Inverse Trigonometric Function

Domain and Range of Inverse Trigonometric Functions :-

f(x)	Domain	Range
$\sin^{-1}x$	$x \in [-1, 1]$ or $\lvert x \rvert \leq 1$	$[-\pi/2, \pi/2]$
$\cos^{-1}x$	$x \in [-1, 1]$ or $\lvert x \rvert \leq 1$	$[0, \pi]$
$\tan^{-1}x$	$x \in R$	$(-\pi/2, \pi/2)$
$\cot^{-1}x$	$x \in R$	$(0, \pi)$
$\sec^{-1}x$	$\lvert x \rvert \geq 1$ or $x \in R - (-1, 1)$	$[0, \pi] - \{\pi/2\}$
$\text{cosec}^{-1}x$	$\lvert x \rvert \geq 1$ or $x \in (-\infty, -1] \cup [1, \infty)$	$[-\pi/2, \pi/2] - \{0\}$

Graph of Inverse Trigonometric Functions :-

1. $\sin^{-1}x$

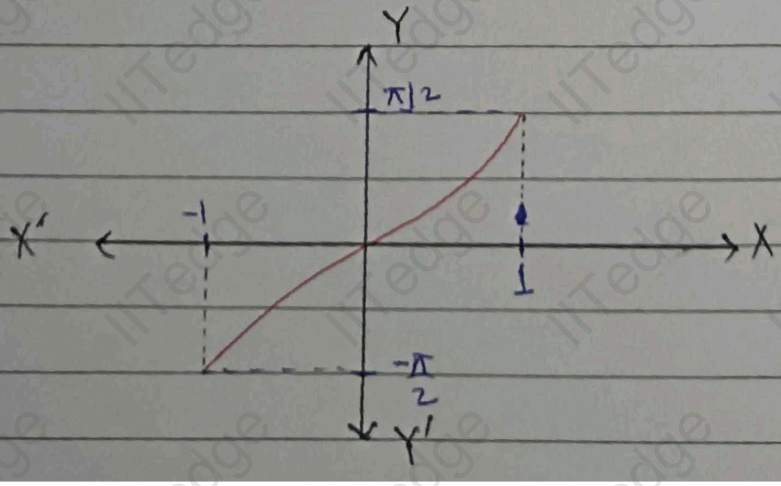
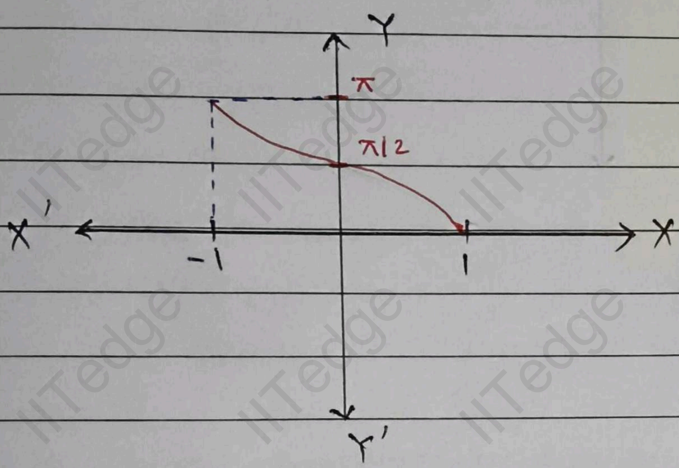


Image of $y = \sin x$ about $y = x$

Domain:: $x \in [-1, 1]$

Range: $[-\pi/2, \pi/2]$

2. $\cos^{-1}x$



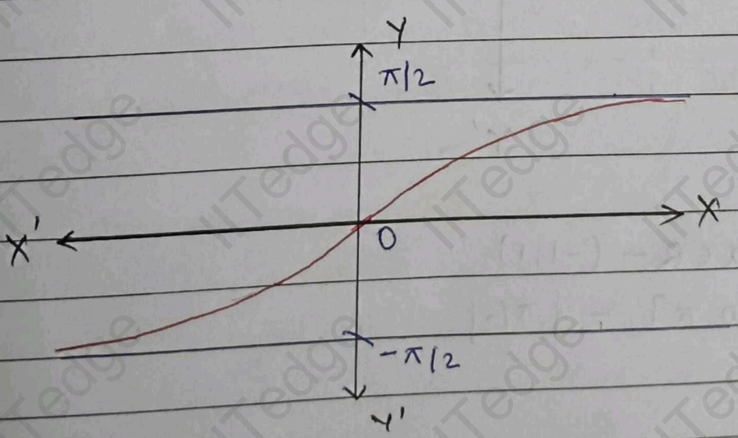
$-1 < x < \frac{1}{\sqrt{2}}$, $\cos^{-1}x > \sin^{-1}x$

$\frac{1}{\sqrt{2}} < x \le 1$, $\sin^{-1}x > \cos^{-1}x$

Domain: $[-1, 1]$

Range: $[0, \pi]$

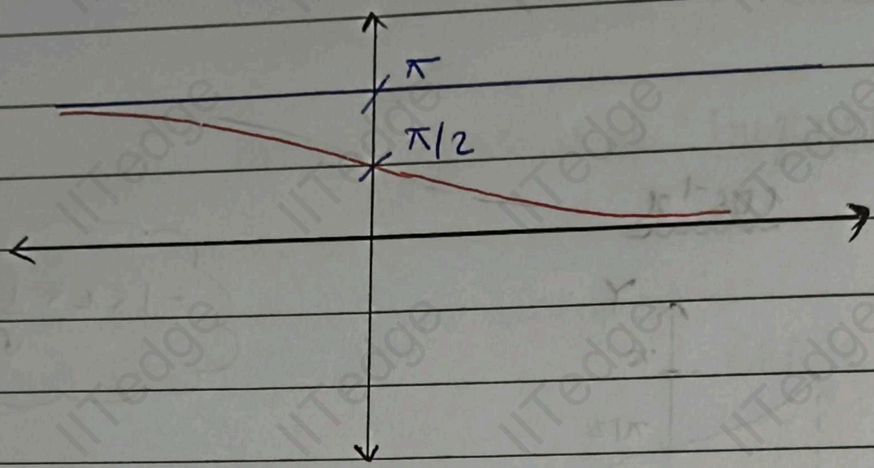
3. $\tan^{-1}x$



Domain: $x \in R$

Range: $(-\pi/2, \pi/2)$

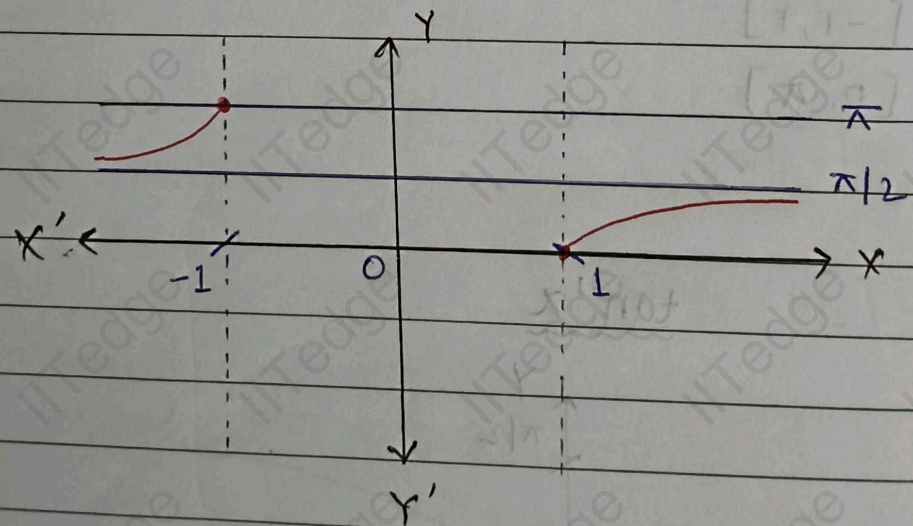
4. $\cot^{-1}x$



Domain :- $x \in R$

Range :- $(0, \pi)$

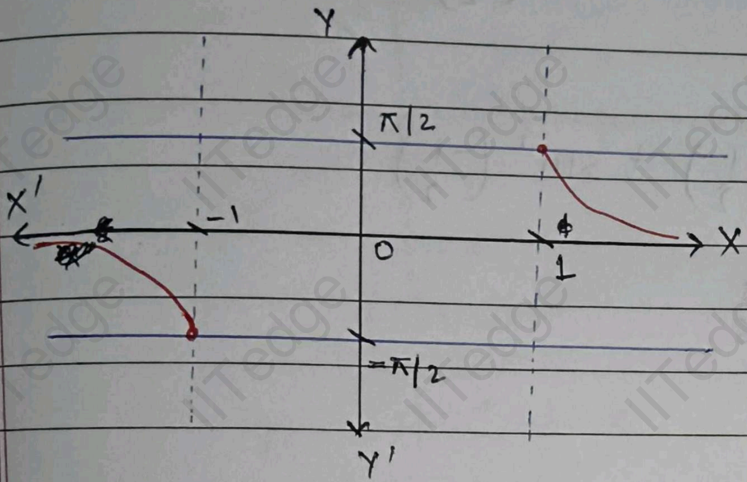
5. $\sec^{-1}x$

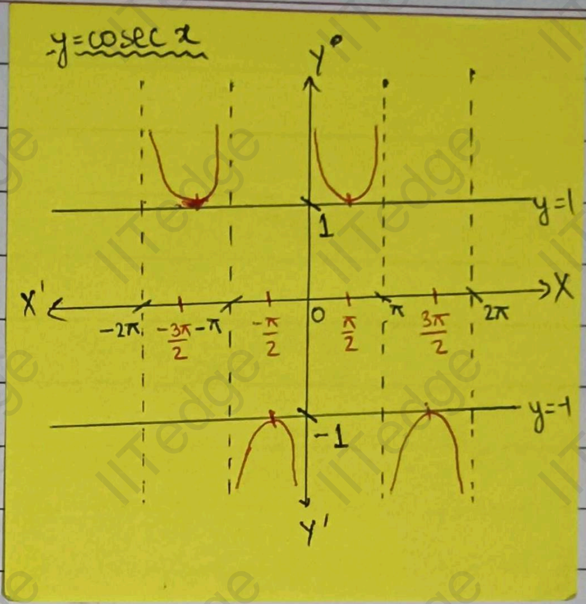


Domain :- $x \in R - (-1, 1)$

Range :- $[0, \pi] - \{\pi/2\}$

6. $\cosec^{-1}x$





Note :

1. All the Inverse Trigonometric functions represent an angle.

2. If $x > 0$, then all six ITF i.e., $\sin^{-1}x$, $\cos^{-1}x$, $\tan^{-1}x$, $\cot^{-1}x$, $\sec^{-1}x$, $\cosec^{-1}x$, represent an acute angle.

