QBLD - GSoC 2020

Ayush Agarwal

24/07/2020

Quantile Regression and the Asymmetric Lalplace Distribution

The Model

The QBLD model can be conveniently expressed in the latent variable formulation (Albert & Chib, 1993) as follows:

$$\begin{aligned} z_{it} &= x_{it}^{'}\beta + s_{it}^{'}\alpha_i + \epsilon_{it}, \qquad \forall i = 1, ..., n; t = 1, ..., T_i \\ y_{it} &= \begin{cases} 1 & if\, z_{it} > 0 \\ 0 & \text{otherwise}, \end{cases} \end{aligned} \tag{1}$$

y_{it} = response variable y at t^{th} time period for the i^{th} individual,
z_{it} = unobserved latent variable z at t^{th} time period for the i^{th} individual,
$x_{it}^{'} = 1 \times k$ vector of fixed-effects covariates,
$\beta = k \times 1$ vector of fixed-effects parameters,
$s_{it}^{'} = 1 \times l$ vector of covariates that have individual-specific effects,
$\alpha_i = l \times 1$ vector of individual-specific parameters, and
ϵ_{it} = the error term $\overset{\text{iid}}{\sim} AL(0, 1, p)$.

ALD Mixture

While working directly with the AL density is an option, the resulting posterior will not yield the full set of tractable conditional distributions necessary for a Gibbs sampler. The mixture representation gives access to the appealing properties of the normal distribution. Thus, we utilize the normal-exponential mixture representation of the AL distribution, presented in Kozumi and Kobayashi (2011) :

$$\epsilon_{it} = w_{it}\theta + \tau\sqrt{w_{it}}u_{it} \qquad \forall i = 1, ..., n; t = 1, ..., T_i \tag{2}$$

$u_{it} \sim N(0, 1)$, is mutually inependent of $w_{it} \sim \exp(1)$,
$\theta = \frac{1-2p}{p(1-p)}$, and $\tau = \sqrt{\frac{2}{p(1-p)}}$.

Model with Priors

Longitudinal data models often involve a moderately large amount of data, so it is important to take advantage of any opportunity to reduce the computational burden. One such trick is to stack the model for each individual i (Hendricks, Koenker, & Poirier, 1979).

We define, $z_i = (z_{i1}, ..., z_{iT_i})^{'}$, $X_i = (x^{'}_{i1}, ..., x^{'}_{iT_i})^{'}$, $S_i = (s^{'}_{i1}, ..., s^{'}_{iT_i})^{'}$, $w_i = (w_{i1}, ..., w_{iT_i})^{'}$, $D_{\tau\sqrt{w_i}} = diag(\tau\sqrt{w_{i1}}, ..., \tau\sqrt{w_{iT_i}})^{'}$, and $u_i = (u_{i1}, ..., u_{iT_i})^{'}$.

Building on Eqs. (1) and (2),

$$\begin{gathered} z_i = X_i\beta + S_i\alpha_i + w_i\theta + D_{\tau\sqrt{w_i}}u_i, \\ y_{it} = \begin{cases} 1 & if\; z_{it} > 0 \\ 0 & \text{otherwise,} \end{cases} \\ \alpha_i|\varphi^2 \sim N_l(0, \varphi^2 I_l), w_{it} \sim \exp(1), u_{it} \sim N(0, 1) \\ \beta \sim N_k(\beta_0, B_0), \varphi^2 \sim IG(c1/2, d1/2) \end{gathered} \tag{3}$$

Algorithm

Blocked Sampling

- Sample (β, z_i) in one block. These are sampled in following two substeps.

  (1) Sample β

$$\begin{gathered} \beta|z, w, \varphi^2 \sim N(\tilde{\beta}, \tilde{B}), \\ where, \quad \tilde{B}^{-1} = (\sum_{i=1}^{n} X_i^{'}\Omega_i^{-1}X_i + B_0^{-1}), \\ \tilde{\beta} = (\sum_{i=1}^{n} X_i^{'}\Omega_i^{-1}(z_i - w_i\theta) + B_0^{-1}\beta_0), \\ \Omega_i = (\varphi^2 S_i S_i^{'} + D^2_{\tau\sqrt{w_i}}). \end{gathered} \tag{4}$$

