Application Of Derivative

Rate Measure:-

Whenever one quantity 'y' varies with another quantity 'x', satisfying some rules $y = f(x)$, then $\frac{dy}{dx}$ or $f'(x)$ represents the rate of change of y w.r.t. 'x'. and $\frac{dy}{dx}$ at $x = a$ (or $f'(a)$) represents the rate of change of y w.r.t. x at $x = a$.

Q. If the radius of a circle is increasing at a uniform rate of 2cm/sec., find the rate of ~~change of~~ increase of area of circle at the instant when radius is 20 cm.

Sol^n. ~~Ar~~ Area $= \pi r^2$

$\frac{dA}{dt} = ?$

$\frac{dr}{dt} = 2 \text{ cm/s}$.

$$\frac{dA}{dt} = \frac{d}{dt}(\pi r^2)$$

$$\frac{dA}{dt} = \pi \frac{d}{dr}(r^2) \cdot \frac{dr}{dt}$$

$$\frac{dA}{dt} = \pi \cdot 2r \cdot \frac{dr}{dt}$$

$$\frac{dA}{dt} = \pi \times 2 \times 20 \times 2$$

$$\boxed{\frac{dA}{dt} = 80\pi \ cm^2/s.}$$

Q. The volume of a cube is increasing at a rate of $9 cm^3/sec$. How fast is the surface area increase when the length of an edge is 10 cm?

Sol^n:

$$\frac{dV}{dt} = 9 \ cm^3/s.$$

$$V = a^3$$

$$\frac{dV}{dt} = \frac{d}{dt} a^3$$

$$\frac{dV}{dt} = 3a^2 \cdot \frac{da}{dt}$$

$$\cancel{9} 3 = \cancel{300}^{100} \cdot \frac{da}{dt}$$

$$\frac{da}{dt} = 0.03$$

$$A = 6a^2$$

$$\frac{dA}{dt} = 6 \times \frac{d}{dt}(a^2)$$

$$\frac{dA}{dt} = 6 \times 2a \times \frac{da}{dt}$$

$$\frac{dA}{dt} = 6 \times 20 \times 0.03$$

M T W T F S S

Page No.:

Date:

YOUVA

$$\boxed{\frac{dA}{dt} = 3.6\ cm^2/sec}$$

Q. If the displacement of a particle is given by $s = \left(\frac{1}{2}t^2 + 4\sqrt{t}\right)$ m. Find the velocity and acceleration at $t = 4$ sec?

Solⁿ:- $S = \frac{1}{2}t^2 + 4\sqrt{t}$

$$\frac{ds}{dt} = t + \frac{2}{\sqrt{t}}$$

$v_4 = 4 + 1$

$$\boxed{v_4 = 5 m/s}$$

$v = t + \frac{2}{\sqrt{t}}$

$\frac{dv}{dt} = 1 + \frac{2}{(t)^{-3/2}} \times \left(\frac{-1}{2}\right)$

$= 1 - \frac{1}{(2)^{-3}}$

$= 1 - 8$

$= -7 m/s$

$v = t + \frac{2}{\sqrt{t}}$

$v = t + 2(t)^{-1/2}$

$\frac{dv}{dt} = 1 + 2 \times \left(\frac{-1}{2}\right) t^{-3/2}$

$\frac{dv}{dt} = 1 - 1(t)^{-3/2}$

$\left.\frac{dv}{dt}\right|_{t=4} = 1 - \frac{1}{8}$

$$\boxed{a = \frac{7}{8} m/s^2}$$

Date

Q. If $s = \frac{1}{2}t^3 - 6t$. find, the acceleration at time when the velocity tends to zero.

Soln:- $s = \frac{1}{2}t^3 - 6t$

$v = \frac{ds}{dt} = \frac{3t^2}{2} - 6$

$v = 0$

$\frac{3t^2}{2} = 6$ 2

$\boxed{t = 2}$

$a = \frac{dv}{dt}$

$a = 3t$

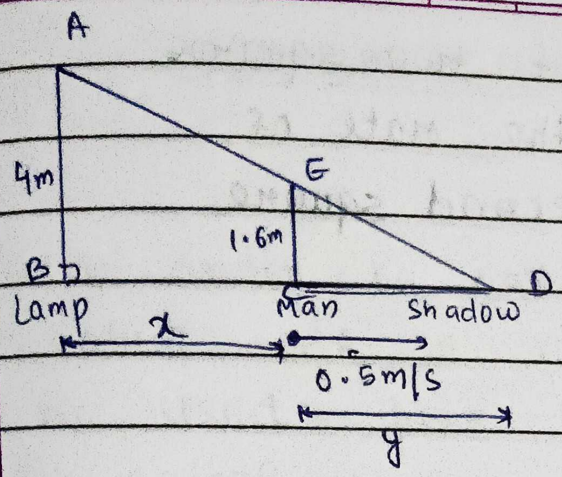
At $t = 2sec$, $\boxed{a = 6\ m/s^2}$

Q. A man who is 1.6m tall walks away from a lamp which is 4m above ground at the rate of 30 m/min. How fast is the man's shadow lengthening.

Soln:- $\frac{dx}{dt} = 30\ m/min.$

$= \cancel{\frac{30}{60}}$

$\cancel{\frac{dx}{dt} = 0.5\ m/s}$



$\triangle ABD \approx \triangle ECD$

$$\frac{AB}{CE} = \frac{BD}{CD}$$

$$\frac{4}{1.6} = \frac{x+y}{y}$$

$$\frac{5}{2} = \frac{x+y}{y}$$

$$5y = 2x + 2y$$

$$3y = 2x$$

Diff. w.r.t. 't'

$$3\frac{dy}{dt} = 2\frac{dx}{dt}$$

$$\cancel{3} \times \frac{dy}{dt} = 2 \times \cancel{30}^{10}$$

$$\boxed{\frac{dy}{dt} = 20 m/min.}$$

Q. If x and y are the size of two squares such that $y = x - x^2$, find the rate of the change of area of second square w.r.t. first square.

~~Sol^n:~~ $\cancel{\dfrac{dy}{dx}}$

$y = x - x^2$

$\cancel{\dfrac{dy}{dx} = 1 - 2x}$

$A = |x|^2$

$\cancel{\dfrac{dA}{dt}}$

$\cancel{\dfrac{dy^2}{dx^2} = ?}$

Sol^n: $\dfrac{d(y^2)}{d(x^2)} = \dfrac{2y \cdot dy/dx}{2x}$

$y = x - x^2$

$\dfrac{dy}{dx} = 1 - 2x$

$\dfrac{dy^2}{dx^2} = \dfrac{\cancel{2}y\,(1-2x)}{\cancel{2}x}$

$= \dfrac{(x-x^2)(1-2x)}{x}$

$= (1-x)(1-2x)$

$\dfrac{d(y^2)}{d(x^2)} = 2x^2 - 3x + 1$

Page No ·

Date ·

YOUVA

Approximation Using Differentials (change)

In order to calculate the approximate value of a function, differentials may be used where the differential of a function is equal to its derivative multiplied by the differential of the independent variables.

Approximate change in the value of y, called its differential, is given by

$$\Delta y = f'(x) \cdot \Delta x$$

Approximate value of y when incrimant Δx is given to independent variable 'x' in $y = f(x)$ is $y + \Delta y = f(x + \Delta x) = f(x) + f'(x) \cdot \Delta x$

Q. Find the approximate value of $\sqrt{25.2}$.

Sol. $f(x) = \sqrt{x}$

$x = 25$, $\Delta x = 0.2$

$$\Delta y = f'(x) \cdot \Delta x$$

$$f(x + \Delta x) = f(x) + f'(x) \cdot \Delta x$$

$$= \sqrt{x} + \frac{1}{2\sqrt{x}} \cdot \Delta x$$

$$= \sqrt{25} + \frac{1}{2\sqrt{25}} \times 0.2$$

$$= 5 + 0.02$$

$$= 5.02$$

Q. Use differential to approximate the value of $\sqrt{101}$

Sol^n:- $x = 100$, $\Delta x = 1$

$$f(x+\Delta x) = f(x) + f'(x)\cdot \Delta x$$

$$= \sqrt{x} + \frac{1}{2\sqrt{x}}\cdot \Delta x$$

$$= \sqrt{100} + \frac{1}{2\sqrt{100}}\cdot 1$$

$$= 10 + \frac{1}{20}$$

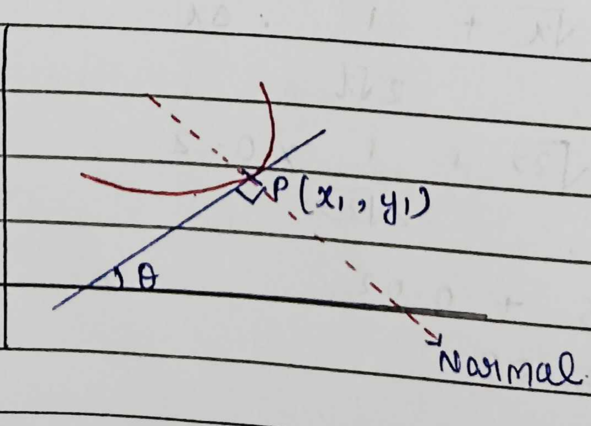
$$= 10 + 0.05$$

$$= 10.05$$

Tangent and Normal

I. Slope of Tangent :-

The value of the derivative at $P(x_1, y_1)$ gives the slope of the tangent to the curve at point P.



$$\left(\frac{dy}{dx}\right)_P = \text{slope of tangent} = \tan\theta.$$

Note :-

1. If the tangent at any point P on the curve is parallel to x-axis, then $\left(\frac{dy}{dx}\right)_P = 0$ or $\tan\theta = 0$. $(\theta = 0°)$

2. If the tangent at any point P on the curve is perpendicular to x-axis (or parallel to y-axis), then $\theta = 90°$, $\tan\theta = \infty$ or $\cot\theta = 0°$.

$$\left(\frac{dx}{dy}\right)_P = 0$$

3. If the tangent at any point on the curve is equally inclined to both the axis, then

$$\left(\frac{dy}{dx}\right)_P = \pm 1$$

II. Slope of Normal :-

Normal to the curve at $P(x_1, y_1)$ is a line perpendicular to tangent at $P(x_1, y_1)$ and passes through P.

$$\text{slope of normal at } P = -\frac{1}{(dy/dx)_P} = -\left(\frac{dx}{dy}\right)_P$$

Note:-

If normal is parallel to x-axis, then

$$\left(\frac{dx}{dy}\right)_P = 0$$

If normal is perpendicular to x-axis (or parallel to y axis), $\left(\frac{dy}{dx}\right)_P = 0$

Equation of tangent and normal

1. Equation of tangent at $P(x_1, y_1)$ to the curve is

$$y - y_1 = \left(\frac{dy}{dx}\right)_P (x - x_1)$$

2. Equation of normal at $P(x_1, y_1)$ to the curve is

$$y - y_1 = -\left(\frac{dx}{dy}\right)_P (x - x_1)$$

Note:-

(a) If a curve passes through origin, then the equation of the tangent at the origin can be directly written by equating the lowest degree terms appearing in the equation of the curve is zero.

eg, In $x^2 + y^2 + 2gx + 2fy = 0$

↳ eq^n of tangent at origin ⇒ $gx + fy = 0$

M T W T F S S
Page No.:
Date:
YOUVA

eg→ $x^3 + y^3 - 3x^2y + 3x^2y^2 + x^2 - y^2 = 0$

Degree→ 3 3 3 3 2 2

Eqn of tangent at origin→

$x^2 - y^2 = 0$

$x - y = 0 \qquad x + y = 0$

eg→

2. Equation of tangent to the curve $x^3 + y^3 - 3xy = 0$

$-3xy = 0$

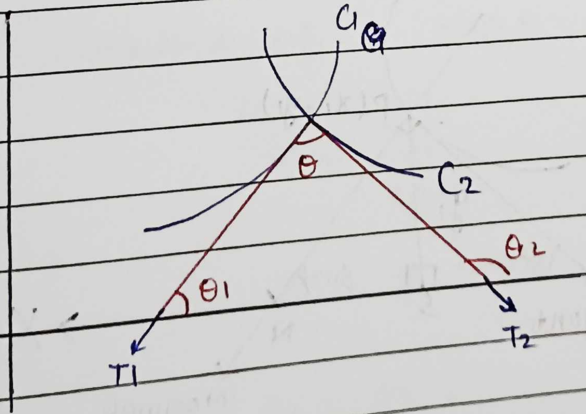
$xy = 0$

$\cancel{y^2 - 3xy = 0}$

$\cancel{y(y-3x) = 0}$

Angle of intersection between two curves:-

Angle of intersection b/w two curves is defined as angle b/w the two tangents drawn to the two curves at their point of intersection.



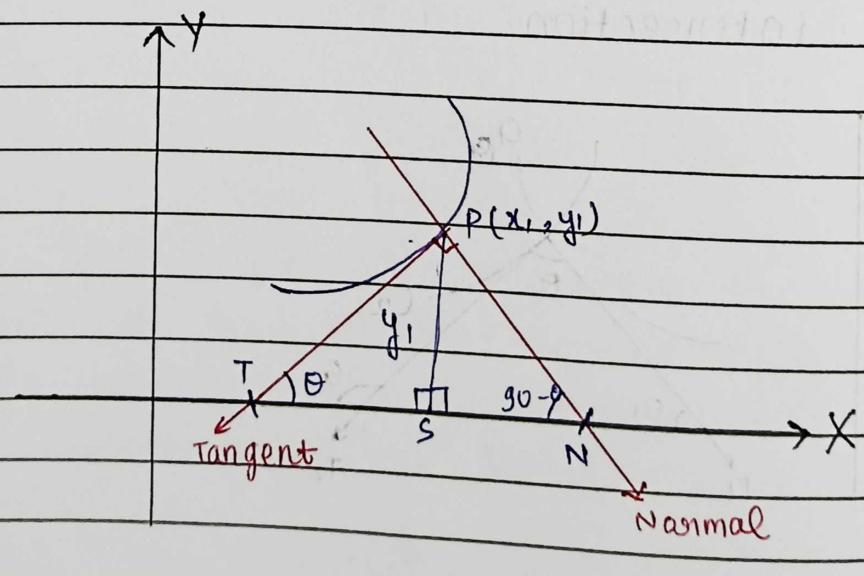
Angle of intersection of ~~two~~ these curves is defined as acute angle b/w the tangents.

Orthogonal curves:-

If the angle of intersection of two curves is a right angle, then two curves are said to be orthogonal and curves are called orthogonal curves.

$$\left(\frac{dy}{dx}\right)_{C_1} \cdot \left(\frac{dy}{dx}\right)_{C_2} = -1$$

Length of Tangent, Subtangent, Normal and Sub-Normal



PT = length of Tangent
ST = Sub-Tangent
PN = length of Normal
SN = Sub-Normal

1. Length of Tangent -

The length of segment PT of the tangent b/w the point of tangent and x-axis is called the length of tangent.

$$\sin\theta = \frac{y_1}{PT}$$

$$PT = \frac{y_1}{\sin\theta}$$

$$PT = y_1 \operatorname{cosec}\theta$$

$$PT = y_1\sqrt{1+\cot^2\theta}$$

$$\boxed{PT = \left| y\sqrt{1+\left(\frac{dx}{dy}\right)^2_{(x_1,y_1)}} \right|}$$

2. Length of Subtangent :-

The projection of segment PT along x-axis (ST) is called the sub-tangent.

~~cos θ = ...~~ $\tan\theta = \frac{y_1}{ST}$

$$ST = \frac{y_1}{\tan\theta}$$

$$ST = y_1 \cot\theta$$

$$ST = \left| y_1\left(\frac{dx}{dy}\right)_{(x_1,y_1)} \right|$$

3. Length of Normal :-

Length of segment PN i.e. the portion of the normal intercepted b/w the point on the curves and x-axis is called the length of normal.

$$PN = \left| y_1 \sqrt{1 + \left(\frac{dy}{dx}\right)^2_{(x_1, y_1)}} \right|$$

4. Length of subnormal :-

The projection of the segment PN along x-axis (SN) is called the subnormal.

$$SN = \left| y_1 \left(\frac{dy}{dx}\right)_{(x_1, y_1)} \right|$$

Q. Find the equation of tangent and normal to the curve $2y = 3 - x^2$ at $(1, 1)$

Solⁿ: $2y = 3 - x^2$

$$2\frac{dy}{dx} = -2x$$

$$\frac{dy}{dx} = -x$$

$$\left(\frac{dy}{dx}\right)_P = -1$$

Eqⁿ of tangent -

$$y - 1 = -1(x - 1)$$

$$y - 1 = -x + 1$$

$$x + y - 2 = 0$$

M T W T F S S

Page No :

Date:

YOUVA

~~Eqn of~~ $2y = 3 - x^2$

$$\cancel{2} = -\cancel{2}x \cdot \frac{dx}{dy}$$

$$\frac{-1}{x} = \frac{dx}{dy}$$

$$\left(\frac{dx}{dy}\right)_P = -1$$

Eqn of normal

$$y - 1 = -(-1)(x-1)$$

$$y - 1 = x - 1$$

$$x - y = 0$$

Q. Find the point on the curve $y - e^{xy} + x = 0$ at which we have vertical tangent.

Soln:

$$\frac{dy}{dx} = \infty$$

$$\frac{dx}{dy} = 0$$

$$y - e^{xy} + x = 0$$

$$1 - e^{xy} \cdot \left(x + y \cdot \frac{dx}{dy}\right) + \frac{dx}{dy} = 0$$

$$1 - e^{xy} \cdot x - e^{xy} \cdot y \cdot \frac{dx}{dy} + \frac{dx}{dy} = 0$$

$$\frac{dx}{dy}\left(1 - e^{xy} y\right) = e^{xy} x - 1$$

$$\frac{dx}{dy} = \frac{e^{xy} \cdot x - 1}{1 - e^{xy} \cdot y}$$

$$e^{xy} \cdot x - 1 = 0$$

$$e^{xy} \cdot x = 1$$

$x = 1$

$y = 0$

$(x, y) = (1, 0)$

Q. If the tangent at a point (x_1, y_1) on the curve $y = x^3 + 3x^2 + 5$ passes through the origin then (x_1, y_1) does not lie on the curve
[JEE Main 2022]

(a) $\cancel{\frac{x^2+y^2}{8}}$ $x^2 + \frac{y^2}{81} = 2$

(b) $\frac{y^2}{9} - x^2 = 8$

(c) $y = 4x^2 + 5$

(d) $\frac{x}{3} - y^2 = 2$

~~Sol~~^{n}:-

$y = x^3 + 3x^2 + 5$

$\left(\frac{dy}{dx}\right) = ?$

$y' = 3x^2 + 6x$

$y' = 3x_1^2 + 6x_1$

$y - y_1 = (3x_1^2 + 6x_1)(x - x_1)$

$(0, 0)$

$+y_1 = (3x_1^2 + 6x_1)(+x_1)$

$y_1 = 3x_1^3 + 6x_1^2$ —— (i)

(x_1, y_1) on the curve

$y = x^3 + 3x^2 + 5$

$y_1 = x_1^3 + 3x_1^2 + 5$ —— (ii)

M T W T F
Page No.:
Date:

From eq^n (i) and (ii)

$$3x_1^3 + 6x_1^2 = x_1^3 + 3x_1^2 + 5$$
$$2x_1^3 + 3x_1^2 - 5 = 0$$
$$2x_1^3 - 2x_1^2 + 5x_1^2 - 5 = 0$$
$$2x_1^2(x_1 - 1) + 5(x_1^2 - 1) = 0$$
$$2x_1^2(x_1-1) + 5(x_1-1)(x_1+1) = 0$$
$$(x_1-1)(2x_1^2 + 5x_1 + 5) = 0$$
$$\downarrow$$
$$D < 0$$
$\therefore$ Imaginary roots

$$\boxed{x_1 = 1}$$

Put in eq^n (i)

$$\boxed{y_1 = 9}$$

Satisfying points in options,
'd' will not satisfy.