3. If $x < 0$, then $\sin^{-1}x$, $\tan^{-1}x$ and $\cosec^{-1}x$ represent an angle from $-\pi/2$ to 0. (4^{th} quadrant).

4. If $x < 0$, then $\cos^{-1}x$, $\cot^{-1}x$ and $\sec^{-1}x$ represent an obtuse angle (2^{nd} quandrant).

Third quadrant is never used in range of ITF.

Q. The value of $\tan^{-1}1 + \cos^{-1}\left(-\frac{1}{2}\right) + \sin^{-1}\left(-\frac{1}{2}\right)$ is.

Sol^n:- $\tan^{-1}1 + \cos^{-1}\left(-\frac{1}{2}\right) + \sin^{-1}\left(-\frac{1}{2}\right)$

$$= \frac{\pi}{4} + \frac{2\pi}{3} - \frac{\pi}{6}$$

$$= \frac{3\pi + 8\pi - 2\pi}{12}$$

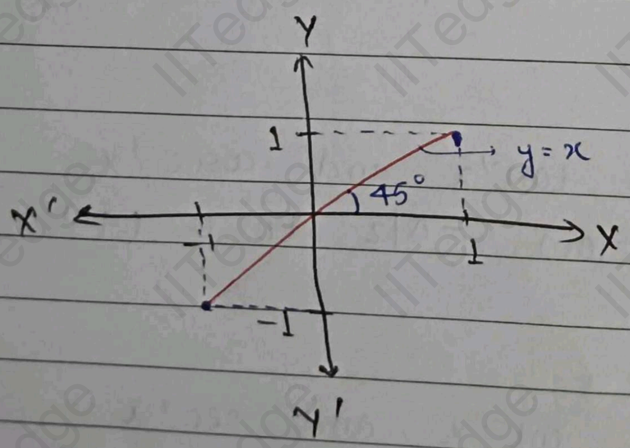
$$= \frac{9\pi}{12} = \frac{3\pi}{4}$$

Properties of Inverse Circular Functions:-

Property 1

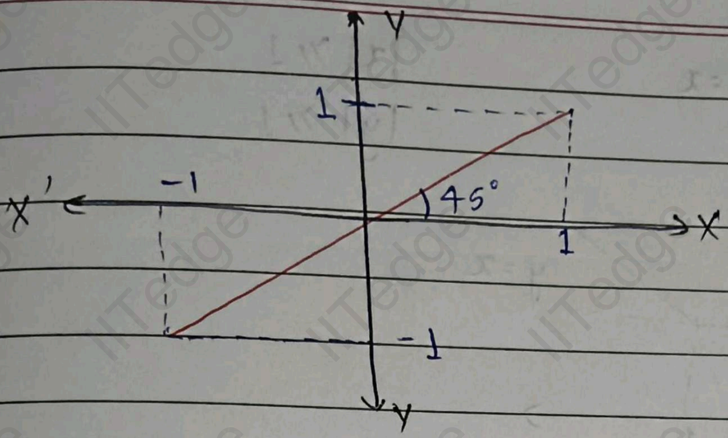
1. $y = \sin(\sin^{-1}x) = x \quad , x \in [-1, 1]$

$y \in [-1, 1]$



2. $y = \cos(\cos^{-1}x) = x \quad , x \in [-1, 1]$

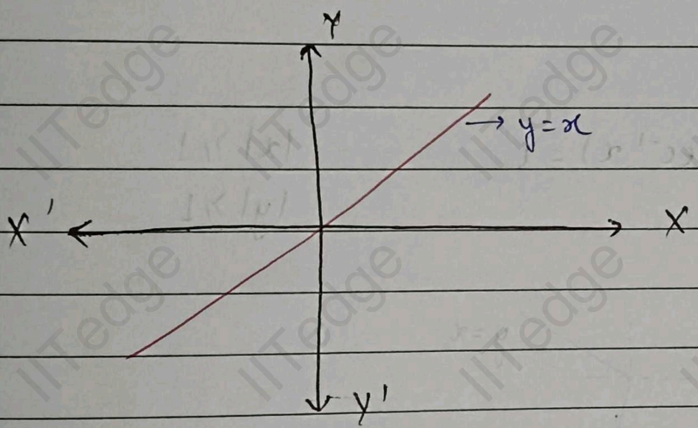
$y \in [-1, 1]$



3. $y = \tan(\tan^{-1}x) = x$

$x \in R$

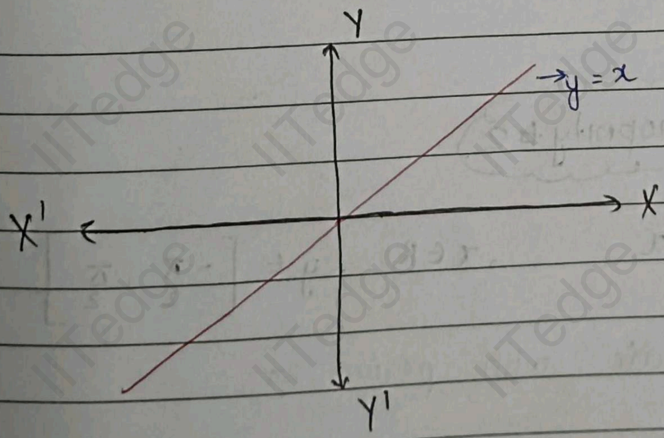
$y \in R$



4. $y = \cot(\cot^{-1}x) = x$

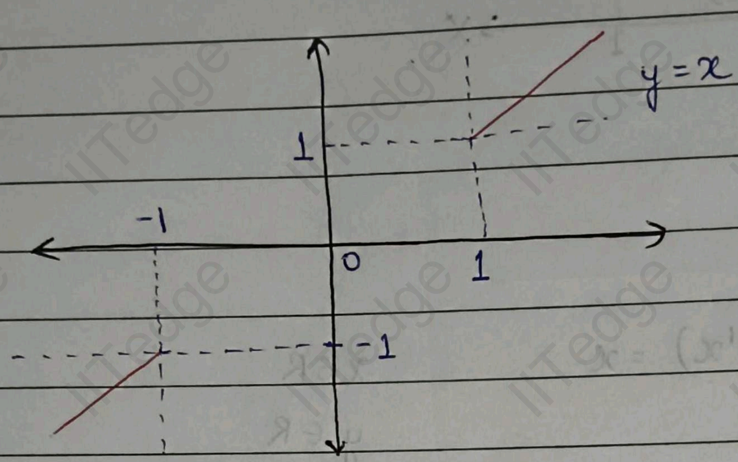
$x \in R$

$y \in R$



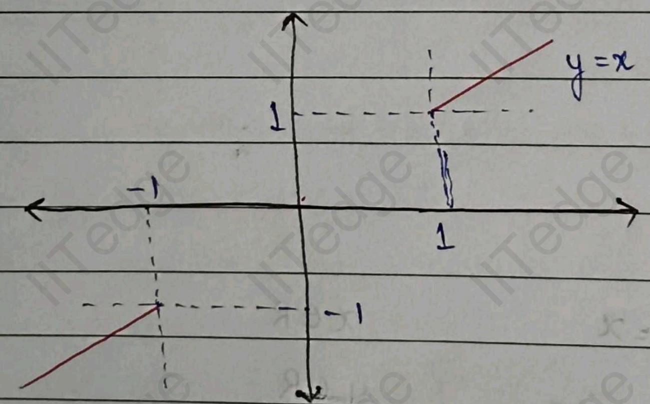
5. $y = \sec(\sec^{-1}x) = x$ $\quad |x| \geq 1$

