Date 17/05/23

Method Of Differentiation

Derivative of f(x) from first principle :-

$$\boxed{f'(x) = \frac{dy}{dx} = \cancel{f(y_1)} = y' = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}}$$

↳ Derivative by first principle

eg. Differentiate $f(x) = \tan x$ by first principle.

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

$$f'(x) = \lim_{h\to 0} \frac{\tan(x+h) - \tan x}{h}$$

$$f'(x) = \lim_{h\to 0} \frac{\frac{\tan x + \tan h}{1 - \tan x \tan h} - \tan x}{h}$$

$$f'(x) = \lim_{h\to 0} \frac{\cancel{\tan x} + \tan h - \cancel{\tan x} + \tan^2 x \tan h}{h(1 - \tan x \tan h)}$$

$$f'(x) = \lim_{h\to 0} \frac{\tan h (1 + \tan^2 x)}{h(1 - \tan x \tan h)}$$

$$f'(x) = \lim_{h\to 0} \frac{\tan h \sec^2 x}{h(1 - \tan x \tan h)}$$

$$f'(x) = \sec^2 x$$

- Derivative of Standard Functions :-

$f(x)$	$f'(x)$
constant	0
x^n	nx^{n-1}
e^x	e^x
a^x	$a^x \ln a$
$\ln\lvert x\rvert$	$\frac{1}{x}$
$\log_a\lvert x\rvert$	$\frac{1}{x\ln a}$ or $\frac{1}{x}\log_a e$
$\sin x$	$\cos x$
$\cos x$	$-\sin x$
$\tan x$	$\sec^2 x$
$\cot x$	$-\operatorname{cosec}^2 x$
$\sec x$	$\sec x \cdot \tan x$
$\operatorname{cosec} x$	$-\operatorname{cosec} x \cot x$
$\sin^{-1} x$	$\frac{1}{\sqrt{1-x^2}}$, $\lvert x\rvert < 1$
$\cos^{-1} x$	$\frac{-1}{\sqrt{1-x^2}}$, $\lvert x\rvert < 1$
$\tan^{-1} x$	$\frac{1}{1+x^2}$
$\cot^{-1} x$	$\frac{-1}{1+x^2}$
$\sec^{-1} x$	$\frac{1}{\lvert x\rvert\sqrt{x^2-1}}$, $\lvert x\rvert > 1$
$\operatorname{cosec}^{-1} x$	$\frac{-1}{\lvert x\rvert\sqrt{x^2-1}}$, $\lvert x\rvert > 1$

Fundamental Theorem :-

1. Term by term Differentiation

$$y = u(x) \pm v(x)$$

$$y' = \frac{d}{dx}\left(u(x)\right) \pm \frac{d}{dx}\left(v(x)\right)$$

2. If $y = K\, f(x)$

$$\frac{dy}{dx} = K \cdot f'(x)$$, where K is constant

3. Product Rule

If $y = u(x) \cdot v(x)$
then,

$$\frac{dy}{dx} = v(x) \cdot \frac{d}{dx}\left(u(x)\right) + u(x) \cdot \frac{d}{dx}\left(v(x)\right)$$

* Generalisation of Product Rule :-

$$\frac{d}{dx}(f \cdot g \cdot h) = gh\frac{df}{dx} + fh\frac{dg}{dx} + fg\frac{dh}{dx}$$

4. Quotient Rule

If $y = \frac{u}{v}$

then,

$$\frac{dy}{dx} = \frac{v.\frac{du}{dx} - u.\frac{dv}{dx}}{v^2}$$

Q. If $y = x^2 + \sin^{-1}x + \ln x$, $\frac{dy}{dx} = ?$

Sol^n:- $$\frac{dy}{dx} = 2x + \frac{1}{\sqrt{1-x^2}} + \frac{1}{x}$$

Q. If $y = \log x^3 + 3\sin^{-1}x + kx^2$, $\frac{dy}{dx} = ?$

Sol^n: $$\cancel{\frac{dy}{dx}} \quad \frac{dy}{dx} = \frac{1}{x^3} \times 3x^2 + \frac{3}{\sqrt{1-x^2}} + 2kx$$

$$\frac{dy}{dx} = \frac{3}{x} + \frac{3}{\sqrt{1-x^2}} + 2kx$$

Q. $y = e^x \sin x$, $\frac{dy}{dx} = ?$

Sol^n: $$\frac{dy}{dx} = e^x \sin x + e^x \cos x$$

$$= e^x(\sin x + \cos x)$$

Q. If $y = e^x \tan x + x \ln x$, then find $\frac{dy}{dx}$.

Sol^n: $$\frac{dy}{dx} = e^x \tan x + e^x \sec^2 x + \ln x + 1$$

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Q. If $y = \dfrac{e^x - \tan x}{x^n + \cot x}$, then $\dfrac{dy}{dx} = ?$

Soln:- $y = \dfrac{(x^n + \cot x)(e^x - \tan x)' - (e^x - \tan x)(x^n + \cot x)'}{(x^n + \cot x)^2}$

$= \dfrac{(x^n + \cot x)(e^x - \sec^2 x) - (e^x - \tan x)(nx^{n-1} - \operatorname{cosec}^2 x)}{(x^n + \cot x)^2}$

Q. If $y = \dfrac{\log x}{x} + \dfrac{e^x}{\sin x} + \log_5 x$, then find $\dfrac{dy}{dx}$.

Soln:- $\dfrac{dy}{dx} = \dfrac{1 - \log x}{x^2} + \dfrac{\sin x\, e^x - e^x \cos x}{\sin^2 x} + \dfrac{1}{x \ln 5}$

Q. If $y = \dfrac{x^4 + x^2 + 1}{x^2 + x + 1}$ and $\dfrac{dy}{dx} = ax + b$, then find a and b.

Soln:- $\cancel{x^4 + x^2 + x}$ $x^4 + x^2 + 1 = (x^2 + x + 1)(x^2 + 1 - x)$

$y = \dfrac{(x^2 + x + 1)(x^2 - x + 1)}{(x^2 + x + 1)}$

$y = x^2 - x + 1$

$\dfrac{dy}{dx} = 2x - 1$

$\boxed{a = 2}$

$\boxed{b = -1}$

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Q. If $y = \dfrac{\sec x + \tan x - 1}{\cancel{\sec}\ \tan x - \sec x + 1}$, then find $\dfrac{dy}{dx}$ at $x = \pi/4$

Sol^n.

$$y = \frac{\frac{1}{\cos x} + \frac{\sin x}{\cos x} - 1}{\frac{\sin x}{\cos x} - \frac{1}{\cos x} + 1}$$

$$y = \frac{\frac{1}{\cos x}(1 + \sin x) - 1}{\frac{1}{\cos x}(\sin x - 1) + 1}$$

$$\frac{dy}{dx} = \frac{(\tan x - \sec x + 1)(\sec x \tan x + \sec^2 x) - (\sec x + \tan x - 1)(\sec^2 x - \sec x \tan x)}{(\tan x - \sec x + 1)^2}$$

$$\frac{dy}{dx} = (\tan x - \sec x + 1)\sec x(\tan x + \sec x) - \sec x(\sec x - \tan x)(\sec x + \tan x - 1)$$

Sol^n.

$$y = \frac{\sec x + \tan x - (\sec^2 x - \tan^2 x)}{\tan x - \sec x + 1}$$

$$y = \frac{(\sec x + \tan x) - (\sec x + \tan x)(\sec x - \tan x)}{(\tan x - \sec x + 1)}$$

$$y = \frac{(\sec x + \tan x)(-\sec x + \tan x + 1)}{(\tan x - \sec x + 1)}$$

$$y = \sec x + \tan x$$

$$\frac{dy}{dx} = \sec x \tan x + \sec^2 x$$

$x = \pi/4$

$$\frac{dy}{dx} = \sqrt{2} + 2$$

Q. If $y = \dfrac{\tan^{-1}x - \cot^{-1}x}{\tan^{-1}x + \cot^{-1}x}$, then find $\dfrac{dy}{dx}$ at $x = -1$.

Solⁿ:-

$$y = \frac{\tan^{-1}x - \tan^{-1}\left(\frac{1}{x}\right)}{\tan^{-1}x + \tan^{-1}\left(\frac{1}{x}\right)}$$

$$y = \frac{\tan^{-1}\left(\frac{x - \frac{1}{x}}{1+1}\right)}{\pi/2}$$

$$y = \frac{2}{\pi}\tan^{-1}\left(\frac{x^2-1}{2x}\right)$$

$$\frac{dy}{dx} = \frac{2}{\pi}$$

$$y = \frac{\tan^{-1}x - \cot^{-1}x}{\pi/2}$$

$$y = \frac{2}{\pi}\left(\tan^{-1}x - \cot^{-1}x\right)$$

$$\frac{dy}{dx} = \frac{2}{\pi}\left[\frac{2}{1+x^2}\right]$$

$$\frac{dy}{dx} = \frac{4}{\pi(1+x^2)}$$

at $x = 1$

$$\frac{dy}{dx} = \frac{2}{\pi}$$

Q. If $y = x^n \log_a (x e^x)$, $\dfrac{dy}{dx} = ?$

Solⁿ:

$$y = x^n\left(\log_a x + \log_a e^x\right)$$

$$y = x^n\left(\log_a x + x\log_a e\right)$$

$$\frac{dy}{dx} = nx^{n-1}\left(\log_a x + x\log_a e\right) + x^n\left(\frac{1}{x\ln a} + \log_a e\right)$$

$$\frac{dy}{dx} = nx^{n-1}\left(\log_a e\, x + \log_a x\right) + x^n\left(\frac{1}{x}\log_a e + \log_a e\right)$$

$$\frac{dy}{dx} = nx^{n-1}\left[\log_a x + x\log_a e\right] + \log_a e\left[x^{n-1} + x^n\right]$$

$$\frac{dy}{dx} = nx^{n-1}\log_a x + \log_a e\left[nx^n + x^{n-1} + x^n\right]$$

5. Chain Rule:-

If $y = f(u)$ and $u = g(x)$

then, $\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$

Q. If $y = \log(\sin x)$, then $\frac{dy}{dx} = ?$

Sol^n: $\frac{dy}{dx} = \frac{1}{\sin x} \cdot \cos x$ # $\left\{ \frac{d(\log \sin x)}{d(\sin x)} \cdot \frac{d(\sin x)}{d(x)} \right\}$

$= \cot x$

Q. If $y = e^{(\tan^{-1}x)^3}$

Sol^n: $\frac{dy}{dx} = e^{(\tan^{-1}x)^3} \cdot 3(\tan^{-1}x)^2 \cdot \frac{1}{1+x^2}$

Right method: {For Boards}

$$\frac{dy}{dx} = \frac{d\left[e^{(\tan^{-1}x)^3}\right]}{d\left[(\tan^{-1}x)^3\right]} \cdot \frac{d\left[(\tan^{-1}x)^3\right]}{d\left[\tan^{-1}x\right]} \cdot \frac{d\left[\tan^{-1}x\right]}{d(x)}$$

Q. If $f'(x) = \sqrt{2x^2-1}$ and $y = f(x^2)$, then $\frac{dy}{dx}$ at $x = 1$ is :

Sol^n:- $y = f(x^2)$

$$\frac{dy}{dx} = f'(x^2) \cdot 2x$$

$$f'(x) = \sqrt{2x^2-1}$$

$$f'(x^2) = \sqrt{2x^4-1}$$

$$\frac{dy}{dx} = \sqrt{2x^4-1} \cdot 2x$$

@ $x=1$

$$\frac{dy}{dx} = 2$$

Logarithmic Differentiation

To find the derivative of a function:-

(a) which is the product or quotient of a number of functions or

(b) of the form $[f(x)]^{g(x)}$, where f and g are both derivable function.

