Matrices and Determinants

Matrix

A set of mn numbers (real or complex) arranged in the form of rectangular array having m rows and n columns is called a matrix of order $m \times n$ (read as m by n matrix)

An $m \times n$ matrix is usually written as

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{bmatrix}_{m \times n}$$

In compact form, the above matrix is represented by $[a_{ij}]_{m \times n}$
where symbol a_{ij} represent any number [a_{ij} lies in the i^{th} row (from top) and j^{th} column (from left)]

Matrix is essentially an arrangement of elements and has no value.

Q. If a matrix has 12 elements, what are the possible order it can have.

Ans: Possible orders are
$1 \times 12,\ 2 \times 6,\ 3 \times 4,\ 4 \times 3,\ 6 \times 2,\ 12 \times 1$

Q. Construct a 2×3 matrix $A = [a_{ij}]$ whose elements are given by $a_{ij} = \dfrac{(i+2j)^2}{2}$.

Solⁿ:

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}_{2 \times 3}$$

$$a_{ij} = \frac{(i+2j)^2}{2}$$

$$A = \begin{bmatrix} 9/2 & 25/2 & 49/2 \\ 8 & 18 & 32 \end{bmatrix}$$

Types of Matrices

1. **Row Matrix** :- A matrix is said to be row matrix if it contains only one row i.e. a matrix $A = [a_{ij}]_{m \times n}$ is said to be row matrix if $m = 1$

eg, $A = [3\ 5\ 8]_{1 \times 3}$

2. Column Matrix:- A matrix is said to be column matrix if it contains only 1 column. i.e. a matrix $A = [a_{ij}]_{m \times n}$ is said to be column matrix if $n = 1$.

eg, $A = \begin{bmatrix} 1 \\ 2 \end{bmatrix}_{2 \times 1}$

3. Rectangular Matrix:- A matrix is said to be rectangular matrix if the number of rows and number of columns are not equal. i.e. a matrix $A = [a_{ij}]_{m \times n}$ is called a rectangular matrix iff $m \neq n$.

eg→ $A = \begin{bmatrix} 2 & 3 & 5 \\ 1 & 2 & 6 \end{bmatrix}$

4. Square Matrix:- A matrix is said to be a square matrix if the number of rows and number of columns are equal i.e. a matrix $A = [a_{ij}]_{m \times n}$ is called a square matrix iff $m = n$

eg→ $A = \begin{bmatrix} 2 & 5 \\ 9 & 7 \end{bmatrix}$

Note:-

If $A = [a_{ij}]$ is a square matrix of order n, then elements (entries), $a_{11}, a_{22}, a_{33}, \dots a_{nn}$ are said to constitute the diagonal of the matrix. The line along the diagonal elements lie is called principal diagonal.

eg→ $\left.\begin{array}{l} a_{11} = 2 \\ a_{22} = 7 \end{array}\right\}$ principal diagonal : 2, 7

5. **Diagonal Matrix** :- A square matrix is said to be diagonal matrix if all its non-diagonal elements are zero. Thus, $a = [a_{ij}]_{m \times m}$ is called a diagonal matrix ~~when~~ if $a_{ij} = 0$ when $i \neq j$.

eg. $A = [2]_{1\times 1}$

$$B = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}_{2\times 2}$$

$$C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}_{3\times 3}$$

A diagonal matrix of order n having $d_1, d_2, \ldots, d_n$ as diagonal elements may be denoted by ~~d~~ $\text{diag}(d_1, d_2, \ldots, d_n)$.

eg. $A = \text{diag}(2)$

$B = \text{diag}(1,2)$

$C = \text{diag}(1,2,3)$

Note:

* No element of principal diagonal in a diagonal matrix is zero.

* Minimum number of zero in a diagonal matrix is given by $n(n-1)$. where n is order of matrix.

6. <u>Scalar Matrix</u>:- A diagonal matrix is said to be a scalar matrix if its diagonal elements are equal. Thus, $a = [a_{ij}]_{m\times m}$ is called scalar matrix if $a_{ij} = \begin{cases} 0 & \text{, if } i \neq j \\ K & \text{, if } i = j \end{cases}$

eg, $A = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}_{2\times 2} = \text{diag}(2,2)$

$B = [2]_{1\times 1} = \text{diag}(2)$

7. <u>Unit or Identity Matrix</u>:- A diagonal matrix is said to be an identity matrix if its diagonal elements equal to 1. Thus, ~~a~~ $A = [a_{ij}]_{m\times m}$ is called unit matrix if

$$a_{ij} = \begin{cases} 0 & \text{, if } i \neq j \\ 1 & \text{, if } i = j \end{cases}$$

Note:

A unit matrix of order n is denoted by I_n or I.

eg, $I_1 = [1]$

$I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

8. **Singleton Matrix**:- A matrix is said to be singleton matrix if it has only 1 element i.e. $A = [a_{ij}]_{m \times n}$ is said to be singleton matrix if $m = n = 1$.

eg. $[2]_{1 \times 1}$

9. **Triangular Matrix**:- A square matrix is called a triangular matrix if its each element above or below principal diagonals is zero. It is of two types:

(a) **Upper Triangular Matrix**:- A square matrix in which all the elements below the principal diagonal are zero is called an upper triangular matrix i.e. $A = [a_{ij}]_{m \times n}$ is said to be an UTM if $a_{ij} = 0$ when $i > j$.

eg. $$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 5 & 6 \\ 0 & 0 & 9 \end{bmatrix}_{3 \times 3}$$

(b) **Lower Triangular Matrix**:- A square matrix in which all elements above the principal diagonal are zero is called a lower triangular matrix i.e. a matrix $A = [a_{ij}]_{m \times n}$ is said to be LTM if $a_{ij} = 0$ when $i < j$

eg. $$\begin{bmatrix} 1 & 0 & 0 \\ 6 & 9 & 0 \\ 9 & 9 & 9 \end{bmatrix}_{3 \times 3}$$

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Note:-

Minimum number of zeroes in a triangular matrix is given by $\frac{n(n-1)}{2}$ where n is order of matrix.

10. Horizontal Matrix:- A matrix is said to be a horizontal matrix if the number of rows is less than number of columns. i.e. a matrix $A = [a_{ij}]_{m \times n}$ is said to be horizontal matrix iff $m < n$

eg. $\begin{bmatrix} 2 & 6 & 3 \\ 9 & 9 & 9 \end{bmatrix}_{2 \times 3}$

11. Vertical Matrix:- A matrix is said to be a vertical matrix if the number of rows is greater than number of columns i.e. a matrix $A = [a_{ij}]_{m \times n}$ is said to be vertical matrix iff $m > n$.

eg. $\begin{bmatrix} 2 & 9 \\ 1 & 6 \\ 3 & 9 \end{bmatrix}_{3 \times 2}$

12. Null or Zero Matrix:- A matrix is said to be null matrix or zero matrix if all elements are zero i.e. a matrix $A = [a_{ij}]_{m \times n}$ is said to be null matrix if $a_{ij} = 0 \;\forall\; i, j$. It is denoted by O.

eg. $O_{2 \times 3} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

13. **Sub-Matrix** :: A matrix which is obtained from a given matrix by deleting any number of rows or number of columns is called a sub matrix of given matrix.

eg, $\begin{bmatrix} 3 & 4 \\ -2 & 5 \end{bmatrix}$ is a submatrix of $\begin{bmatrix} 1 & 2 & 3 \\ 9 & 3 & 4 \\ 6 & -2 & 5 \end{bmatrix}$

Trace of Matrix

The sum of all diagonal element of a square matrix $A = [a_{ij}]_{n \times n}$ is called the trace of matrix A. It is denoted by $Tr(A)$.

$$Tr(A) = \sum_{i=1}^{n} a_{ii} = a_{11} + a_{22} + \dots + a_{nn}$$

eg, $A = \begin{bmatrix} 1 & 2 \\ 3 & 5 \end{bmatrix}$

$Tr(A) = 1 + 5 = 6$

Properties of Trace of a Matrix ::

Let $A = [a_{ij}]_{n \times n}$, $B = [b_{ij}]_{n \times n}$ and K is a scalar then,

(i) $Tr(KA) = K \cdot Tr(A)$

(ii) $Tr(A \pm B) = Tr(A) \pm Tr(B)$

(iii) $Tr(AB) = Tr(BA)$

(iv) $Tr(A) = Tr(A')$

(v) $Tr(I_n) = n$

Equal Matrices

Two matrices are said to be equal, if

(i) They are of the same order, i.e., they have same number of rows and columns.

(ii) The elements in the corresponding position of the two matrices are equal.

Q. If $\begin{bmatrix} x+3 & 2y+x \\ z-1 & 4w-8 \end{bmatrix} = \begin{bmatrix} -x-1 & 0 \\ 3 & 2w \end{bmatrix}$

then find the value of $|x+y| + |z+w|$

Soln:-

$z-1 = 3$
$z = 4$

$x+3 = -x-1$
$2x = -4$
$x = -2$

$2y+x = 0$
$2y-2 = 0$
$y = 1$

$4w-8 = 2w$
$2w = 8$
$w = 4$

$|x+y| + |z+w| = |-1| + |8| = 9$

Q. If $\begin{bmatrix} 2\alpha+1 & 3\beta \\ 0 & \beta^2-5\beta \end{bmatrix} = \begin{bmatrix} \alpha+3 & \beta^2+2 \\ 0 & -6 \end{bmatrix}$

find the eqⁿ whose roots are α and β.

Solⁿ:
$$2\alpha+1 = \alpha+3$$
$$\boxed{\alpha = 2}$$

$$3\beta = \beta^2+2$$
$$\beta^2 - 3\beta + 2 = 0 \quad -(1)$$

$$\beta^2 - 5\beta = -6$$
$$\beta^2 - 5\beta + 6 = 0 \quad -(2)$$

$$\beta^2 - 3\beta + 2 = 0$$
$$-\ \beta^2 - 5\beta + 6 = 0$$
$$2\beta - 4 = 0$$

$$\boxed{\beta = 2}$$

$$x^2 - (\alpha+\beta)x + \alpha\beta = 0$$
$$x^2 - 4x + 4 = 0$$

Algebra of Matrices

(i) Addition :-

$A+B = [a_{ij} + b_{ij}]$ where A and B are of same order. obtained by adding the corresponding elements of A and B.

(a) Addition of matrices is commutative, i.e.

$$A + B = B + A$$

(b) Addition of matrices is associative, i.e.,

$$(A+B) + C = A + (B+C)$$

(c) Existence of Additive Identity, i.e. $A + O = O + A = A$
The null matrix O is the additive identity for matrix addition.

(d) Existence of Additive Inverse, i.e. $A + B = O = B + A$, B is called additive inverse of A. (O is null matrix).

(e) Cancellation laws: If A, B and C are matrices of the same order $m \times n$, then $A + B = A + C \Rightarrow B = C$
(Left cancellation law)

$B + A = C + A \Rightarrow B = C$ (Right cancellation law)

Scalar Multiplication

If $A = [a_{ij}]_{m \times n}$, then $KA = [Ka_{ij}]_{m \times n}$, where K is a scalar.

Properties of Scalar Multiplication:-

If $A = [a_{ij}]_{m \times n}$ and $B = [b_{ij}]_{m \times n}$ are two matrices and K, l are scalars, then (i) $K(A+B) = K(A) + K(B)$

(ii) $(K + l)A = KA + lA$

(iii) $(Kl)A = K(lA) = l(KA)$

Note:

If two matrices A and B are of same order, then only then addition and subtraction is possible. Then, these matrices are said to be conformable for addition and subtraction.

* Multiplication

Let $A = [a_{ij}]_{m \times n}$ and $B = [b_{ij}]_{p \times q}$, then the matrix multiplication AB is possible iff. $n = p$ and matrices are said to be conformable for multiplication.

Note:

In the product $AB = \begin{cases} A = \text{prefactor} \\ B = \text{post factor} \end{cases}$

eg. $A = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 2 & 3 & 4 \end{bmatrix}_{3 \times 3}$ and $B = \begin{bmatrix} 1 & -2 \\ -1 & 0 \\ 2 & -1 \end{bmatrix}_{3 \times 2}$

$n = p$

$$AB = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix}_{\underbrace{3 \times 2}_{m \times q}}$$

$a_{11} = I^{st}$ row (A) $\times$ I^{st} column (B)

$a_{12} = I^{st}$ row (A) $\times$ II^{nd} column (B)

$$AB = \begin{bmatrix} (0-1+4) & (0+0-2) \\ (1-2+6) & (-2+0-3) \\ (2-3+8) & (-4+0-4) \end{bmatrix}$$

$$AB = \begin{bmatrix} 3 & -2 \\ 5 & -5 \\ 7 & -8 \end{bmatrix}$$

Q. If $A = \begin{bmatrix} 1 & 2 \\ -2 & 3 \end{bmatrix}_{2\times2}$ and $B = \begin{bmatrix} 2 & 1 \\ 2 & 3 \end{bmatrix}_{2\times2}$ and $C \begin{bmatrix} -3 & 1 \\ 1 & 0 \end{bmatrix}_{2\times2}$.

Verify that (i) $(AB)C = A(BC)$

(ii) $A(B+C) = AB + AC$

Sol^n: (i) $AB = \begin{bmatrix} 6 & 7 \\ 2 & 7 \end{bmatrix}_{2\times2}$

$$(AB)C = \begin{bmatrix} -4 & 6 \\ 8 & 2 \end{bmatrix}_{2\times2}$$

$$BC = \begin{bmatrix} -4 & 2 \\ 0 & 2 \end{bmatrix}_{2\times2}$$

$$A(BC) = \begin{bmatrix} -4 & 6 \\ 8 & 2 \end{bmatrix}$$

$(AB)C = A(BC)$

Proved

Properties of Multiplication of Matrices:

(i) Multiplication of matrices is not commutative, i.e.,
$AB \neq BA$.

(ii) Matrix multiplication associative if conformability is assumed.

$$(AB)C = A(BC)$$

(iii) Matrix multiplication is distributive w.r.t. addition i.e., $A(B+C) = AB + AC$, whenever both sides of equality are defined.

(iv) If A is $m \times n$ matrix, then $I_m A = A = I_n A$

(v) If product of two matrices is a zero matrix, it is not necessary that one of the matrices is a zero matrix.

Note:

If A and B are two non-zero matrices such that $AB = 0$, then A and B are called the divisors of zero. Also, if $AB=0 \Rightarrow |AB| = 0 \Rightarrow |A|\,|B| = 0$
$\Rightarrow |A| = 0$ or $|B| = 0$ but not the converse.

$|\,|$ → Determinant.

(vi) Multiplication of a matrix A by a null matrix confirmable with A for multiplication.

eg. $A = \begin{bmatrix} 3 & 4 \\ 5 & 6 \\ 7 & 8 \end{bmatrix}_{3\times 2}$ $\quad B = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}_{2\times 3}$

$$AB = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}_{3\times 3}$$

(vii) Multiplication of a matrix by itself

→ Product of $A \cdot A$ m times $= A^m$
→ $(A^m)^n = A^{mn}$

Note:

Let I be unit matrix, then $I^2 = I^3 = I^4 \ldots = I^n = I$
$(n \in I^+)$

If A and B are two matrices of same order, then

(i) $(A+B)^2 = A^2 + AB + BA + B^2$

(ii) $(A-B)^2 = A^2 - AB - BA + B^2$

(iii) $(A-B)(A+B) = A^2 + AB - BA - B^2$

(iv) $(A+B)(A-B) = A^2 - AB + BA - B^2$

(v) $A(-B) = (-A)B = -AB$

Q. Find x so that $\begin{bmatrix} 1 & x & 1 \end{bmatrix}_{1\times 3} \begin{bmatrix} 1 & 3 & 2 \\ 0 & 5 & 1 \\ 0 & 3 & 2 \end{bmatrix}_{3\times 3} \begin{bmatrix} 1 \\ 1 \\ x \end{bmatrix}_{3\times 1} = 0$

Solⁿ:-

$$\begin{bmatrix} 1 & 5x+6 & x+4 \end{bmatrix}_{1\times 3} \begin{bmatrix} 1 \\ 1 \\ x \end{bmatrix}_{3\times 1} = 0$$

$$\begin{bmatrix} 1 + 5x + 6 + x(x+4) \end{bmatrix} = 0$$

$$1 + 5x + 6 + x^2 + 4x = 0$$

$$x^2 + 9x + 7 = 0$$

$$x = \frac{-9 \pm \sqrt{53}}{2}$$

Q. If $A = \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}$ and $A^2 - \lambda A - I_2 = 0$, then $\lambda = ?$

Solⁿ:- $A^2 = \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}$

$$A^2 = \begin{bmatrix} 5 & 8 \\ 8 & 13 \end{bmatrix}$$

$$\lambda A = \begin{bmatrix} \lambda & 2\lambda \\ 2\lambda & 3\lambda \end{bmatrix}$$

$$A^2 - \lambda A - I_2 = 0$$

$$\begin{bmatrix} 5 & 8 \\ 8 & 13 \end{bmatrix} - \begin{bmatrix} \lambda & 2\lambda \\ 2\lambda & 3\lambda \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = 0$$

$$5 - \lambda - 1 = 0$$
$$\boxed{\lambda = 4}$$

Verification → $8 - 2\lambda = 0$ $\boxed{\lambda = 4}$ $8 - 2\lambda = 0$ $\boxed{\lambda = 4}$

$$13 - 3\lambda - 1 = 0$$
$$\boxed{\lambda = 4}$$

Q. Assume X, Y, Z, W and P are matrices of order $2 \times n$, $3 \times K$, $2 \times p$, $n \times 3$ and $p \times K$ respectively. Choose the correct answer in the below question.

