Integration

Indefinite Integration

Definition:

If f and g are functions of x such that $g'(x) = f(x)$, then the f^n g is called an anti-derivative or primitive f^n or integral of f w.r.t. 'x'

It is written symbalically $\int f(x) \cdot dx = g(x)$

where $\frac{d}{dx}(g(x)) = f(x)$

Summary:

If $\frac{d}{dx}[F(x) + c] = f(x)$, then $F(x) + c$ is called an anti-derivative of $f(x)$.

$$\int f(x) \cdot dx = F(x) + c$$

c is called constant of Integration.

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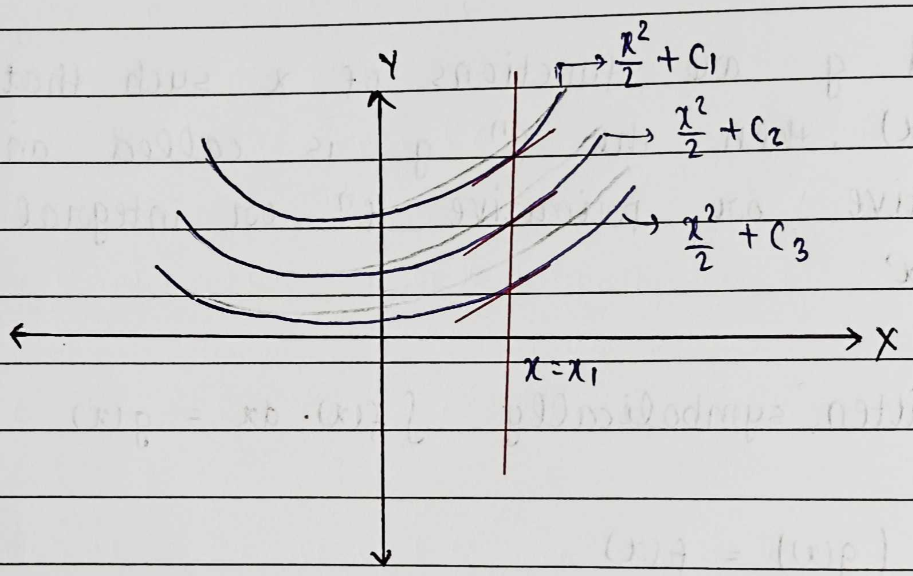
Geometrical Interpretation

$y = \int x \cdot dx = \frac{x^2}{2} + c$

$y = \int f(x) \cdot dx$ denotes a family of curves such that the slope of tangent at $x = x_1$ on every member is same.



$F'(x) = f(x)$

$F'(x_1) = f(x_1)$

where x_1 lies in the domain of $f(x)$.

Hence, anti-derivative of a f^n is not unique.

(i) If $g_1(x)$ and $g_2(x)$ are two anti-derivatives of a f^n $f(x)$ on $[a, b]$, then they differ only by a constant, i.e., $g_1(x) - g_2(x) = c$.

(ii) Anti-derivative of a continuous f^n is differentiable. If $f(x)$ is continuous, then $\int f(x) \cdot dx = F(x) + c \Rightarrow F'(x) = f(x)$ that implies $F'(x)$ is always exist. that implies $F(x)$ is differentiable.

(iii) If integrand is discontinuous at $x = x_1$, then its ~~a~~ anti-derivative at $x = x_1$ need not be discontinuous.

eg, $x^{-1/3}$

$$\int x^{-1/3} = \frac{x^{2/3}}{2/3} + c$$

$x^{-1/3}$ is discontinuous at $x = 0$ but $\frac{x^{2/3}}{2/3} + c$ is continuous at $x = 0$

(iv) Anti-derivative of a ~~pe~~ periodic f^n need not be a periodic f^n.

eg, $f(x) = \cos x + 1$

$$\int (\cos x + 1)\, dx$$

$$= \sin x + x + C$$

$\cos x + 1$ is periodic but $\sin x + x$ is not a periodic f^n.

Fundamental Integration Formula

(i) $\int x^n \cdot dx = \frac{x^{n+1}}{n+1} + c$, $n \neq -1$

(ii) $\int \frac{1}{x} \cdot dx = \ln|x| + c$

(iii) $\int e^x \cdot dx = e^x + c$

(iv) $\int a^x \cdot dx = \frac{a^x}{\ln a}$

(v) $\int \sin x \cdot dx = -\cos x + c$

(vi) $\int \cos x \cdot dx = \sin x + c$

(vii) $\int \sec^2 x \cdot dx = \tan x + c$

(viii) $\int \operatorname{cosec}^2 x \cdot dx = -\cot x + c$

(ix) $\int \sec x \cdot \tan x \cdot dx = \sec x + c$

(x) $\int \operatorname{cosec} x \cot x \cdot dx = -\operatorname{cosec} x + c$

(xi) $\int \cot x \, dx = \ln \sin x + c$

(xii) $\int \tan x \cdot dx = \ln \sec x + c$ or $-\ln \cos x + c$

(xiii) $\int \sec x \cdot dx = \ln|\sec x + \tan x| + c$ OR $\ln\left|\tan\left(\frac{\pi}{4} + \frac{x}{2}\right)\right| + c$

(xiv) $\int \operatorname{cosec} x \cdot dx = \ln|\operatorname{cosec} x - \cot x| + c$

$= \ln\left|\tan \frac{x}{2}\right| + c$

(xv) $\int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1}\left(\frac{x}{a}\right) + c$

(xvi) $\int \frac{-dx}{\sqrt{a^2 - x^2}} = \cos^{-1}\left(\frac{x}{a}\right) + c$

(xvii) $\int \frac{dx}{a^2 + x^2} = \frac{1}{a} \tan^{-1}\left(\frac{x}{a}\right) + c$

(xviii) $\int \frac{-dx}{x^2+a^2} = \frac{1}{a}\cot^{-1}\left(\frac{x}{a}\right) + c$

(xix)

$\int \frac{dx}{|x|\sqrt{x^2-a^2}} = \frac{1}{a}\sec^{-1}\left(\frac{x}{a}\right) + c$

(xx) $\int \frac{-dx}{|x|\sqrt{x^2-a^2}} = \frac{1}{a}\operatorname{cosec}^{-1}\left(\frac{x}{a}\right) + c$

(xxi) $\int \frac{dx}{\sqrt{a^2+x^2}} = \ln\left|x+\sqrt{a^2+x^2}\right| + c$

(xxii) $\int \frac{dx}{\sqrt{x^2-a^2}} = \ln\left|x+\sqrt{x^2-a^2}\right| + c$

(xxiii) $\int \frac{dx}{x^2-a^2} = \frac{1}{2a}\ln\left(\frac{x-a}{x+a}\right) + c$

(xxiv) $\int \frac{dx}{a^2-x^2} = \frac{1}{2a}\ln\left(\frac{a+x}{a-x}\right) + c$

(xxv) $\int \sqrt{x^2+a^2}\,dx = \frac{x}{2}\sqrt{x^2+a^2} + \frac{a^2}{2}\ln\left|x+\sqrt{x^2+a^2}\right| + c$

(xxvi) $\int \sqrt{x^2-a^2}\,.\,dx = \frac{x}{2}\sqrt{x^2-a^2} - \frac{a^2}{2}\ln\left|x+\sqrt{x^2-a^2}\right| + c$

(xxvii) $\int \sqrt{a^2-x^2}\,.\,dx = \frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right) + c$

(xxviii) $\int e^{ax} \sin bx \cdot dx = \frac{e^{ax}}{a^2+b^2}(a \sin bx - b \cos bx)$

(xxix) $\int e^{ax} \cos bx \cdot dx = \frac{e^{ax}}{a^2+b^2}(a \cos bx + b \sin bx)$

(xxx) $\int \ln x \cdot dx = x(\ln x - 1) + c$

Techniques of Integration

1. Substitution or change of independent variable.

If $g(x)$ is continuously differentiable f^n, then to evaluate integrals of the form

$$I = \int f(g(x)) \cdot g'(x)\, dx$$

Put $g(x) = t$

Diff. w.r.t. 'x'

$$g'(x) = \frac{dt}{dx}$$

$$g'(x) \cdot dx = dt$$

$$I = \int f(t) \cdot dt$$

Then, we substitute back the value of t.

Q. $\int e^{m\tan^{-1}x} \cdot \frac{1}{1+x^2} \cdot dx$

Sol^n:- put $\tan^{-1}x = t$

$\frac{1}{1+x^2} = \frac{dt}{dx}$ (Diff. w.r.t. 'x')

$$\frac{1}{1+x^2} \cdot dx = dt$$

$$\int e^{mt} \cdot dt$$

$$= \frac{e^{mt}}{m} + c$$

$$= \frac{e^{m\tan^{-1}x}}{m} + c$$

Q. $\int x \sin(4x^2+7)\, dx$

Sol^n: Put $4x^2+7 = t$

Diff. w.r.t 'x'

$$8x = \frac{dt}{dx}$$

$$x \cdot dx = \frac{dt}{8}$$

$$\int \frac{\sin t}{8} \cdot dt$$

$$= \frac{1}{8} \int \sin t \cdot dt$$

$$= -\frac{1}{8} \cos t + c$$

$$= \frac{-1}{8} \cos(4x^2+7) + c$$

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Q. $\int \frac{x^2}{9+16x^6} \cdot dx$

Sol^n:- Put $x^3 = t$

Diff. w.r.t. 'x'

$3x^2 = \frac{dt}{dx}$

$3x^2 dx = dt$

$x^2 . dx = \frac{dt}{3}$

$\frac{1}{3} \int \frac{dt}{3^2 + (4t)^2}$

$= \frac{1}{3} \times \frac{1}{3} \times \frac{1}{4} \tan^{-1}\left(\frac{4t}{3}\right) + c$

↳ Divide by coeff. of variable

$= \frac{1}{36} \tan^{-1}\left(\frac{4x^3}{3}\right) + c$

Q. $\int \frac{x^2-1}{(x^4+3x^2+1)\tan^{-1}\left(x+\frac{1}{x}\right)} \cdot dx$

Sol^n:- Put $\tan^{-1}\left(x+\frac{1}{x}\right) = t$

Diff. w.r.t. 'x'

$\frac{1}{1+\left(x+\frac{1}{x}\right)^2} \cdot \left(1-\frac{1}{x^2}\right) = \frac{dt}{dx}$

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$$\frac{1}{1+\left(\frac{x^2+1}{x}\right)^2}\left(1-\frac{1}{x^2}\right) = \frac{dt}{dx}$$

$$\frac{\cancel{x^2}}{x^2+(x^2+1)^2}\cdot\frac{(x^2-1)}{\cancel{x^2}} = \frac{dt}{dx}$$

$$\frac{(x^2-1)}{x^2+x^4+2x^2+1}\cdot dx = dt$$

$$\frac{(x^2-1)}{(x^4+3x^2+1)}\cdot dx = dt$$

$$\int \frac{1}{t}\cdot dt$$

$$= \ln t + c$$

$$= \ln\left|\tan^{-1}\left(x+\frac{1}{x}\right)\right| + c$$

Q. $\int \frac{\tan^4\sqrt{x}\cdot\sec^2\sqrt{x}}{\sqrt{x}}\cdot dx$

Sol^n:- Put $\tan\sqrt{x} = t$

Diff. w.r.t. 'x'

$$\sec^2\sqrt{x}\cdot\frac{1}{2\sqrt{x}} = \frac{dt}{dx}$$

$$\frac{\sec^2\sqrt{x}}{\sqrt{x}} = 2\cdot dt$$

$$\int 2t^4\cdot dt$$

$$= \frac{2t^5}{5} + c$$

$$= \frac{2(\tan\sqrt{x})^5}{5} + c$$

$$= \frac{2\tan^5\sqrt{x}}{5} + c$$

Fundamental deduction of method of substitution:-

(i) $$\int (f(x))^n \cdot f'(x)\,dx = \frac{f(x)^{n+1}}{n+1} + c$$

Put $f(x) = t$

$f'(x)\,dx = dt$

$$\int t^n\,dt = \frac{t^{n+1}}{n+1} + c$$

eg, $$\int \tan^5 x \sec^2 x \cdot dx = \frac{\tan^6 x}{6} + c$$

(ii) $$\int \frac{f'(x)}{(f(x))^n}\,dx = \frac{(f(x))^{1-n}}{1-n} + c$$

Put $f(x) = t$

$f'(x) \cdot dx = dt$

$$\int (t)^{-n}\,dt = \frac{t^{1-n}}{1-n} + c$$

Note:

If $\int f(x) \cdot dx = g(x) + c$, then

$$\int f(ax+b)\, dx = \frac{1}{a} g(ax+b) + c$$

(Divide by coefficient of x or derivative of $ax+b$ i.e. 'a'

Q. $\int \cos^3 x \cdot dx$

Sol^n: $\cos 3x = 4\cos^3 x - 3\cos x$

$$\cos^3 x = \frac{\cos 3x + 3\cos x}{4}$$

$$= \frac{1}{4}\int (\cos 3x + 3\cos x)\, dx$$

$$= \frac{1}{4}\left[\frac{\sin 3x}{3} + 3\sin x\right] + c$$

Method:2:-

$\int \cos^2 x \cdot \cos x \, dx$

$\int (1-\sin^2 x) \cos x \cdot dx$

Put $\sin x = t$

$\cos x \cdot dx = dt$

$$= \int (1-t^2)\, dt = t - \frac{t^3}{3}$$

$$= \sin x - \frac{\sin^3 x}{3} + c$$

Q. $\int \frac{\cos^3 x}{\sin^2 x + \sin x} dx$

Sol^n:- Put $\sin x = t$

$\cos x \, dx = t$

$= \int \frac{1-t^2}{t^2+t} dt$

$= \int \frac{(1+t)(1-t)}{t(1+t)} dt$

$= \int \left(\frac{1}{t} - 1\right) dt$

$= \ln t - t + c$

$= \ln(\sin x) - \sin x + c$

Standard substitution:-

(i) $\int \frac{dx}{\sqrt{a^2+x^2}}$ or $\int \sqrt{a^2+x^2} \, dx$

Put $x = a\tan\theta$ or $a\cot\theta$

$1 + \tan^2\theta = \sec^2\theta$

$\sec\theta = \sqrt{1+\tan^2\theta}$

$$= \sqrt{1 + \frac{x^2}{a^2}}$$

$$= \frac{1}{a} \sqrt{a^2 + x^2}$$

$x = a \tan\theta$

$dx = a \sec^2\theta \cdot d\theta$

$$\int \frac{a \sec^2\theta \cdot d\theta}{a \sec\theta}$$

$$= \int \sec\theta \cdot d\theta$$

$$= \ln|\sec\theta + \tan\theta| + c$$

$$= \ln\left|\frac{\sqrt{x^2+a^2}}{a} + \frac{x}{a}\right| + c$$

$$= \ln\left|x + \sqrt{x^2+a^2}\right| - \ln a + c$$

$$= \ln\left|x + \sqrt{x^2+a^2}\right| + c$$

(ii) $\int \frac{dx}{\sqrt{a^2-x^2}}$ or $\int \sqrt{a^2-x^2} \cdot dx$

Put $x = a\cos\theta$ or $a\sin\theta$

(iii) $\int \frac{dx}{\sqrt{x^2-a^2}}$ or $\int \sqrt{x^2-a^2} \cdot dx$

Put $x = a\sec\theta$ or $a\,\mathrm{cosec}\,\theta$

(iv) $\int \sqrt{\frac{a-x}{a+x}}\, dx$

Put $x = a\cos 2\theta$

(v) $\int \sqrt{\dfrac{x-\alpha}{\beta-x}}\, dx$ or $\int \sqrt{(x-\alpha)(\beta-x)}\, dx$.

Put $x = \alpha\cos^2\theta + \beta\sin^2\theta$

(vi) $\int \sqrt{\dfrac{x-\alpha}{x-\beta}}\, dx$. or $\int \sqrt{(x-\alpha)(x-\beta)}\,.\, dx$.

Put $x = \alpha\sec^2\theta - \beta\tan^2\theta$

(vii) $\int \dfrac{dx}{\sqrt{(x-\alpha)(x-\beta)}}\,.\,dx$

Put $x-\alpha = t^2$

or $x+\beta = t^2$

Q. $\int \dfrac{dx}{\sqrt{(x-a)(b-x)}}$

Soln:- Put $x = a\cos^2\theta + b\sin^2\theta$

$dx = [a \cdot 2\cos\theta\,(-\sin\theta) + 2b\sin\theta\cos\theta]\, d\theta$

$= \sin 2\theta\,(b-a)\, d\theta$

$x-a = a\cos^2\theta + b\sin^2\theta - a$

$= a(\cos^2\theta - 1) + b\sin^2\theta$

$= -a\sin^2\theta + b\sin^2\theta$

$= \sin^2\theta\,(b-a)$

$b-x = b - a\cos^2\theta - b\sin^2\theta$

$= (b-a)\cos^2\theta$

$$= \int \frac{(b-a)\sin 2\theta \cdot d\theta}{\sqrt{(b-a)^2 \sin^2\theta \cos^2\theta}}$$

$$= \int \frac{(b-a)\sin 2\theta \cdot d\theta}{(b-a)\sin\theta \cos\theta}$$

$$= 2\theta$$

$$x = a - a\sin^2\theta + b\sin^2\theta$$
$$= a + (b-a)\sin^2\theta$$

$$\frac{x-a}{b-a} = \sin^2\theta$$

$$\sin\theta = \sqrt{\frac{x-a}{b-a}}$$

$$\theta = \sin^{-1}\sqrt{\frac{x-a}{b-a}}$$

$$= 2\sin^{-1}\sqrt{\frac{x-a}{b-a}}$$

8. $\int \frac{\sin x}{\sin(x+a)} \cdot dx$

Sol^n:- Put $x+a=t \Rightarrow x = t-a$

$dx = dt$

$$\int \frac{\sin(t-a)}{\sin t} dt$$

$$= \int \frac{\sin t \cos a - \cos t \sin a}{\sin t} \cdot dt$$

$$= \int \cos a \cdot dt - \sin a \int \cot t \cdot dt$$

$$= \cos a \cdot t - \sin a \ \ln(\sin t) + c$$

$$= (x+a)\cos a - \sin a \cdot \ln(\sin(x+a)) + c$$

Q. $\int x\sqrt{x-5} \ dx$

Solⁿ:- Put $x-5 = t^2$

$$1 = 2t \cdot \frac{dt}{dx}$$

$$dx = 2t \cdot dt$$

$$= \int (t^2+5)\sqrt{t^2} \cdot 2t \cdot dt$$

$$= \int (t^2+5)2t^2 \cdot dt$$

$$= \int 2t^4 + 10t^2 \cdot dt$$

$$= \frac{2t^5}{5} + \frac{10t^3}{3} + c$$

$$= \frac{2(\sqrt{x-5})^5}{5} + \frac{10(\sqrt{x-5})^3}{3} + c$$

Q. $\int \frac{8x+3}{\sqrt{4x+7}} \cdot dx$

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Sol^n:- $4x+7 = t^2$ $\Rightarrow x = \frac{t^2-7}{4}$

$4\,dx = 2t\cdot dt$

$dx = \frac{2t\cdot dt}{4}$

$$\int \frac{\cancel{8}^{2}\frac{(t^2-7)}{\cancel{4}} + 3}{t} \cdot \frac{2t}{4}\cdot dt$$

$$\int \frac{2t^2-14+3}{\cancel{t}} \cdot \frac{\cancel{2t}}{\cancel{4}\,2}\cdot dt$$

$$= \frac{1}{\cancel{4}2}\int (2t^2-11)\,dt$$

$$= \frac{1}{\cancel{4}2}\left[\frac{2t^3}{3} - 11t\right] + c$$

$$= \frac{1}{\cancel{4}2}\left[\frac{2(\sqrt{4x+7})^3}{3} - 11\sqrt{4x+7}\right] + c$$

$$= \frac{1}{2}\left[\frac{2(4x+7)^{3/2}}{3} - 11(4x+7)^{1/2}\right] + c$$

Q. $\int \frac{dx}{\sqrt{x+1}-\sqrt{x}}$

Sol^n:- $$\int\left(\frac{1}{\sqrt{x+1}-\sqrt{x}} \times \frac{\sqrt{x+1}+\sqrt{x}}{\sqrt{x+1}+\sqrt{x}}\right) dx$$

$$\int (\sqrt{x+1}+\sqrt{x})\,dx$$

$$\int (x+1)^{1/2} + x^{1/2} \cdot dx$$

$$= \frac{2(x+1)^{3/2}}{3} + \frac{2x^{3/2}}{3} + C$$

* <u>Integration of the form :-</u>

(i) $\int f\left(x+\frac{1}{x}\right)\left(1-\frac{1}{x^2}\right) \cdot dx$

Put $x + \frac{1}{x} = t$

$$\left(1-\frac{1}{x^2}\right) dx = dt$$

(ii) $\int f\left(x-\frac{1}{x}\right)\left(1+\frac{1}{x^2}\right) dx.$

Put $x - \frac{1}{x} = t$

$$\left(1+\frac{1}{x^2}\right) dx = dt$$

(iii) $\int \frac{x^2+1}{x^4+Kx^2+1} dx$

<u>Divide numerator and denominator by x^2</u>

$$\int \frac{1+\frac{1}{x^2}}{x^2+\frac{1}{x^2}+K}$$

Put $x - \frac{1}{x} = t$

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(iv) $\int \frac{x^2-1}{x^4+Kx^2+1} dx$