Q. If the tangent to the curve $y = x^3 + ax - b$ at the point $(1, -5)$ is perpendicular to the line $-x + y + 4 = 0$, then which one of the following points lies on the curve?

[JEE Main 2019]

(a) $(-2, 2)$ ~~(b) $(2, -2)$~~ (c) $(-2, 1)$ (d) $(2, -1)$

Sol^n

~~$-x + y + 4 = 0$~~

~~$\perp$ line $\rightarrow -x - y + c = 0$~~

~~$(1, -5)$~~

~~$1 - (-5) + c = 0$~~

~~$c = -6$~~

~~Eq$^n \Rightarrow x - y - 6 = 0$~~

~~$x - y = 6$~~

$-1 + 5 + c = 0$

$c = -4$

$-x - y - 4 = 0$

$\downarrow$ eq^n of tangent $x + y + 4$

$y = -4 - x$

Page No.:

Date:

$$y = x^3 + ax - b$$

$(1,-5)$ $\quad -5 = 1 + a - b$

$$a - b = -6$$

$$\frac{dy}{dx} = 3x^2 + a$$

$$y - y_1 = (3x^2 + a)(x - x_1)$$

$$y + 5 = (3x^2 + a)(x - 1)$$

$$y + 5 = 3x^3 - 3x^2 + ax - a$$

$y + 5 = (3+a)(x-1)$

$y + 5 = 3x - 3 + ax - a$

$4 - x + 5 = 3x - 3 + ax - a$

$9 - x = 3x - 3 + ax - a$

$ax - a =$

$y = x \cancel{4} - 6$

$x - 1 = 3x^3 - 3x^2 + ax - a$

sol^n:-

$$y = x^3 + ax - b$$

$$\left.\frac{dy}{dx}\right|_{x=5} = 3x^2 + a$$

$$= 3 + a$$

$$-x + y + 4 = 0$$

$$-1 + \frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = 1$$

$$(3+a)(1) = -1$$

$$\boxed{a = -4}$$

$$y = x^3 + ax - b$$

$(1,-5)$ $\quad -5 = 1 + a - b$

$$-5 = 1 - 4 - b$$

$$\boxed{b = 2}$$

M T W T F S S

Page No.:

Date:

YOUVA

Curve → $y = x^3 - 4x - 2$

checking points,

'b' will satisfy.

Q. The tangent to the curve $y = x^2 - 5x + 5$, ~~the~~ parallel to the line $2y = 4x + 1$, also passes through the point

[JEE Main 2019]

(a) $\left(\frac{1}{4}, \frac{7}{2}\right)$ (b) $\left(\frac{7}{2}, \frac{1}{4}\right)$ (c) $\left(\frac{-1}{8}, 7\right)$

(d) $\left(\frac{1}{8}, -7\right)$

Solⁿ: $4x - 2y + 1 = 0 \longrightarrow \text{Slope} = 2$

$y = x^2 - 5x + 5$

$\frac{dy}{dx} = 2x - 5$

$2x - 5 = 2$

$2x = 7$

$\boxed{x = \frac{7}{2}}$

$y = \frac{49}{4} - \frac{35}{2} + 5$

$y = \frac{49 - 70 + 20}{4}$

$y = -1/4$

$\frac{28}{2} - 2y + 1 = 0$

$2y = 15$

$(x_1, y_1) = \left(\frac{7}{2}, \frac{-1}{4}\right)$

$y = \frac{49}{4} - \frac{35}{2} + 5$

$\boxed{y = -1/4}$

Date

$y + \frac{1}{4} = 2\left(x - \frac{7}{2}\right)$

$y + \frac{1}{4} = 2x - 7$

~~2x − y~~

Checking options

'd' will satisfy.

Q. If θ denotes the acute angle b/w the curves $y = 10 - x^2$ and $y = 2 + x^2$ at the point of their intersection, then $|\tan\theta|$ is.

[JEE Main 2019].

Sol^n:-

$y = 10 - x^2$

$m_1 = -2x$

$\tan\theta_1 = -2x$

$y = 2 + x^2$

$y = 2x$

$\tan\theta_2 = 2x$

$10 - x^2 = 2 + x^2$

$8 = 2x^2$

$x^2 = 4$

$x = \pm 2$

$\left.\begin{matrix} x = 2 \\ y = 6 \end{matrix}\right\}$ $\left.\begin{matrix} x = -2 \\ y = 6 \end{matrix}\right\}$

$\left.\begin{matrix} m_1 = -4 \text{ or } 4 \\ m_2 = 4 \text{ or } -4 \end{matrix}\right\}$ can take any one.

$\tan\theta = \left\{ \frac{m_1 - m_2}{1 + m_1 m_2} \right.$

$\tan\theta = \frac{-4-4}{1-16}$

$|\tan\theta| = \frac{8}{15}$

Page No.: Date: YOUVA

Q. If the curves $y^2 = 6x$, $9x^2 + by^2 = 16$ intersect each other at right angles, then find b.

[JEE Main 2018]

Solⁿ: $y^2 = 6x$

$$2y \cdot \frac{dy}{dx} = \cancel{6}\,3$$

$$\left(\frac{dy}{dx}\right)_{C_1} = \frac{\cancel{6}\,3}{y}$$

$$\cancel{\frac{dy}{dx} = \frac{3}{\sqrt{6x}}}$$

$$9x^2 + 6y^2 = 16$$

$$18x + 12y \cdot \frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = \frac{-18x}{12y}$$

$$\frac{dy}{dx} = \frac{-3x}{2y}$$

$$9x^2 + by^2 = 16$$

$$18x + 2by \cdot y' = 0$$

$$\cancel{2}by \cdot \frac{dy}{dx} = -\cancel{18}\,9x$$

$$\left(\frac{dy}{dx}\right)_{C_2} = \frac{-9x}{by}$$

$$\cancel{\frac{dy}{dx}}$$

$$6x = \frac{16 - 9x^2}{b}$$

$$6bx = 16 - 9x^2$$

$$\frac{3}{y} \cdot \left(\frac{+9x}{by}\right) = +1$$

$$27x = by^2$$

$$27x = b \cdot 6x$$

$$\boxed{b = \frac{9}{2}}$$

Q. The normal to the curve $y(x-2)(x-3) = x + 6$ at the point where the curve intersects the y-axis passes through the point

[JEE Main 2017]

(a) ... (b) $(1/2, 1/2)$ (c) $(1/2, -1/3)$ (d) $(1/2, 1/3)$

Page No:

Date

Sol^n :-

$$y(x-2)(x-3) = x+6$$

$$\frac{dy}{dx}(x-2)(x-3) + y(x-3) + y(x-2) = \cancel{x+6}\ 1$$

$$\frac{dy}{dx}(x-2)(x-3) + \frac{(x+6)\cancel{(x-3)}}{(x-2)\cancel{(x-3)}} + \frac{(x+6)\cancel{(x-2)}}{\cancel{(x-2)}(x-3)} = \cancel{(x+6)}\ 1$$

$$\frac{dy}{dx}(x-2)(x-3) + \frac{x+6}{x-2} + \frac{x+6}{x-3} = \cancel{x+6}\ 1$$

$$\frac{dy}{dx}(x-2)(x-3) = (x+6)\left[1 - \frac{1}{(x-2)} - \frac{1}{(x-3)}\right]$$

$$\frac{dy}{dx}(x-2)(x-3) = (x+6)\left[\frac{(x-2)(x-3)-1}{(x-2)(x-3)}\right]$$

$(0, y)$

$$\frac{dy}{dx}(6) - 3 - 2 = 1$$

At $x = 0$

$\boxed{y = 1}$

$$\boxed{\frac{dy}{dx} = 1}$$

$$\frac{dx}{dy} = 1$$

Eq^n of normal :-

$$y - y_1 = -\left(\frac{dx}{dy}\right)(x - x_1)$$

$$y - 1 = -1(x-0)$$

$$y - 1 = -x$$

$$x + y = 1$$

satisfying options 'b' is correct.

Q. Consider : $f(x) = \tan^{-1}\left(\sqrt{\frac{1+\sin x}{1-\sin x}}\right)$, $x \in (0, \pi/2)$.

A normal to $y = f(x)$ at $x = \pi/6$ also passes through the point

(a) $(0,0)$ (b) $(0, 2\pi/3)$ (c) $(\pi/6, 0)$ (d) $(\pi/4, 0)$

Sol: $f(x) = \tan^{-1}\left(\frac{\sin x/2 + \cos x/2}{\cos x/2 - \sin x/2}\right)$ $\left\{ \begin{array}{c} \text{In } (0, \pi/4) \\ \cos x > \sin x \end{array} \right\}$

$= \tan^{-1}\left(\tan\left(\frac{\pi}{4} + \frac{x}{2}\right)\right)$

$= \frac{\pi}{4} + \frac{x}{2}$

$y = \frac{\pi}{4} + \frac{x}{2}$

$\frac{dy}{dx} = \frac{1}{2}$

$\frac{dx}{dy} = 2$

At $\left\{ \begin{array}{l} x = \pi/6 \\ y = \pi/3 \end{array} \right\}$

$y - \frac{\pi}{3} = -2(x - \pi/6)$

$y - \frac{\pi}{3} = -2x + \frac{\pi}{3}$

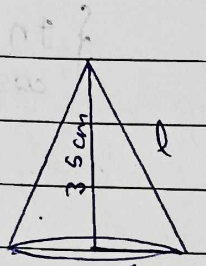
Satisfying options

'b' is correct.

Q. Water is being filled at the rate of $1 cm^3/sec$ in a right circular conical vessel (vertex downwards) of height 35 cm and diameter 14 cm. When the height of the water level is 10 cm, the rate (in cm^2/sec) at which the ~~weight~~ wet conical surface area of the vessel increases is

[JEE Main 2022]

sol^n:-



$$\cancel{l = \sqrt{49 + 1225}}$$

$$\cancel{l = \sqrt{1274}}$$

$$\frac{dV}{dt} = 1 \ cm^3/s.$$

$$V = \frac{1}{3}\pi r^2 h$$

$$\frac{dV}{dt} = \frac{1}{3}\pi \frac{d}{dt}(r^2 h)$$

$$h = 35 cm$$

$$r = 7 cm$$

$$\frac{h}{r} = 5$$

$$r = \frac{h}{5}$$

Page No.:
Date:

$$\frac{dV}{dt} = \frac{1}{3}\pi \left\{ \frac{d}{dt}\left(\frac{h^3}{25}\right)\right.$$

$$1 = \frac{1}{\cancel{3} \times 25}\pi \; \cancel{3}h^2 \cdot \frac{dh}{dt}$$

$$\frac{dh}{dt} = \frac{25}{\pi h^2}$$

$$A = \pi r l$$

$$A = \pi \frac{h}{5} \times \sqrt{\frac{h^2}{25} + h^2}$$

$$A = \frac{\pi \times h^2 \sqrt{26}}{5 \times 5}$$

$$A = \frac{\pi \sqrt{26}\, h^2}{25}$$

$$\frac{dA}{dt} = \frac{\sqrt{26}}{25}\pi \frac{d}{dt}(h^2)$$

$$= \frac{\sqrt{26}}{25}\pi \times 2h \frac{dh}{dt}$$

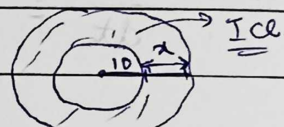
$$= \frac{\sqrt{26}}{\cancel{25}} \cancel{\pi} \times 2\cancel{h} \times \frac{\cancel{25}}{\cancel{\pi} h^2}$$

$$\frac{dA}{dt} = \frac{2\sqrt{26}}{h}$$

$$\left.\frac{dA}{dt}\right|_{h=10} = \frac{\sqrt{26}}{5}$$

Q. A spherical iron ball of radius 10 cm is coated with the layer of ice of uniform thickness that melts at a rate of $50\ cm^3/min$. When the thickness of ice is 5 cm, then the rate at which the thickness (in cm/min.) of the ice decreases is. [JEE Main 2021]

Solⁿ: $\because R = 10\ cm$



$\frac{dV}{dt} = -50\ cm^3/min.$ {rate is decreasing}

$V = \frac{2}{3}\pi r^3$

$\frac{dV}{dt} = \frac{2\pi}{3}\frac{d}{dt}(r^3)$

$\cancel{50} = \frac{\cancel{2}}{\cancel{3}}\pi \times (\cancel{3}r^2) \cdot \frac{dr}{dt}$

$\frac{\cancel{25 \times 3}}{\pi} = \cancel{3 \times 25} \cdot \frac{dr}{dt}$

$V = \frac{4}{3}\pi\left[(10+x)^3 - 10^3\right]$

$\frac{dV}{dt} = \frac{4}{3}\pi\frac{d}{dt}\left[(10+x)^3 - 10^3\right]$

$\frac{dV}{dt} = \frac{4}{3}\pi\, 3(10+x)^2 \cdot \frac{dx}{dt}$

$-50 = \frac{4}{\cancel{3}}\pi \times \cancel{3}(10+5)^2 \cdot \frac{dx}{dt}$

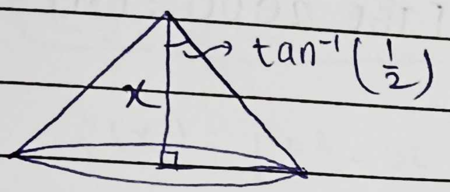
$\frac{dx}{dt} = \frac{-\cancel{50}^{\,2}}{\cancel{4}_{2}\pi \times \cancel{225}_{9}}$

$\boxed{\frac{dx}{dt} = \frac{-1}{18\pi}}$ cm/min.

Q. A water tank has a shape of an inverted right circular cone whose semi-vertical angle is $\tan^{-1}\left(\frac{1}{2}\right)$. Water is powered into it at a constant rate of $5\ cm^3/min$. then the rate (m/min.) at which the level

of water is rising at the instant when the depth of the water in the tank is 10 m is-
[JEE Main 2019]

solⁿ:-



$$\frac{dV}{dt} = 5\,cm^3/min$$

$$\tan\left(\tan^{-1}\left(\frac{1}{2}\right)\right) = \frac{r}{\cancel{x}h}$$

$$\frac{1}{2} = \frac{r}{h}$$

$$r = \frac{h}{2}$$

$$V = \frac{1}{3}\pi r^2 h$$

$$V = \frac{1}{3}\pi \times \frac{h^2}{4} \times h$$

$$V = \frac{1}{\cancel{3}}\frac{\pi}{12} \cdot h^3$$

$$\frac{dV}{dt} = \frac{\pi}{\cancel{12}4} \cdot 3h^2 \cdot \frac{dh}{dt}$$

$$5 = \frac{\pi}{\cancel{12}4} \times h^2 \times \frac{dh}{dt}$$

$$\cancel{5} = \frac{\pi}{\cancel{4}} \times \cancel{(100)}^{\cancel{25}\,5} \times \frac{dh}{dt}$$

$$\frac{dh}{dt} = \frac{1}{5\pi} \quad m/min.$$

Q. If the normal to the curve $y = f(x)$ at the point $(3,4)$ makes an angle $\frac{3\pi}{4}$ with the positive x-axis, then $f'(3)$ is equal to-

[JEE 2000, 1M]

Solⁿ:

$m = \tan\theta$

$m = \tan\left(\frac{3\pi}{4}\right)$

$m = -1$ → Normal.

$\cancel{y-4 = -1(x-3)}$

$\cancel{y-4 = -x+3}$

$\cancel{x+y-7=0}$

$\cancel{y = -x+7}$

$\cancel{\frac{dy}{dx} = -1}$

$y-4 = -(-1)(x-3)$

$y-4 = x-3$

$y = x+1$

$f(x) = x+1$

$f'(x) = 1$

$\boxed{f'(3) = 1}$

Q. The angle b/w the tangent lines to the graph of the function $f(x) = \int_2^x (2t-5)\,dt$ at the points where the graph cuts the x-axis is.

Solⁿ:

$f(x) = \int_2^x (2t-5)\,dt$

$f'(x) = (2x-5).1$

$\frac{dy}{dx} = 2x-5$

$f(x) = 0$

$x = 2$

$(2,0)$

Page No.: Date:

~~y~~ Eqn of tangent

$y - 0 = \left(\frac{dy}{dx}\right)_2 \cdot (x-2)$

$y = -x + 2$

$f(x) = [t^2 - 5t]_2^x$

$f(x) = [x^2 - 5x - 4 + 10]$

$0 = x^2 - 5x + 6$

$0 = (x-2)(x-3)$

$x = 2$

$\left.\frac{dy}{dx}\right|_{x=2} = -1$

$m_1 = -1$

$x = 3$

$\left.\frac{dy}{dx}\right|_{x=3} = 1$

$m_2 = 1$

$m_1 m_2 = -1$

Angle b/w tangents $= \frac{\pi}{2}$

The curve $x^2 - y^2 = 5$ and $\frac{x^2}{18} + \frac{y^2}{8} = 1$ cut each other at any common point at an angle is.

$x^2 - y^2 = 5$

$x^2 = 5 + y^2$

$\frac{x^2}{18} + \frac{y^2}{8} = 1$

$\frac{4x^2 + 9y^2}{72} = 1$

$4x^2 + 9y^2 = 72$

$$4(y^2+5)+9y^2=72$$
$$4y^2+20+9y^2=72$$
$$13y^2=52$$
$$y^2=4$$
$$y=\pm 2$$

~~yea~~ $x^2=5+y^2$
$$x^2=9$$
$$x=\pm 3$$

$$(x,y)=(2,3)$$

$$y^2=x^2-5$$
$$2y\cdot\left(\frac{dy}{dx}\right)_{c_1}=2x$$

$$\left(\frac{dy}{dx}\right)_{c_1}=\frac{x}{y}$$

$$\frac{x^2}{18}+\frac{y^2}{8}=1$$

$$\frac{x}{9}+\frac{y}{4}\cdot\frac{dy}{dx}=0$$

$$\frac{y}{4}\cdot\frac{dy}{dx}=\frac{-x}{9}$$

$$\left(\frac{dy}{dx}\right)_{c_2}=-\frac{4x}{9y}.$$

$$m_1m_2=-\frac{4x^2}{9y^2}$$
$$(x,y)=(\pm 3,\pm 2)$$

$$m_1 m_2 = \frac{-4 \times 9}{9 \times 4}$$

$$m_1 m_2 = -1$$

Angle b/w curves $= \frac{\pi}{2}$

Q. The lines tangent to the curve $y^3 - x^2y + 5y - 2x = 0$ and $x^4 - x^3y^2 + 5x + 2y = 0$ at the origin intersect at an angle θ equal to.

Sol:-

$$y^3 - x^2y + 5y - 2x = 0$$

$$3y^2 \cdot \frac{dy}{dx} - \left(2xy + x^2\frac{dy}{dx}\right) + 5\frac{dy}{dx} - 2 = 0$$

$$3y^2 \cdot \frac{dy}{dx} - 2xy - x^2\frac{dy}{dx} + 5\frac{dy}{dx} - 2 = 0$$

$$\frac{dy}{dx}(3y^2 - x^2 + 5) = 2xy + 2$$

$(0,0)$

$$\frac{dy}{dx} \cdot 5 = 2$$

$$\left(\frac{dy}{dx}\right)_{c_1} = \frac{2}{5}$$

$$x^4 - x^3y^2 + 5x + 2y = 0$$

$$4x^3 - y^2 \cdot 3x^2 - x^3 \cdot 2y \cdot \frac{dy}{dx} + 5 + 2\frac{dy}{dx} = 0$$

$$\frac{dy}{dx}(2 - x^3 \cdot 2y) = y^2 3x^2 - 4x^3 - 5$$

$(0,0)$

$$\left(\frac{dy}{dx}\right)_{c_2} = -\frac{5}{2}$$

Page No.:
Date:

$$m_1 m_2 = \frac{2}{5} \times \left(\frac{-5}{2}\right)$$
$$= -1$$

Angle b/w tangents $= \frac{\pi}{2}$

Method : 2 →

If a curve passes through origin, then lowest degree terms are eqⁿ of tangent at origin.

$$x^4 - x^3y^2 + 5x - 2y = 0$$
$$5x - 2y = 0 \rightarrow \text{Eq}^n \text{ of tangent}$$
$$\left(\frac{dy}{dx}\right)_{c_1} = \cancel{\frac{2}{5}}\ \frac{-5}{2}$$

$$y^3 - x^2y + 5y - 2x = 0$$
$$5y - 2x = 0 \longrightarrow \text{Eq}^n \text{ of tangent}$$
$$\left(\frac{dy}{dx}\right)_{c_2} = \frac{2}{5}$$

$$m_1 m_2 = -1$$
$$\boxed{\theta = \pi/2}$$

Q. If the tangent to the curve $2y^3 = ax^2 + x^3$ at a point (a, a) cuts-off intercepts p and q on the coordinate axes, where $p^2 + q^2 = 61$, then a equals.