It is convineant to take the logarithm of the function first, and then differentiate eg, $y = x^x$, then find $\frac{dy}{dx}$.

Sol^n:- Taking log both sides

$$\log y = x \log x$$

Diff. w.r.t. x.

$$\frac{d}{dx}(\log y) = \frac{d}{dx}(x \log x)$$

$$\frac{d}{dy}(\log y) \cdot \frac{dy}{dx} = \frac{d}{dx}(x \log x)$$

$$\frac{1}{y} \cdot \frac{dy}{dx} = \left(\log x + x \cdot \frac{1}{x}\right)$$

$$\frac{dy}{dx} = y(\log x + 1)$$

$$\boxed{\frac{dy}{dx} = x^x(1 + \log x)}$$

Q. $y = (\sin x)^{\ln x}$, then find $\frac{dy}{dx}$.

Sol^n. Taking ln both sides

$$\ln y = \ln x \cdot \ln \sin x$$

$$\ln y = \ln(x + \sin x)$$

Diff. w.r.t. x

$$\frac{1}{y} \frac{dy}{dx} = \frac{1}{x + \sin x}(1 + \cos x)$$

$$\frac{dy}{dx} = y\left[\frac{1 + \cos x}{x + \sin x}\right]$$

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Diff. w.r.t. x

$$\frac{1}{y}\frac{dy}{dx} = \frac{\ln \sin x}{x} + \ln x \frac{1}{\sin x} \cos x$$

$$\frac{dy}{dx} = y\left[\frac{\ln \sin x}{x} + \cot x \ln x\right]$$

$$\frac{dy}{dx} = (\sin x)^{\ln x}\left[\frac{\ln \sin x}{x} + \cot x \ln x\right]$$

Parametric Differentiation

If $y = f(\theta)$ and $x = g(\theta)$, where θ is a parameter, ~~they~~ then,

$$\boxed{\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta}} = \frac{f'(\theta)}{g'(\theta)}$$

Q. If $y = a\cos t$ and $x = a(t - \sin t)$, then find the value of $\frac{dy}{dx}$ at $t = \pi/2$

Sol^n:- $y = a\cos t$

$$\frac{dy}{dt} = -a\sin t$$

$x = a(t - \sin t)$

$$\frac{dx}{dt} = a(1 - \cos t)$$

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$$

$$\frac{dy}{dx} = \frac{-\cancel{a}\sin t}{\cancel{a}(1-\cos t)}$$

$$\frac{dy}{dx} = \frac{-\sin t}{1-\cos t}$$

At $t = \pi/2$

$$\frac{dy}{dx} = -1$$

Q. Find $\frac{dy}{dx}$ at $t = \pi/4$ if $y = \cos^4 t$ and $x = \sin^4 t$.

Sol^n:-

$y = \cos^4 t$

$$\frac{dy}{dt} = -4\cos^3 t \cdot \sin t$$

$x = \sin^4 t$

$$\frac{dx}{dt} = 4\sin^3 t \cdot \cos t$$

$$\frac{dy}{dx} = \frac{-\cancel{4}\cos^3 t \sin t}{\cancel{4}\sin^3 t \cdot \cos t}$$

$$= -\cot^3 t \cdot \tan t$$

$t = \frac{\pi}{4}$

$$\frac{dy}{dx} = -1$$

Derivative of a function w.r.t. another function

Let $y = f(x)$ and $z = g(x)$

then

$$\boxed{\frac{dy}{dz} = \frac{dy/dx}{dz/dx}}$$

Q. Differentiate $\ln(\tan x)$ w.r.t. $\sin^{-1} e^x$.

Soln:-

$$y = \ln(\tan x)$$

$$\frac{dy}{dx} = \frac{1}{\tan x} \cdot \sec^2 x$$

$$z = \sin^{-1} e^x$$

$$\frac{dz}{dx} = \frac{1}{\sqrt{1-(e^x)^2}} \cdot e^x$$

$$\frac{dy}{dz} = \frac{\sec^2 x \sqrt{1-e^{2x}}}{\tan x \cdot e^x}$$

$$\frac{dy}{dz} = \frac{1}{\cos^2 x} \cdot \frac{\cos x}{\sin x} \cdot \frac{\sqrt{1-e^{2x}}}{e^x}$$

$$\frac{dy}{dz} = \frac{\sqrt{1-e^{2x}}}{\sin x \cos x \, e^x}$$

Q. The value of $\ln 2 \frac{d}{dx}\left(\log_{\cos x} \operatorname{cosec} x\right)$ at $x = \frac{\pi}{4}$ is

Soln:-

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Soln.: $\ln 2 \frac{d}{dx}\left(\log_e \operatorname{cosec} x \cdot \log_{\cos x} e\right)$

$$\frac{d}{dx}\left(\ln \operatorname{cosec} x \cdot \log_{\cos x} e\right)$$

$$\left(\frac{1}{\operatorname{cosec} x} \cdot -\operatorname{cosec} x \cdot \cot x\right) \log_{\cos x} e + \left(\ln \operatorname{cosec} x\right)\left(\right.$$

$$\frac{d}{dx}\left(\frac{\ln \operatorname{cosec} x}{\ln \cos x}\right)$$

$$= \frac{\ln \cos x \cdot \frac{1}{\operatorname{cosec} x} \cdot -\operatorname{cosec} x \cdot \cot x + \ln \operatorname{cosec} x \cdot \frac{1}{\cos x} \sin x}{(\ln \cos x)^2}$$

$$= \frac{\ln \cos x(-\cot x) + \ln \operatorname{cosec} x \cdot \tan x}{(\ln \cos x)^2}$$

$$\ln 2 \quad \frac{\ln \cos x(-\cot x) + \ln \operatorname{cosec} x (\tan x)}{(\ln \cos x)^2}$$

$x = \pi/4$

$$\ln 2\left(\frac{-\ln \frac{1}{\sqrt{2}} + \ln \sqrt{2} \cdot}{\left(\ln \frac{1}{\sqrt{2}}\right)^2}\right)$$

$$\ln 2 \left(\frac{2(\ln \sqrt{2})}{(-\ln \sqrt{2})^2}\right.$$

$$\ln 2 \left(\frac{2\cancel{(\ln \sqrt{2})}}{(-\ln \sqrt{2}) \cdot \cancel{(-\ln \sqrt{2})}}\right)$$

$$\frac{2 \ln 2}{\ln \sqrt{2}} = \frac{2 \ln (\sqrt{2})^2}{\ln \sqrt{2}} = 4$$

Q. If $y(x) = \cot^{-1}\left[\frac{\sqrt{1+\sin x} + \sqrt{1-\sin x}}{\sqrt{1+\sin x} - \sqrt{1-\sin x}}\right]$, $x \in \left(\frac{\pi}{2}, \pi\right)$

then $\frac{dy}{dx}$ at $x = \frac{5\pi}{6}$. [JEE Main 2021]

Solⁿ:

$$\frac{\sqrt{1+\sin x} + \sqrt{1-\sin x}}{\sqrt{1+\sin x} - \sqrt{1-\sin x}} \times \frac{\sqrt{1+\sin x} + \sqrt{1-\sin x}}{\sqrt{1+\sin x} + \sqrt{1-\sin x}}$$

$$= \frac{(\sqrt{1+\sin x} + \sqrt{1-\sin x})^2}{\cancel{1} + \sin x - \cancel{1} + \sin x}$$

$$= \frac{1 + \cancel{\sin x} + 1 - \cancel{\sin x} + 2\sqrt{(1+\sin x)(1-\sin x)}}{2\sin x}$$

$$= \frac{2 + 2\sqrt{1-\sin^2 x}}{2\sin x}$$

$1+\sin x = (\sin x/2 + \cos x/2)^2$

$1 - \sin x = (\sin x/2 - \cos x/2)^2$

$$= \frac{2 + 2|\cos x|}{2\sin x}$$

$\rightarrow |\cos x| = -\cos x \quad x \in (\pi/2, \pi)$

$$= \frac{1 - \cos x}{\sin x}$$

$$= \frac{2\sin^2 x/2}{\sin x} = \frac{2\sin^2 x/2}{\sin x} = \frac{\cancel{2}\sin^2 x/2}{\cancel{2}\cancel{\sin x/2}\cos x/2}$$

$$= \tan x/2$$

$$\boxed{y(x) = \cot^{-1}\left[\frac{1-\cos x}{\sin x}\right]}$$

$$y(x) = \cot^{-1}\left(\tan x/2\right)$$

~~$y(x) = \tan^{-1}$~~

$$\frac{dy}{dx} = \frac{-1}{1+\tan^2 x/2} \cdot \sec^2 x/2 \cdot \frac{1}{2}$$

$$\frac{dy}{dx} \, P = \frac{-1}{\sec^2 x/2} \cdot \sec^2 x/2 \cdot \frac{1}{2}$$

$$\boxed{\frac{dy}{dx} = \frac{-1}{2}}$$

$\cos 2x = 1 - 2\sin^2 x$

$2\sin^2 x = 1 - \cos 2x$

Q. The derivative of $\tan^{-1}\left(\frac{\sin x - \cos x}{\sin x + \cos x}\right)$ w.r.t. $\frac{x}{2}$, $x \in (0, \pi/2)$ [JEE Main 2019]

Solⁿ:

$$\frac{\sin x - \cos x}{\sin x + \cos x} \times \frac{\sin x - \cos x}{\sin x - \cos x} = \frac{(\sin x - \cos x)^2}{\sin^2 x - \cos^2 x}$$

$$= \frac{\sin^2 x + \cos^2 x - 2\sin x\cos x}{-\cos 2x}$$

$$= \frac{1 - \sin 2x}{-\cos 2x}$$

Solⁿ:-

$$\frac{\sin x - \cos x}{\sin x + \cos x}$$

Div. by. $\cos x$

$$\frac{\tan x - 1}{\tan x + 1}$$

$$\tan^{-1}\left(\frac{\tan x - 1}{1 + \tan x}\right) = \tan^{-1}(\tan x) - \tan^{-1}(1)$$

$$= x - \frac{\pi}{4}$$

$y = x - \pi/4$

$\frac{dy}{dx} = 1$

$z = x/2$

$\frac{dz}{dx} = \frac{1}{2}$

$$\frac{dy}{dz} = \frac{1}{1/2} = 2$$

Q. If $f(1) = 1$ and $f'(1) = 3$, then the derivative of $f(f(f(x))) + (f(x))^2$ at $x = 1$.

Sol^n:- $y = f(f(f(x))) + (f(x))^2$

$y' = f'(f(f(x))) \cdot f'(f(x)) \cdot f'(x) + 2f(x) \cdot f'(x)$

$=$ at $x = 1$

$y' = 3 \times 3 \times 3 + 2 \times 3$

$y' = 33$

Q. If $2y = \left(\cot^{-1}\left(\frac{\sqrt{3}\cos x + \sin x}{\cos x - \sqrt{3}\sin x}\right)\right)^2$, $x \in \left(0, \frac{\pi}{2}\right)$

then $\frac{dy}{dx} = ?$

Sol^n:- $\frac{\sqrt{3}\cos x + \sin x}{\cos x - \sqrt{3}\sin x} = \frac{\frac{\sqrt{3}}{2}\cos x + \frac{1}{2}\sin x}{\frac{1}{2}\cos x - \frac{\sqrt{3}}{2}\sin x}$

