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CL 249 Computational Methods Lab
Assignment 6 : Numerical Integration

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Q. Obtain the value of the integration :

$$I = \int_0^{30} \left(\frac{250x}{x+6} \right) \cdot e^{-\frac{x}{10}} \, dx$$

using (a) trapezoidal rule

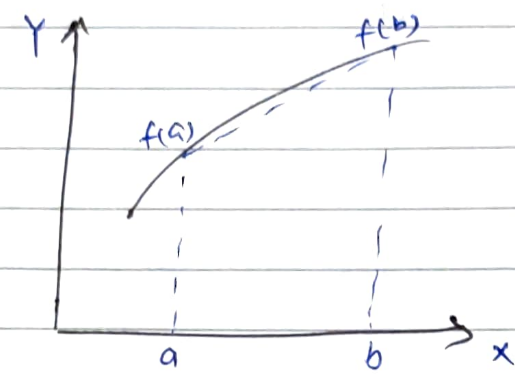
(b) Gauss - Quadrature rule.

Submit plot of of step size (h) vs integral value (I) using the two techniques.

Method

(i) Trapezoidal Rule

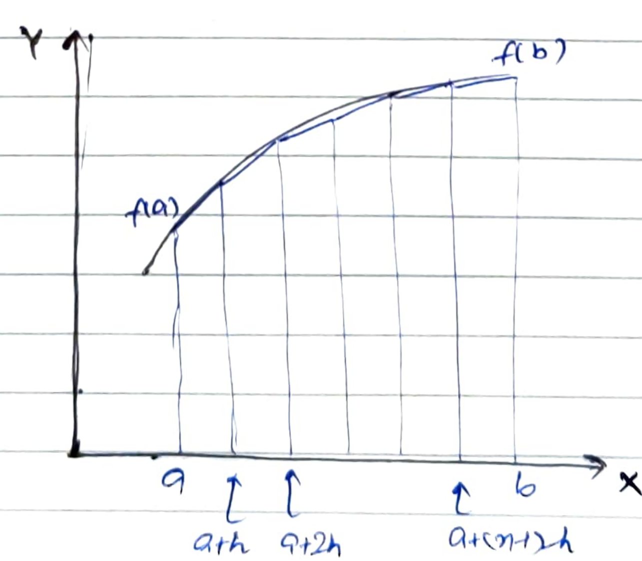
Assume a function, $y = f(x)$ as shown below:



For an approximate of $\int f(x)dx$, we calculate the area of the trapezium $= \left(\frac{b-a}{2}\right) \times (f(a) + f(b))$

i.e. $\frac{(\text{height of interval})}{2} (f(x_i) + f(x_{i+1}))$

For better approximation, divide ~~interval~~ domain into equal intervals of h.



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Now, calculate area of each trapezium.

$$\text{Area} = \frac{h}{2}\Big[f(a) + f(a+h) \quad \text{Ist Interval}$$

$$+ f(a+h) + f(a+2h) \quad \text{IInd Interval}$$

$$+ \ldots$$

$$\vdots$$

$$+ f(a+(n-1)h) + f(b) \Big] \quad \text{Nth Interval}$$

~~Here~~ $\Rightarrow \quad I = \frac{h}{2}\left(f(a) + 2\sum_{i=1}^{n-1} f(a+ih) + f(b) \right)$

Here, $\quad n = \frac{(b-a)}{h}$

(ii) Gauss-Quadrature Rule.

For a function, $y = f(x)$

$$I = \int_a^b f(x)\,dx$$

we write it as $\quad I = \int_{-1}^{1} g(t)\,dt \quad$ using appropriate ~~xo~~ x-t relation.

Now, let's approximate $g(t)$ as a third degree polynomial

$$g(t) = mt^3 + nt^2 + pt + q$$

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$\therefore \int_{-1}^{1} g(t)\, dt = \frac{2n}{3} + 2q$

$\Rightarrow \frac{2n}{3} + 2q = C_0\, g(t_0) + C_1 g(t_1)$

$\because$ It's a general case, regardless of coefficient value, the answer remains same.