Divide numerator and denominator by x^2

sol: Put $x + \frac{1}{x} = t$

Q. $\int \frac{x^x(x^{2x}+1)(\ln x+1)}{(x^{4x}+1)} dx$

soln:- Put $x^x = t$

$x^x(1+\ln x)\ dx = dt$

$\int \frac{(t^2+1)}{(t^4+1)} dt$

Divide by t^2

$\int \frac{1+\frac{1}{t^2}}{t^2+\frac{1}{t^2}} \cdot dt$

$t - \frac{1}{t} = y$

$\left(1+\frac{1}{t^2}\right) \cdot dt = dy$

$= \int \frac{dy}{\left(t-\frac{1}{t}\right)^2+2}$

$= \int \frac{dy}{y^2+(\sqrt{2})^2}$

$$= \frac{1}{\sqrt{2}} \tan^{-1}\left(\frac{y}{\sqrt{2}}\right) + c$$

$$= \frac{1}{\sqrt{2}} \tan^{-1}\left(\frac{t - \frac{1}{t}}{\sqrt{2}}\right) + c$$

$$= \frac{1}{\sqrt{2}} \tan^{-1}\left(\frac{x^x - \frac{1}{x^x}}{\sqrt{2}}\right) + c$$

$$= \frac{1}{\sqrt{2}} \tan^{-1}\left(\frac{x^{2x} - 1}{\sqrt{2}\, x^x}\right) + c$$

Q. $$\int \frac{x^{-7/6} - x^{5/6}}{x^{1/3}(x^2+x+1)^{1/2} - x^{1/2}(x^2+x+1)^{1/3}} \cdot dx$$

Sol^n:- { Multiply by $x^{7/6}$

$$\int \frac{1 - x^2}{x^{3/2}(x^2+x+1)^{1/2} - x^{5/3}(x^2+x+1)^{1/3}}\, dx$$

$$x^{3/2} = \frac{x^2}{x^{1/2}} \qquad x^{5/3} = \frac{x^2}{x^{1/3}}$$

$$\int \frac{1 - x^2}{x^2\left(\frac{x^2+x+1}{x}\right)^{1/2} - x^2\left(\frac{x^2+x+1}{x}\right)^{1/3}}\, dx$$

Divide by x^2

$$\int \frac{\frac{1}{x^2} - 1}{\left(x + \frac{1}{x} + 1\right)^{1/2} - \left(x + \frac{1}{x} + 1\right)^{1/3}}$$

Put $x + \frac{1}{x} = t$

$\left(1 - \frac{1}{x^2}\right) dx = dt$

$= -\int \frac{dt}{(t+1)^{1/2} - (t+1)^{1/3}}$

$=$ Put $t+1 = y^6$

$dt = 6y^5 . dy$

$= -6 \int \frac{y^3 - 1 + 1}{y - 1} dy$

$= -6 \left[\int \frac{\cancel{(y-1)}(y^2+y+1)}{\cancel{(y-1)}} dy + \ln(y-1) \right]$

$= -6 \left[\frac{y^3}{3} + \frac{y^2}{2} + y + \ln(y-1) \right] + c$

where $y = \left(x + \frac{1}{x} + 1 \right)^{1/6}$

Q. $\int \frac{(\sqrt{x})^5}{(\sqrt{x})^7 + x^6} dx = a \log \left(\frac{x^k}{x^k + 1} \right) + c$,

then a and k are.

(a) $\left(\frac{2}{5}, \frac{5}{2}\right)$ (b) $\left(\frac{1}{5}, \frac{2}{5}\right)$ (c) $\left(\frac{5}{2}, \frac{1}{2}\right)$ (d) $\left(\frac{2}{5}, \frac{1}{2}\right)$

Soln: Divide by $(\sqrt{x})^5$

$$= \int \frac{1}{x + (\sqrt{x})^7}\, dx$$

$$= \int \frac{1}{x(1 + x^{5/2})}\, dx$$

$$= \frac{x^{3/2}}{x^{5/2}(1 + x^{5/2})} \cdot dx$$

Put ~~let~~ $x^{5/2} = t$

$$\frac{5}{2} x^{3/2} dx = dt$$

$$x^{3/2} \cdot dx = \frac{2}{5} \cdot dt$$

$$= \int \frac{2/5\, dt}{t(t+1)}$$

$$= -\frac{2}{5} \int \frac{1}{t+1} - \frac{1}{t} \qquad = -\frac{2}{5}\left[\int \frac{1}{t-1} - \int \frac{1}{t}\right]$$

$$= -\frac{2}{5}\left[\ln(t+1) - \ln t\right] + c$$

$$= -\frac{2}{5} \ln\left(\frac{t+1}{t}\right) + c$$

$$= -\frac{2}{5} \ln\left(\frac{x^{5/2}+1}{x^{5/2}}\right) + c = \frac{2}{5} \ln\left(\frac{x^{5/2}}{x^{5/2}+1}\right)$$

$(a = 2/5 \;,\; K = 5/2)$

Q. $\displaystyle\int \frac{\cos 5x + \cos 4x}{1 - 2\cos 3x} \cdot dx$

Sol^n: $\displaystyle\int \frac{2\cos\left(\frac{9x}{2}\right)\cos\left(\frac{x}{2}\right)}{1 - 2\left(2\cos^2 \frac{3x}{2} - 1\right)} \cdot dx$

Multiply and divide by $\cos(3x/2)$

$$= \int \frac{2\cos(x/2)\cos(3x/2)\cos(9x/2)}{3\cos(3x/2) - 4\cos^3(3x/2)} \cdot dx$$

$$= \int \frac{2\cos(x/2)\cos(3x/2)\cancel{\cos(9x/2)}}{-\cancel{\cos(9x/2)}} \cdot dx$$

$$= -\int 2\cos(x/2)\cos(3x/2) \cdot dx$$

$$= -\int (\cos 2x + \cos x)\, dx$$

$$= -\left[\frac{\sin 2x}{2} + \sin x\right] + C$$

Q. $\int \frac{\cos 7x - \cos 8x}{1 + 2\cos 5x} \cdot dx$

Sol^n: $\int \frac{-2\sin\left(\frac{15x}{2}\right)\sin\left(\frac{-x}{2}\right)}{1 + 2\cos 5x} \cdot dx$

$$\int \frac{2\sin\left(\frac{15x}{2}\right)\sin\left(\frac{x}{2}\right)}{1 + 2\cos 5x} \cdot dx$$

$$\int \frac{2\sin(15x/2)\sin(x/2) \cdot dx}{1 + 2\cdot \cancel{(2\cos^2\frac{5x}{2} - 1)} \left(1 - 2\sin^2\frac{5x}{2}\right)}$$

$$\int \frac{2\sin\left(\frac{15x}{2}\right)\sin(x/2)}{3 - 4\sin^2(5x/2)} \, dx$$

Multiply and divide by $\sin(5x/2)$

$$\int \frac{2\sin(15x/2)\sin(x/2)\sin(5x/2)}{3\sin(5x/2) - 4\sin^3(5x/2)} \, dx$$

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$$= \int \frac{2\cancel{\sin(15x/2)} \sin(x/2) \sin(5x/2)}{\cancel{\sin(15x/2)}} \cdot dx$$

$$= \int 2\sin(x/2)\sin(5x/2) \cdot dx \quad \{2\sin A \sin B = \cos(A-B) - \cos(A+B)\}$$

$$= \int \cos(2x) - \cos(3x) \quad dx$$

$$= \frac{\sin 2x}{2} - \frac{\sin 3x}{3} + C$$

Integration by Parts

$$\int (u \cdot v)\, dx = u \cdot \int v \cdot dx - \int \left[\left(\frac{du}{dx}\right) \cdot \int v \cdot dx \right] \cdot dx$$

Integral of product of two f^n equal to first f^n into integration of second f^n minus integral of (differentiation of 1^{st} f^n into integration of 2^{nd} f^n).

For the selection of first f^n, we use ILATE rule.

I = Inverse
L = Logarithmic
A = Algebraic
T = Trigonometric
E = Exponential

Q. $\int \frac{x}{1+\sin x} \cdot dx$

Sol^n:- $\int \frac{x(1-\sin x)}{(1+\sin x)(1-\sin x)} \cdot dx$

$\int \frac{x(1-\sin x)}{\cos^2 x} \cdot dx$

$\int x(\sec^2 x - \sec x \tan x) \cdot dx$

$= \int x \sec^2 x \cdot dx - \int x \sec x \tan x$

$= x \tan x - \int \tan x \cdot dx - \left[x \cdot \sec x - \int \sec x \, dx\right]$

$= x \tan x + \log(\cos x) - x \sec x + \ln|\sec x + \tan x| + C$

Note:

If there is no other f^n then, 1 is taken as second f^n.

Q. $\int \frac{\sin^{-1}\sqrt{x} - \cos^{-1}\sqrt{x}}{\sin^{-1}\sqrt{x} + \cos^{-1}\sqrt{x}} \cdot dx$

Sol^n:- $\sin^{-1}\sqrt{x} + \cos^{-1}\sqrt{x} = \frac{\pi}{2}$

$= \frac{2}{\pi} \int \sin^{-1}\sqrt{x} - (\pi/2 - \sin^{-1}\sqrt{x}) \cdot dx$

$= \frac{2}{\pi} \int \left(2\sin^{-1}\sqrt{x} - \frac{\pi}{2}\right) \cdot dx$

$= \frac{2}{\pi} \left[2 \int \sin^{-1}\sqrt{x} . dx - \frac{\pi x}{2} \right]$ —— (i)

$\int \sin^{-1}\sqrt{x} \cdot 1 \; dx$

second f^n

$= x \cdot \sin^{-1}\sqrt{x} - \int \frac{1}{\sqrt{1-x}} \cdot \frac{1}{2\sqrt{x}} \cdot x \, dx$

$= x \sin^{-1}\sqrt{x} - \frac{1}{2} \int \sqrt{\frac{x}{1-x}} \cdot dx$

$x = \sin^2\theta$ $\qquad dx = 2\sin\theta\cos\theta \cdot d\theta$

$= x \sin^{-1}\sqrt{x} - \frac{1}{2} \int \tan\theta \cdot 2\sin\theta\cos\theta \cdot d\theta$

$= x \sin^{-1}\sqrt{x} - \frac{1}{2} \int 2\sin^2\theta \cdot d\theta$

$= x \sin^{-1}\sqrt{x} - \frac{1}{2} \int (1-\cos 2\theta) \, d\theta$

$= x \sin^{-1}\sqrt{x} - \frac{1}{2} \quad \theta - \frac{\sin 2\theta}{2} + c$

$= x \sin^{-1}\sqrt{x} - \frac{1}{2} \sin^{-1}\sqrt{x} - \frac{1}{2} \sin(2\sin^{-1}\sqrt{x}) + c$

Putting this in (i)

$$= \frac{2}{\pi}\left[2\left(x\sin^{-1}\sqrt{x} - \frac{1}{2}\sin^{-1}\sqrt{x} - \frac{1}{2}\sin(2\sin^{-1}\sqrt{x})\right) - \frac{\pi}{2}x\right] + C$$

$$= \frac{4x}{\pi}\sin^{-1}\sqrt{x} - \frac{2}{\pi}\sin^{-1}\sqrt{x} - \frac{2}{\pi}\sin(2\sin^{-1}\sqrt{x}) - x + C$$

Q. $\int \ln x \cdot dx$

Solⁿ: $\int \ln x \cdot 1 \, dx$

$$= \ln x \int 1 \cdot dx - \int\left[\frac{1}{x} \cdot \int 1 \cdot dx\right] \cdot dx$$

$$= x\ln x - \int \frac{1}{\cancel{x}} \cdot \cancel{x} \cdot dx$$

$$= x\ln x - x + C$$

$$= x(\ln x - 1) + C$$

Q. $\int x^3 \sin(x^2) \cdot dx$

Solⁿ: Put $x^2 = t$

$2x \cdot dx = dt$

$$\frac{1}{2}\int \underset{(I)}{t}\ \underset{II}{\sin t}\ dt$$

$$= \frac{1}{2}\left[-t \cdot \cos t - \int -1 \cos t \cdot dt\right]$$

$$= \frac{1}{2}\left[-t\cos t + \sin t\right] + C$$

$$= \frac{1}{2}\left[\sin(x^2) - x^2\cos(x^2)\right] + C$$

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Integral of the form

* $\int e^x [f(x) + f'(x)] . dx = e^x f(x) + c$

Proof: $\int e^x [f(x) + f'(x)]$

$= \int \overset{II}{e^x} \overset{I}{f(x)} . dx + \int e^x f'(x) dx$

$= e^x f(x) - \cancel{\int e^x f'(x) . dx} + \cancel{\int e^x f'(x) . dx}$

$= e^x f(x) + c$

Note :-

$\int e^{kx} [f(kx) + f'(kx)] dx = e^{kx} f(kx) + c$

Q. Evaluate

$\int e^x \left(\frac{1-x}{1+x^2}\right)^2 . dx$

Soln :- $\int e^x \frac{(1-x)^2}{(1+x^2)^2} . dx$

$= \int e^x \frac{(1+x^2-2x)}{(1+x^2)^2} . dx$

$= \int e^x \frac{\cancel{(1+x^2)}}{(1+x^2)^2} - \frac{2x}{(1+x^2)^2} . dx$

$$= \int e^x \underbrace{\frac{1}{1+x^2}}_{f(x)} + \underbrace{\frac{-2x}{(1+x^2)^2}}_{f'(x)} \cdot dx$$

$$= \frac{e^x}{1+x^2} + c$$

Q. $\int e^x \left(\frac{1+\sin x \cos x}{\cos^2 x} \right) dx$

sol^n: $\int e^x \left(\frac{1}{\cos^2 x} + \frac{\sin x \cos x}{\cos^2 x} \right) dx$

$$\int e^x (\overset{f'(x)}{\sec^2 x} + \overset{f(x)}{\tan x}) \cdot dx$$

$$= e^x \tan x + c$$

Q. $\int e^{2x} \left(\frac{1+\sin 2x}{1+\cos 2x} \right)$

sol^n: $\int e^{2x} \left(\frac{1+\sin 2x}{2\cos^2 x} \right) dx$

$$\frac{1}{2} \int e^{2x} \left(\frac{1+2\sin x \cos x}{\cos^2 x} \right) dx$$

$$\frac{1}{2} \int e^{2x} (\sec^2 x + 2\tan x) dx$$

Put $2x = t$

$$\frac{1}{2} \int e^t \left[\underbrace{\sec^2 \left(\frac{t}{2}\right)}_{f'(x)} + \underbrace{2\tan\left(\frac{t}{2}\right)}_{f(x)} \right] dt$$

$= \frac{1}{2} e^t \tan\left(\frac{t}{2}\right) + c$

$= \frac{1}{2} e^{2x} \tan x + c$

Q. $\int x e^{x^2} (\sin x^2 + \cos x^2)\, dx$

Soln:- Put $x^2 = t$

$2x \cdot dx = dt$

$\frac{1}{2} \int e^t (\underbrace{\sin t}_{f(x)} + \underbrace{\cos t}_{f'(x)})$

$= \frac{1}{2} e^t \cancel{\cos t} \sin t + c$

$= \frac{1}{2} e^{x^2} \sin(x^2) + c$

★ $\int [f(x) + x f'(x)]\, dx = x f(x) + c$

Q. Evaluate :

$\int \frac{x + \sin x}{1 + \cos x} \cdot dx$

Soln: $\int \frac{x + 2\sin x/2 \cos x/2}{2\cos^2 x/2}\, dx$

$= \int \left[\frac{x}{2\cos^2 x/2} + \tan(x/2) \right] \cdot dx$

$$= \int \Big[\underbrace{\tfrac{1}{2} x \sec^2 x/2}_{x f'(x)} + \underbrace{\tan(x/2)}_{f(x)} \Big] dx$$

$$= x \tan(x/2) + c$$

Q. $\int \Big[\underbrace{\tan(e^x)}_{f(x)} + \underbrace{x e^x \sec^2(e^x)}_{f'(x)} \Big] dx.$

Ans = $x \tan(e^x) + c$

Q. $\int (\ln x + 1)\, dx$

Solⁿ:- $\int \Big(\underbrace{\ln x}_{f(x)} + \underbrace{\frac{x}{x}}_{x.f'(x)} \Big) . dx$

$$= x \ln x + c$$

★ Integral of the form

* $\int e^{ax} \sin bx . dx = \frac{e^{ax}}{a^2 + b^2} (a \sin bx - b \cos bx)$

* $\int e^{ax} \cos bx . dx = \frac{e^{ax}}{a^2 + b^2} (a \cos bx + b \sin bx)$

$$\star \quad \int e^{ax} \sin bx \cdot dx = \frac{e^{ax}}{a^2+b^2}\left[a \sin bx - b \cos bx\right]$$

Proof:

$$\int e^{ax} \sin bx \cdot dx$$

$$I = \int \underset{(II)}{e^{ax}} \underset{(I)}{\sin bx} \cdot dx$$

$$= \frac{\sin bx}{a} e^{ax} - \int \frac{b \cos bx \cdot e^{ax}}{a} \cdot dx$$

$$= \frac{\sin bx}{a} - \frac{b}{a} \int \underset{(II)}{e^{ax}} \underset{(I)}{\cos bx} \cdot dx$$

$$= \frac{\sin bx}{a} - \frac{b}{a}\left[\frac{\cos bx \cdot e^{ax}}{a} + \int \frac{b \sin bx \; e^{ax}}{a} \cdot dx\right.$$

$$I = \frac{\sin bx}{a} - \frac{b}{a^2} \cos bx \cdot e^{ax} - \frac{b^2}{a^2} I$$

$$I\left(1 + \frac{b^2}{a^2}\right) = \frac{e^{ax}}{a^2}\left[a \sin bx - b \cos bx\right]$$

$$I\left(\frac{a^2+b^2}{\cancel{a^2}}\right) = \frac{e^{ax}}{\cancel{a^2}}\left(a \sin bx - b \cos bx\right)$$

$$I = \frac{e^{ax}}{a^2+b^2}\left(a \sin bx - b \cos bx\right)$$

Q. $\int \underset{II}{e^x} \underset{I}{\cos^2 x} \cdot dx$

Sol^n:- $\cos^2 x \cdot e^x - \int \left[-\sin 2x \cdot e^x \right] dx$

$\cos^2 x \cdot e^x + \int (\sin 2x \cdot e^x)\, dx$

$\downarrow$

$a = 1, \; b = 2$

$\cos^2 x \cdot e^x + \frac{e^x}{1+4} (\sin 2x - 2\cos 2x) + c$.

$\frac{(1+\cos 2x)}{2} e^x + \frac{e^x}{5} \cdot (\sin 2x - 2\cos 2x) + c$

$\frac{e^x}{2} + \frac{e^x \cos 2x}{2} + \frac{e^x}{5} (\sin 2x - 2\cos 2x) + c$

$\frac{e^x}{2} + \frac{e^x}{10} (\cos 2x + 2\sin 2x) + c$

Q. $\int \sin(\log x) \cdot dx$

Sol^n:- Put $\log x = t$

$\frac{1}{x} dx = dt$

$= \int e^t \sin t.\, dx$

$\downarrow$

$a = 1, \; b = 1$

$= \frac{e^t}{2} (\sin t - \cos t)$

Q. Evaluate:

$$\int \frac{x^2 \cdot dx}{(x\sin x + \cos x)^2}$$

Sol^n:- Multiply and divide by $x\cos x$

$$\int \frac{x^2 \cdot \cancel{x}\cos x \cdot dx}{\cancel{x}\cos x\,(x\sin x + \cos x)^2}$$

$$= \int \underset{(I)}{x\sec x} \; \underset{(II)}{\frac{x\cos x}{(x\sin x+\cos x)^2}} \cdot dx$$

$$= x\sec x \int \frac{x\cos x \cdot dx}{(x\sin x+\cos x)^2} - \int \left[\frac{d}{dx} x\sec x \cdot \int \frac{x\cos x}{(x\sin x+\cos x)^2}\right] dx$$

$$\int \frac{x\cos x}{(x\sin x+\cos x)^2} \cdot dx$$

$x\sec x$

$\sec x + x\sec x\tan$

$(x\sin x+\cos x).dx$

$= x\cos x + \sin x - \sin x$

$= x\cos x$

$$\int \frac{f'(x)}{(f(x))^n} = \frac{f(x)^{1-n}}{1-n} + C$$

$$= \frac{(x\sin x+\cos x)^{1-2}}{1-2}$$

$$= \frac{-1}{x\sin x+\cos x}$$

Putting in above ...