(i) The restriction on n, K and p so that $PY + WY$ will be defined are

(a) $K = 3, p = n$ (b) K is arbitary, $p = 2$

(c) p is arbitary, $K = 3$ (d) $K = 2, p = 3$

(ii) If $n = p$, then the order of the matrix $7X - 5Z$ is

(a) $p \times 2$ (b) $2 \times n$

(c) $n \times 3$ (d) $p \times n$

Soln :-
$X \to 2 \times n$

$Y \to 3 \times K$

$Z \to 2 \times p$

$W \to n \times 3$

$P \to p \times K$

(i) $PY + WY$

For definition of PY

$\boxed{K = 3}$ Order $\to p \times K$

For definition of WY

Order $\to n \times K$

Now,

Only same order can be added

$p \times K = n \times K$

$\boxed{p = n}$

(ii) $7X - 5Z$

$2 \times n = 2 \times p$

$\boxed{n = p}$

Order of $7X - 5Z \Rightarrow \begin{cases} 2 \times n \\ 2 \times p \end{cases}$

Important Types of Matrices:

1. **Idempotent matrix**:- A square matrix A is called idempotent provided it satisfy the relation

$A \quad A^2 = A$

Note:-

$A^n = A \quad \forall \quad n \geq 2, n \in N$

2. **Periodic Matrix**:- A square matrix A is called periodic if $A^{K+1} = A$ where $K \in I^+$. If K is the least positive integer for which $A^{K+1} = A$, then K is said to be period of A.

Note: For $K = 1$, we get $A^2 = A$ i.e. A is idempotent matrix. Period of an idempotent matrix is 1.

3. **Nilpotent Matrix**:- A square matrix A is called nilpotent matrix of order m provided it satisfies the relation, $A^K = O$ and $A^{K-1} \neq O$, where $K \in I^+$ and O is null matrix. and K is the order of the nilpotent matrix A.

4. **Involutory Matrix**:- A square matrix A is called involutory provided it satisfy the relation $A^2 = I$ where I is identity matrix.

Note:- $A = A^{-1}$ for involutory matrix.

Q. If $A = \begin{bmatrix} 1 & 0 \\ 1/2 & 1 \end{bmatrix}$, then $A^{100} = ?$

Soln:- $A^2 = \begin{bmatrix} 1 & 0 \\ 1/2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1/2 & 1 \end{bmatrix}$

$A^2 = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$

$A^3 = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1/2 & 1 \end{bmatrix}$

$A^3 = \begin{bmatrix} 1 & 0 \\ 3/2 & 1 \end{bmatrix}$

$\frac{1}{2}, 1, \frac{3}{2}, \ldots$

$a = 1/2 \qquad d = 1/2$

$n = 100$

$a_n = 1/2 + \frac{99}{2}$

$a_n = 50$

$A^{100} = \begin{bmatrix} 1 & 0 \\ 50 & 1 \end{bmatrix}$

Q. If $A = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$ and $A^2 = I$ is true for $\theta =$

~~(a)~~ 0 (b) $\pi/4$ (c) $\pi/2$ (d) None of these

Soln:-

Q.

Soln:-

Sol^n: $\cancel{A^2 = I}$ $\quad$ $\cancel{A = A^{-1}}$

$$A^2 = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

$$A^2 = \begin{bmatrix} \cos^2\theta - \sin^2\theta & 2\sin\theta\cos\theta \\ -\sin 2\theta & \cos^2\theta - \sin^2\theta \end{bmatrix} \qquad I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\hookrightarrow A^n = \begin{bmatrix} \cos n\theta & \sin n\theta \\ -\sin n\theta & \cos n\theta \end{bmatrix}$$

$A^2 = I$

$\sin 2\theta = 0$ $\qquad$ $\cos^2\theta - \sin^2\theta = 1$

$\theta = 0°$ $\qquad$ $\boxed{\theta = 0°}$

Q. If the product of n matrices

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix} \cdots \begin{bmatrix} 1 & n \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 378 \\ 0 & 1 \end{bmatrix}$$

then $n = ?$

$\cancel{1 \cdot 2 \cdot 3 \cdots n = 378}$

$\cancel{A^1 = 378}$

Sol^n:

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & \overset{1+2}{3} \\ 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & \overset{1+2+3}{6} \\ 0 & 1 \end{bmatrix}$$

$$\frac{n(n+1)}{2} = 378$$

$n(n+1) = 756$

$n(n+1) = 27 \times 28$

$\boxed{n = 27}$

Q. Let $A = \begin{bmatrix} 0 & \alpha \\ 0 & 0 \end{bmatrix}$ and $(A+I)^{50} - 50A = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$

then $A+B+C+D = ?$

Solⁿ:- $A = \begin{bmatrix} 0 & \alpha \\ 0 & 0 \end{bmatrix}$ $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$$A + I = \begin{bmatrix} 1 & \alpha \\ 0 & 1 \end{bmatrix}$$

$$(A+I)^2 = \begin{bmatrix} 1 & 2\alpha \\ 0 & 1 \end{bmatrix}$$

$$(A+I)^{50} = \begin{bmatrix} 1 & 50\alpha \\ 0 & 1 \end{bmatrix}$$

$$50A = \begin{bmatrix} 0 & 50\alpha \\ 0 & 0 \end{bmatrix}$$

$$(A+I)^{50} - 50A = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$$

$$\begin{bmatrix} 1 & 50\alpha \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 50\alpha \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$$

$$A + B + C + D = 2$$

Q. Let $A = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}$ $(\alpha \in R)$ such that $A^{32} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ then value of α is - [JEE Main 2019]

Sol^n:- $A^2 = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix} \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}$

$A^2 = \begin{bmatrix} \cos 2\alpha & -\sin 2\alpha \\ \sin 2\alpha & \cos 2\alpha \end{bmatrix}$

$A^{32} = \begin{bmatrix} \cos 32\alpha & -\sin 32\alpha \\ \sin 32\alpha & \cos 2\alpha \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

$\cos 32\alpha = \cos \pi/2$

$\boxed{\alpha = \frac{\pi}{64}}$

Q Let $P = \begin{bmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 16 & 4 & 1 \end{bmatrix}$ and I be the identity matrix of order three. If $Q = [q_{ij}]$ is a matrix such that $P^{50} - Q = I$, then $\frac{q_{31} + q_{32}}{q_{21}} = ?$ [JEE Advance 2016]

Sol^n:- $P^2 = \begin{bmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 16 & 4 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 16 & 4 & 1 \end{bmatrix}$

$P^2 = \begin{bmatrix} 1 & 0 & 0 \\ 8 & 1 & 0 \\ 48 & 8 & 1 \end{bmatrix}$

$$P^3 = \begin{bmatrix} 1 & 0 & 0 \\ 8 & 1 & 0 \\ 48 & 8 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 16 & 4 & 1 \end{bmatrix}$$

$$P^3 = \begin{bmatrix} 1 & 0 & 0 \\ 12 & 1 & 0 \\ 96 & 12 & 1 \end{bmatrix}$$

$\overset{\times 3}{16, 48}, 96, \ldots$ ($\times 6$)

$16 \times$ Sum of n terms

$16 \times \frac{n(n+1)}{2}$

$16 \times \frac{50 \times 51}{2}$

$= 20400$

$$P^{50} = \begin{bmatrix} 1 & 0 & 0 \\ 200 & 1 & 0 \\ 20400 & 200 & 1 \end{bmatrix}$$

$$P^{50} - Q = I$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 200 & 1 & 0 \\ 20400 & 200 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = Q$$

$$Q = \begin{bmatrix} 0 & 0 & 0 \\ 200 & 0 & 0 \\ 20400 & 200 & 0 \end{bmatrix}$$

$$q_{31} = 20400$$

$$\frac{q_{31} + q_{32}}{q_{21}} = \frac{20600}{200} = 103$$

Transpose of a Matrix

Let $A = [a_{ij}]_{m\times n}$ be any given matrix then the matrix obtained by interchanging the rows and columns of A is called the transpose of A. Transpose of matrix A is denoted by A' or A^T or A^t. In other words, if $A = [a_{ij}]_{m\times n}$, then $A' = [a_{ji}]_{n\times m}$

Properties of Transpose of a Matrices :-

If A' and B' denotes the transpose of A and B respectively, then

$(A')' = A$

$(A \pm B)' = A' \pm B'$

$(AB)' = B'A'$ $\longrightarrow$ { In general $(A_1 A_2 \dots A_n)' = A_n' \dots A_2' A_1'$ }
Reverse Law for Transpose

$(KA)' = KA'$ (k is scalar)

Note :- $I' = I$ (I = Identity matrix)

Symmetric Matrix

A square matrix $A = [a_{ij}]_{m\times n}$ is said to be symmetric if $A' = A$ i.e., $a_{ij} = a_{ji}$ $\forall$ i,j.

eg. $\begin{bmatrix} 1 & 2 \\ 2 & 9 \end{bmatrix}$ $\begin{bmatrix} 1 & 5 & 4 \\ 5 & 3 & 9 \\ 4 & 9 & 6 \end{bmatrix}$

Note:

Maximum number of distinct entries in any symmetric matrix of order n is $\frac{n(n+1)}{2}$.

For any symmetric matrix A with real number entries, then $A + A'$ is a symmetric matrix.

Proof: $(A+A')' = A' + (A')'$

$= A' + A$

$= A + A'$

$\because (A+A')' = (A+A')$

$\therefore A+A'$ is a symmetric matrix.

Skew Symmetric Matrix

A square matrix $A = [a_{ij}]_{n\times n}$ is said to be skew symmetric matrix, if $A' = -A$, i.e. $a_{ij} = -a_{ji}, \forall\, i,j$ (the pair of conjugate elements are additive inverse of each other).

If we put $i=j$, then

$$a_{ii} = -a_{ii}$$

$$2a_{ii} = 0$$

$$a_{ii} = 0$$

This means that all the diagonal elements of a skew symmetric matrix are zero, but not the converse.

eg $\begin{bmatrix} 0 & 5 & 4 \\ -5 & 0 & 6 \\ -4 & -6 & 0 \end{bmatrix}$

Note :

Trace of a skew symmetric matrix is 0.

For any square matrix A with real number entries, then $A - A'$ is a skew symmetric matrix

Proof: $(A-A')' = A' - (A')'$

$= A' - A$

$(A-A')' = -(A-A')$

$\therefore$ $A - A'$ is skew symmetric matrix.

Every square matrix can be uniquely expressed as sum of a symettric and a skew symmetric matrix.

$$A = \underbrace{\frac{1}{2}(A+A')}_{\text{Symmetric matrix}} + \underbrace{\frac{1}{2}(A-A')}_{\text{Skew symmetric Matrix}}$$

Q. Express A as the sum of symmetric and skew symmetric matrix, $A = \begin{bmatrix} 3 & 5 \\ -1 & 2 \end{bmatrix}$

Sol^n:

$$A' = \begin{bmatrix} 3 & -1 \\ 5 & 2 \end{bmatrix}$$

$$P = \frac{1}{2}(A + A')$$

$$P = \frac{1}{2}\left\{\begin{bmatrix} 3 & 5 \\ -1 & 2 \end{bmatrix} + \begin{bmatrix} 3 & -1 \\ 5 & 2 \end{bmatrix}\right\}$$

$$P = \frac{1}{2}\left\{\begin{bmatrix} 6 & 4 \\ 4 & 4 \end{bmatrix}\right\}$$

$$P = \begin{bmatrix} 3 & 2 \\ 2 & 2 \end{bmatrix}$$

$P = \frac{1}{2}(A + A')$ is a symmetrix matrix.

$$Q = \frac{1}{2}(A - A')$$

$$Q = \frac{1}{2}\left\{\begin{bmatrix} 3 & 5 \\ -1 & 2 \end{bmatrix} - \begin{bmatrix} 3 & -1 \\ 5 & 2 \end{bmatrix}\right\}$$

$$Q = \frac{1}{2}\left\{\begin{bmatrix} 0 & 6 \\ -6 & 0 \end{bmatrix}\right\}$$

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$$Q = \begin{bmatrix} 0 & 3 \\ -3 & 0 \end{bmatrix}$$

$Q = \frac{1}{2}(A - A')$ is skew symmetric matrix

$$P + Q = \begin{bmatrix} 3 & 5 \\ -1 & 2 \end{bmatrix}$$

$P + Q = A$

Hence,

A is represented as the sum of symmetric and unsymmetric matrix.

<u>Properties of Symmetric and skew symmetric Matrices:</u>

1. If A be a square matrix then AA' and $A'A$ are symmetric matrices.

2. All positive integral power of a symmetric matrix are symmetric, because $(A^n)' = (A')^n$

3. All positive odd integral powers of the skew symmetric matrix are skew symmetric and all positive even integral powers of a skew symmetric matrix is symmetric because $(A^n)' = (A')^n$.

4. If A be a symmetric matrix and B be square matrix of order that of A, then $-A$, KA, A', A^{-1}, A^n and $B'AB$ are also symmetric matrix where $n \in N$ and K is a scalar.

5. If A be a skew symmetric matrix, then

(a) $(A)^{2n}$ is symmetric matrix, $\forall n \in N$

(b) $(A)^{2n+1}$ is skew symmetric matrix, $\forall n \in N$

(c) $B'AB$ is skew symmetric matrix, where B is a square matrix of order same as of A.

6. If A and B are two symmetric matrices, then

(a) $A \pm B$, $AB + BA$ are symmetric matrices.

(b) $AB - BA$ are skew symmetric matrices.

(c) AB is a symmetric matrix iff. $AB = BA$

7. If A and B are two skew symmetric matrices, then

(a) $A \pm B$, $AB - BA$ are skew symmetric matrices

(b) $AB + BA$ is a symmetric matrix.

Proof: Property 6

$$\begin{aligned}(AB+BA)' &= (AB)' + (BA)' \\ &= B'A' + A'B' \\ &= BA + AB \qquad \{A' = A \quad B' = B\} \\ &= AB + BA\end{aligned}$$

$$\begin{aligned}(AB-BA)' &= (AB)' - (BA)' \\ &= B'A' - A'B' \\ &= BA - AB \\ &= -(AB - BA)\end{aligned}$$

$AB - BA$ is skew symmetric matrix

Orthogonal Matrix

A square matrix A is said to be a ~~square~~ orthogonal matrix iff $AA' = I$ where I is an identity matrix.

