

Hyperbola

Abstract—This document contains solution of Problem Loney(314, 7)

Download latex-tikz codes from

<https://github.com/ayushkesh/Matrix-Theory-EE5609/tree/master/A7>

1 QUESTION

Find the asymptotes of the hyperbola given below and also the equations to their conjugate hyperbolas.
 $8x^2 + 10xy - 3y^2 - 2x + 4y - 2 = 0$

2 SOLUTION

The above equation can be expressed in the form

$$\mathbf{x}^T \mathbf{V} \mathbf{x} + 2\mathbf{u}^T \mathbf{x} + f = 0 \quad (2.0.1)$$

Comparing equation we get

$$\mathbf{V} = \mathbf{V}^T = \begin{pmatrix} 8 & 5 \\ 5 & -3 \end{pmatrix} \quad (2.0.2)$$

$$\mathbf{u} = \begin{pmatrix} -1 \\ 2 \end{pmatrix} \quad (2.0.3)$$

$$f = -2 \quad (2.0.4)$$

Expanding the Determinant of V.

$$\Delta_V = \begin{vmatrix} 8 & 5 \\ 5 & -3 \end{vmatrix} < 0 \quad (2.0.5)$$

Hence from (2.0.5) given equation represents the hyperbola The characteristic equation of \mathbf{V} is obtained by evaluating the determinant

$$|V - \lambda \mathbf{I}| = 0 \quad (2.0.6)$$

$$\begin{vmatrix} 8 - \lambda & 5 \\ 5 & -3 - \lambda \end{vmatrix} = 0 \quad (2.0.7)$$

$$(8 - \lambda)(-3 - \lambda) - 25 = 0 \quad (2.0.8)$$

$$\lambda_1 = \frac{5 + \sqrt{221}}{2} \quad (2.0.9)$$

$$\lambda_2 = \frac{5 - \sqrt{221}}{2} \quad (2.0.10)$$

The eigenvector \mathbf{p} is defined as

$$\mathbf{V}\mathbf{p} = \lambda\mathbf{p} \quad (2.0.11)$$

$$\Rightarrow (\mathbf{V} - \lambda\mathbf{I})\mathbf{p} = 0 \quad (2.0.12)$$

For $\lambda_1 = \frac{5 + \sqrt{221}}{2}$,

$$(\mathbf{V} - \lambda_1 \mathbf{I}) = \begin{pmatrix} \frac{11 - \sqrt{221}}{2} & 5 \\ 5 & \frac{-11 - \sqrt{221}}{2} \end{pmatrix} \quad (2.0.13)$$

By row reduction ,

$$\begin{pmatrix} \frac{11 - \sqrt{221}}{2} & 5 \\ 5 & \frac{-11 - \sqrt{221}}{2} \end{pmatrix} \quad (2.0.14)$$

$$\xleftrightarrow{R_1 \leftrightarrow R_2} \begin{pmatrix} \frac{-11 - \sqrt{221}}{2} & 5 \\ \frac{11 - \sqrt{221}}{2} & 5 \end{pmatrix} \quad (2.0.15)$$

$$\xleftrightarrow{R_2 \leftarrow R_2 - \frac{11 - \sqrt{221}}{10} R_1} \begin{pmatrix} 5 & \frac{-11 - \sqrt{221}}{2} \\ 0 & 0 \end{pmatrix} \quad (2.0.16)$$

$$\xleftrightarrow{R_1 \leftarrow R_1 / 5} \begin{pmatrix} 1 & \frac{-11 - \sqrt{221}}{10} \\ 0 & 0 \end{pmatrix} \quad (2.0.17)$$

Substituting equation 2.0.17 in equation 2.0.12 we get

$$\begin{pmatrix} 1 & \frac{-11 - \sqrt{221}}{10} \\ 0 & 0 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad (2.0.18)$$

Where, $\mathbf{p} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$ Let $v_2 = t$

$$v_1 = \frac{t(11 + \sqrt{221})}{10} \quad (2.0.19)$$

Eigen vector \mathbf{p}_1 is given by

$$\mathbf{p}_1 = \begin{pmatrix} \frac{t(11 + \sqrt{221})}{10} \\ t \end{pmatrix} \quad (2.0.20)$$

