Assignment-4

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***Abstract*—This document contains solution of Problem Geolin**(1.12)

Download latex-tikz codes from

https://github.com/ayushkesh/Matrix−Theory−EE5609/tree/master/A4

1 Question

Show that $\sin 30° = \frac{1}{2}$ and $\cos 30° = \frac{\sqrt{3}}{2}$.

2 Solution

Consider an equilateral $\triangle ABC$ as shown in figure:1. Let the angle bisector of $\angle$A intersect side **BC** at a point D between B and C.

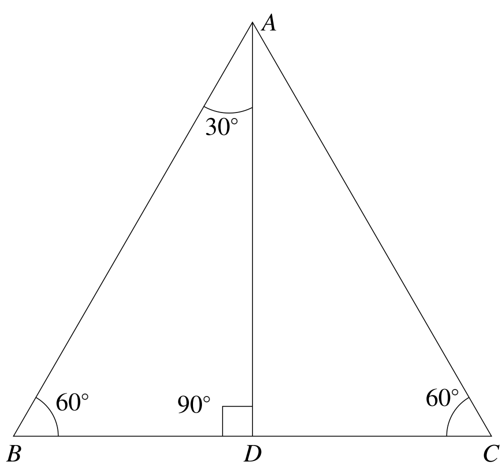

Fig. 1: Equilateral $\triangle ABC$

$$\|\mathbf{A}-\mathbf{B}\| = \|\mathbf{B}-\mathbf{C}\| = \|\mathbf{A}-\mathbf{C}\| \tag{2.0.1}$$

$$\angle ABD = \angle BAC = \angle ACD = 60° \tag{2.0.2}$$

$$\angle BAD = \angle CAD = 30° \tag{2.0.3}$$

Using angle bisector theorem ie. triangle will divide the opposite side into two segments that are proportional to the other two sides of the triangle.

$$\frac{\|\mathbf{A}-\mathbf{B}\|}{\|\mathbf{B}-\mathbf{D}\|} = \frac{\|\mathbf{A}-\mathbf{C}\|}{\|\mathbf{B}-\mathbf{C}\| - \|\mathbf{B}-\mathbf{D}\|} \tag{2.0.4}$$

Using (2.0.1)

$$\frac{\|\mathbf{A}-\mathbf{B}\|}{\|\mathbf{B}-\mathbf{D}\|} = \frac{\|\mathbf{A}-\mathbf{B}\|}{\|\mathbf{A}-\mathbf{B}\| - \|\mathbf{B}-\mathbf{D}\|} \tag{2.0.5}$$

$$\implies \|\mathbf{A}-\mathbf{B}\| = 2\|\mathbf{B}-\mathbf{D}\| \tag{2.0.6}$$

Taking the inner product of sides BA and AD.

$$(\mathbf{B}-\mathbf{A})^T(\mathbf{A}-\mathbf{D}) = \|\mathbf{B}-\mathbf{A}\|\|\mathbf{A}-\mathbf{D}\|\cos\theta \tag{2.0.7}$$

$$\cos\angle BAD = \frac{(\mathbf{B}-\mathbf{A})^T(\mathbf{A}-\mathbf{D})}{\|\mathbf{B}-\mathbf{A}\|\|\mathbf{A}-\mathbf{D}\|} \tag{2.0.8}$$

Now To Find AD.

$$\begin{aligned}
&(\mathbf{B}-\mathbf{A})^T(\mathbf{B}-\mathbf{A}) \\
&= (\mathbf{B}-\mathbf{D}+\mathbf{D}-\mathbf{A})^T(\mathbf{B}-\mathbf{D}+\mathbf{D}-\mathbf{A}) \\
&= [(\mathbf{B}-\mathbf{D})^T + (\mathbf{D}-\mathbf{A})^T][(\mathbf{B}-\mathbf{D}) + (\mathbf{D}-\mathbf{A})] \\
&= (\mathbf{B}-\mathbf{D})^T(\mathbf{B}-\mathbf{D}) + (\mathbf{B}-\mathbf{D})^T(\mathbf{D}-\mathbf{A}) + \\
&(\mathbf{D}-\mathbf{A})^T(\mathbf{B}-\mathbf{D}) + (\mathbf{D}-\mathbf{A})^T(\mathbf{D}-\mathbf{A})
\end{aligned} \tag{2.0.9}$$

