Assignment-6

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***Abstract*—This document contains solution of Problem Lonet**(314, 5)

Download latex-tikz codes from

https://github.com/ayushkesh/Matrix−Theory−EE5609/tree/master/A6

1 Question

What conics do the given equations represent? $6x^2 - 5xy - 6y^2 + 14x + 5y + 4 = 0$

2 Solution

The above equation can be expressed in the form

$$\mathbf{x}^T\mathbf{V}\mathbf{x} + 2\mathbf{u}^T\mathbf{x} + f = 0 \tag{2.0.1}$$

Comparing equation we get

$$\mathbf{V} = \mathbf{V}^T = \begin{pmatrix} 6 & \frac{-5}{2} \\ \frac{-5}{2} & -6 \end{pmatrix} \tag{2.0.2}$$

$$\mathbf{u} = \begin{pmatrix} 7 \\ \frac{5}{2} \end{pmatrix} \tag{2.0.3}$$

$$f = 4 \tag{2.0.4}$$

The above equation (2.0.1) represents a pair of straight lines if

$$\begin{vmatrix} \mathbf{V} & \mathbf{u} \\ \mathbf{u}^T & f \end{vmatrix} = 0 \tag{2.0.5}$$

Verify the given equation as if it is pair of straight lines

$$\Delta = \begin{vmatrix} 6 & \frac{-5}{2} & 7 \\ \frac{-5}{2} & -6 & \frac{5}{2} \\ 7 & \frac{5}{2} & 4 \end{vmatrix} \tag{2.0.6}$$

$$\implies 6\begin{vmatrix} -6 & \frac{5}{2} \\ \frac{5}{2} & 4 \end{vmatrix} - \frac{-5}{2}\begin{vmatrix} -\frac{5}{2} & \frac{5}{2} \\ 7 & 4 \end{vmatrix} + 7\begin{vmatrix} -\frac{5}{2} & -6 \\ 7 & \frac{5}{2} \end{vmatrix} = 0 \tag{2.0.7}$$

$$\implies \Delta = 0 \tag{2.0.8}$$

Since equation (2.0.5) is satisfied, we could say that the given equation represents two straight lines

$$\Delta_V = \begin{vmatrix} 6 & \frac{-5}{2} \\ \frac{-5}{2} & -6 \end{vmatrix} < 0 \tag{2.0.9}$$

Let the equations of lines be,

$$\left(\mathbf{n_1}^T\mathbf{x} - c_1\right)\left(\mathbf{n_1}^T\mathbf{x} - c_1\right) = \mathbf{x}^T\mathbf{V}\mathbf{x} + 2\mathbf{u}^T\mathbf{x} + f = 0 \tag{2.0.10}$$

$$\left(\mathbf{n_1}^T\mathbf{x} - c_1\right)\left(\mathbf{n_2}^T\mathbf{x} - c_2\right) = \mathbf{x}^T\begin{pmatrix} 6 & \frac{-5}{2} \\ \frac{-5}{2} & -6 \end{pmatrix}\mathbf{x} + 2\begin{pmatrix} 7 & \frac{5}{2} \end{pmatrix}\mathbf{x} + 4 \tag{2.0.11}$$

$$\mathbf{n_1} * \mathbf{n_2} = \begin{pmatrix} a \\ 2b \\ c \end{pmatrix} = \begin{pmatrix} 6 \\ -5 \\ -6 \end{pmatrix} \tag{2.0.12}$$

$$c_2\mathbf{n_1} + c_1\mathbf{n_2} = -2\begin{pmatrix} 7 \\ \frac{5}{2} \end{pmatrix} \tag{2.0.13}$$

$$c_1c_2 = 4 \tag{2.0.14}$$

The slopes of the lines are given by the roots of the polynomial

$$cm^2 + 2bm + a = 0 \tag{2.0.15}$$

$$\implies m_i = \frac{-b \pm \sqrt{-\Delta_V}}{c} \tag{2.0.16}$$

$$\mathbf{n_i} = k\begin{pmatrix} -m_i \\ 1 \end{pmatrix} \tag{2.0.17}$$

Substituting the given data in above equations (2.0.15) we get,

$$-6m^2 - 5m + 6 = 0 \tag{2.0.18}$$

$$\implies m_i = \frac{\frac{-5}{2} \pm \sqrt{-(\frac{-169}{4})}}{-6} \tag{2.0.19}$$

Solving equation (2.0.19) we get,

$$m_1 = -\frac{3}{2}, m_2 = \frac{2}{3} \tag{2.0.20}$$

$$= \mathbf{n_1} = \begin{pmatrix} -3 \\ -2 \end{pmatrix}, \mathbf{n_2} = \begin{pmatrix} -2 \\ 3 \end{pmatrix} \tag{2.0.21}$$