Let $t = 1$, we get

$$\mathbf{p}_1 = \begin{pmatrix} \frac{11 + \sqrt{221}}{10} \\ 1 \end{pmatrix} \quad (2.0.21)$$

For $\lambda_2 = \frac{5-\sqrt{221}}{2}$,

$$(\mathbf{V} - \lambda_2 \mathbf{I}) = \begin{pmatrix} \frac{11+\sqrt{221}}{2} & 5 \\ 5 & \frac{-11+\sqrt{221}}{2} \end{pmatrix} \quad (2.0.22)$$

By row reduction ,

$$\begin{pmatrix} \frac{11+\sqrt{221}}{2} & 5 \\ 5 & \frac{-11+\sqrt{221}}{2} \end{pmatrix} \xrightarrow{R_1 \leftarrow R_2 + \frac{11-\sqrt{221}}{10} R_1} \begin{pmatrix} \frac{11+\sqrt{221}}{2} & 5 \\ 0 & 0 \end{pmatrix} \quad (2.0.23)$$

$$\xrightarrow{R_1 \leftarrow \frac{R_1}{\frac{11+\sqrt{221}}{10}}} \begin{pmatrix} 1 & \frac{10}{11+\sqrt{221}} \\ 0 & 0 \end{pmatrix} \quad (2.0.24)$$

Substituting equation 2.0.24 in equation 2.0.12 we get

$$\begin{pmatrix} 1 & \frac{10}{11+\sqrt{221}} \\ 0 & 0 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad (2.0.25)$$

Where, $\mathbf{p} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$ Let $v_2 = t$

$$v_1 = \frac{-t(10)}{11 + \sqrt{221}} \quad (2.0.26)$$

Eigen vector \mathbf{p}_2 is given by

$$\mathbf{p}_2 = \begin{pmatrix} \frac{-t(10)}{11+\sqrt{221}} \\ t \end{pmatrix} \quad (2.0.27)$$

Let $t = 1$, we get

$$\mathbf{p}_2 = \begin{pmatrix} \frac{(-10)}{11+\sqrt{221}} \\ 1 \end{pmatrix} \quad (2.0.28)$$

By eigen decomposition \mathbf{V} can be represented by

$$\mathbf{V} = \mathbf{P} \mathbf{D} \mathbf{P}^T \quad (2.0.29)$$

where

$$\mathbf{P} = (\mathbf{p}_1 \quad \mathbf{p}_2) \quad (2.0.30)$$

$$\mathbf{D} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \quad (2.0.31)$$

Substituting equations 2.0.21, 2.0.28 in equation 2.0.30 we get

$$\mathbf{P} = \begin{pmatrix} \frac{11+\sqrt{221}}{10} & \frac{-10}{11+\sqrt{221}} \\ 1 & 1 \end{pmatrix} \quad (2.0.32)$$

Substituting equations 2.0.9, 2.0.10 in 2.0.31 we get

$$\mathbf{D} = \begin{pmatrix} \frac{5+\sqrt{221}}{2} & 0 \\ 0 & \frac{5-\sqrt{221}}{2} \end{pmatrix} \quad (2.0.33)$$

Centre of the hyperbola is given by

$$\mathbf{c} = -\mathbf{V}^{-1} \mathbf{u} \quad (2.0.34)$$

$$\Rightarrow \mathbf{c} = -\begin{pmatrix} \frac{3}{49} & \frac{5}{49} \\ \frac{5}{49} & \frac{-8}{49} \end{pmatrix} \begin{pmatrix} -1 \\ 2 \end{pmatrix} \quad (2.0.35)$$

$$\Rightarrow \mathbf{c} = \begin{pmatrix} \frac{-3}{49} & \frac{-5}{49} \\ \frac{-5}{49} & \frac{8}{49} \end{pmatrix} \begin{pmatrix} -1 \\ 2 \end{pmatrix} \quad (2.0.36)$$

$$\Rightarrow \mathbf{c} = \begin{pmatrix} \frac{-1}{7} \\ \frac{3}{7} \end{pmatrix} \quad (2.0.37)$$