$= \frac{\cos \pi/6 \cos x + \sin \pi/6 \sin x}{\cos x \sin \pi/6 - \cos \pi/6 \sin x}$

$= \frac{\cos(\pi/6 - x)}{\sin(\pi/6 - x)} = \cot\left(\frac{\pi}{6} - x\right)$

$2y = \left(\cot^{-1}\left(\cot\left(\frac{\pi}{6} - x\right)\right)\right)^2$

$2y = \left(\frac{\pi}{6} - x\right)^2$

$$2y = x^2 + \frac{\pi^2}{36} - \frac{2x\pi}{3}$$

$$y = \frac{x^2}{2} + \frac{\pi^2}{72} + \frac{\pi}{6}x$$

$$y' = 2x + \pi/6$$

$$y = \frac{1}{2}\left(\frac{\pi}{6} - x\right)^2$$

$$y' = \frac{1}{2} \times 2\left(\frac{\pi}{6} - x\right) \cdot (-1)$$

$$y' = -\left(\frac{\pi}{6} - x\right)$$

$$y' = x - \pi/6$$

Q. $y = \sqrt{\cos x + \sqrt{\cos x + \sqrt{\cos x \ldots \infty}}}$

Sol^n:-

$$y^2 = \cos x + \sqrt{\cos x + \sqrt{\cos x + \ldots \infty}}$$

$$y^2 = \cos x + y$$

Differentiate w.r.t. x

$$2y \cdot \frac{dy}{dx} = -\sin x + \frac{dy}{dx}$$

$$(2y-1)\frac{dy}{dx} = -\sin x$$

$$\frac{dy}{dx} = \frac{-\sin x}{2y-1}$$

Q. If $y = (\tan x)^{(\tan x)^{(\tan x)^{\infty}}}$, then $\frac{dy}{dx} = ?$

Sol^n:

$$y = (\tan x)^{y}$$

$$\log y = y \log \tan x$$

Differentiate w.r.t. 'x'

$$\frac{1}{y}\frac{dy}{dx} = y \frac{1}{\tan x} \cdot \sec^2 x + \log \tan x \cdot \frac{dy}{dx}$$

$$\frac{dy}{dx}\left(\frac{1}{y} - \log(\tan x)\right) = y \frac{2}{\sin 2x}$$

$$\frac{dy}{dx}\left(\frac{1 - y\log(\tan x)}{y}\right) = \frac{2y}{\sin 2x}$$

$$\frac{dy}{dx} = \frac{2y^2}{\sin 2x(1 - y\log(\tan x))}$$

Q. $\frac{d}{dx}\left[\left(\frac{\tan^2 2x - \tan^2 x}{1 - \tan^2 2x \tan^2 x}\right)\cot 3x\right] = ?$

Sol^n:

$$y = \left[\frac{(\tan 2x - \tan x)}{(1 + \tan 2x \tan x)} \frac{(\tan 2x + \tan x)}{(1 - \tan 2x \cdot \tan x)} \cdot \right] \cot 3x$$

$$y = \tan x \cdot \tan 3x \cdot \cot 3x$$

$$y = \tan x$$

$$\frac{dy}{dx} = \sec^2 x$$

Q. $\frac{d}{dx}\left[\tan^{-1}\left(\frac{\sqrt{x}\,(3-x)}{1-3x}\right)\right] = ?$

$$\tan 3x = \frac{3\tan x - \tan^3 x}{1-3\tan^2 x}$$

Sol^n: Let $\sqrt{x} = \tan\theta$

$$y = \tan^{-1}\left(\frac{3\sqrt{x} - x\sqrt{x}}{1-3x}\right)$$

$$= \tan^{-1}\left(\frac{3\sqrt{x} - (\sqrt{x})^3}{1-3x}\right)$$

$$= \tan^{-1}\left(\frac{3\tan\theta - \tan^3\theta}{1-3\tan^2\theta}\right)$$

$$= \tan^{-1}(\tan 3\theta)$$

$$= 3\theta$$

$$y = 3\tan^{-1}\sqrt{x}$$

$$\frac{dy}{dx} = \frac{3}{1+x} \cdot \frac{1}{2\sqrt{x}}$$

$$= \frac{3}{2(1+x)\sqrt{x}}$$

Q. If $x = a\sin 2\theta\,(1+\cos 2\theta)$, $y = b\cos 2\theta\,(1-\cos 2\theta)$, then $\frac{dy}{dx} = ?$

Sol^n:

$x = a\sin 2\theta \cdot 2\cos^2\theta$ $\qquad$ $y = b\cos 2\theta \cdot 2\sin^2\theta$

$$\frac{dx}{d\theta} = 2a\cos 2\theta \cdot 2\cos^2\theta - a\sin 2\theta \cdot 2 \cdot 2\cos\theta \cdot \sin\theta$$

$$= 2a\left[2\cos 2\theta\cos^2\theta - \sin 2\theta \cdot \sin 2\theta\right]$$

Sol^n: $x = a\sin 2\theta(1+\cos 2\theta)$

$$\frac{dx}{d\theta} = a\left[2\cos 2\theta(1+\cos 2\theta) - 2\sin 2\theta \cdot \sin 2\theta\right]$$

$$= 2a\left[\cos 2\theta + \cos^2 2\theta - \sin^2 2\theta\right]$$

$$= 2a\left[\cos 2\theta + \cos 4\theta\right]$$

$$= 2a\left[\cos 4\theta + \cos 2\theta\right]$$

$y = b\cos 2\theta(1-\cos 2\theta)$

$$\frac{dy}{d\theta} = b\left[-2\sin 2\theta(1-\cos 2\theta) + 2\cos 2\theta \cdot \sin 2\theta\right]$$

$$= 2b\left[-\sin 2\theta + \sin 2\theta\cos 2\theta + \sin 2\theta\cos 2\theta\right]$$

$$= 2b\left[2\sin 2\theta\cos 2\theta - \sin 2\theta\right]$$

$$= 2b\left[\sin 4\theta - \sin 2\theta\right]$$

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta}$$

$$= \frac{\cancel{2}b\left[\sin 4\theta - \sin 2\theta\right]}{\cancel{2}a\left[\cos 4\theta + \cos 2\theta\right]} \qquad \left\{ \begin{array}{l} \to \sin C - \sin D \\ \to \cos C + \cos D. \end{array} \right\}$$

$$= \frac{b}{a} \cdot \frac{\cancel{2\cos 3\theta} \cdot \sin\theta}{\cancel{2\cos 3\theta} \cdot \cos\theta}$$

$$= \frac{b}{a}\tan\theta$$

Differentiation of Implicit Functions

$\phi(x, y) = 0$

(a) To find $\frac{dy}{dx}$ of implicit functions, we differentiate each term w.r.t. 'x' regarding y as a function of x. and then collect terms with $\frac{dy}{dx}$ together on one side.

(b) Also, ~~dy~~ $\frac{dy}{dx} = \frac{-\frac{\partial \phi}{\partial x}}{\frac{\partial \phi}{\partial y}}$, where ~~$\frac{\partial \phi}{\partial x}$~~ =

$\frac{\partial \phi}{\partial x}$ = Partial derivative of $\phi(x, y)$ w.r.t. 'x' taking y as a constant.

and

$\frac{\partial \phi}{\partial y}$ = Partial derivative of $\phi(x, y)$ w.r.t. 'y' taking x as a constant.

(c) In case of implicit function, generally both x and y are present in answers of $\frac{dy}{dx}$.

Q. If $x^y + y^x = 2$, then find $\frac{dy}{dx}$.

Soln:- $\frac{\partial \phi}{\partial x} = yx^{y-1} + y^x \ln y$

$\frac{\partial \phi}{\partial y} = x^y \ln x + xy^{x-1}$

$\frac{dy}{dx} = \frac{-(yx^{y-1} + y^x \ln y)}{(x^y \ln x + xy^{x-1})}$

<u>Method 2:</u> <u>General method:-</u>

$x^y + y^x = 2$

$\downarrow \quad \downarrow$

$u \quad v$

$u + v = 2$

$\frac{du}{dx} + \frac{dv}{dx} = 2$ —— (i)

$u = x^y$

$\ln u = y \ln x$

Diff. w.r.t. x

$\frac{1}{u} \cdot \frac{du}{dx} = \frac{dy}{dx} \ln x + \frac{y}{x}$

$\frac{du}{dx} = x^y \left[\frac{dy}{dx} \ln x + \frac{y}{x} \right]$

$v = y^x$

$\log v = x \log y$

diff w.r.t. 'x'

$$\frac{1}{v} \cdot \frac{dv}{dx} = \log y + \frac{x}{y} \frac{dy}{dx}$$

$$\frac{dv}{dx} = y^x \left[\log y + \frac{x}{y} \frac{dy}{dx} \right]$$

Put in eqn (i)

$$x^y \left[\frac{dy}{dx} \ln x + \frac{y}{x} \right] + y^x \left[\log y + \frac{x}{y} \frac{dy}{dx} \right] = 0$$

$$\frac{dy}{dx} \left[x^y \log x + y^x \frac{x}{y} \right] = - \left[x^y \frac{y}{x} + y^x \log y \right]$$

$$\frac{dy}{dx} = \frac{-\left[x^{y-1} \cdot y + y^x \log y \right]}{\left[x^y \log x + x y^{x-1} \right]}$$

Q. If $y = \cfrac{\sin x}{1 + \cfrac{\cos x}{1 + \cfrac{\sin x}{1+\cos x \cdots \infty}}}$, then find $\frac{dy}{dx}$.

Soln:

$$y = \cfrac{\sin x}{1 + \cfrac{\cos x}{1+y}}$$

$$y = \frac{\sin x \cdot (1+y)}{1+y+\cos x}$$

$$y(1+y+\cos x) - \sin x(1+y) = 0$$

$$\frac{dy}{dx} = \frac{-\frac{\partial \phi}{\partial x}}{\frac{\partial \phi}{\partial y}}$$

$$= \frac{-[y(-\sin x) - \cos x(1+y)]}{1 + y + \cos x + y(1) - \sin x}$$

$$= \frac{y\sin x + \cos x(1+y)}{1 + 2y + \cos x - \sin x}$$

Q. Find $\frac{dy}{dx}$ if $x+y = \sin(x-y)$

Soln:-

$$x + y = \sin x \cos y - \sin y \cos x$$

$$x + y - \sin x \cos y + \sin y \cos x = 0$$

$$\frac{dy}{dx} = \frac{-\frac{\partial \phi}{\partial x}}{\frac{\partial \phi}{\partial y}}$$

$$= -\frac{[1 - \cos y \cos x - \sin y \sin x]}{1 + \sin x \sin y + \cos y \cos x}$$

$$= \frac{-[1 - [\cos x \cos y + \sin x \sin y]]}{1 + \cos x \cos y + \sin x \sin y}$$

$$= \frac{\cos(x-y) - 1}{\cos(x-y) + 1}$$

Q. If $x^2 + xe^y + y = 0$, find $\frac{dy}{dx}$ and $\frac{dy}{dx}$ at $(0,0)$

Soln:- $\frac{dy}{dx} = \frac{-[2x + e^y]}{[e^y \cdot x + 1]}$

$\left.\frac{dy}{dx}\right|_{(0,0)} = -1$

Differentiation by Trigonometric Transformation

Some standard substitutions:-

Expression.	Substitution
$\sqrt{a^2 - x^2}$	$x = a\cos\theta$, $a\sin\theta$
$\sqrt{a^2 + x^2}$	$x = a\tan\theta$, $a\cot\theta$
$\sqrt{x^2 - a^2}$	$x = \sec\theta$, $\operatorname{cosec}\theta$
$\sqrt{\frac{a-x}{a+x}}$ or $\sqrt{\frac{a+x}{a-x}}$	$x = a\cos 2\theta$
$\sqrt{2ax - x^2}$	$x = a(1 - \cos\theta)$