Sol^n:

$$2y^3 = ax^2 + x^3$$

$$2 \times 3y^2 \cdot \frac{dy}{dx} = 2ax + 3x^2$$

$$\frac{dy}{dx} = \frac{2ax+3x^2}{6y^2}$$

$$\left.\frac{dy}{dx}\right|_{(a,a)} = \frac{2a^2+3a^2}{6a^2}$$

$$\left.\frac{dy}{dx}\right|_{(a,a)} = \frac{5}{6}$$

Eq^n of tangent

$$y-a = \frac{5}{6}(x-a)$$

$$6y-6a = 5x-5a$$

$$6y = 5x+a$$

$(p,0)$

$$\cancel{0 = 5p + a}$$

$$\cancel{a = 6 - 5p}$$

$$5p+a=0$$

$$p = -\frac{a}{5}$$

$(0,q)$

$$6q = a$$

$$q = \frac{a}{6}$$

$$p^2+q^2 = 61$$

$$\frac{a^2}{25} + \frac{a^2}{36} = 61$$

$$\frac{36a^2+25a^2}{900} = 61$$

$$a^2 = 900$$

$$\boxed{a = \pm 30}$$

Page No.:
Date:

Q. The tangent to $y = ax^2 + bx + \frac{7}{2}$ at $(1,2)$ is parallel to the normal at the point $(-2,2)$ on the curve $y = x^2 + 6x + 10$, find the value of a and b.

sol^n:-

$$y = x^2 + 6x + 10$$

$$\frac{dy}{dx} = 2x + 6$$

$$\left.\frac{dy}{dx}\right|_{(-2,2)} = 2$$

$$\left.\frac{dx}{dy}\right|_{(-2,2)} = \frac{1}{2}$$

$$y - 2 = -\frac{1}{2}(x+2)$$

$$2y - 4 = -x - 2$$

$$x + 2y - 2 = 0 \rightarrow (L_1)$$

$$2y = -x + 2$$

$$y = -\frac{x}{2} + 1$$

$$y = ax^2 + bx + \frac{7}{2}$$

$$\frac{dy}{dx} = 2ax + b$$

$$\left.\frac{dy}{dx}\right|_{(1,2)} = 2a + b$$

$$y - 2 = (2a+b)(x-1)$$

$$y - 2 = 2ax - 2a + bx - b$$

$$y = (2a+b)x - 2a - b + 2 \rightarrow (L_2)$$

$$L_1 \parallel L_2$$

$-\frac{1}{2} = 2a + b$

$4a + 2b = -1 \quad — ①$

$2 = a + b + \frac{7}{2}$

$-\frac{3}{2} = a + b$

$2a + 2b = -3 \quad — ②$

eqⁿ ① − ②

$2a = 2$

$\boxed{a = 1}$

$\boxed{b = -5/2}$

Monotonocity

There are two types of monotonic functions:-

(i) Increasing functions:-

~~Strictly Increasing Functions:~~

Strictly Increasing functions
Increasing or Non

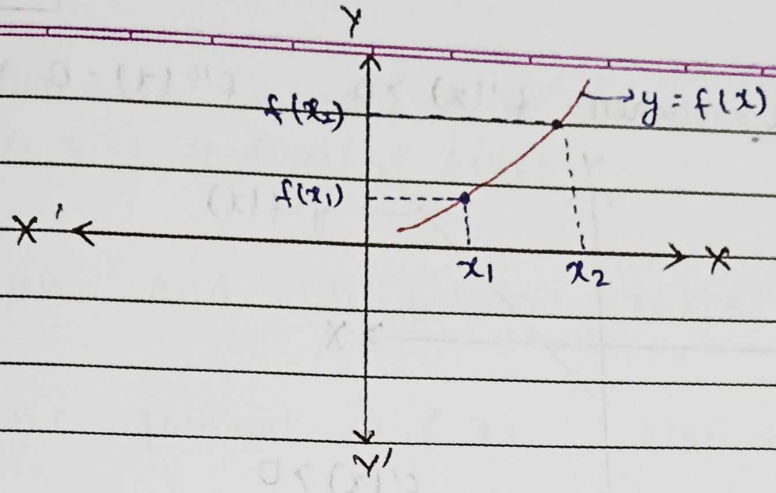
(ii) Decreasing functions:-

strictly Decreasing Functions
Decreasing or non-increasing functions.

Monotonocity:-
Functions are said to be monotonic if they are either increasing or decreasing in their entire domain.
eg → e^x, $\ln x$

(i) Increasing Functions:-

(A) strictly Increasing Functions:- A function $f(x)$ is known as strictly increasing function in its domain if $x_1 < x_2 \Rightarrow f(x_1) < f(x_2)$



Here, $x_1 < x_2$

$f(x_1) < f(x_2)$

Thus, f(x) is strictly increasing function.

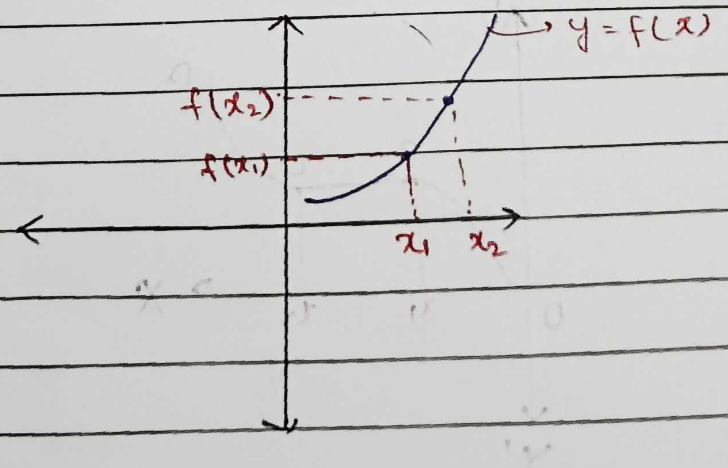
Note:-

condition for strictly increasing function, $f'(x) > 0$; $\forall x \in$ Domain

classification of strictly Increasing Functions:

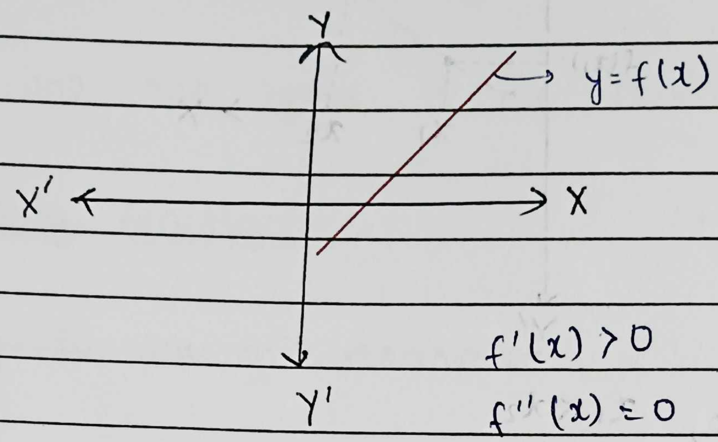
Increasing Functions can be classified as:

i) Concave up when $f'(x) > 0$ and $f''(x) > 0$ $\forall x \in$ Domain



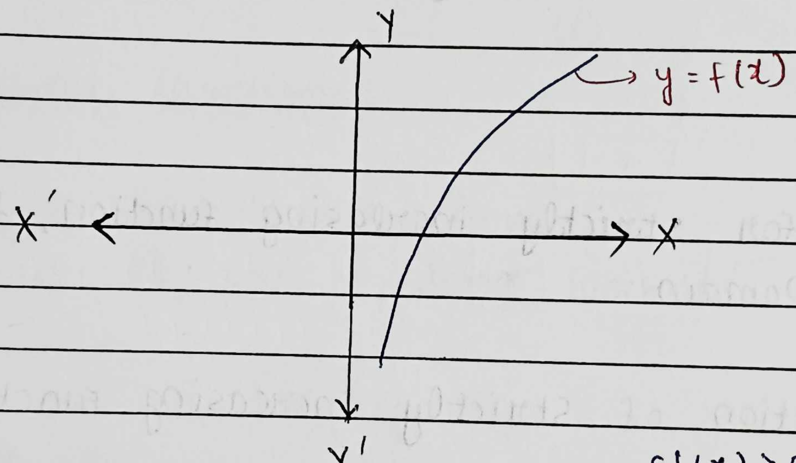
M T W T F S S
Page No.:
Date:
YOUVA

(ii) Straight Line when $f'(x) > 0$, $f''(x) = 0$ $\forall x \in$ Domain



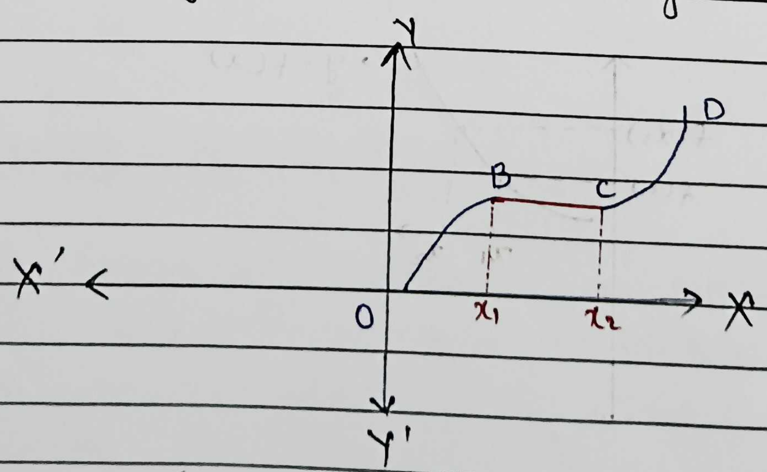
$f'(x) > 0$
$f''(x) = 0$

(iii) Concave Down when $f'(x) > 0$, $f''(x) < 0$ $\forall x \in$ Domain



$f'(x) > 0$
$f''(x) < 0$

(B) Only Increasing or Non-Decreasing Functions :-



A function $f(x)$ is said to be non-decreasing if

~~for~~ $x_1 < x_2 \Rightarrow f(x_1) \leq f(x_2)$

For AB and CD portion, $x_1 < x_2 \Rightarrow f(x_1) < f(x_2)$

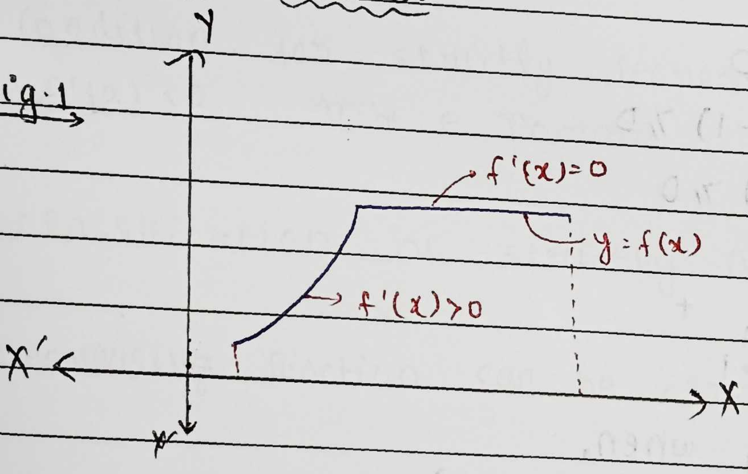
For BC portion, $x_1 < x_2$, $f(x_1) = f(x_2)$

Note :-

Condition for Increasing or Non-Decreasing Function, $f'(x) \geq 0 \ \forall x \in$ Domain

Point to consider :-

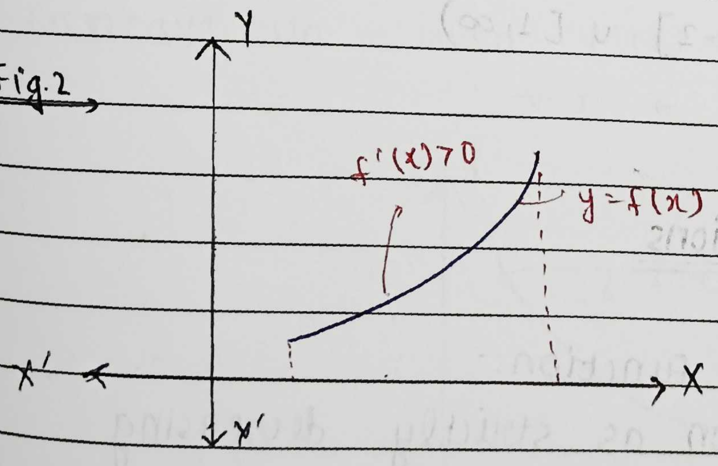
Fig.1 →



Increasing

$f'(x) \geq 0$

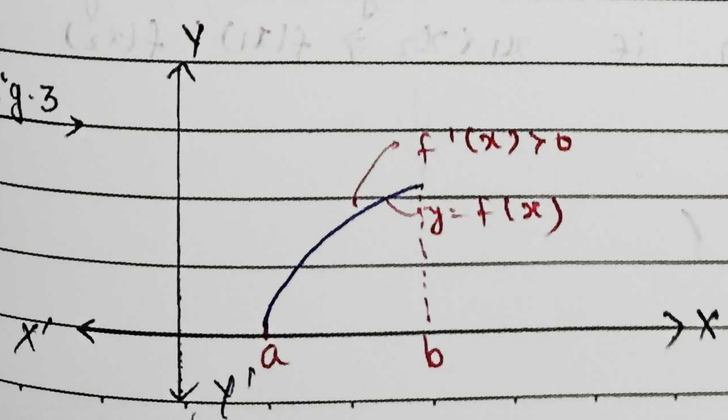
Fig.2 →



strictly increasing

$f'(x) > 0$

Fig.3 →



strictly increasing

$f'(x) > 0$

In the above figure 1, function is only increasing or it is non-decreasing function and figure 2 and figure 3, show strictly increasing function on $[a,b]$.

Q. Find the interval interval in which $f(x) = 2x^3 + 3x^2 - 12x + 1$ is increasing.

Solⁿ:-

$$f'(x) \geq 0$$

$$6x^2 + 6x - 12 \geq 0$$

$$x^2 + x - 2 \geq 0$$

$$\cancel{x^2 + x - 2x}$$

$$x^2 - x + 2x - 2 \geq 0$$

$$x(x-1) + 2(x-1) \geq 0$$

$$(x-1)(x+2) \geq 0$$

$$\overset{+}{} \quad \underset{-2}{|} \quad \overset{-}{} \quad \underset{1}{|} \quad \overset{+}{}$$

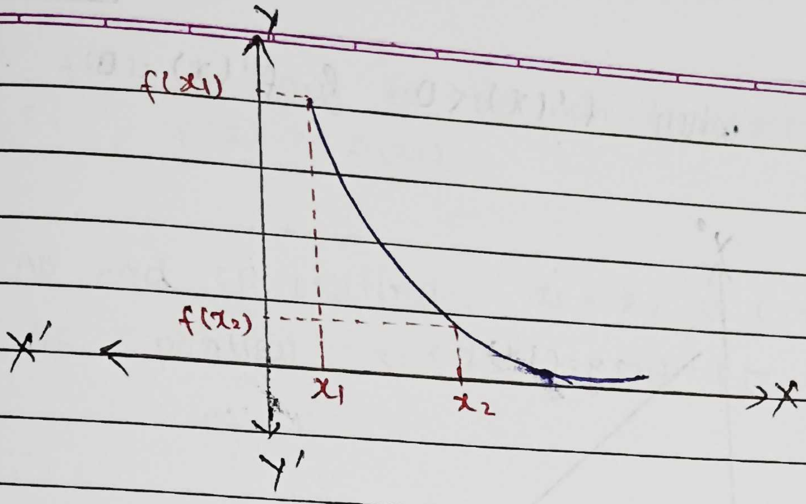
$f(x)$ in increasing when,

$$x \in (-\infty, -2] \cup [1, \infty)$$

(ii) <u>Decreasing Functions</u>

(A) Strictly Decreasing Function:-

A f^n $f(x)$ is known as strictly decreasing f^n in its domain if $x_1 < x_2 \Rightarrow f(x_1) > f(x_2)$



Here, $x_1 < x_2$
$f(x_1) > f(x_2)$

Thus, $f(x)$ is strictly decreasing function.

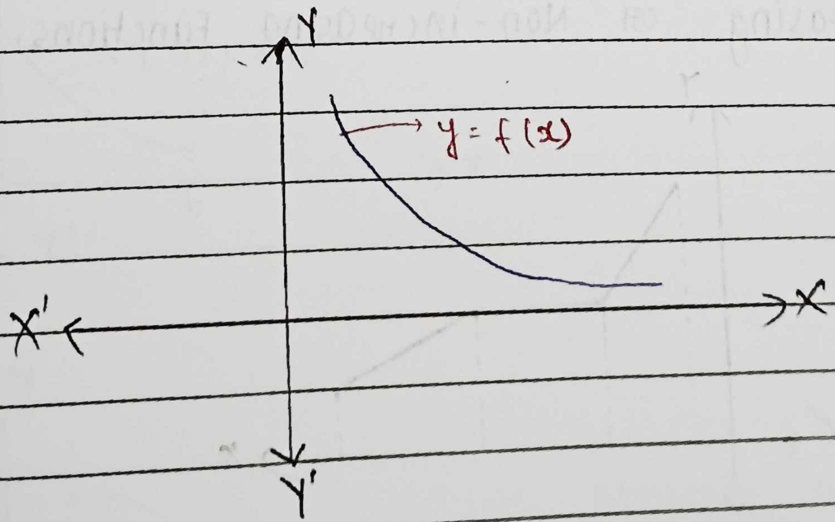
Note:-

Condition for strictly decreasing function,
$f'(x) < 0 \quad \forall x \in$ Domain.

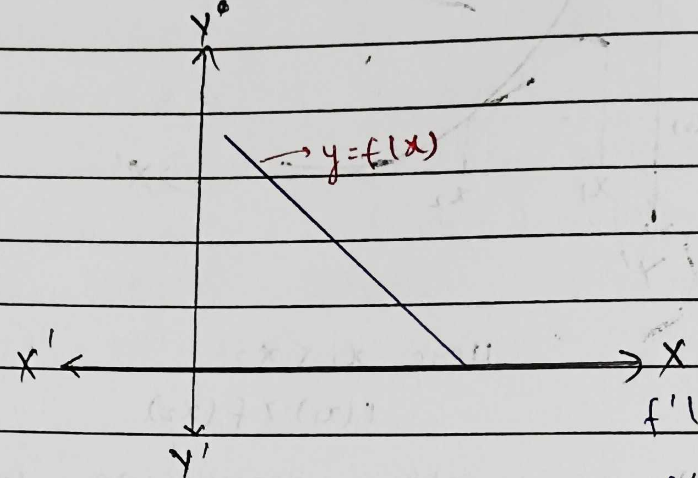
Classification of strictly Decreasing Functions:-

Decreasing function can be classified as:

(i) Concave up when $f'(x) < 0$ and $f''(x) > 0$
$\forall x \in$ Domain.