Since,

$$\mathbf{u}^T \mathbf{V}^{-1} \mathbf{u} - f = 1 > 0 \quad (2.0.38)$$

there isn't a need to swap axes In hyperbola,

$$axes = \begin{cases} \sqrt{\frac{\mathbf{u}^T \mathbf{V}^{-1} \mathbf{u} - f}{\lambda_1}} \\ \sqrt{\frac{f - \mathbf{u}^T \mathbf{V}^{-1} \mathbf{u}}{\lambda_2}} \end{cases} \quad (2.0.39)$$

From above equations we can say that,

$$\sqrt{\frac{\mathbf{u}^T \mathbf{V}^{-1} \mathbf{u} - f}{\lambda_1}} = \sqrt{\frac{2}{5 + \sqrt{221}}} \quad (2.0.40)$$

$$\sqrt{\frac{f - \mathbf{u}^T \mathbf{V}^{-1} \mathbf{u}}{\lambda_2}} = \sqrt{\frac{2}{5 - \sqrt{221}}} \quad (2.0.41)$$

Now we have,

$$\mathbf{y}^T \mathbf{D} \mathbf{y} = \mathbf{u}^T \mathbf{V}^{-1} \mathbf{u} - f \quad (2.0.42)$$

where ,

$$\mathbf{y} = \mathbf{P}^T (\mathbf{x} - \mathbf{c}) \quad (2.0.43)$$

To get \mathbf{y} ,

$$\mathbf{y} = \mathbf{P}^T \mathbf{x} - \mathbf{P}^T \mathbf{c} \quad (2.0.44)$$

$$\mathbf{y} = \begin{pmatrix} \frac{11+\sqrt{221}}{10} & 1 \\ \frac{-10}{11+\sqrt{221}} & 1 \end{pmatrix} \mathbf{x} - \begin{pmatrix} \frac{11+\sqrt{221}}{10} & 1 \\ \frac{-10}{11+\sqrt{221}} & 1 \end{pmatrix} \begin{pmatrix} \frac{-1}{7} \\ \frac{3}{7} \end{pmatrix} \quad (2.0.45)$$

$$\mathbf{y} = \begin{pmatrix} \frac{11+\sqrt{221}}{10} & 1 \\ \frac{-10}{11+\sqrt{221}} & 1 \end{pmatrix} \mathbf{x} - \begin{pmatrix} \frac{-11-\sqrt{221}}{70} + \frac{3}{7} \\ \frac{70}{(7)11+(7)\sqrt{221}} + \frac{3}{7} \end{pmatrix} \quad (2.0.46)$$

Substituting the equations (2.0.38), (2.0.33) in equation (2.0.42)

$$\Rightarrow \mathbf{y}^T \begin{pmatrix} \frac{5+\sqrt{221}}{2} & 0 \\ 0 & \frac{5-\sqrt{221}}{2} \end{pmatrix} \mathbf{y} + 2 = 0 \quad (2.0.47)$$

2.1 Asymptotes of hyperbola

Equation of a hyperbola and the combined equation of the Asymptotes differ only in the constant term.

$$8x^2 + 10xy - 3y^2 - 2x + 4y + K = 0 \quad (2.1.1)$$

The above equation can be expressed in the form

$$\mathbf{x}^T \mathbf{V} \mathbf{x} + 2\mathbf{u}^T \mathbf{x} + f = 0 \quad (2.1.2)$$

Comparing equation we get

$$\mathbf{V} = \mathbf{V}^T = \begin{pmatrix} 8 & 5 \\ 5 & -3 \end{pmatrix} \quad (2.1.3)$$

$$\mathbf{u} = \begin{pmatrix} -1 \\ 2 \end{pmatrix} \quad (2.1.4)$$

$$f = K \quad (2.1.5)$$

$$\Delta = \begin{vmatrix} 8 & 5 & -1 \\ 5 & -3 & 2 \\ -1 & 2 & K \end{vmatrix} \quad (2.1.6)$$

$$\Rightarrow K = -1 \quad (2.1.7)$$

Similar way expanding the Determinant of V.