$$\because \triangle ABD = \angle BAD + \angle ADB + \angle DBA \tag{2.0.10}$$

$$180° = 30° + \angle ADB + 60°. \tag{2.0.11}$$

$$\angle ADB = 180° - (60° + 30°) = 90°. \tag{2.0.12}$$

$$\implies (\mathbf{B}-\mathbf{D})^T(\mathbf{D}-\mathbf{A}) = 0 \tag{2.0.13}$$

$$\implies (\mathbf{D}-\mathbf{A})^T(\mathbf{B}-\mathbf{D}) = 0 \tag{2.0.14}$$

which gives

$$\begin{aligned}
&(\mathbf{B}-\mathbf{A})^T(\mathbf{B}-\mathbf{A}) = \\
&(\mathbf{B}-\mathbf{D})^T(\mathbf{B}-\mathbf{D}) + (\mathbf{D}-\mathbf{A})^T(\mathbf{D}-\mathbf{A}) \\
&\|\mathbf{B}-\mathbf{A}\|^2 = \|\mathbf{B}-\mathbf{D}\|^2 + \|\mathbf{D}-\mathbf{A}\|^2 \\
&\implies \|\mathbf{D}-\mathbf{A}\|^2 = \|\mathbf{B}-\mathbf{A}\|^2 - \|\mathbf{B}-\mathbf{D}\|^2
\end{aligned} \tag{2.0.15}$$

Using Eq (2.0.6) we get,

$$\|\mathbf{D}-\mathbf{A}\|^2 = \|\mathbf{B}-\mathbf{A}\|^2 - \frac{1}{4}\|\mathbf{B}-\mathbf{A}\|^2 \tag{2.0.16}$$

$$\|\mathbf{D}-\mathbf{A}\| = \frac{\sqrt{3}}{2}\|\mathbf{B}-\mathbf{A}\| \tag{2.0.17}$$

$$\implies \|\mathbf{B}-\mathbf{A}\| = \frac{2}{\sqrt{3}}\|\mathbf{D}-\mathbf{A}\| \tag{2.0.18}$$

Let $\mathbf{A} = 0$. Substituting in (2.0.6) and (2.0.18)

$$\|\mathbf{B}\| = 2\|\mathbf{B} - \mathbf{D}\| \tag{2.0.19}$$

$$\|\mathbf{B}\| = \frac{2}{\sqrt{3}}\|\mathbf{D}\| \tag{2.0.20}$$

Square on both sides in (2.0.19) and (2.0.20), we get,

$$\|\mathbf{B}\|^2 = 4\|\mathbf{B} - \mathbf{D}\|^2 \tag{2.0.21}$$

$$\frac{1}{4}\|\mathbf{B}\|^2 = \|\mathbf{B}\|^2 + \|\mathbf{D}\|^2 - 2\mathbf{B}^T\mathbf{D} \tag{2.0.22}$$

$$\|\mathbf{B}\|^2 = \frac{4}{3}\|\mathbf{D}\|^2 \tag{2.0.23}$$

Solving (2.0.22) and (2.0.23) we get,

$$\frac{1}{3}\|\mathbf{D}\|^2 = \frac{4}{3}\|\mathbf{D}\|^2 + \|\mathbf{D}\|^2 - 2\mathbf{D}^T\mathbf{D} \tag{2.0.24}$$

$$0 = \|\mathbf{D}\|^2 - 2\mathbf{B}^T\mathbf{D} \tag{2.0.25}$$

$$\implies \mathbf{B}^T\mathbf{D} = \|\mathbf{D}\|^2 \tag{2.0.26}$$

Substitute $\mathbf{A} = 0$ in (2.0.8) we get,

$$\cos\angle BAD = \frac{\mathbf{B}^T\mathbf{D}}{\|\mathbf{B}\|\,\|\mathbf{D}\|} \tag{2.0.27}$$

$$\because \|\mathbf{B}\| = \frac{2}{\sqrt{3}}\|\mathbf{D}\| \tag{2.0.28}$$

$$\implies \cos\angle BAD = \frac{\mathbf{B}^T\mathbf{D}}{\frac{2}{\sqrt{3}}\|\mathbf{D}\|\,\|\mathbf{D}\|} = \frac{\mathbf{B}^T\mathbf{D}}{\frac{2}{\sqrt{3}}\|\mathbf{D}\|^2} \tag{2.0.29}$$

Substitute (2.0.26) in (2.0.29)

$$\cos\angle BAD = \frac{\frac{\sqrt{3}}{2}\|\mathbf{D}\|^2}{\|\mathbf{D}\|^2} = \frac{\sqrt{3}}{2} \tag{2.0.30}$$

$$\because \angle BAD = 30° \tag{2.0.31}$$

$$\implies \cos 30° = \frac{\sqrt{3}}{2} \tag{2.0.32}$$

$$\because \cos^2\theta + \sin^2\theta = 1 \tag{2.0.33}$$

$$\sin 30° = \sqrt{1 - \cos^2 30°} \tag{2.0.34}$$

$$\implies \sin 30° = \frac{1}{2}. \tag{2.0.35}$$