Note:

1. If $AA' = I \Rightarrow A' = A^{-1}$

2. If A and B are orthogonal, then AB is also orthogonal.

3. If A is orthogonal, then A^{-1} and A' is also orthogonal.

Proof: $AA' = I$

$BB' = I$

$$AB\,(AB)' = AB \cdot B'A'$$
$$= AIA'$$
$$= AA'$$
$$= I$$

Q. If $A = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ a & 2 & b \end{bmatrix}$ satisfying $AA' = 9I_3$.

find the value of $|a| + |b|$

Sol^n: $A' = \begin{bmatrix} 1 & 2 & a \\ 2 & 1 & 2 \\ 2 & -2 & b \end{bmatrix}$

Q. Let A and B be any two 3×3 symmetric and skew symmetric matrices, then which of the following is not true? [JEE Main 2022]

(a) $A^4 - B^4$ is symmetric matrix.
(b) $AB - BA$ is a symmetric matrix
(c) $B^5 - A^5$ is a skew symmetric matrix
(d) $AB + BA$ is a skew symmetric matrix

Sol^n: $A' = A$, $B' = -B$

$$\cancel{(AB)'} = (AB-BA)' = (AB)' - (BA)'$$
$$= B'A' - A'B'$$
$$= -BA + AB$$
$$= AB - BA$$
$$\downarrow$$
Symmetric matrix

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$$(AB+BA)' = (AB)' + (BA)'$$
$$= B'A' + A'B'$$
$$= -BA - AB$$
$$= -(BA+AB)$$
$$= -(AB+BA)$$
$$\downarrow$$

skew symmetric matrix.

$$(B^5 - A^5)' = (B^5)' - (A^5)'$$
$$= (B')^5 - (A')^5$$
$$= -B^5 - A^5$$
$$= -(B^5 + A^5)$$
$$\downarrow$$

Neither symmetric nor skew symmetric

Q. Let X and Y be two arbitary 3 × 3 non-zero skew symmetric matrices and Z be an arbitary 3 × 3 non-zero symmetric matrix. Then, which of the following matrices is/are skew symmetric.

[JEE Advance 2015]

(a) $Y^3Z^4 - Z^4Y^3$ (b) $X^{44} + Y^{44}$

~~(c)~~ $X^4Z^3 - Z^3X^4$ ~~(d)~~ $X^{23} + Y^{23}$

Soln. $X' = -X$, $Y' = -Y$, $Z' = Z$

(a) $(Y^3Z^4 - Z^4Y^3)' = (Y^3Z^4)' - (Z^4Y^3)'$
$$= (Z^4)'(Y^3)' - (Y^3)'(Z^4)'$$
$$= -Z^4Y^3 + Y^3Z^4$$
$$= \text{symmetric}$$

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(b) $x^{44}+y^{44}$ = even power $\Rightarrow$ symmetric.

(c) $(x^4z^3-z^3x^4)' = (x^4z^3)' - (z^3x^4)'$
$= (z^3)'(x^4)' - (x^4)'(z^3)'$
$= z^3x^4 - x^4z^3$
$= -(x^4z^3 - z^3x^4)$
= skew symmetric.

(d) $x^{23}+y^{23}$ = odd power
= skew symmetric.

Q. Let $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix}$, then $A^{2025} - A^{2020}$ is equal to

[JEE Main 2022]

(a) $A^6 - A$ (b) A^5 (c) $A^5 - A$ (d) A^6

Solⁿ. $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix}$

$A^2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix}$

$A^2 = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix}$

$$A^3 = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

$$A^3 = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

$$A^{2025} - A^{2020} = \begin{bmatrix} 1 & 0 & 0 \\ 2024 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 2019 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

$$A^{2025} - A^{2020} = \begin{bmatrix} 0 & 0 & 0 \\ 5 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Checking options:

$$A^6 - A = \begin{bmatrix} 1 & 0 & 0 \\ 5 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 5 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Determinant of Matrix:

Let A be a square matrix, then the determinant formed by the elements of A without changing their respective positions, is called the determinant of A and is denoted by $|A|$ or $det(A)$.

eg. If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

$$|A| = \begin{vmatrix} a & b \\ c & d \end{vmatrix}$$

Properties of the determinant of a matrix:

If A and B are square matrices of same order, then

(i) $|A|$ exist $\Leftrightarrow$ A is a square matrix.

(ii) $|A'| = |A|$

(iii) $|AB| = |A|\,|B|$ and $|AB| = |BA|$

(iv) If A is orthogonal matrix, then $|A| = \pm 1$

Proof: $AA^T = I$

$$|AA^T| = |I|$$

$$|A|\,|A^T| = 1$$

$$|A|\,|A| = 1$$

$$|A|^2 = 1$$

$|A| = \pm 1$

(v) If A is a skew symmetric matrix of odd order, then $|A| = 0$

(vi) If A is a skew symmetric matrix of even order, then $|A|$ is a perfect square.

(vii) $|KA| = K^n |A|$, where n is the order of matrix (A) and K is a scalar.

(viii) $|A^n| = |A|^n$

(ix) The determinant of diagonal matrix is the product of its elements.

$\rightarrow \quad A = \text{diag}\,(a_{11} \quad a_{22} \quad a_{33})$

$|A| = a_{11} \cdot a_{22} \cdot a_{33}$

Calculation of 2×2 matrix

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$|A| = ad - bc$$

eg. $A = \begin{bmatrix} 4 & 2 \\ 6 & 3 \end{bmatrix}$

$|A| = 12 - 12$

$= 0$

Q. $A = \begin{bmatrix} 1 & -1 & 5 \\ 2 & 4 & 3 \\ 5 & 0 & 3 \end{bmatrix}$ $\quad |A| = ?$

Soln:- $|A| = 1\begin{vmatrix} 4 & 3 \\ 0 & 3 \end{vmatrix} - (-1)\begin{vmatrix} 2 & 3 \\ 5 & 3 \end{vmatrix} + 5\begin{vmatrix} 2 & 4 \\ 5 & 0 \end{vmatrix}$

$= 1(12-0) + 1(6-15) + 5(0-20)$

$= \cancel{9+9-90}\ 12 - 9 - 100$

$= \cancel{90}\ -97$

Q. If A, B, C are square matrices of order n and $|A| = 2$, $|B| = 3$, $|C| = 5$, then find the value of $10 \det(A^3B^2C^{-1})$.

Soln:- $10|A^3B^2C^{-1}| = 10|A^3||B^2||C^{-1}|$

$= 10|A|^3|B|^2|C|^{-1}$

$= \cancel{10}^{2} \times 8 \times 9 \times \frac{1}{5}$

$= 144$

Singular and Non-singular matrices :-

A square matrix A is said to be singular if $|A| = 0$ and a square matrix A is said to be non-singular if $|A| \neq 0$

Adjoint of a Matrix :-

Let $A = [a_{ij}]$ be a square matrix of order n and C_{ij} be co-factor of a_{ij} in A, then the transpose of the matrix of co-factors of elements of A is called the adjoint of A and is denoted by adj (A).

$$\text{adj}(A) = [C_{ij}]'$$

$$(\text{adj}(A))_{ij} = [C_{ji}] : \text{co-factor of } a_{ji} \text{ in } A.$$

Minors and co-factors :-

Minors ↓ M_{ij}

co-factors ↓ C_{ij}

$$C_{ij} = (-1)^{i+j} M_{ij}$$

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

~~Eg~~ M_{11} = Minor of $a_{11} = \begin{vmatrix} a_{22} & a_{32} \\ a_{23} & a_{33} \end{vmatrix}$

$$= (a_{22} \cdot a_{33} - a_{23} \cdot a_{32})$$

Note:-

The adjoint of a square matrix of order 2 is obtained by interchanging the diagonal elements and changing signs of off diagonal elements

eg, $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

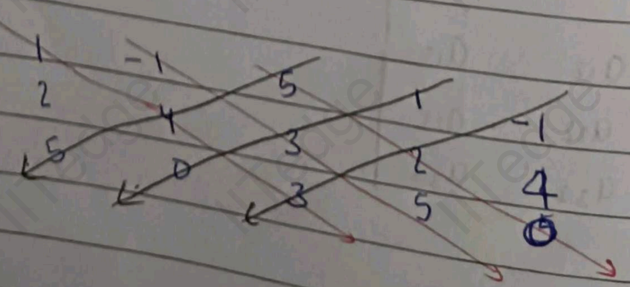
$$adj(A) = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Sarrus Diagram

↓

To calculate determinant

$$A = \begin{bmatrix} 1 & -1 & 5 \\ 2 & 4 & 3 \\ 5 & 0 & 3 \end{bmatrix}$$



$$|A| = (12-15+0) - (100+0-6)$$
$$= -3-94$$
$$= -97$$

Short-cut method to determine the adjoint of 3×3 matrix :-

$$A = \begin{bmatrix} 1 & -1 & 5 \\ 2 & 4 & 3 \\ 5 & 0 & 3 \end{bmatrix}$$

1	-1	5	1	-1
2	4	3	2	4
5	0	3	5	0
1	-1	5	1	-1
2	4	3	2	4

$$\text{adj } A = \begin{bmatrix} (4\times3-3\times0) & (0\times5-3\times-1) & (-1\times3-5\times4) \\ (3\times5-3\times2) & (3\times1-5\times5) & (5\times2-3\times1) \\ (2\times0-5\times4) & (5\times(-1)-1\times0) & (1\times4-2(-1)) \end{bmatrix}$$

$$\text{adj } A = \begin{bmatrix} 12 & 3 & -23 \\ 9 & -22 & 7 \\ -20 & -5 & 6 \end{bmatrix}$$

Steps:

1. Write down the three rows of A and rewrite ~~fist~~ first two columns in right.

2. After step 1, rewrite first two rows.

3. After step 2; deleting first row and first column, then we get all elements of adjoint A.

4. Expand row-wise and write coloumn-wise or expand column-wise and write row-wise.

Q. Find the adjoint of the matrix $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 5 & 0 \\ 2 & 4 & 3 \end{bmatrix}$

1	2	3	1	2
0	5	0	0	5
2	4	3	2	4
1	2	3	1	2
0	5	0	0	5

$$\text{adj}(A) = \begin{bmatrix} 15 & 6 & -15 \\ 0 & -3 & 0 \\ -10 & 0 & 5 \end{bmatrix}$$

Properties of Adjoint Matrix :-

1. If A be a square matrix of order n, then
$A(\text{adj} A) = (\text{Adj} A).A = |A| I_n$

- If A be a square singular matrix of order n, then
$A(\text{adj} A) = (\text{Adj} A).A = O$ (O = null matrix)

* • If A be a square non-singular matrix of order n, then
$|\text{adj} A| = |A|^{n-1}$

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Proof :- $A(\text{adj } A) = |A| I_n$

Let $|A| = K$

$$|A(\text{adj } A)| = |KI|$$
$$|A|\,|\text{adj } A| = K^n |I|$$
$$|A|\,|\text{adj } A| = |A|^n |I|$$
$$|\text{adj } A| = |A|^{n-1}$$

Note :-

$$|\underbrace{\text{adj}(\text{adj}(\text{adj}(\cdots \text{adj}}_{\text{adj repeated } m \text{ times}}(A))))| = |A|^{(n-1)^m}$$ * Most Imp

If A and B are square matrices of order n, then ~~adj~~ $\text{adj}(AB) = \text{adj}(B) \cdot \text{adj}(A)$

If A be a square matrix of order n, then
$(\text{adj } A)' = \text{adj } A'$

If A be a square non-singular matrix of order n, then
$$\text{adj}(\text{adj } A) = |A|^{n-2} \cdot A$$

If A be a square non-singular matrix of order n, then
$$|\text{adj}(\text{adj } A)| = |A|^{(n-1)^2}$$

If A be a square matrix of order n and K be scalar, then
$$\text{adj}(KA) = K^{n-1} \text{adj}(A)$$

If A be a square matrix of order m and $m \in N$, then

$(adj\ A^m) = (adj\ A)^m$

If A and B be two square matrices of order n such that ~~ad~~ B is the adj (A) and K is scalar, then

$|AB + KI_n| = (|A| + K)^n$

Adjoint of a ~~digon~~ diagonal matrix is a diagonal matrix.

eg → $A = \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix}$

$adj\ (A) = \begin{bmatrix} bc & 0 & 0 \\ 0 & ca & 0 \\ 0 & 0 & ab \end{bmatrix}$

$adj\ (I_n) = I_n$

Find

Q. (i) $|adj\ (3A)|$, if $A = \begin{bmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix}$

(ii) $|A|\ |adj\ A|$

(iii) $adj\ (adj\ A)$

(iv) $|adj\ (adj\ (adj\ A))|$

Sol^n. $|A| = 4$

↓

A is non-singular matrix

(i) $|\text{adj}(3A)| = |3A|^2$
$= (3^3|A|)^2$
$= 3^6 \; |A|^2$
$= 3^6 \times 4^2$
$= 11664$

(ii) $|A| \; |\text{adj}(A)| = |A| \; |A|^{3-1}$
$= |A| \; |A|^2$
$= |A|^3 = 4^3 = 64$

(iii) $\text{adj}(\text{adj} A) = |A|^{n-2} \cdot A$
$= |A|^{3-2} \cdot A$
$= |A| \cdot A$
$= 4A$

(iv) $|\text{adj}(\text{adj}(\text{adj} A))| = |A|^{(n-1)^3}$
$= |A|^{2^3}$
$= |A|^8$
$= (4)^8$

Q. Let A be a matrix of order 3x3 such that $|A| = 2$, $B = 2A^{-1}$, $C = \dfrac{\text{adj} A}{\sqrt[3]{16}}$, then value of $\det(A^3 B^2 C^3)$ is.

Sol^n. $B = 2A^{-1}$
$|B| = |2A^{-1}|$
$|B| = 2^3 \; |A|^{-1}$
$|B| = \dfrac{8}{|A|}$
$|B| = 4$

$|C| = \left|\dfrac{\text{adj} A}{\sqrt[3]{16}}\right|$
$|C| = \dfrac{1}{16} \; |\text{adj} A|$
$|C| = \dfrac{1}{16} \cdot |A|^2$
$|C| = 1/4$

$$|A^3B^2C^3| = |A|^3\,|B|^2\,|C|^3$$
$$= \cancel{8}^{2} \times \cancel{16} \times \frac{1}{\cancel{64}} \times$$

$$= 2$$

Q. If A and B are square matrices of order 3 such that $|A|=3$, $|B|=2$, then value of $|A^{-1}\,\text{adj}(B^{-1}).\,\text{adj}(3A^{-1})|$

Soln:- $|A^{-1}|\;|\text{adj}(B^{-1})|\;|\text{adj}(3A^{-1})|$

$$= \frac{1}{|A|}\;|B^{-1}|^2.\;|3A^{-1}|^2$$

$$= \frac{1}{|A|}\cdot\frac{1}{|B|^2}\cdot(3^3)^2\,\frac{1}{|A|^2}$$

$$= \frac{1}{3}\times\frac{1}{4}\times 27\times\frac{1}{9}$$

$$= \frac{1}{12} = \frac{1}{4}$$

$$= \frac{1}{3}\times\frac{1}{4}\times 27\times \cancel{27}^{3}\times\frac{1}{9}$$

$$= \frac{27}{4}$$

Q. If A be a invertible matrix of order 3 and B is another matrix of same order as of A such that $|B|=2$, $A^T|A|B = A|B|B^T$, if $|AB^{-1}\,\text{adj}\,(A^TB)^{-1}| = K$ then value of 4K is.