$$\Delta_V = \begin{vmatrix} 8 & 5 \\ 5 & -3 \end{vmatrix} < 0 \quad (2.1.8)$$

From (2.1.8) we could say that the given equation represents two straight lines Let the equations of lines be,

$$(\mathbf{n}_1^T \mathbf{x} - c_1)(\mathbf{n}_2^T \mathbf{x} - c_2) = \mathbf{x}^T \mathbf{V} \mathbf{x} + 2\mathbf{u}^T \mathbf{x} + f = 0 \quad (2.1.9)$$

$$(\mathbf{n}_1^T \mathbf{x} - c_1)(\mathbf{n}_2^T \mathbf{x} - c_2) = \mathbf{x}^T \begin{pmatrix} 8 & 5 \\ 5 & -3 \end{pmatrix} \mathbf{x} + 2(-1 \ 2)\mathbf{x} - 1 \quad (2.1.10)$$

$$\mathbf{n}_1 * \mathbf{n}_2 = \begin{pmatrix} a \\ 2b \\ c \end{pmatrix} = \begin{pmatrix} 8 \\ 10 \\ -3 \end{pmatrix} \quad (2.1.11)$$

$$c_2 \mathbf{n}_1 + c_1 \mathbf{n}_2 = -2 \begin{pmatrix} -1 \\ 2 \end{pmatrix} \quad (2.1.12)$$

$$c_1 c_2 = -1 \quad (2.1.13)$$

The slopes of the lines are given by the roots of the polynomial

$$cm^2 + 2bm + a = 0 \quad (2.1.14)$$

$$\Rightarrow m_i = \frac{-b \pm \sqrt{-\Delta_V}}{c} \quad (2.1.15)$$

$$\mathbf{n}_i = k \begin{pmatrix} -m_i \\ 1 \end{pmatrix} \quad (2.1.16)$$

Substituting the given data in above equations (2.1.14) we get,

$$-3m^2 + 10m + 8 = 0 \quad (2.1.17)$$

$$m_1 = 4, m_2 = \frac{-2}{3} \quad (2.1.18)$$

$$= \mathbf{n}_1 = \begin{pmatrix} -4 \\ 1 \end{pmatrix}, \mathbf{n}_2 = \begin{pmatrix} -2 \\ -3 \end{pmatrix} \quad (2.1.19)$$

We know that,

$$\mathbf{n}_1 * \mathbf{n}_2 = \begin{pmatrix} a \\ 2b \\ c \end{pmatrix} \quad (2.1.20)$$

Verification using Toeplitz matrix, From equation (2.1.19)

$$\mathbf{n}_1 = \begin{pmatrix} -4 & 0 \\ 1 & -4 \\ 0 & -1 \end{pmatrix} \mathbf{n}_2 = \begin{pmatrix} -2 \\ -3 \end{pmatrix} \quad (2.1.21)$$

$$\Rightarrow \begin{pmatrix} -4 & 0 \\ 1 & -4 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ -1 \end{pmatrix} = \begin{pmatrix} 8 \\ 10 \\ -3 \end{pmatrix} = \begin{pmatrix} a \\ 2b \\ c \end{pmatrix} \quad (2.1.22)$$

\Rightarrow Equation (2.1.19) satisfies (2.1.20) c_1 and c_2 can be obtained as,

$$(\mathbf{n}_1 \ \mathbf{n}_2) \begin{pmatrix} c_2 \\ c_1 \end{pmatrix} = -2\mathbf{u} \quad (2.1.23)$$