$|y| \geq 1$



6. $y = \operatorname{cosec}(\operatorname{cosec}^{-1}x) = x$ $\quad |x| \geq 1$

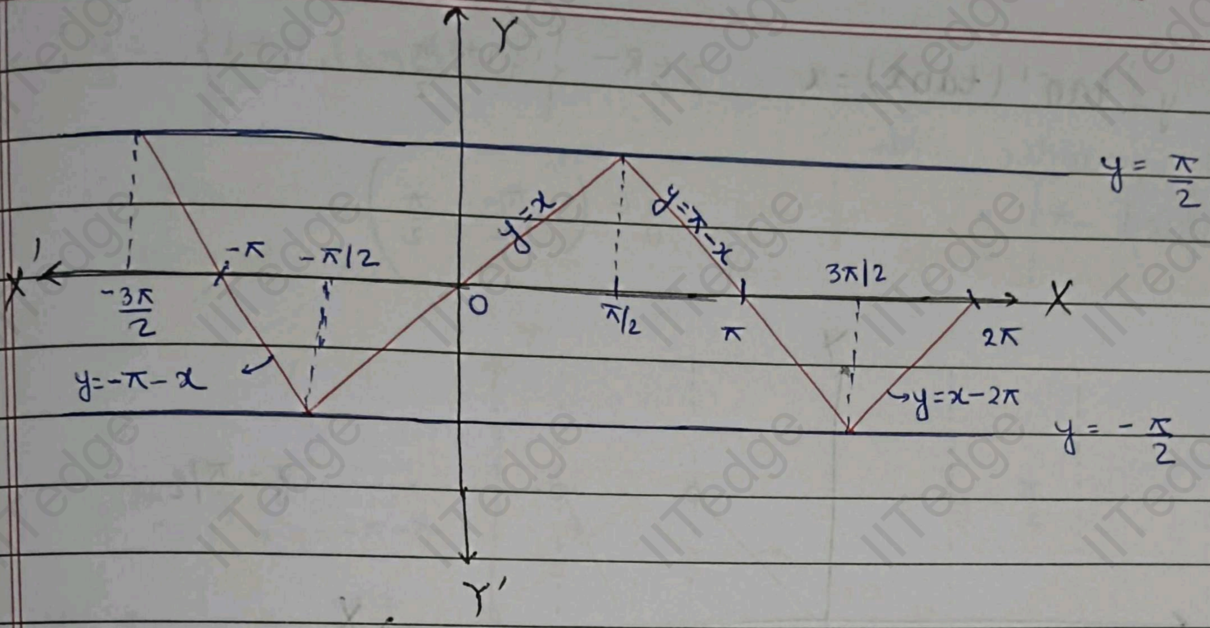
$|y| \geq 1$



{Property #2}

1. $y = \sin^{-1}(\sin x) = x$ $\quad , x \in R \quad , y \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

is periodic with period 2π.



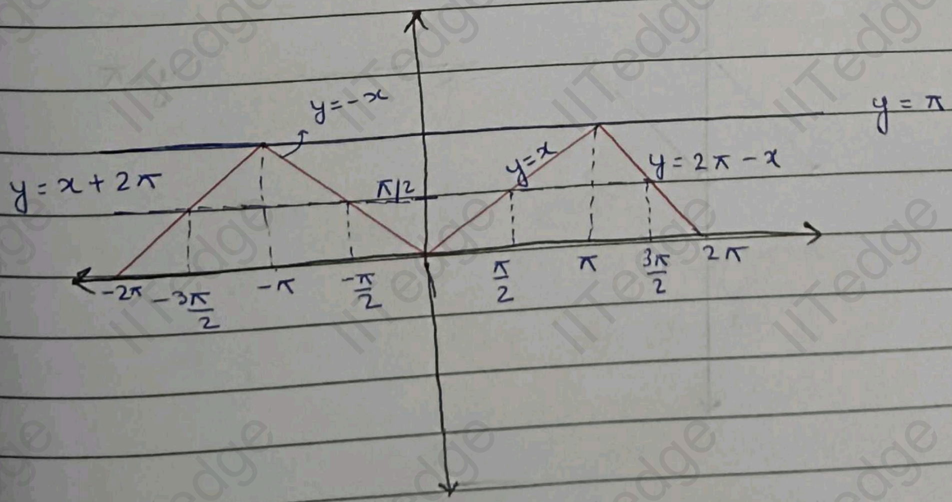
eg. $\sin^{-1}(\sin 5)$ ↦ radian

B/w $\frac{3\pi}{2}$ ~~to~~ and 2π

$\sin^{-1}(\sin 5) = 5 - 2\pi$

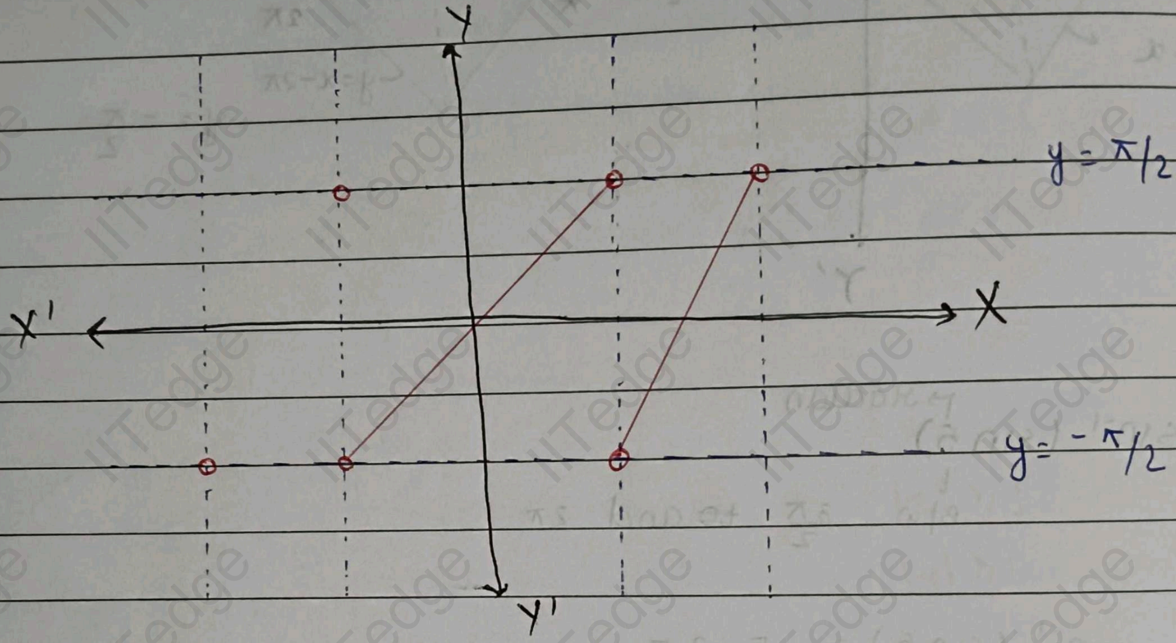
2. $y = \cos^{-1}(\cos x) = x$, $x \in R$, $y \in [0, \pi]$

is periodic with period 2π.



3. $y = \tan^{-1}(\tan x) = x \qquad x \in R - \left\{(2n+1)\frac{\pi}{2}, n \in I\right\}$

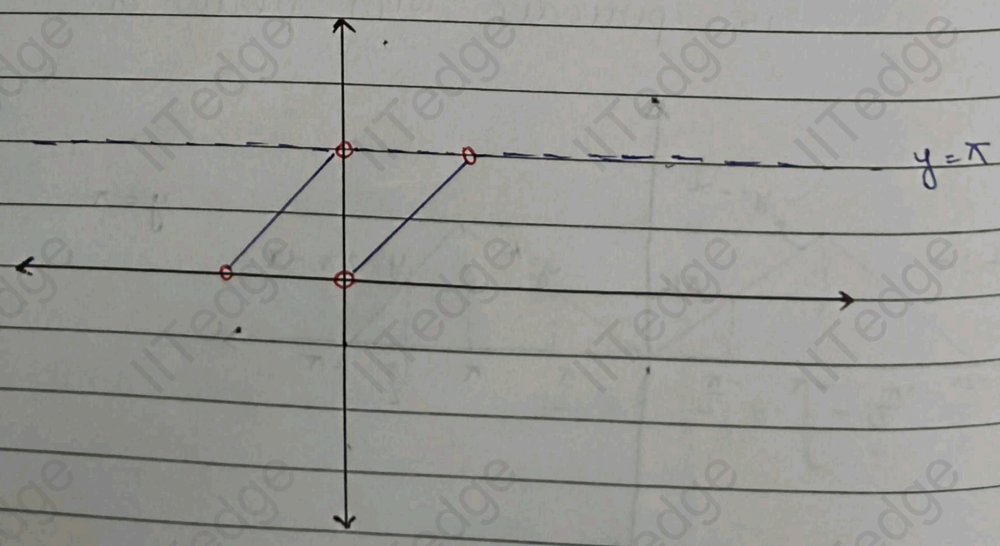
is periodic with period π $\qquad y \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$