$\therefore$ After trying with different m, n, p, q values

$\Rightarrow C_0 = C_1 = 1 \quad t_0 = \frac{1}{\sqrt{3}} \quad t_1 = -\frac{1}{\sqrt{3}}$

After manipulations, we get

$$x = \frac{(b-a)t}{2} + \frac{(b+a)}{2}$$

$$\Rightarrow g(t) = \frac{(b-a)}{2} f\left(\frac{(b-a)t}{2} + \frac{(a+b)}{2}\right)$$

Divide ~~dom~~ limits into intervals of height h, and apply Gauss Quadrature separately, and add to get a better approximation.

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PSEUDOCODE

main.m

```
a=0 ;    b=30;

loop i = 1 to 300
        h(i) = (b-a)/i
End loop.

Call trapezoid_rule (h,a,b)
Call gauss_quadrature (h,a,b)
```

trapezoidal_rule.m

```
loop i = 1 to length of h

        I_temp = f(a) + f(b)
        loop j = 1 to i-1
                I_temp = I_temp + 2f(a+jh(i))
        end loop
        I(i) = I_temp . h(i)/2

end loop
```

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gauss_quadrature.m

loop $i = 1$ to length of h

$x_l = a$

$t_0 = 1/\sqrt{3}$

$t_1 = -1/\sqrt{3}$

$I_{temp} = 0$

loop $j = 1$ to i

$x_u = x_l + h(i)$

$$I_{temp} = I_{temp} + \frac{(x_u - x_l)}{2} * \left[f\left(\left(\frac{x_u - x_l}{2}\right)t_0 + \left(\frac{x_u + x_l}{2}\right)\right) + f\left(\left(\frac{x_u - x_l}{2}\right)t_1 + \left(\frac{x_u + x_l}{2}\right)\right)\right]$$

$x_l = x_u$

end loop

$I(i) = I_{temp}$

end loop

main.m

plot graph h vs I (trapezoidal)

plot graph h vs I (gauss Quadrature)

Define fn for returning function value at input x.

```
clc;
clear;
clear all;
```

Main.m

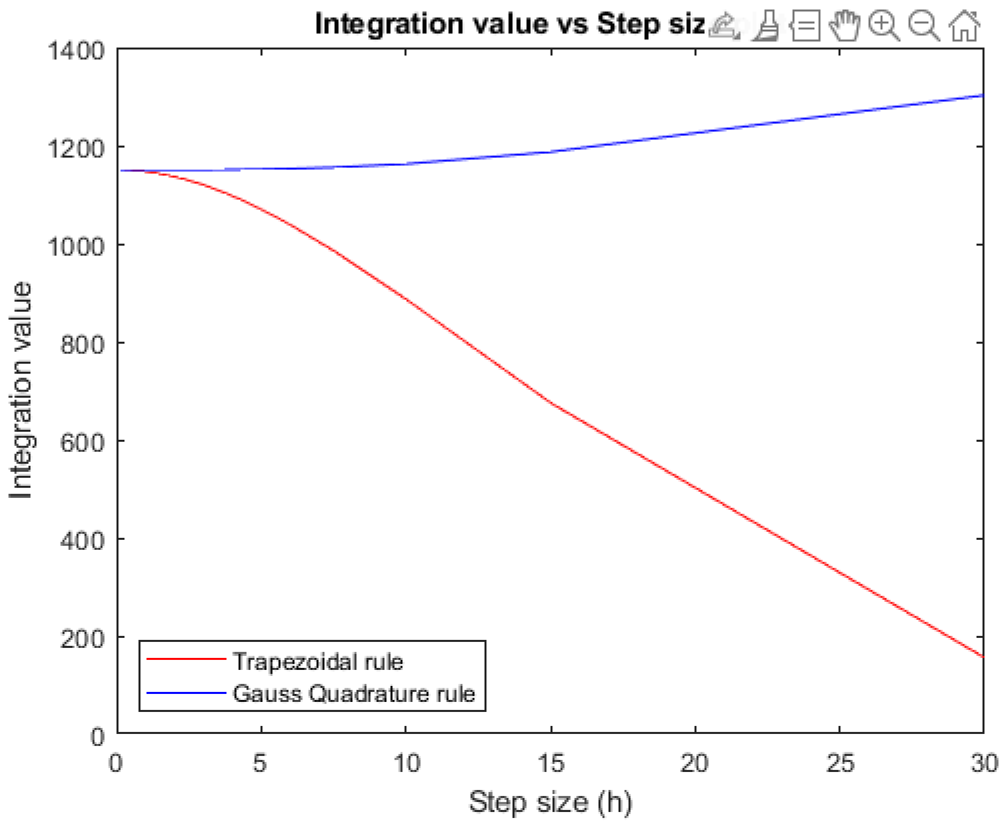
```
a=0; b=30; % integration limits

for i= 1:300
    h(i)= (b-a)./i; % defining step size
end

I_trapezoidal = trapezoidal_rule(h,a,b); % returning integral vector
 using Trapezoidal rule
I_gauss_quadrature = gauss_quadrature(h,a,b); % returning integral
 vector using Gauss Quadrature rule
```