Substituting (2.1.19) in (2.1.23), the augmented matrix is,

$$\begin{pmatrix} -4 & -2 & -2 \\ 1 & -3 & 4 \end{pmatrix} \xrightarrow[R_2 \leftarrow R_2 - R_1]{R_1 \leftarrow -R_1/4} \begin{pmatrix} 1 & \frac{1}{2} & \frac{1}{2} \\ 0 & -\frac{7}{2} & \frac{7}{2} \end{pmatrix} \quad (2.1.24)$$

$$\xrightarrow[R_1 \leftarrow R_1 - \frac{1}{2}R_2]{R_2 \leftarrow -\frac{2}{7}R_2} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \end{pmatrix} \quad (2.1.25)$$

$$\Rightarrow c_1 = 1, c_2 = -1 \quad (2.1.26)$$

Equations (2.1.9), can be modified as, from (2.1.19) and (2.1.26) in we get,

$$\begin{pmatrix} -4 & 1 \end{pmatrix} \mathbf{x} = 1 \quad (2.1.27)$$

$$\begin{pmatrix} -2 & -3 \end{pmatrix} \mathbf{x} = -1 \quad (2.1.28)$$

$$\begin{aligned} \Rightarrow & (-4x + y - 1)(-2x - 3y + 1) = 0 \\ \Rightarrow & \boxed{(4x - y + 1)(2x + 3y - 1) = 0} \end{aligned} \quad (2.1.29)$$

The angle between the lines can be expressed as,

$$\mathbf{n}_1 = \begin{pmatrix} -4 \\ 1 \end{pmatrix}, \quad \mathbf{n}_2 = \begin{pmatrix} -2 \\ -3 \end{pmatrix} \quad (2.1.30)$$

$$\cos \theta = \frac{\mathbf{n}_1^T \mathbf{n}_2}{\|\mathbf{n}_1\| \|\mathbf{n}_2\|} \quad (2.1.31)$$

$$\Rightarrow \theta = \cos^{-1}\left(\frac{0}{\sqrt{221}}\right) = 90^\circ. \quad (2.1.32)$$

By row reduction ,

$$\begin{pmatrix} \frac{11-\sqrt{221}}{2} & 5 \\ 5 & \frac{-11-\sqrt{221}}{2} \end{pmatrix} \quad (2.2.9)$$

$$\xleftrightarrow{R_1 \leftrightarrow R_2} \begin{pmatrix} \frac{-11-\sqrt{221}}{2} & 5 \\ \frac{11-\sqrt{221}}{2} & 5 \end{pmatrix} \quad (2.2.10)$$

$$\xleftrightarrow{R_2 \leftarrow R_2 - \frac{11-\sqrt{221}}{10} R_1} \begin{pmatrix} 5 & \frac{-11-\sqrt{221}}{2} \\ 0 & 0 \end{pmatrix} \quad (2.2.11)$$

$$\xleftrightarrow{R_1 \leftarrow R_1/5} \begin{pmatrix} 1 & \frac{-11-\sqrt{221}}{10} \\ 0 & 0 \end{pmatrix} \quad (2.2.12)$$

Substituting equation 2.2.12 in equation 2.2.7 we get

$$\begin{pmatrix} 1 & \frac{-11-\sqrt{221}}{10} \\ 0 & 0 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad (2.2.13)$$

Where, $\mathbf{p} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$ Let $v_2 = t$

$$v_1 = \frac{t(11 + \sqrt{221})}{10} \quad (2.2.14)$$

Eigen vector \mathbf{p}_1 is given by

$$\mathbf{p}_1 = \begin{pmatrix} \frac{t(11+\sqrt{221})}{10} \\ t \end{pmatrix} \quad (2.2.15)$$

2.2 Equation of Asymptotes

The characteristic equation of \mathbf{V} is obtained by evaluating the determinant (2.1.3)

$$|V - \lambda \mathbf{I}| = 0 \quad (2.2.1)$$

$$\begin{vmatrix} 8 - \lambda & 5 \\ 5 & -3 - \lambda \end{vmatrix} = 0 \quad (2.2.2)$$

$$(8 - \lambda)(-3 - \lambda) - 25 = 0 \quad (2.2.3)$$

$$\lambda_1 = \frac{5 + \sqrt{221}}{2} \quad (2.2.4)$$

$$\lambda_2 = \frac{5 - \sqrt{221}}{2} \quad (2.2.5)$$

The eigenvector \mathbf{p} is defined as

$$\mathbf{V}\mathbf{p} = \lambda \mathbf{p} \quad (2.2.6)$$

$$\Rightarrow (\mathbf{V} - \lambda \mathbf{I})\mathbf{p} = 0 \quad (2.2.7)$$

For $\lambda_1 = \frac{5 + \sqrt{221}}{2}$,

$$(\mathbf{V} - \lambda_1 \mathbf{I}) = \begin{pmatrix} \frac{11-\sqrt{221}}{2} & 5 \\ 5 & \frac{-11-\sqrt{221}}{2} \end{pmatrix} \quad (2.2.8)$$