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$$= x\sec x \cdot \left(\frac{-1}{x\sin x+\cos x}\right) + \int \frac{(\sec x + x\sec x\tan x)}{(x\sin x+\cos x)} \cdot dx$$

$$= \frac{-x\sec x}{(x\sin x+\cos x)} + \int \frac{\frac{1}{\cos x} + \frac{x\sin x}{\cos^2 x}}{(x\sin x+\cos x)} \cdot dx$$

$$= \frac{-x\sec x}{x\sin x+\cos x} + \int \sec^2 x \cdot dx$$

$$= \frac{-x\sec x}{x\sin x+\cos x} + \tan x + c$$

Q. Evaluate:

$$\int \frac{\sin 2x}{\sin 5x \sin 3x} dx$$

Soln:-

$$\int \frac{\sin(5x-3x)}{\sin 5x \sin 3x} dx$$

$$\int \frac{\cancel{\sin 5x}\cos 3x}{\cancel{\sin 5x}\sin 3x} - \frac{\cos 5x \cancel{\sin 3x}}{\sin 5x \cancel{\sin 3x}} dx$$

$$\int (\cot 3x - \cot 5x)\, dx$$

$$= \frac{1}{3}\log \sin 3x - \frac{1}{5}\log \sin 5x + c$$

Q. Evaluate:

$$\int \frac{\log(x+\sqrt{1+x^2})}{\sqrt{1+x^2}}\,dx$$

Soln:- $\log(x+\sqrt{1+x^2}) = t$

$$\frac{1}{x+\sqrt{1+x^2}} \cdot \left(1+\frac{1}{\cancel{2}\sqrt{1+x^2}} \cdot \cancel{2}x\right) dx = dt$$

$$\frac{1}{\cancel{(x+\sqrt{1+x^2})}} \cdot \frac{\cancel{(\sqrt{1+x^2}+x)}}{\sqrt{1+x^2}}\,dx = dt$$

$$\frac{dx}{\sqrt{1+x^2}} = dt$$

$$\int t \cdot dt$$

$$= \left[\frac{t^2}{2}\right] + C$$

$$= \frac{1}{2}\left[\log(x+\sqrt{1+x^2})\right]^2 + C$$

Q. Evaluate:

$$\int \sqrt{\frac{1-x}{1+x}}\,dx$$

Soln:- Method 1

Multiply and divide by $\sqrt{1-x}$

$$\int \frac{1-x}{\sqrt{1-x^2}} \cdot dx$$

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$$\int\left(\frac{1}{\sqrt{1-x^2}} - \frac{2x}{2\sqrt{1-x^2}}\right) dx$$

$$\int \frac{1}{\sqrt{1-x^2}}\, dx + \int \frac{-2x}{2\sqrt{1-x^2}}\, dx$$

$$\sin^{-1}x + \int \frac{-2x}{2\sqrt{1-x^2}}\, dx$$

$$1-x^2 = t$$

$$-2x\,dx = dt$$

$$= \sin^{-1}x + \int \frac{dt}{2\sqrt{t}}$$

$$= \sin^{-1}x + \sqrt{t}\ dt + C$$

$$= \sin^{-1}x + \sqrt{1-x^2} + C$$

Q. $\int \frac{\sin x}{\sin x - \cos x} \cdot dx$

Sol^n:- $\int \frac{\sin x - \cos x + \cos x + \sin x}{2(\sin x - \cos x)} dx$

$\frac{1}{2}\left[\int \frac{\cancel{\sin x - \cos x}}{\cancel{\sin x - \cos x}} dx + \int \frac{\cos x + \sin x}{\sin x - \cos x} dx\right.$

$= \frac{1}{2}\left[\int dx + \int \frac{f'(x)}{f(x)} dx\right]$

$= \frac{1}{2}\left[x + \ln f(x)\right] + c$

$= \frac{1}{2}\left[x + \ln(\sin x - \cos x)\right] + c$

Q. $\int \frac{(\log x - 1)^2}{[1 + (\log x)^2]^2} dx$ is equal to :

Sol^n:- $\int \frac{(\log x)^2 + 1 - 2\log x}{[1 + (\log x)^2]^2} dx$

$\int \frac{1}{1 + (\log x)^2} - \frac{2\log x}{[1 + (\log x)^2]^2}$

$$\frac{d}{dx}\left[1+(\log x)^2\right]^{-1} = \frac{-1}{\left[1+(\log x)^2\right]^2} \cdot 2\log x \cdot \frac{1}{x}$$

$$\int \left[f(x) + x f'(x)\right] dx$$

$$= x f(x) + C$$

$$= \cancel{x} \frac{x}{(\log x)^2+1} + C$$

Integration of Trigonometric Functions

$$\int \sin^m x \cdot \cos^n x \, dx$$

Case I :

When m and $n \in N$

(i) If one of them is odd, then substitute for the term of even power.

(ii) If both are odd, substitute either of them.

(iii) If both are even, use trigonometric identities to convert integrand into cosines of multiple angles.

eg, $\sin^2 x = \frac{1-\cos 2x}{2}$

Case II :

If $m+n$ is a negative even integer,

Put $\tan x = t$

Case III :

If $(m+n)$ is negative odd integer, replace 1 by $(\sin^2 x + \cos^2 x)$ or multiply by $(\sin^2 x + \cos^2 x)$.

Q. $\int \sin^3 x \cdot \cos^5 x \, dx$

Sol^n:- Put $\cos x = t$

$-\sin x \, dx = dt$

$-\int t^5 (1-\cos^2 x) \, dt$

$-\int t^5 (1-t^2) \, dt$

$-\int (t^5 - t^{\cancel{8}7}) \, dt$

$-\left[\frac{t^6}{6} - \frac{t^8}{8}\right] + C$

$\frac{\cos^8 x}{8} - \frac{\cos^6 x}{6} + C$

Method 2:-

Put $\sin x = t$

$\cos x \, dx = dt$

$\int t^3 (1-t^2)^2 \, dt$

$= \int t^3 (1 + t^4 - 2t^2) \, dt$

$$= \int (t^3 + t^7 - 2t^5)\, dt$$

$$= \frac{t^4}{4} + \frac{t^8}{8} - \frac{t^6}{3} + C$$

$$= \frac{\sin^4 x}{4} + \frac{\sin^8 x}{8} - \frac{\sin^6 x}{3} + C$$

Q. $\int \sin^{-11/3} x \cos^{-1/3} x \cdot dx$

Solⁿ: Multiply and Divide by $\cos^{-11/3}$

$$\int \left(\frac{\sin x}{\cos x}\right)^{-11/3} \cdot \cancel{\sec} \cos^{-1/3} \cdot \cos^{-11/3} \cdot dx$$

$$\int (\tan x)^{-11/3} \frac{1}{\cos^4 x}\, dx$$

$$\int (\tan x)^{-11/3} \cdot \sec^4 x \, dx$$

$$\int (\tan x)^{-11/3} (1 + \tan^2 x) \sec^2 x \, dx$$

Put $\tan x = t$

$\sec^2 x \, dx = dt$

$$\int t^{-11/3} (1 + t^2)\, dt$$

$$\int t^{-11/3}\, dt + \int t^{-5/3}\, dt$$

$$\frac{t^{-8/3}}{(-8/3)} + \frac{t^{-2/3}}{-2/3}$$

$$= -\frac{3}{8} t^{-8/3} - \frac{3}{2} t^{-2/3}$$

$$= -\frac{3}{8} \tan^{-8/3} x - \frac{3}{2} \tan^{-2/3} x + C$$

Q. $\displaystyle\int \frac{dx}{\sin x \cdot \cos^2 x}$

Soln:- $\displaystyle\int \left(\frac{\sin^2 x + \cos^2 x}{\sin x \cos^2 x} \right) dx$

$$\int \sec x \tan x \, dx + \int \operatorname{cosec} x \, dx$$

$$= \sec x + \ln|\operatorname{cosec} x - \cot x| + C$$

Q. $\displaystyle\int \frac{(\sin x)^{1/2}}{\cos^{9/2} x} dx$

Soln:- $\displaystyle\int \sin^{1/2} \cos^{-9/2} x \cdot dx$

Divide and multiply by $\cos^{1/2} x$

$$\int \left(\frac{\sin x}{\cos x}\right)^{1/2} \cos^{-4} x \, dx$$

$$\int \tan^{1/2} x \sec^4 x \, dx$$

$$\int \tan^{1/2} x \, (1 + \tan^2 x) \sec^2 x \cdot dx$$

$\tan x = t$
$\sec^2 x \cdot dx = dt$

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$$\int t^{1/2}(1+t^2)\,dt$$

$$\int (t^{1/2}+t^{5/2})\,dt$$

$$= \frac{2t^{3/2}}{3} + \frac{2t^{7/2}}{7} + c = \frac{2(\tan x)^{3/2}}{3} + \frac{2(\tan x)^{7/2}}{7} + c$$

Q. Solve the following:

(i) $\int \frac{dx}{\sin(x-a)\cos(x-b)}$

Sol^n:-

$$\frac{1}{\cos(a-b)} \cdot \int \frac{\cos[(x-b)-(x-a)]\,dx}{\sin(x-a)\cos(x-b)}$$

$$\frac{1}{\cos(a-b)} \int \frac{\cancel{\cos x \cos b}\ \cos(x-b)\cos(x-a) + \sin(x-b)\sin(x-a)}{\sin(x-a)\cos(x-b)}\,dx$$

$$\frac{1}{\cos(a-b)} \int \frac{\cos(x-a)}{\sin(x-a)} + \frac{\sin(x-b)}{\cos(x-b)}\,dx$$

$$\frac{1}{\cos(a-b)} \int [\cot(x-a) + \tan(x-b)]\,dx$$

$$\frac{1}{\cos(a-b)} [\ln \sin(x-a) - \ln \cos(x-b)] + C$$

$$\frac{1}{\cos(a-b)} \ln \frac{\sin(x-a)}{\cos(x-b)} + C$$

(ii) $\int \frac{dx}{\cos(x-a)\cdot\cos(x-b)}$

Sol^n:-

$$\frac{1}{\sin(a-b)} \int \frac{\sin[(x-b)-(x-a)]}{\cos(x-a)\cdot\cos(x-b)}\,dx$$

$$\frac{1}{\sin(a-b)} \int \frac{\sin(x-b)\cos(x-a) - \sin(x-a)\cos(x-b)}{\cos(x-a)\cdot\cos(x-b)}\,dx$$

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$$\frac{1}{\sin(a-b)}\int \frac{\sin(x-b)}{\cos(x-b)} - \frac{\sin(x-a)}{\cos(x-a)}\, dx$$

$$\frac{1}{\sin(a-b)}\int \tan(x-b) - \tan(x-a)\quad dx$$

$$\frac{1}{\sin(a-b)}\left[-\ln\cos(x-b) + \ln\cos(x-a)\right] + C$$

$$\frac{1}{\sin(a-b)}\quad \ln\frac{\cos(x-a)}{\cos(x-b)} + c$$

(iii) Q. $\int \frac{\sin(x+a)}{\sin(x+b)}\, dx.$

Sol^n:- Put $x+b=t$

$dx = dt$

$x = t-b$

$$\int \frac{\sin(t-b+a)}{\sin t}\cdot dt$$

$$= \int \frac{\sin(t+(a-b))}{\sin t}\, dt$$

$$= \int \frac{\sin t\cos(a-b) + \cos t\sin(a-b)}{\sin t}\, dt$$

$$= \int\left[\cos(a-b) + \cot t\sin(a-b)\right] dt$$

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$= \cos(a-b)\, t + \sin(a-b) \cdot \ln \sin t + c$

$= \cos(a-b)\,(x+b) + \sin(a-b) \ln \sin(x+b) + c$

Integration of the form :

$$\int \frac{dx}{a\sin^2 x + b\cos^2 x} \quad \text{or} \quad \int \frac{dx}{a + b\sin^2 x} \quad \text{or} \quad \int \frac{dx}{a + b\cos^2 x}$$

$$\text{or} \int \frac{dx}{a\sin^2 x + b\sin x\cos x + c\cos^2 x} \quad \text{or} \quad \int \frac{dx}{(a\sin x + b\cos x)^2}$$

$$\text{or} \int \frac{dx}{a\sin^2 x + b\cos^2 x + c}$$

Procedure : Divide ~~by~~ numerator and denominator by $\cos^2 x$ then substitute $\tan x = t$

Q. $\int \frac{dx}{(2\sin x + 3\cos x)^2}$

Soln :- $\int \frac{dx}{4\sin^2 x + 9\cos^2 x + 12\sin x\cos x}$

$= \int \frac{dx \, \sec^2 x}{4\tan^2 x + 9 + 12\tan x}$

Put $\tan x = t$

$\sec^2 x \, dx = dt$

$$= \int \frac{dt}{4t^2 + 12t + 9}$$

$$= \int \frac{dt}{(2t+3)^2}$$

$$= \int (2t+3)^{-2} \, dt$$

$$= \frac{-1}{2t+3} + c$$

$$= \frac{-1}{2\tan x + 3} + c$$

Q. $\int \frac{dx}{4\sin^2 x + 9\cos^2 x}$

Sol^n:- $\int \frac{dx \sec^2 x}{4\tan^2 x + 9}$

~~tan t~~ Put $\tan x = t$

$\sec^2 x \, dx = dt$

$$\int \frac{dt}{4t^2 + 9}$$

$$\int \frac{dt}{(2t)^3 + (3)^3}$$

$$= \frac{1}{3} \times \frac{1}{2} \tan^{-1}\left(\frac{2t}{3}\right) + c$$

$$= \frac{1}{6} \tan^{-1}\left(\frac{2\tan x}{3}\right) + c$$

Q. $\int \frac{\sin x}{\sin 3x}\,dx$

Sol^n:- $\int \frac{\sin x}{3\sin x - 4\sin^3 x}\,dx$

$$= \int \frac{dx}{3 - 4\sin^2 x}$$

$$= \int \frac{\sec^2 x}{3\sec^2 x - 4\tan^2 x}\,dx$$

$\Rightarrow \quad \tan x = t$

$\sec^2 x\,dx = dt$

$$= \int \frac{dt}{3(1+\tan^2 x) - 4\tan^2 x}$$

$$= \int \frac{dt}{3(1+t^2) - 4t^2}$$

$$= \int \frac{dt}{3 - t^2} = \int \frac{dt}{(\sqrt{3})^2 - (t)^2}$$

$$= \frac{1}{2\sqrt{3}} \ln\left(\frac{\cancel{\tan}\ \sqrt{3} + t}{\sqrt{3} - t}\right) + c$$

$$= \frac{1}{2\sqrt{3}} \ln\left(\frac{\sqrt{3} + \tan x}{\sqrt{3} - \tan x}\right) + c$$

Integration of the farm

$$\int \frac{dx}{a\sin x + b\cos x} \quad \text{or} \quad \int \frac{dx}{a + b\sin x} \quad \text{or} \int \frac{dx}{a + b\cos x}$$

or $$\int \frac{dx}{a\sin x + b\cos x + c}$$

Procedure:

Put $\sin x = \dfrac{2\tan x/2}{1+\tan^2 x/2}$

$\cos x = \dfrac{1-\tan^2 x/2}{1+\tan^2 x/2}$

Then,

Put $\tan x/2 = t$

Q. $$\int \frac{dx}{2+\sin x + \cos x}$$

Sol^n:-

$$\int \frac{dx}{2+\left(\frac{2\tan x/2}{1+\tan^2 x/2}\right) + \left(\frac{1-\tan^2 x/2}{1+\tan^2 x/2}\right)}$$

$$\int \frac{dx}{\frac{2(1+\tan^2 x/2) + 2\tan x/2 + 1 - \tan^2 x/2}{1+\tan^2 x/2}}$$

$$\int \frac{\sec^2 x/2 \quad dx}{2+2\tan^2 x/2 + 2\tan x/2 + 1 - \tan^2 x/2}$$

$$\int \frac{\sec^2 x/2 \, dx}{\tan^2 x/2 + 2\tan x/2 + 3}$$

Put $\tan x/2 = t$

$$\frac{1}{2}\sec^2 x/2 \, dx = dt$$

$$2 \int \frac{dt}{t^2 + 2t + 3}$$

$$2 \int \frac{dt}{t^2 + 2t + 1 + 2}$$

$$2 \int \frac{dt}{(t+1)^2 + 2} = \frac{1}{2}\int \frac{dt}{(t+1)^2 + (\sqrt{2})^2}$$

$$= 2 \times \frac{1}{\sqrt{2}} \tan^{-1}\left(\frac{t+1}{\sqrt{2}}\right) + c$$

$$= \sqrt{2} \tan^{-1}\left(\frac{\tan x/2 + 1}{\sqrt{2}}\right) + c$$

Q. $\displaystyle\int \frac{dx}{\sqrt{3}\sin x + \cos x}$

Soln:- $$\int \frac{dx}{\sqrt{3}\left(\frac{2\tan x/2}{1+\tan^2 x/2}\right) + \left(\frac{1-\tan^2 x/2}{1+\tan^2 x/2}\right)}$$

$$= \int \frac{\sec^2 x/2 \, dx}{2\sqrt{3}\tan x/2 + 1 - \tan^2 x/2}$$

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$\tan x/2 = t$

$\frac{1}{2} \sec^2 x/2 \, dx = dt$

$= \frac{2}{2} \int \frac{dt}{2\sqrt{3}t + 1 - t^2}$

$= -2 \int \frac{dt}{t^2 - 2\sqrt{3}t - 1}$

$= -2 \int \frac{dt}{t^2 - 2\sqrt{3}t + (\sqrt{3})^2 - 1 - 3}$

$= -2 \int \frac{dt}{(t - \sqrt{3})^2 - 4}$

$= -2 \int \frac{dt}{(t - \sqrt{3})^2 - (2)^2}$

$= -2 \times \frac{1}{4} \ln \left(\frac{t - \sqrt{3} - 2}{t - \sqrt{3} + 2} \right) + c$

$= -\frac{1}{2} \ln \left(\frac{\tan x/2 - \sqrt{3} - 2}{\tan x/2 - \sqrt{3} + 2} \right) + c$

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Integration of the form

$$\int \frac{a\sin x + b\cos x + c}{p\sin x + q\cos x + r}\,dx \quad \text{or} \quad \int \frac{a\sin x + b\cos x}{p\sin x + q\cos x}\,dx$$

Procedure:

Express numerator $= \lambda$ denominator $+ \mu \frac{d}{dx}$ (Denominator) + r

Find λ, μ and γ by comparing the ~~c~~ coefficient of $\sin x$, $\cos x$ and ~~contsc~~ constant term and split into sum of three integrals.

$$N^r = \lambda(D^r) + \mu \frac{d}{dx}(D^r) + \gamma$$

$$I = \lambda\int dx + \mu \int \frac{\frac{d}{dx}(D^r)}{D^r}\,dx + \gamma \int \frac{dx}{D^r}$$

Q. Evaluate $\int \frac{2 + 3\cos x}{\sin x + 2\cos x + 3}\,dx$

Sol^n: $2 + 3\cos x = \lambda(\sin x + 2\cos x + 3) + \mu(\cos x - 2\sin x) + \gamma$

$0 = \lambda - 2\mu$

$3 = 2\lambda + \mu$

$$= \int \frac{\left[\frac{6}{5}(\sin x + 2\cos x + 3) + \frac{3}{5}(\cos x - 2\sin x) - \frac{8}{5}\right]dx}{\sin x + 2\cos x + 3}$$

$$= \frac{6}{5}\int dx + \frac{3}{5}\int \frac{\cos x - 2\sin x}{\sin x + 2\cos x + 3}\,dx - \frac{8}{5}\int \frac{dx}{\sin x + 2\cos x + 3}$$

$$= \frac{6}{5}x + \frac{3}{5}\ln|\sin x + 2\cos x + 3| - \frac{8}{5}\underbrace{\int \frac{dx}{\sin x + 2\cos x + 3}}_{I}$$

$$I = \int \frac{dx}{\sin x + 2\cos x + 3}$$

$$\int \frac{dx}{\frac{2\tan x/2}{1+\tan^2 x/2} + 2\left(\frac{1-\tan^2 x/2}{1+\tan^2 x/2}\right) + 3}$$

$$\int \frac{\sec^2 x/2 \;\; dx}{2\tan x/2 + 2 - 2\tan^2 x/2 + 3 + 3\tan^2 x/2}$$

Put $\tan x/2 = t$

$\sec^2 x/2 \; dx = 2dt$

$$I = 2\int \frac{dt}{\cancel{2t}\; t^2 + 2t + 5}$$

$$I = 2\int \frac{dt}{(t+1)^2 + (2)^2}$$

$$I = 2 \times \frac{1}{2}\tan^{-1}\left(\frac{\tan x/2 + 1}{2}\right)$$

$$\Rightarrow \frac{6}{5}x + \frac{3}{5} \ln|\sin x + 2\cos x + 3| - \frac{8}{5} \tan^{-1}\left(\frac{\tan x/2 + 1}{2}\right)$$

Q. $\int \frac{3\sin x + 2\cos x}{3\cos x + 2\sin x} \cdot dx$

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Integral of the form

Type A → $\int \frac{dx}{ax^2+bx+c}$ or $\int \sqrt{ax^2+bx+c}\, dx$ or $\int \frac{dx}{\sqrt{ax^2+bx+c}}$

Note:-

If ax^2+bx+c can be factorised, then the integration is done by the method of partial fraction.