Plot

```
figure()
plot(h, I_trapezoidal,'r');
hold on;
plot(h,I_gauss_quadrature,'b');
title('Integration value vs Step size plot');
legend('Trapezoidal rule','Gauss Quadrature
 rule','Location','southwest');
xlabel('Step size (h)');
ylabel('Integration value');
```



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f.m

```
function y=f(x)
% y as a function of x
y = ((250.*x)./(x+6).*exp(-x./10));
end
```

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trapezoidal_rule.m

```
function I = trapezoidal_rule(h,a,b)
    for i= 1:length(h) % looping through all step size
        I_temp = f(a) + f(b);
        for j= 1:i-1
            I_temp = I_temp + 2.*f(a + j.*h(i)); % trapezoidal formula
        end
        I(i) = I_temp.*(h(i)./2); % mutiplying by height of
 trapezium/2
    end
end
```

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gauss_quadrature.m

```
function I= gauss_quadrature(h,a,b)
% Two point method
    for i= 1:length(h) % looping through all step size
        x_l=a;
        t0= 1./sqrt(3);
        t1= -1./sqrt(3);
        I_temp=0;
        for j=1:i
            x_u= x_l + h(i); % upper bound of interval
            I_temp= I_temp + (f((x_u - x_l).*t0./2 + (x_l + x_u)./2)
 + f((x_u - x_l).*t1./2 + (x_l + x_u)./2)).*(x_u - x_l)./2; % Gauss
 Quadrature formula

            x_l= x_u; % updating lower bound of next interval
        end
        I(i) = I_temp; % storing integral value for diff. step size
    end
end
```

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Comments & Remarks.

The trapezoidal rule returns integral value which decreases as step size increases.

$$\text{Error} = \frac{-h^3}{12}\left(f''(\xi_1) + f''(\xi_2) + \cdots + f''(\xi_n)\right)$$

$$= \frac{-h^3}{12} \cdot n f''(\xi_0) = \frac{-h^3}{12} \cdot \frac{(b-a)}{h} f''(\xi_0) = \frac{-h^2}{12}(b-a) f''(\xi_0)$$

$$\Rightarrow \text{Error} \Rightarrow O(h^2)$$

The Gauss Quadrature is a cubic level approximation, and from plot, it is seen that, integral value increases as step size increases.

$$\text{Error} = O(h^5) f^{4}(\xi)$$ → for cubic expressions result are accurate.

hence, it gives better results.