Let $t = 1$, we get

$$\mathbf{p}_1 = \begin{pmatrix} \frac{11+\sqrt{221}}{10} \\ 1 \end{pmatrix} \quad (2.2.16)$$

For $\lambda_2 = \frac{5 - \sqrt{221}}{2}$,

$$(\mathbf{V} - \lambda_2 \mathbf{I}) = \begin{pmatrix} \frac{11+\sqrt{221}}{2} & 5 \\ 5 & \frac{-11+\sqrt{221}}{2} \end{pmatrix} \quad (2.2.17)$$

By row reduction ,

$$\begin{pmatrix} \frac{11+\sqrt{221}}{2} & 5 \\ 5 & \frac{-11+\sqrt{221}}{2} \end{pmatrix} \xleftrightarrow{R_1 \leftarrow R_2 + \frac{11-\sqrt{221}}{10} R_1} \begin{pmatrix} \frac{11+\sqrt{221}}{2} & 5 \\ 0 & 0 \end{pmatrix} \quad (2.2.18)$$

$$\xleftrightarrow{R_1 \leftarrow \frac{R_1}{11+\sqrt{221}}} \begin{pmatrix} 1 & \frac{10}{11+\sqrt{221}} \\ 0 & 0 \end{pmatrix} \quad (2.2.19)$$

Substituting equation 2.2.19 in equation 2.2.7 we get

$$\begin{pmatrix} 1 & \frac{10}{11+\sqrt{221}} \\ 0 & 0 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad (2.2.20)$$

Where, $\mathbf{p} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$ Let $v_2 = t$

$$v_1 = \frac{-t(10)}{11 + \sqrt{221}} \quad (2.2.21)$$

Eigen vector \mathbf{p}_2 is given by

$$\mathbf{p}_2 = \begin{pmatrix} \frac{-t(10)}{11 + \sqrt{221}} \\ t \end{pmatrix} \quad (2.2.22)$$

Let $t = 1$, we get

$$\mathbf{p}_2 = \begin{pmatrix} \frac{(-10)}{11 + \sqrt{221}} \\ 1 \end{pmatrix} \quad (2.2.23)$$

By eigen decompostion \mathbf{V} can be represented by

$$\mathbf{V} = \mathbf{PDP}^T \quad (2.2.24)$$

where

$$\mathbf{P} = (\mathbf{p}_1 \quad \mathbf{p}_2) \quad (2.2.25)$$

$$\mathbf{D} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \quad (2.2.26)$$

Substituting equations 2.2.16, 2.2.23 in equation 2.2.25 we get

$$\mathbf{P} = \begin{pmatrix} \frac{11 + \sqrt{221}}{10} & \frac{-10}{11 + \sqrt{221}} \\ 1 & 1 \end{pmatrix} \quad (2.2.27)$$

$$\mathbf{D} = \begin{pmatrix} \frac{5 + \sqrt{221}}{2} & 0 \\ 0 & \frac{5 - \sqrt{221}}{2} \end{pmatrix} \quad (2.2.28)$$

Centre of the hyperbola is given by

$$\mathbf{c} = -\mathbf{V}^{-1}\mathbf{u} \quad (2.2.29)$$

$$\Rightarrow \mathbf{c} = -\begin{pmatrix} \frac{3}{49} & \frac{5}{49} \\ \frac{5}{49} & \frac{8}{49} \end{pmatrix} \begin{pmatrix} -1 \\ 2 \end{pmatrix} \quad (2.2.30)$$

$$\Rightarrow \mathbf{c} = \begin{pmatrix} \frac{-3}{49} & \frac{-5}{49} \\ \frac{-5}{49} & \frac{-8}{49} \end{pmatrix} \begin{pmatrix} -1 \\ 2 \end{pmatrix} \quad (2.2.31)$$

$$\Rightarrow \mathbf{c} = \begin{pmatrix} \frac{-1}{7} \\ \frac{3}{7} \end{pmatrix} \quad (2.2.32)$$