4. $y = \cot^{-1}(\cot x) = x \qquad x \in R - \{n\pi, n \in I\}$

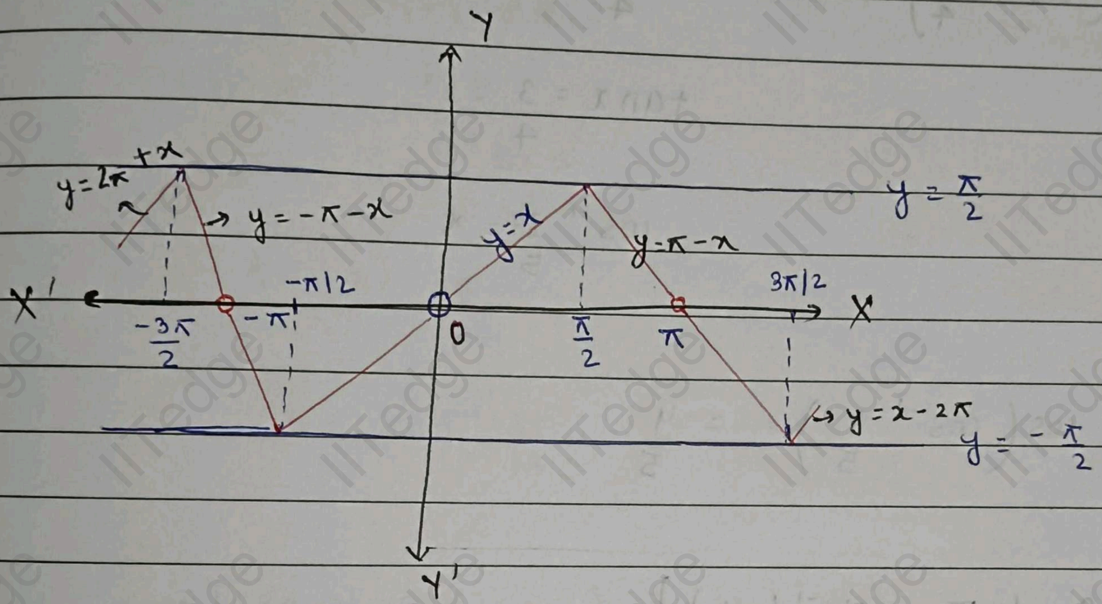
$y \in (0, \pi)$

is periodic with period π



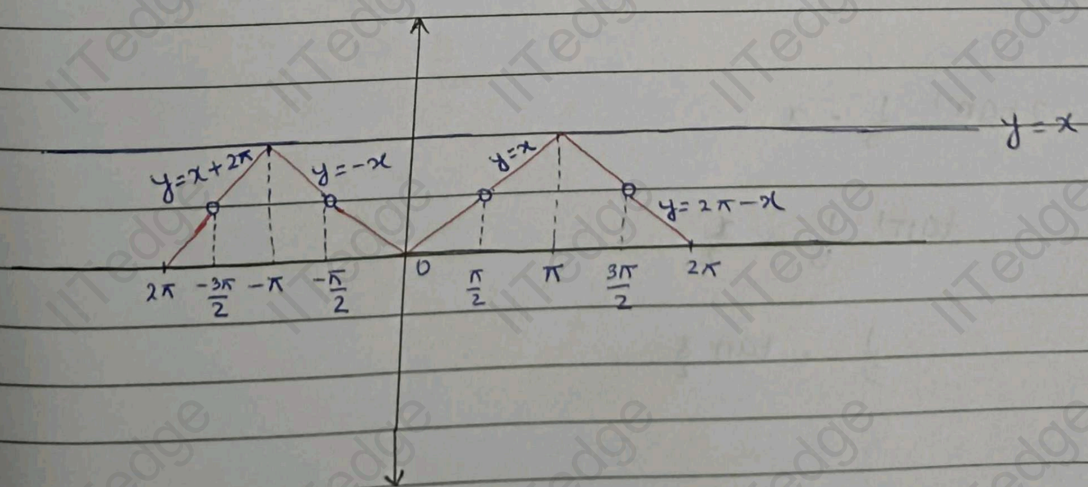
5. $y = \operatorname{cosec}^{-1}(\operatorname{cosec} x)$ $\qquad x \in R - \{n\pi, n \in I\}$

is periodic with period 2π. $\qquad y \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] - \{0\}$



6. $y = \sec^{-1}(\sec x) = x$ $\quad , x \in R - \left\{(2n+1)\frac{\pi}{2}, n \in I\right\}$

is periodic with period 2π. $\qquad y \in [0, \pi] - \left\{\frac{\pi}{2}\right\}$



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Page

Q. $\sin\left(\cos^{-1}\frac{3}{5}\right) \Rightarrow \sin\left(\sin^{-1}\frac{4}{5}\right) = \frac{4}{5}$

Q. $\cos\left(\tan^{-1}\frac{3}{4}\right) \Rightarrow \tan^{-1}\frac{3}{4} = x$

$\tan x = \frac{3}{4}$

$\cos^{-1}\frac{4}{5} = x$

$\cos\left(\cos^{-1}\frac{4}{5}\right) = \frac{4}{5}$

Q. $\sin\left(\frac{\pi}{2} - \sin^{-1}\left(-\frac{1}{2}\right)\right)$

Soln:- $\cancel{\sin\left(\frac{\pi}{2} - \sin^{-1}\left(\right.\right.}$ $\sin\left(\frac{\pi}{2} + \frac{\pi}{6}\right) = \frac{\sqrt{3}}{2}$

Q. $\tan\left(2\tan^{-1}\frac{1}{5} - \frac{\pi}{4}\right)$

Soln:- $2\tan^{-1}\frac{1}{5} = x$

$\tan^{-1}\frac{1}{5} = \frac{x}{2}$

$\frac{1}{5} = \tan\frac{x}{2}$

$\therefore \tan 2x = \dfrac{2\tan x}{1-\tan^2 x}$

$\because \tan x = \dfrac{2\tan x/2}{1-\tan^2 x}$

$\tan x = \dfrac{\cancel{2}/5}{\frac{24}{12}\cancel{/25}\ 5}$

$\tan x = \dfrac{5}{12}$

$x = \tan^{-1}\dfrac{5}{12}$

$\tan\left(\underset{\downarrow A}{\tan^{-1}\dfrac{5}{12}} - \underset{\downarrow B}{\dfrac{\pi}{4}}\right)$

$= \dfrac{\tan(\tan^{-1}5/12) - \tan\pi/4)}{1+\tan(\tan^{-1}5/12)\tan\pi/4}$

$= \dfrac{5/12 - 1}{1+\frac{5}{12}} = \dfrac{-7}{17}$

Formulae :-

P-1 :-

1. $\sin(\sin^{-1}x) = x$, $|x| \leq 1 \xrightarrow{or} x \in [-1, 1]$

2. $\cos(\cos^{-1}x) = x$, $|x| \leq 1$

3. $\tan(\tan^{-1}x) = x$, $x \in R$

4. $\cot(\cot^{-1}x) = x$, $x \in R$

P-2 :

1. $\sin^{-1}(\sin x) = x$, $x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

2. $\cancel{\sin} \cos^{-1}(\cos x) = x$, $x \in [0, \pi]$

3. $\tan^{-1}(\tan x) = x$, $x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

P-3 :-

1. $\operatorname{cosec}^{-1}x = \sin^{-1}\left(\frac{1}{x}\right)$, $|x| \geq 1$

2. $\sec^{-1}x = \cos^{-1}\left(\frac{1}{x}\right)$, $|x| \geq 1$

3. $$\cot^{-1}x = \begin{cases} \tan^{-1}\left(\frac{1}{x}\right) & , x > 0 \\ \pi + \tan^{-1}\left(\frac{1}{x}\right) & , x < 0 \end{cases}$$

P-4 :-

1. $\sin^{-1}(-x) = -\sin^{-1}x$, $|x| \le 1$
2. $\operatorname{cosec}^{-1}(-x) = -\operatorname{cosec}^{-1}x$, $|x| \ge 1$
3. $\tan^{-1}(-x) = -\tan^{-1}x$, $\cancel{\forall}\ x \in R$
4. $\cot^{-1}(-x) = \pi - \cot^{-1}x$, $x \in R$
5. $\sec^{-1}(-x) = \pi - \sec^{-1}x$, $|x| \ge 1$
6. $\cos^{-1}(-x) = \pi - \cos^{-1}x$, $|x| \le 1$

P-5 :-

1. $\sin^{-1}x + \cos^{-1}x = \pi/2$, $|x| \le 1$
2. $\tan^{-1}x + \cot^{-1}x = \pi/2$, $x \in R$
3. $\sec^{-1}x + \operatorname{cosec}^{-1}x = \pi/2$, $|x| \ge 1$

P-6 :-

1. $$\tan^{-1}x + \tan^{-1}y = \tan^{-1}\left(\frac{x+y}{1-xy}\right), \begin{Bmatrix} x > 0, y > 0 \\ xy < 1 \end{Bmatrix}$$

$$= \pi + \tan^{-1}\left(\frac{x+y}{1-xy}\right) \begin{Bmatrix} x > 0, y > 0 \\ xy > 1 \end{Bmatrix}$$

2. $$\tan^{-1}x - \tan^{-1}y = \tan^{-1}\left(\frac{x-y}{1+xy}\right) \quad \begin{Bmatrix} x > 0, y > 0 \end{Bmatrix}$$

P-7 :-

1. $$\sin^{-1}x + \sin^{-1}y = \sin^{-1}\left(x\sqrt{1-y^2} + y\sqrt{1-x^2}\right), \ x \ge 0, \ y \ge 0 \text{ and } x^2+y^2 \le 1$$

$\sin^{-1}x + \sin^{-1}y = \pi - \sin^{-1}\left(x\sqrt{1-y^2} + y\sqrt{1-x^2}\right)$ $\{x \geq 0, y \geq 0, x^2+y^2 \geq 1\}$

2. $\sin^{-1}x - \sin^{-1}y = \sin^{-1}\left(x\sqrt{1-y^2} - y\sqrt{1-x^2}\right)$ $\{x \geq 0, y \geq 0\}$

3. $\cos^{-1}x + \cos^{-1}y = \cos^{-1}\left(xy - \sqrt{1-x^2}\sqrt{1-y^2}\right)$ $\{x \geq 0, y \geq 0\}$

$\cancel{\cos^{-1}x - \cos^{-1}y = \cos^{-1}\left(xy + \sqrt{1-x^2}\sqrt{1-y^2}\right)}$ $\{$

If $x < y$, $\cancel{xy > 0}$ $x > 0$, $y > 0$

4. $\cos^{-1}x - \cos^{-1}y = \cos^{-1}\left(xy + \sqrt{1-x^2}\sqrt{1-y^2}\right)$ $\{x < y, x > 0, y > 0\}$