Soln:. $|B| = 2$

$$A^T\underbrace{|A|}_{m}B = A\underbrace{|B|}_{n}B^T$$
$$|A^T m B| = |A n B^T|$$
$$|A^T|\,|m|\,|B| = |A|\,|n|\,|B^T|$$
$$m^3\,|A^T|\,|B| = n^3\,|A|\,|B^T|$$
$$|A|^3\;|A|\,|B|$$

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$$|A|^3 = |B|^3$$
$$|A| = |B| = 2$$

Now,

$$K = |AB^{-1} \, adj(A^TB)^{-1}|$$

$$K = |A| \cdot |B|^{-1} \, |adj(A^TB)^{-1}|$$

$$K = 2 \times \frac{1}{2} \cdot |(A^TB)|^{-2}$$

$$K = |A^T|^{-2} \, |B|^{-2}$$

$$K = \frac{1}{4} \times \frac{1}{4}$$

$$\boxed{K = \frac{1}{16}}$$

$$4K = 4 \times \frac{1}{16}$$

$$\boxed{4K = 1/4}$$

Q. If $|A| = 2$, $|3 \, adj(|3A| A^2)| = ?$ Order = 3

{JEE Main 2023 April}

Soln: $|3 \, adj(|3A| A^2)|$

$= 3^3 \, |adj(|3A| A^2)|$

$= 3^3 \, ||3A| A^2|^2$

$= 3^3 \, |K|^2 \, |A^2|^2$

$= 3^3 \, K^6 \, |A|^4$

$= 3^3 \, |3A|^6 \, |A^4|$

$= 3^3 \cdot 3^{18} \, |A|^6 \, |A|^4$

$= 3^{21} \cdot 2^{10}$

Inverse of a Matrix :- (Reciprocal Matrix)

A square matrix A (non-singular) of order n is said to be invertible if there exist a square matrix B of the same order such that $AB = I_n = BA$, then B is called the inverse (reciprocal) of A and is denoted by A^{-1}.

$$A^{-1} = B \iff AB = I_n = BA$$

$$A(\text{adj} A) = |A| I_n$$

Pre-multiply by A^{-1}

$$A^{-1}A(\text{adj} A) = A^{-1}|A| I_n$$

$$I_n(\text{adj} A) = A^{-1}|A| I_n$$

$$\text{adj} A = A^{-1}|A|$$

$$\boxed{A^{-1} = \frac{\text{adj}(A)}{|A|}} \rightarrow \text{provided } |A| \neq 0$$

Note :-

~~If~~ The necessary and sufficient condition for a square matrix A to be invertible is that $|A| \neq 0$.

Properties of Inverse of a Matrix :-

1. Uniqueness of Inverse.
   - Every invertible matrix posses a unique inverse.

2. Reversal Law
   - If A and B are invertible matrices of order $n \times n$, then AB is invertible and $(AB)^{-1} = B^{-1}A^{-1}$

Note:-
If A, B, C, ~~and~~ ..., Y, Z are invertible matrices, then $(ABC \dots YZ)^{-1} = Z^{-1}Y^{-1} \dots C^{-1}B^{-1}A^{-1}$

3. Let A be an invertible matrix of order n, then A^T is also invertible and $(A^T)^{-1} = (A^{-1})^T$

4. Let A be an invertible matrix of order n and $K \in N$, then ~~$(A^K)^{-1}$~~ $(A^K)^{-1} = (A^{-1})^K = A^{-K}$

5. Let A be an invertible matrix of order n. then $(A^{-1})^{-1} = A$

Note:
$I_n^{-1} = I_n$ as $I_n \cdot I_n^{-1} = I_n$

6. Let A be an invertible matrix of order n, then
$$|A^{-1}| = \frac{1}{|A|}$$

7. Inverse of a non-singular diagonal matrix is a diagonal matrix.

i.e. $A = \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix}$

$A^{-1} = \begin{bmatrix} 1/a & 0 & 0 \\ 0 & 1/b & 0 \\ 0 & 0 & 1/c \end{bmatrix}$

Note:-

The inverse of a non-singular square matrix A of order 2 is obtained by interchanging the diagonal elements and changing signs of off diagonal elements and dividing by $|A|$

eg, $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

$|A| = ad - bc \neq 0$

$adj(A) = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

$$A^{-1} = \frac{\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}}{|A|}$$

$$A^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Q. Compute the inverse of the matrix $A = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & 1 & 1 \end{bmatrix}$

Sol^n:- adj A =

0	1	2	0	1
1	2	3	1	2
3	1	1	3	1
0	1	2	0	1
1	2	3	1	2

~~adj A~~

$$\text{adj } A = \begin{bmatrix} -1 & 1 & -1 \\ 8 & -6 & 2 \\ -5 & 3 & -1 \end{bmatrix}$$

$$|A| = \cancel{8}-6 \qquad 8-10$$
$$= -2 \qquad -2$$

$$\cancel{A^{-1} = \begin{bmatrix} -1/2 & 1 \\ 4 & -3 \\ -5/2 & \end{bmatrix}}$$

$$A^{-1} = \begin{bmatrix} 1/2 & -1/2 & 1/2 \\ -4 & 3 & -1 \\ 5/2 & -3/2 & 1/2 \end{bmatrix}$$

Q. Matrices A and B satisfy $AB = B^{-1}$, where $B = \begin{bmatrix} 2 & -2 \\ -1 & 0 \end{bmatrix}$, find the value of λ for which $\lambda A - 2B^{-1} + I = 0$, without finding B^{-1}.

Sol^n:-

$$\lambda A - 2B^{-1} + I = 0$$

Post multiply by B

$$\lambda AB - 2I + IB = 0$$
$$\lambda B^{-1} - 2I + IB = 0$$

post multiply by B

$$\lambda I - 2B + B^2 = 0$$

~~$B^2 - 2B + \lambda = 0$~~

$$B^2 - 2B + \lambda I = 0$$

$$\lambda I = 2B - B^2$$

$$\begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} = \begin{bmatrix} 4 & -4 \\ -2 & 0 \end{bmatrix} - \begin{bmatrix} 6 & -4 \\ -2 & 2 \end{bmatrix}$$

$$\begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} = \begin{bmatrix} -2 & 0 \\ 0 & -2 \end{bmatrix}$$

$$\boxed{\lambda = -2}$$

Q. Let A and B be two 3X3 matrices such that $AB = I$ and $|A| = \frac{1}{8}$, then $|\text{adj}(B\,\text{adj}(2A))| = ?$ {JEE Main 2022}

Soln:-

$$|B\,\text{adj}(2A)|^{(2)^2}$$
$$= |B\,\text{adj}(2A)|^4$$
$$= B^{12}\,|\text{adj}(2A)|^4$$
$$= B^{12}$$
$$= |B|^4\,|\text{adj}(2A)|^4$$
$$= (A^{-1})^4\,|2A|^8$$
$$= \frac{1}{|A|^4}\,2^7 \cdot |A|^8$$
$$= 2^7 \cdot |A|^4$$
$$= 2^7$$

$$AB = I$$
$$B = A^{-1}$$
$$|B| = 8$$

$$= |B\,\text{adj}(2A)|^4$$
$$= |B|^2\,|\text{adj}(2A)|^2$$

$$|\text{adj}(B\,\text{adj}(2A))|$$
$$= |B\,\text{adj}(2A)|^2$$
$$= |B|^2\,|\text{adj}\,2A|^2$$
$$= 64\,|2A|^4$$
$$= 64 \times 2^{12} \times \frac{1}{8^4}$$
$$= \frac{2^{18}}{2^{12}}$$
$$= 2^6$$
$$= 64$$

Elementary Row Operation

The following three types of operations (transformations) on the rows of a given matrix are known as elementary row operation.

(i) The interchange of i^{th} and j^{th} row is denoted by $R_i \longleftrightarrow R_j$ or R_{ij}

(ii) The multiplication of i^{th} row by a constant K $(K \neq 0)$ is denoted by $R_i \longleftrightarrow KR_i$

(iii) The addition of i^{th} row to the elements of j^{th} row multiplied by constant K is denoted by $R_i \longleftrightarrow R_i + KR_j$

Note :-

Similarily, we can define the three column operations.

(i) $C_i \longleftrightarrow C_j$

(ii) $C_i \longleftrightarrow KC_j$

(iii) $C_i \longleftrightarrow C_i + KC_j$

Equivalent Matrices

Two matrices are said to be equivalent if one is obtained from the other by elementary operations.

$\sim$ is used for equivalence.

Properties of Equivalent Matrices:-

1. If A and B are equivalent matrices, there exist non-singular matrices P and Q such that $B = PAQ$.

2. If A and B are equivalent matrices such that $B = PAQ$, then $P^{-1} B Q^{-1} = A$

3. Every non-singular square matrix can be expressed as the product of elementary matrices.

To compute the inverse of a non-singular matrix by elementary operations (Gauss-Jordan Method):-

If A be a non-singular matrix of order n, then write $A = I_n A$.

If A is reduced to I_n by elementary operations (LHS), then suppose I_n is reduced to P (RHS) and not change A in RHS, then after elementary operations, we get $I_n = PA$, then P is the inverse of A. $P = A^{-1}$

Q. Find the inverse of the matrix $A = \begin{bmatrix} 1 & 2 & 5 \\ 2 & 3 & 1 \\ -1 & 1 & 1 \end{bmatrix}$ using elementary row operations.

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Sol^n:

$$A = \begin{bmatrix} 1 & 2 & 5 \\ 2 & 3 & 1 \\ -1 & 1 & 1 \end{bmatrix}$$

$$A = I_3 A$$

~~Pre-multiply~~

$$\begin{bmatrix} 1 & 2 & 5 \\ 2 & 3 & 1 \\ -1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A$$

$R_2 \rightarrow R_2 + 2R_3$ and $R_3 \rightarrow R_3 + R_1$

$$\begin{bmatrix} 1 & 2 & 5 \\ 0 & 5 & 3 \\ 0 & 3 & 6 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 1 & 0 & 1 \end{bmatrix} A$$

$R_3 \rightarrow \frac{2}{3} R_3$

$$\begin{bmatrix} 1 & 2 & 5 \\ 0 & 5 & 3 \\ 0 & 2 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 2/3 & 0 & 2/3 \end{bmatrix} A$$

$R_3 \rightarrow R_3 - R_2$

$$\begin{bmatrix} 1 & 2 & 5 \\ 0 & 5 & 3 \\ 0 & -3 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 2/3 & -1 & -4/3 \end{bmatrix} A$$

$R_1 \rightarrow R_1 - 5R_3$

$$\begin{bmatrix} 1 & 17 & 0 \\ 0 & 5 & 3 \\ 0 & -3 & 1 \end{bmatrix} = \begin{bmatrix} -7/3 & 5 & 20/3 \\ 0 & 4 & 6 \\ 2/3 & -1 & -4/3 \end{bmatrix} A$$

$R_2 \leftrightarrow R_2 - 3R_3$

$$\begin{bmatrix} 1 & 17 & 0 \\ 0 & 14 & 0 \\ 0 & -3 & 1 \end{bmatrix} = \begin{bmatrix} -7/3 & 5 & 20/3 \\ -2 & 4 & 6 \\ 2/3 & -1 & -4/3 \end{bmatrix} A$$

$R_2 \leftrightarrow \frac{1}{14} R_2$

$$\begin{bmatrix} 1 & 17 & 0 \\ 0 & 1 & 0 \\ 0 & -3 & 1 \end{bmatrix} = \begin{bmatrix} -7/3 & 5 & 20/3 \\ -1/7 & 2/7 & 3/7 \\ 2/3 & -1 & -4/3 \end{bmatrix} A$$

$R_3 \leftrightarrow R_3 + 3R_2$

$$\begin{bmatrix} 1 & 17 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -7/3 & 5 & 20/3 \\ -1/7 & 2/7 & 3/7 \\ 5/21 & -1/7 & -1/21 \end{bmatrix} A$$

$R_1 \leftrightarrow R_1 - 17R_2$

$$\underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{I} = \underbrace{\begin{bmatrix} 2/21 & 1/7 & -13/21 \\ -1/7 & 2/7 & 3/7 \\ 5/21 & -1/7 & -1/21 \end{bmatrix}}_{P} \underbrace{A}_{A}$$

$AA^{-1} = I$

$$A^{-1} = \begin{bmatrix} 2/21 & 1/7 & -13/21 \\ -1/7 & 2/7 & 3/7 \\ 5/21 & -1/7 & -1/21 \end{bmatrix}$$

Matrix Polynomial

Let $f(x) = a_0 x^m + a_1 x^{m-1} + \dots + a_m$ be a polynomial in x. Let $A = [a_{ij}]_{n \times n}$, then the expression of the form ~~f(a)~~ ~~$f(A) = a_0 x^m + a_1 m$~~

$$f(A) = a_0 A^m + a_1 A^{m-1} + \dots\ a^m I_n$$

is called a matrix polynomial.

Solutions of Linear simultaneous Equation using matrix method :-

$$a_1 x + b_1 y + c_1 z = d_1$$
$$a_2 x + b_2 y + c_2 z = d_2$$
$$a_3 x + b_3 y + c_3 z = d_3$$

$$AX = B$$

$$A = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}, \quad X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, \quad B = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$$

$$AX = B$$

Pre multiply by A^{-1}

$$A^{-1} A X = A^{-1} B$$
$$I X = A^{-1} B$$
$$X = A^{-1} B$$

$$X = \frac{(\text{adj } A) B}{|A|}$$

Types of Equations :

1. When system of equations is Non-Homogenous

If $|A| \neq 0$, then the system of equations is consistent and has a unique solution given by $X = A^{-1}B$

(a) If $|A| \neq 0$ and $(adj\ A)B \neq O$, system is consistent having unique non-trivial solution. (O = Null matrix)

Trivial → If all variables equal to zero

* (b) If $|A| \neq 0$ and $(adj\ A)B = O$, system is consistent having trivial solution.

(c) If $|A| = 0$ and $(adj\ A)B \neq O$, then the system of equation is inconsistent and has no solution.

(d) If $|A| = 0$ and $(adj\ A)B = O$, then the system of equation is consistent an has infinite number of solutions.

2. When system of equations is Homogenous :

(a) If $|A| \neq 0$, then the system of equations has only trivial solution and it has one solution.

(b) If $|A| = 0$, then the system of equation has non-trivial solution and it has infinite number of sol^n.

(c) If no. of eq^n less than no. of unknowns, then it has non-trivial solution.

Q. Solve the system of equations

$$x + 2y + 3z = 1$$
$$2x + 3y + 2z = 2$$
$$3x + 3y + 4z = 1$$

with the help of matrix method.

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 2 \\ 3 & 3 & 4 \end{bmatrix} \qquad B = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$$

$$1(12-6) - 2(8-6) + 3(-3)$$

$$AX = B$$
$$X = A^{-1}B$$

$$A = \begin{array}{c|cccc} 1 & 2 & 3 & 1 & 2 \\ \hline 2 & 3 & 2 & 2 & 3 \\ 3 & 3 & 4 & 3 & 3 \\ 1 & 2 & 3 & 1 & 2 \\ 2 & 3 & 2 & 2 & 3 \end{array}$$

$$\text{adj}A = \begin{bmatrix} 6 & 1 & -5 \\ -2 & -5 & 4 \\ -3 & 3 & -1 \end{bmatrix}$$

$\cancel{|A| = 6 - 12 - 3}$ $\qquad |A| = -7$

$$A^{-1} = \begin{bmatrix} -6/7 & -1/7 & 5/7 \\ 2/7 & 5/7 & -4/7 \\ 3/7 & -3/7 & 1/7 \end{bmatrix}$$

$$X = A^{-1}B$$

$$= \begin{bmatrix} -6/7 & -1/7 & 5/7 \\ 2/7 & 5/7 & -4/7 \\ 3/7 & -3/7 & 1/7 \end{bmatrix}_{3\times3} \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}_{3\times1}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -3/7 \\ 8/7 \\ -2/7 \end{bmatrix}$$

$$x = \frac{-3}{7}, \quad y = \frac{8}{7}, \quad z = \frac{-2}{7}$$

Q. The system of equation
$$x+y+z=2$$
$$2x+y-z=3$$
$$3x+2y+\lambda z=4$$
has unique sol^n if $\lambda = ?$

Sol^n:- $A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & -1 \\ 3 & 2 & \lambda \end{bmatrix}$

for unique sol^n:

$|A| \neq 0$

$$1(\lambda+2) - 1(2\lambda+3) + 1(4-3) \neq 0$$
$$\lambda+2-2\lambda-3+1 \neq 0$$
$$-\lambda \neq 0$$
$$\boxed{\lambda \neq 0}$$

Q. The value of a for which the following system of equations

$$a^3x + (a+1)^3 y + (a+2)^3 z = 0$$
$$ax + (a+1) y + (a+2) z = 0$$
$$x + y + z = 0$$

has a non-trivial solⁿ is equal to.

solⁿ:

$$A = \begin{bmatrix} a^3 & (a+1)^3 & (a+2)^3 \\ a & (a+1) & (a+2) \\ 1 & 1 & 1 \end{bmatrix}$$

$$|A| = 0$$

$$\cancel{a^3(a+1-1) - (a+1)^3(a-a+2)}$$

$$a^3(\cancel{a}+1-\cancel{a}-2) - (a+1)^3(\cancel{a}-\cancel{a}-2) + (a+2)^3(\cancel{a}-\cancel{a}-1) = 0$$

$$-a^3 - (a+1)^3(-2) + (a+2)^3(-1) = 0$$

$$\cancel{a^3 = a^3 + 1 - 3a(a+1)}$$

$$-a^3 + 2(a^3+1+3a(a+1)) - 1(a^3+8+6a(a+2)) = 0$$

$$-a^3 + 2[a^3+1+3a^2+3a] - 1[a^3+8+6a^2+12a] = 0$$

$$-\cancel{a^3} + \cancel{2a^3} + 2 + \cancel{6a^2} + 6a - \cancel{a^3} - 8 - \cancel{6a^2} - 12a = 0$$

$$-6a - 6 = 0$$

$\boxed{\cancel{a \neq -1}}$, $\boxed{a = -1}$

Cayley Hamilton Theorem

Every square matrix A satisfies its characteristic equation $|A-\lambda I|=0$, i.e., $a_0\lambda^n + a_1\lambda^{n-1} + \dots + a_n = 0$

By cayley Hamilton Theorem:-

$a_0 A^n + a_1 A^{n-1} + \dots + a_n I = O$ → Null matrix

Q. Find the characteristic eqⁿ of the matrix $A=\begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix}$ and hence find its inverse using cayley Hamilton Theorem.