If the denominator can't be factorised, then express it as the sum or difference of two squares by the method of completing square.

Type B:-

$\int \frac{px+q}{ax^2+bx+c}\, dx$ or $\int \frac{px+q}{\sqrt{\underbrace{ax^2+bx+c}_{A}}}\, dx$ or

$\int (px+q)\sqrt{\overbrace{ax^2+bx+c}^{A}}\, .dx$

"$A = ax^2+bx+c$"

Procedure:- $px+q = \lambda \frac{d}{dx}(A) + \mu$

Find λ and μ by comparing the coefficient of $\cancel{x^2}$, x and constant term.

Q. $\int \frac{dx}{x^2-x+1}$

Soln:- $\int \cancel{\frac{dx}{x^2-x+1}}$ $\int \frac{dx}{x^2-2\times\frac{1}{2}x+\left(\frac{1}{2}\right)^2-\left(\frac{1}{2}\right)^2+1}$

$\int \frac{dx}{(x-1/2)^2-\frac{1}{4}+1} = \int \frac{dx}{\left(x-\frac{1}{2}\right)^2+\frac{3}{4}}$

$\Rightarrow \int \frac{dx}{(x-1/2)^2+(\sqrt{3}/2)^2}$

$\Rightarrow \frac{2}{\sqrt{3}} \tan^{-1}\left(\frac{(x-1/2)2}{\sqrt{3}}\right) \Rightarrow \frac{2}{\sqrt{3}} \tan^{-1}\left(\frac{2x-1}{\sqrt{3}}\right)+c$

Q. $\int \frac{dx}{\sqrt{x^2-2x+3}}$

Soln:- $\int \frac{dx}{\sqrt{x^2-2x+1+2}} = \int \frac{dx}{\sqrt{(x-1)^2+(\sqrt{2})^2}}$

$= \ln\left| x-1+\sqrt{(x-1)^2+2} \right| + c$

Q. $\int \sqrt{2x^2-3x+1}\ dx$

Soln:- $\int \sqrt{(\sqrt{2}x)^2-2\times\sqrt{2}\times\frac{3}{2\sqrt{2}}x+\left(\frac{3}{2\sqrt{2}}\right)^2+1-\left(\frac{3}{2\sqrt{2}}\right)^2}\ dx$

$\int \sqrt{\left(\sqrt{2}x-\frac{3}{2\sqrt{2}}\right)^2+1-\frac{9}{8}}\ dx$

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$$= \int \sqrt{\left(\sqrt{2}x - \frac{3}{2\sqrt{2}}\right)^2 - \frac{1}{8}}\ dx$$

$$= \int \sqrt{\left(\sqrt{2}x - \frac{3}{2\sqrt{2}}\right)^2 - \left(\frac{1}{2\sqrt{2}}\right)^2}\ dx$$

$$= \frac{\sqrt{2}x - \frac{3}{2\sqrt{2}}}{2} \sqrt{2x^2 - 3x + 1} - \frac{1}{8\times 2} \ln\left|\sqrt{2}x - \frac{3}{2\sqrt{2}} + \sqrt{2x^2-3x+1}\right|$$

$$\Rightarrow \frac{\frac{4x-3}{2\sqrt{2}}}{\frac{2}{1}} \sqrt{2x^2-3x+1} - \frac{1}{16} \ln\left|\frac{4x-3}{2\sqrt{2}} + \sqrt{2x^2-3x+1}\right| + C$$

$$\Rightarrow \frac{4x-3}{4\sqrt{2}} \sqrt{2x^2-3x+1} - \frac{1}{16} \ln\left|\frac{4x-3}{2\sqrt{2}} + \sqrt{2x^2-3x+1}\right| + C$$

Q. $\int \frac{3x+2}{4x^2+4x+5}\ dx$

Solⁿ:- $3x+2 = \lambda(8x+4) + \mu$

$3x+2 = 8\lambda x + 4\lambda + \mu$

$8\lambda = 3$

$\boxed{\lambda = 3/8}$

$4\lambda + \mu = 2$

$\frac{3}{2} + \mu = 2$

$\boxed{\mu = 1/2}$

$$\int \frac{\frac{3}{8}(8x+4) + 1/2}{4x^2+4x+5}$$

$$= \frac{3}{8}\int \frac{\overset{\to f'(x)}{8x+4}}{\underset{\to f(x)}{4x^2+4x+5}}\,dx + \frac{1}{2}\int \frac{dx}{4x^2+4x+5}$$

$$= \frac{3}{8}\ln(4x^2+4x+5) + \frac{1}{2}\int \frac{dx}{(2x+1)^2+2^2}$$

$$= \frac{3}{8}\ln(4x^2+4x+5) + \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2}\tan^{-1}\left(\frac{2x+1}{2}\right) + C$$

$$= \frac{3}{8}\ln(4x^2+4x+5) + \frac{1}{8}\tan^{-1}\left(\frac{2x+1}{2}\right) + C$$

Q. $\displaystyle\int \frac{(2\sin 2x - \cos x)}{6 - \cos^2 x - 4\sin x}\,dx$

Solⁿ:-

$\cancel{2\sin 2x - \cos x = \lambda(\sin 2x - 4\cos x) + \mu}$

$\cancel{2\sin 2x - \cos x = \lambda\sin 2x - 4\lambda\cos x + \mu}$

$$\int \frac{4\sin x\cos x - \cos x}{6-(1-\sin^2 x) - 4\sin x}\,dx$$

$$\int \frac{\cos x(4\sin x - 1)}{\sin^2 x - 4\sin x + 5}\,dx$$

Put $\sin x = t$

$\cos x\,dx = dt$

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$$\int \frac{4t-1}{t^2-4t+5}\, dt$$

$4t-1 = \lambda(2t-4) + \mu$

$4t-1 = 2\lambda t - 4\lambda + \mu$

$2\lambda = 4$

$\boxed{\lambda = 2}$

$\mu - 4\lambda = -1$

$\boxed{\mu = 7}$

$$= \int \frac{2(2t-4)+7}{t^2-4t+5}\, dt$$

$$= 2\int \frac{\overbrace{2t-4}^{f'(x)}}{\underbrace{t^2-4t+5}_{f(x)}}\, dt + 7\int \frac{dt}{t^2-4t+5}\, dt$$

$$= 2\ln(t^2-4t+5) + 7\int \frac{dt}{(t-2)^2+1}$$

$$= 2\ln(t^2-4t+5) + 7 \times \frac{1}{1} \times \tan^{-1}\left(\frac{t-2}{1}\right)$$

$$= 2\ln(\sin^2 x - 4\sin x + 5) + 7\tan^{-1}(\sin x - 2) + C$$

Q. $\int (x+1)\sqrt{1-x-x^2}\, dx$

Sol^n: $x+1 = \lambda(-1-2x) + \mu$

$x+1 = -\lambda - 2\lambda x + \mu$

$-2\lambda = 1$

$\boxed{\lambda = -1/2}$

$\mu - \lambda = +1$

$\mu + \frac{1}{2} = +1$

$\boxed{\cancel{\mu = -3/2}}$ $\boxed{\mu = 1/2}$

$$\int \left[-\frac{1}{2}(-2x-1) + \frac{1}{2} \right] \sqrt{1-x-x^2}\, dx$$

$$\frac{-1}{2} \int \underset{f'(x)}{\underset{\downarrow}{(-2x-1)}} \underset{f(x)^{1/2}}{\underset{\downarrow}{\sqrt{1-x-x^2}}}\, dx + \frac{1}{2} \int \sqrt{1-x-x^2}\, dx$$

$$\frac{-1}{2} \frac{f(x)^{3/2}}{3/2} + \frac{1}{2} \int \sqrt{1-x-x^2}\, dx$$

$$-\frac{1}{\cancel{2}} \frac{(1-x-x^2)^{3/2}}{3} + \frac{1}{2} \int \sqrt{-(x^2+x-1)}\, dx$$

$$-\frac{(1-x-x^2)^{3/2}}{3} + \frac{1}{2} \int \sqrt{-\left(x^2+x+(1/2)^2\right)+1+(1/2)^2}\, dx$$

$$-\frac{(1-x-x^2)^{3/2}}{3} + \frac{1}{2} \int \sqrt{-(x+1/2)^2 + 5/4}\, dx$$

$$-\frac{(1-x-x^2)^{3/2}}{3} + \frac{1}{2} \int \sqrt{\left(\frac{\sqrt{5}}{2}\right)^2 - (x+1/2)^2}\, dx$$

$$-\frac{(1-x-x^2)^{3/2}}{3} + \frac{1}{2} \times \frac{(x+1/2)}{2} \sqrt{1-x-x^2} + \frac{5}{4 \times 2} \sin^{-1}\left(\frac{(x+1/2)2}{\sqrt{5}}\right) + c$$

$$= -\frac{(1-x-x^2)^{3/2}}{3} + \frac{(2x+1)}{8}\sqrt{1-x-x^2} + \frac{5}{8}\sin^{-1}\left(\frac{2x+1}{\sqrt{5}}\right) + c$$

Integration of Irrational Algebraic ~~Indefinite~~ Function

1. $$\int \frac{dx}{(ax^2+bx+c)\sqrt{px+q}} \quad \text{or} \quad \int \frac{dx}{(ax+b)\sqrt{px+q}}$$

$ax^2+bx+c \downarrow$ Quadratic, $\sqrt{px+q} \downarrow$ Linear; $(ax+b) \downarrow$ Linear, $\sqrt{px+q} \downarrow$ Linear

Put $px+q = t^2$

2. $$\int \frac{dx}{(ax+b)\sqrt{px^2+qx+r}}$$

Put $ax+b = \frac{1}{t}$

3. $$\int \frac{dx}{(ax^2+b)\sqrt{px^2+qx+r}}$$

Put $x = \frac{1}{t}$

Q. $$\int \frac{dx}{(x+1)\sqrt{x-2}}$$

Sol^n:-

$x-2 = t^2$

$dx = 2t\,dt$

$$\int \frac{2t\,dt}{(t^2+3)\sqrt{t^2}} = \int \frac{2\cancel{t}\,dt}{(t^2+3)\cancel{t}}$$

$$= 2\int \frac{dt}{(t^2+3)}$$

$$= 2\int \frac{dt}{(t)^2 + (\sqrt{3})^2}$$

$$= \frac{2}{\sqrt{3}} \tan^{-1}\left(\frac{t}{\sqrt{3}}\right) + c$$

$$= \frac{2}{\sqrt{3}} \tan^{-1}\left(\frac{x-2}{\sqrt{3}}\right) + c$$

Q. $\int \frac{(x+2)\, dx}{(x^2+3x+3)\sqrt{x+1}}$

Solⁿ:- $x+1 = t^2$

$dx = 2t\, dt$

$$\int \frac{(t^2+1)\, 2t\, dt}{[(t^2-1)^2 + 3(t^2-1) + 3]\, t}$$

$$\int \frac{2(t^2+1)\, dt}{t^4 - 2t^2 + 1 + 3t^2 - 3 + 3}$$

$$2\int \frac{\cancel{2}(t^2+1)\, dt}{t^4 + t^2 + 1}$$

Divide by t^2

$$2\int \frac{1 + \frac{1}{t^2}}{t^2 + 1 + \frac{1}{t^2}}$$

$$t - \frac{1}{t} = y$$

$$\left(1 + \frac{1}{t^2}\right) dt = dy$$

$$= 2 \int \frac{dy}{\left(t - \frac{1}{t}\right)^2 + 3}$$

$$= 2 \int \frac{dy}{y^2 + 3} = 2 \int \frac{dy}{(y)^2 + (\sqrt{3})^2}$$

$$= 2 \, \frac{1}{\sqrt{3}} \tan^{-1}\left(\frac{y}{\sqrt{3}}\right)$$

$$\cancel{= \sqrt{2} \tan^{-1}\left(\frac{t - \frac{1}{t}}{2}\right)}$$

$$\cancel{= \sqrt{2} \tan^{-1}\left(\frac{t^2 - 1}{2t}\right) + c} = \cancel{\sqrt{2} \tan^{-1}\left(\frac{x}{2\sqrt{x+1}}\right) + c}$$

$$\cancel{\Rightarrow \sqrt{2} \tan^{-1}\left(\frac{x}{2\sqrt{x+1}}\right) + c}$$

$$= \frac{2}{\sqrt{3}} \tan^{-1}\left(\frac{t^2 - 1}{\sqrt{3}\, t}\right) + c$$

$$= \frac{2}{\sqrt{3}} \tan^{-1}\left(\frac{x}{\sqrt{3x+3}}\right) + c$$

Q. $\displaystyle\int \frac{dx}{(x-1)\sqrt{x^2+x+1}}$

sol^n:- $x - 1 = 1/t \Rightarrow x = 1/t + 1$

$dx = dt \, \frac{-1}{t^2}$

$$-\int \cancel{\frac{dt}{t\sqrt{(t+1)^2+t+1+1}}} \quad \cancel{\frac{1}{t^2}}$$

$$\cancel{=\int \frac{dt}{t\sqrt{t^2+2t+1+t+2}}} = \cancel{\int \frac{dt}{t\sqrt{t^2+3t+3}}}$$

$$= -\int \frac{1\ t \cdot dt}{t^2\sqrt{\left(1+\frac{1}{t}\right)^2+1+\frac{1}{t}+1}}$$

$$= -\int \frac{dt}{t\sqrt{\frac{3t^2+3t+1}{t^2}}}$$

$$= -\int \frac{dt}{\sqrt{3t^2+3t+1}} = \frac{-1}{\sqrt{3}}\int \frac{dt}{\sqrt{t^2+8t+1/3}}$$

$$= \frac{-1}{\sqrt{3}}\int \frac{dt}{\sqrt{(t)^2+2\times t\times\frac{1}{2}+\left(\frac{1}{2}\right)^2-\left(\frac{1}{2}\right)^2+\frac{1}{3}}}$$

$$= -\frac{1}{\sqrt{3}}\int \frac{dt}{\sqrt{\left(t+\frac{1}{2}\right)^2+\frac{1}{12}}}$$

$$= -\frac{1}{\sqrt{3}}\int \frac{dt}{\sqrt{\left(t+\frac{1}{2}\right)^2+\left(\frac{1}{2\sqrt{3}}\right)^2}}$$

$$= \frac{-1}{\sqrt{3}}\ \ln\left|t+\frac{1}{2}+\sqrt{t^2+t+1/3}\right| + c$$

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$$= -\frac{1}{3} \ln \left| \frac{1}{x-1} + \frac{1}{2} + \sqrt{\left(\frac{1}{x-1}\right)^2 + \left(\frac{1}{x-1}\right) + \frac{1}{3}} \right| + C$$

Q. $\int \frac{dx}{(1+x^2)\sqrt{1-x^2}}$

sol^n:- Put $x = \cos\theta$

$dx = -\sin\theta . d\theta$

$$= -\int \frac{\cancel{\sin\theta}\, d\theta}{(1+\cos^2\theta)\cancel{\sin\theta}} = -\int \frac{d\theta}{1+\cos^2\theta}$$

Divide by $\cos^2\theta$

$$= -\int \frac{\sec^2\theta . d\theta}{\sec^2\theta + 1} = -\int \frac{\sec^2\theta . d\theta}{2 + \tan^2\theta}$$

Put $\tan\theta = t$

$\sec^2\theta . d\theta = dt$

$$-\int \frac{dt}{(\sqrt{2})^2 + t^2}$$

$$= -\frac{1}{\sqrt{2}} \tan^{-1}\left(\frac{t}{\sqrt{2}}\right)$$

$$= -\frac{1}{\sqrt{2}} \tan^{-1}\left(\frac{\tan\theta}{2}\right)$$

$$= -\frac{1}{\sqrt{2}} \tan^{-1}\left(\frac{\tan \cos^{-1}x}{2}\right)$$

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Integral of the form:-

$$\int \frac{K(x)}{ax^2+bx+c} dx$$

where $K(x)$ is a polynomial of degree greater than or equal to 2.

Q. $$\int \frac{x^2+x+3}{x^2-x-2} dx$$

Sol^n:- $$\int \frac{x^2-x-2+2x+5}{x^2-x-2} dx$$

$$= \int \frac{x^2-x-2}{x^2-x-2} dx + \int \frac{2x+5}{x^2-x-2} dx$$

$$= \int dx + \int \frac{2x+5}{x^2-x-2} dx$$

$$= x + \text{(I)}$$

For (I):-

$2x+5 = \lambda(2x-1)+\mu$

$2x+5 = 2\lambda x - \lambda + \mu$

$2\lambda = 2$

$\boxed{\lambda = 1}$

$\mu - \lambda = 5$

$\boxed{\mu = 6}$

$$I = \int \frac{1(2x-1)+6}{x^2-x-2}\,dx$$

$$I = \int \frac{2x-1}{x^2-x-2}\,dx + \int \frac{6}{x^2-x-2}\,dx$$

$$I = \ln(x^2-x-2) + 6\int \frac{dx}{(x+1)(x-2)}$$

$$I = \ln(x^2-x-2) + 6\times\frac{1}{3}\int\left(\frac{1}{x-2} - \frac{1}{x+1}\right)dx$$

$$I = \ln(x^2-x-2) + 2\left[\ln(x-2) - \ln(x+1)\right] + c$$

$$I = \ln(x^2-x-2) + 2\ln\left(\frac{x-2}{x+1}\right)$$

Hence,

$$\int f(x) = x + \ln(x^2-x-2) + 2\ln\left(\frac{x-2}{x+1}\right) + c$$

Integral of the form

$$\int \frac{ax^2+bx+c}{px^2+qx+r}\,dx \quad \text{or} \quad \int \frac{ax^2+bx+c}{\sqrt{px^2+qx+r}}\,dx$$

$$\text{or} \quad \int (ax^2+bx+c)\sqrt{px^2+qx+r}\;.\,dx$$

Express $ax^2+bx+c = \lambda(px^2+qx+r) + \mu\frac{d}{dx}(px^2+qx+r)$

$+\ \gamma$

Find λ, μ and γ by comparing the coefficients of x^2, x and constant term

Q. $\displaystyle\int \frac{2x^2+5x+4}{\sqrt{x^2+x+1}}\,dx$

Solⁿ: $2x^2+5x+4 = \lambda(x^2+x+1) + \mu(2x+1) + \gamma$

$\boxed{\lambda = 2}$

$\lambda + 2\mu = 5$

$\boxed{\mu = 3/2}$

$\lambda + \mu + \gamma = 4$

$\boxed{\gamma = 1/2}$

$$= \int \frac{2(x^2+x+1) + 3/2(2x+1) + 1/2}{\sqrt{x^2+x+1}}\,dx$$

$$= \int \frac{2(x^2+x+1)}{\sqrt{x^2+x+1}}\,dx + \int \frac{3/2(2x+1)\,dx}{\sqrt{x^2+x+1}} + \int \frac{1/2\,dx}{\sqrt{x^2+x+1}}$$

$$I = \underbrace{2\int \sqrt{x^2+x+1}\,dx}_{I_1} + \underbrace{\frac{3}{2}\int \frac{(2x+1)}{\sqrt{x^2+x+1}}\,dx}_{I_2} + \underbrace{\frac{1}{2}\int \frac{dx}{\sqrt{x^2+x+1}}}_{I_3}$$

Now, x^2+x+1

$(x)^2 + 2 \times x \times \frac{1}{2} + \left(\frac{1}{2}\right)^2 - \left(\frac{1}{2}\right)^2 + 1$

$\left(x+\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2$

$$I_1 = 2\int \sqrt{\left(x+\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2}$$

$$I_1 = 2\left[\frac{(x+1/2)}{2}\sqrt{x^2+x+1} + \frac{3}{4\times 2}\ln\left|\left(x+\frac{1}{2}\right)+\sqrt{x^2+x+1}\right|\right]$$

$$I_1 = \left(x+\frac{1}{2}\right)\sqrt{x^2+x+1} + \frac{3}{4}\ln\left|\left(x+\frac{1}{2}\right)+\sqrt{x^2+x+1}\right|$$

$$I_3 = \frac{1}{2}\int\frac{dx}{\sqrt{x^2+x+1}}$$

$$I_3 = \frac{1}{2}\ln\left|\left(x+\frac{1}{2}\right)+\sqrt{x^2+x+1}\right|$$

$$I_2 = \frac{3}{2}\int\frac{(2x+1)}{\sqrt{x^2+x+1}}\,dx$$

Put $x^2+x+1 = t$

$(2x+1)\,dx = dt$

$$I_2 = \frac{3}{2}\int\frac{dt}{\sqrt{t}}$$

$$I_2 = \frac{3}{2}\times 2t^{1/2}$$

$$I_2 = 3(x^2+x+1)^{1/2}$$

$$I = I_1 + I_2 + I_3$$

$$I = \left(x+\frac{1}{2}\right)\sqrt{x^2+x+1} + \frac{3}{4}\ln\left|\left(x+\frac{1}{2}\right)+\sqrt{x^2+x+1}\right|$$

$$+ 3\sqrt{x^2+x+1} + \frac{1}{2}\ln\left|\left(x+\frac{1}{2}\right)+\sqrt{x^2+x+1}\right| + C$$

Partial Fraction

This technique is used if a rational f^n is being integrated when denominator can be factorised.