$= -\cos^{-1}\left(xy + \sqrt{1-x^2}\sqrt{1-y^2}\right)$ $\{x > y, x > 0, y > 0\}$

P-8:-

If $\tan^{-1}x + \tan^{-1}y + \tan^{-1}z = \tan^{-1}\left(\dfrac{x+y+z-xyz}{1-xy-yz-xz}\right)$,

$x > 0, y > 0, z > 0$ and $xy+yz+zx < 1$

(i) If $\tan^{-1}x + \tan^{-1}y + \tan^{-1}z = \pi$, then
$x+y+z = xyz$

(ii) If $\tan^{-1}x + \tan^{-1}y + \tan^{-1}z = \pi/2$, then
$xy+yz+xz = 1$

P-9:-

* $2\tan^{-1}x = \sin^{-1}\left(\frac{2x}{1+x^2}\right) = \cos^{-1}\left(\frac{1-x^2}{1+x^2}\right) = \tan^{-1}\left(\frac{2x}{1-x^2}\right)$

1. $\sin^{-1}\left(\frac{2x}{1+x^2}\right) = \begin{cases} 2\tan^{-1}x, & |x| \leq 1 \\ \pi - 2\tan^{-1}x, & x > 1 \\ -(\pi + 2\tan^{-1}x), & x < -1 \end{cases}$

2. $\cos^{-1}\left(\frac{1-x^2}{1+x^2}\right) = \begin{cases} 2\tan^{-1}x, & x \geq 0 \\ -2\tan^{-1}x, & x < 0 \end{cases}$

3. $\tan^{-1}\left(\frac{2x}{1-x^2}\right) = \begin{cases} 2\tan^{-1}x, & |x| < 1 \\ \pi + 2\tan^{-1}x, & x < -1 \\ -(\pi - 2\tan^{-1}x), & x > 1 \end{cases}$

P-10:-

$\sin^{-1}(3x-4x^3) = \begin{cases} -(\pi + 3\sin^{-1}x), & -1 \leq x \leq -1/2 \\ 3\sin^{-1}x, & -1/2 \leq x \leq 1/2 \\ \pi - 3\sin^{-1}x, & 1/2 \leq x \leq 1 \end{cases}$

$\cos^{-1}(4x^3-3x) = \begin{cases} 3\cos^{-1}x - 2\pi, & -1 \leq x \leq -1/2 \\ 2\pi - 3\cos^{-1}x, & -1/2 \leq x \leq 1/2 \\ 3\cos^{-1}x, & 1/2 \leq x \leq 1 \end{cases}$

$\cos^{-1}(2x^2-1) = \begin{cases} 2\cos^{-1}x, & 0 \leq x \leq 1 \\ 2\pi - 2\cos^{-1}x, & -1 \leq x \leq 0 \end{cases}$

Q. The value of $\sin^{-1}\left(-\frac{\sqrt{3}}{2}\right) + \cos^{-1}\left(\cos\frac{7\pi}{6}\right)$ is.

Sol^n:
$$\sin^{-1}\left(-\frac{\sqrt{3}}{2}\right) + \cos^{-1}\left(\cos\left(2\pi - \frac{5\pi}{6}\right)\right)$$
$$-\sin^{-1}\left(\frac{\sqrt{3}}{2}\right) + \cos^{-1}\left(\cos\frac{5\pi}{6}\right)$$
$$-\frac{\pi}{3} + \frac{5\pi}{6} = \frac{-2\pi+5\pi}{6} = \frac{\pi}{2}$$

Q. (i) $\sin^{-1}(\sin 10)$ (ii) $\tan^{-1}(\tan(-6))$
(iii) $\cot^{-1}(\cot 4)$

Sol^n: (i) $\frac{5\pi}{2} \leq 10 \leq \frac{7\pi}{2}$

For odd multiples of π, $m = -1$

$$y = -x + 3\pi$$
$$y = 3\pi - 10$$

(ii) $\tan^{-1}(\tan(-6))$

$$-\frac{5\pi}{2} < -6 < -\frac{3\pi}{2}$$

For tan, ~~Even multiple of~~ π,
$m = 1$

$$y = x + 2\pi$$
$$y = 2\pi - 6$$

(iii) $\cot^{-1}(\cot 4)$

$$\pi < 4 < 2\pi$$

odd multiple of π

$m = 1$

$y = x - \pi$

$y = 4 - \pi$

Q. The value of $\cos^{-1}(\cos 10)$ is

Soln:- Y-coordinate is zero @ even multiples of π

$3\pi < 10 < 4\pi$ $\quad m = -1$

$y = -x + 4\pi$

$y = 4\pi - 10$

Q. The value of (i) $\sin^{-1}(\sin \theta)$ (ii) $\cos^{-1}(\cos\theta)$ is.

Soln:- New method (short cut method) for solving these problems:-

First check nearest multiple of π which is nearest to given number, such that

$n\pi \pm x \in [-1.57, 1.57]$ {for sin and tan}

[$n\pi \pm x \in$ ~~Domain~~ Range ~~of~~ Trigonometric Ratio]

eg, $\theta \rightarrow \theta - 2\pi \quad\quad \theta - 3\pi$

$1.72 \quad\quad -1.42$ ✓

Two possibility : $(\theta - 3\pi)$ and $(3\pi - \theta)$.

both will give same answer.

Verification :-

$\sin^{-1}(\sin[3\pi + \overbrace{(\theta - 3\pi)}^{A}])$

$= -A$

$= 3\pi - \theta$

$\sin^{-1}(\sin(3\pi - \overbrace{(3\pi - \theta)}^{B}))$

$= B$

$= 3\pi - \theta$

(ii) $\cos^{-1}(\cos 8)$

Sol^n:- $n\pi \pm x \in [0, \pi]$

$n\pi \pm x \in [0, 3.14]$

$8 \rightarrow 8 - 2\pi$ $\qquad$ $3\pi - 8$

1.72 ✓ $\qquad$ 1.42 ✓

$\cos^{-1}(\cos[2\pi + (8-2\pi)]$ $\qquad$ $\cos^{-1}(\cos[3\pi - (3\pi - 8)]$

$= -2\pi + 8$ $\qquad$ $= \pi - (3\pi - 8)$

$= -2\pi + 8$

Q. (i) $\cos^{-1}\left(\cos\left(\frac{13\pi}{6}\right)\right)$ $\qquad$ (ii) $\tan^{-1}\left(\tan\left(\frac{7\pi}{6}\right)\right)$

(iii) $\sin^{-1}\left(\sin\left(\frac{5\pi}{6}\right)\right)$

Sol^n:- $n\pi \pm x \in [0, \pi]$

$\frac{13\pi}{6} \rightarrow \frac{13\pi}{6} - 2\pi$

$\cos^{-1}\left(\cos\left[2\pi + \left(\frac{13\pi}{6} - 2\pi\right)\right]\right)$

$= -2\pi + \frac{13\pi}{6}$

$= \frac{\pi}{6}$

(ii) $\tan^{-1}\left(\tan\left(\frac{7\pi}{6}\right)\right)$

$n\pi \pm x \in \left[-\pi/2 \;, \pi/2\right]$

$\frac{7\pi}{6} \rightarrow \quad \frac{7\pi}{6} - \pi$

$= \tan^{-1}\left(\tan\left[\pi + \left(\frac{7\pi}{6} - \pi\right)\right]\right)$

$= \frac{7\pi}{6} - \pi \quad = \frac{\pi}{6}$

(iii) $\sin^{-1}\left(\sin\left(\frac{5\pi}{6}\right)\right)$

$n\pi \pm x \in \left[-\pi/2 \,, \pi/2\right]$

$\frac{5\pi}{6} \longrightarrow \quad \frac{5\pi}{6} - \pi$

$\sin^{-1}\left(\sin\left[\pi + \left(\frac{5\pi}{6} - \pi\right)\right]\right)$

$\cancel{\sin^{-1}} = - \left(\frac{5\pi}{6} - \pi\right)$

$= \frac{\pi}{6}$

Q. Prove that $\tan^{-1}\left(\frac{1}{7}\right) + \tan^{-1}\left(\frac{1}{13}\right) = \tan^{-1}\left(\frac{2}{9}\right)$

Soln:- LHS $\Rightarrow \tan^{-1}x + \tan^{-1}y = \tan^{-1}\left(\frac{x+y}{1-xy}\right) \quad \{xy < 1\}$

$$= \tan^{-1}\left(\frac{\frac{1}{7}+\frac{1}{13}}{1-\frac{1}{91}}\right)$$

$$= \tan^{-1}\left(\frac{\frac{2\cancel{0}}{\cancel{91}}}{\frac{9\cancel{0}}{\cancel{91}}}\right)$$

$$= \tan^{-1}\left(\frac{2}{9}\right)$$

LHS = RHS.