Q. If $y = \sin^2\left(\cot^{-1}\left(\sqrt{\frac{1+x}{1-x}}\right)\right)$, then $\frac{dy}{dx} = ?$

Sol^n:- Put $x = \cos 2\theta$ $\quad \theta = \frac{\cos^{-1}(x)}{2}$

$$\sin^2\left(\cot^{-1}\left(\sqrt{\frac{1+\cos 2\theta}{1-\cos 2\theta}}\right)\right)$$

$$\sin^2 \cot^{-1}\sqrt{\frac{\cancel{2}\cos^2\theta}{\cancel{2}\sin^2\theta}}$$

$$\sin^2 \cot^{-1} \cot\theta$$

$$y = \sin^2\theta$$

$$\cancel{\frac{d}{d}}\ \cancel{y = \sin^2 \frac{\cos^{-1}(x)}{2}}$$

$$\cancel{\frac{dy}{dx} = 2\sin\left(\frac{\cos^{-1}(x)}{2}\right) \cdot \cos\frac{x}{2} \cdot \frac{1}{x}}$$

$$x = \cos 2\theta$$

$$x = 1 - 2\sin^2\theta$$

$$\frac{1-x}{2} = \sin^2\theta$$

$$y = \frac{1-x}{2}$$

$$\frac{dy}{dx} = \frac{-1}{2}$$

Q. If $f(x) = \sin^{-1}\dfrac{2x}{1+x^2}$, then find $f'(2)$, $f'\left(\frac{1}{2}\right)$, and $f'(1)$

Soln: Put $x = \tan\theta$

$$y = \sin^{-1}\frac{2\tan\theta}{1+\tan^2\theta}$$

$$y = \sin^{-1}(\sin 2\theta)$$

$\cancel{y|F| \quad 2\theta}$

$\cancel{y = 1\backslash 2(\tan^{-1}x}$

$$y = \begin{cases} \cancel{\pi - 2\theta}, & \cancel{2\theta > \pi/2} \\ \cancel{2\theta}, & \cancel{-\frac{\pi}{2} \le \theta \le \pi/2} \\ -(\pi + 2\theta), & \theta < -\pi/ \end{cases}$$

$$f(x) = y = \begin{cases} \pi - 2\tan^{-1}x, & x > 1 \\ 2\tan^{-1}x, & -1 \le x \le 1 \\ -(\pi + 2\tan^{-1}x), & x < -1 \end{cases}$$

$$f'(x) = \begin{cases} \dfrac{-2}{1+x^2}, & x > 1 \\ \dfrac{2}{1+x^2}, & -1 < x < 1 \\ \dfrac{-2}{1+x^2}, & x < -1 \end{cases}$$

$f'(2) = \dfrac{-2}{5}$

$f'\left(\frac{1}{2}\right) = \dfrac{8}{5}$

$f'(1)$ does not exist.

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Derivative of a function and its Inverse function

If g is the inverse of f, then

(a) $g(f(x)) = x \Rightarrow g'(f(x)) \cdot f'(x) = 1$

(b) $f(g(x)) = x \Rightarrow f'(g(x)) \cdot g'(x) = 1$

Q. If g is the inverse of f and $f'(x) = \frac{1}{1+x^n}$, then $g'(x)$ equals-

Solⁿ:

$$f(g(x)) = x$$
$$f'(g(x)) \cdot g'(x) = 1$$
$$\frac{1}{1+(g(x))^n} \cdot g'(x) = 1$$

$$g'(x) = 1 + (g(x))^n$$

Q. If g is inverse of f and $f(x) = 2x + \sin x$, then $g'(x)$ equals.

Solⁿ:

$$f(x) = 2x + \sin x$$
$$f'(x) = 2 + \cos x$$

$$f'(g(x)) \cdot g'(x) = 1$$
$$g'(x) = \frac{1}{2 + \cos(g(x))}$$

Differentiation of Determinants

If ~~f(x)~~ $F(x) = \begin{vmatrix} f(x) & g(x) & h(x) \\ l(x) & m(x) & n(x) \\ u(x) & v(x) & w(x) \end{vmatrix}$,

where f, g, h, l, m, n, u, v and w are differentiable functions of x, then

$$F'(x) = \begin{vmatrix} f'(x) & g'(x) & h'(x) \\ l(x) & m(x) & n(x) \\ u(x) & v(x) & w(x) \end{vmatrix} + \begin{vmatrix} f(x) & g(x) & h(x) \\ l'(x) & m'(x) & n'(x) \\ u(x) & v(x) & w(x) \end{vmatrix}$$

$$+ \begin{vmatrix} f(x) & g(x) & h(x) \\ l(x) & m(x) & n(x) \\ u'(x) & v'(x) & w'(x) \end{vmatrix}$$

Note :-

Sometimes, it is better to expand the determinant first and then differentiate.

Q. If $f(x) = \begin{vmatrix} x & x^2 & x^3 \\ 1 & 2x & 3x^2 \\ 0 & 2 & 6x \end{vmatrix}$, then find $f'(x)$.

Sol^n:

$$f(x) = x(12x^2 - 6x^2) - x^2(6x - 0) + x^3(2)$$
$$= 6x^3 - 6x^3 + 2x^3$$
$$= 2x^3$$

$$f'(x) = 6x^2$$

Method 2

$$f'(x) = \begin{vmatrix} 1 & 2x & 3x^2 \\ 1 & 2x & 3x^2 \\ 0 & 2 & 6x \end{vmatrix} + \begin{vmatrix} x & x^2 & x^3 \\ 0 & 2 & 6x \\ 0 & 2 & 6x \end{vmatrix} + \begin{vmatrix} x & x^2 & x^3 \\ 1 & 2x & 3x^2 \\ 0 & 0 & 6 \end{vmatrix}$$

$$= 6(2x^2 - x^2)$$

$$= 6x^2$$

Q. If $x\log_e(\log_e x) - x^2 + y^2 = 4$ $(y > 0)$, then $\frac{dy}{dx}$ at $x = e$

~~Sol^n:~~

$$x \ln(\ln x) - x^2 + y^2 - 4 = 0$$

$$\frac{\partial \phi}{\partial x} = \ln(\ln x) + \frac{1}{\ln x} - 2x$$

$$\frac{\partial \phi}{\partial y} = 2y$$

$$\frac{dy}{dx} = -\left(\ln(\ln x) + \frac{1}{\ln x} - 2x\right)$$

Sol^n:

$$x \ln(\ln x) - x^2 + y^2 = 4$$

Diff. w.r.t. x

$$\ln(\ln x) + \frac{1}{\ln x} - 2x + 2y \cdot \frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = \frac{-\ln(\ln x) - \frac{1}{\ln x} + 2x}{2y}$$

$$y = \sqrt{4 - x\ln(\ln x) + x^2}$$

$x = e$

$$\left.\frac{dy}{dx}\right|_{x=e} = \frac{-1+2e}{2\sqrt{4+e^2}}$$

Q. If g is the inverse of a function f and $f'(x) = \frac{1}{1+x^5}$, then $g'(x)$ is

Soln:-

$$f'(g(x)) \cdot g'(x) = 1$$

$$\frac{1}{1+(g(x))^5} \cdot g'(x) = 1$$

$$g'(x) = 1 + (g(x))^5$$

Q. If $y = \sec(\tan^{-1}x)$, then $\frac{dy}{dx}$ at $x = 1$.

Soln:-

$$y = \sec(\tan^{-1}x)$$

$$\frac{dy}{dx} = \sec(\tan^{-1}x) \cdot \tan(\tan^{-1}x) \cdot \frac{1}{1+x^2}$$

$$\left.\frac{dy}{dx}\right|_{x=1} = \frac{\sqrt{2}}{2}$$

$$= \frac{1}{\sqrt{2}}$$

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Q. For $x > 1$, if $(2x)^{2y} = 4e^{2x-2y}$, then

$(1 + \log_e 2x)^2 \frac{dy}{dx} = ?$

sol^n.

$(2x)^{2y} = 4e^{2x-2y}$

$2y \ln 2x = (2x-2y) \ln 4e$

$2y \ln 2x = (2x-2y)(\ln 4 + 1)$

$2y \ln 2x = 2\ln 4\, x + 2x - 2y\ln 4 - 2y$

$2y\ln 2x + 2y\ln 4 + 2y = 2\ln 4 x + 2x$

$2y(\ln 2x + \ln 4 + 1) = 2\ln 4 x + 2x$

$\frac{dy}{dx}$ Diff w.r.t. x

$2y\left(\frac{1\times 2}{2x}\right) + (\ln 2x + \ln 4 + 1).2\frac{dy}{dx} = \frac{2}{x} + 2$

$\frac{dy}{dx} 2(\ln 2x + \ln 4 + 1) = \frac{2}{x} + 2 - \frac{2y}{x}$

$\frac{dy}{dx} 2(\ln 2x + \ln 4 + 1) = \frac{2 + 2x - 2y}{x}$

sol^n:-

$(2x)^{2y} = 4e^{2x-2y}$

$(2x)^{2y} = \frac{4e^{2x}}{e^{2y}}$

$(2x)^{2y} \cdot e^{2y} = 4e^{2x}$

$(2xe)^{2y} = 4e^{2x}$

$2y \ln(2xe) = \ln(4e^{2x})$

$$y = \frac{\ln(4e^{2x})}{2\ln(2xe)}$$

$$y = \frac{\ln 4 + 2x}{2[\ln 2 + \ln x + 1)}$$

$$\frac{dy}{dx} = \frac{4(\ln 2 + \ln x + 1) - 2(\ln 4 + 2x)\left(\frac{1}{x}\right)}{4(\ln x + \ln 2 + 1)^2}$$

$$\frac{dy}{dx} = \frac{4(\ln 2x + 1) - 2(\ln 4 + 2x)\frac{1}{x}}{4(\ln 2x + 1)^2}$$

$$(\ln 2x + 1)^2 \frac{dy}{dx} = \frac{4x(\ln 2x + 1) - 2(\ln 4 + 2x)}{4x}$$

$$= \frac{4x\ln 2x + \cancel{4x} - 2\ln 4 - \cancel{4x}}{4x}$$

$$= \frac{4x\ln 2x - 2\ln 4}{4x}$$

$$(\ln 2x + 1)^2 \frac{dy}{dx} = \frac{2x\ln 2x - \ln 4}{2x}$$

$$(\ln 2x + 1)^2 \frac{dy}{dx} = \frac{x\ln 2x - \ln 2}{x}$$

Q. Let $f: R \to R$ be a function such that $f(x) = x^3 + x^2 f'(1) + x f''(2) + f'''(3)$, $x \in R$, then $f(2)$ equals. [JEE Main 2019].