If degree of numerator is greater than the degree of denominator, then first divide numerator by denominator.

Note :-

Before decomposing into partial fraction, we must ensure that degree of N^r is less than degree of D^r.

	Form of rational f^n	Form of partial fraction.
1.	$\dfrac{px^2 + qx + r}{(x-a)(x-b)(x-c)}$	$\dfrac{A}{(x-a)} + \dfrac{B}{(x-b)} + \dfrac{C}{(x-c)}$
2.	$\dfrac{px^2 + qx + r}{(x-a)^2(x-b)}$	$\dfrac{A}{(x-a)} + \dfrac{B}{(x-a)^2} + \dfrac{C}{(x-b)}$
3.	$\dfrac{px^2+qx+r}{(x-a)\underbrace{(x^2+bx+c)}_{\text{can't be factorised further}}}$	$\dfrac{A}{(x-a)} + \dfrac{Bx+C}{(x^2+bx+c)}$

Q. Evaluate :

1. $\int \frac{x}{(x-2)(x+5)} dx$

Soln :- $\frac{x}{(x-2)(x+5)} = \frac{A}{(x-2)} + \frac{B}{(x+5)}$

Put $x=2$ in $\frac{x}{x+5}$

Put $x=-5$ in $\frac{x}{x+2}$

$A = 2/7$, $B = 5/7$

$= \frac{2}{7} \frac{1}{(x-2)} + \frac{5}{7(x+5)}$

$= \frac{2}{7} \int \frac{1}{(x-2)} dx + \frac{5}{7} \int \frac{dx}{(x+5)}$

$= \frac{2}{7} \ln(x-2) + \frac{5}{7} \ln(x+5) + C$

Proper method to find A & B :-

$\frac{x}{\cancel{(x-2)(x+5)}} = \frac{A(x+5) + B(x-2)}{\cancel{(x-2)(x+5)}}$

$A + B = 1$

$5A = 2B$

$A = 2B/5$

$A = 2/7$, $B = 5/7$

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2. $$\int \frac{x^4}{(x+2)(x^2+1)} \cdot dx$$

sol^n: $$\int \frac{x^4-1+1}{(x+2)(x^2+1)} dx$$

$$\Rightarrow \int \frac{x^4-1}{(x+2)(x^2+1)} dx + \int \frac{dx}{(x+2)(x^2+1)}$$

$$= \int \frac{(x^2-1)\cancel{(x^2+1)}}{(x+2)\cancel{(x^2+1)}} dx + \int \frac{dx}{(x+2)(x^2+1)}$$

$$= \int \frac{x^2-1}{x+2} dx + \underbrace{\int \frac{dx}{(x+2)(x^2+1)}}_{I}$$

$$= \int \frac{x^2-4}{x+2} + \frac{3}{x+2} dx + I$$

$$= \int (x-2)\, dx + \int \frac{3}{x+2} dx + I$$

$$X = \frac{x^2}{2} - 2x + 3 \ln (x+2) + I$$

$$I = \frac{1}{(x+2)(x^2+1)} = \frac{A}{x+2} + \frac{Bx+C}{x^2+1}$$

$$\boxed{A = 1/5}$$

$$1 = A(x^2+1) + (Bx+C)(x+2)$$

$$A + B = 0$$

$$\boxed{B = -1/5}$$

$$C + 2B = 0$$

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$\boxed{c = 2/5}$

$$I = \int \frac{dx}{5(x+2)} + \int \frac{-1/5\, x + 2/5}{x^2+1}\, dx$$

$$I = \frac{1}{5} \ln(x+2) - \frac{1}{5} \int \frac{x-2}{x^2+1}\, dx$$

$$= \frac{1}{5} \ln(x+2) - \frac{1}{10} \int \frac{2x}{x^2+1}\, dx + \frac{2}{5} \int \frac{dx}{x^2+1}$$

$$= \frac{1}{5} \ln(x+2) - \frac{1}{10} \ln(x^2+1) + \frac{2}{5} \tan^{-1} x + c$$

$$X = \frac{x^2}{2} - 2x + 3 \ln(x+2) + \frac{1}{5} \ln(x+2) - \frac{1}{10} \ln(x^2+1)$$

$$+ \frac{2}{5} \tan^{-1} x + C$$

Indirect and Direct Substitution

Indirect substitution :-

If the integral is of the farm $f(x) \cdot g(x)$. where $g(x)$ is a f^n of the integral of $f(x)$, then put integral of $f(x) = t$.

Q. Evaluate :

$$\int \frac{\sqrt{x}}{\sqrt{x^3+a^3}} dx$$

Sol^n :-

Put $x^{3/2} = t$ {By integrating N^r)

$\frac{3}{2} x^{1/2} dx = dt$

$$\frac{2}{3} \int \frac{dt}{\sqrt{t^2+(a^{3/2})^2}}$$

$$= \frac{2}{3} \ln \left| t + \sqrt{t^2+(a^{3/2})^2} \right| + c$$

$$= \frac{2}{3} \ln \left| x^{3/2} + \sqrt{x^3+a^3} \right| + c$$

Q. $\int \frac{(\sin x + \cos x)}{9 + 16 \sin 2x} dx$

Sol^n :. $\int \sin x + \cos x = -\cos x + \sin x$

Put $\sin x - \cos x = t$

$\cos x + \sin x \ dx = dt$

$$\int \frac{dt}{} \qquad (\sin x - \cos x)^2 = 1 - \sin 2x$$

$$\sin 2x = 1 - (\sin x - \cos x)^2$$

$$\int \frac{dt}{9 + 16\,[1 - (\sin x - \cos x)^2]}$$

$$\int \frac{dt}{9 + 16(1 - t^2)} = \int \frac{dt}{9 + 16 - 16t^2}$$

$$= -\int \frac{dt}{25 + 16t^2} = \int \frac{dt}{(5)^2 + (4t)^2} = \int \frac{dt}{(5)^2 - (4t)^2}$$

$$= -\frac{1}{5} \times \frac{1}{4} \tan^{-1}\left(\frac{4t}{5}\right) \qquad = \frac{1}{2} \times \frac{1}{5} \times \frac{1}{4} \ln\left(\frac{5+4t}{5-4t}\right) + c$$

$$= -\frac{1}{20} \tan^{-1} \qquad = \frac{1}{40} \ln\left(\frac{5 + 4\sin x - 4\cos x}{5 - 4\sin x + 4\cos x}\right) + c$$

Derived Substitution

Sometimes, it is useful to write the integral as a sum of two related integrals which can be evaluated by making suitable substitution.

1. Algebraic Twins :-

(i) $$\int \frac{2x^2}{x^4+1}\,dx = \int \frac{x^2+1}{x^4+1}\,dx + \int \frac{x^2-1}{x^4+1}\,dx$$

Divide N^r and D^r by x^2 and proceed.

Always ensure coefficient of $x^2 = 2$.

(ii) $\int \frac{2}{x^4+1} dx \Rightarrow \int \frac{x^2+1}{x^4+1} dx - \int \frac{x^2-1}{x^4+1} dx$

~~Re~~ Divide N^r and D^r by x^2 and proceed.

(iii) $\int \frac{2x^2}{x^4+Kx^2+1} dx \Rightarrow \int \frac{x^2+1}{x^4+Kx^2+1} dx + \int \frac{x^2-1}{x^4+Kx^2+1} dx$

Divide N^r and D^r by x^2 and proceed.

* conclusion:

$2x^2 = (x^2+1) + (x^2-1)$

$2 = (x^2+1) - (x^2-1)$

2. Trigonometric Twins:

(i) $\int \sqrt{\tan x} \cdot dx$

Put $\tan x = t^2$, then proceed.

$\sec^2 x \, dx = 2t \, dt$

$(1+\tan^2 x) \, dx = 2t \, dt$

$(1+t^4) \, dx = 2t \, dt$

$dx = \frac{2t}{1+t^4} dt$

$\int \frac{2t^2}{1+t^4} \Rightarrow$ Now proceed with method of Algebraic twin.

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(ii) $\sqrt{\cot x} \cdot dx$

Put $\cot x = t^2$ and proceed

(iii) • $\int \frac{dx}{\sin^4 x + \cos^4 x}$

Divide ~~by~~ N^r and D^r by $\cos^4 x$.

then, put $\tan x = t$

• $\int \frac{dx}{\sin^6 x + \cos^6 x}$

Divide N^r and D^r by $\cos^6 x$

then, put $\tan x = t$

(iv) $\int \frac{\pm \sin x \pm \cos x}{a + b \sin x \cos x} \, dx$

Proceed with method of indirect substitution.

Q. $\int \frac{5}{1+x^4} \, dx$

Sol^n:- $\frac{5}{2} \int \frac{2}{1+x^4} \, dx$

$\frac{5}{2} \left[\int \frac{x^2+1}{1+x^4} \, dx - \int \frac{x^2-1}{1+x^4} \, dx \right]$

Divide by x^2

$\frac{5}{2} \left[\int \frac{1 + \frac{1}{x^2}}{\frac{1}{x^2} + x^2} \, dx - \int \frac{1 - \frac{1}{x^2}}{\frac{1}{x^2} + x^2} \, dx \right]$

$$= \frac{5}{2}\left[\int \frac{1+\frac{1}{x^2}}{\frac{1}{x^2}+x^2}\,dx - \int \frac{1-\frac{1}{x^2}\,dx}{x^2+\frac{1}{x^2}}\right]$$

Put $x - \frac{1}{x} = t$

Put $x + \frac{1}{x} = y$

$1 + \frac{1}{x^2}\,dx = dt$

$1 - \frac{1}{x^2}\,dx = dy$

$=$

$$= \frac{5}{2}\int \frac{dt}{\left(x-\frac{1}{x}\right)^2+2} - \frac{5}{2}\int \frac{dy}{\left(x+\frac{1}{x}\right)^2-2}$$

$$= \frac{5}{2}\int \frac{dt}{t^2+2} - \frac{5}{2}\int \frac{dy}{y^2-2}$$

$$= \frac{5}{2}\int \frac{dt}{t^2+(\sqrt{2})^2} - \frac{5}{2}\int \frac{dy}{y^2-(\sqrt{2})^2}$$

$$= \frac{5}{2\sqrt{2}}\tan^{-1}\left(\frac{t}{\sqrt{2}}\right) - \frac{5}{2}\times\frac{1}{2\sqrt{2}}\ln\left(\frac{y-\sqrt{2}}{y+\sqrt{2}}\right) + C$$

$$= \frac{5}{2\sqrt{2}}\tan^{-1}\left(\frac{x^2-1}{\sqrt{2}x}\right) - \frac{5}{4\sqrt{2}}\ln\left(\frac{\frac{x^2+1}{x}-\sqrt{2}}{\frac{x^2+1}{x}+\sqrt{2}}\right) + C$$

$$= \frac{5}{2\sqrt{2}}\tan^{-1}\left(\frac{x^2-1}{\sqrt{2}x}\right) - \frac{5}{4\sqrt{2}}\ln\left(\frac{x^2-\sqrt{2}x+1}{x^2+\sqrt{2}x+1}\right) + C$$

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Q. $\int \frac{dx}{x^4 + 5x^2+1}$

Solⁿ:- $\frac{1}{2} \int \frac{2}{x^4+5x^2+1} dx$

$\frac{1}{2}\left[\int \frac{x^2+1}{x^4+5x^2+1} dx - \int \frac{x^2-1}{x^4+5x^2+1} dx\right]$

$\frac{1}{2}\left[\int \frac{1+\frac{1}{x^2}}{x^2+\frac{1}{x^2}+5} dx - \int \frac{1-\frac{1}{x^2}}{x^2+\frac{1}{x^2}+5} dx\right]$

Put $x - \frac{1}{x} = t$

$1 + \frac{1}{x^2} dx = dt$

Put $x + \frac{1}{x} = y$

$1 - \frac{1}{x^2} dx = dy$

$\frac{1}{2}\int \frac{dt}{t^2+7} - \frac{1}{2}\int \frac{dt}{y^2+3}$

$= \frac{1}{2} \times \frac{1}{\sqrt{7}} \tan^{-1}\left(\frac{t}{\sqrt{7}}\right) - \frac{1}{2} \times \frac{1}{\sqrt{3}} \tan^{-1}\left(\frac{y}{\sqrt{3}}\right) + C$

$= \frac{1}{2\sqrt{7}} \tan^{-1}\left(\frac{x^2-1}{\sqrt{7}x}\right) - \frac{1}{2\sqrt{3}} \tan^{-1}\left(\frac{x^2+1}{\sqrt{3}x}\right) + C$

Q. $\int \frac{4}{\sin^4 x + \cos^4 x} dx$

Sol^n:- Divide by $\cos^4 x$

$4 \int \frac{\sec^4 x}{\tan^4 x + 1} dx$

Put $\tan x = t$

$\sec^2 x = dt$

$4 \int \frac{(1+\tan^2 x)\sec^2 x \, dx}{\tan^4 x + 1}$

$4 \int \frac{(1+t^2)\, dt}{t^4 + 1}$

Divide by t^2

$4 \int \frac{\frac{1}{t^2} + 1}{t^2 + \frac{1}{t^2}} dt$

$t - \frac{1}{t} = y$

$1 + \frac{1}{t^2} dt = dy$

$4 \int \frac{dy}{y^2 + (\sqrt{2})^2}$

$$= \frac{4}{\sqrt{2}} \tan^{-1}\left(\frac{y}{\sqrt{2}}\right) + c$$

$$= \frac{4}{\sqrt{2}} \tan^{-1}\left(\frac{t^2-1}{\sqrt{2}t}\right) + c$$

$$= \frac{4}{\sqrt{2}} \tan^{-1}\left(\frac{\tan^2 x-1}{\sqrt{2}\tan x}\right) + c$$

Q. $\int \frac{dx}{2\sin x + \sec x}$

Solⁿ: $\int \frac{\cos x \, dx}{\sin 2x + 1}$

$$\frac{1}{2}\int \frac{2\cos x \, dx}{1+\sin 2x}$$

$$\frac{1}{2}\int \frac{\sin x + \cos x \, dx}{(\sin x + \cos x)^2} + \frac{1}{2}\int \frac{\cos x - \sin x}{(\sin x + \cos x)^2} dx$$

$$\frac{1}{2}\int \frac{dx}{\sin x + \cos x} + \frac{1}{2}\int \frac{\cos x - \sin x}{(\sin x + \cos x)^2} dx$$

Divide by $\sqrt{2}$

Put $\sin x + \cos x = t$

$\cos x - \sin x \, dx = dt$

$$\frac{1}{2\sqrt{2}}\int \frac{dx}{\frac{1}{\sqrt{2}}\sin x + \frac{1}{\sqrt{2}}\cos x} + \frac{1}{2}\int \frac{dt}{t^2}$$

$$\frac{1}{2\sqrt{2}}\int \frac{dx}{\sin(x+\pi/4)} + \frac{1}{2} \times \left(\frac{-1}{t}\right)$$

$= \frac{1}{2\sqrt{2}} \int \operatorname{cosec}\left(x + \frac{\pi}{4}\right) dx - \frac{1}{2(\sin x + \cos x)} + C$

$= \frac{1}{2\sqrt{2}} \ln\left(\frac{\tan\left(x + 1/\sqrt{2}\right)}{2} \right) - \frac{1}{2(\sin x + \cos x)} + C$

$\frac{1}{2\sqrt{2}} \ln\left| \tan\left(\frac{\sqrt{2}x + 1}{2\sqrt{2}}\right) \right| - \frac{1}{2(\sin x + \cos x)} + C$

$= \frac{1}{2\sqrt{2}} \ln\left| \tan\left(\frac{\sqrt{2}x + 1}{2\sqrt{2}}\right) \right| - \frac{1}{2(\sin x + \cos x)} + C$

Q. ~~$\int (\sqrt{\tan x} + \sqrt{\cot x}) . dx$~~

$= \frac{1}{2\sqrt{2}} \ln\left| \frac{\tan (x + \pi/4)}{2} \right| - \frac{1}{2(\sin x + \cos x)} + C$

Q. $\int (\sqrt{\tan x} + \sqrt{\cot x}) . dx$

Sol^n:- $\int \left(\frac{\sqrt{\sin x}}{\sqrt{\cos x}} + \frac{\sqrt{\cos x}}{\sqrt{\sin x}} \right) dx$

$= \int \frac{\sin x + \cos x}{\sqrt{\sin x \cos x}} dx$

Put $\sin x - \cos x = t$

$\cos x + \sin x \, dx = dt$

$= \int \frac{\sin x + \cos x}{\sqrt{\frac{1 - (\sin x - \cos x)^2}{2}}} dx$

$= \int \frac{\sqrt{2}\, dt}{\sqrt{1-t^2}}$

$= \sqrt{2} \sin^{-1} t + C$

$= \sqrt{2} \sin^{-1} (\sin x - \cos x) + C$

Q. $\int \sqrt{\tan x}\, . dx$

Sol^n:- Put $\tan x = t^2$

$\sec^2 x\, dx = 2t\, dt$

$(1+\tan^2 x)\, dx = 2t\, dt$

$dx = \frac{2t\, dt}{1+t^4}$

$\int \frac{2t^2}{1+t^4} dt$

$\Rightarrow \int \frac{t^2+1}{1+t^4} dt + \int \frac{t^2-1}{1+t^4} dt$

Divide by t^2

$= \int \frac{1+\frac{1}{t^2}}{\frac{1}{t^2}+t^2} dt + \int \frac{1-\frac{1}{t^2}}{\frac{1}{t^2}+t^2} dt$

Put $t - \frac{1}{t} = y$

Put $t + \frac{1}{t} = z$

$1 + \frac{1}{t^2} dt = dy$

$1 - \frac{1}{t^2} dt = dz$

$= \int \frac{dy}{y^2+2} + \int \frac{dz}{z^2-2}$

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$$= \int \frac{dy}{y^2 + (\sqrt{2})^2} + \int \frac{dz}{z^2 - (\sqrt{2})^2}$$

$$= \frac{1}{\sqrt{2}} \tan^{-1}\left(\frac{y}{\sqrt{2}}\right) + \frac{1}{2\sqrt{2}} \ln\left(\frac{z - \sqrt{2}}{z + \sqrt{2}}\right) + C$$

$$= \frac{1}{\sqrt{2}} \tan^{-1}\left(\frac{t^2 - 1}{\sqrt{2}t}\right) + \frac{1}{2\sqrt{2}} \ln\left(\frac{\frac{t^2+1}{t} - \sqrt{2}}{\frac{t^2+1}{t} + \sqrt{2}}\right) + C$$

$$= \frac{1}{\sqrt{2}} \tan^{-1}\left(\frac{t^2 - 1}{\sqrt{2}t}\right) + \frac{1}{2\sqrt{2}} \ln\left(\frac{t^2 - \sqrt{2}t + 1}{t^2 + \sqrt{2}t + 1}\right) + C$$

$$= \frac{1}{\sqrt{2}} \tan^{-1}\left(\frac{\tan x - 1}{\sqrt{2\tan x}}\right) + \frac{1}{2\sqrt{2}} \ln\left(\frac{\tan x - \sqrt{2\tan x} + 1}{\tan x + \sqrt{2\tan x} + 1}\right) + C$$

Integration of the form

$\int \tan^m x \sec^n x \cdot dx$ or $\int \cot^m x \ \mathrm{cosec}^n x \cdot dx$

(i) If n is even positive integer, then put $\tan x = t$ and proceed.

(ii) If n is not even positive integer, then see

$m \to$ odd $I^+ \to \{$put $\sec x = t\}$

$m \to$ Even $I^+ \to$ convert $\tan^2 x$ into $(\sec^2 x - 1)$. then, put $\tan x = t$ if required.