Q. Compute $\tan^{-1}\left(\frac{1}{5}\right) + \tan^{-1}\left(\frac{1}{7}\right) + \tan^{-1}\left(\frac{1}{3}\right) + \tan^{-1}\left(\frac{1}{8}\right)$

Soln:-

$$\tan^{-1}\left(\frac{\frac{1}{5}+\frac{1}{7}}{1-\frac{1}{35}}\right) + \tan^{-1}\left(\frac{\frac{1}{3}+\frac{1}{8}}{1-\frac{1}{24}}\right)$$

$$\tan^{-1}\left(\frac{\frac{\cancel{12}\,6}{\cancel{35}}}{\frac{\cancel{34}\,17}{\cancel{35}}}\right) + \tan^{-1}\left(\frac{\frac{11}{\cancel{24}}}{\frac{23}{\cancel{24}}}\right)$$

$$\tan^{-1}\left(\frac{6}{17}\right) + \tan^{-1}\left(\frac{11}{23}\right)$$

$$\tan^{-1}\left(\frac{\frac{6}{17}+\frac{11}{23}}{1-\frac{66}{17\times 23}}\right) = \tan^{-1}\left(\frac{\frac{138+187}{\cancel{391}}}{\frac{391-66}{\cancel{391}}}\right)$$

$= \tan^{-1}\left(\frac{325}{325}\right)$

$= \tan^{-1}(1)$

$= \frac{\pi}{4}$

Q. Compute $\sin^{-1}\left(\frac{12}{13}\right) + \cot^{-1}\left(\frac{4}{3}\right) + \tan^{-1}\left(\frac{63}{16}\right) = ?$

Sol^n:- $\cot^{-1}\left(\frac{4}{3}\right) = x$

$\cot x = \frac{4 \to B}{3 \to P}$ $\quad\quad \sin x = \frac{3}{5}$

$H = 5$

$\sin^{-1}\left(\frac{12}{13}\right) + \sin^{-1}\left(\frac{3}{5}\right) + \tan^{-1}\left(\frac{63}{16}\right)$

$= \tan^{-1}\left(\frac{12}{5}\right) + \tan^{-1}\left(\frac{3}{4}\right) + \tan^{-1}\left(\frac{63}{16}\right)$

$xy > 1 \leftarrow$

$= \left[\pi + \tan^{-1}\left(\frac{\frac{12}{5} + \frac{3}{4}}{1 - \frac{36}{20}}\right)\right] + \tan^{-1}\left(\frac{63}{16}\right)$

$= \left[\pi + \tan^{-1}\left(-\frac{63}{16}\right)\right] + \tan^{-1}\left(\frac{63}{16}\right)$

$= \pi - \tan^{-1}\left(\frac{63}{16}\right) + \tan^{-1}\left(\frac{63}{16}\right)$

$= \pi$

Q. Prove that $\cos^{-1}\left(\frac{12}{13}\right) + \sin^{-1}\left(\frac{3}{5}\right) = \sin^{-1}\left(\frac{56}{65}\right)$

Sol^n:- $LHS = \cos^{-1}\left(\frac{12}{13}\right) + \sin^{-1}\left(\frac{3}{5}\right)$

$$= \sin^{-1}\left(\frac{5}{13}\right) + \sin^{-1}\left(\frac{3}{5}\right)$$

$$= \sin^{-1}\left[\frac{5}{13}\sqrt{1-\frac{9}{25}} + \frac{3}{5}\sqrt{1-\frac{25}{169}}\right]$$

$$= \sin^{-1}\left[\frac{5}{13} \times \frac{\cancel{16}4}{5} + \frac{3}{5} \times \frac{12}{13}\right]$$

$$= \sin^{-1}\left(\frac{56}{65}\right)$$

Trick to get Triplets

For odd:-

First find square of that number, then break the square into two consecutive terms:

eg, $3 \xrightarrow{3^2} 9 \begin{cases} 4 \\ 5 \end{cases}$

$5 \xrightarrow{5^2} 25 \begin{cases} 12 \\ 13 \end{cases}$

classmate

Date

Page

<u>For even</u>

Find square.

Divide square term by 2.

split into two terms such that difference is 2.

$8 \xrightarrow{8^2} 64 \xrightarrow{/2} 32 \begin{cases} 15 \\ 17 \end{cases}$

$6 \xrightarrow{6^2} 36 \xrightarrow{/2} 18 \begin{cases} 8 \\ 10 \end{cases}$

$10 \xrightarrow{10^2} 100 \xrightarrow{/2} 50 \begin{cases} 24 \\ 26 \end{cases}$

Q $\sin^{-1}\left(\sin \frac{2\pi}{3}\right) + \cos^{-1}\left(\cos \frac{7\pi}{6}\right) + \tan^{-1}\left(\tan \frac{3\pi}{4}\right)$

[JEE Main 2022]

Soln:- $\sin^{-1} \sin\left(\pi - \frac{\pi}{3}\right) + \cos^{-1}\left(\cos 2\pi - \frac{5\pi}{6}\right) + \tan^{-1} \tan\left(\pi - \frac{\pi}{4}\right)$

$\sin^{-1} \sin\left(\frac{\pi}{3}\right) + \cos^{-1} \cos\left(\frac{5\pi}{6}\right) + \tan^{-1} \tan\left(\frac{\pi}{4}\right)$

$\frac{\pi}{3} + \frac{5\pi}{6} - \frac{\pi}{4}$

$\frac{4\pi + 10\pi - 3\pi}{12} = \frac{11\pi}{12}$

Q. $2\pi - \left(\sin^{-1}\frac{4}{5} + \sin^{-1}\frac{5}{13} + \sin^{-1}\frac{16}{65}\right)$ [JEE Main 2020]

Sol^{n}:- $2\pi - \left(\sin^{-1}\left(\frac{4}{5}\times\frac{12}{13} + \frac{5}{13}\times\frac{3}{5}\right) + \sin^{-1}\frac{16}{65}\right)$

$$2\pi - \left(\sin^{-1}\left(\frac{63}{65}\right) + \sin^{-1}\frac{16}{65}\right)$$

$$2\pi - \sin^{-1}\left(\frac{63}{65}\sqrt{1-256}\right.$$

$$2\pi - \sin^{-1}\left(\frac{63}{65}\times\frac{16}{65} + \frac{16}{65}\times\frac{63}{65}\right)$$

$2\pi -$

Sol^{n}:- $2\pi - \left(\tan^{-1}\frac{4}{3} + \tan^{-1}\frac{5}{12} + \tan^{-1}\frac{16}{63}\right)$

$$2\pi - \left(\tan^{-1}\frac{\frac{4}{3}+\frac{5}{12}}{1-\frac{20}{36}} + \tan^{-1}\frac{16}{63}\right)$$

$$2\pi - \left(\tan^{-1}\frac{\frac{63}{36}}{\frac{16}{36}} + \tan^{-1}\frac{16}{63}\right)$$

$$2\pi - \left(\tan^{-1}\frac{63}{16} + \tan^{-1}\frac{16}{63}\right)$$

$$= 2\pi - \left(\cot^{-1}\frac{16}{63} + \tan^{-1}\frac{16}{63}\right)$$

$$= 2\pi - \frac{\pi}{2}$$

$$= \frac{3\pi}{2}$$

Q. Considering only the principal values of the ITF, the domain of the function ~~f(x) = cos^{-1}~~ $f(x) = \cos^{-1}\left(\frac{x^2-4x+2}{x^2+3}\right)$ is.

[JEE Main 2022]

Sol^n:

$$-1 \le \frac{x^2-4x+2}{x^2+3} \le 1$$

$\cancel{x^2 - 4x + 3 \ge -x^2 - 3}$

$\cancel{2x^2 - 4x + 6 \ge 0}$

$\cancel{x^2 - 2x + 3 \ge 0}$

$\cancel{x^2 - 3x + x + 3 \ge 0}$

$\cancel{x(x-3) + 1(x-}$

$\cancel{x^2 - 4x + 3 \le x^2 + 3}$

$\cancel{-4x \le 0}$

$\cancel{x \le 0}$

$$-x^2 - 3 \le x^2 - 4x + 2 \le x^2 + 3$$

$x^2 - 4x + 2 \le -x^2 - 3$

$2x^2 - 4x + 5 \le 0$

$a > 0$, $D < 0$

$x \in R$

$x^2 - 4x + 2 \le x^2 + 3$

$-4x \le 1$

$x \ge -\frac{1}{4}$

$$x \in \left[-\frac{1}{4}, \infty\right)$$

Q. $\cos^{-1}x - \cos^{-1}\frac{y}{2} = \alpha$, then $4x^2 - 4xy\cos\alpha + y^2 = ?$

Soln.: $\cos^{-1}\left(x\frac{y}{2} + \sqrt{1-x^2}\sqrt{1-\frac{y^2}{4}}\right) = \alpha$

$$\cos^{-1}\left(\frac{xy}{2} + \frac{\sqrt{1-x^2}\sqrt{4-y^2}}{2}\right) = \alpha$$

$$\cos\alpha = \frac{xy}{2} + \frac{\sqrt{1-x^2}\sqrt{4-y^2}}{2}$$

$$\frac{\sqrt{1-x^2}\sqrt{4-y^2}}{2} = \cos\alpha - \frac{xy}{2}$$

Squaring both sides

$$\frac{(1-x^2)(4-y^2)}{4} = \cos^2\alpha + \frac{x^2y^2}{4} - xy\cos\alpha$$

$$(1-x^2)(4-y^2) = 4\cos^2\alpha + x^2y^2 - 4xy\cos\alpha$$

$$4 - y^2 - 4x^2 + \cancel{x^2y^2} = 4\cos^2\alpha + \cancel{x^2y^2} - 4xy\cos\alpha$$

$$4x^2 - 4xy\cos\alpha + y^2 = 4(1-\cos^2\alpha)$$
$$= 4\sin^2\alpha$$

Q. If α, β are the roots of the eqn $x^2 - 4x + 1 = 0$ then value of

$f(\alpha,\beta) = \frac{\beta^3}{2}\operatorname{cosec}^2\left(\frac{1}{2}\tan^{-1}\frac{\beta}{\alpha}\right) + \frac{\alpha^3}{2}\sec^2\left(\frac{1}{2}\tan^{-1}\frac{\alpha}{\beta}\right)$ is-