Solⁿ:

$$A-\lambda I = \begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}$$

$$= \begin{bmatrix} 2-\lambda & 1 \\ 3 & 2-\lambda \end{bmatrix}$$

$|A-\lambda I| = 0$

$(2-\lambda)(2-\lambda) - 3 = 0$

$4-4\lambda+\lambda^2-3=0$

$\lambda^2-4\lambda+1=0$

$\lambda = \dfrac{4 \pm \sqrt{16+4}}{2}$

$\lambda = \dfrac{4 \pm 2\sqrt{5}}{2}$

$\lambda = 2 \pm \sqrt{5}$

By CHT,

$A^2 - 4A + I = O$ → Null matrix

$A^2 - 4A = -I$

$I = 4A - A^2$

Post Multiply by A^{-1}

$IA^{-1} = 4AA^{-1} - AAA^{-1}$

$A^{-1} = 4I - A$

$$A^{-1} = \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix} - \begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} 2 & -1 \\ -3 & 2 \end{bmatrix}$$

Q. Let $A = \begin{bmatrix} 4 & -2 \\ \alpha & \beta \end{bmatrix}$. If $A^2 + 4A + 18I = 0$, then $|A| = ?$

[JEE Main 2022]

Soln:

$$\left[\begin{array}{l} A^2 + 4A + 18I = 0 \\ \lambda^2 + 4\lambda + 18 = 0 \end{array}\right] \qquad A^2 + 4A + 18I = 0$$

$$A^2 = \begin{bmatrix} 4 & -2 \\ \alpha & \beta \end{bmatrix}\begin{bmatrix} 4 & -2 \\ \alpha & \beta \end{bmatrix}$$

$$A^2 = \begin{bmatrix} 16 - 2\alpha & -8 - 2\beta \\ 4\alpha + \alpha\beta & -2\alpha + \beta^2 \end{bmatrix}$$

$$A^2 + 4A + 18I = 0$$

$$\begin{bmatrix} 16 - 2\alpha & -8 - 2\beta \\ 4\alpha + \alpha\beta & -2\alpha + \beta^2 \end{bmatrix} + \begin{bmatrix} 16 & -8 \\ 4\alpha & 4\beta \end{bmatrix} + \begin{bmatrix} 18 & 0 \\ 0 & 18 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$16 - 2\alpha + 16 + 18 = 0$$
$$2\alpha = 50$$
$$\boxed{\alpha = 25}$$

$$-8 - 2\beta - 8 = 0$$
$$\boxed{\beta = -8}$$

$$A = \begin{bmatrix} 4 & -2 \\ 25 & -8 \end{bmatrix}$$

$$|A| = -32 + 50$$
$$|A| = 18$$

Q. Let $A = \begin{bmatrix} 2 & -1 \\ 0 & 2 \end{bmatrix}$. If $B = I - {}^5C_1 (\text{adj} A) + {}^5C_2 (\text{adj} A)^2 - \ldots - {}^5C_5 (\text{adj} A)^5$. Then sum of all elements of matrix B is: [JEE Main 2022]

Sol^n:- $B = I - {}^5C_1 (\text{adj}) + {}^5C_2 (\text{adj} A)^2 - \ldots - {}^5C_5 (\text{adj} A)^5$

$$B = (I - \text{adj} A)^5$$

$$\text{adj} A = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}$$

$$B = \left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix} \right)^5$$

$$B = \begin{bmatrix} -1 & -1 \\ 0 & -1 \end{bmatrix}^5$$

$$[x]^2 \begin{bmatrix} -1 & -1 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} -1 & -1 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$$

$$[x]^3 = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -1 & -1 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} -1 & -3 \\ 0 & -1 \end{bmatrix}$$

$$B = \begin{bmatrix} -1 & -1 \\ 0 & -1 \end{bmatrix}^5$$

$$B = \begin{bmatrix} -1 & -5 \\ 0 & -1 \end{bmatrix}$$

sum of elements $= -1 - 1 - 5 = -7$

Q. The positive value of the determinant of matrix A whose $adj(adj\ A) = \begin{bmatrix} 14 & 28 & -14 \\ -14 & 14 & 28 \\ 28 & -14 & 14 \end{bmatrix}$

Soln:- $adj(adj\ A) = 14\begin{bmatrix} 1 & 2 & -1 \\ -1 & 1 & 2 \\ 2 & -1 & 1 \end{bmatrix}$

$$|adj(adj\ A)| = |14A|$$

$$|adj(adj A)| = 14^3 |A|$$

$$|adj(adj\ A)| = 14^3 [1(1+2) - 2(-1-4) - 1(1-2)]$$

$$|adj(adj A)| = 14^3 [3+10+1]$$

$$|adj(adj A)| = 14^4$$

$$|adj(adj\ A)| = |A|^{2^2}$$

$$14^4 = |A|^4$$

$$|A| = 14$$

Q. Let A and B be two invertible matrices of order 3x3. If ~~|A|~~ $|ABA^T| = 8$, $|AB^{-1}| = 8$, then $|BA^{-1}B^T| = ?$

Soln:- $(AB)^{-1} = B^{-1}A^{-1}$

$$|A||B||A^T| = 8 \qquad \frac{|A|}{|B|} = 8 \quad -②$$

$$|A|^2 |B| = 8 \quad -①$$

$$|A|^2 \frac{|A|}{8} = 8$$

$$|A| = 4$$

$$|B| = \frac{1}{2}$$

Then,

$$|BA^{-1}B^T| = |B|\,|A^{-1}|\,|B^T|$$

$$= \frac{|B|^2}{|A|}$$

$$= \frac{1}{4 \times 4}$$

$$= \frac{1}{16}$$

Determinants

* $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

$|A| = \det A = ad - bc.$

Note:-

1. A determinant is generally denoted by D or Δ.
2. Shape of every determinant is square.

Important Results for co-factors:-

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$|A| = a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

$$|A| = a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} + a_{12}\left(-\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}\right) + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

$$|A| = a_{11}A_{11} + a_{12}A_{12} + a_{13}A_{13}$$

$$|A| = \Delta$$

(i) The sum of the products of the elements of any row or column with their corresponding co-factors is equal to the value of the determinant.

(ii) The sum of the products of the elements of any row (or column) with corresponding co-factors of another row (or column) is zero.

eg→ $a_{11}C_{21} + a_{12}C_{22} + a_{13}C_{23} = 0$

(iii) If the value of a n order determinant is Δ, then the value of the determinant formed by the co-factors of ~~the~~ corresponding elements of the given determinant is given by :

$$\Delta^c = \Delta^{n-1}$$

eg→ For order 3

$$\Delta^c = \Delta^2$$

Q. If the value of a 3rd order determinant is 11 Find the value of square of the determinant formed by the cofactors.

Soln:

$$\Delta^c = \Delta^{3-1} \qquad \Delta = 11$$

$$\Delta^c = \Delta^2$$

$$(\Delta^c)^2 = (\Delta^2)^2 = (11)^4$$

Use of Determinants in Co-ordinate Geometry:-

1. Area of triangle whose vertices are (x_1, y_1), (x_2, y_3) and (x_3, y_3) is given by

$$\Delta = \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$$

2. If points (x_1, y_1), (x_2, y_2), (x_3, y_3) are ~~coll~~ co-linear, then

$$\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = 0$$

3. Eqn of a straight line passing through two points (x_1, y_1) and (x_2, y_2) is:

$$\begin{vmatrix} x & y & 1 \\ x_1 & y_2 & 1 \\ x_2 & y_3 & 1 \end{vmatrix} = 0$$

4. If the lines ~~or~~ $a_1x + b_1y + c_1 = 0$
$a_2x + b_2y + c_2 = 0$
$a_3x + b_3y + c_3 = 0$
are ~~collinee~~ concurrent:

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0$$

If $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ represents a pair of straight lines, then:

$$\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = 0$$

Equation of circle through three non-colinear points (x_1, y_1), (x_2, y_2), (x_3, y_3) is given by

$$\begin{vmatrix} x^2+y^2 & x & y & 1 \\ x_1^2+y_1^2 & x_1 & y_1 & 1 \\ x_2^2+y_2^2 & x_2 & y_2 & 1 \\ x_3^2+y_3^2 & x_3 & y_3 & 1 \end{vmatrix} = 0$$

Some Useful operations:-

1. The interchange of i^{th} and j^{th} row is denoted by $R_i \longleftrightarrow R_j$ (In case of column, $C_i \longleftrightarrow C_j$)

2. Addition of m times the ~~add~~ elements of j^{th} row to the corresponding i^{th} row is denoted by $R_i \longrightarrow R_i + mR_j$ (In case of column, $C_i \longrightarrow C_i + mC_j$)

Properties of Determinants :-

Property 1 :- The value of a determinant remains unaltered when rows are change into corresponding columns and vice versa.

$$|A| = |A^T|$$

Property 2 :- If any two rows (or two columns) of a determinant are interchanged, then the sign of determinant is changed and the numerical value remains unaltered.

Property 3 :- If two rows (or columns) of a determinant are identical, then value of determinant is zero.

eg,
$$\begin{vmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \\ 4 & 5 & 6 \end{vmatrix} = 0 \quad , \quad \begin{vmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 4 & 5 & 6 \end{vmatrix} = 0$$

Property 4 :- If the elements of any row (or any column) of a determinant be each multiplied by the same factor K, then the value of the determinant is multiplied by K.

Property 5 :- If every element of some column (or row) is the sum of two items, then the determinant is equal to the sum of two determinants, one containing one the first term in place of each sum, the other only the second term. The remaining elements of both determinants are the same as in the given determinant.

i.e., $\begin{vmatrix} a_1+x & b_1 & c_1 \\ a_2+y & b_2 & c_2 \\ a_3+z & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} x & b_1 & c_1 \\ y & b_2 & c_2 \\ z & b_3 & c_3 \end{vmatrix}$

$$\begin{vmatrix} a_1+b_1+c_1 & d_1+e_1 & f_1 \\ a_2+b_2+c_2 & d_2+e_2 & f_2 \\ a_3+b_3+c_3 & d_3+e_3 & f_3 \end{vmatrix} = \begin{vmatrix} a_1 & d_1 & f_1 \\ a_2 & d_2 & f_2 \\ a_3 & d_3 & f_3 \end{vmatrix}$$

$$+ \begin{vmatrix} a_1 & e_1 & f_1 \\ a_2 & e_2 & f_2 \\ a_3 & e_3 & f_3 \end{vmatrix} + \begin{vmatrix} b_1 & d_1 & f_1 \\ b_2 & d_2 & f_2 \\ b_3 & d_3 & f_3 \end{vmatrix} + \begin{vmatrix} b_1 & e_1 & f_1 \\ b_2 & e_2 & f_2 \\ b_3 & e_3 & f_3 \end{vmatrix}$$

$$+ \begin{vmatrix} c_1 & d_1 & f_1 \\ c_2 & d_2 & f_2 \\ c_3 & d_3 & f_3 \end{vmatrix} + \begin{vmatrix} c_1 & e_1 & f_1 \\ c_2 & e_2 & f_2 \\ c_3 & e_3 & f_3 \end{vmatrix}$$

Property 6:-

$$\begin{vmatrix} a_1+mb_1+nc_1 & b_1 & c_1 \\ a_2+mb_2+nc_2 & b_2 & c_2 \\ a_3+mb_3+nc_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

Property 7:- Determinant of diagonal matrix, upper triangular matrix and lower triangular matrix will be the product of the diagonal elements.

$$\begin{vmatrix} a & 0 & 0 \\ d & b & 0 \\ e & f & c \end{vmatrix} = \begin{vmatrix} a & d & e \\ 0 & b & f \\ 0 & 0 & c \end{vmatrix} = \begin{vmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{vmatrix} = abc$$

Property 8:- If determinant Δ becomes zero on putting $x = \alpha$, then we can conclude that $x - \alpha$ is a factor of Δ.

Note :-

(i) It should be noted that while applying operations on determinant that atleast one row (or column) must remain unchanged.

(ii) Maximum no. of operations at a time $= n-1$ where n is the order of the matrix.

(iii) It should be noted that, if the row (or column) which is changed by multipying a non-zero number, then the determinant will be divided by that number.

Q. Evaluate $\begin{vmatrix} 13 & 16 & 19 \\ 14 & 17 & 20 \\ 15 & 18 & 21 \end{vmatrix}$

Sol^n:- $R_2 \longrightarrow R_2 - R_1$ $\quad$ $R_3 \longrightarrow R_3 - R_1$

$$\begin{vmatrix} 13 & 16 & 19 \\ 1 & 1 & 1 \\ 2 & 2 & 2 \end{vmatrix} \Big] \text{Identical}$$

$|A| = 0$

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Q. with expanding as far as possible, prove that

$$\begin{vmatrix} 1 & 1 & 1 \\ x & y & z \\ x^3 & y^3 & z^3 \end{vmatrix} = (x-y)(y-z)(z-x)(x+y+z)$$

Sol^n:- $C_1 \longrightarrow C_1 - C_3$ $\quad C_2 \longrightarrow C_2 - C_3$

$$\begin{vmatrix} 0 & 0 & 1 \\ x-z & y-z & z \\ x^3-z^3 & y^3-z^3 & z^3 \end{vmatrix}$$

$$= \begin{vmatrix} 0 & 0 & 1 \\ x-z & y-z & z \\ (x-z)(x^2+xz+z^2) & (y-z)(y^2+yz+z^2) & z^3 \end{vmatrix}$$

$$= (x-z)(y-z)\begin{vmatrix} 0 & 0 & 1 \\ 1 & 1 & z \\ x^2+z^2+zx & y^2+yz+z^2 & z^3 \end{vmatrix}$$

$= (x-z)(y-z)(y^2+yz+\cancel{z^2} - x^2 - \cancel{z^2} - zx)$

$= (x-z)(y-z)(y^2-x^2+yz-zx)$

$= (x-z)(y-z)[(y-x)(y+x)+z(y-x)]$

$= (x-z)(y-z)(y-x)(x+y+z)$

$= (x-y)(y-z)(z-x)(x+y+z)$

System of Linear Equations

1. **Consistent Equation (Intersecting Lines)** :- Definite and unique solution.

A system of (linear) equations is said to be consistent if it has at least one solution.