(iii) If $m = 0$, $n =$ odd I^+ use 'By parts' or 'reduction formula'

Q. $\int \sqrt{\tan x} \cdot \sec^4 x \ dx$

Sol^n:-

$\tan x = t$

$\sec^2 x \ dx = dt$

$$\int t^{1/2} (1 + t^2) \cdot dt$$

$$= \int (t^{1/2} + t^{5/2}) \ dt$$

$$= \frac{2t^{3/2}}{3} + \frac{2t^{7/2}}{7} + C$$

$$= \frac{2(\tan x)^{3/2}}{3} + \frac{2(\tan x)^{7/2}}{7} + C$$

Q. $\int \operatorname{cosec}^4 x \cdot dx$

Sol^n:- put $\cot x = t$

$-\operatorname{cosec}^2 x \, dx = dt$

$-\int (1+t^2)\, dt$

$-\left[t + \frac{t^3}{3}\right] + c$

$-\cot x - \frac{\cot^3 x}{3} + c$

Q. $\int \tan^4 x \cdot dx$

Sol^n:- $\int (\sec^2 x - 1)^2 \, dx$

$\int \sec^4 x \, dx - 2\int \sec^2 x \, dx + \int 1 \cdot dx$

$\tan x = t$

$\sec^2 x \, dx = dt$

$\int (1+t^2)\, dt - 2\tan x + x + c$

$t + \frac{t^3}{3} - 2\tan x + x + c$

$\tan x + \frac{\tan^3 x}{3} - 2\tan x + x + c.$

Q. $\int \sec^3 x \cdot dx$

Sol^n: $\int \sec x \cdot \sec^2 x \cdot dx$

$\int \sec x\,(1+\tan^2 x)\, dx$

~~Put~~ $\cancel{\int \sec x +}\ \cancel{\int}$

$\int \sec x\, dx + \int \sec x \tan^2 x\, dx$

Put $\sec x = t$

$\sec x \tan x\, dx = dt$

$\ln|\sec x + \tan x| + \int \tan x \cdot dt$

$\ln|\sec x + \tan x| + \int \sqrt{\sec^2 x - 1}\, dt$

$\ln|\sec x + \tan x| + \int \sqrt{t^2-1}\, dt$

$\cancel{\ln|\sec x + \tan x| + \ln\left|t + \sqrt{t^2-1}\right|}$

$\ln|\sec x + \tan x| + \frac{t}{2}\sqrt{t^2-1} - \frac{1}{2}\ln\left|t+\sqrt{t^2-1}\right| + c$

$\ln|\sec x + \tan x| + \frac{\sec x}{2}\cdot \tan x - \frac{1}{2}\ln|\sec x + \tan x| + c$

$\frac{1}{2}\ln|\sec x + \tan x| + \frac{\sec x \tan x}{2} + c$

~~Q.~~

Integration of the form ::

$\int \sqrt{\sec^2 x - a}\, dx$ or $\int \sqrt{\operatorname{cosec}^2 x - a}$ or

$\int \sqrt{\tan^2 x - a}\, dx$. or $\int \sqrt{\cot^2 x - a}\, dx$

Method :

$$\int \frac{\sqrt{\sec^2 x - a} \cdot \sqrt{\sec^2 x - a}}{\sqrt{\sec^2 x - a}}\, dx$$

$$\int \frac{\sec^2 x}{\sqrt{\sec^2 x - a}}\, dx - a \int \frac{dx}{\sqrt{\sec^2 x - a}}$$

$$\int \frac{\sec^2 x}{\sqrt{1 + \tan^2 x - a}}\, dx - a\, I$$

Put $\tan x = t$

and proceed.

$$I = \int \frac{\cos x\, dx}{\sqrt{1 - a\underbrace{\cos^2 x}_{(1-\sin^2 x)}}}$$

Put $\sin x = y$

and proceed.

Q. $\int \sqrt{\operatorname{cosec}^2 x - 2}$ or $\int \frac{\sqrt{\cos 2x}}{\sin x}\, dx$.

Sol^n: $\int \frac{\operatorname{cosec}^2 x - 2}{\sqrt{\operatorname{cosec}^2 x - 2}}\, dx$

$$I = \int \frac{\operatorname{cosec}^2 x}{\sqrt{\operatorname{cosec}^2 x - 2}}\,dx - 2\int \frac{dx}{\sqrt{\operatorname{cosec}^2 x - 2}}$$

$$\downarrow \qquad\qquad\qquad \downarrow$$

$$I_1 \qquad\qquad\qquad I_2$$

$$I_1 = \int \frac{\operatorname{cosec}^2 x\,dx}{\sqrt{\cot^2 x - 1}}$$

$$\cot x = t$$

$$-\operatorname{cosec}^2 x\,dx = dt$$

$$I_1 = -\int \frac{dt}{\sqrt{t^2 - 1}}$$

$$I_1 = -\ln\left|t + \sqrt{t^2 - 1}\right|$$

$$I_1 = -\ln\left|\cot x + \sqrt{\cot^2 x - 1}\right|$$

$$I_2 = \int \frac{\sin x\,dx}{\sqrt{1 - 2\sin^2 x}}$$

$$I_2 = \int \frac{\sin x\,dx}{\sqrt{1 - 2(1 - \cos^2 x)}}$$

$$I_2 = \int \frac{\sin x\,dx}{\sqrt{1 - 2 + 2\cos^2 x}}$$

$$I_2 = \int \frac{\sin x\,dx}{\sqrt{2\cos^2 x - 1}}$$

Put $\cos x = t$

$$-\sin x\,dx = dt$$

$$I_2 = -\int \frac{dt}{\sqrt{2t^2-1}} = -\frac{1}{\sqrt{2}}\left\{ \ln\left|\sqrt{2}\,t + \sqrt{2t^2-1}\right| \right\}$$

Hence,

$$I = \ln\left|\frac{1}{\cot x + \sqrt{\cot^2 x - 1}}\right| + \frac{1}{\sqrt{2}} \ln\left|\frac{1}{\sqrt{2}\cos x + \sqrt{\cos 2x}}\right|$$

Integration of the form:

(i) $$\int \frac{dx}{(x-a)^l (x-b)^m}$$

~~Su~~ Sum of power = -2

Put $\underline{x - a = t(x-b)}$

Q. $$\int \frac{dx}{\sqrt[4]{(x-1)^3 (x+2)^5}}$$

Solⁿ :- $$\int \frac{dx}{(x-1)^{3/4} (x+2)^{5/4}}$$

Put $x - 1 = t(x+2)$

$dx = dt(x+2) + t\cdot dx$

$dx(1-t) = dt(x+2)$

$x - 1 = tx + 2t$

$x(1-t) = 2t+1$

$$x = \frac{2t+1}{1-t}$$

$$dx = \frac{(1-t)2 - (2t+1)(-1)}{(1-t)^2}\,dt$$

$$dx = \frac{2 - 2t + 2t + 1}{(1-t)^2}\,dt$$

$$dx = \frac{3}{(1-t)^2}\, dt$$

$$x - 1 = \frac{3t}{1-t}$$

$$x + 2 = \frac{3}{1-t}$$

$$= \int \frac{\frac{3}{(1-t)^2}\, dt}{\left(\frac{3t}{1-t}\right)^{3/4}\left(\frac{3}{1-t}\right)^{5/4}}$$

$$= \int \frac{\frac{3}{\cancel{(1-t)^2}}\, dt}{9 \cdot t^{3/4} \cancel{\left(\frac{1}{1-t}\right)^2}}$$

$$= \frac{1}{3}\int t^{-3/4}\, dt$$

$$= \frac{1}{3}\left[\frac{t^{1/4}}{1/4}\right] + c$$

$$= \frac{4}{3} t^{1/4} + c$$

$$= \frac{4}{3}\left(\frac{x-1}{x+2}\right)^{1/4} + c$$

(ii) $\int \sqrt[n]{\frac{x-a}{x-b}}\, dx$

Put $\frac{x-a}{x-b} = t^n$

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(iii) $\int \frac{p(x)}{(x-a)^m} dx$

put $x-a=t$

Integral of the form:

$\int x^m (a+bx^n)^p dx$

Case I: If $p \in N$, we expand using Binomial and Integrate.

Case II: If $p \in I^-$, put $x = t^k$ (k is LCM of D^r of m and n)

Case III: If $\frac{m+1}{n}$ is an integer and p is fraction

$(a+bx^n) = t^k$ where k is the denominator of p.

Case IV: If $\frac{m+1}{n} + p$ is an integer and $p \in$ fraction,

put $(a+bx)^n = t^k . x^n$.

Q. $\int x^{1/3} (2+x^{1/2})^2 dx$

Sol^n: $\int x^{1/3} (4 + x + 4x^{1/2}) dx$

$\int 4x^{1/3} . dx + \int x^{4/3} dx + 4\int x^{5/6} dx$

$\frac{\cancel{4}.3x^{4/3}}{\cancel{4}} + \frac{3x^{7/3}}{7} + \frac{4 \times 6 x^{11/6}}{11} + c$

$$= 3x^{4/3} + \frac{3}{7}x^{7/3} + \frac{24}{11}x^{11/6} + c$$

Q. $\int x^{-2/3}(1+x^{2/3})^{-1}\,dx$

Sol^n:- Put $x = t^3$

$dx = 3t^2 \cdot dt$

$$= \int t^{-2}(1+t^2)^{-1} \cdot 3t^2\,dt$$

$$= \int \frac{3}{1+t^2}\,dt$$

$$= 3\tan^{-1} t + c$$

$$= 3\tan^{-1}(x)^{1/3} + c$$

Q. $\int x^{-2/3}(1+x^{1/3})^{1/2}\,dx$

Sol^n:- $1 + x^{1/3} = t^2$

$\frac{1}{3}x^{-2/3}dx = 2t\,dt$

$x^{-2/3}dx = 6t\,dt$

$$\int 6t^2 \cdot dt$$

$$6 \cdot \frac{t^3}{3} + c$$

$$= 2t^3 + c$$

$$= 2(1+x^{1/3})^{3/2} + c$$

Q. $\int x^{1/2} (1+x^{1/3})^4 \, dx$

Soln:- $\int x^{1/2} [{}^4C_0 (x^{1/3})^0 + {}^4C_1 (x^{1/3})^1 + {}^4C_2 (x^{1/3})^2 + {}^4C_3 (x^{1/3})^3 + {}^4C_4 (x^{1/3})^4] \, dx$

$= \int x^{1/2} (1 + 4x^{1/3} + 6x^{2/3} + 4x + x^{4/3}) \, dx$

$= \int x^{1/2} dx + 4\int x^{5/6} dx + 6\int x^{7/6} dx + 4\int x^{3/2} dx + \int x^{11/6} dx$

$= \frac{2x^{3/2}}{3} + \frac{4.6x^{11/6}}{11} + \frac{6 \times 6x^{13/6}}{13} + \frac{4 \times 2x^{5/2}}{5} + \frac{6x^{17/6}}{17} + C$

$= \frac{2x^{3/2}}{3} + \frac{24}{11}x^{11/6} + \frac{36}{13}x^{13/6} + \frac{8}{5}x^{5/2} + \frac{6}{17}x^{17/6} + C$

Q. $\int x^{-11} (1+x^4)^{-1/2} \, dx.$

Soln:. $\frac{m+1}{n} + p = \frac{-11+1}{4} - \frac{1}{2} = \frac{-10}{4} - \frac{1}{2} = -3$

$1+x^4 = t^2 x^4$

$x^{-4} + 1 = t^2 \Rightarrow x^{-4} = t^2 - 1$

$-4x^{-5} \, dx = 2t dt$

$-2x^{-5} \, dx = t \, dt$

$\Rightarrow \int x^{-11} (t^2 x^4)^{-1/2} \cdot \frac{-t \, dt}{2x^{-5}}$

$= \frac{-1}{2} \int x^{-6} \cdot t^{-1} t x^{-2} \, dt$

$$= -\frac{1}{2}\int x^{-8}\, dt$$

$$= -\frac{1}{2}\int (t^2-1)^2\, dt$$

$$= -\frac{1}{2}\int (t^4 - 2t^2 + 1)\, dt$$

$$= -\frac{1}{2}\left[\frac{t^5}{5} - \frac{2t^3}{3} + t\right] + C$$

$$= -\frac{1}{10}(x^{-4}+1)^{5/2} + \frac{1}{3}(x^{-4}+1)^{3/2} - \frac{1}{2}(x^{-4}+1)^{1/2} + C$$

Integration of the form

$$\int f(x \cdot x^{p_1/q_1} \cdot x^{p_2/q_2} \ldots)\, dx$$

Put $x = t^k$

where k is the LCM of D^r of powers.

Q. $\int \frac{x^{2/3} - x^{1/3}}{6x^{1/4}}\, dx$

Soln:

3, 3, 4

LCM = 12

Put $x = t^{12}$

$dx = 12t^{11}\, dt$

$$= \frac{12}{6}\int \frac{(t^8 - t^4)\, t^{11}\, dt}{t^3}$$

$$= 2\int (t^{16} - t^{12})\, dt$$

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$$= 2\left[\frac{t^{17}}{17} - \frac{t^{13}}{13}\right] + c$$

$$= 2\,\frac{x^{17/12}}{17} - 2\,\frac{x^{13/12}}{13} + c$$

Q. $\displaystyle\int \frac{x^{1/2}\,dx}{1+x^{3/4}}$

Solⁿ:- Put $x = t^4$

$dx = 4t^3\,dt$

$$= \int \frac{t^2}{1+t^3} \cdot 4t^3\,dt$$

$$= 4\int \frac{t^5}{1+t^3}\,dt$$

Put $1+t^3 = y$

$3t^2\,dt = dy$

$$= \frac{4}{3}\int \frac{t^3\,dy}{\cancel{t^2}\, y}$$

$$= \frac{4}{3}\int \frac{y-1}{y}\,dy$$

$$= \frac{4}{3}\int\left(1 - \frac{1}{y}\right) dy$$

$$= \frac{4}{3}\left[y - \ln y\right] + c$$

$$= \frac{4}{3}\left[1+t^3 - \ln(1+t^3)\right] + c$$

$$= \frac{4}{3}\left[1+x^{3/4} - \ln(1+x^{3/4})\right] + c$$

Integration of the form

$$\int f\left(x, (ax+b)^{p_i/q_i}\right) dx \longrightarrow \text{Put } (ax+b) = t^{q_i}$$

OR

$$\int f\left(x, \left(\frac{ax+b}{cx+d}\right)^{p_i/q_i}\right) dx$$

$\hookrightarrow$ Put $\left(\frac{ax+b}{cx+d}\right) = t^{k}$

k is the denominator of fraction.

Q. $$\int \frac{2}{(2-x)^2}\sqrt[3]{\frac{2-x}{2+x}}\, dx$$

Sol^n: Put $\frac{2-x}{2+x} = t^3$

$$\frac{4^{2}}{-2x} = \frac{t^3+1}{t^3-1}$$

$$\frac{x}{2} = \frac{1-t^3}{1+t^3}$$

$$x = \frac{2(1-t^3)}{(1+t^3)}$$

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$$x = \frac{2-2t^3}{1+t^3}$$

$$dx = \frac{(1+t^3)\cdot(-6t^2) - (2-2t^3)(3t^2)}{(1+t^3)^2}\,dt$$

$$dx = \frac{-6t^2 - 6t^5 - 6t^2 + 6t^5}{(1+t^3)^2}\,dt$$

$$dx = \frac{-12t^2}{(1+t^3)^2}\,dt$$

$$2-x = 2 - \frac{2-2t^3}{1+t^3}$$

$$= \frac{2+2t^3-2+2t^3}{1+t^3}$$

$$= \frac{4t^3}{(1+t^3)}$$

$$\int \frac{2}{\frac{(4t^3)^2}{\cancel{(1+t^3)^2}}} \cdot t \cdot \frac{-12t^2}{\cancel{(1+t^3)^2}}\,dt$$

$$-\frac{24}{16}\int \frac{t^3}{t^6}\,dt$$

$$-\frac{3}{2}\int t^{-3}\,dt$$

$$+\frac{3}{2}\cdot\frac{t^{-2}}{+2} + c$$

$$= \frac{3}{4t^2} + c$$

$$= \frac{3}{4}\left(\frac{2-x}{2+x}\right)^{-2/3} + c$$

Integration of the form:

$$\int \frac{dx}{(x-K)^r \sqrt{ax^2+bx+c}}, \quad r \geq 2, \quad r \in I$$

Put $x - K = \frac{1}{t}$

Integral of the form

$$\int \frac{ax^2 + bx + c}{(kx+l)\sqrt{px^2 + qx + r}}\, dx.$$

write $ax^2 + bx + c = \lambda (kx + l) \cdot \frac{d}{dx}(px^2 + qx + r)$

$+ \mu (kx + l) + \gamma$

where λ, μ and γ are constants which can be obtained by comparing the coefficients of like terms on both sides.

Q. $$\int \frac{2x^2 + 5x + 9}{(x+1)\sqrt{x^2 + x + 1}}\, dx$$

Sol^n: $2x^2 + 5x + 9 = \lambda (x+1)(2x+1) + \mu (x+1) + \gamma$

$2\lambda = 2$

$\boxed{\lambda = 1}$

$3\lambda + \mu = 5$

$\boxed{\mu = 2}$

$\lambda + \mu + \gamma = 9$

$\boxed{\gamma = 6}$

$$\int \frac{(x+1)(2x+1) + 2(x+1) + 6}{(x+1)\sqrt{x^2+x+1}}\,dx$$

$$I = \underbrace{\int \frac{2x+1}{\sqrt{x^2+x+1}}\,dx}_{I_1} + 2\underbrace{\int \frac{dx}{\cancel{(x+1)}\sqrt{x^2+x+1}}}_{I_2} + 6\underbrace{\int \frac{dx}{(x+1)\sqrt{x^2+x+1}}}_{I_3}$$

$$I_1 = \int \frac{2x+1}{\sqrt{x^2+x+1}}\,dx.$$

$$x^2+x+1 = t$$

$$2x+1 \; dx = dt$$

$$I_1 = \int t^{-1/2} \,.\, dt$$

$$I_1 = 2t^{1/2}$$

$$I_1 = 2(x^2+x+1)^{1/2}$$

$$I_2 = \int \frac{dx}{\sqrt{x^2+x+1}}$$

$$I_2 = \int \frac{dx}{\sqrt{x^2+x+\left(\frac{1}{2}\right)^2 - \left(\frac{1}{2}\right)^2 + 1}}$$

$$I_2 = \int \frac{dx}{\sqrt{\left(x+\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2}}$$

$$I_2 = \ln\left| x + \frac{1}{2} + \sqrt{x^2+x+1} \right|$$

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$$I_3 = \int \frac{dx}{(x+1)\sqrt{x^2+x+1}}$$ $x = \frac{1}{t} - 1$

$x+1 = \frac{1}{t} \Rightarrow dx = -\frac{1}{t^2}\,dt$

$$= -\int \frac{dt}{t^2 \times \left(\frac{1}{t}\right)\sqrt{\left(\frac{1}{t}-1\right)^2 + \frac{1}{t} - \cancel{1} + \cancel{1}}}$$

$$= -\int \frac{dt}{t\sqrt{\frac{1}{t^2} + 1 - \frac{2}{t} + \frac{1}{t}}} = -\int \frac{dt}{t\left(\sqrt{\frac{1+t^2-t}{t^2}}\right)}$$

$$= -\int \frac{dt}{\sqrt{t^2-t+1}} = -\int \frac{dt}{\sqrt{\left(t-\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2}}$$

$$= -\ln\left| t - \frac{1}{2} + \sqrt{t^2-t+1} \right|$$

Hence,

$$I = 2(x^2+x+1)^{1/2} + 2\ln\left| x + \frac{1}{2} + \sqrt{x^2+x+1} \right| - \ln\left| \frac{1-x}{2(x+1)} + \frac{\sqrt{x^2+x+1}}{x+1} \right|$$

Manipulating Integrands

1. $\int \frac{dx}{x(x^n+1)}$

$n \in N$

Take x^n common and put $1 + x^{-n} = t$

2. $\int \frac{dx}{x^2(x^n+1)^{\frac{n-1}{n}}}$

$n \in N$

Take x^n common and put $1 + x^{-n} = t^n$

3. $\int \frac{dx}{x^n(1+x^n)^{1/n}}$

$n \in N$

Take x^n common and put $1+x^{-n} = t^n$

Q. $\int \frac{dx}{x(x^2+1)}$

Sol^n: $\int \frac{x\,dx}{x^2(x^2+1)}$

put $x^2+1 = t$

$2x\,dx = dt$

$x\,dx = \frac{dt}{2}$

$\frac{1}{2}\int \frac{dt}{t(t-1)}$

$= \frac{1}{2}\int \left(\frac{1}{t-1} - \frac{1}{t}\right) dt$

$= \frac{1}{2}\left[\ln(t-1) - \ln t\right] + c$

$= \frac{1}{2}\left[\ln x^2 - \ln(x^2+1)\right] + c$

Q. $\int \frac{dx}{x^2(x^3+1)^{2/3}}$

Sol^n. $\cancel{\int \frac{x}{x^3(x^3+1)^{2/3}}\,dx}$ $\int \frac{dx}{x^4(1+x^{-3})^{2/3}}$

$$1 + \cancel{x^3} = \cancel{t^3}$$
$$\cancel{3}x^2\, dx = \cancel{3}t^2\, dt$$

Put $1 + x^{-3} = t^3$

$$-\cancel{3}x^{-4}\, dx = \cancel{3}t^2\, dt$$
$$x^{-4}\, dx = -t^2\, dt$$

$$= -\int \frac{\cancel{t^2}\, dt}{\cancel{t^2}}$$
$$= -t + C$$
$$= -(1+x^{-3})^{1/3} + C$$
$$= -\frac{(x^3+1)^{1/3}}{x} + C$$

Q. $\displaystyle\int \frac{dx}{x^3(x^3+1)^{1/3}}$

Sol^n:- $\displaystyle\int \frac{dx}{x^4(1+x^{-3})^{1/3}}$

Put $1 + x^{-3} = t^3$

$$-3x^{-4}\, dx = 3t^2\, dt$$
$$x^{-4}\, dx = -t^2\, dt$$

$$= -\int \frac{t^2\, dt}{t}$$
$$= -\int t \cdot dt$$
$$= -\frac{t^2}{2} + C$$

$$= -\frac{(1+x^{-3})^{2/3}}{2} + C$$

Integration Using Euler's Substitution

Integral of the form

$\int f(x), \sqrt{ax^2+bx+c}\ dx$

are calculated with the aid of one of the three Euler's substitution:-

(i) $\sqrt{ax^2+bx+c} = t \pm x\sqrt{a}$ if $a > 0$

(ii) $\sqrt{ax^2+bx+c} = tx - \sqrt{c}$ if $c > 0$

(iii) $\sqrt{ax^2+bx+c} = (x-\alpha)t$ if $ax^2+bx+c = a(x-\alpha)(x-\beta)$

i.e. α is the real root of ax^2+bx+c.