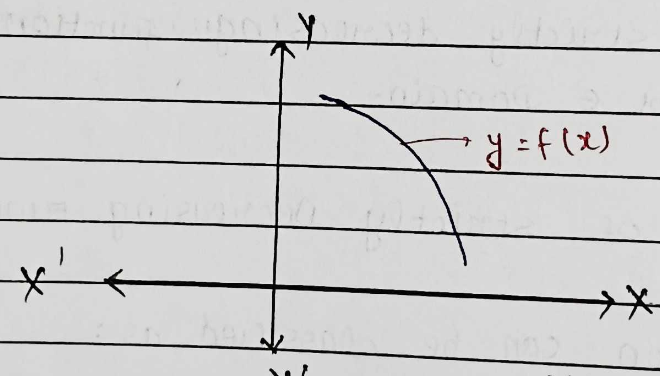
(ii) Straight Line when $f'(x) < 0$ & $f''(x) = 0$ $\forall x \in$ Domain.



$f'(x) < 0$

$f''(x) = 0$

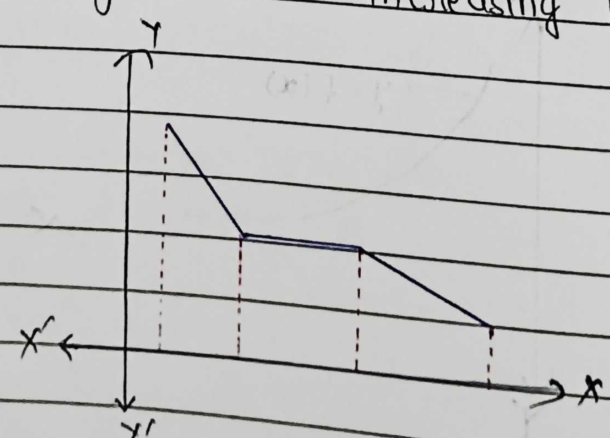
Concave Down when $f'(x) < 0$ and $f''(x) < 0$ $\forall x \in$ Domain



$f'(x) < 0$

$f''(x) < 0$

(B) Only Decreasing or Non-increasing Functions:-



A f^n $f(x)$ is said to be non-increasing if for $x_1 < x_2$, $f(x_1) \geq f(x_2)$

For AB and CD portion, $x_1 < x_2 \Rightarrow f(x_1) > f(x_2)$

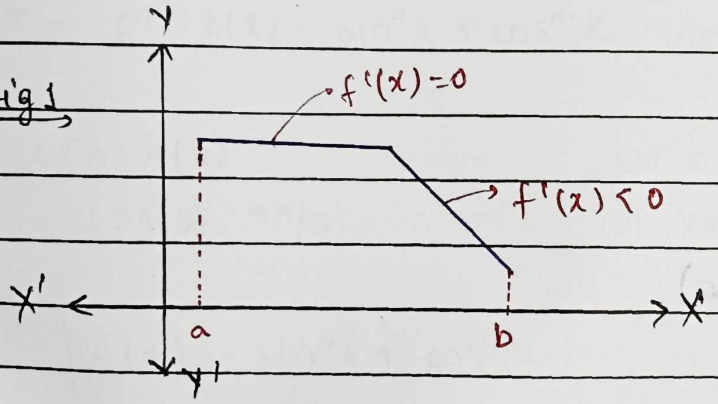
For BC portion, $x_1 < x_2 \Rightarrow f(x_1) = f(x_2)$

Note:-

condition for only decreasing or non-increasing functions, $f'(x) \leq 0$

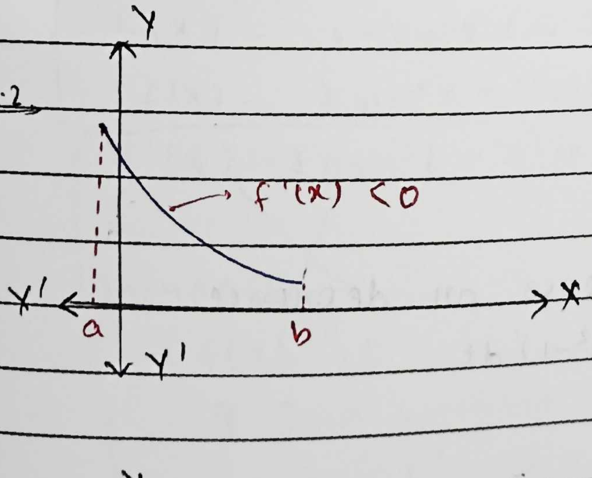
Points to consider:-

Fig 1

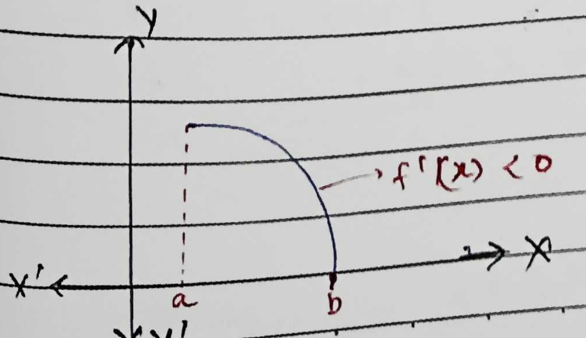


Only decreasing or non-increasing

Fig. 2



strictly decreasing



strictly decreasing

In the above, figure 1 is only decreasing or non-increasing and figure 2, figure 3 is strictly decreasing in interval $[a,b]$.

Q. Find the interval in which $f(x) = x^3 - 3x^2 - 9x + 20$ is strictly increasing or strictly decreasing.

Sol^n:-

$f'(x) = 3x^2 - 6x - 9$

$f'(x) = 3(x^2 - 2x - 3)$

~~strictly increasing~~

$f'(x) = 3(x^2 + x - 3x - 3)$

$f'(x) = 3(x+1)(x-3)$

$+ \quad -1 \quad - \quad 3 \quad +$

strictly increasing

$f'(x) > 0$

$x \in (-\infty, -1) \cup (3, \infty)$

Strictly decreasing

$f'(x) < 0$

$(x-3)(x+1) < 0$

$x \in (-1, 3)$

Q. Find the interval of increase or decrease of the $f(x) = \int_{-1}^{x} (t^2 + 2t)(t^2 - 1)\, dt$.

Sol^n:-

$f(x) = \int_{-1}^{x} (t^4 +$

Page No.: Date: YOUVA

Solⁿ: $f'(x) = (x^2+2x)(x^2-1)$

$f'(x) = x(x+2)(x+1)(x-1)$

$+ \quad - \quad + \quad - \quad +$

$-2 \quad -1 \quad 0 \quad 1$

Increasing , $f'(x) \geq 0$

$x \in (-\infty,-2] \cup [-1,0] \cup [1,\infty)$

Decreasing ; $f'(x) \leq 0$

$x \in [-2,-1] \cup [0,1]$

Q. The f^n $f(x) = \sin^4 x + \cos^4 x$, increasing if

(a) $x \in (0, \pi/8)$ ~~(b)~~ $x \in (\pi/4, 3\pi/8)$

(c) $x \in (3\pi/8, 5\pi/8)$ (d) $x \in (5\pi/8, 3\pi/4)$

Solⁿ:

$f(x) = \sin^4 x + \cos^4 x$

$f(x) = 1 - 2\sin^2 x \cos^2 x$

$f(x) = 1 - 2\sin^2 x(1-\sin^2 x)$

$f(x) = 1 - 2\sin^2 x + 2\sin^4 x$

$f(x) = 2\sin^4 x - 2\sin^2 x - 1$

$f'(x) = 8\sin^3 x - 4\sin x$

~~Increasing~~

~~$f(x) > 0$~~

$8\sin^3 x - 4\sin x > 0$

$4\sin x(2\sin^2 x - 1) > 0$

$-4\sin x \cdot \cos 2x > 0$

$f'(x) =$

$4\sin^3 x \cdot \cos x - 4\cos^3 x \cdot \sin x$

$2\sin^2 x \cdot \sin 2x - 2\cos^2 x \cdot \sin 2x$

$-2\sin 2x(\cos^2 x - \sin^2 x)$

$-2\sin 2x \cos 2x$

$-\sin 4x$

$f(x) = \cancel{8}\sin^3 x \cdot \cos x - 4\sin x \cdot \cos$

$f'(x) = -\sin 4x$

Increasing

$-\sin 4x \geq 0$

$\sin 4x \leq 0$

$\pi \leq 4x \leq 2\pi$

$\frac{\pi}{4} \leq x \leq \frac{\pi}{2}$

$x \in [\pi/4, \pi/2]$

$(\pi/4, 3\pi/8) \subset [\pi/4, \pi/2]$

Hence,

'b' is correct

Q. let $f(x) = \int_0^x e^t (t-1)(t-2) \, dt$, then f decreases in the interval.

Sol^n: $f'(x) = e^x (x-1)(x-2)$

~~$f'(x) > 0$~~ Decreasing f^n

$f'(x) \le 0$

$+ \quad 1 \quad - \quad 2 \quad +$

$x \in [1, 2]$

Q. The f^n $f(x) = 2\log(x-2) - x^2 + 4x + 1$ increases in the interval

(a) $(1, 2)$ (b) $(2, 3)$ (c) $(5/2, 3)$
(d) $(2, 4)$

Sol^n: $f'(x) = \dfrac{2}{x-2} - 2x + 4$

$x - 2 > 0$

$x > 2$

~~$= 2 - (2$~~

$= \dfrac{2 - 2x(x-2) + 4(x-2)}{x-2}$

$= \dfrac{2 - 2x^2 + 4x + 4x - 8}{x-2}$

$f'(x) \geq 0$

$x \neq 2$

$2-2x^2+4x+4x-8 \geq 0$

$-2x^2+8x-6 \geq 0$

$2x^2-8x+6 \leq 0$

$x^2-4x+3 \leq 0$

$(x-1)(x-3) \leq 0$

$+ \quad 1 \quad - \quad 3 \quad +$

$\cancel{x \in (-\infty, 1) \cup (3, \infty)}$

$x \in (1,2) \cup (2,3)$

But $\log(x-2)$

$x \in (2,3)$

$f'(x) \geq 0$

$$\frac{-2(x^2-4x+3)}{(x-2)} \geq 0$$

$$\frac{x^2-4x+3}{(x-2)} \leq 0$$

$$\frac{(x-1)(x-3)}{(x-2)} \leq 0$$

$- \quad 1 \quad + \quad 2 \quad - \quad 3 \quad +$

$x \in [-\infty, 1] \cup (2, 3]$

For existence of log

$x-2>0$

$x>2$

Hence,

$x \in (2,3]$

The f^n x^x decreases on the interval

(a) $(0,e)$ (b) $(0,1)$ (c) $(0, 1/e)$ (d) NOT

$f(x) = x^x$

$f'(x) = x^x(1+\ln x)$

$f'(x) \leq 0$

$x^x(1+\ln x) \leq 0$

$\ln x \leq -1$

$x \in (0, 1/e)$

$x \leq 1/e$

$\ln x = -1$

$(x = 1/e)$

Page No :
Date

Q. If the f^n $f(x) = 2x^2 - kx + 5$ is increasing in $[1,2]$, then k lies in the interval.

~~(a)~~ $(-\infty, 4)$ (b) $(4, \infty)$ (c) $(-\infty, 8]$
(d) $(8, \infty)$

Sol^n:

$f(x) = 2x^2 - kx + 5$
$f'(x) = 4x - k$

Increasing
$f'(x) \geq 0$
$4x - k \geq 0$
$k \leq 4x$
$k \leq 4$, $k \leq 8$

Hence,

$k \leq 4$
$k \in (-\infty, 4]$

Q. The f^n $f(x) = \tan^{-1}(\sin x + \cos x)$ is an increasing f^n in

(a) $(\pi/4, \pi/2)$ ~~(b)~~ $(-\pi/2, \pi/4)$ (c) $(0, \pi/2)$
(d) $(-\pi/2, \pi/2)$

Sol^n: $f(x) = \tan^{-1}(\sin x + \cos x)$

$$f'(x) = \frac{1}{1 + \sin^2 x + \cos^2 x + \sin 2x} \cdot (\cos x - \sin x)$$

$$f'(x) = \frac{\cos x - \sin x}{\sin 2x + 2}$$

$f'(x) \geq 0$

$\frac{\cos x - \sin x}{\sin 2x + 2} \geq 0$

+ve

$\cos x - \sin x \geq 0$

~~so~~ $\cos x \geq \sin x$

$x \in (-\pi/2, \pi/4)$

Q. Let $f(x) = e^x - x$ and $g(x) = x^2 - x$, $\forall x \in R$, then the set of all $x \in R$, where the f^n $h(x) = (f \circ g)(x)$ is increasing, is

[JEE Main 2019]

Sol^n:

$h(x) = f(g(x))$

$= e^{g(x)} - g(x)$

$= e^{x^2 - x} - (x^2 - x)$

$= e^{x^2 - x} - x^2 + x$

$h'(x) = e^{x^2 - x} \cdot (2x - 1) - 2x + 1$

$= e^{x^2 - x}(2x - 1) - 1(2x - 1)$

$= (2x - 1)(e^{x^2 - x} - 1)$

$h'(x) \geq 0$

$(2x - 1)(e^{x^2 - x} - 1) \geq 0$

$e^{x^2 - x} \geq 1$

$x^2 - x \geq 0$

$x(x-1) \geq 0$

$-\quad + \quad - \quad +$

$0 \quad \frac{1}{2} \quad 1$

$x \in [0, 1/2] \cup [1, \infty)$

Q. Let $f:[0,2] \to R$ be a twice differentiable f^n such that $f''(x) > 0$, $\forall\ x \in (0,2)$, if $\phi(x) = f(x) + f(2-x)$, then ϕ is.

[JEE Main 2019]

(a) Increasing on $(0,1)$ and decreasing on $(1,2)$
(b) Decreasing on $(0,2)$
(c) ✓ Decreasing on $(0,1)$ and increasing on $(1,2)$
(d) Increasing on $(0,2)$

Sol^n:

$$\phi(x) = f(x) + f(2-x)$$
$$\phi'(x) = f'(x) - f'(2-x)$$

$$f''(x) > 0 \quad \forall\ x \in (0,2)$$

It means $f'(x)$ is strictly increasing in $(0,2)$

For increasing

$$\phi'(x) \geq 0$$
$$f'(x) - f'(2-x) \geq 0$$
$$f'(x) \geq f'(2-x)$$

Increasing

$$x > 2-x$$
$$x > 1$$
$$x \in (1,2)$$

For decreasing

$$\phi'(x) \leq 0$$
$$f'(x) \leq f'(2-x)$$
$$x < 2-x$$
$$x < 1$$
$$x \in (0,1)$$

Q. If the f^n f given by $f(x) = x^3 - 3(a-2)x^2 + 3ax + 7$, for some $a \in R$ is increasing in $(0,1]$ and decreasing in $[1,5)$, then the root of the eq^n $\frac{f(x) - 14}{(x-1)^2} = 0$ $\{x \neq 1\}$ is

[JEE Main 2019]

M T W T F S S

Page No.:

Date:

YOUVA

(a) -7 (b) 6 (c) 7 (d) 5

Sol^n:
$$f(x) = x^3 - 3(a-2)x^2 + 3ax + 7$$
$$f'(x) = 3x^2 - 6(a-2)x + 3a$$

Increasing f^n

$f'(x) \geq 0$

$3x^2 - 6(a-2)x + 3a \geq 0$

$3x^2 + 12x - 6ax + 3a \geq 0$

Decreasing f^n

$f'(x) \leq 0$

$3x^2 + 12x - 6ax + 3a \leq 0$

$f'(1) = 0$

$3x^2 + 12x - 6ax + 3a = 0$

$x = 1$

$\boxed{a = 5}$

$$f(x) = x^3 - 9x^2 + 15x + 7$$

$$\frac{f(x) - 14}{(x-1)^2} = 0$$

$$\frac{x^3 - 9x^2 + 15x - 7}{(x-1)^2} = 0$$

$\cancel{x^2(x-9)}$ $x \neq 1$

$x^3 - 9x^2 + 15x - 7 = 0$

Satisfying option,

(a) $-343 - 441 - 105 - 7 \neq 0$

(c) $343 - 441 + 105 - 7$

$= 448 - 448$

$= 0$

'c' is correct

Method 2

How to calculate root of cubic eq^n?

Hit and Trial find 1^{st} root

$x = 1$ will satisfy

$(x-1)$ is one root

Divide by $(x-1)$ whole polynomial, we'll get $(x-1)^2(x-7) = 0$

Q. A spherical balloon is being inflated at the rate of 35 cm³/ min. The rate of increase in the surface are (in cm²/min.) of the balloon when its diameter is 14 cm is.

[JEE Main 2013]

Solⁿ:-

$$\frac{dV}{dt} = 35\, cm^3/min$$

$$V = \frac{4}{3}\pi r^3$$

$$\frac{dV}{dt} = \frac{4}{\cancel{3}}\pi \times \cancel{3}r^2 \cdot \frac{dr}{dt}$$

$$\frac{35}{4\pi r^2} = \frac{dr}{dt}$$

$$A = 4\pi r^2$$

$$\frac{dA}{dt} = 4\pi \cdot 2r \cdot \frac{dr}{dt}$$

$$\frac{dA}{dt} = \cancel{4\pi} \cdot 2\cancel{r} \cdot \frac{35}{\cancel{4\pi} r^{\cancel{2}}}$$

$$\frac{dA}{dt} = \frac{70}{r}$$

$$r = 7$$

$$\frac{dA}{dt} = 10\, cm^2/min.$$

Page No.:
Date:

A spherical balloon is filled with $4500\pi\ m^3$ of helium gas. If a leak in the balloon causes the gas to escape at the rate of $72\pi\ m^3/min$, then the rate (in m/min) at which the radius of the balloon decreases 49 minutes after the leakage began is:

Sol?

$1\ min. = 72\pi\ m^3$

$49\ min. = 72\pi \times 49$

$= 3528\pi$

Vol. of balloon after 49 min. $= 4500\pi - 3528\pi$

$= 972\pi$

$\frac{4}{3}\cancel{\pi} r^3 = \overset{243}{\cancel{972}}\cancel{\pi}$

$r^3 = 729$

$\boxed{r = 9\ m.}$

$V = \frac{4}{3}\pi r^3$

$\frac{dV}{dt} = \frac{4}{3}\pi \times 3r^2 \cdot \frac{dr}{dt}$

$18\ \cancel{72\pi} = \frac{4\cancel{\pi}}{\cancel{3}} \times \cancel{3} \times (9)^2 \cdot \frac{dr}{dt}$

$\frac{18}{81} = \frac{dr}{dt}$

$\boxed{\frac{dr}{dt} = \frac{2}{9}}$

Q. If the line joining the points $(0,3)$ and $(5,-2)$ is a tangent to the curve $y = \frac{c}{x+1}$, then find the value of c?

Solⁿ:- $(0,3)$ $(5,-2)$

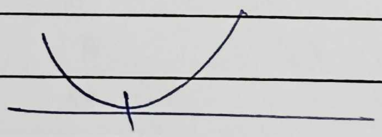
$$m = \frac{-2-3}{5} = -1$$

$$y = \frac{c}{x+1}$$

$$\frac{dy}{dx} = \frac{-c}{(x+1)^2}$$

$$-1 = \frac{-c}{(x+1)^2}$$

$$c = (x+1)^2 \quad \text{—— (i)}$$



$B(x, 3-x)$

$$y - 3 = -1(x-0)$$

$$x + y = 3$$

$$y = \frac{c}{x+1}$$

$$3 - x = \frac{c}{x+1}$$

$$\boxed{x \neq -1}$$

$$(3-x)(x+1) = c$$

$$(3-x)\cancel{(x+1)} = (x+1)^{\cancel{2}} \quad \{\text{from eq}^n \text{ (i)}\}$$

$$3 - x = x + 1$$

$$2 = 2x$$

$$\boxed{x = 1}$$

M T W T F S S

Page No.