We know that,

$$\mathbf{n_1} * \mathbf{n_2} = \begin{pmatrix} a \\ 2b \\ c \end{pmatrix} \tag{2.0.22}$$

Verification using Toeplitz matrix, From equation (2.0.21)

$$\mathbf{n_1} = \begin{pmatrix} -3 & 0 \\ -2 & -3 \\ 0 & -2 \end{pmatrix} \mathbf{n_2} = \begin{pmatrix} -2 \\ 3 \end{pmatrix} \tag{2.0.23}$$

$$\implies \begin{pmatrix} -3 & 0 \\ -2 & -3 \\ 0 & -2 \end{pmatrix} \begin{pmatrix} -2 \\ 3 \end{pmatrix} = \begin{pmatrix} 6 \\ -5 \\ -6 \end{pmatrix} = \begin{pmatrix} a \\ 2b \\ c \end{pmatrix} \tag{2.0.24}$$

$\implies$ Equation (2.0.21) satisfies (2.0.22)
c_1 and c_2 can be obtained as,

$$\begin{pmatrix} \mathbf{n_1} & \mathbf{n_2} \end{pmatrix} \begin{pmatrix} c_2 \\ c_1 \end{pmatrix} = -2\mathbf{u} \tag{2.0.25}$$

Substituting (2.0.21) in (2.0.25), the augmented matrix is,

$$\begin{pmatrix} -3 & -2 & 14 \\ -2 & 3 & 5 \end{pmatrix} \xleftrightarrow[R_2 \leftarrow R_2 + 2R_1]{R_1 \leftarrow -R_1/3} \begin{pmatrix} 1 & \frac{2}{3} & -\frac{14}{3} \\ 0 & \frac{13}{3} & -\frac{13}{3} \end{pmatrix} \tag{2.0.26}$$

$$\xleftrightarrow[R_1 \leftarrow R_1 - \frac{2}{3}R_2]{R_2 \leftarrow \frac{3}{13}R_2} \begin{pmatrix} 1 & 0 & -4 \\ 0 & 1 & -1 \end{pmatrix} \tag{2.0.27}$$

$$\implies c_1 = -4, c_2 = -1 \tag{2.0.28}$$

Equations (2.0.10), can be modified as,from (2.0.21) and (2.0.28) in we get,

$$\begin{pmatrix} -3 & -2 \end{pmatrix} \mathbf{x} = -4 \tag{2.0.29}$$

$$\begin{pmatrix} -2 & 3 \end{pmatrix} \mathbf{x} = -1 \tag{2.0.30}$$

$$\implies (-3x - 2y + 4)(-2x + 3y + 1) = 0$$

$$\implies \boxed{(3x + 2y - 4)(2x - 3y - 1) = 0} \tag{2.0.31}$$

The angle between the lines can be expressed as,

$$\mathbf{n_1} = \begin{pmatrix} -3 \\ -2 \end{pmatrix}, \quad \mathbf{n_2} = \begin{pmatrix} -2 \\ 3 \end{pmatrix} \tag{2.0.32}$$

$$\cos\theta = \frac{\mathbf{n_1}^T \mathbf{n_2}}{||\mathbf{n_1}|| \, ||\mathbf{n_2}||} \tag{2.0.33}$$

$$\implies \quad \theta = \cos^{-1}\left(\frac{0}{\sqrt{169}}\right) = 90°. \tag{2.0.34}$$

Python Code

https://github.com/ayushkesh/Matrix−Theory−EE5609/blob/master/A6/A6.ipynb

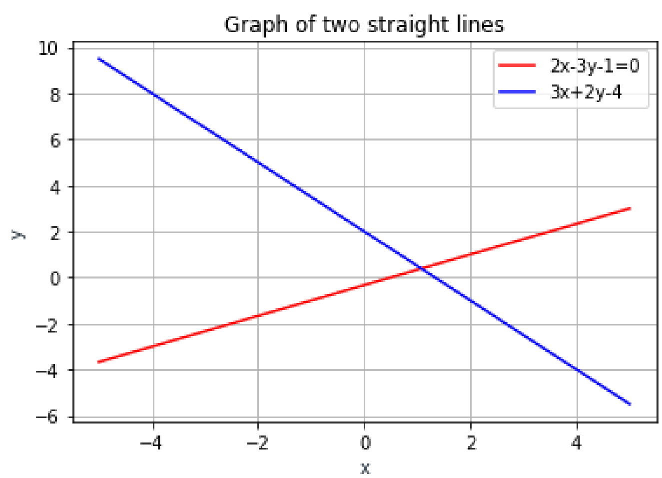

Fig. 1: Pair of straight lines