Q. $\displaystyle\int \frac{x\,dx}{(\sqrt{7x-10-x^2})^3}$

Soln:-

$-x^2+7x-10$

$-x^2+2x+5x-10$

$-x(x-2)+5(x-2)$

$(x-2)(5-x)$

$\sqrt{7x-10-x^2} = (x-2)t$

$7x-10-x^2 = (x-2)^2 t^2$

$(x-2)(5-x) = (x-2)^2 t^2$

$\cancel{-x^2+7x-10 = t^2x^2 - 4t^2x + 4t^2}$

$5-x = (x-2)t^2$

$5-x = t^2x - 2t^2$

$t^2x + x - 2t^2 - 5 = 0$

$$x(t^2+1) = 5+2t^2$$

$$x = \frac{5+2t^2}{1+t^2}$$

$$x-2 = \frac{5+2t^2-2-2t^2}{1+t^2}$$

$$x-2 = \frac{3}{1+t^2}$$

$$dx = \frac{(1+t^2)(4t)-(5+2t^2)(2t)}{(1+t^2)^2}\,dt$$

$$dx = \frac{-6t}{(1+t^2)^2}\,dt$$

$$\int \frac{\left(\frac{5+2t^2}{\cancel{1+t^2}}\right)\frac{\overset{2}{\cancel{-6t}}}{\cancel{(1+t^2)^2}}\,dt}{t^{\cancel{3}\,2}\cdot\frac{\overset{9}{\cancel{27}}}{\cancel{(1+t^2)^3}}}$$

$$= -\frac{2}{9}\int \frac{5+2t^2}{t^2}\,dt$$

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Q. $\int \frac{dx}{x+\sqrt{x^2-x+1}}$

Sol^n:- $\int \cancel{\frac{x-\sqrt{x^2-x+1}}{x-1}} \, \cancel{dx}$

$$\sqrt{x^2-x+1} = tx-1$$

$$x^2-x+1 = (tx-1)^2$$

$$x^2-x+\cancel{1} = t^2x^2-2tx+\cancel{1}$$

$$t^2x^2-x^2-2tx+x=0$$

$$\cancel{x^2(t^2-1)} = \cancel{x(1-2t)}$$

$$x^2(t^2-1) = x(2t-1)$$

$$x = \frac{2t-1}{t^2-1}$$

$$dx = \frac{(t^2-1).2-(2t-1).2t}{(t^2-1)^2} dt$$

$$dx = \frac{2t^2-2-4t^2+2t}{(t^2-1)^2} dt$$

$$dx = \frac{-2t^2+2t-2}{(t^2-1)^2} dt$$

$$\int \frac{-2t^2+2t-2}{\left(\frac{2t-1}{t^2-1}+tx-1\right)(t^2-1)^2} dt$$

$$tx-1 = t\left(\frac{2t-1}{t^2-1}\right)-1$$

$$= \frac{2t^2-t-t^2+1}{t^2-1} = \frac{t^2-t+1}{t^2-1}$$

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Q. $\int \frac{\sqrt{x^2+10x+24}}{x+5}\, dx$

Sol^n:- $\cancel{x+5}$ $\int \frac{\sqrt{(x+5)^2-1}}{x+5}\, dx$

Put $x+5 = \sec\theta$

$dx = \sec\theta \cdot \tan\theta \cdot d\theta$

$= \int \frac{\sqrt{\sec^2\theta - 1}}{\sec\theta} \cdot \sec\theta \tan\theta\, d\theta$

$= \int \tan\theta \cdot \tan\theta \cdot d\theta$

$= \int \tan^2\theta\, d\theta$

$= \int (\sec^2\theta - 1)\, d\theta$

$= \tan\theta - \theta + C$

$= \cancel{\tan (\sec^{-1}(x+5))}$

$\tan\theta = \sqrt{\sec^2\theta - 1}$

$= \sqrt{x^2+10x+24}$

$= \sqrt{x^2+10x+24} - \sec^{-1}(x+5) + C$

Q. $\int \frac{p x^{p+2q-1} - q x^{q-1}}{x^{2p+2q} + 2x^{p+q} + 1}\, dx$

Sol^n:- Divide by x^{2q} in N^r and D^r

$$\int \frac{p x^{p-1} - q x^{-q-1}}{x^{2p} + 2x^{p-q} + x^{-2q}} \, dx$$

$$\cancel{x^{2p} + 2x^{p-q} + x^{-2q} = t}$$

$$\cancel{2p x^{2p-1} + 2(p-q) x^{p-q-1} - 2q x^{-2q-1} = dt/dx}$$

$$\cancel{2px^{2p-1} + 2p x^{p-q-1} - 2q x^{p-q-1} - 2q x^{-2q-1} = dt/dx}$$

$$\cancel{2px^{3p-q-2} - 2q x^{p-3q-2} = dt/dx}$$

$$\int \frac{p x^{p-1} - q x^{-q-1}}{(x^p + x^{-q})^2} \, dx$$

$$x^p + x^{-q} = t$$

$$p x^{p-1} - q x^{-q-1} \, dx = dt$$

$$= \int \frac{dt}{t^2}$$

$$= -\frac{1}{t} + c$$

$$= -\frac{1}{x^p + x^{-q}} + c = \frac{-1}{x^p + \frac{1}{x^q}} + c$$

$$= \frac{-x^q}{x^{p+q} + 1} + c$$

Q. $$\int \frac{x^2 (1 - \ln x)}{(\ln x)^4 - x^4} \, dx$$

Sol^n: $$\int \frac{x^2 (1 - \ln x)}{x^4 \left[\left(\frac{\ln x}{x} \right)^4 - 1 \right]} \cdot dx$$

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Put $\frac{\ln x}{x} = t$

$\frac{1-\ln x}{x^2}\,dx = dt$

$= \int \frac{dt}{t^4-1}$

$= \int \frac{dt}{(t^2+1)(t+1)(t-1)}$

By partial fraction,

$\frac{A}{t+1} + \frac{B}{t-1} + \frac{Ct+D}{t^2+1}$

$A = -\frac{1}{4}$, $B = \frac{1}{4}$, $C=0$, $D = -\frac{1}{2}$

$= -\frac{1}{4}\int \frac{dt}{t+1} + \frac{1}{4}\int \frac{dt}{t-1} - \frac{1}{2}\int \frac{dt}{t^2+1}$

$= -\frac{1}{4}\ln(t+1) + \frac{1}{4}\ln(t-1) - \frac{1}{2}\tan^{-1}(t) + C$

$= -\frac{1}{4}\ln\left(\frac{\ln x}{x}+1\right) + \frac{1}{4}\ln\left(\frac{\ln x}{x}-1\right) - \frac{1}{2}\tan^{-1}\left(\frac{\ln x}{x}\right) + C$

Q. $\int \sec^2\theta\,(\sec\theta+\tan\theta)^2\,d\theta$

Sol^n:- Put $\sec\theta + \tan\theta = t$

$\sec\theta(\sec\theta+\tan\theta)\,d\theta = dt$

$\int t \cdot \sec\theta \cdot dt$

$\sec\theta + \tan\theta = t$

$\sec\theta - \tan\theta = \frac{1}{t}$

$+$

$2\sec\theta = \frac{t^2+1}{t}$

$\Rightarrow \sec\theta = \frac{t^2+1}{2t}$

$= \int \cancel{t}\left(\frac{t^2+1}{2\cancel{t}}\right) dt$

$= \frac{1}{2}\times\frac{t^3}{3} + \frac{1}{2}t + c$

$= \frac{t^3}{6} + \frac{1}{2}t + c$

$= \frac{t}{2}\left[\frac{t^2}{3} + 1\right] + c$

$= \frac{\sec\theta + \tan\theta}{2}\left[\frac{(\sec\theta + \tan\theta)^2 + 3}{3}\right] + c$

$= \frac{\sec\theta + \tan\theta}{6}\left[\sec^2\theta + \tan^2\theta + 2\sec\theta\tan\theta + 3\right] + c$

$= \frac{\sec\theta + \tan\theta}{6}\left[4 + 2\tan^2\theta + 2\sec\theta\tan\theta\right] + c$

$= \frac{\sec\theta + \tan\theta}{3}\left[2 + \tan\theta(\tan\theta + \sec\theta\right] + c$

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Q. $\int \frac{3x^2-1}{2x\sqrt{x}} \tan^{-1} x \, dx$

Sol^n: $\frac{3}{2}\int x^{1/2} \tan^{-1}x \, dx - \frac{1}{2}\int x^{-3/2} \tan^{-1}x \, dx$

$\cancel{\frac{3}{2}}\left[\frac{\tan^{-1}x \times x^{3/2}}{\cancel{3/2}} - \int \frac{1}{1+x^2} \cdot \frac{x^{3/2}}{\cancel{3/2}} dx\right] - \frac{1}{2}\int x^{-3/2} \tan^{-1}x \, dx$

$x^{3/2} \tan^{-1}x - \underbrace{\int \frac{x^{3/2} \, dx}{1+x^2}}_{\text{I}} - \frac{1}{2}\int x^{-3/2} \tan^{-1}x \, dx$

$I = \int \frac{x^{3/2} \cdot x^{1/2}}{x^{1/2}(1+x^2)} \cdot dx$

$= \int \frac{x^2}{\sqrt{x}(1+x^2)} dx$

$= \int \frac{x^2+1-1}{\sqrt{x}(1+x^2)} dx$

$= \int \frac{1}{\sqrt{x}} dx - \int \frac{1}{\underbrace{\sqrt{x}}_{\text{I}} \underbrace{(1+x^2)}_{\text{II}}} dx$

$= 2\sqrt{x} - \left[\frac{1}{\sqrt{x}} \tan^{-1}x + \frac{1}{2}\int x^{-3/2} \tan^{-1}x \cdot dx\right]$

$\Rightarrow x^{3/2} \tan^{-1}x - 2\sqrt{x} + \frac{1}{\sqrt{x}} \tan^{-1}x + \cancel{\frac{1}{2}\int x^{-3/2} \tan^{-1}x} - \cancel{\frac{1}{2}\int x^{-3/2} \tan^{-1}x \, dx}$

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$$= x^{3/2}\tan^{-1}x - 2\sqrt{x} + \frac{1}{\sqrt{x}}\tan^{-1}x + c$$

Q. $\int (x^{7m}+x^{2m}+x^{m})(2x^{6m}+7x^{m}+14)^{1/m}\,dx$, $m\in N$

Soln:- Multiply and Divide by x

$$\int \frac{(x^{7m}+x^{2m}+x^{m})(2x^{6m}+7x^{m}+14)^{1/m}\cdot x}{x}\cdot dx$$

$$\int (x^{7m-1}+x^{2m-1}+x^{m-1})(2x^{7m}+7x^{2m}+14x^{m})^{1/m}\,dx$$

Put $2x^{7m}+7x^{2m}+14x^{m} = t$

$14m(x^{7m-1}+x^{2m-1}+x^{m-1})\,dx = dt$

$$\frac{1}{14m}\int t^{1/m}\cdot dt$$

$$= \frac{1}{14m}\times\frac{t^{1/m+1}}{\frac{1}{m}+1}$$

$$= \frac{1}{14(m+1)}(2x^{7m}+7x^{2m}+14x^{m})^{\frac{m+1}{m}} + C$$

Q. $\int \frac{(2x+3)\,dx}{x(x+1)(x+2)(x+3)+1} = C - \frac{1}{f(x)}$ where $f(x)$ is of the form of ax^2+bx+c, then $(a+b+c)$ equals

Soln: $\int \frac{(2x+3)\,dx}{x(x+3)(x+1)(x+2)+1}$

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$$= \int \frac{(2x+3)\,dx}{(x^2+3x)(x^2+3x+2)+1}$$

Put $x^2+3x = t$

$(2x+3)\,dx = dt$

$$= \int \frac{dt}{t(t+2)+1}$$

$$= \int \frac{dt}{t^2+2t+1} = \int \frac{dt}{(t+1)^2}$$

~~ttk easy~~ $= \dfrac{-1}{t+1} + C$

$$= \frac{-1}{x^2+3x+1} + C$$

$a = 1, \; b = 3, \; c = 1$

$a + b + c = 5$

Q: $$\int \frac{dt}{(1+\sqrt{t})^8} = \frac{-1}{3(1+\sqrt{t})^{p_1}} + \frac{2}{7(1+\sqrt{t})^{p_2}} + C, \text{ then}$$

(a) $p_1 = 5$ (b) $p_1 = 6$ (c) $p_2 = 7$ (d) $p_2 = 8$

Sol^n: $$\int \frac{dt}{(1+\sqrt{t})^8}$$

$1 + \sqrt{t} = x$

$\dfrac{1}{2\sqrt{t}}\,dt = dx$

$dt = 2\sqrt{t}\, dx$

$dt = 2(x-1)\, dx$

$$\int \frac{2(x-1)\,dx}{x^8}$$

$$= \int \frac{2}{x^7}\,dx - \int \frac{2}{x^8}\,dx$$

$$= \frac{2 \times x^{-7+1}}{-7+1} - 2 \times \frac{x^{-8+1}}{-8+1} + C$$

$$= -\frac{2}{6 \times x^6} + \frac{2}{7x^7} + C$$

$$= \frac{-1}{3(1+\sqrt{t})^6} + \frac{2}{7(1+\sqrt{t})^7} + C$$

$p_1 = 6 \quad , \quad p_2 = 7$

Q. $\displaystyle\int \frac{3+x^3}{\sqrt{2+2x^2}}\,dx$

sol^n:- $\displaystyle\frac{3}{\sqrt{2}} \int \frac{dx}{\sqrt{1+x^2}} + \frac{1}{\sqrt{2}} \int \frac{x^3\,dx}{\sqrt{1+x^2}}$

$$\frac{3}{\sqrt{2}} \times \ln\left|x + \sqrt{1+x^2}\right| + \underbrace{\frac{1}{\sqrt{2}} \int \frac{x^3\,dx}{\sqrt{1+x^2}}}_{I_1} = I$$

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$$I_1 = \int \frac{x^3\,dx}{\sqrt{1+x^2}}$$

$$1+x^2 = t^2$$

$$\cancel{2}x\,dx = \cancel{2}t\,dt$$

$$I_1 = \int \frac{(t^2-1)\,\cancel{t}\,dt}{\cancel{t}}$$

$$I_1 = \frac{t^3}{3} + t + C$$

$$I = \frac{3}{\sqrt{2}} \ln\left|x + \sqrt{1+x^2}\right| + \frac{1}{\sqrt{2}}\left[\frac{(1+x^2)^{3/2}}{3} + (1+x^2)^{1/2}\right] + C$$

Q. $$\int \frac{x^2(x^6+x^5-1)}{(2x^6+3x^5+2)^2}\,dx$$

Soln:- $$\int \frac{\cancel{x^2}(x^6+x^5-1)}{x^{\cancel{6}4}(2x^3+3x^2+2x^{-3})^2}\,dx$$

$$\int \frac{x^2+x-x^{-4}}{(2x^3+3x^2+2x^{-3})^2}\,dx$$

Put $2x^3+3x^2+2x^{-3} = t$

$$(6x^2+6x-6x^{-4})\,dx = dt$$

$$(x^2+x-x^{-4})\,dx = \frac{dt}{6}$$

$$\frac{1}{6}\int \frac{dt}{t^2}$$

$$= \frac{1}{6} \cdot \left(-\frac{1}{t}\right) + C$$

$$= -\frac{1}{6t} + C$$

$$= -\frac{1}{6} \left(2x^3 + 3x^2 + 2x^{-3}\right)^{-1} + C$$

Q. $$\int \frac{(x^5 + x^4 + x^2)\, dx}{\sqrt{4x^7 + 5x^6 + 10x^4}}$$

Sol^n: $$\int \frac{x^5 + x^4 + x^2 \; dx}{x\left(4x^5 + 5x^4 + 10x^2\right)^{1/2}}$$

$$= \int \frac{x^4 + x^3 + x \; dx}{(4x^5 + 5x^4 + 10x^2)^{1/2}}$$

Put $4x^5 + 5x^4 + 10x^2 = t$

$20(x^4 + x^3 + x)\, dx = dt$

$$\frac{1}{20} \int \frac{dt}{\sqrt{t}}$$

$$= \frac{1}{20} \times 2\sqrt{t} + c$$

$$= \frac{1}{10} \left(4x^5 + 5x^4 + 10x^2\right)^{1/2} + c$$

$$= x \left(\frac{x^3}{25} + \frac{x^2}{20} + \frac{1}{10}\right) + c$$

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Q. $\int \frac{x^4-1}{x^2\sqrt{x^4+x^2+1}}\,dx$

sol^n:- $\int \frac{x^4-1}{x^3\,(x^2+x^{-2}+1)^{1/2}}\,dx$

$\int \frac{x+x^{-3}}{\sqrt{x^2+x^{-2}+1}}\,dx$

Put $x^2+x^{-2}+1 = t^2$

$(2x-2x^{-3})\,dx = 2t\,dt$

$x-x^{-3}\;dx = t\,dt$

$\int \frac{t\,dt}{t}$

$= t + c$

$= \sqrt{x^2+x^{-2}+1} + c$

$= \frac{\sqrt{x^4+x^2+1}}{x} + c$

Q. $\int \frac{x^2-1}{x\sqrt{1+x^4}}\,dx$

sol^n:- $\int \frac{x^2-1}{x^2\sqrt{x^{-2}+x^2}}\,dx$

$\int \frac{\left(1-\frac{1}{x^2}\right)dx}{\sqrt{x^2+\frac{1}{x^2}}}$

$$x + \frac{1}{x} = t$$

$$1 - \frac{1}{x^2} \quad dx = dt$$

$$= \int \frac{dt}{\sqrt{t^2 - (\sqrt{2})^2}} \quad \cancel{dt}$$

$$= \ln \left| t + \sqrt{t^2 - 2} \right| + C$$

$$= \ln \left| \frac{x^2 + 1 + \sqrt{x^4 + 1}}{x} \right| + C$$

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$$\int \frac{dt}{}$$

sol^n:- $\displaystyle\int \frac{2-(1-x^2)}{(1-x)^2\sqrt{1+x^4}}\,dx$

$$\int \frac{2}{(1-x)^2\sqrt{1+x^4}}\,dx - \int \frac{(1-x^2)}{(1-x)^2\sqrt{1+x^4}}\,dx$$

