{n-1}. \tag{2.0.5}$$

$$D\left(f\right) \in \mathbf{W}, dim(\mathbf{W}) = (n-1). \tag{2.0.6}$$

As **W** is a real space of real polynomial with degree (n-1)

$$\tag{2.0.7}$$

Given	**D** :R[x]→ **R**[**x**] denote the map $\mathbf{D}f = \frac{df}{dx}, \forall f$
Statement 1	D is one-one.
	If D is one-one $\implies \mathbf{N}(\mathbf{D}(\mathbf{f})) = 0$ $\implies dim(\mathbf{N}(\mathbf{D}(\mathbf{f})) = 0$ According to rank-Nullity theorem :- $dim(\mathbf{N}(\mathbf{D})) + dim(\mathbf{R}(\mathbf{D}) = dim(\mathbf{V})$

	$\implies 0 + dim(\mathbf{R}(\mathbf{D}) = \text{n}$
	$\implies$ dim(R(D))= n
	As $dim(\mathbf{R}(\mathbf{D}) \leq dim(\mathbf{W})$
	$*\ dim\mathbf{R}(\mathbf{D}) = dim(\mathbf{W})$, when **D** is one-one
	$*\ dim\mathbf{R}(\mathbf{D}) < dim(\mathbf{W})$, when **D** is not onto
	So $dim(\mathbf{R}(\mathbf{D}))$= n not possible
	False Statement
Statement 2	D is onto.
	$dim(\mathbf{R}(\mathbf{D})) = dim(\mathbf{W})$
	$dim(\mathbf{R}(\mathbf{D})) = n - 1$
	As range **W** will be same for f,
	So there will be no nullspace
	$\implies dim(\mathbf{N}(\mathbf{D})) = 0$
	But $dim(\mathbf{R}(\mathbf{D})) = dim(\mathbf{V})$.
	False Statement
Statement 3	There exist E :R[x] $\rightarrow \mathbf{R}[\mathbf{x}]$ so that $D(E(f)) = f, \forall f$.
	Let $E(f) = \int f dx$
	$\because$E(f)= $\int \left(f_0 + f_1 x + f_2 x^2 + ...f_n x^n\right) dx$
	$= f_0 x + \frac{f_1 x^2}{2} + ...\frac{f_n x^{n+1}}{n+1}$ + a, where a is any constant.
	Now, $D(E(f)) = \frac{d}{dx}(E(f))$
	$= f_0 + m f_1 x + ...f_n x^n$
	$\implies f$
	True Statement
Statement 4	There exist $E : \mathbf{R}[\mathbf{x}] \rightarrow \mathbf{R}[\mathbf{x}]$ so that $E(D(f)) = f, \forall f$.
	This is Possible only when $E = D^{-1}$

	and in that case $\mathbf{V}$ and $\mathbf{W}$ must be isomorphic. $\implies D$ should be onto But from statement 2, we get D is not onto. So there will be no such $\mathbf{E}$ st. $\mathbf{E}(\mathbf{D}(\mathbf{f})) = f$
	True Statement

TABLE 1: Solution