Date

YOUVA

$c = (x+1)^2$

$c = (1+1)^2$

$\boxed{c = 4}$

Q. The co-ordinate of the points on the graph of the function $f(x) = \frac{x^3}{3} - \frac{5x^2}{2} + 7x - 4$ where the tangent drawn cuts off intercepts from the co-ordinate axes which are equal in magnitude but opposite in sign is.

~~(a)~~ $(2, 8/3)$ ~~(b)~~ $(3, 7/2)$ (c) $(1, 5/6)$ (d) NOT

Solⁿ:- $f(x) = \frac{x^3}{3} - \frac{5x^2}{2} + 7x - 4$

$f'(x) = x^2 - 5x + 7$

$\frac{dy}{dx} = x^2 - 5x + 7$

Intercepts are equal in magnitude but opposite in sign means, $m = 1$

M T W T F S S
Page No
Date
YOUVA

$$\frac{dy}{dx} = 1$$

$$x^2 - 5x + 7 = 1$$

$$x^2 - 5x + 6 = 0$$

$$(x-2)(x-3) = 0$$

$x = 2$

$y = \frac{8}{3}$

$x = 3$

$y = \frac{7}{2}$

$\frac{8}{3} - 10 + 14 - 4$

$\frac{8 - 30 + 42 - 12}{3}$

$9 - \frac{45}{2} + 21 - 4$

$\frac{18 - 45 + 42 - 8}{2}$

Q. The curve $x + y - \ln(x+y) = 2x + 5$ has a vertical tangent at the point (α, β), then find $\alpha + \beta$.

Sol.

$$x + y - \ln(x+y) = 2x + 5$$

Diff. w.r.t. 'y'

$$\frac{dx}{dy} + 1 - \frac{1}{x+y}\left(\frac{dx}{dy} + 1\right) = 2\frac{dx}{dy}$$

$$\frac{dx}{dy}\left(1 - 2 - \frac{1}{x+y}\right) = -1 + \frac{1}{x+y}$$

$$\frac{dx}{dy}\left(-1 - \frac{1}{x+y}\right) = \frac{\cancel{-x-y} - x - y + 1}{x+y}$$

$$\frac{dx}{dy} = \frac{-(x+y)+1}{\cancel{(x+y)}} \cdot \frac{(x+y)}{-x-y-1}$$

$$\frac{dx}{dy} = \frac{-x-y+1}{-x-y-1}$$

$$\frac{dx}{dy} = 0$$

$$-x - y + 1 = 0$$

$$x + y = 1$$

(α, β) $\boxed{\alpha + \beta = 1}$

M T W T F S S
Page No.:
Date:
YOUVA

Q. Coffee is coming out from a conical filter, with height and diameter both are 15cm into a cylinderical coffee pot with a diameter 15cm. The rate at which coffee comes out from the filter into the pot is $100\ cm^3/min$. The rate (in cm/min.) at which the level in the pot is rising at the instant when the coffee in the pot is 10cm, is.

Sol^n: ~~conical filter~~

$$\cancel{V = \frac{1}{3}\pi r^2 h}$$

$$\cancel{h = 15\ cm}$$
$$\cancel{r = \frac{15}{2}\ cm}$$
$$\cancel{\boxed{r = \frac{h}{2}}}$$

$$\frac{dV}{dt} = 100\ cm^3/min.$$

$$V = \pi r^2 h$$

$$\frac{dV}{dt} = \pi r^2 . \frac{dh}{dt}$$

$$100 = \pi \times \left(\frac{15}{2}\right)^2 . \frac{dh}{dt}$$

$$\frac{dh}{dt} = \frac{100 \times 4}{225\pi}$$

$$\boxed{\frac{dh}{dt} = \frac{16}{9\pi}}$$

Q. Water run into an inverted conical tent at the rate of $20\ cm^3/min$. and leaks out at the rate of $5\ cm^3/min$. The height of the water is three times the radius of the water's surface. The radius of the water surface is increasing when the radius is 5cm, is

$V = \frac{1}{3}\pi r^2 h$

$h = 3r$

$V = \frac{1}{\cancel{3}}\pi r^3 \cdot \cancel{3}$

$V = \pi r^3$

$\frac{dV}{dt} = \pi \cdot 3r^2 \cdot \frac{dr}{dt}$

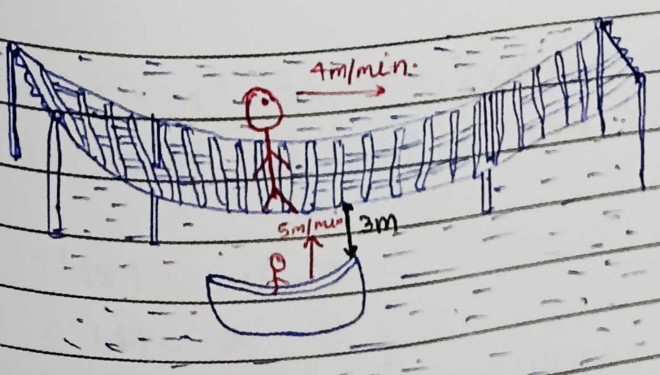
$5 \; t5 = \pi \times \cancel{3} r^2 \cdot \frac{dr}{dt}$

$5 = 25\pi \cdot \frac{dr}{dt}$

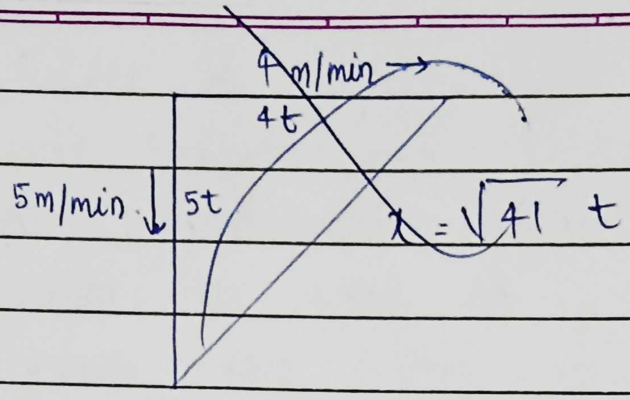
$$\boxed{\frac{dr}{dt} = \frac{1}{5\pi}}$$

Q. A man is standing on a straight bridge over a river and another man on ~~the~~ a boat on the river just below the man on the bridge. If the first man starts walking at the uniform speed of 4 m/min. and the boat moves perpendicularly towards the bridge at the speed of 5m/min., then at what rate are the separating after 4 minutes, if the height of the bridge above the boat is 3m.

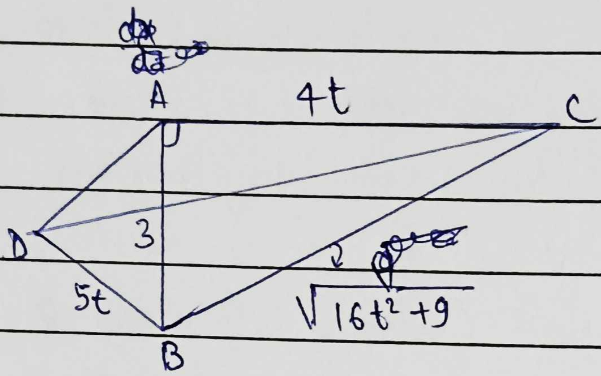
Sol^n:



~~Sol^n~~ :-



Sol^n :-



$$DC = \sqrt{(AC)^2 + (AD)^2}$$

$$DC = \sqrt{16t^2 + 9 + 25t^2}$$

$$DC = \sqrt{41t^2 + 9}$$

$$x = \sqrt{41t^2 + 9}$$

$$\frac{dx}{dt} = \frac{1}{\cancel{2}\sqrt{41t^2+9}} \cdot \cancel{2} \times 41t$$

$$\left.\frac{dx}{dt}\right|_{t=4} = \frac{164}{\sqrt{665}}$$

Non-monotonic function

Functions which are increasing as well as decreasing in their domain are said to be non-monotonic.

eg, $y = \sin x$, $y = |x|$

Monotonocity of a function at a point:-

1. A function is said to be monotonically increasing at $x = a$ if $f'(a) > 0$

2. A function is said to be monotonically decreasing at $x = a$ if $f'(a) < 0$

3. If $f'(a) = 0$, then check the sign of $f'(a^+)$ and $f'(a^-)$

(a) If $f'(a^+) > 0$ and $f'(a^-) > 0 \Rightarrow$ Increasing

(b) If $f'(a^+) < 0$ and $f'(a^-) < 0 \Rightarrow$ Decreasing

(c) Otherwise, neither increasing nor decreasing

Q. Let $f(x) = x^3 - 3x + 2$. Examine the nature of f^n at point at $x = 0, 1$ and 2.

Sol:-

$f'(x) = 3x^2 - 3$

$f'(0) = -3$] Decreasing.

$f'(2) = 9 \rightarrow$ Increasing

$f'(1) = 0$

$f'(1^+) > 0$

$f'(1^-) < 0$

At $x = 1$, neither increasing nor decreasing.

Q. If function $f(x) = x^3 + \lambda x^2 - \lambda x + 1$ is increasing at $x = 0$ and decreasing at $x = 1$, then find the greatest integral value of λ.

Sol^n:- $f(x) = x^3 + \lambda x^2 - \lambda x + 1$

$f'(x) = 3x^2 + 2\lambda x - \lambda$

$f'(0) = -\lambda$

$f'(1) = 3 + \lambda$

$(-\lambda) > 0 \qquad\qquad 3 + \lambda < 0$

$\lambda < 0 \qquad\qquad \lambda < -3$

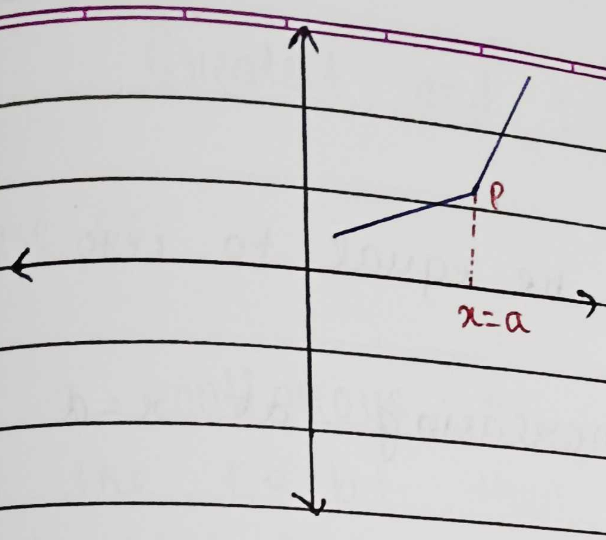
$\lambda \in (-\infty, -3)$

Greatest integral value of $\lambda = -4$

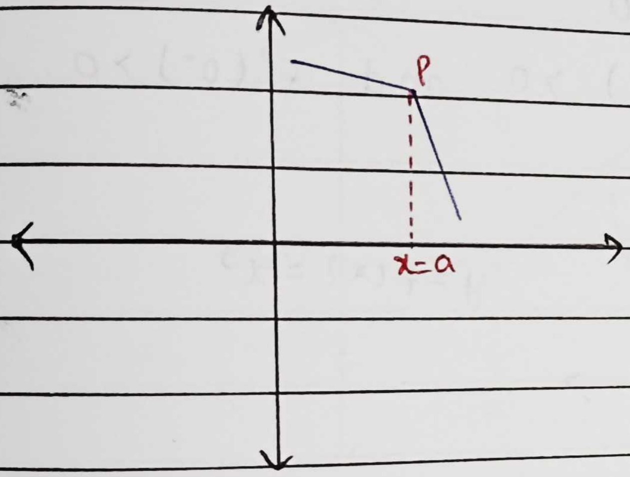
Note:

It should be noted that we can talk of monotonocity of $f(x)$ at $x = a$ only if $x = a$ lies in the domain of $f(x)$, without any consideration of continuity or differentiability of $f(x)$ at $x = a$.

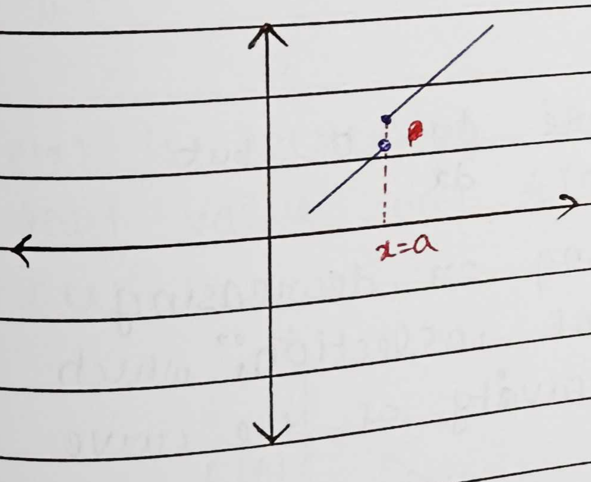
M T W T F S S
Page No.:
Date:
YOUVA



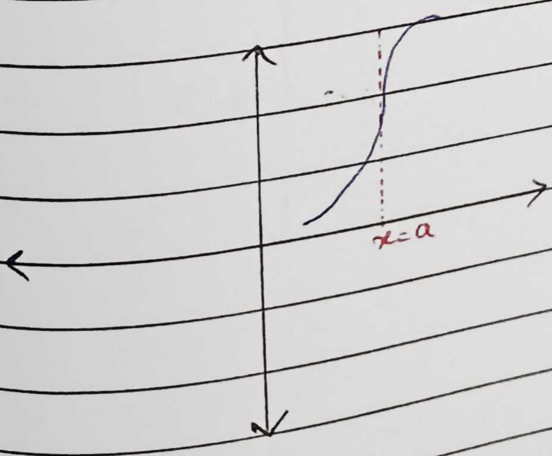
Increasing at
x=a



Decreasing at
x=a



Increasing at
x=a



Increasing at
x=a

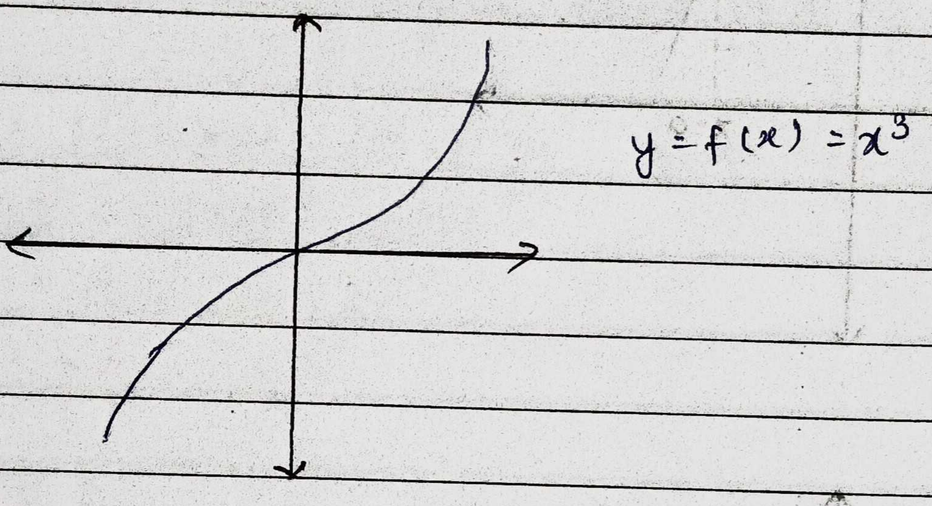
Note:-

- $\frac{dy}{dx}$ at some point may be equal to zero but f(x) may still be increasing at $x=a$.

eg,

consider $f(x) = x^3$, which is increasing at $x=0$ although $f'(x)=0$

This is because $f'(0^+) > 0$ and $f'(0^-) > 0$.

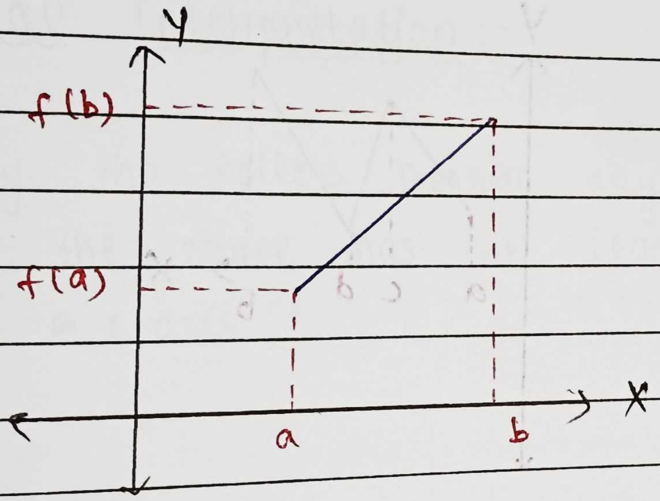


At all such points, where $\frac{dy}{dx} = 0$ but f(x) is still increasing or decreasing are known as "point of inflection," which indicate the change concavity of the curve.

Greatest and least value of a f^n

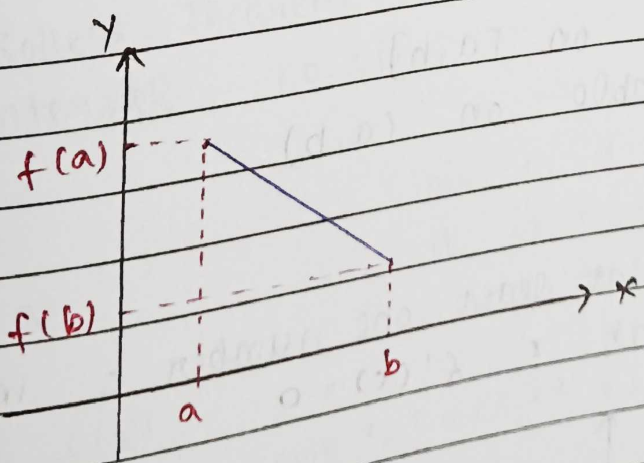
Case I :-

If a continuous f^n $y = f(x)$ is strictly increasing in the $[a, b]$, then $f(a)$ is the least value and $f(b)$ is greatest value.



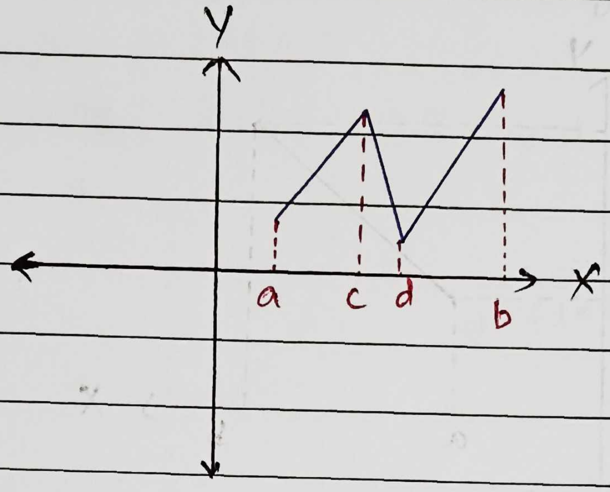
Case II :-

If $f(x)$ is decreasing in $[a, b]$, then $f(b)$ is the greatest value and $f(a)$ is the least value and $f(a)$ is the greatest value of $f(x)$.



Case III :-

If $f(x)$ is non-monotonic in $[a,b]$ and is continuous, then the greatest and least value of $f(x)$ in $[a,b]$ are those where $f'(x)=0$ or $f'(x)$ does not exist. or at the extreme values.