Solⁿ:- $f(x) = x^3 + x^2 f'(1) + x f''(2) + f'''(3)$

$f(1) = 1 + f'(1) + f''(2) + f'''(3)$

$f(2) = 8 + 4f'(1) + 2f''(2) + f'''(3)$

$f(2) - f(1) = 7 + 3f'(1) + f''(2)$ —— (i)

$f(3) = 27 + 9f'(1) + 3f''(2) + f'''(3)$

$f(3) - f(2) = 19 + 5f'(1) + f''(2)$ —— (ii)

Solⁿ:- $f(x) = x^3 + x^2 f'(1) + x f''(2) + f'''(3)$

$f'(x) = 3x^2 + 2x f'(1) + f''(2)$

$f''(x) = 6x + 2f'(1)$

$f'''(x) = 6$

$\boxed{f'''(3) = 6}$

$f'(1) = 3 + 2f'(1) + f''(2)$

$f''(2) = 12 + 2f'(1)$

$f'(1) = 3 + 2f'(1) + 12 + 2f'(1)$

$-3f'(1) = 15$

$\boxed{f'(1) = -5}$

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$$\boxed{f''(2) = 2}$$

$f(x) = x^3 + x^2 \times (-5) + x 2 + 6$

$f(x) = x^3 - 5x^2 + 2x + 6$

$f(2) = 8 - 20 + 4 + 6$

$$\boxed{f(2) = -2}$$

Higher Order Derivatives :-

Let a function $y = f(x)$ be defined on an interval (a, b). If $f(x)$ is differentiable function, then its derivative $f'(x)$ is called the first derivative of y w.r.t. 'x'. If $f'(x)$ is again differentiable function on (a,b), then its derivative $f''(x)$ $\left\{ \frac{d^2y}{dx^2} \text{ or } y'' \right\}$ is called second derivative of y w.r.t. 'x'. similarily, the third order derivative of y w.r.t. 'x', if it exists is defined by $\frac{d^3y}{dx^3} = \frac{d}{dx}\left(\frac{d^2y}{dx^2}\right)$ and denoted by $f'''(x)$ or y''' and so on.

Note :-

If $x = f(\theta)$ and $y = g(\theta)$, where θ is a parameter, then $\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{g'(\theta)}{f'(\theta)}$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right)$$

$$= \frac{d}{d\theta}\left(\frac{dy}{dx}\right) \cdot \frac{d\theta}{dx}$$

$$\boxed{\frac{d^2y}{dx^2} = \frac{\frac{d}{d\theta}\left(\frac{dy}{dx}\right)}{\left(\frac{dx}{d\theta}\right)}}$$

In general,

$$\boxed{\frac{d^n y}{dx^n} = \frac{\frac{d}{d\theta}\left(\frac{d^{n-1}y}{dx^{n-1}}\right)}{\frac{dx}{d\theta}}}$$

Q. If $x = a(t + \sin t)$ and $y = a(1-\cos t)$, then find $\frac{d^2y}{dx^2}$.

Sol:

$$\frac{dx}{dt} = a(1+\cos t)$$

$$\frac{dy}{dt} = a(\sin t)$$

$$\frac{dy}{dx} = \frac{\sin t}{1+\cos t}$$

Method 1 -

$$\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left(\frac{\sin t}{1+\cos t}\right)}{a(1+\cos t)}$$

$$= \frac{\cos t(1+\cos t) - \sin t(-\sin t)}{a(1+\cos t)^2 \cdot (1+\cos t)}$$

$$= \frac{\cos t + \cos^2 t + \sin^2 t}{a(1+\cos t)^3}$$

$$= \frac{(\cos t + 1)}{a(1+\cos t)^3}$$

$$= \frac{1}{a(1+\cos t)^2}$$

$$= \frac{1}{4a}(\sec^2 t/2)$$

Method 2

$$\frac{dy}{dx} = \frac{2\sin t/2 \cos t/2}{2\cos^2 t/2}$$

$$= \tan(t/2)$$

$$\frac{d^2y}{dx^2} = \frac{\sec^2(t/2)\cdot 1/2}{a(1+\cos t)}$$

$$\frac{d^2y}{dx^2} = \frac{1}{4a}\sec^4(t/2)$$

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Q. If $e^y + xy = e$, then ordered pair $\left(\frac{dy}{dx}, \frac{d^2y}{dx^2} \right)$ at $x = 0$ is

[JEE Main 2019]

Soln:

$$e^y + xy - e = 0$$

At $x = 0$

$e^y = e$

$\boxed{y = 1}$

$$\frac{dy}{dx} = -\frac{y}{e^y + x}$$

$$\left.\frac{dy}{dx}\right|_{(0,1)} = \frac{-1}{e}$$

$$y'(e^y + x) = -y$$

Diff. w.r.t. 'x'

$$y''(e^y + x) + y'(e^y \cdot y' + 1) = -y'$$

$$y''e = \frac{-y' - y'(e^y \cdot y' + 1)}{(e^y + x)}$$

$$\left. y'' \right|_{(0,1)} = \frac{\frac{1}{e} + \frac{1}{e}(0)}{(e)}$$

$$\left.\frac{d^2y}{dx^2}\right|_{(0,1)} = \frac{1}{e^2}$$

Q. If $x = 3\tan t$ and $y = 3\sec t$, find $\frac{d^2y}{dx^2}$ at $t = \pi/4$.

[JEE Main 2019]

Sol^n $\frac{dx}{dt} = 3\sec^2 t$

$\frac{dy}{dt} = 3\sec t \tan t$

$\frac{dy}{dx} = \frac{\cancel{3\sec t}\tan t}{\cancel{3}\sec^{\cancel{2}} t} = \frac{\tan t}{\sec t} = \frac{\sin t \cancel{\cos t}}{\cos t}$

$= \sin t$

$\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}}$

$= \frac{\cos t}{3\sec^2 t}$

$\frac{d^2y}{dx^2} = \frac{\cos^3 t}{3}$

$\left.\frac{d^2y}{dx^2}\right|_{t=\pi/4} = \frac{1}{6\sqrt{2}}$

Q. Let $f(x) = \begin{vmatrix} x^3 & \sin x & \cos x \\ 6 & -1 & 0 \\ p & p^2 & p^3 \end{vmatrix}$, where p = constant.

then $\frac{d^3}{dx^3} f(x)$ at $x = 0$ [JEE Main 1997]

Sol^n:- $f(x) = x^3(-p^3) - \sin x(6p^3) + \cos x(6p^2+p)$

$$f(x) = -p^3x^3 - 6p^3\sin x + 6p^2\cos x + p\cos x$$

$p = $ cons.

$$\frac{df(x)}{dx} = -3p^3x^2 - 6p^3\cos x - 6p^2\sin x - p\sin x$$

$$\frac{d^2f(x)}{dx^2} = -6p^3x + 6p^3\sin x - 6p^2\cos x - p\cos x$$

$$\frac{d^3f(x)}{dx^3} = -6p^3 + 6p^3\cos x + 6p^2\sin x + p\sin x$$

$$\left.\frac{d^3f(x)}{dx^3}\right|_{x=0} = -6p^3 + 6p^3$$

$$= 0$$

Q. If $y(x) = (x^x)^x$, $x > 0$, then $\frac{d^2x}{dy^2} + 20$ at $x = 1$ is equal to.

[JEE main 2022]

sol^n:- $\frac{d^2x}{dy^2} = \frac{d}{dy}\left(\frac{dx}{dy}\right)$

$y = (x)^{x^2}$

$\ln y = x^2 \ln x$

Diff. w.r.t. 'x'

$\frac{1}{y}\cdot\frac{dy}{dx} = 2x \cdot \ln x + x$

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$$\frac{dy}{dx} = (x^x)^x \left[2x \ln x + x \right]$$

$$\frac{dx}{dy} = \frac{1}{(x^x)^x (2x\ln x + x)}$$

$$\frac{d^2x}{dy^2} = \frac{d}{dx}\left(\frac{dx}{dy}\right) \cdot \frac{dx}{dy}$$

$$= \frac{d}{dx}\left(\frac{1}{(x^x)^x (2x\ln x + x)}\right) \cdot \frac{dx}{dy}$$

$$= \frac{d}{dx}\left(y\,(2x\ln x + x)\right)^{-1} \cdot \frac{dx}{dy}$$

$$= \frac{-1}{[y(2x\ln x + x)]^2} \cdot \frac{dy}{dx}(2x\ln x + x) + y(2\ln x + 2 + 1) \cdot \frac{dx}{dy}$$

$$= \cancel{\frac{-(2x\ln x + x)\,y\,(2\ln x + 3)}{[y(2x\ln x + x)]^2}}$$

$$\cancel{= -1\,(x^x)^x}$$

$$\frac{d^2x}{dy^2} = \frac{-1\left[y(2x\ln x + x)(2x\ln x + x) + \cancel{y}(2\ln x + 3) \cdot \frac{1}{\cancel{y(2x\ln x + x)}}\right]}{[y(2x\ln x + x)]^2}$$

$$\cancel{= -[y(2x\ln x + x)^2 +}$$

$$\left.\frac{d^2x}{dy^2}\right|_{x=1} = \frac{-1\left[1(0+1)(1) + (3)\right]}{[1(1)^2]}$$

$$= -4$$

$$\frac{d^2x}{dy^2} + 20 = -4+20 = 16$$

Q. $\frac{d^2x}{dy^2}$ is equivalent to [AIEEE 2007]

(a) $\left(\frac{d^2y}{dx^2}\right)^{-1}$ (b) $-\left(\frac{d^2y}{dx^2}\right)^{-1}\left(\frac{dy}{dx}\right)^{-3}$

(c) $\left(\frac{d^2y}{dx^2}\right)\left(\frac{dy}{dx}\right)^{-2}$ ~~(d)~~ $-\left(\frac{d^2y}{dx^2}\right)\left(\frac{dy}{dx}\right)^{-3}$

Solⁿ:- $\frac{d^2x}{dy^2} = \frac{d}{dy}\left(\frac{dx}{dy}\right)$

$$= \frac{d}{dx}\left(\frac{dx}{dy}\right)\left(\frac{dx}{dy}\right)$$

$$= \frac{d}{dx}\left(\frac{dy}{dx}\right)^{-1} \cdot \left(\frac{dy}{dx}\right)^{-1}$$

$$= \frac{-1}{\left(\frac{dy}{dx}\right)^2} \cdot \frac{d^2y}{dx^2}\left(\frac{dy}{dx}\right)^{-1}$$

$$\frac{d^2x}{dy^2} = -\frac{d^2y}{dx^2} \cdot \left(\frac{dy}{dx}\right)^{-3}$$

Q. If $f''(x) = -f(x)$ where $f(x)$ is continuous, double differentiable function and $g(x) = f'(x)$. ~~If~~ $F(x) = \left\{f\left(\frac{x}{2}\right)\right\}^2 + \left\{g\left(\frac{x}{2}\right)\right\}^2$ and $F(5) = 5$, then $F(10) = ?$ [AIEEE 2006]

Solⁿ $F(x) = \left\{f\left(\frac{x}{2}\right)\right\}^2 + \left\{g\left(\frac{x}{2}\right)\right\}^2$

$F(5) = \left\{f\left(\frac{5}{2}\right)\right\}^2 + \left\{g\left(\frac{5}{2}\right)\right\}^2$

$5 = \left\{f\left(\frac{5}{2}\right)\right\}^2 + \left\{g\left(\frac{5}{2}\right)\right\}^2$

Solⁿ $f''(x) = \frac{d}{dx}(f'(x)) = -f(x)$

$\frac{d}{dx}(g(x)) = -f(x)$

$g'(x) = -f(x)$

$F'(x) = \cancel{x} f\left(\frac{x}{2}\right) \cdot f'\left(\frac{x}{2}\right) \cdot \frac{1}{\cancel{x}} + \cancel{x} \cdot g\left(\frac{x}{2}\right) \cdot g'\left(\frac{x}{2}\right) \frac{1}{\cancel{x}}$