Since,

$$\mathbf{u}^T \mathbf{V}^{-1} \mathbf{u} - f = 0 \quad (2.2.33)$$

Now,

$$\mathbf{y}^T \mathbf{D} \mathbf{y} = \mathbf{u}^T \mathbf{V}^{-1} \mathbf{u} - f \quad (2.2.34)$$

where ,

$$\mathbf{y} = \mathbf{P}^T (\mathbf{x} - \mathbf{c}) \quad (2.2.35)$$

To get \mathbf{y} ,

$$\mathbf{y} = \mathbf{P}^T \mathbf{x} - \mathbf{P}^T \mathbf{c} \quad (2.2.36)$$

$$\mathbf{y} = \begin{pmatrix} \frac{11 + \sqrt{221}}{10} & 1 \\ \frac{-10}{11 + \sqrt{221}} & 1 \end{pmatrix} \mathbf{x} - \begin{pmatrix} \frac{11 + \sqrt{221}}{10} & 1 \\ \frac{-10}{11 + \sqrt{221}} & 1 \end{pmatrix} \begin{pmatrix} \frac{-1}{7} \\ \frac{3}{7} \end{pmatrix} \quad (2.2.37)$$

$$\mathbf{y} = \begin{pmatrix} \frac{11 + \sqrt{221}}{10} & 1 \\ \frac{-10}{11 + \sqrt{221}} & 1 \end{pmatrix} \mathbf{x} - \begin{pmatrix} \frac{-11 - \sqrt{221}}{70} + \frac{3}{7} \\ \frac{70}{(7)11 + (7)\sqrt{221}} + \frac{3}{7} \end{pmatrix} \quad (2.2.38)$$

Substituting the equations (2.2.33), (2.2.28) in equation (2.2.34) Equation of asymptotes is

$$\Rightarrow \mathbf{y}^T \begin{pmatrix} \frac{5 + \sqrt{221}}{2} & 0 \\ 0 & \frac{5 - \sqrt{221}}{2} \end{pmatrix} \mathbf{y} + 1 = 0 \quad (2.2.39)$$

And the Equations of Conjugate hyperbola is 2(Equation of Asymptotes)- Equation of hyperbola.

$$\Rightarrow \mathbf{y}^T \begin{pmatrix} \frac{5 + \sqrt{221}}{2} & 0 \\ 0 & \frac{5 - \sqrt{221}}{2} \end{pmatrix} \mathbf{y} = 0 \quad (2.2.40)$$

Python Code

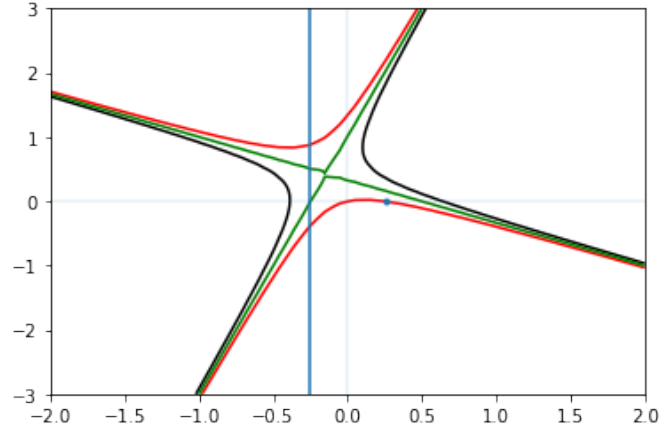


Fig. 1: Hyperbola with asymptotes and its conjugate

https://github.com/ayushkesh/Matrix-Theory-EE5609/blob/master/A7/codes/A7_4.ipynb