2. **Inconsistent Equation** – No solution (Parallel lines)

A system of (linear) equations is said to be inconsistent if it has no solution.

$$a_1x + b_1y + c_1 = 0$$
$$a_2x + b_2y + c_2 = 0$$

$\frac{a_1}{a_2} \neq \frac{b_1}{b_2} \longrightarrow$ Unique sol^n (Intersecting Lines)

$\frac{a_1}{a_2} = \frac{b_1}{b_2} \neq \frac{c_1}{c_2} \longrightarrow$ No solution (Parallel Lines)

3. **Dependent Equation** :- Infinite solutions (Consistent Equation)

A system of equations is said to be dependent if it has infinite solutions

$$\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$$

System of linear equations in three variables:

Let's consider the system of linear equation be

$a_1x + b_1y + c_1z = d_1$

$a_2x + b_2y + c_2z = d_2$

$a_3x + b_3y + c_3z = d_3$

$$\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \qquad \Delta_1 = \begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix}$$

$$\Delta_2 = \begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix} \qquad \Delta_3 = \begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}$$

If $\Delta \neq 0$

$x = \frac{\Delta_1}{\Delta}$, $y = \frac{\Delta_2}{\Delta}$, $z = \frac{\Delta_3}{\Delta}$ {Cramer's Rule}

Note:-

Δ_i is obtained by replacing elements of i^{th} column by d_1, d_2, d_3 where $\{i = 1, 2, 3\}$

Cramer's Rule can be used only when $\Delta \neq 0$

Nature of solution of system of linear equation :

Let's consider the system of linear equation be:

$$a_1x + b_1y + c_1 = d_1$$
$$a_2x + b_2y + c_2 = d_2$$
$$a_3x + b_3y + c_3 = d_3$$

Case I : If $\Delta \neq 0$, then system will have unique finite solutions and so, equations are consistent.

(i) If $\Delta \neq 0$ and at least one of $\Delta_1, \Delta_2, \Delta_3 \neq 0$, then the given system of equations is consistent and has unique non-trivial solutions.

(ii) If $\Delta \neq 0$ and $\Delta_1 = \Delta_2 = \Delta_3 = 0$, then the given system of equations is consistent and has trivial solution only.

Case II : If $\Delta = 0$,

(i) If $\Delta = 0$, but at least one of $\Delta_1, \Delta_2, \Delta_3$ is non-zero, then the equation is inconsistent and have no solution.

(ii) If $\Delta = \Delta_1 = \Delta_2 = \Delta_3 = 0$, then the system of equations will have infinite number of solutions i.e. equations are consistent.

System of Homogenous Linear Equations :

Let's consider a system of homogenous linear equations of variables x, y, z be.

$$a_1 x + b_1 y + c_1 z = 0$$
$$a_2 x + b_2 y + c_2 z = 0$$
$$a_3 x + b_3 y + c_3 z = 0$$

$$\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

Case I: If $\Delta \neq 0$, then $x = y = z = 0$ is the only solution of the above system. This solution is called a trivial solution.

Case II: If $\Delta = 0$, at least one of x, y, z is non-zero. This solution is called a non-trivial solution.

Q. Solve the following system of equations by Cramer's Rule:

$$x + y + z = 9$$
$$2x + 5y + 7z = 52$$
$$2x + y - z = 0$$

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Soln:

$$\Delta = \begin{vmatrix} 1 & 1 & 1 \\ 2 & 5 & 7 \\ 2 & 1 & -1 \end{vmatrix} \qquad \Delta_1 = \begin{vmatrix} 9 & 1 & 1 \\ 52 & 5 & 7 \\ 0 & 1 & -1 \end{vmatrix}$$

$$\Delta_2 = \begin{vmatrix} 1 & 9 & 1 \\ 2 & 52 & 7 \\ 2 & 0 & -1 \end{vmatrix} \qquad \Delta_3 = \begin{vmatrix} 1 & 1 & 9 \\ 2 & 5 & 52 \\ 2 & 1 & 0 \end{vmatrix}$$

$\cancel{x = \frac{\Delta_1}{\Delta}}$

$$\Delta = 1(-5-7) - 1(-2-14) + 1(2-10)$$
$$= -12 + 16 - 8$$
$$= -4$$

$$\Delta_1 = 9(-5-7) - 1(-52) + 1(52)$$
$$= -108 + 52 + 52$$
$$= -4$$

$$\Delta_2 = 1(-52) - 9(-2-14) + 1(-104)$$
$$= -52 + 144 - 104$$
$$= -12$$

$$\Delta_3 = -52 - 1(-104) + 9(2-10)$$
$$= -52 + 104 - 72$$
$$= -124 + 104$$
$$= -20$$

$$x = \frac{\Delta_1}{\Delta} = 1$$

$$y = \frac{\Delta_2}{\Delta} = 3$$

$$z = \frac{\Delta_3}{\Delta} = 5$$

Q. For what values of p and q, the system of equations

$$x + y + z = 6$$
$$x + 2y + 3z = 10$$
$$x + 2y + pz = q$$

has

(i) unique solution

(ii) infinitely many solutions

(iii) No solution.

Sol^n:

$$\Delta = \begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 2 & p \end{vmatrix} \qquad \Delta_1 = \begin{vmatrix} 6 & 1 & 1 \\ 10 & 2 & 3 \\ q & 2 & p \end{vmatrix}$$

$$\Delta_2 = \begin{vmatrix} 1 & 6 & 1 \\ 1 & 10 & 3 \\ 1 & q & p \end{vmatrix} \qquad \Delta_3 = \begin{vmatrix} 1 & 1 & 6 \\ 1 & 2 & 10 \\ 1 & 2 & q \end{vmatrix}$$

$$\Delta = 1(2p-6) - 1(p-3) + 1(2-2)$$
$$= 2p - 6 - p + 3$$
$$= p - 3$$

$$\Delta_1 = 6(2p-6) - 1(10p - 3q) + 1(20 - 2q)$$
$$= 12p - 36 - 10p + 3q + 20 - 2q$$
$$= 2p + q - 16$$

$$\Delta_2 = 1(10p - 3q) - 6(p-3) + 1(q - 10)$$
$$= 10p - 3q - 6p + 18 + q - 10$$
$$= 4p - 2q + 8$$

$$\Delta_3 = 1(2q-20) - 1(q-10) + 6(2-2)$$
$$= 2q - 20 - q + 10$$
$$= q - 10$$

(i) For unique solⁿ

$\Delta \neq 0$

$p - 3 \neq 0$

$p \neq 3$

$q \in R$

(ii) For infinitely many solution

$\Delta = \Delta_1 = \Delta_2 = \Delta_3 = 0$

$\Delta = 0$

$p = 3$

$\Delta_1 = 0$ $\quad$ $\Delta_2 = 0$

$2p + q = 16$ $\quad$ $4p - 2q = -8$

$q - 10 = 0$

$\boxed{q = 10}$

$2p = 6$

$\boxed{p = 3}$

(iii) For no solution

$\Delta = 0$ and at least one of $\Delta_1, \Delta_2, \Delta_3$ is non-zero.

$\Delta_1 \neq 0$ $\quad$ $\Delta_2 \neq 0$ $\quad$ $\Delta_3 \neq 0$

$\boxed{p = 3}$ $\quad$ $q \neq 10$ $\quad$ $q \neq 10$ $\quad$ $q \neq 10$

$\boxed{q \neq 10}$

Condition for consistency of three linear equations in two variables:-

Let's consider a system of linear equations in x and y,

$$a_1x + b_1y + c_1 = 0$$
$$a_2x + b_2y + c_2 = 0$$
$$a_3x + b_3y + c_3 = 0$$

will be consistent, the values of x and y obtained from any two equations satisfy the third equation, then the required condition is

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0$$

Note:-

For consistency of three linear equation in two variables, the number of solution is 1.

Q. Find the value of λ if the following equations are consistent.

$$x + y - 3 = 0$$
$$(1+\lambda)x + (2+\lambda)y - 8 = 0$$
$$x - (1+\lambda)y + 2 + \lambda = 0$$

Sol^n:- $\Delta = \begin{vmatrix} 1 & 1 & -3 \\ 1+\lambda & 2+\lambda & -8 \\ 1 & -(1+\lambda) & 2+\lambda \end{vmatrix}$

$C_1 \longrightarrow C_1 - C_2$

$$\Delta = \begin{vmatrix} 0 & 1 & -3 \\ -1 & 2+\lambda & -8 \\ 2+\lambda & -(1+\lambda) & 2+\lambda \end{vmatrix}$$

$$\Delta = -1\left[-2-\lambda+16+8\lambda\right] - 3\left[1+\lambda - 4 - \lambda^2 - 4\lambda\right]$$
$$= -1(7\lambda+14) - 3(-\lambda^2 - 3\lambda - 3)$$
$$= -7\lambda - 14 + 3\lambda^2 + 9\lambda + 9$$
$$\cancel{= -3\lambda^2 - 22\lambda - 29} \qquad = 3\lambda^2 + 2\lambda - 5$$

$\Delta = 0$

$3\lambda^2 + 22\lambda + 29 = 0$

$\lambda = \dfrac{-22 \pm \sqrt{484 - 348}}{6}$

$\lambda = \dfrac{-22 \pm \sqrt{36}}{6}$

$\lambda = \dfrac{-22 \pm 6}{6}$

$\lambda = -\dfrac{14}{3}$ $\qquad \cancel{\lambda = -\dfrac{8}{3}}$

$3\lambda^2 + 2\lambda - 5 = 0$

$3\lambda^2 - 3\lambda + 5\lambda - 5 = 0$

$3\lambda(\lambda - 1) + 5(\lambda - 1) = 0$

$(\lambda - 1)(3\lambda + 5) = 0$

$\boxed{\lambda = 1}$ $\qquad \boxed{\lambda = -\dfrac{5}{3}}$

Special Determinants

1. Cyclic Determinant

The elements of rows (or columns) are in cyclic arrangement

$$\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} = -(a^3+b^3+c^3-3abc)$$

$$= -(a+b+c)(a^2+b^2+c^2-ab-bc-ca)$$

$$= -\frac{1}{2}(a+b+c)\left[(a-b)^2+(b-c)^2+(c-a)^2\right]$$

$$= -(a+b+c)(a+b\omega+c\omega^2)(a+b\omega^2+c\omega)$$, where ω, ω^2 are cube roots of unity

2. Other Important Determinants

(i) $$\begin{vmatrix} 0 & b & -c \\ -b & 0 & a \\ c & -a & 0 \end{vmatrix} = 0$$ skew symmetric matrix of odd order

(ii) $$\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ bc & ac & ab \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix} = (a-b)(b-c)(c-a)$$

(iii) $$\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^3 & b^3 & c^3 \end{vmatrix} = (a-b)(b-c)(c-a)(a+b+c)$$

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(iv) $\begin{vmatrix} 1 & 1 & 1 \\ a^2 & b^2 & c^2 \\ a^3 & b^3 & c^3 \end{vmatrix} = (a-b)(b-c)(c-a)(ab+bc+ca)$

(v) $\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^4 & b^4 & c^4 \end{vmatrix} = (a-b)(b-c)(c-a)(a^2+b^2+c^2-ab-bc-ca)$

Q. If the minimum and the maximum values of the ~~fn~~ $f:\left[\frac{\pi}{4}, \frac{\pi}{2}\right] \to R$, defined by ~~f(θ) =~~

$$f(\theta) = \begin{vmatrix} -\sin^2\theta & -1-\sin^2\theta & 1 \\ -\cos^2\theta & -1-\cos^2\theta & 1 \\ 12 & 10 & -2 \end{vmatrix}$$

are m and M, then the ordered pair $(m, M) = ?$

Soln: $R_1 \longrightarrow R_1 - R_2$

$$f(\theta) = \begin{vmatrix} \cos^2\theta - \sin^2\theta & \cos^2\theta - \sin^2\theta & 0 \\ -\cos^2\theta & -1-\cos^2\theta & 1 \\ 12 & 10 & -2 \end{vmatrix}$$

$$f(\theta) = \begin{vmatrix} \cos 2\theta & \cos 2\theta & 0 \\ -\cos^2\theta & -1-\cos^2\theta & 1 \\ 12 & 10 & -2 \end{vmatrix}$$

$C_2 \longrightarrow \cancel{C_2 - C_2}\; C_2 - C_1$

$$f(\theta) = \begin{vmatrix} \cos 2\theta & 0 & 0 \\ -\cos^2\theta & -1 & 1 \\ 12 & -2 & -2 \end{vmatrix}$$

$= \cos 2\theta (2+2)$

$= 4\cos 2\theta$

$\cancel{m = 0}$ $\quad f : \left[\frac{\pi}{4}, \frac{\pi}{2}\right] \longrightarrow R$

$\cancel{M =}$

$2\theta \to \left[\frac{\pi}{2}, \pi\right]$

$m = 4\cos\pi$ at $\theta = \pi/2$ $\quad M = 4\cos\pi/2$ at $\theta = \pi/4$

$= -4$ $\quad M = 0$

$(m, M) = (-4, 0)$

Q. Let the system of linear equations

$$x + 2y + z = 2$$
$$\alpha x + 3y - z = \alpha$$
$$-\alpha x + y + 2z = -\alpha$$

be inconsistent, then $\alpha = ?$ {JEE Main 2022}

Sol^n:

$$\Delta = \begin{vmatrix} 1 & 2 & 1 \\ \alpha & 3 & -1 \\ -\alpha & 1 & 2 \end{vmatrix}$$

$\Delta = 0$

$1(6+1) - 2(2\alpha - \alpha) + 1(\alpha + 3\alpha) = 0$

$7 - 2\alpha + 4\alpha = 0$

$7 = -2\alpha$

$$\boxed{\alpha = -7/2}$$

Q. The system of equations
$$-Kx + 3y - 14z = 25$$
$$-15x + 4y - Kz = 3$$
$$-4x + y + 3z = 4$$
is consistent for all K in the set [JEE main 2022]

(a) R (b) $R - \{-11, 13\}$ (c) $R - \{13\}$ ✓(d) $R - \{-11, 11\}$

Solⁿ:-
$$\Delta = \begin{vmatrix} -K & 3 & -14 \\ -15 & 4 & -K \\ -4 & 1 & 3 \end{vmatrix}$$

$\Delta = -K(12+K) - 3(-45-4K) - 14(-15+16)$

$= -\cancel{12K} - K^2 + 135 + \cancel{12K} - 14$

$= -K^2 + 121$

$\Delta \neq 0$

$-K^2 + 121 \neq 0$

$K^2 \neq 11$

$K \neq \pm 11$

$K \in R - \{-11, 11\}$

Q. If the system of linear equations

$$2x + y - z = 7$$
$$x - 3y + 2z = 1$$
$$x + 4y + \delta z = k$$

where, $\delta, k \in R$ has infinitely many solutions, then $\delta + k$ is {JEE Main 2022}

Sol^n:-

$$\Delta = \begin{vmatrix} 2 & 1 & -1 \\ 1 & -3 & 2 \\ 1 & 4 & \delta \end{vmatrix} \qquad \Delta_1 = \begin{vmatrix} 7 & 1 & -1 \\ 1 & -3 & 2 \\ k & 4 & \delta \end{vmatrix}$$

$$\Delta_2 = \begin{vmatrix} 2 & 7 & -1 \\ 1 & 1 & 2 \\ 1 & k & \delta \end{vmatrix} \qquad \Delta_3 = \begin{vmatrix} 2 & 1 & 7 \\ 1 & -3 & 1 \\ 1 & 4 & k \end{vmatrix}$$

$$\Delta = \Delta_1 = \Delta_2 = \Delta_3 = 0$$

$$\Delta = 0$$
$$2(-3\delta - 8) - 1(\delta - 2) - 1(4+3) = 0$$
$$-6\delta - 16 - \delta + 2 - 7 = 0$$
$$-7\delta - 21 = 0$$
$$\delta = -3$$

$$\Delta_3 = 0$$
$$2(-3k-4) - 1(k-1) + 7(4+3) = 0$$
$$-6k - 8 - k + 1 + 49 = 0$$
$$-7k + 42 = 0$$
$$\boxed{k = 6}$$

$$\delta + k = -3 + 6 = 3$$

Q. Let the system of linear equations $4x + \lambda y + 2z = 0$

$2x - y + z = 0$

$\mu x + 2y + 3z = 0$

$\lambda, \mu \in R$ has a non-trivial solution, then which of the following is true? [JEE Main 2021]