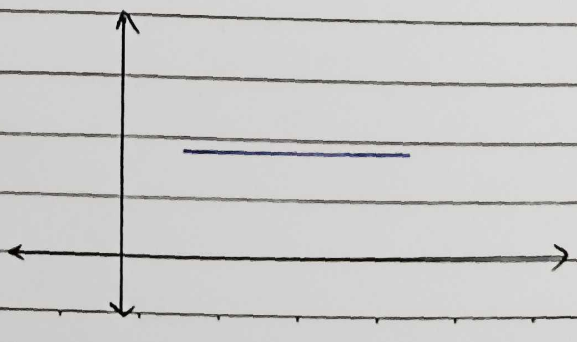


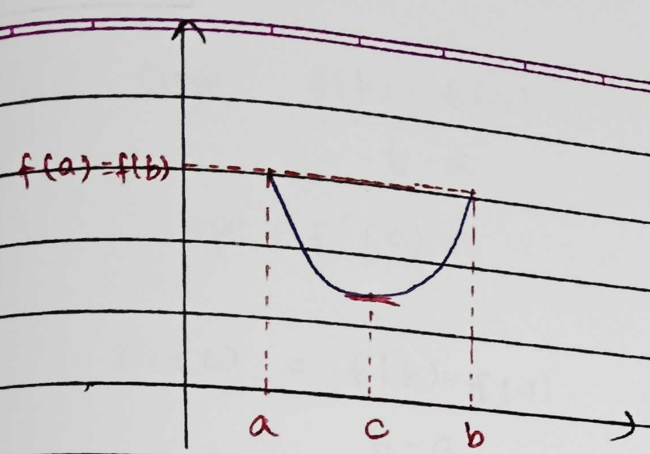
ROLLE'S THEOREM

Let f be a function that satisfies the following three conditions:

(a) f is continuous on $[a,b]$.
(b) f is differentiable on (a,b)
(c) $f(a) = f(b)$

then, there exist at least one number c in (a,b) such that $f'(c)=0$





Geometrical Interpretation :-

Geometrically, the Rolle's Theorem say that b/w A and B, the curve has at least on tangent parallel to x-axis.

Note :-

If f is differentiable function, then b/w any two consecutive roots of $f(x) = 0$, there is at least one root of the eq^n $f'(x) = 0$.

Q. Verify Rolle's Theorem for the f^n $f(x) = x^3 - 3x^2 + 2x$ in the interval $[0, 2]$.

Sol^n:-

$f(0) = 0$

$f(2) = 0$

This is a polynomial f^n, it is always continuous and differentiable.

$f(0) = f(2)$

There must exist some "c" $\in (0, 2)$ such that $f'(c) = 0$

$f'(x) = 3x^2 - 6x + 2$

$f'(c) = 0$

$3c^2 - 6c + 2 = 0$

$$c = \frac{6 \pm \sqrt{12}}{6}$$

$$c = \frac{6 \pm 2\sqrt{3}}{6}$$

$$c = \frac{3 \pm \sqrt{3}}{3}$$

$$\boxed{c = 1 \pm \frac{1}{\sqrt{3}}}$$

LAGRANGE THEOREM

or

LAGRANGE MEAN VALUE THEOREM (LMVT)

Let f be a function that satisfies the following conditions:-

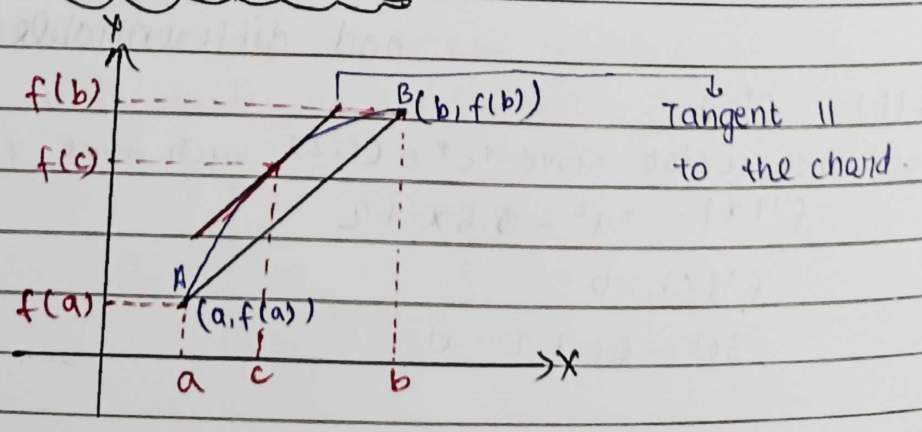
(a) f is continuous in $[a,b]$

(b) f is differentiable in (a,b)

(c) $f(a) \neq f(b)$

then, there is a number c in (a,b) such that

$$f'(c) = \frac{f(b)-f(a)}{b-a}$$



M T W T
Page No.:
Date:

$$\text{Slope} = \frac{f(b)-f(a)}{b-a}$$

$$\text{Slope} = f'(c)$$

$$f'(c) = \frac{f(b)-f(a)}{b-a}$$

Geometrical Interporetation :-

Geometrically, LMVT says that somewhere, b/w A and B, the curve has at least one tangent parallel to chord AB.

Q. Find c of the LMVT for the f^n $f(x) = 3x^2 + 5x + 7$ in the interval $[1,3]$

Sol^n :-

$f'(x) = 6x + 5$

~~$f'(3) = 23$~~ $\quad f(3) = 49$

$f(1) = 15$

$$f'(c) = \frac{49-15}{2}$$

$$f'(c) = \frac{34}{2}$$

$$f'(c) = 17$$

$f'(c) = 6c + 5$

$17 = 6c + 5$

$\boxed{c = 2}$

Q. If $f(x)$ is continuous and differentiable over $[-2,5]$ and ~~f'~~ $-4 \leq f'(x) \leq 3$ $\forall x$ in $(-2,5)$, then the greatest possible value of $f(5) - f(-2)$.

Sol^n:- $f'(c) \in [-4, 3]$

$$f'(c) = \frac{f(5) - f(-2)}{7}$$

$$3 = \frac{f(5) - f(-2)}{7}$$ {Greatest value of c = 3}

$$f(5) - f(-2) = 21$$

Q. the value ~~The value~~ of 'a' for which the function $(a+2)x^3 - 3ax^2 + 9ax - 1$ decreases monotonically throughout for all real x, are.

Sol^n:-

$$f(x) = (a+2)x^3 - 3ax^2 + 9ax - 1$$
$$f'(x) = 3(a+2)x^2 - 6ax + 9a$$

$$D \leq 0$$
$$36a^2 - 4 \times 9a(3a+6) \leq 0$$
$$36a^2 - 108a^2 - 216 \leq 0$$
$$-72a^2 - 216 \leq 0$$
$$72a^2 + 216 \geq 0$$
$$2a^2 + 6 \geq 0$$
$$2a^2 \geq -6$$
$$a^2 \geq -3$$
$$a^2 + 3 \geq 0$$

M T W T F S S

Page No.:

Date:

YOUVA

Q. If $f''(x) > 0$, $\forall\ x \in R$, $f'(3) = 0$ and ~~g(x)~~ $g(x) = f(\tan^2 x - 2\tan x - 4)$, $0 < x < \frac{\pi}{2}$, then $g(x)$ is increasing in

(a) $(0, \pi/4)$ (b) $(\pi/6, \pi/3)$ (c) $(0, \pi/3)$

✓(d) NOT

Sol^n:

$x \in (0, \pi/2)$

$f'(3) = 0$

$g(x) = f(\tan^2 x - 2\tan x - 4)$

$g'(x) = f'(\tan^2 x - 2\tan x - 4) \cdot [2\tan x \cdot \sec^2 x - 2\sec^2 x]$

Increasing

$g'(x) \geq 0$

$f'(\tan^2 x - 2\tan x - 4)\ \{2\sec^2 x(\tan x - 1) \geq 0$

$\downarrow$ zero at $\pi/4$ $\downarrow$ can't be zero $\downarrow$ zero at $\pi/4$

$$\begin{array}{ccc} - & | & + \\ & \pi/4 & \end{array}$$

$x \in (\pi/4, \pi/2)$

Q. Let $f(x) = x^{\alpha} \log x$ for $x > 0$ and $f(0) = 0$ follow Rolle's theorem for $[0, 1]$, then α is -

(a) -2 (b) -1 (c) 0 ✓(d) $1/2$

Page No :
Date

Sol^n: $f(x) = x^{\alpha} \log x$

$f'(x) = \alpha x^{\alpha-1} \log x + x^{\alpha-1}$

$f'(x) = x^{\alpha-1} (\alpha \log x + 1)$

$f'(c) = 0$

$c^{\alpha-1} (\alpha \log c + 1) = 0$

$\alpha \log c + 1 = 0$

$\alpha \log c = -1$

$\log c = \frac{-1}{\alpha}$

$c = e^{-1/\alpha}$

$0 < c < 1$

$0 < e^{-1/\alpha} < 1$

$-\infty < -1/\alpha < 0$

$0 < \frac{1}{\alpha} < \infty$

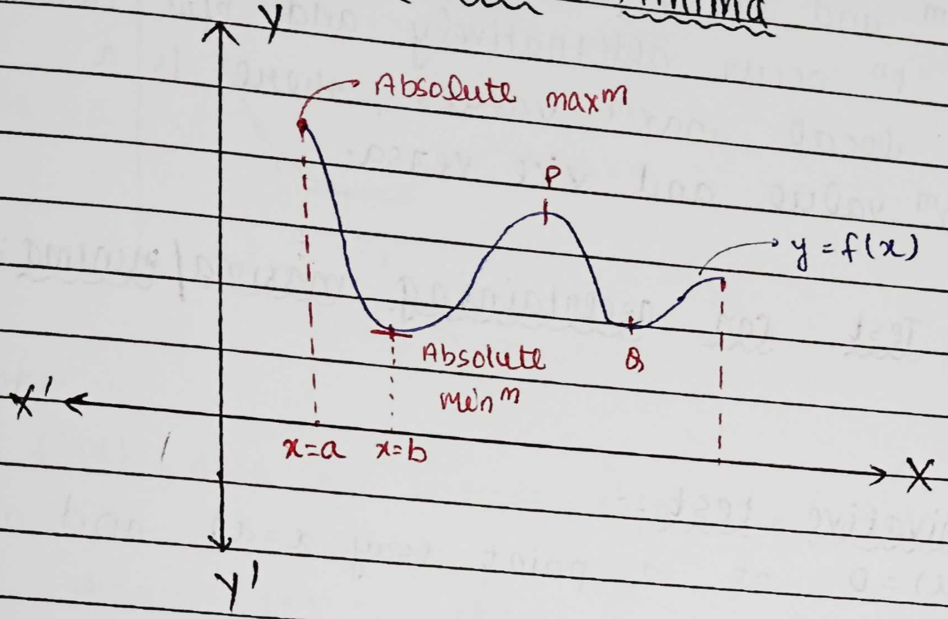
$0 < \alpha < \infty$

$\alpha \in (0, \infty)$

$\boxed{\alpha = 1/2}$

M T W T F S S
Page No.:
Date:
YOUVA

Maxima and Minima



P = Local ~~Maxima~~ Minima (~~only~~ minima / Relative minima)

Q = Local Maxima (Maxima/ Relative maxima)

Note :

(i) Max^m and Min^m values of a function are also known as local/relative maxima or local/ relative minima respectively.

~~Method~~

(ii) The term extrema is used both for maxima or minima.

(iii) A max^m (min^m) value of function may not be the greatest (least) value in a finite interval.

(iv) A f^n can have several extreme values and a local min^m value may even be greater than a ~~to~~ local max^m value.

Local max^m and local min^m values of a continuous f^n occur alternatively and b/w two consecutive local max^m values, there is a local min^m value and vice-versa.

Derivative Test for ascertaining maxima/minima :-

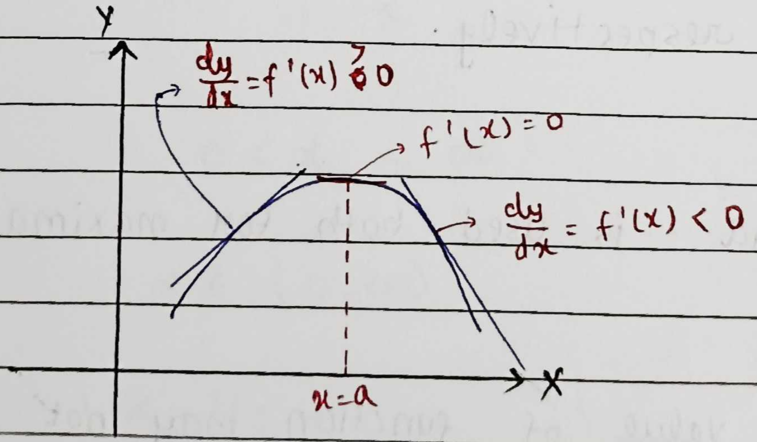
~~Maxima~~/

(a) First derivative test :-

If $f'(x) = 0$ at a point (say $x = a$) and

(i) If $f'(x)$ changes sign from +ve to -ve in the neighbourhood of $x = a$, then $x = a$ is said to be a point of local maxima.

(ii) If $f'(x)$ changes sign from -ve to +ve in the neighbourhood of $x = a$, then $x = a$ is said to be a point of local minima.



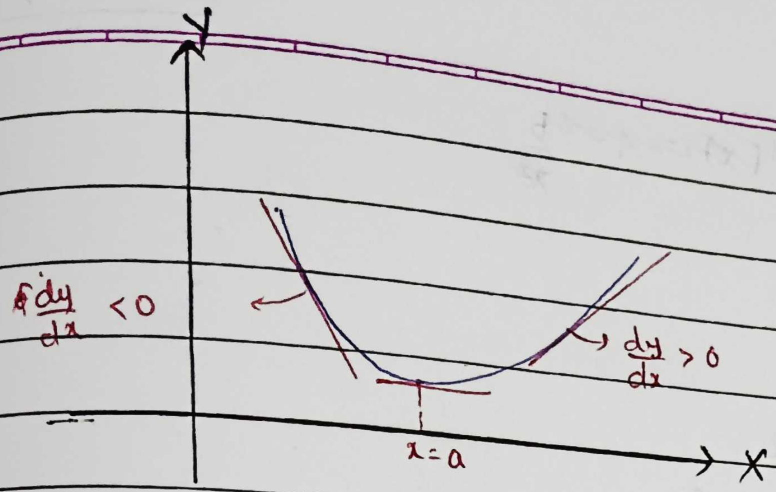
(Point of maxima)

M	T	W	T	F	S	S

Page No.:

Date

YOUVA



(Point of minima)

Note:-

If $f'(x)$ does not change sign, i.e., has the same sign in a certain complete neighbourhood of a, then $f(x)$ is either increasing or decreasing throughout this neighbourhood implying that $x = a$ is not a point of extrema of f.

Q. Let $f(x) = x + \frac{1}{x}$, $x \neq 0$. Discuss the local max^m and local min^m of $f(x)$.

Sol^n:

$f(x) = x + \frac{1}{x}$

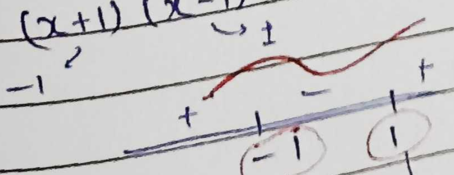
$f'(x) = 1 - \frac{1}{x^2}$

$f'(x) = 0$

$x^2 - 1 = 0$

$(x+1)(x-1) = 0$

-1, 1



Point of local minima

Point of Local maxima

$\cancel{f(x) = x + \frac{1}{x}}$ $\quad \cancel{f'(x) = 1 - \frac{1}{x^2}}$

$f(x) = x + \frac{1}{x}$

local max^m value

$f(-1) = -1 - 1 = -2$

local min^m value

$f(1) = 1 + 1 = 2$

Q. Find local maxima and local minima for the f^n $f(x) = x^3 - 3x$.

Sol^n:- $f(x) = x^3 - 3x$

$f'(x) = 3x^2 - 3$

$f'(x) = 3(x^2 - 1)$

$f'(x) = 0$

$3(x+1)(x-1) = 0$

$+ \quad -1 \quad - \quad 1 \quad +$

-1 Point of local maxima

1 Point of local minima

Local max^m value :

$f(-1) = -1 + 3 = 2$

Local max^m value :

$f(1) = 1 - 3 = -2$

Q. If f^n $f(x) = x^3 - 62x^2 + ax + 9$ has local maxima at $x = 1$, then find the value of a.

Sol^n:- $f(x) = x^3 - 62x^2 + ax + 9$

$$f'(x) = 3x^2 - 124x + a$$

$$f'(x) = 0$$

$$3x^2 - 124x + a = 0$$

$x = 1$

$$3 - 124 + a = 0$$

$$\boxed{a = 121}$$

(b) Second derivative test :-

If $f(x)$ is continuous and differentiable at $x = a$ where $f'(a) = 0$ and $f''(a)$ also exist, then for finding maxima/minima at $x = a$, second derivative test can be used.

(i) $f''(a) < 0 \Rightarrow x = a$ is a point of local maxima.

(ii) $f''(a) > 0 \Rightarrow x = a$ is a point of local minima.

(iii) If $f''(a) \Rightarrow$ second derivative test fails. To identify maxima/minima at this point either first derivative test or higher derivative test can be used.

Date:

Q. If $f(x) = 2x^3 - 3x^2 - 36x + 6$ has local max^m and min^m at $x = a$ and $x = b$ respectively then ordered pair (a, b) is.

Sol^n:- $f(x) = 2x^3 - 3x^2 - 36x + 6$

$f'(x) = 6x^2 - 6x - 36$

$f''(x) = 12x - 6$

$f'(x) = 0$

$6x^2 - 6x - 36 = 0$

$x^2 - x - 6 = 0$

$x^2 + 2x - 3x - 6 = 0$

$x(x+2) - 3(x+2) = 0$

$(x-3)(x+2) = 0$

$+ \quad -2 \quad - \quad 3 \quad +$

$a = -2$

$b = 3$

$(a, b) \equiv (-2, 3)$

verification by $f''(x)$ ↓

$f''(-2) = -30$

$f''(-2) < 0$

↳ -2 is point of local maxima

$f(3) = 30$

$f(3) > 0$

↳ 3 is point of local minima.

Q. Find the point of local maxima of f^n
$f(x) = \sin x(1+\cos x)$ in $x \in (0, \pi/2)$.

Sol:- $f(x) = \sin x(1+\cos x)$

$f'(x) = \cos x(1+\cos x) + \sin x(-\sin x)$

$= \cos x + \cos^2 x \cancel{\sin x} - \sin^2 x$

$= \cos x + \cos 2x$

$f'(x) = 0$

$\cos x + \cos 2x = 0$

$\cos x + 2\cos^2 x - 1 = 0$

$2\cos^2 x + \cos x - 1 = 0$

$2\cos^2 x + 2\cos x - \cos x - 1 = 0$

$2\cos x(\cos x + 1) - 1(\cos x + 1) = 0$

$(\cos x + 1)(2\cos x - 1) = 0$

$\cos x = -1$ — N.P. in $(0, \pi/2)$

$\cos x = 1/2$

$x = \pi/3$

$f''(x) = -\sin x - 2\sin 2x$

$f''(\pi/3) = -\frac{\sqrt{3}}{2} - 2 \times \frac{\sqrt{3}}{2}$

$= -\frac{3\sqrt{3}}{2} < 0$

$x = \pi/3$ is point of local maxima.

Q. Find point local maxima and minima of $f(x) = x^5 - 5x^4 + 5x^3 - 1$.

Sol^n:

$f(x) = x^5 - 5x^4 + 5x^3 - 1$

$f'(x) = 5x^4 - 20x^3 + 15x^2$

$f'(x) = 0$

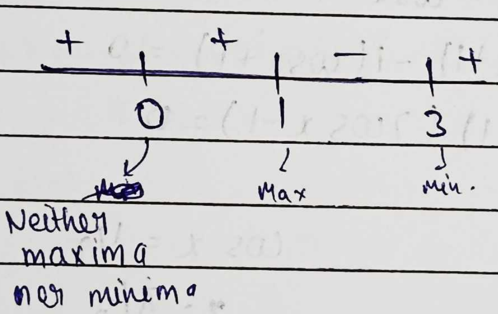
$5x^4 - 20x^3 + 15x^2 = 0$

$5x^2(x^2 - 4x + 3) = 0$

$5x^2(x^2 - x - 3x + 3) = 0$

$5x^2(x(x-1) - 3(x-1)) = 0$

$5x^2(x-1)(x-3) = 0$



Point of maxima → 1

Point of minima → 3

0 → Neither maxima nor minima. {Also $f''(0) = 0$}

M T W T F S S
Page No.:
Date:
YOUVA

Global max^m and Global min^m :-

1. **Global maximum / Global maxima (Absolute max^m) :-**

A f^n f has an absolute maxima (or global maxima) at c if $\boxed{f(c) \geq f(x)}$ $\forall x$ in the domain of f.