Q. $\displaystyle\int \frac{1+x^2}{(1-x^2)\sqrt{1+x^4}}\,dx$

sol^n:- $\displaystyle\int \frac{1+x^2}{\left(\frac{1}{x}-x\right)x^2\left(\frac{1}{x^2}+x^2\right)^{1/2}}\,dx$

$$= -\int \frac{\frac{1}{x^2}+1}{\left(x-\frac{1}{x}\right)\sqrt{\frac{1}{x^2}+x^2}}\,dx$$

$$x-\frac{1}{x}=t$$

$$1+\frac{1}{x^2}\,dx=dt$$

$$= -\int \frac{dt}{t\sqrt{t^2-2}}$$

Put $t=\sqrt{2}\tan\theta \rightarrow \dfrac{t}{\sqrt{2}}=\tan\theta$

$dt=\sqrt{2}\sec^2\theta\cdot d\theta$

$$\theta=\tan^{-1}\frac{t}{\sqrt{2}}$$

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$$= -\int \frac{\cancel{\sqrt{2}}\, \sec^2\theta \, d\theta}{\cancel{\sqrt{2}} \tan\theta \sqrt{2}\, \cancel{\sec\theta}}$$

$$= -\frac{1}{\sqrt{2}} \int \frac{1}{\sin\theta} \, d\theta$$

$$= \frac{-1}{\sqrt{2}} \int \operatorname{cosec}\theta \, d\theta$$

$$= \frac{-1}{\sqrt{2}} \ln\left| \tan\theta/2 \right| + c$$

$$\cancel{t - \frac{\ell}{2} \ln \tan\left(\frac{t}{\sqrt{2}} \tan^{-1}\right)}$$

$$= -\frac{1}{\sqrt{2}} \ln\left| \tan\left(\frac{1}{2} \tan^{-1} \frac{t}{\sqrt{2}} \right) \right| + c$$

$$= -\frac{1}{2} \ln\left| \tan\left(\frac{1}{2} \tan^{-1} \frac{x^2-1}{\sqrt{2}x} \right) \right| + c$$

Q. $\displaystyle\int \frac{1}{(1+x^4)\sqrt{\sqrt{1+x^4} - x^2}} \, dx$

solⁿ: $\displaystyle\int \frac{dx}{(1+x^4)\,x\sqrt{\sqrt{1+\frac{1}{x^4}} - 1}}$

$$= \frac{x^{-5}\, dx}{\left(1+\frac{1}{x^4}\right)\sqrt{\sqrt{1+\frac{1}{x^4}} - 1}}$$

$$\sqrt{1+\frac{1}{x^4}} - 1 = t^2$$

$$\frac{1}{\cancel{x}\sqrt{1+\frac{1}{x^4}}} \cdot -\cancel{4}x^{-5}\,dx = \cancel{2}t\,dt$$

$$x^{-5}\,dx = -t\,(t^2+1)\,dt$$

$$\int \frac{-\cancel{t}\,\cancel{(t^2+1)}}{(t^2+1)^{\cancel{2}}\cdot\cancel{t}}\,dt$$

$$= -\tan^{-1} t + c$$

$$= -\tan^{-1}\sqrt{\sqrt{1+\frac{1}{x^4}}-1} + c$$

Q. $\int \sin(101x)\cdot\sin^{99}x\,dx$

Soln:- $\int \sin(x+100x)\cdot\sin^{99}x\,dx$

$$= \int [\sin x\cos 100x + \cos x\sin 100x]\,\sin^{99}x\,dx$$

$$= \int \sin^{100}x\cos 100x\,dx + \int \underset{\text{I}}{\sin 100x}\cdot\underset{\text{II}}{\underbrace{\cos x\sin^{99}x}}\,dx$$

$$= \int \cancel{\sin^{100}x\cos 100x\,dx} + \sin 100x\int\cos x\sin^{99}x\,dx - \int 100\cos\sin 100$$

$$= \int \sin^{100}x\cos 100x\,dx + \sin 100x\int\cos x\sin^{99}x - \int\left[100\cos 100x\cdot\int\cos x\sin^{99}x\,dx\right]dx$$

$$= \int \sin^{100}x \cos 100x \, dx + \sin 100x \frac{\sin^{100}x}{100} - \int \frac{\cancel{100} \cos 100x}{\cancel{100}} \sin^{100}x \, dx$$

$$= \int \cancel{\sin^{100}x \cos 100x \, dx} + \frac{\sin 100x \sin^{100}x}{100} - \int \cancel{\sin^{100}x \cdot \cos 100x \, dx} + C$$

$$= \frac{\sin(100x) \cdot \sin^{100}x}{100} + C$$

Q. Let $f(x)$ satisfies $xf^2(x) - f(x) = x - 1 \ \forall x \in R$, $f(1) \neq 0$.

If $h(x) = \int \left(\frac{f(x) + f(-x) - f''(x)}{2 - f'(x)} \right) f(x) \, dx$,

then $h(3) - h(2)$ equals :

(a) $\tan^{-1}(\pi/4)$ (b) $\tan^{-1}(1)$ (c) $\tan^{-1}(\tan 1)$

(d) $\tan^{-1}(-\tan 1)$

Sol^n:- $x f^2(x) - f(x) - (x-1) = 0$

$$f(x) = \frac{1 \pm \sqrt{1 + 4x(x-1)}}{2x}$$

$$f(x) = \frac{1 \pm (2x-1)}{2x}$$

$$f(x) = 1, \ \frac{1}{x} - 1$$

$f(1) \neq 0$

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$f'(x) = 0$

$f''(x) = 0$

$$h(x) = \int \frac{1 + 1 - 0}{2 - 0} \cdot 1 \, dx$$

$$h(x) = \int dx$$

$h(x) = x$

$h(3) = 3$

$h(2) = 2$

$h(3) - h(2) = 1$

$\tan^{-1}(\tan 1) = 1$ ✓

Q. If $f(x) = \int 2e^x \cos^2 x(-\tan^2 x + \tan x + 1)\, dx$ and $f(x)$ passes through $(\pi, 0)$, then $(f(0) + f'(0))$ equals

Sol^n^:

$$f(x) = \int e^x(-2\sin^2 x + 2\cos^2 x + 2\sin x \cos x)\, dx$$

$$= \int e^x(\underbrace{2\cos 2x}_{f'(x)} + \underbrace{\sin 2x}_{f(x)})\, dx$$

$$= e^x \sin 2x + C$$

$f(0) = 0$

$f'(0) = 2$

$f(0) + f'(0) = 2$

Q. $\int \frac{\sin 2x - \sin 2K}{\sin x - \sin K + \cos x - \cos K} dx.$

Sol^n:- $\int \frac{1 + \sin 2x - \sin 2K - 1}{\sin x - \sin K + \cos x - \cos K} dx$

$\int \frac{(\sin x + \cos x)^2 - (\sin K + \cos K)^2}{\sin x + \cos x - \sin K - \cos K} dx$

$\int \frac{(\sin x + \cos x + \sin K + \cos K)\cancel{(\sin x + \cos x - \sin K - \cos K)}}{\cancel{(\sin x + \cos x - \sin K - \cos K)}} dx$

$\int (\sin x + \cos x + \sin K + \cos K) dx$

$= -\cos x + \sin x + x(\sin K + \cos K) + C$

$= \sin x - \cos x + x(\sin K + \cos K) + C$

Q. $\int x \cdot 2^{\ln(x^2+1)} dx$

Sol^n:- $\ln(x^2+1) = t$

$\frac{1}{x^2+1} \cdot 2x\, dx = dt$

$x\, dx = \frac{x^2+1}{2} dt$

$x\, dx = \frac{e^t}{2} dt$

$= \frac{1}{2} \int e^t \cdot 2^t dt$

$= \frac{1}{2} \int (2e)^t \cdot dt$

$$= \frac{1}{2}\,\frac{(2e)^t}{\ln 2e} + c$$

$$= \frac{(2e)^{\ln(x^2+1)}}{2(1+\ln 2)} +$$

$$= \frac{(x^2+1)\;2^{\ln(x^2+1)}}{2(1+\ln 2)} + c$$

$$a^{\log_b c} = c^{\log_b a}$$

$$2^{\ln(x^2+1)} = (x^2+1)^{\ln 2}$$

$$= \frac{(x^2+1)\,(x^2+1)^{\ln 2}}{2(1+\ln 2)} + c$$

$$= \frac{(x^2+1)^{\ln 2+1}}{2(1+\ln 2)} + c$$

Q. $$\int \frac{3x^4-1}{(x^4+x+1)^2}\,dx$$

Soln: $$\cancel{\int \frac{3x^4-1}{x^8(1+x^{-3}+x^{-4})}\,dx} \qquad \int \frac{x^2(3x^2-x^{-2})}{x^2(x^3+1+x^{-1})^2}\,dx$$

$$x^3+1+x^{-1} = t$$

$$(3x^2-x^{-2})\,dx = dt$$

$$\int \frac{dt}{t^2} = -\frac{1}{t} + c$$

$$= -\frac{x}{x^4+x+1} + c$$

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Q. $\int e^x \left(\frac{x^2-3}{(x-1)^2}\right) dx$

Sol^n:- $\int e^x \left(\frac{x^2-3}{x^2-2x+1}\right) dx$ $\int e^x \left(\frac{x^2-1-2}{(x-1)^2}\right) dx$

$\int \cancel{e^x} \frac{(x^2-3)}{x^2(1-2x^{-1}+x^{-2})} \, dx$ $\int e^x \left[\underbrace{\frac{x+1}{x-1}}_{f(x)} - \underbrace{\frac{2}{(x-1)^2}}_{f'(x)} \right] dx$

$= e^x \left(\frac{x+1}{x-1}\right) + c$

Q. $\int e^x \left(\underbrace{\log x}_{f(x)} + \underbrace{\frac{1}{x^2}}_{f''(x)}\right) dx$

Sol^n:- $\int e^x \left(\underbrace{\log x - \frac{1}{x}}_{f(x)} + \underbrace{\frac{1}{x} + \frac{1}{x^2}}_{f'(x)}\right) dx$

$= e^x \left(\log x - \frac{1}{x}\right) + c$

Q. $\int x \cdot \frac{\ln(x+\sqrt{1+x^2})}{\sqrt{1+x^2}} \, dx$

Sol^n:- ~~Let~~ Put $\sqrt{1+x^2} = t$

$1+x^2 = t^2$

$x = \sqrt{t^2-1}$

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$$\frac{1}{2\sqrt{x^2+1}} \cdot 2x \ dx = dt$$

$$\frac{x \ dx}{\sqrt{1+x^2}} = dt$$

$$\int \ln\left(t + \sqrt{t^2-1}\right) dt$$

$$\int \ln\left(t + \sqrt{t^2-1}\right) \cdot 1 \ dt$$

$$\ln\left(t + \sqrt{t^2-1}\right) \cdot \cancel{x} t \cdot - \int \frac{1}{t+\sqrt{t^2-1}} \left(1 + \frac{t}{\sqrt{t^2-1}}\right) \cdot t \ dt$$

$$= t \cdot \ln\left(t + \sqrt{t^2-1}\right) - \int \frac{t}{\underbrace{\sqrt{t^2-1}}_{y}} dt$$

$$y^2 = t^2 - 1$$

$$2y \ dy = 2t \ dt$$

$$= t \cdot \ln\left(t + \sqrt{t^2-1}\right) - \frac{1}{2} \int \frac{\cancel{y} \, dy}{\cancel{y}}$$

$$= t \cdot \ln\left(t + \sqrt{t^2-1}\right) - \frac{1}{2} y + c$$

$$= t \cdot \ln\left(t + \sqrt{t^2-1}\right) - \frac{1}{2} \sqrt{(t^2-1)} + c$$

$$= \sqrt{1+x^2} \ \ln\left(\sqrt{1+x^2} + x\right) - \frac{1}{2} \sqrt{x^2} + c$$

$$= \sqrt{1+x^2} \ \ln\left(\sqrt{1+x^2} + x - \cancel{x} \ x + c\right.$$

Q. If $\int \frac{\sqrt{1-x^2}}{x^4}\, dx = A(x)\left(\sqrt{1-x^2}\right)^m + C$, then

$(A(x))^m = ?$

sol^n:

$$\int \frac{\sqrt{1-x^2}}{x^4}\, dx$$

Put $x = \cos\theta$ $\qquad \tan\theta = \frac{\sqrt{1-x^2}}{x}$

$dx = -\sin\theta \cdot d\theta$

$$\int \frac{\sin\theta}{\cos^4\theta}\, dx$$

$$= -\int \frac{\sin^2\theta\, d\theta}{\cos^4\theta}$$

$$= -\int \tan^2\theta \cdot \sec^2\theta \cdot d\theta$$

$\tan\theta = t$

$\sec^2\theta \cdot d\theta = dt$

$$= -\int t^2 \cdot dt$$

$$= -\frac{t^3}{3} + C$$

$$= -\frac{\tan^3\theta}{3} + C$$

$$= -\frac{(1-x^2)\cdot\sqrt{1-x^2}}{3x^3} + C$$

$$= -\frac{1}{3x^3}\left(\sqrt{1-x^2}\right)^3$$

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Now,

$$-\frac{1}{3x^3}\left(\sqrt{1-x^2}\right)^3 = A(x)\left(\sqrt{1-x^2}\right)^m$$

$$m = 3$$

$$A(x) = \frac{-1}{3x^3}$$

$$(A(x))^m = \left(-\frac{1}{3x^3}\right)^3 = \frac{-1}{27x^9}$$

Q. $\displaystyle\int \frac{d\theta}{\cos^2\theta(\tan 2\theta + \sec 2\theta)} = \lambda \tan\theta + 2\ln|f(\theta)| + c$,

then ordered pair $(\lambda, f(\theta))$ is equal to.

solⁿ: $\displaystyle\int \frac{\sec^2\theta \; d\theta}{\frac{\cancel{\tan 2\theta}\ \sin 2\theta + 1}{\cos 2\theta}}$

$$\int \frac{\sec^2\theta \cdot (\cos^2\theta - \sin^2\theta)}{(\sin\theta + \cos\theta)^2} \, d\theta$$

$$\int \frac{\cancel{\sec^2\theta}\,(\sin\theta + \cos\theta)}{(\cos\theta - \sin\theta)} \qquad \int \frac{\sec^2\theta\,(\cos\theta - \sin\theta)}{(\sin\theta + \cos\theta)}\, d\theta$$

Divide by $\cos\theta$

$$\int \frac{\sec^2\theta\,(1 - \tan\theta)}{(1 + \tan\theta)}\, d\theta$$

$1 + \tan\theta = t$

$\sec^2\theta\, d\theta = dt$

$$= \int \frac{2-t}{t}\,dt$$

$$= \int \frac{2}{t}\,dt - \int dt$$

$$= 2\ln t - t + C$$

$$= 2\ln(1+\tan\theta) - 1 - \tan\theta + C$$

$$= 2\ln(1+\tan\theta) - \tan\theta + C$$

$$2\ln(1+\tan\theta) - \tan\theta + C = \lambda\tan\theta + 2\ln|f(\theta)| + C$$

$$\boxed{\lambda = -1}$$

$$f(\theta) = 1+\tan\theta$$

$$(\lambda, f(\theta)) = (-1, 1+\tan\theta)$$

Q. $\displaystyle\int \frac{x^2-1}{x^3\sqrt{2x^4-2x^2+1}}\,dx$

Soln:

$$\int \frac{x^2-1}{x^4 \cancel{x}(2x^2-2+x^{-2})^{1/2}}\,\cancel{dx} \qquad \int \frac{x^2-1}{x^5(2-2x^{-2}+x^{-4})^{1/2}}\,dx$$

$$\int \frac{x^{-2}-x^{-3}}{(2x^2-2+x^{-2})^{1/2}} \qquad \int \frac{x^{-3}-x^{-5}}{(2-2x^{-2}+x^{-4})^{1/2}}\,dx$$

$$2-2x^{-2}+x^{-4} = t^2$$

$$4x^{-3} - 4x^{-5} = 2t\,dt$$

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$$x^{-3} - x^{-5} = \frac{t}{2}\,dt$$

$$= \frac{1}{2}\int \frac{\cancel{t}\,dt}{\cancel{t}}$$

$$= \frac{1}{2}t + c$$

$$= \frac{1}{2} \times (2 - 2x^{-2} + x^{-4})^{1/2} + c$$

$$= \frac{1}{2}\frac{\sqrt{2x^4 - 2x^2 + 1}}{x^2} + c$$

Q. If $I = \int \frac{e^x}{e^{4x} + e^{2x} + 1}\,dx$ and $J = \int \frac{e^{-x}}{e^{-4x} + e^{-2x} + 1}\,dx$

then, $J - I$ equals.

Sol^n:- $I = \int \frac{e^x}{e^{4x} + e^{2x} + 1}\,dx$ $\quad J = \int \frac{e^{3x}}{e^{4x} + e^{2x} + 1}\,dx$

$$J - I = \int \frac{e^{3x} - e^x}{e^{4x} + e^{2x} + 1}\,dx$$

$$J - I = \int \frac{e^x(e^{2x} - 1)}{e^{4x} + e^{2x} + 1}\,dx$$

$e^x = t$

$e^x\,dx = dt$

$$J - I = \int \frac{t^2 - 1}{t^4 + t^2 + 1}\,dt$$

Divide N^r and D^r by t^2

$$J-I = \int \frac{1-\frac{1}{t^2}}{t^2+\frac{1}{t^2}+1}\,dt$$

Put $t+\frac{1}{t} = y$

$$\left(1-\frac{1}{t^2}\right)dt = dy$$

$$J-I = \int \frac{dy}{y^2-1}$$

$$= \frac{1}{2}\ln\left(\frac{y-1}{y+1}\right)+c$$

$$= \frac{1}{2}\ln\left(\frac{\frac{t^2+1}{t}-1}{\frac{t^2+1}{t}+1}\right)+c$$

$$= \frac{1}{2}\ln\left(\frac{t^2-t+1}{t^2+t+1}\right)+c$$

$$= \frac{1}{2}\ln\left(\frac{e^{2x}-e^x+1}{e^{2x}+e^x+1}\right)+c$$

Q. $$\int \frac{\sec^2 x\,dx}{(\sec x+\tan x)^{9/2}}$$

Sol^n: Put $\sec x+\tan x = t^2$

$$\sec x(\tan x+\sec x)\,dx = 2t\,dt$$

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$$\sec x \, dx = \frac{2t \, dt}{(\tan x + \sec x)}$$

$$= 2\int \frac{\sec x \cdot t \, dt}{(\sec x + \tan x)^{11/2}}$$

$$= 2\int \frac{\sec x \, t \, dt}{t^{11}}$$

$$\sec x + \tan x = t^2$$

$$\sec x - \tan x = \frac{1}{t^2}$$

$+$

$$2\sec x = t^2 + \frac{1}{t^2}$$

$$\sec x = \frac{t^4+1}{2t^2}$$

$$= \frac{\cancel{2}}{\cancel{2}}\int \frac{(t^4+1)\, t \, dt}{t^{13}}$$

$$= \int \frac{t^5 + t}{t^{13}} \, dt$$

$$= \int \frac{1}{t^8} \, dt + \int \frac{1}{t^{12}} \, dt$$

$$= \frac{-1}{7t^7} - \frac{1}{11t^{11}} + C$$

$$= \frac{-1}{7} \frac{1}{(\sec x + \tan x)^{7/2}} - \frac{1}{11(\sec x + \tan x)^{11/2}} + C$$

$$= \frac{-1}{(\sec x + \tan x)^{11/2}} \left\{ \frac{1}{11} + \frac{1}{7}(\sec x + \tan x)^2 \right\} + C$$

Q. $\int \frac{\cos x + x\sin x}{x(x+\cos x)}\,dx$

Soln:- $\int \frac{\cos x + x\sin x}{x^2\left(\frac{x+\cos x}{x}\right)}\,dx$

$\frac{x+\cos x}{x} = t$

$\frac{\cancel{x}(\cancel{1}-\sin x) - (\cancel{x}+\cos x)}{x^2}\,dx = dt$

$-\frac{(\cos x + x\sin x)}{x^2}\,dx = dt$

$-\int \frac{dt}{t}$

$= -\ln t + C$

$= \ln\left(\frac{1}{t}\right) + C$

$= -\ln\left(\frac{x+\cos x}{x}\right) + C$

$= \ln\left(\frac{x}{x+\cos x}\right) + C$

Q. $\int \frac{(x^4 - x)^{1/4}}{x^5}\,dx$

Soln:- $\int \frac{x(1-x^{-3})^{1/4}}{x^5}\,dx$

$$= \int \frac{(1-x^{-3})^{1/4}}{x^4}\, dx$$

$$1 - x^{-3} = t^4$$

$$\cancel{4x^{-4}\, dx = 4t^3\, dt}$$

$$\cancel{\frac{dx}{x^4} = t^3\, dt}$$

$$3x^{-4}\, dx = 4t^3\, dt$$

$$\frac{dx}{x^{-4}} = \frac{4}{3} t^3\, dt$$

$$= \frac{4}{3} \int t \cdot t^3\, dt$$

$$= \frac{4}{3} \int t^4\, dt$$

$$= \frac{4}{3} \times \frac{t^5}{5} \cancel{dt} + c$$

$$= \frac{4}{3} \times \frac{(1-x^{-3})^{5/4}}{5} + c$$

$$= \frac{4}{15}\left(1 - \frac{1}{x^3}\right)^{5/4} + c$$