Sol^n. $\Delta = 0$

✓(a) $\mu = 6, \lambda \in R$

(b) $\lambda = 2, \mu \in R$

(c) $\lambda = 3, \mu \in R$

(d) $\mu = -6, \lambda \in R$

$$\begin{vmatrix} 4 & \lambda & 2 \\ 2 & -1 & 1 \\ \mu & 2 & 3 \end{vmatrix} = 0$$

$$4(-3-2) - \lambda(6-\mu) + 2(4+\mu) = 0$$
$$-20 - 6\lambda + \mu\lambda + 8 + 2\mu = 0$$
$$\mu\lambda - 6\lambda + 2\mu - 12 = 0$$
$$\lambda(\mu-6) + 2(\mu-6) = 0$$
$$(\lambda+2)(\mu-6) = 0 \quad -①$$

$\boxed{\mu = 6}$ $\boxed{\lambda = -2}$

~~If both are~~ At least one can satisfy eq^n ①

Q. The system of equations $kx + y + z = 1$

$x + ky + z = k$

$x + y + kz = k^2$

has no solution if $k = ?$ [JEE Main 2021]

Sol^n. $$\Delta = \begin{vmatrix} k & 1 & 1 \\ 1 & k & 1 \\ 1 & 1 & k \end{vmatrix}$$

$\Delta = K(K^2-1) - 1(K-1) + 1(1-K)$

$~~= K^3 - K - K + 1 + 1 - K~~$

$\Delta = (K-1)(K^2 - K - 1 \cdot 1) = 0$

~~$\Delta = 0$~~

~~$K^3 - 3K + 2 = 0$~~ $(K-1)(K-1)(K+2) = 0$

$K = 1, -2$

$K = 1$ will make & infinitely many solⁿ

Hence,

$\boxed{K = -2}$

Q. An ordered pair (α, β) for which the system of linear equations

$$(1+\alpha)x + \beta y + z = 2$$
$$\alpha x + (1+\beta)y + z = 3$$
$$\alpha x + \beta y + 2z = 2$$

has a unique solⁿ, is: [JEE main 2019]

(a) $(2,4)$ (b) $(-4,2)$ (c) $(1,-3)$ (d) $(-3,1)$

Solⁿ: $\Delta \neq 0$

$$\begin{vmatrix} 1+\alpha & \beta & 1 \\ \alpha & 1+\beta & 1 \\ \alpha & \beta & 2 \end{vmatrix} \neq 0$$

$R_1 \to R_1 - R_2$

$$\begin{vmatrix} 1 & -1 & 0 \\ \alpha & 1+\beta & 1 \\ \alpha & \beta & 2 \end{vmatrix} \neq 0$$

$$C_1 \longrightarrow C_1 + C_2$$

$$\begin{vmatrix} 0 & -1 & 0 \\ 1+\alpha+\beta & \cancel{\alpha}+\beta & 1 \\ \alpha+\beta & \beta & 2 \end{vmatrix} \neq 0$$

$$1\left[(1+\alpha+\beta)2 - \alpha-\beta\right] \neq 0$$

$$2+2\alpha+2\beta-\alpha-\beta \neq 0$$

$$\alpha+\beta \neq -2$$

$$(\alpha,\beta) = (2,4)$$

Q. The number of values of K for which the system of equation $(K+1)x + 8y = 4K$

$Kx + (K+3)y = 3K-1$

has infinitely many solutions is / are :-

[JEE Main 2019]

Solⁿ. $\dfrac{a_1}{a_2} = \dfrac{b_1}{b_2} = \dfrac{c_1}{c_2}$

$$\frac{K+1}{K} = \frac{8}{K+3} = \frac{4K}{3K-1}$$

$$4K(K+3) = 8(3K-1)$$

$$4K^2+12K = 24K-8$$

$$4K^2-12K+8=0$$

$$4K^2-4K-8K+8=0$$

$$4K(K-1)-8(K-1)=0$$

$$\boxed{K=1} \qquad \boxed{K=2}$$

$\cancel{(k+1)}\cancel{(k+3)}$ $\cancel{(k+1)(k+3)}$

$k=2$ will not satisfy above eqn $\left(\frac{k+1}{k} = \frac{8}{k+3}\right)$

Hence,

$\boxed{k=1}$

No. of values of $k = 1$

For $k=2$, system has no solution.

Q. which of the following values of α satisfy the equation [JEE Advance 2015]

$$\begin{vmatrix} (1+\alpha)^2 & (1+2\alpha)^2 & (1+3\alpha)^2 \\ (2+\alpha)^2 & (2+2\alpha)^2 & (2+3\alpha)^2 \\ (3+\alpha)^2 & (3+2\alpha)^2 & (3+3\alpha)^2 \end{vmatrix} = -648\alpha$$

(a) -4 (b) 9 (c) -9 (d) 4

Soln:

$$\begin{vmatrix} \alpha^2+2\alpha+1 & 4\alpha^2+4\alpha+1 & 9\alpha^2+6\alpha+1 \\ \alpha^2+4\alpha+4 & 4\alpha^2+8\alpha+4 & 9\alpha^2+12\alpha+4 \\ \alpha^2+6\alpha+9 & 4\alpha^2+12\alpha+9 & 9\alpha^2+18\alpha+9 \end{vmatrix} = -648\alpha$$

$R_1 \rightarrow R_1 - R_2$ and $R_2 \rightarrow R_2 - R_3$

$$\begin{vmatrix} -2\alpha-3 & -4\alpha-3 & -6\alpha-3 \\ -2\alpha-5 & -4\alpha-5 & -6\alpha-5 \\ \alpha^2+6\alpha+9 & 4\alpha^2+12\alpha+9 & 9\alpha^2+18\alpha+9 \end{vmatrix} = -648\alpha$$

$R_1 \longrightarrow R_1 - R_2$

$$\begin{vmatrix} 2 & 2 & 2 \\ -2\alpha-5 & -4\alpha-5 & -6\alpha-5 \\ \alpha^2+6\alpha+9 & 4\alpha^2+12\alpha+9 & 9\alpha^2+18\alpha+9 \end{vmatrix} = -648\alpha$$

$C_1 \longrightarrow C_1 - C_2$ and $C_2 \longrightarrow C_2 - C_3$

$$\begin{vmatrix} 0 & 0 & 2 \\ 2\alpha & 2\alpha & -6\alpha-5 \\ -6\alpha-3\alpha^2 & -5\alpha^2-6\alpha & 9\alpha^2+18+9 \end{vmatrix} = -648\alpha$$

$$2\left[-2\alpha(5\alpha^2+6\alpha)+2\alpha(6\alpha+3\alpha^2)\right] = -648\alpha$$

$$-10\alpha^3 - \cancel{12\alpha^2} + \cancel{12\alpha^2} + 6\alpha^3 = -324\alpha$$

$$+4\alpha^3 = +324\alpha$$

$$4\alpha^3 - 324\alpha = 0$$

$$\alpha^3 - 81\alpha = 0$$

$$\alpha(\alpha^2-81) = 0$$

$$\alpha(\alpha-9)(\alpha+9) = 0$$

Matrix multiplication is same as Determinant Multiplication

$$\alpha = 0, \pm 9$$

Q. Let $z = \dfrac{-1+\sqrt{3}i}{2}$, where $i = \sqrt{-1}$, and $r, s \in \{1, 2, 3\}$. Let $P = \begin{bmatrix} (-z)^r & z^{2s} \\ z^{2s} & z^r \end{bmatrix}$ and I be the identity matrix of order 2. Then, the total number of ordered pair (r, s) for which $P^2 = -I$ is

{JEE Advance 2016}

$r = \{1, 2, 3\}$

Q.

Soln:- $z = \dfrac{-1+\sqrt{3}i}{2} = \omega$

$$P = \begin{bmatrix} (-\omega)^r & (\omega)^{2s} \\ (\omega)^{2s} & (\omega)^r \end{bmatrix}$$

$$P^2 = \begin{bmatrix} (-\omega)^{2r} + (\omega)^{4s} & (-\omega)^r(\omega)^{2s} + (\omega)^{2s}(\omega)^r \\ (-\omega)^r(\omega)^{2s} + (\omega)^r(\omega)^{2s} & (\omega)^{4s} + (\omega)^{2r} \end{bmatrix}$$

Soln:-

$P^2 = -I$

$(-\omega)^{2r} + (\omega)^{4s} = -1$

$\omega^2 [(\omega)^r + (\omega)^{2s}] = -1$

From here, r can't be 3.

$\omega^3 = 1$.

$(-\omega)^r(\omega)^{2s} + (\omega)^{2s}(\omega)^r = 0$

$\omega^{r+2s}[(-1)^r + 1] = 0$

Q.

$(-1)^r + 1 = 0 \quad \Rightarrow \quad (-1)^r = -1$

$r = 1, 3$

But,

r can't be 3

Soln:

Hence,

$\boxed{r = 1}$

$\omega^{2r} + \omega^{4s} = -1$

$1 + \omega^2 + \omega^{4s} = 0$

$\boxed{s = 1}$

$\left\{ \begin{array}{l} \omega^4 = \omega \\ 1 + \omega + \omega^2 = 0 \end{array} \right\}$

Q. Let A be a 3×3 invertible matrix, if $|adj(24A)| = |adj(3\,adj(2A))|$, then $|A|^2$ is

[JEE Main 2022]

Sol^n:- $|adj(24A)| = |24A|^2 = (24)^{6}|A|^2$.

$$|adj(3\,adj(2A))| = |3\,adj(2A)|^2$$
$$= (3)^6\,|adj(2A)|^2$$
$$= (3)^6\,|2A|^4$$
$$= (3)^6\,(2)^{12}\,|A|^4$$

$$(24)^6\,|A|^2 = (3)^6\,(2)^{12}\,|A|^4$$
$$\cancel{3^6}\times 8^6\,|A|^2 = \cancel{(3)^6}\,(2)^{12}\,|A|^4$$
$$2^{18} = 2^{12}\,|A|^2$$
$$|A|^2 = 2^6$$

$$\boxed{|A|^2 = 2^6}$$

Q. Let $A = \begin{bmatrix} 2 & b & 1 \\ b & b^2+1 & b \\ 1 & b & 2 \end{bmatrix}$, where $b>0$, the min. value of $\frac{|A|}{b}$ is.

[JEE Main 2019].

Sol^n:
$$|A| = 2\left[2b^2+2-b^2\right] - b\left[2b-b\right] + 1\left[b^2-b^2-1\right]$$
$$= 2\left[b^2+2\right] - b\left[b\right] + 1\left[-1\right]$$
$$= 2b^2+4-b^2-1$$
$$= b^2+3$$

$$\frac{|A|}{b} = \frac{b^2+3}{b} = b + \frac{3}{b}$$

AM $\geq$ GM

$$\frac{\frac{b^2+3}{b}}{2} \geq \left(b \times \frac{3}{b}\right)^{1/2}$$

$$\frac{b^2+3}{b} \geq 2\sqrt{3}$$

when reciprocal is given and minima and is asked, then use AM $\geq$ GM

Min. value of $\frac{b^2+3}{b} = 2\sqrt{3}$

Q. Let $|M|$ denotes the determinant of square matrix M. Let $g: [0, \pi/2] \to R$ be the function defined by $g(\theta) = \sqrt{f(\theta)-1} + \sqrt{f\left(\frac{\pi}{2}-\theta\right)-1}$

where $f(\theta) = \frac{1}{2}\begin{vmatrix} 1 & \sin\theta & 1 \\ -\sin\theta & 1 & \sin\theta \\ -1 & -\sin\theta & 1 \end{vmatrix}$

$$+ \begin{vmatrix} \sin\pi & \cos(\theta+\pi/4) & \tan(\theta-\pi/4) \\ \sin(\theta-\pi/4) & -\cos\pi/2 & \ln(4/\pi) \\ \cot(\theta+\pi/4) & \ln\pi/4 & \tan\pi \end{vmatrix}$$

Let $p(x)$ be a quadratic polynomial whose roots are maximum and minimum value of the function $g(\theta)$ and $p(2) = 2-\sqrt{2}$, then which of the following is/are true?

[JEE Advance 20

(a) $p\left(\frac{3+\sqrt{2}}{4}\right) < 0$

(c) $p\left(\frac{5\sqrt{2}-1}{4}\right) > 0$

(b) $p\left(\frac{1+3\sqrt{2}}{4}\right) > 0$

(d) $p\left(\frac{5-\sqrt{2}}{4}\right) < 0$

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$$f(\theta) = \frac{1}{2}\begin{vmatrix} 1 & \sin\theta & 1 \\ -\sin\theta & 1 & \sin\theta \\ -1 & -\sin\theta & 1 \end{vmatrix} + \begin{vmatrix} \sin\pi & \cos\left(\theta+\frac{\pi}{4}\right) \\ \sin\left(\theta-\frac{\pi}{4}\right) & -\cos\pi/2 \\ \cot\left(\theta+\frac{\pi}{4}\right) & \ln\pi/4 \end{vmatrix}$$

$\downarrow$ A $\qquad\qquad\qquad\qquad$ $\downarrow$ B

$$B = \begin{vmatrix} \cancel{\sin\pi} & \cos(\theta+\pi/4) & \tan(\theta-\pi/4) \\ \sin(\theta-\pi/4) & -\cos\pi/2 & \ln 4/\pi \\ \cot(\theta+\pi/4) & \ln\pi/4 & \tan\pi \end{vmatrix}$$

$\cancel{B = -\cos\left(\theta+\frac{\pi}{4}\right)}$

$$B = \begin{vmatrix} \sin\pi & \cos(\theta+\pi/4) & \tan(\theta-\pi/4) \\ \sin(\theta-\pi/4) & -\cos\pi/2 & \ln 4/\pi \\ \cot(\theta+\pi/4) & \ln\pi/4 & \tan\pi \end{vmatrix}$$

$$\cos\left(\theta+\frac{\pi}{4}\right) = \cos\left(\frac{\pi}{2}-\left(\frac{\pi}{4}-\theta\right)\right) = \sin\left(\frac{\pi}{4}-\theta\right)$$

$$= -\sin\left(\theta-\frac{\pi}{4}\right)$$

B is skew symmetric matrix of odd order

$|B| = 0$

$$A = \begin{vmatrix} 1 & \sin\theta & 1 \\ -\sin\theta & 1 & \sin\theta \\ -1 & -\sin\theta & 1 \end{vmatrix}$$

$A = 1(1+\sin^2\theta) - \sin\theta(-\sin\theta+\sin\theta) + 1(\sin^2\theta+1)$

$= 2 + 2\sin^2\theta$

$$f(\theta) = \frac{1}{2}(A+B)$$

$$f(\theta) = \frac{1}{2}(2 + 2\sin^2\theta)$$

$$f(\theta) = 1 + \sin^2\theta$$

$$g(\theta) = \sqrt{f(\theta)-1} + \sqrt{f\left(\frac{\pi}{2}-\theta\right)-1}$$

$$g(\theta) = \sqrt{\sin^2\theta} + \sqrt{\cos^2\theta}$$

$$g(\theta) = |\sin\theta| + |\cos\theta|$$

$| \; | \to$ Mod

$\theta \in \left[0, \frac{\pi}{2}\right]$

$$g(\theta) = \sin\theta + \cos\theta$$

$\text{Min}^m = 1$

$\text{Max}^m = \sqrt{2}$

1 and $\sqrt{2}$ are roots of $p(x)$

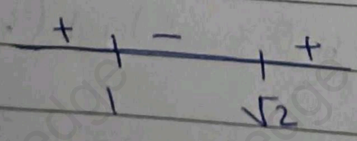
$$p(x) = x^2 - (\sqrt{2}+1)x + \sqrt{2}$$

To verify,

$p(2) = 2 - \sqrt{2}$

Hence,

roots $= 1, \sqrt{2}$



(a) $p\left(\frac{3+\sqrt{2}}{4}\right) < 0$

$\downarrow$

$\frac{4.4}{4} = 1.1 \rightarrow -ve$

$p\left(\frac{3+\sqrt{2}}{4}\right) < 0$ {(a) is correct}

(b) $p\left(\frac{1+3\sqrt{2}}{4}\right) > 0$

$\downarrow$

$\frac{5.2}{4} = 1.3 \rightarrow -ve$

Hence, $p\left(\frac{1+3\sqrt{2}}{4}\right) < 0$ {'b' is incorrect}

(c) $p\left(\frac{5\sqrt{2}-1}{4}\right) > 0$

$\downarrow$

$\frac{6}{4} = 1.5 \rightarrow +ve$

Hence, {c is correct}

(d) $p\left(\frac{5-\sqrt{2}}{4}\right) < 0$

$\downarrow$

$\frac{3.6}{4} = 0.9 \rightarrow +ve$

'D' is wrong.