~~The~~

f(c) is called the max^m value of f on domain.

2. **Global minima / Absolute minima :-**

A f^n f has an absolute minima at c if $\boxed{f(c) \leq f(x)}$ $\forall x$ in the domain and f(c) is called the min^m value of f on domain.

Q. Let $f(x) = 2x^3 - 9x^2 + 12x + 6$. Discuss the global maxima and minima of f(x) in [0, 2].

Sol^n:- $f'(x) = 6x^2 - 18x + 12$

$f'(x) = 0$

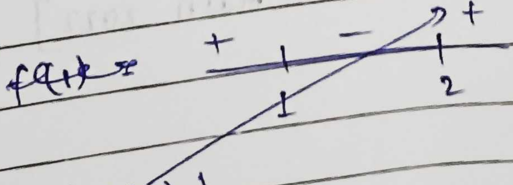
$6x^2 - 18x + 12 = 0$

$x^2 - 3x + 2 = 0$

$x^2 - x - 2x + 2 = 0$

$x(x-1) - 2(x-1) = 0$

$(x-1)(x-2) = 0$



Don't check local maxima and minima.

maxima → 1

Max^m value   checking boundary pt.

$f(1) = 11$ $f(0) = 6$

$f(2) = 10$

~~Min^m value~~

~~$\cancel{f(2) = 6}$~~

~~$f(2) = 10$~~

Min^m value

$f(0) = 6$

Q. Let $f(x) = 2x^3 - 9x^2 + 12x + 6$. Discuss the global maxima and global minima of $f(x)$ in $(1,3)$.

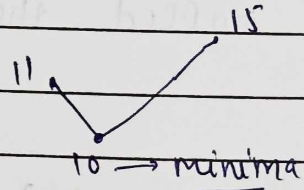
Sol^n:- $f'(x) = 0$

$(x-1)(x-2) = 0$

$f(1^+) = 11$

$f(2) = 10$

$f(3) = 15$



Global maxima does not exist.

$x = 2$ is point of Global minima in $(1,3)$

Min^m value $= 10$

Q. The sum of absolute max^m and absolute min^m value of the f^n $f(x) = |2x^2 + 3x - 2| + \sin x \cos x$ in the interval $[0, 1]$.

[JEE Main 2022]

Sol^n: $2x^2 + 3x - 2 = 0$

$(x+2)(2x-1) = 0$

$x = \frac{1}{2}, -2$

$$\begin{array}{ccccc} + & & - & & + \\ -2 & & 0 & & 1/2 \end{array}$$

$[0, \tfrac{1}{2}] \rightarrow$ -ve

$$f(x) = \begin{cases} -(2x^2+3x-2) + \sin x \cos x, & 0 \le x < 1/2 \\ 2x^2+3x-2 + \sin x \cos x, & 1/2 < x \le 1 \end{cases}$$

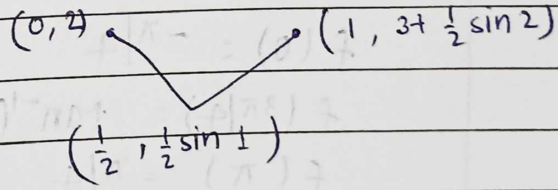
$$f'(x) = \begin{cases} -(4x+3) + \cos^2 x - \sin^2 x, & 0 < x < 1/2 \\ 4x+3 + \cos^2 x - \sin^2 x, & 1/2 < x < 1 \end{cases}$$

$$f'(x) = \begin{cases} -(4x+3) + \cos 2x, & 0 < x < 1/2 \rightarrow \text{always -ve} \downarrow\text{ing} \\ 4x+3 + \cos 2x, & 1/2 < x < 1 \rightarrow \text{always +ve} \uparrow\text{ing} \end{cases}$$

$f(0) = 2$

$f(1/2) = \frac{1}{2}\sin 1$

$f(1) = 3 + \frac{1}{2}\sin 2$



Min^m value $= \frac{1}{2}\sin 1$

Max^m value $= 3 + \frac{1}{2}\sin 2$

Sum

$= \frac{1}{2}\sin 1 + 3 + \frac{1}{2}\sin 2$

$= 3 + \frac{1}{2}(\sin 1 + \sin 2)$

Date:

Q. Sum of the absolute max^m and absolute min^m values of the function $f(x) = \tan^{-1}(\sin x - \cos x)$ in the interval $[0, \pi]$

Sol^n:

$$f(x) = \tan^{-1}(\sin x - \cos x)$$

$$f'(x) = \frac{1}{1 + (\sin x - \cos x)^2} \cdot \cos x + \sin x$$

$$f'(x) = \frac{\sin x + \cos x}{1 + \sin^2 x + \cos^2 x - \sin 2x}$$

$$f'(x) = \frac{\sin x + \cos x}{2 - \sin 2x}$$

$$f'(x) = 0$$

$$\sin x + \cos x = 0$$

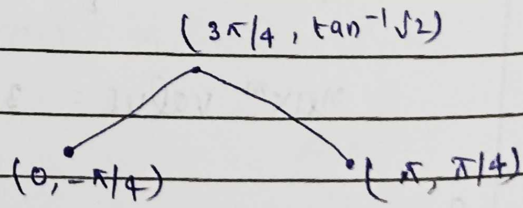
~~Max~~ ⇒

$$x = 3\pi/4$$

$$f(0) = -\pi/4$$

$$f(3\pi/4) = \tan^{-1}(\sqrt{2})$$

$$f(\pi) = \pi/4$$



Max^m value $= \tan^{-1}\sqrt{2}$

Min^m value $= -\pi/4$

Sum $= \tan^{-1}\sqrt{2} - \frac{\pi}{4}$

M	T	W	T	F	S	S

Page No :
Date
YOUVA

Q. The local max^m value of the f^n $f(x) = \left(\frac{2}{x}\right)^{x^2}$, $x > 0$ is

[JEE Mains 2021]

Sol^n:

$$f(x) = \left(\frac{2}{x}\right)^{x^2}$$

~~Take~~ $y = \left(\frac{2}{x}\right)^{x^2}$

$$\ln y = x^2 . \ln \frac{2}{x}$$

$$\frac{1}{y} . \frac{dy}{dx} = 2x \ln \frac{2}{x} + x^2 . \frac{x}{2} . \left(-\frac{2}{x^2}\right)$$

$$\frac{dy}{dx} = \left(\frac{2}{x}\right)^{x^2} \left[2x \ln \left(\frac{2}{x}\right) - x\right]$$

$$\frac{dy}{dx} = 0$$

$$\left(\frac{2}{x}\right)^{x^2} . \left[2x \ln \left(\frac{2}{x}\right) - x\right] = 0$$

$$2x \ln \left(\frac{2}{x}\right) - x = 0$$

$$2 \ln \frac{2}{x} = 1$$

$$\ln \frac{2}{x} = \frac{1}{2}$$

$$\frac{2}{x} = e^{1/2}$$

$$\boxed{x = \frac{2}{e^{1/2}}}$$

$$\boxed{x = 2e^{-1/2}}$$

$$f(2e^{-1/2}) = \left(\frac{\cancel{2}}{\cancel{2}e^{-1/2}}\right)^{(2e^{-1/2})^2}$$

$$= \left(e^{1/2}\right)^{4e^{-1}}$$

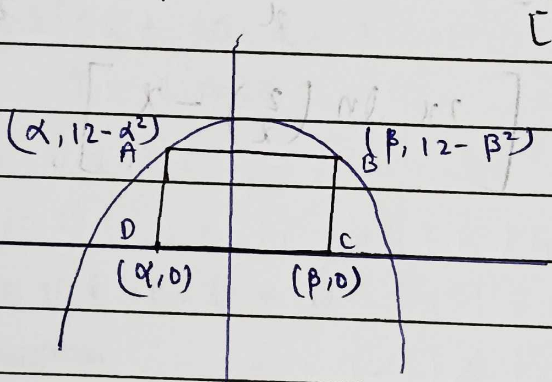
$$= e^{2e^{-1}}$$

$$= e^{2/e}$$

Q. The max^m area (in sq. units) of a rectangle having its base on the x-axis and other two vertices on the parabola $y = 12 - x^2$ such that rectangle lies inside the parabola.

[JEE Main 2019]

Sol^n:-



ABCD is a rectangle

$\alpha = \beta$

$l = 2\alpha$

$b = 12 - \alpha^2$

$A = l \times b$

$A = 2\alpha(12 - \alpha^2)$

$A = 24\alpha - 2\alpha^3$

$$\frac{dA}{d\alpha} = 24 - 6\alpha^2$$

$$\frac{dA}{d\alpha} = 0$$

M T W T F S
Page No.:
Date:
YOU

$24 - 6\alpha^2 = 0$

$\alpha = \pm 2$

$\dfrac{d^2A}{d\alpha^2} = -12\alpha$

$\alpha = 2$

$\dfrac{d^2A}{d\alpha^2} = -ve$

$\alpha = 2 \rightarrow$ pt. of maxima

Max^m area $= 48 - 16$

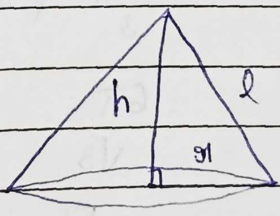
$= 32$

Q. The max^m volume of the right circular cone having slant height 3m is

[JEE Main 2019]

Sol^n:-

$V = \dfrac{1}{3}\pi r^2 h$



$l^2 = h^2 + r^2$

$9 = h^2 + r^2$

$r^2 = 9 - h^2$

$V = \dfrac{1}{3}\pi(9 - h^2)h$

$\dfrac{dV}{dh} = \dfrac{1}{3}\pi\left[9 - h^2 + h(-2h)\right]$

$= \dfrac{1}{3}\pi\left[9 - h^2 - 2h^2\right]$

$= \dfrac{1}{3}\pi\left[9 - 3h^2\right]$

M T W T F S S
Page No :
Date
YOUVA

$$\frac{dV}{dh} = \frac{1}{\cancel{3}}\pi \times \cancel{3}(3-h^2)$$

$$= \pi(3-h^2)$$

$$\frac{dV}{dh} = 0$$

$$h^2 = 3$$

$$h = \pm\sqrt{3}$$

$$\frac{d^2V}{dh^2} = \pi(-2h)$$

Pt. of $max^m = \sqrt{3}$

max^m volume

$$V = \frac{1}{3} \times \pi(9-h^2)h$$

$$= \frac{1}{3}\pi(9-3)\sqrt{3}$$

$$= \frac{6\pi}{\sqrt{3}}$$

$$V_{max.} = 2\sqrt{3}\pi$$

Q. Let $f(x) = x^2 + \frac{1}{x^2}$, $g(x) = x - \frac{1}{x}$, $x \in R - \{-1, 0, 1\}$

If $h(x) = \frac{f(x)}{g(x)}$, then the local min^m value of $h(x)$.

[JEE Main 2018]

Solⁿ:-

$$f(x) = x^2 + \frac{1}{x^2}$$

$$= \left(x - \frac{1}{x}\right)^2 + 2$$

$$f(x) = \left(x - \frac{1}{x}\right)^2 + 2$$

$$\cancel{h(x) = \frac{f(x)}{g(x)}}$$

$$h(x) = \frac{f(x)}{g(x)}$$

$$h'(x) = \frac{f'(x)\, g(x) - f(x) \cdot g'(x)}{(g(x))^2} = 0$$

$$2\left(x - \frac{1}{x}\right) \cdot \left(1 + \frac{1}{x^2}\right) - \left[\left(x - \frac{1}{x}\right)^2 + 2\right] \cdot \left(1 + \frac{1}{x^2}\right) = 0$$

$$\left(1 + \frac{1}{x^2}\right)\left[2\left(x - \frac{1}{x}\right) - \left(x - \frac{1}{x}\right)^2 - 2\right] = 0$$

$$h(x) = \frac{\left(x - \frac{1}{x}\right)^2 + 2}{\left(x - \frac{1}{x}\right)}$$

$$h(x) = \underbrace{\left(x - \frac{1}{x}\right)}_{a} + \frac{2}{\left(x - \frac{1}{x}\right)}$$

$$= a + \frac{2}{a}$$

AM $\geq$ GM

$$\frac{a + \frac{2}{a}}{2} \geq \left(a \cdot \frac{2}{a}\right)^{1/2}$$

Page No.:
Date

$$a + \frac{2}{a} \geq 2\sqrt{2}$$

$$\left(x-\frac{1}{x}\right) + \frac{2}{\left(x-\frac{1}{x}\right)} \geq 2\sqrt{2}$$

$$h(x) \geq 2\sqrt{2}$$

min^m value of $h(x) = 2\sqrt{2}$

Q. The max^m value of the f^n $f(x) = 3x^3 - 18x^2 + 27x - 4$ on the set $S = \{x \in R : x^2 + 30 \leq 11x\}$ is:

[JEE Main 2019]

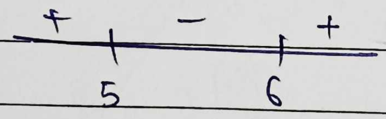
Sol^n:-

$$x^2 - 11x + 30 \leq 0$$

$$x^2 - 5x - 6x + 30 \leq 0$$

$$x(x-5) - 6(x-5) \leq 0$$

$$(x-6)(x-5) \leq 0$$



$$x \in [5, 6]$$

$$f(x) = 3x^3 - 18x^2 + 27x - 40$$

$$f'(x) = 9x^2 - 36x + 27 = 0$$

$$x^2 - 4x + 3 = 0$$

$$x^2 - x - 3x + 3 = 0$$

$$x(x-1) - 3(x-1) = 0$$

$$(x-1)(x-3) = 0$$

$$\underbrace{1 \quad 3}_{\text{Not in domain.}}$$

M T W T F S S
Page No.:
Date:
YOUVA

$f(1) = -3 - 18 + 27 - 40$
$= -28$

$f(3) = 81 - 162 + 81$

$$\begin{array}{r} 36 \\ \times 18 \\ \hline 288 \\ 360 \\ \hline 648 \end{array}$$

$f(5) = 125 \times 3 - 18 \times 25 + 25 \times 5 - 40$
$= 375 - 450 + 135 - 40$
$= 500 - 490$
$= 20$

$f(6) = 3 \times 216 - 18 \times 36 + 27 \times 6 - 40$
$= 648 - 648 + 162 - 40$
$= 122$

Max^m value $= 122$

Q. The largest term in the sequence $a_n = \frac{n^2}{n^3 + 200}$ is given by:

Sol^n:-

$a_n = \frac{n^2}{n^3 + 200}$

$\frac{da_n}{dn} = \frac{(n^3 + 200)\,2n - n^2 \cdot 3n^2}{(n^3 + 200)^2} = 0$

$n(2n^3 + 400 - 3n^3) = 0$

$n(400 - n^3) = 0$

Q. $\dfrac{x}{1+x\tan x}$ is maxima at

Sol^n:- $f'(x) = \dfrac{1+\cancel{x\tan x} - x(\cancel{\tan x}+x\sec^2 x)}{(1+x\tan x)^2} = 0$

$\Rightarrow 1 - x^2\sec^2 x = 0$

$x^2\sec^2 x = 1$

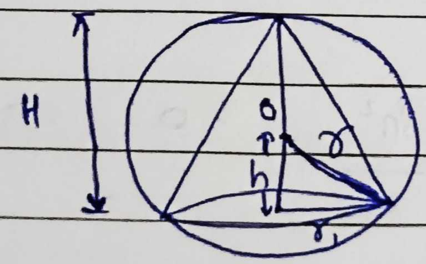
$x\sec x = \pm 1$

$x = \pm \dfrac{1}{\sec x}$

$\boxed{x = \pm\cos x}$

Q. The ratio of height of cone of max^m volume inscribed in a sphere to its radius is:-

Sol^n:



O = centre of sphere

r = radius of sphere.

$H = r + h$

$r^2 = r_1^2 + h^2$

$r_1^2 = r^2 - h^2$

Page No.:
Date:
YOUVA

$V = \frac{1}{3}\pi r_1^2 H$

$V = \frac{1}{3}\pi (r^2 - h^2)(r + h)$

$r = \text{constant}$

$\frac{dV}{dh} = \frac{\pi}{3}\left[-2h(r+h) + r^2 - h^2\right]$

$= \frac{\pi}{3}\left[-2rh - 2h^2 + r^2 - h^2\right]$

$= \frac{\pi}{3}\left[r^2 - 2rh - 3h^2\right]$

$\frac{dV}{dh} = 0$

$r^2 - 2rh - 3h^2 = 0$

$r^2 + rh - 3rh - 3h^2 = 0$

$r(r+h) - 3h(r+h) = 0$

$(r+h)(r-3h) = 0$

$\cancel{r = -h}$ $\quad r = 3h$

$H = r + h$

$H = 3h + h$

$H = 4h$

Hence,

$\frac{H}{r} = \frac{4h}{3h} = \frac{4}{3}$

Date

Q. Min^m value of $2^{\sin x} + 2^{\cos x}$.

[JEE Main 2020]

~~Solⁿ:~~

~~AM $\geq$ GM~~

~~$\frac{2^{\sin x} + 2^{\cos x}}{2} \geq \left(2^{\sin x + \cos x}\right)$~~

Solⁿ:-

$$f(x) = 2^{\sin x} + 2^{\cos x}$$

$$f'(x) = 2^{\sin x} \cdot \ln 2 \cdot \cos x - 2^{\cos x} \ln 2 \cdot \sin x$$

$$f'(x) = \ln 2 \left(2^{\sin x} \cos x - 2^{\cos x} \cdot \sin x\right) = 0$$

$$2^{\sin x} \cos x = 2^{\cos x} \cdot \sin x$$

$$2^{\sin x - \cos x} = \tan x$$

$\boxed{x = \pi/4}$ or $\boxed{x = \pi + \pi/4}$

$$f(\pi/4) = 2^{1/\sqrt{2}} + 2^{1/\sqrt{2}} = 2^{1+\frac{1}{\sqrt{2}}} \rightarrow \text{Max}^m$$

~~f(π+~~ $f(5\pi/4) = 2^{-1/\sqrt{2}} + 2^{-1/\sqrt{2}} = 2^{1-\frac{1}{\sqrt{2}}} \rightarrow \text{Min}^m$

Q. Let x, y be positive real numbers and m, n positive integers, the max^m value of the expression $\frac{x^m \cdot y^n}{(1+x^{2m})(1+y^{2n})}$.

[JEE Main 2019]

Solⁿ:-

~~$\frac{1 + x^m + x^m}{3} \geq$~~

AM $\geq$ GM

$$\frac{1 + x^{2m}}{2} \geq \left(x^{2m}\right)^{1/2}$$

$$1+x^{2m} \geq 2x^m \qquad \frac{x^m}{1+x^{2m}} \leq \frac{1}{2}$$

↳ Max. value

$$\frac{1+y^{2n}}{2} \geq (y^{2n})^{1/2}$$

$$1+y^{2n} \geq 2y^n \qquad \frac{y^n}{1+y^{2n}} \leq \frac{1}{2}$$

↳ Max. value

$$\frac{x^m \cdot y^n}{(1+x^{2m})(1+y^{2n})}$$

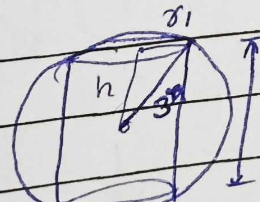
Max^m

$$\frac{(1+x^{2m}) \cdot (1+y^{2n})}{2 \cdot 2 \cdot (1+x^{2m})(1+y^{2n})}$$

Max^m value $= \frac{1}{4}$

Q. The height of the right circular cylinder of max^m volume inscribed in a sphere of radius 3 is :- [JEE Main 2019]

sol^n:-



$$r_1^2 + h^2 = 9$$
$$r_1^2 = 9 - h^2$$

$$H = 2h$$

$$V = \pi r_1^2 H$$
$$V = \pi \cdot (9-h^2)\, 2h$$
$$V = 18\pi h - 2\pi h^3$$

$$\frac{dV}{dh} = 18\pi - 6\pi h^2$$

$$18\pi - 6\pi h^2 = 0$$

$$\cancel{6\pi} h^2 = \cancel{18}^{3}\cancel{\pi}$$

$$h = \sqrt{3}$$

$$H = 2h$$

$$\boxed{H = 2\sqrt{3}}$$

Q. A helicopter is flying along the curve given by $y - x^{3/2} = 7$, $(x \geq 0)$. A soldier positioned at the point $\left(\frac{1}{2}, 7\right)$ wants to shoot down the helicopter when it is nearest to him. Then this nearest distance is.