Sol^n: $x^2-4x+1=0$

$\alpha+\beta=4$

$\alpha\beta=1$

$\alpha = 2+\sqrt{3}$

$\beta = 2-\sqrt{3}$

$\alpha-\beta = 2\sqrt{3}$

$$\frac{\beta^3}{2}\operatorname{cosec}^2\left(\frac{1}{2}\tan^{-1}\left(\frac{\beta}{\alpha}\right)\right)+\frac{\alpha^3}{2}\sec^2\left(\frac{1}{2}\tan^{-1}\frac{\alpha}{\beta}\right)$$

$$=\frac{\beta^3}{2\sin^2\left(\frac{1}{2}\tan^{-1}\beta/\alpha\right)}+\frac{\alpha^3}{2\cos^2\left(\frac{1}{2}\tan^{-1}\alpha/\beta\right)}$$

$$=\frac{\beta^2}{1-\cos\left(\tan^{-1}\beta/\alpha\right)}+\frac{\alpha^3}{1+\cos\left(\tan^{-1}\alpha/\beta\right)}$$

$$=\frac{\beta^3}{1-\frac{\alpha}{\sqrt{\alpha^2+\beta^2}}}+\frac{\alpha^3}{1+\frac{\beta}{\sqrt{\alpha^2+\beta^2}}}$$

$$=\sqrt{\alpha^2+\beta^2}\left(\frac{\beta^3}{\sqrt{\alpha^2+\beta^2}-\alpha}+\frac{\alpha^3}{\sqrt{\alpha^2+\beta^2}+\beta}\right)$$

$$=\sqrt{\alpha^2+\beta^2}\left(\frac{\beta^3\left(\sqrt{\alpha^2+\beta^2}\right)+\beta^4+\alpha^3\left(\sqrt{\alpha^2+\beta^2}\right)-\alpha^4}{\left(\sqrt{\alpha^2+\beta^2}-\alpha\right)\left(\sqrt{\alpha^2+\beta^2}+\beta\right)}\right)$$

$$=\sqrt{\alpha^2+\beta^2}\left(\frac{\sqrt{\alpha^2+\beta^2}\left(\beta^3+\alpha^3\right)+\beta^4-\alpha^4}{\left(\alpha^2+\beta^2\right)-\sqrt{\alpha^2+\beta^2}\left(\alpha-\beta\right)-\alpha\beta}\right)$$

$$=\sqrt{14}\left(\frac{\sqrt{14}\times 52-112\sqrt{3}}{14-\sqrt{14}\cdot 2\sqrt{3}-1}\right)$$

$$=\frac{14\times 52-112\sqrt{42}}{13-2\sqrt{42}}$$

$$= 56\frac{(13-2\sqrt{42})}{(13-2\sqrt{42})}$$

$$= 56$$

Q. $\cot^{-1}7 + \cot^{-1}13 + \cot^{-1}21 + \cot^{-1}31 + \ldots$ n terms.

Soln:

$$\underbrace{7 + 13}_{6} + \underbrace{13 + 21}_{8} + \underbrace{21 + 31}_{10} + \ldots$$

$$\tan^{-1}\left(\frac{1}{7}\right) + \tan^{-1}\left(\frac{1}{13}\right) + \tan^{-1}\left(\frac{1}{21}\right) + \tan^{-1}\left(\frac{1}{31}\right) + \ldots$$

$$\tan^{-1}\left(\frac{3-2}{1+2\times 3}\right) + \tan^{-1}\left(\frac{4-3}{1+4\times 3}\right) + \tan^{-1}\left(\frac{5-4}{1+5\times 4}\right) + \ldots$$

$$= \cancel{\tan^{-1}3} - \tan^{-1}2 + \cancel{\tan^{-1}4} - \cancel{\tan^{-1}3} + \ldots + \tan^{-1}(n+2) - \cancel{\tan^{-1}(n+1)}$$

$$\tan^{-1}(n+2) - \tan^{-1}2$$

$$= \tan^{-1}\left(\frac{n}{1+2n+4}\right)$$

$$= \tan^{-1}\left(\frac{n}{2n+5}\right)$$

Q. $\tan^{-1}\left(\frac{1}{3}\right) + \tan^{-1}\left(\frac{2}{9}\right) + \ldots + \tan^{-1}\left(\frac{2^{n-1}}{1+2^{2n-1}}\right)$

~~upto~~ upto infinity.

Date ____
Page ____

Sol^n. $\tan^{-1}\left(\frac{2-1}{1+1\times 2}\right) + \tan^{-1}\left(\frac{4-2}{1+8}\right) + \cdots + \tan^{-1}\left(\frac{2^n-2^{n-1}}{1+2^n \cdot 2^{n-1}}\right)$

$$T_n = \tan^{-1}\left(\frac{2^n-2^{n-1}}{1+2^n \cdot 2^{n-1}}\right)$$

$$\cancel{\tan^{-1}} 2 - \tan^{-1} 1 + \cancel{\tan^{-1}} 2^2 - \cancel{\tan^{-1} 2} + \cdots + \tan^{-1} 2^n - \cancel{\tan^{-1}} 2^{n-1}$$

$$\tan^{-1} 2^n - \tan^{-1} 1$$

$n \to \infty$

$$\tan^{-1}\infty - \tan^{-1} 1$$

$$= \frac{\pi}{2} - \frac{\pi}{4}$$

$$= \frac{\pi}{4}$$

Q. $\sin^{-1}\frac{1}{\sqrt{5}} + \sin^{-1}\frac{1}{\sqrt{65}} + \sin^{-1}\frac{1}{\sqrt{325}} + \cdots + \sin^{-1}\left(\frac{1}{\sqrt{(4n^4)+1}}\right)$

upto infinity.

Sol^n: $\tan^{-1}\frac{1}{2} + \tan^{-1}\frac{1}{8} + \tan^{-1}\frac{1}{18} + \cdots \tan^{-1}\frac{1}{2n^2}$

$$T_n: \cancel{\tan^{-1}\frac{1}{1+2n^2+1}} \quad \tan^{-1}\frac{2}{1+4n^2-1}$$

$$T_n = \tan^{-1}\frac{2}{1+(2n-1)(2n+1)}$$

$$= \tan^{-1}\frac{(2n+1)-(2n-1)}{1+(2n-1)(2n+1)}$$

$$T_n = \tan^{-1}(2n+1) - \tan^{-1}(2n-1)$$

$S_n = \cancel{\tan^{-1} 3} - \tan^{-1} 1 + \cancel{\tan^{-1} 5} - \cancel{\tan^{-1} 3} + \dots + \tan^{-1}(2n+1) - \cancel{\tan^{-1}(2n-1)}$

$$S_n = \lim_{n\to\infty} \tan^{-1}(2n+1) - \tan^{-1} 1$$

$$= \frac{\pi}{2} - \frac{\pi}{4}$$

$$= \frac{\pi}{4}$$

Q. If $0 < a, b < 1$ and $\tan^{-1} a + \tan^{-1} b = \pi/4$, then the value of

$(a+b) - \left(\frac{a^2+b^2}{2}\right) + \left(\frac{a^3+b^3}{3}\right) - \left(\frac{a^4+b^4}{4}\right) + \dots$ is

Sol^n:-

$$\tan^{-1} a + \tan^{-1} b = \pi/4$$

$$\tan^{-1} \frac{a+b}{1-ab} = \frac{\pi}{4}$$

$$a+b = 1-ab$$

$$a+b+ab-1 = 0$$

$$(a+b) - \left(\frac{a^2+b^2}{2}\right) + \left(\frac{a^3+b^3}{3}\right) - \left(\frac{a^4+b^4}{4}\right) + \dots$$

$$\left(a - \frac{a^2}{2} + \frac{a^3}{3} - \frac{b^4}{4} - \dots\right) + \left(b - \frac{b^2}{2} + \frac{b^3}{3} - \frac{b^4}{4} + \dots\right)$$

$$= \ln(1+a) + \ln(1+b)$$

$$= \ln(1+a)(1+b)$$

$$= \ln(1+a+b+ab)$$

$$= \ln 2$$

$$= \log_e 2$$

Q. $\sec^{-1}\left(\frac{1}{4}\sum_{k=0}^{10}\sec\left(\frac{7\pi}{12}+\frac{k\pi}{2}\right)\cdot\sec\left(\frac{7\pi}{12}+\frac{(k+1)\pi}{2}\right)\right)$ in the interval $[-\pi/4, 3\pi/4]$ [JEE Advanced 2019]

Solⁿ: $\sec^{-1}\left(\frac{1}{4}\sum_{k=0}^{10}\frac{\sin\left\{\left[\frac{7\pi}{12}+\frac{(k+1)\pi}{2}\right]-\left[\frac{7\pi}{12}+\frac{k\pi}{12}\right]\right\}}{\cos\left(\frac{7\pi}{12}+\frac{k\pi}{2}\right)\cos\left(\frac{7\pi}{12}+\frac{(k+1)\pi}{2}\right)}\right)$

$\cancel{\sec^{-1}}\ \frac{7\pi}{12}+\frac{(k+1)\pi}{2}=A$

$\frac{7\pi}{12}+\frac{k\pi}{2}=B$

$\frac{\sin A\cos B-\cos A\sin B}{\cos B\cdot\cos A}$

$\tan A-\tan B$

$\sec^{-1}\left(\frac{1}{4}\sum_{k=0}^{10}\tan\left(\frac{7\pi}{12}+\frac{(k+1)\pi}{2}\right)-\tan\left(\frac{7\pi}{12}+\frac{k\pi}{2}\right)\right)$