$= f(x/2) \cdot g(x/2) - g(x/2)\, f(x/2)$

$F'(x) = 0$

↓

It means $F(x)$ is constant i.e. 5

Hence

$F(10) = 5$

Q. If $(a+\sqrt{2}\,b\cos x)(a-\sqrt{2}\,b\cos y) = a^2-b^2$, where $a>b>0$, then $\frac{dx}{dy}$ at $(\pi/4, \pi/4)$ is

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Sol^n:- $(a+\sqrt{2}b\cos x)(a-\sqrt{2}b\cos y) = a^2-b^2$

$\cancel{a^2} - \sqrt{2}ab\cos y + \sqrt{2}ab\cos x - 2b^2\cos y = \cancel{a^2} - b^2$

$\sqrt{2}ab(\cos x - \cos y) - 2b^2\cos y + b^2 = 0$

$\sqrt{2}ab\cos x - \sqrt{2}ab\cos y - 2b^2\cos y + b^2 = 0$

Sol^n:- $(a+\sqrt{2}b\cos x)(a-\sqrt{2}b\cos y) = a^2-b^2$

$\frac{dx}{dy} \Rightarrow \left(-\sqrt{2}b\sin x \cdot \frac{dx}{dy}\right)\left(\sqrt{2}b\sin y\right) = 0$

$\sqrt{2}b\sin x \cdot \frac{dx}{dy} = \sqrt{2}b\sin y$

$\frac{dx}{dy} = \frac{\sqrt{2}b\sin y}{\sqrt{2}b\sin x}$

$\left.\frac{dx}{dy}\right|_{(\pi/4,\pi/4)} = \frac{\sqrt{2}b}{}$

Sol^n:- $(a+\sqrt{2}b\cos x)(a-\sqrt{2}b\cos y) = a^2-b^2$

Diff. w.r.t. y.

$(a+\sqrt{2}b\cos x)(\sqrt{2}b\sin y) + (a-\sqrt{2}b\cos y)(-\sqrt{2}b\sin x \cdot \frac{dx}{dy})$

$(a-\sqrt{2}b\cos y)\left(-\sqrt{2}b\sin x\,\frac{dx}{dy}\right) = -(a+\sqrt{2}b\cos x)(\sqrt{2}b\sin y)$

$(\pi/4, \pi/4)$

$(a-b)(\cancel{t}) \frac{dy}{dx} = +(a+b)(\cancel{t})$

$\frac{dy}{dx} = \frac{a+b}{a-b}$

Q. If $f(x) = \frac{1}{1-x}$, then the derivative of the function $f[f\{f(x)\}]$ is equal to

Sol: $\frac{d}{dx} f[f\{f(x)\}] = f'(f(f(x))) . f'f(x) . f'(x)$

$= f'\left(f\left(\frac{1}{1-x}\right)\right) . f'\left(\frac{1}{1-x}\right) . \frac{1}{(1-x)^2}$

$= f'\left(\frac{x-1}{x}\right) . \frac{(1-x)^2}{x^2} . \frac{1}{(1-x)^2}$

$= \cancel{x^2} . \frac{\cancel{(1-x)^2}}{\cancel{x^2}} . \frac{1}{\cancel{(1-x)^2}}$

$= \underline{1}$

Q. $x\sqrt{1+y} + y\sqrt{1+x} = 0$, then $\frac{dy}{dx}$.

Soln:

$x\sqrt{1+y} = -y\sqrt{1+x}$
$x^2(1+y) = y^2(1+x)$
$x^2 + x^2y - y^2 - xy^2 = 0$
$(x-y)(x+y) + xy(x-y) = 0$
$\cancel{(x-y)}(x+y+xy) = 0$
This will not satisfy the eqn

$$x+y+xy=0$$

$$y(1+x)=-x$$

$$y=\frac{-x}{1+x}$$

$$\frac{dy}{dx}=\frac{(-1)(1+x)+x.1}{(1+x)^2}$$

$$=\frac{-1-\cancel{x}+\cancel{x}}{(1+x)^2}$$

$$=-(1+x)^{-2}$$

20. $(x-y)e^{x/x-y}=K$ then

Sol^n:

$$\log(x-y).\ e^{x/x-y}=\ln k$$

$$\log(x-y).\ \frac{x}{x-y}=\ln K$$

Diff. w.r.t. 'x'

$$\frac{1}{(x-y)}.(1-y').\left[\frac{x-y-x(1-y')}{(x-y)^2}\right]=0$$

$$\frac{(1-y')[\cancel{x}-y-\cancel{x}+xy']}{(x-y)^3}=0$$

$$(1-y')(-y+xy')=0$$

$$-y+xy'+yy'-xy'^2=0$$

~~$y'(x-xy'y+x)-y=0$~~

$$y'(x+y-xy')-y=0$$

y

$$\frac{x}{x-y}\left[\frac{1}{x-y}\cdot(1-y')\right]+\log(x-y)\left[\frac{-y+xy'}{(x-y)^2}\right]=0$$

$$\frac{x(1-y')}{(x-y)^2}+\frac{\log(x-y)\,[xy'-y]}{(x-y)^2}=0$$

$$x-xy'+xy'\cdot\log(x-y)-y\log(x-y)=0$$

$$xy'(\log(x-y)-1)-y\log(x-y)+x=0$$

$$y'(x\log(x-y)-x)-$$

Q. $y=(\tan)^{(\tan x)^{\tan x}}$, then at $x=\frac{\pi}{4}$, the value of $\frac{dy}{dx}=?$

Soln. $\log y=\underbrace{(\tan x)^{\tan x}}_{z}\cdot\log(\tan x)$

Diff. w.r.t. x

$$\frac{1}{y}\,y'=\log(\tan x)\cdot\left(\frac{dz}{dx}\right)+(\tan x)^{\tan x}\cdot\frac{1}{\tan x}\cdot\sec^2 x$$

0

Put $x=\pi/4 \rightarrow y=1$

$$\left.\frac{dy}{dx}\right|_{\pi/4}=0+(1)\times\frac{1}{(1)}\times(2)$$

Q. If $y = \tan^{-1}\dfrac{x}{1+\sqrt{1-x^2}} + \sin\left\{2\tan^{-1}\sqrt{\left(\dfrac{1-x}{1+x}\right)}\right\}$, then

$\dfrac{dy}{dx} = ?$

Solⁿ:- Put $x = \cos\theta$

$$y = \tan^{-1}\frac{\cos\theta}{1+\sin\theta} + \sin\left\{2\tan^{-1}\sqrt{\frac{1-\cos\theta}{1+\cos\theta}}\right\}$$

$$y = \tan^{-1}\frac{\cos 2\theta/2}{1+\sin 2\cdot\theta/2} + \sin\left\{2\tan^{-1}\tan\theta/2\right\}$$

$$y = \tan^{-1}\frac{\cos^2\frac{\theta}{2} - \sin^2\theta/2}{(\cos\theta/2 + \sin\theta/2)^2} + \sin\theta$$

$$y = \tan^{-1}\frac{\cos\theta/2 - \sin\theta/2}{\cos\theta/2 + \sin\theta/2} + \sin\theta$$

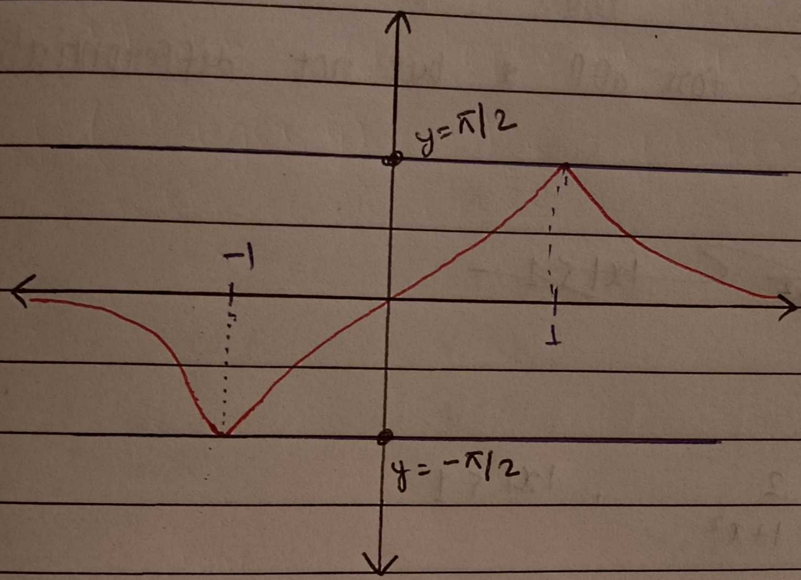
$$y = \tan^{-1}\tan\left(\frac{\pi}{4} - \frac{\theta}{2}\right) + \sin\theta$$

$$y = \frac{\pi}{4} - \frac{\theta}{2} + \sin\theta$$

$$y = \frac{\pi}{4} - \frac{\cos^{-1}x}{2} + \sin(\cos^{-1}x)$$

Analysis and Graphs of Some ITFs

(a) $y = f(x) = \sin^{-1}\left(\frac{2x}{1+x^2}\right) = \begin{cases} 2\tan^{-1}x, & -1 \le x \le 1 \\ -(\pi + 2\tan^{-1}x), & x < -1 \\ (\pi - 2\tan^{-1}x), & x > 1 \end{cases}$



sharp edge @ 1 and -1. Hence, f^n is non-differentiable at 1, -1

$\sin^{-1}(\sin 2\theta) = 2\theta$

when

$-\frac{\pi}{2} \le 2\theta \le \frac{\pi}{2}$

$-\frac{\pi}{4} \le \theta \le \frac{\pi}{4}$

$-1 \le \tan\theta \le 1$

$-1 \le x \le 1$

between $(\pi/2, 3\pi/2) \longrightarrow -x + \pi$

$\Rightarrow \pi - 2\tan^{-1}x$

Important Points :-

1. D : $x \in R$

R : $y \in \left[-\frac{\pi}{2}, \frac{\pi}{2} \right]$

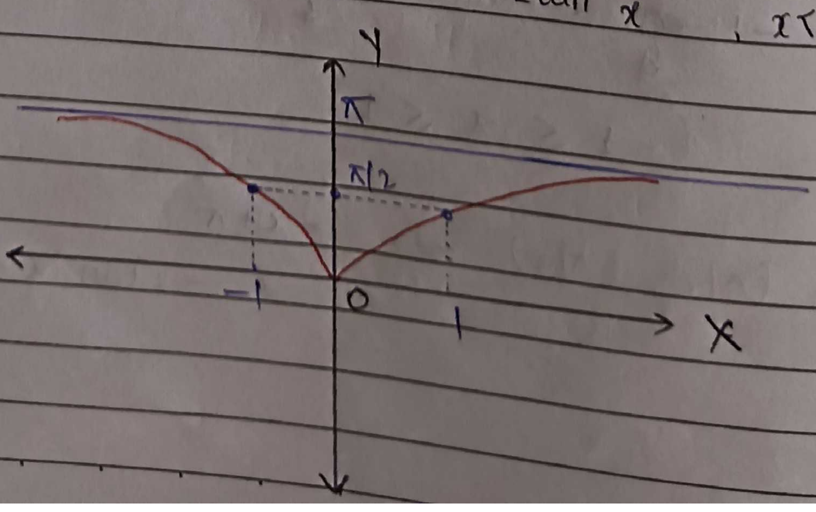
2. f is continuous for all x but not differentiable at $x = \pm 1$

3. $\cancel{\frac{dy}{dx} = \begin{cases} \frac{2}{1+x^2} & |x| \le 1 \end{cases}}$