[JEE Main 2019]

Sol^n:

$$y - x^{3/2} = 7$$

$$\frac{dy}{dx} - \frac{3}{2} x^{1/2} = 0$$

$$\frac{dy}{dx} = \frac{3}{2}\sqrt{x}$$

$x = x_1$, $y = x_1^{3/2} + 7$

$(x_1, x_1^{3/2} + 7)$ $(1/2, 7)$

$$m_2 = \frac{x_1^{3/2}}{x_1 - \frac{1}{2}}$$

$$m_1 m_2 = -1$$

$$\left(\frac{3}{\cancel{2}}\sqrt{x_1}\right)\left(\frac{\cancel{2}x_1^{3/2}}{2x_1-1}\right) = -1$$

$$\frac{3x_1^{1/2} \cdot x_1^{3/2}}{2x_1-1} = -1$$

$$3x_1^2 = -2x_1+1$$

$$3x_1^2+2x_1-1=0$$

$$3x_1^2+3x_1-x_1-1=0$$

$$3x_1(x_1+1)-1(x_1+1)=0$$

$$(3x_1-1)(x_1+1)=0 \qquad \{x \geq 0\}$$

$$x_1 = \frac{1}{3}, \quad y_1 = \left(\frac{1}{3}\right)^{3/2}+7$$

$$A\left(\frac{1}{3}, \left(\frac{1}{3}\right)^{3/2}+7\right) \qquad B\left(\frac{1}{2}, 7\right)$$

$$AB = \sqrt{\left(\frac{1}{3}-\frac{1}{2}\right)^2 + \left(\left(\frac{1}{3}\right)^{3/2}-\frac{1}{2}\right)^2}$$

$$AB = \sqrt{\frac{1}{36} + \left(\frac{1}{3}\right)^3 + \left(\frac{1}{2}\right)^2 - \left(\frac{1}{3}\right)^{3/2}}$$

$$AB = \sqrt{\frac{1}{36} + \frac{1}{27} + \frac{1}{4} - \frac{1}{3\sqrt{3}}}$$

$$AB = \sqrt{}$$

$$AB = \sqrt{\frac{1}{36} + \frac{1}{27}}$$

$$AB = \sqrt{\frac{3+4}{108}}$$

$$AB = \sqrt{\frac{7}{36 \times 3}}$$

$$AB = \frac{1}{6}\sqrt{\frac{7}{3}}$$

Q. The shortest distance b/w the point $\left(\frac{3}{2}, 0\right)$ and the curve $y = \sqrt{x}$ $(x > 0)$, is

[JEE Main 2019]

solⁿ:-

$$y = \sqrt{x}$$

$$\frac{dy}{dx} = \frac{1}{2\sqrt{x}}$$

$$m_1 = \frac{1}{2\sqrt{x_1}}$$

$$(x_1, \sqrt{x_1}) \qquad \left(\frac{3}{2}, 0\right)$$

$$m_2 = \frac{\sqrt{x_1}}{x_1 - \frac{3}{2}}$$

$$m_2 = \frac{2\sqrt{x_1}}{2x_1 - 3}$$

$$m_1 m_2 = -1$$

$$\frac{1}{\cancel{2\sqrt{x_1}}} \cdot \frac{\cancel{2\sqrt{x_1}}}{2x_1 - 3} = -1$$

$1 = -2x_1 + 3$

$2x_1 - 3 + 1 = 0$

$x_1 = 1$

$y_1 = 1$

$(1, 1) \qquad \left(\frac{3}{2}, 0\right)$

$$AB = \sqrt{\left(\frac{3}{2}-1\right)^2 + (0-1)^2}$$

$$= \sqrt{\frac{1}{4} + 1}$$

$$= \frac{\sqrt{5}}{2}$$

Important Note:-

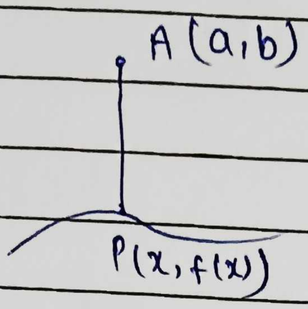
If the sum of two real number x ~~and~~ and y is constant, then their product is maximum when they are equal.

If product of two +ve number is constant, then their sum is least if they are equal.

Least / Greatest Distance b/w two curves :-

Least (Greatest) distance b/w two non-intersecting curves always lies along common normal (whenever defined)

Given a fixed point A (a, b) and moving point P (x, f(x)) on curve y = f(x), then AP will be max^m / min^m if it is normal to curve at P.



$AP^2 = F(x) = (x-a)^2 + (f(x)-b)^2$

$F'(x) = 2(x-a) + 2(f(x)-b).\ f'(x) = 0$

$$f'(x) = \frac{a-x}{f(x)-b}$$

$$m_{AP} = \frac{f(x)-b}{x-a}$$

$$\boxed{f'(x) \cdot m_{AP} = -1}$$

Q. Find co-ordinate of points on the curve $x^2 = 4y$ which is at least distance from line $y = x - 4$.

Sol^n:

$y = x - 4$

$m = 1$

slope of normal $= -1$

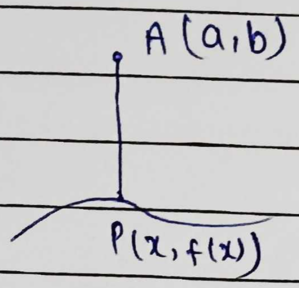
Normal to straight line is common to curve.

$x^2 = 4y$

$2x = 4\dfrac{dy}{dx}$

$$\frac{dy}{dx} = \frac{x}{2}$$

will be max^m / min^m if it is normal to curve at P.



$$AP^2 = F(x) = (x-a)^2 + (f(x)-b)^2$$

$$F'(x) = 2(x-a) + 2(f(x)-b) \cdot f'(x) = 0$$

$$f'(x) = \frac{a-x}{f(x)-b}$$

$$m_{AP} = \frac{f(x)-b}{x-a}$$

$$\boxed{f'(x) \cdot m_{AP} = -1}$$

Q. Find co-ordinate of points on the curve $x^2 = 4y$ which is at least distance from line $y = x - 4$.

Sol^n:

$$y = x - 4$$

$$m = 1$$

slope of normal $= -1$

Normal to straight line is common to curve.

$$x^2 = 4y$$

$$2x = 4\frac{dy}{dx}$$

$$\frac{dy}{dx} = \frac{x}{2}$$

$y' \cdot m_1 = -1$

$\frac{x}{2} \times (+1) = +1$

$\boxed{x = 2}$

$\boxed{y = 1}$

co-ordinate $(2, 1)$

Some special points on a curve :-

(i) critical points :-

Points on domain for which $f'(x) = 0$ or does not exist are called critical points.

(ii) Stationary Points :-

Stationary Points are points on domain where $f'(x) = 0$.

Note :- Every stationary point is a critical point but converse is not true.

(iii) Point of Inflection :-

A point where the graph of a function has a tangent line and where the concavity changes is called a point of inflection.

If function $y = f(x)$ is double differentiable, then point at which $\frac{d^2y}{dx^2} = 0$ and changes its sign is the point of inflection.

Note:-

If at any point $\frac{d^2y}{dx^2}$ does not exist but sign of $\frac{d^2y}{dx^2}$ changes about the point, then it is also called Point of inflection.

Q. Find critical points and stationary point of $f(x) = (x-2)^{2/3}(2x+1)$

Sol^n:- $f(x) = (x-2)^{2/3}(2x+1)$

$$f'(x) = \frac{2}{3}(x-2)^{-1/3}(2x+1) + 2(x-2)^{2/3}$$

$$f'(x) = 0$$

$$\frac{2}{3}(x-2)^{-1/3}(2x+1) + 2(x-2)^{2/3} = 0$$

$$2(x-2)^{-1/3}(2x+1) + 6(x-2)^{2/3} = 0$$

$$\frac{1}{(x-2)^{1/3}}(2x+1) + 3(x-2)^{2/3} = 0$$

$$\frac{2x+1+3(x-2)}{(x-2)^{1/3}} = 0$$

$$\frac{5x-5}{(x-2)^{1/3}} = 0$$

$$\frac{x-1}{(x-2)^{1/3}} = 0$$

At $x=1 \rightarrow f'(x)=0$

At $x=2 \rightarrow f'(x) = DNE$

stationary pt: $x=1$

Q. Point of inflection for curve $y=x^{5/3}$ is :

Sol^n:-

$y = x^{5/3}$

$\frac{dy}{dx} = \frac{5}{3} x^{2/3}$

At $x=0$:

$\frac{dy}{dx}$ = DNE

At $x=-1$

$dy/dx = -ve$

At $x=1$

$dy/dx = +ve$

So,

$x=0$ is the point of inflection.

point $\equiv (0,0)$

Q. Find inflection point of $3x^4 - 4x^3$.

Sol^n:-

$f(x) = 3x^4 - 4x^3$

$f'(x) = 12x^3 - 12x^2$

$f''(x) = 36x^2 - 24x$

$f''(x) = 0$

$24x = 36x^2$

$3x^2 - 2x = 0$

$x(3x-2) = 0$

Date:

$$\begin{array}{ccccc} + & & - & & + \\ \hline & 0 & & 2/3 & \end{array}$$

Hence,

Both are point of inflection.

Q. If $x=1$ is the critical point of $f(x)=(3x^2+ax-2-a)e^x$, then

[JEE 2008]

(a) $x=1$, $x=-2/3$ are local minima of f^n.

(b) $x=1$, $x=-2/3$ are local maxima of f^n.

(c) $x=1$ is local maxima and $x=-2/3$ is local minima

(d) $x=1$ is local minima and $x=-2/3$ is local maxima.

Soln: $f(x)=(3x^2+ax-2-a)e^x$

$$f'(x)=e^x(3x^2+ax-2-a)+e^x(6x+a)$$

$$f'(x)=0$$

$$\cancel{e^x}(3x^2+ax-2-a+6x+a)=0$$

$$3x^2+ax+6x-2=0$$

$x=1$ will satisfy this

$$3+a+6-2=0$$

$$\boxed{a=-7}$$

M T W T F S S
Page No.
Date
YOUVA

$$f(x) = (3x^2+7x-9)\, e^x$$

$$f'(x) = e^x(3x^2-x-2) = 0$$

$$3x^2 - x - 2 = 0$$

$$3x^2 - 3x + 2x - 2 = 0$$

$$3x(x-1) + 2(x-1) = 0$$

$$(x-1)(3x+2) = 0$$

$+$	$-$	$+$

$-2/3$ $\quad$ 1

maxima $\quad$ minima

Q. Find critical and stationary point of $f(x) = \dfrac{e^x}{x}$.

sol^n:-

$$f(x) = \frac{e^x}{x}$$

$$f'(x) = \frac{xe^x - e^x}{x^2} = 0$$

$$x \neq 0$$

$$e^x(x-1) = 0$$

$$x = 1$$

critical Point $= 0, 1$

stationary Point $= 1$

Page No.:

Date:

Q. A wire of length 22 m is to be cut into two pieces. One of the pieces is to be made into square and other into an equilateral Δ. Then, the length of the side of the equilateral Δ so that the combined area of the square and equi. Δ is $\min^m$, is?

[JEE Main 2022, 2021]

Sol^n:

22 m

x $\quad$ $22-x$

$$f(x) = (22-x)^2 + \frac{\sqrt{3}}{4} x^2$$

$$f'(x) = 2(22-x)\cdot(-1) + \frac{\sqrt{3}}{2}x = 0$$

$$\frac{\sqrt{3}}{2}x = 2(22-x)$$

$$\sqrt{3} = 4(22-x)$$

$$\sqrt{3} = 88-4x$$

$$\cancel{x(4+\sqrt{3}) = 88}$$

$$\cancel{x = \frac{88}{4+\sqrt{3}}}$$

$$\frac{\sqrt{3}}{2}x = 44-2x$$

$$\sqrt{3}x = 88-4x$$

$$x = \frac{88}{4+\sqrt{3}}$$

Sol^n:-

22m

Perimeter →	x	$22-x$
side	$x/3$	$\frac{22-x}{4}$

$$f(x) = \frac{(22-x)^2}{16} + \frac{\sqrt{3}}{4} \times \frac{x^2}{9}$$

Page No.:
Date

$$f'(x) = \frac{(22-x)(-1)}{8} + \frac{\sqrt{3}}{18}x = 0$$

$$\frac{\sqrt{3}}{\cancel{18}\,9}x = \frac{22-x}{\cancel{8}\,4}$$

$$4\sqrt{3}x = 198 - 9x$$

$$(4\sqrt{3}+9)x = 198$$

$$x = \frac{198}{9+4\sqrt{3}}$$

side of equi. $\Delta = \frac{x}{3} = \frac{66}{9+4\sqrt{3}}$

Q. Let the eccentricity of the ellipse $x^2 + a^2y^2 = 25a^2$ be 'b' times the eccentricity of the hyperbola $x^2 - a^2y^2 = 5$ where a is the min^m distance b/w the curves $y = e^x$ and $y = \ln x$, then $a^2 + \frac{1}{b^2}$ is :

[JEE Main 2022]

Sol^n:

$y = e^x$

$\left(\frac{dy}{dx}\right)_{c_1} = e^x$

$y = \ln x$

$\left(\frac{dy}{dx}\right)_{c_2} = \frac{1}{x}$

$$\frac{e^x}{x} = -1$$

$$e^x = -x$$

$$x^2 + a^2y^2 = 25a^2$$

$$\frac{x^2}{25a^2} + \frac{y^2}{25} = \cancel{25}\,1$$

$e_1^2 = 1 - \frac{1}{a^2}$

$e_1^2 = \frac{a^2-1}{a^2}$

$\frac{5}{2}$

$\frac{x^2}{5} - \frac{a^2y^2}{5} = 1$

$e_2^2 = 1 + a^2$

$\sqrt{\frac{a^2-1}{a^2}} = b\sqrt{(1+a^2)}$

$b = \sqrt{\frac{a^2-1}{a^2(a^2+1)}}$

$e_1^2 = 1 - \frac{25}{25a^2}$

$e_1^2 = 1 - \frac{1}{a^2}$

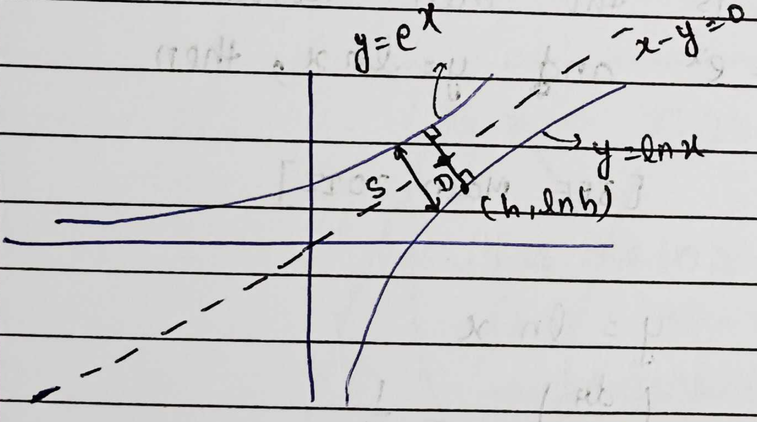
$e_2^2 = 1 + \frac{1}{a^2}$

$e_1 = b e_2$

$e_1^2 = b^2 e_2^2$

$\frac{a^2-1}{\cancel{a^2}} = b^2 \cdot \frac{a^2+1}{\cancel{a^2}}$

$b^2 = \frac{a^2-1}{a^2+1}$



$D = \frac{h - \ln h}{\sqrt{2}}$

$S = 2D = \sqrt{2}(h - \ln h)$

$\frac{dS}{dh} = \sqrt{2}\left(1 - \frac{1}{h}\right) = 0$

$\boxed{h=1}$

$a = S = \sqrt{2}$

$b^2 = \frac{1}{3}$

$a^2 + \frac{1}{b^2} = 2 + 3 = 5$

M T W T F S S
Page No.:
Date:
YOUVA

Q. The sum of the absolute max^m values of the absolute min^m and the absolute in the interval $[-1,2]$ of the f^n $f(x) = |3x - x^2 + 2| - x$ is.

Sol^n:-

~~$3x - x^2 + 2$~~

~~$-x^2 + 3x + 2$~~

~~$f(-1) = |-3-1| + 2 = 6 + 1 = 7$~~

$\cancel{f(2)} = 2 + \cancel{2} = 4 - \cancel{2} = 2$

$f(x) = |3x - x^2 + 2| - x$

$3x - x^2 + 2 = 0$

$x^2 - 3x - 2 = 0$

$x = \dfrac{3 \pm \sqrt{17}}{2}$

$x = \dfrac{3 + \sqrt{17}}{2}$ $\longrightarrow$ Not in domain

$x = \dfrac{3 - \sqrt{17}}{2}$

$+ \quad - \quad +$

$\dfrac{3-\sqrt{17}}{2} \qquad \dfrac{3+\sqrt{17}}{2}$

~~$-1 < x < \dfrac{3 - \sqrt{17}}{2}$~~

$$f(x) = \begin{cases} x^2 - 4x - 2, & -1 \leq x < \dfrac{3 - \sqrt{17}}{2} \\ 2x - x^2 + 2, & \dfrac{3 - \sqrt{17}}{2} < x \leq 2 \end{cases}$$

$$f'(x) = \begin{cases} 2x - 4, & -1 < x < \dfrac{3 - \sqrt{17}}{2} \\ 2 - 2x, & \dfrac{3 - \sqrt{17}}{2} < x < 2 \end{cases}$$

Date:

$f(-1) = 3 \longrightarrow$ Global max^m.

$f(1) = 3$

$f(2) = 2$

$f\left(\frac{3-\sqrt{17}}{2}\right) = \frac{\sqrt{17}-3}{2} \longrightarrow$ Global min^m.

$$\frac{\sqrt{17}-3}{2} + 3 = \frac{\sqrt{17}+3}{2}$$

Q. The sum of the max^m and min^m values of the f^n $f(x) = |5x-7| + [x^2+2x]$ in the interval $\left[\frac{5}{4}, 2\right]$, where $[\cdot] \to$ GIF.

Sol^n:-

$5x - 7 = 0$

$x = 7/5$

$- \quad | \quad +$

$\quad 7/5$

$$f(x) = \begin{cases} -(5x-7) + [x^2+2x] & , \ 5/4 \leq x < 7/5 \\ 5x-7 + [x^2+2x] & , \ 7/5 \leq x \leq 2 \end{cases}$$

$$f(x) = \begin{cases} -5 \\ 5 \end{cases}$$

$$f(5/4) = \left|\frac{25}{4} - 7\right| + \left[\frac{25}{16} + \frac{10}{4}\right]$$

$$= \frac{3}{4} + 4$$

$$f(5/4) = \frac{19}{4}$$

Page No.:
Date:

$f(7/5) = 0 + \left[\frac{49}{25} + \frac{14}{5}\right]$

$= 4 \rightarrow$ Global min^m $\{f'[x] = 0\}$

$f(2) = 3 + [4 + 4]$

$= 11 \rightarrow$ Global max^m.

Sum of Max^m and $Min^m = 11 + 4 = 15$