$\sec^{-1}\frac{1}{4}\left(\tan\left(\frac{7\pi}{12}+\frac{\pi}{2}\right)-\tan\left(\frac{7\pi}{12}\right)+\tan\left(\frac{7\pi}{12}+\pi\right)\right.$

$-\tan\left(\frac{7\pi}{12}+\frac{\pi}{2}\right)+\tan\left(\frac{7\pi}{12}+\frac{3\pi}{12}\right)-\tan\left(\frac{7\pi}{12}+\pi\right)$

$+\ldots+\tan\left(\frac{7\pi}{12}+5\pi\right)-\tan\left(\frac{7\pi}{12}+\frac{9\pi}{2}\right)$

$\left.+\tan\left(\frac{7\pi}{12}+\frac{11\pi}{2}\right)-\tan\left(\frac{7\pi}{12}+5\pi\right)\right)$

$=\sec^{-1}\left(\frac{1}{4}\left[\tan\frac{73\pi}{12}-\tan\frac{7\pi}{12}\right]\right)$

$$= \sec^{-1}\left(\frac{1}{4}\left[\tan\frac{\pi}{12} - \tan\frac{7\pi}{12}\right]\right)$$

$$= \sec^{-1}\left(\frac{1}{4}\left(2-\sqrt{3}+2+\sqrt{3}\right)\right)$$

$$= \sec^{-1}\left(\frac{1}{4}\times 4\right)$$

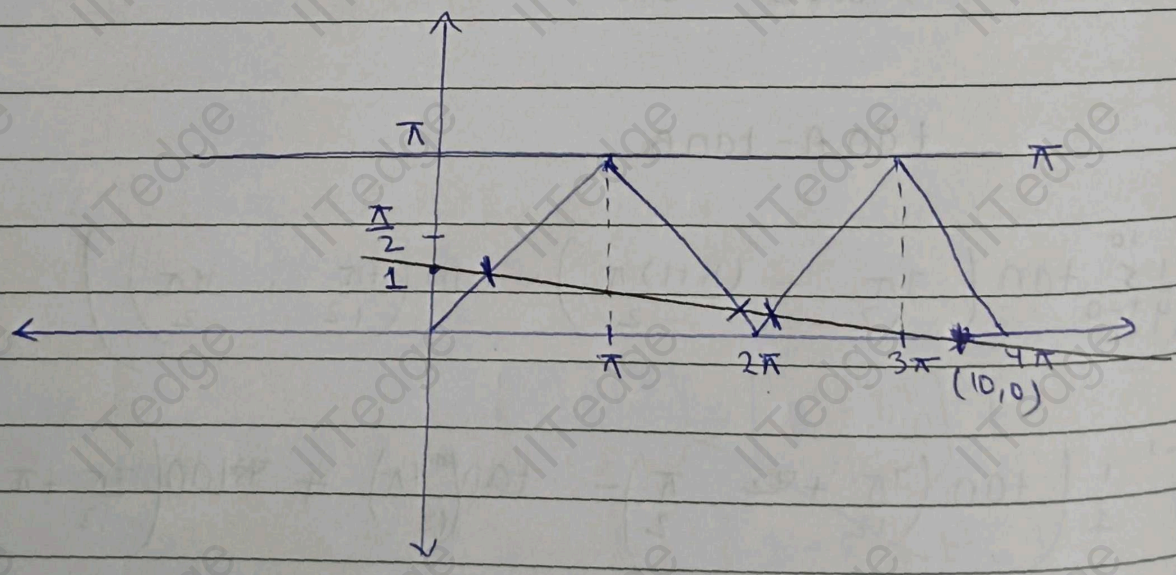
$$= \sec^{-1}(1)$$

$$= 0$$

Q. If $f : [0, 4\pi] \to [0, \pi]$ be defined by $f(x) = \cos^{-1}(\cos x)$. Then the number of points $x \in [0, 4\pi]$ satisfying the eqn $f(x) = \dfrac{10-x}{10}$ is

[JEE Advanced 2014]

Soln:



$$f(x) = \frac{10-x}{10}$$

$$f(x) = 1 - \frac{x}{10}$$

No. of solutions = 3

~~Rem~~ Remember:-

1. $\sin^{-1}x + \sin^{-1}y + \sin^{-1}z = \frac{3\pi}{2} \Rightarrow x = y = z = 1$

2. $\cos^{-1}x + \sin^{-1}y + \sin^{-1}z = 3\pi \Rightarrow x = y = z = -1$

3. $\tan^{-1}1 + \tan^{-1}2 + \tan^{-1}3 = \pi$ and

4. $\tan^{-1}1 + \tan^{-1}1/2 + \tan^{-1}1/3 = \frac{\pi}{2}$

Q. Find the value of $\cos(2\cos^{-1}x) + \sin^{-1}x)$ at $x = \frac{1}{5}$, where $0 \le \cos^{-1}x \le \pi$ and $\sin^{-1}x \in [-\pi/2, \pi/2]$. [1981]

Sol:- $\cos[2\cos^{-1}x + \sin^{-1}x]$

~~$\cos(2\cos^{-1}x +$~~

~~$\cos \cos^{-1}$~~ $= \cos(\cos^{-1}x + \pi/2)$

$= \cos(-\sin^{-1}x)$ $\qquad = -\sin(\cos^{-1}x)$

$= \cos\left(-\sin^{-1}\left(\frac{1}{5}\right)\right)$ $\qquad = -\sin(\cos^{-1}1/5)$

$= \cos\left(-\cos^{-1}\frac{2\sqrt{6}}{5}\right)$ $\qquad = -\sin\left(\sin^{-1}\frac{2\sqrt{6}}{5}\right)$

$=$ $\qquad = -\frac{2\sqrt{6}}{5}$

Date
Page

Q. considering only the principal values of the ITF, the value of

$$\frac{3}{2}\cos^{-1}\sqrt{\frac{2}{2+\pi^2}} + \frac{1}{4}\sin^{-1}\frac{2\sqrt{2}\pi}{2+\pi^2} + \tan^{-1}\frac{\sqrt{2}}{\pi}$$

[JEE Advanced 2022]

sol^n:-

$\cos x = \sqrt{\frac{2}{2+\pi^2}}$ $\frac{\to B}{\to H}$

$P^2 = 2+\pi^2-2$

$P = \pi$

$\tan x = \frac{\pi}{\sqrt{2}}$

Wrong approach

$\sin x = \frac{2\sqrt{2}\pi}{2+\pi^2}$ $\frac{\to P}{\to H}$

$B^2 = (2+\pi^2)^2 - (2\sqrt{2}\pi)^2$

$B^2 = 4+\pi^4+4\pi^2-8\pi^2$

$B^2 = \pi^4-4\pi^2+4$

$B^2 = \pi^4-2\pi^2-2\pi^2+4$

$B^2 = \pi^2(\pi^2-2)-2(\pi^2-2)$

$B = \pi^2-2$

$\tan x = \frac{2\sqrt{2}\pi}{\pi^2-2}$

$$\frac{3}{2}\tan^{-1}\frac{\pi}{\sqrt{2}} + \frac{1}{4}\tan^{-1}\frac{2\sqrt{2}\pi}{\pi^2-2} + \tan^{-1}\frac{\sqrt{2}}{\pi}$$

sol^n:-

$$\frac{3}{2}\cos^{-1}\sqrt{\frac{1}{1+\frac{\pi^2}{2}}} + \frac{1}{4}\sin^{-1}\frac{\sqrt{2}\pi}{1+\frac{\pi^2}{2}} + \tan^{-1}\left(\frac{\sqrt{2}}{\pi}\right)$$

$$\frac{3}{2}\cos^{-1}\sqrt{\frac{1}{1+\left(\frac{\pi}{\sqrt{2}}\right)^2}} + \frac{1}{4}\sin^{-1}\frac{\sqrt{2}\pi}{1+\left(\frac{\pi}{\sqrt{2}}\right)^2} + \tan^{-1}\left(\frac{\sqrt{2}}{\pi}\right)$$

$\frac{\pi}{\sqrt{2}} = \tan\theta$

classmate

Date

Page

$$\tan\frac{\pi}{4} < \cancel{\tan}\ \frac{\pi}{\sqrt{2}} < \tan\frac{\pi}{2}$$

$$\theta \in \left(\frac{\pi}{4}, \frac{\pi}{2}\right) \qquad 2\theta \in \left(\frac{\pi}{2}, \pi\right)$$

$$\cancel{\frac{3}{2}\theta}\ \frac{3}{2}\cos^{-1}\sqrt{\frac{1}{1+\tan^2\theta}} + \frac{1}{4}\sin^{-1}\frac{\sqrt{2}\pi}{1+\tan^2\theta} + \tan^{-1}(\cot\theta)$$

$$\frac{3}{2}\cos^{-1}(\cos\theta) + \frac{1}{4}\sin^{-1}\left(\frac{2\tan\theta}{1+\tan^2\theta}\right) + \tan^{-1}(\cot\theta)$$

$$\frac{3}{2}\theta + \frac{1}{4}\sin^{-1}(\sin 2\theta) + \frac{\pi}{2} - \cot^{-1}(\cot\theta)$$

$$2\theta \in \left(\frac{\pi}{2}, \pi\right) \to \pi - 2\theta$$

$$= \frac{3}{2}\theta + \frac{1}{4}(\pi - 2\theta) + \frac{\pi}{2} - \theta$$

$$= \frac{3}{2}\theta - \frac{\theta}{2} - \theta + \frac{\pi}{4} + \frac{\pi}{2}$$

$$= \frac{3\pi}{4}$$

$$= \frac{3 \times 22}{7 \times 4}$$

$$= 2.36$$