3. $$\frac{dy}{dx} = \begin{cases} \frac{2}{1+x^2} , & |x| < 1 \\ \text{Do not exist}, & |x| = 1 \\ \frac{-2}{1+x^2} & |x| > 1 \end{cases}$$

4. Increasing in $(-1, 1)$ and decreasing in $(-\infty, -1) \cup (1, \infty)$

(b) $y = f(x) = \cos^{-1}\left(\frac{1-x^2}{1+x^2}\right) = \begin{cases} 2\tan^{-1}x , & x \geq 0 \\ -2\tan^{-1}x , & x < 0 \end{cases}$



Important Points :-

1. D: $x \in R$

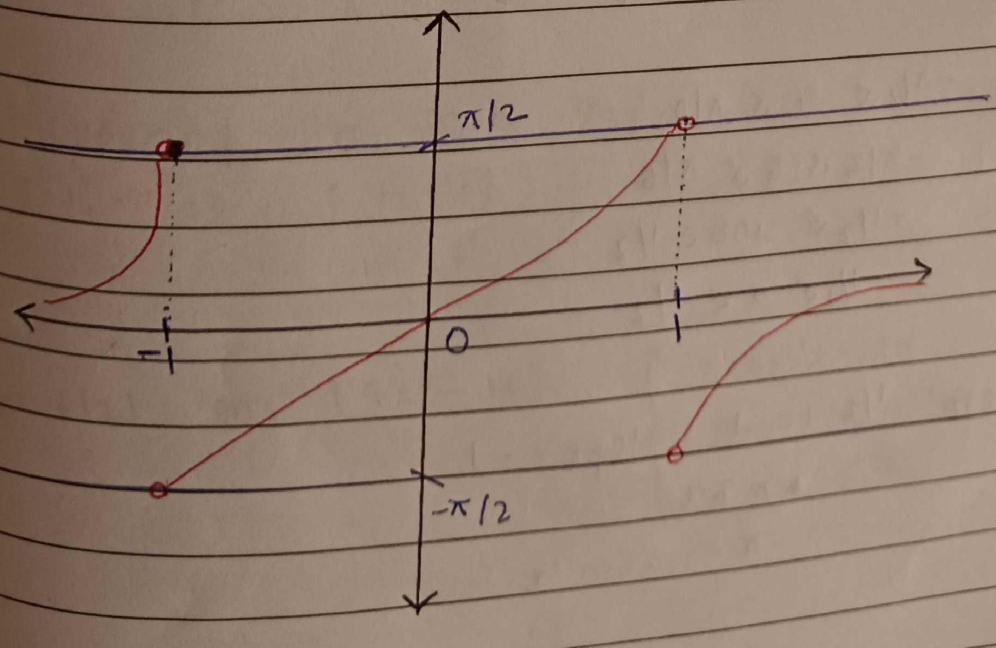
R : $y \in [0, \pi]$

2. f is continuous for all x but not differentiable at $x = 0$

3. $$\frac{dy}{dx} = \begin{cases} \frac{2}{1+x^2}, & x > 0 \\ D.N.E., & x = 0 \\ \frac{-2}{1+x^2}, & x < 0 \end{cases}$$

4. Increasing in $(0, \infty)$ and decreasing in $(-\infty, 0)$

5. $$y = \tan^{-1}\left(\frac{2x}{1-x^2}\right) = \begin{cases} 2\tan^{-1}x, & |x| < 1 \\ \pi + 2\tan^{-1}x, & x < -1 \\ -\pi + 2\tan^{-1}x, & x > 1 \end{cases}$$



Important Points:-

D : $x \in R - \{-1, 1\}$

R : $y \in (-\pi/2, \pi/2)$

It is neither continuous nor differentiable at $x = \pm 1$

$$\frac{dy}{dx} = \begin{cases} \dfrac{2}{1+x^2}, & |x| \neq 1 \\ D.NE, & |x| = 1 \end{cases}$$

Increasing for all x in its domain.

(d) $y = f(x) = \sin^{-1}(3x - 4x^3)$ ~~$3(\pi + 3\sin^{-1}x)\ -1 \le x < -1/2$~~

$$= \begin{cases} -(\pi + 3\sin^{-1}x), & -1 \le x < -1/2 \\ 3\sin^{-1}x, & -1/2 \le x \le 1/2 \\ \pi - 3\sin^{-1}x, & 1/2 < x \le 1 \end{cases}$$

$\sin^{-1}(\sin 3\theta) = 3\theta$ when

$-\pi/2 \le 3\theta \le \pi/2$

$-\pi/6 \le \theta \le \pi/6$

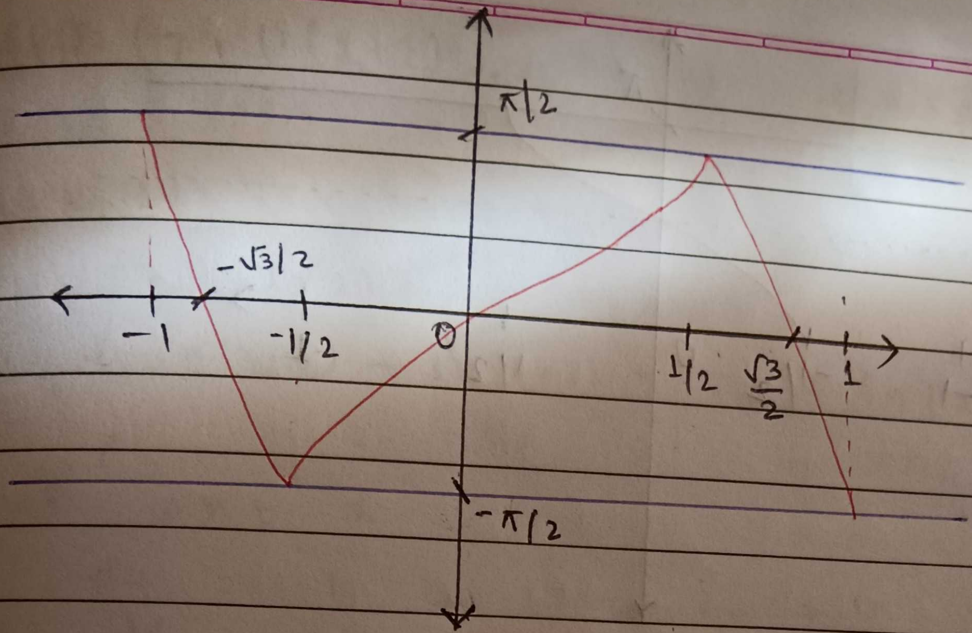
$-1/2 \le \sin\theta \le 1/2$

$-1/2 \le x \le 1/2$

Between $1/2$ to 1 slope $= -1$

$\pi \mp x$

$\pi - 3\sin^{-1}x$.



Important Points:-

D: $x \in [-1, 1]$

R: $y \in [-\pi/2, \pi/2]$

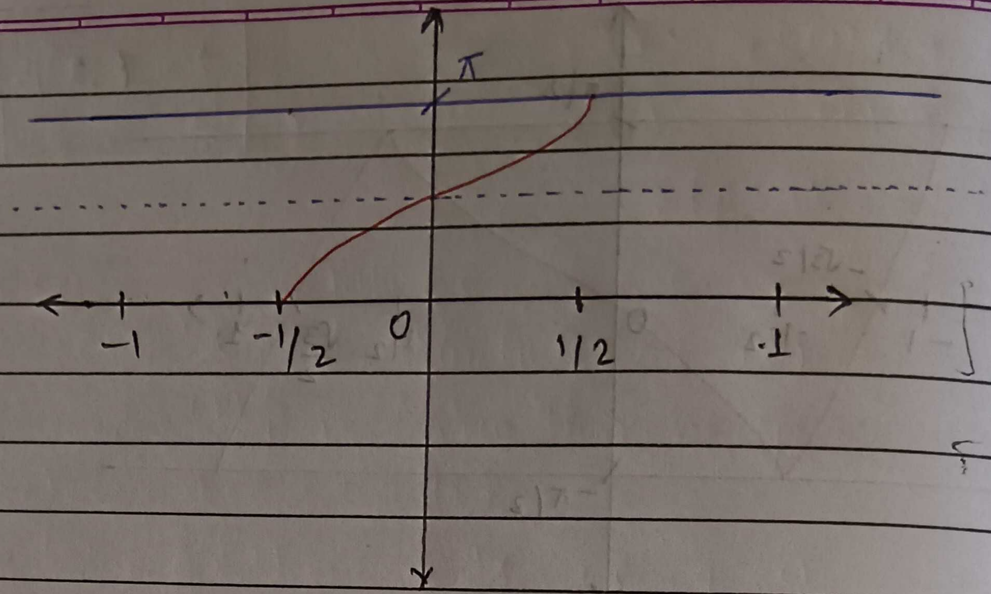
Continuous everywhere in its domain and non-differentiable at $x = \pm 1/2$

$$\frac{dy}{dx} = \begin{cases} \frac{3}{\sqrt{1-x^2}}, & -1/2 < x < 1/2 \\ \frac{-3}{\sqrt{1-x^2}}, & (-1, -1/2) \cup (1/2, 1) \end{cases}$$

Increasing in $(-1/2, 1/2)$ and decreasing in $[-1, -1/2) \cup (1/2, 1]$

(e)

$$y = f(x) = \cos^{-1}(4x^3 - 3x) = \begin{cases} 3\cos^{-1}x - 2\pi, & -1 \le x < -1/2 \\ 2\pi - 3\cos^{-1}x, & -1/2 \le x \le 1/2 \\ 3\cos^{-1}x, & 1/2 < x \le 1 \end{cases}$$



Important points:

1. D: $x \in [-1, 1]$
   R: $y \in [0, \pi]$

2. Continuous everywhere in its domain and non-differentiable at $x = \pm 1/2$.

3. $$\frac{dy}{dx} = \begin{cases} \dfrac{3}{\sqrt{1-x^2}}, & -1/2 < x < 1/2 \\ \dfrac{-3}{\sqrt{1-x^2}}, & x \in (-1, -1/2) \cup (1/2, 1) \end{cases}$$

4. Increasing in $(-1/2, 1/2)$ and decreasing in $(-1, -1/2) \cup (1/2, 1)$

Q. If $f(x) = (x+1)(x+2) \ldots (x+n)$, then find $f'(0)$?

Solⁿ: $\cancel{f'(x) = 1(x+2) \ldots (x+n)}$

$y = (x+1)(x+2) \ldots (x+n)$

$\ln y = \ln(x+1) + \ln(x+2) + \ldots + \ln(x+n)$

Diff. w.r.t. x

$$\frac{1}{y} \cdot \frac{dy}{dx} = \frac{1}{x+1} + \frac{1}{x+2} + \ldots + \frac{1}{x+n}$$

$$y'\Big|_{x=0} = f(0)\left[1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n}\right]$$

$$f'(0) = n!\left[1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n}\right]$$

Q. If $y = f(x) = x^3 + x^5$ and g is the inverse of f then find $g'(2)$.

Solⁿ: $f'(g(x)) \cdot g'(x) = 1$

$f(x) = x^3 + x^5$

$f'(x) = 3x^2 + 5x^4$

$\cancel{f'}\left(3(g(x))^2 + 5(g(x))^4\right) \cdot g'(x) = 1$

$3(g(2))^2 + 5(g(2))^4 \cdot g'(2) = 1$

$f(g(x)) = x$

$f(g(x)) = 2$

$f(x) = x^3 + x^5$

$\boxed{g(x) = 1}$

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$$\boxed{g'(2) = \frac{1}{8}}$$