$5-1.4$

$\frac{3.6}{4}$

Q. Let β be a real number. Consider the matrix $A = \begin{bmatrix} \beta & 0 & 1 \\ 2 & 1 & -2 \\ 3 & 1 & -2 \end{bmatrix}$. If $A^7 - (\beta-1)A^6 - \beta A^5$ is a singular matrix, then value of 9β is

{JEE Advance 2022}

Sol^n:-

$\cancel{|A| = 0}$

$\cancel{\beta(-2+2) + 1(2-3) \neq 0}$

$$|A| = \beta(-2+2) + 1(2-3) = -1$$

Now,

$$A^7 - (\beta-1)A^6 - \beta A^5$$

$$A^5\left[A^2 - (\beta-1)A - \beta I\right]$$

$$A^5\left[A^2 + A - \beta A - \beta I\right]$$

$$A^5\left[A(A+I) - \beta I(A+I)\right]$$

$$A^5\left[(A-\beta I)(A+I)\right]$$

$$\left|A^7 - (\beta-1)A^6 - \beta A^5\right| = 0$$

$$\left|A^5(A-\beta I)(A+I)\right| = 0$$

$$|A|^5\,|A-\beta I|\,|A+I| = 0$$

$$-1\,|A-\beta I|\,|A+I| = 0$$

$$A - \beta I = \begin{bmatrix} \beta & 0 & 1 \\ 2 & 1 & -2 \\ 3 & 1 & -2 \end{bmatrix} - \begin{bmatrix} \beta & 0 & 0 \\ 0 & \beta & 0 \\ 0 & 0 & \beta \end{bmatrix}$$

$$A-\beta I = \begin{bmatrix} 0 & 0 & 1 \\ 2 & 1-\beta & -2 \\ 3 & 1 & -2-\beta \end{bmatrix}.$$

$$|A-\beta I| = 1\,[2-3(1-\beta)]$$
$$= 2-3+3\beta$$
$$= 3\beta-1$$

$$A+I = \begin{bmatrix} \beta+1 & 0 & 1 \\ 2 & 2 & -2 \\ 3 & 1 & -1 \end{bmatrix}$$

$$|A+I| = \beta+1(-2+2) + 1(2-6)$$
$$= -4$$

$$-1\{3\beta-1)(-4) = 0$$
$$4(3\beta-1) = 0$$

$$\boxed{\beta = \frac{1}{3}}$$

$$9\beta = 3 \times \frac{1}{3}$$

$$\boxed{9\beta = 3}$$

Q. If $M = \begin{bmatrix} 5/2 & 3/2 \\ -3/2 & -1/2 \end{bmatrix}$, then $M^{2022} = ?$

{JEE Advance 2022}

Sol^n:- $M^2 = \begin{bmatrix} 5/2 & 3/2 \\ -3/2 & -1/2 \end{bmatrix} \begin{bmatrix} 5/2 & 3/2 \\ -3/2 & -1/2 \end{bmatrix}$

$M^2 = \begin{bmatrix} 16/4 & 12/4 \\ -12/4 & -8/4 \end{bmatrix}$

$M^2 = \begin{bmatrix} 4 & 3 \\ -3 & -2 \end{bmatrix}$

$M^3 = \begin{bmatrix} 11/2 & 9/2 \\ -9/2 & -7/2 \end{bmatrix}$

$\frac{5}{2}, \frac{16}{4}, \frac{44}{8}, \ldots$ or $\frac{5}{2}, \frac{8}{2}, \frac{11}{2}, \ldots$

$\cancel{\frac{12131}{2}}$

$2a_{2022} = 5 + 2021 \times 3$

$a_{2022} = \frac{6068}{2} = 3034$

$\cancel{M^{2022} = [\ 12131 \times 2}$

$M^{2022} = \begin{bmatrix} 3034 & 3033 \\ -3033 & -3032 \end{bmatrix}$

Method:2

$$M = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \frac{3}{2}\begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}$$

$$M = I + \frac{3}{2} A$$

$$M^2 = I + \frac{9}{4} A^2 + 3A$$

$$A^2 = 0$$

$$(M^2)^1 = I + 3A$$

$$M^4 = I + 9A^2 + 6A$$

$$(M^2)^2 = M^4 = I + 6A$$

$$(M^2)^{1011} = I + 3033A$$

$$M^{2022} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 3033 & 3033 \\ -3033 & -3033 \end{bmatrix}$$

$$\cancel{M^{2022} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}} \qquad M^{2022} = \begin{bmatrix} 3034 & 3033 \\ -3033 & -3032 \end{bmatrix}$$

Q. Let $P = \begin{bmatrix} 3 & -1 & -2 \\ 2 & 0 & \alpha \\ 3 & -5 & 0 \end{bmatrix}$, $\alpha \in R$. Suppose $Q = [q_{ij}]$ is a matrix such that $PQ = kI$ where $k \in R$, $k \neq 0$ and I is the identity matrix of order 3. If $q_{23} = -\frac{k}{8}$, $|Q| = \frac{k^2}{2}$, then

[JEE Advance 2016]

(a) $\alpha = 0,\ K = 8$

(b) $4\alpha - K + 8 = 0$

(c) $|P(\text{adj}(Q))| = 2^9$

(d) $|Q(\text{adj}(P))| = 2^{13}$

Sol^n:

$$P = \begin{bmatrix} 3 & -1 & -2 \\ 2 & 0 & \alpha \\ 3 & -5 & 0 \end{bmatrix}$$

$$|P| = 3(5\alpha) + 1(-3\alpha) - 2(-10)$$
$$= 15\alpha - 3\alpha + 20$$
$$= 12\alpha + 20$$

$$PQ = KI$$
$$|PQ| = |KI|$$
$$|P|\,|Q| = K^3 |I|$$
$$(12\alpha + 20)\ \frac{K^2}{2} = K^3$$

$$12\alpha + 20 = 2K$$

$$PQ = KI$$

Pre-multiply by P^{-1}

$$P^{-1}PQ = P^{-1}KI$$
$$Q = \frac{\text{adj}(P)}{|P|}\,K$$

$$Q = \frac{\text{adj}(P)}{2K} \cdot K$$

$$Q = \frac{\text{adj}(P)}{2}$$

$$\text{adj}(P) = \begin{array}{c|ccccc} 3 & -1 & -2 & 3 & -1 \\ \hline 2 & 0 & \alpha & 2 & 0 \\ 3 & -5 & 0 & 3 & -5 \\ 3 & -1 & -2 & 3 & -1 \\ 2 & 0 & \alpha & 2 & 0 \end{array}$$

$$q_{23} = \frac{-4-3\alpha}{2}$$

$$\frac{-k}{8} = \frac{-4-3\alpha}{2}$$

$$k = 16 + 12\alpha$$

$$12\alpha + 20 = 2k \quad -①$$

$$-\quad \cancel{12\alpha} + 16 = k \quad -②$$

$$4 = k$$

$$\therefore \boxed{k=4}$$

$$\boxed{\alpha = -1}$$

Checking options:-

(b) $4(-1) - 4 + 8 = 0$ (Correct)

(c) $|P| = 8 \qquad |Q| = 8$

$$|P(\text{adj } Q)| = |P|\,|\text{adj}(Q)|$$
$$= 8\,|Q|^2$$
$$= 2^3 \cdot 2^6$$
$$= 2^9 \quad \text{(Correct)}$$

(d) $|Q(\text{adj} P)| = |P(\text{adj} Q)| = 2^9$

(Incorrect)

Q. Let $P = \begin{bmatrix} 3 & -1 & -2 \\ 2 & 0 & \alpha \\ 3 & -5 & 0 \end{bmatrix}$

then, $\alpha^2 + k^2 = ?$ [JEE Main 2022]

Sol^n:- Same as previous question

$\alpha = -1 \quad , k = 4$

$\alpha^2 + k^2 = 1 + 16$

$= 17$

Q. The trace of a square matrix is defined to be the sum of its diagonal entries. If A is a 2×2 matrix, such that the trace of A is 3 and the tr (A^3) is -18, then value of $|A|$ is.

[JEE Advance 2020]

Sol^n:- $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

$a + d = 3$

$A^2 = \begin{bmatrix} a^2 + bc & ab + bd \\ ac + cd & bc + d^2 \end{bmatrix}$

$A^3 = \begin{bmatrix} a(a^2+bc) + c(ab+bd) & \\ & b(ac+cd) + d(bc+d^2) \end{bmatrix}$

$$a(a^2+bc) + c(ab+bd) + b(ac+cd) + d(bc+d^2) = -18$$
$$a^3 + abc + abc + bcd + abc + bcd + bcd + d^3 = -18$$
$$a^3 + d^3 + 3abc + 3bcd = -18$$
$$(a+d)^3 - 3ad(a+d) + 3bc(a+d) = -18$$
$$27 - 9ad + 9bc = -18$$
$$27 - 9(ad-bc) = -18$$
$$-9(ad-bc) = -45$$
$$ad-bc = 5$$

$$|A| = ad-bc$$
$$|A| = 5$$

Q. Let $P_1 = I$, $P_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$, $P_3 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$P_4 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$, $P_5 = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$,

& $P_6 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$ and $X = \sum_{k=1}^{6} P_K \begin{bmatrix} 2 & 1 & 3 \\ 1 & 0 & 2 \\ 3 & 2 & 1 \end{bmatrix} P_K^T$

where P_K^T denotes the transpose of matrix P_K. then which of the following option is/are correct? [JEE Advance 2019]

(a) X is a symmetric matrix

(b) The sum of diagonal entries of X is 18

(c) $X - 30I$ is an invertible matrix

(d) If $X \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \alpha \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, then $\alpha = 30$.

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Sol^n:- $X = \left(P_1 A P_1^T + P_2 A P_2^T + P_3 A P_3^T + P_4 A P_4^T + P_5 A P_5^T + P_6 A P_6^T\right)$.

$= \left(P_1 A P_1^T + P_2 A P_2^T + P_3 A P_3^T + P_4 A P_4^T + P_5 A P_5^T + P_6 A P_6^T\right)^T$

$= (P_1 A P_1^T)^T + (P_2 A P_2^T)^T + (P_3 A P_3^T)^T + (P_4 A P_4^T)^T + (P_5 A P_5^T)^T + (P_6 A P_6^T)^T$

$= P_1 A^T P_1^T + P_2 A^T P_2^T + P_3 A^T P_3^T + P_4 A^T P_4^T + P_5 A^T P_5^T + P_6 A^T P_6^T$

$A = A^T$

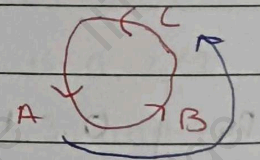
$= P_1 A P_1^T + P_2 A P_2^T + P_3 A P_3^T + P_4 A P_4^T + P_5 A P_5^T + P_6 A P_6^T$

Hence,

X is symmetric.

* $tr(AB) = tr(BA)$

*



$tr(ABC) = tr(BCA)$
$= tr(CAB)$

$Tr(X) = \sum_{k=1}^{6} Tr(\overset{A}{P_k}\ \overset{B}{A}\ \overset{C}{P_k^T})$

$Tr(X) = \sum_{k=1}^{6} Tr(A P_k^T P_k)$

$Tr(X) = \sum_{k=1}^{6} Tr(A)$

$Tr(X) = 6\ Tr(A)$

$= 6 \times 3$

$Tr(X) = 18$

Checking option 'D'

$$X\begin{bmatrix}1\\1\\1\end{bmatrix} = \alpha\begin{bmatrix}1\\1\\1\end{bmatrix}$$

$$X = \sum_{k=1}^{6} (P_k A R)$$

$$= \sum_{k=1}^{6} P_k \begin{bmatrix}6\\3\\6\end{bmatrix}$$

$$= \begin{bmatrix}2 & 2 & 2\\2 & 2 & 2\\2 & 2 & 2\end{bmatrix}\begin{bmatrix}6\\3\\6\end{bmatrix} = \begin{bmatrix}30\\30\\30\end{bmatrix} = 30\begin{bmatrix}1\\1\\1\end{bmatrix}$$

$XR = 30R$

$\boxed{\alpha = 30}$

$XR = 30R$

$R(X - 30I) = 0$

$|X - 30I| = 0$

$|X - 30I| = 0 \rightarrow$ Not invertible matrix

Q. which of the following is/are not the square of a 3×3 matrix with real entries? [JEE Advance 2017]

(a) $\begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & -1\end{bmatrix}$ (b) $\begin{bmatrix}1 & 0 & 0\\0 & -1 & 0\\0 & 0 & -1\end{bmatrix}$

(c) $\begin{bmatrix}-1 & 0 & 0\\0 & -1 & 0\\0 & 0 & -1\end{bmatrix}$ (d) $\begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{bmatrix}$

Sol^n:- For a matrix to be square of other matrix, its determinant should be positive.

$|a| = -1$
$|b| = 1$
$|c| = -1$
$|d| = 1$

Q. For any 3×3 matrix M, let $E = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 8 & 13 & 18 \end{bmatrix}$,

$P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$, $F = \begin{bmatrix} 1 & 3 & 2 \\ 8 & 18 & 13 \\ 2 & 4 & 3 \end{bmatrix}$.

If Q is non-singular matrix of order 3, then which of the following statements is/are true?

[JEE Advance 2021]

(a) $F = PEP$ and $P^2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

(b) $|EQ + PFQ^{-1}| = |EQ| + |PFQ^{-1}|$

(c) $|(EF)^3| > |EF|^2$

(d) sum of the diagonal entries of $(P^{-1}EP + F)$
= sum of diagonal entries of $(E + P^{-1}FP)$.

Soln:- (a) $P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$

$P^2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$F = \begin{bmatrix} 1 & 3 & 2 \\ 8 & 18 & 13 \\ 2 & 4 & 3 \end{bmatrix}$

$PEP = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 8 & 13 & 18 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$

$= \begin{bmatrix} 1 & 2 & 3 \\ 8 & 13 & 18 \\ 2 & 3 & 4 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$

$= \begin{bmatrix} 1 & 3 & 2 \\ 8 & 18 & 13 \\ 2 & 4 & 3 \end{bmatrix}$

$F = PEP$ (a is correct)

~~(b)~~ (c) $|E| = 1(54-52) - 2(36-32) + 3(26-24)$

$= 2 - 8 + 6$

$= 0$

$|EF|^3 = 0$

$|EF|^2 = 0$

'c is incorrect'

$|F| = -2 + 6 - 4$

$= 0$

RHS =
(b) $|EQ| + |PFQ^{-1}|$

$|E||Q| + |P||F|\frac{1}{|Q|}$

$|E| = 0 \quad |F| = 0$

0

LHS = $|EQ + PFQ^{-1}|$

$\downarrow$

A

$A = EQ + PFQ^{-1}$

Post multiply by Q

$AQ = EQ^2 + PF$

$= EQ^2 + P(PEP)$

$= EQ^2 + EP$

$AQ = E(Q^2 + P)$

$|AQ| = |E(Q^2+P)|$

$|A||Q| = |E||Q^2+P|$

$|A| = 0$

$|EQ + PFQ^{-1}| = 0$

LHS = RHS

[b is correct]

(d) $P^2 = I$

$P = P^{-1}$

$$LHS = Tr(P^{-1}EP + F)$$
$$= Tr(PEP + F)$$
$$= Tr(F + F)$$
$$= Tr(2F)$$
$$= 2\,Tr(F)$$
$$= 2(1 + 18 + 3)$$
$$= 44$$

$$RHS = Tr(E + P^{-1}FP)$$
$$= Tr(E + P^{-1}PEP^2)$$
$$= Tr(E + E)$$
$$= 2\,Tr(E)$$
$$= 2(1 + 18 + 3)$$
$$= 44$$

$$LHS = RHS$$

(d is correct)