Method (2) :- g is inverse of f.

f is function of (x)

Hence, g is function of (y)

we have to find $\frac{dx}{dy}$ i.e. $g'(x)$

$y = x^3 + x^5$

Diff. w.r.t. 'y'

$1 = 3x^2 \cdot \frac{dx}{dy} + 5x^4 \cdot \frac{dx}{dy}$

$1 = \frac{dx}{dy}(3x^2 + 5x^4)$

$\frac{dx}{dy} = \frac{1}{3x^2 + 5x^4}$

$\left.\frac{dx}{dy}\right|_{y=2} = ?$

$y = x^3 + x^5$

$y = 2$

$\boxed{x = 1}$

$\left.\frac{dx}{dy}\right|_{y=2} = \frac{1}{8}$

Q. Let $f(x) = \exp(x^3 + x^2 + x)$ for any real no. x and let g be the inverse function for f, then the value of $g'(e^3) = ?$

Sol^n:

$$f(x) = e^{x^3 + x^2 + x}$$

$$y = e^{x^3 + x^2 + x}$$

Diff. w.r.t. y

$$1 = e^{x^3 + x^2 + x} \cdot (3x^2 + 2x + 1) \cdot \frac{dx}{dy}$$

$$\frac{dx}{dy} = \frac{1}{e^{x^3 + x^2 + x}(3x^2 + 2x + 1)}$$

$$y = e^{x^3 + x^2 + x}$$

$$e^3 = e^{x^3 + x^2 + x}$$

$$3 = x^3 + x^2 + x$$

$$\boxed{x = 1}$$

$$\left.\frac{dx}{dy}\right|_{y = e^3} = \frac{1}{6e^3}$$

Q. $f(x) = x^2 + x\, g'(1) + g''(2)$ and $g(x) = f(1)x^2 + f'(x) \cdot x + f''(x)$ then find $f(2)$ and $g(2)$

Sol^n:

$$f(x) = x^2 + x g'(1) + g''(2)$$

$$f'(x) = 2x + g'(1)$$

$$f''(x) = 2$$

$$g(x) = f(1)x^2 + x \cdot f'(x) + \overset{\to 2}{f''(x)}$$

$$g'(x) = 2f(1)x + f'(x)$$

$$g'(x) = 2f(1)x + 2x + g'(1)$$

$$g''(x) = 2f(1) + 2$$

$$g''(2) = 2\overset{\to -1}{f(1)} + 2$$

$$\boxed{g''(2) = 0}$$

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$f'(x) = 2x + g'(1)$

$f'(2) = 4 + g'(1)$

$g'(x) = 2f(1)x + \cancel{f(2x)}\, 2x + g'$

$\cancel{g'(1)} = 2f(1) + 2 + \cancel{g'(1)}$

$\boxed{f(1) = -1}$

$g'(x) = 2f(1)x + f'(x)$

$g'(2) = 4f(1) + f'(2)$

$g'(2) = -4 + 4 + g'(1)$

$g'(2) = g'(1)$

Method 2

$g'(1) = a$ $\qquad g''(2) = b$

$f(x) = x^2 + ax + b$

$g(x) = f(1)x^2 + xf'(x) + f''(x)$

$f'(x) = 2x + a$

$f''(x) = 2$

$g(x) = f(1)x^2 + x(2x+a) + 2$

$g(x) = f(1)x^2 + 2x^2 + ax + 2$

$g'(x) = 2f(1)x + 4x + a$

Put $x = 1$

$g'(1) = 2f(1) + 4 + a$

$a = 2(1 + a + b) + 4 + a$

$a = 2 + 2a + 2b + 4 + a$

$2a + 2b + 6 = 0$

$a + b + 3 = 0$ —— (i)

$g''(x) = 2f(1) + 4$

$= 2(1 + a + b) + 4$

$g''(2) = 2a + 2b + 6$

$b = 2a + 2b + 6$

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$2a+b+6=0$ —— (ii)

eqⁿ (ii) — (i)

$\boxed{a=-3}$

$\boxed{b=0}$

$f(x) = x^2 - 3x$

$f(2) = -2$

$g(x) = -3x+2$

$g(2) = -4$

Q. The function $f(x) = e^x + x$ being differentiable and one to one, has a differentiable inverse $f^{-1}(x)$, then find $\frac{d}{dx}(f^{-1}(x))$ at the point $f(\log 2)$.

Solⁿ: $\frac{d}{dx}(f^{-1}(x)) = \frac{dx}{dy}$

$y = e^x + x$

Diff. w.r.t. y

$1 = e^x \cdot \frac{dx}{dy} + 1\frac{dx}{dy}$

$\frac{dx}{dy} = \frac{1}{e^x+1}$

$\cancel{y = e^x + x}$

$\cancel{\ln x = e^x + x}$

$\cancel{\ln . \ln 2 = x + \ln x}$

$$\left.\frac{dx}{dy}\right|_{x=\ln 2} = \frac{1}{e^{\ln 2}+1}$$

$$\left.\frac{dx}{dy}\right|_{x=\ln 2} = \frac{1}{3}$$

Q. If $y = (1+x)(1+x^2)(1+x^4)\ldots(1+x^{2^n})$, then $\left.\frac{dy}{dx}\right|_{x=0}$ is.

Solⁿ:

$\cancel{y = (1+x)(1+x^2)(1+x^4)\ldots(1+x^{2^n})}$

$\cancel{\ln y = \ln(1+x) + \ln(1+x^2) + \ln(1+x^4) + \ldots \ln(1+x^{2^n})}$

~~Diff. w.r.t. x~~

$\cancel{\frac{1}{y}\cdot\frac{dy}{dx} = \frac{1}{1+x} + \frac{1}{1+x^2} + \frac{1}{1+x^4} + \ldots \frac{1}{1+x^{2^n}}}$

$\cancel{\frac{1}{\ln}\cdot\left.\frac{dy}{dx}\right|_{x=0} =}$

Solⁿ:-

$$y = \frac{(1-x)(1+x)(1+x^2)(1+x^4)\ldots(1+x^{2^n})}{(1-x)}$$

$$y = \frac{(1-x^2)(1+x^2)(1+x^4)\ldots(1+x^{2^n})}{(1-x)}$$

$$y = \frac{(1-x^{2^n})(1+x^{2^n})}{(1-x)}$$

$$y = \frac{1-(x^{2^n})^2}{(1-x)}$$

$$y = \frac{1-x^{2^{n+1}}}{(1-x)}$$

$$y(1-x) = 1-x^{2^{n+1}}$$

Diff. w.r.t. 'x'

$$\cancel{\frac{dy}{dx}(1-x) + y(-1) = 1-x^{2^{n+1}}}$$

$$\cancel{\frac{dy}{dx} =}$$

$$y'(1-x) + y(-1) = -2^{n+1} \cdot x^{2^{n+1}-1}$$

$$y'(1-x) = -2^{n+1} \cdot x^{2^{n+1}-1} + y$$

At $x=0$

$$\boxed{y' = 1}$$

Q. If $y\sqrt{1-x^2} + x\sqrt{1-y^2} = 1$ when $|x|<1$ and $|y|<1$, then find $\frac{dy}{dx}$.

Sol^n:

$$y'\sqrt{1-x^2} + y\frac{1}{2\sqrt{1-x^2}}\cdot(-2x) + \sqrt{1-y^2} + \frac{x}{2\sqrt{1-y^2}}\cdot(-2y)\cdot y' = 0$$

$$y'\sqrt{1-x^2} - \frac{xy}{\sqrt{1-x^2}} + \sqrt{1-y^2} - \frac{xy\,y'}{\sqrt{1-y^2}} = 0$$

$$y'\left(\sqrt{1-x^2} - \frac{xy}{\sqrt{1-y^2}}\right) = \frac{xy}{\sqrt{1-x^2}} - \sqrt{1-y^2}$$

$$y'\left(\frac{\sqrt{(1-x^2)(1-y^2)} - xy}{\sqrt{1-y^2}}\right) = -\left(\frac{\sqrt{(1-x^2)(1-y^2)} - xy}{\sqrt{1-x^2}}\right)$$

$$y' = -\frac{\sqrt{1-y^2}}{\sqrt{1-x^2}}$$

$$\frac{dy}{dx} = -\sqrt{\frac{1-y^2}{1-x^2}}$$

Q. If $\sqrt{1-x^6} + \sqrt{1-y^6} = a^3(x^3-y^3)$, then prove that

$$\frac{dy}{dx} = \frac{x^2\sqrt{1-y^6}}{y^2\sqrt{1-x^6}}$$

Sol^n:- $\sqrt{1-x^6} + \sqrt{1-y^6} = a^3(x^3-y^3)$

Diff. w.r.t. 'x'

$$\frac{1}{\cancel{2}\sqrt{1-x^6}}\cdot(-\overset{3}{\cancel{6}}x^5) + \frac{1}{\cancel{2}\sqrt{1-y^6}}\cdot(-\overset{3}{\cancel{6}}y^5)\cdot y' = a^3(3x^2-3y^2\cdot y')$$

$$\cancel{-\frac{6x^5}{2x}}$$

$$\frac{-3x^5}{\sqrt{1-x^6}} - \frac{3y^5}{\sqrt{1-y^6}}y' = 3a^3x^2 - 3a^3y^2\cdot y'$$

$$y'\left(3a^3y^2 - \frac{3y^5}{\sqrt{1-y^6}}\right) = 3a^3x^2 + \frac{3x^5}{\sqrt{1-x^6}}$$

Q. If $\log(x^2+y^2) = 2\tan^{-1}\left(\frac{y}{x}\right)$, then prove that

$$\frac{dy}{dx} = \frac{x+y}{x-y}$$

Soln.:

$$\log(x^2+y^2) = 2\tan^{-1}\left(\frac{y}{x}\right)$$

Differentiate w.r.t. 'x'

$$\frac{1}{(x^2+y^2)}(2x+2y\cdot y') = \frac{2}{1+\frac{y^2}{x^2}} \cdot \left(\frac{xy'-y}{x^2}\right)$$

$$\frac{2(x+yy')}{\cancel{(x^2+y^2)}} = \frac{2\cancel{x^2}}{\cancel{(x^2+y^2)}} \cdot \frac{(xy'-y)}{\cancel{x^2}}$$

$$2x + 2yy' = 2xy' - 2y$$

$$x + yy' = \cancel{2}xy' - y$$

$$\therefore x + y = \cancel{x}y'(x-y)$$

$$\boxed{\frac{dy}{dx} = \frac{x+y}{x-y}}$$

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Q. If $5f(x) + 3f\left(\frac{1}{x}\right) = x+2$, $y = xf(x)$

then $\left.\frac{dy}{dx}\right|_{x=1} = ?$

Solⁿ:- $5f(x) + 3f\left(\frac{1}{x}\right) = x+2$

$5f(1) + 3f(1) = 3$

$f(1) = \frac{3}{8}$

$5f'(x) - 3f'\left(\frac{1}{x}\right)\cdot\frac{1}{x^2} = 1$

$5f'(1) - 3f'(1) = 1$

$2f'(1) = 1$

$f'(1) = \frac{1}{2}$

$y = xf(x)$

$\frac{dy}{dx} = f(x) + x\cdot f'(x)$

$\left.\frac{dy}{dx}\right|_{x=1} = f(1) + f'(1)$

$= \frac{3}{8} + \frac{1}{2}$

$= \frac{3+4}{8}$

$\left.\frac{dy}{dx}\right|_{x=1} = \frac{7}{8}$