(2) Sample the vector $z_i|y_i, \beta, w_i, \varphi^2 \sim TMVN_{B_i}(X_i\beta + w_i\theta, \Omega_i)$ for all $i = 1, ..., n$, where $B_i = (B_{i1} * B_{i2} * ... * B_{iT_i})$ and B_{it} are interval $(0, \infty)$ if $y_{it} = 1$, and the interval $(-\infty, 0]$ if $y_{it} = 0$. This is done by sampling z_i at the j^{th} pass of the MCMC iteration using a series of conditional posteriors:

$$
\begin{aligned}
z_{it}^{j}|z_{i1}^{j}, ...z_{i(t-1)}^{j}, z_{i(t+1)}^{j-1}, ..., z_{iT_i}^{j-1} &\sim TN_{B_i}(\mu_{t|-t}, \Sigma_{t|-t}), \qquad t = 1, ..., T_i. \\
where, \quad \mu_{t|-t} &= x_{it}^{'}\beta + w_{it}\theta + \Sigma_{t,-t}\Sigma_{-t,-t}^{-1}(z_{i,-t}^{j} + (X_i\beta + w_i\theta)_{-t}), \\
\Sigma_{t|-t} &= \Sigma_{t,t} - \Sigma_{t,-t}\Sigma_{-t,-t}^{-1}\Sigma_{-t,t},
\end{aligned}
\tag{5}
$$

where $z_{i,-t}^{j} = (z_{i1}^{j}, ...z_{i(t-1)}^{j}, z_{i(t+1)}^{j-1}, ..., z_{iT_i}^{j-1})$, $(X_i\beta + w_i\theta)_{-t}$ is column vector with t^{th} element removed, $\Sigma_{t,t}, \Sigma_{t,-t}, \Sigma_{-t,-t}$ are $(t,t)^{th}$ element, t^{th} row with t^{th} element removed, and t^{th} row and column removed respectively.

- Sample α

$$
\begin{aligned}
\alpha_i|z, \beta, w, \varphi^2 &\sim N(\tilde{a}, \tilde{A}), \quad \forall i = 1, ..., n \\
where, \quad \tilde{A^{-1}} &= (S_i^{'}D_{\tau\sqrt{w_i}}^{2}S_i + \frac{1}{\varphi^2}I_l), \\
\tilde{a} &= \tilde{A}(S_i^{'}D_{\tau\sqrt{w_i}}^{2}(z_i - X_i\beta - w_i\theta)).
\end{aligned}
\tag{6}
$$

- Sample w

$$
\begin{aligned}
w_{it}|z_{it}, \beta, \alpha_i &\sim GIG(0.5, \tilde{\lambda_{it}}, \tilde{\eta}) \forall i = 1, ..., n; t = 1, ..., T_i, \\
where, \quad \tilde{\lambda_{it}} &= (\frac{z_{it} - x_{it}^{'}\beta - s_{it}^{'}\alpha_i}{\tau})^2 \\
\tilde{\eta} &= (\frac{\theta^2}{\tau^2} + 2).
\end{aligned}
\tag{7}
$$

- Sample φ^2

$$
\begin{aligned}
\varphi^2|\alpha &\sim IG(\tilde{c_1}/2, \tilde{d_1}/2), \\
where, \quad \tilde{c_1} &= (nl + c_1), \\
\tilde{d_1} &= (\sum_{i=1}^{n} \alpha_i^{'}\alpha_i + d_1).
\end{aligned}
\tag{8}
$$

.

Unblocked Sampling

- Sample β

$$
\begin{aligned}
&\beta|z,w,\varphi^2 \sim N(\tilde{\beta},\tilde{B}),\\
where,\quad &\tilde{B}^{-1} = (\sum_{i=1}^{n} X_i^{'}\Psi_i^{-1}X_i + B_0^{-1}),\\
\tilde{\beta} = &(\sum_{i=1}^{n} X_i^{'}\Psi_i^{-1}(z_i - w_i\theta - S_i\alpha_i) + B_0^{-1}\beta_0),\\
&\Psi_i = D^2_{\tau\sqrt{w_i}}.
\end{aligned}
\tag{9}
$$

- Sample α as in (6).
- Sample w as in (7).
- Sample φ^2 as in (8).
- Sample $z|y,\alpha,w\ \forall i = 1,...,n; t = 1,...,T_i$, from univariate truncated normal as:

$$
z_{it}|y,\beta,w = \begin{cases} TN_{(-\infty,0]}(x_{it}^{'}\beta + s_{it}^{'}\alpha_i + w_{it}\theta, \tau^2 w_{it}) & if\, y_{it} = 0 \\ TN_{(0,\infty)}(x_{it}^{'}\beta + s_{it}^{'}\alpha_i + w_{it}\theta, \tau^2 w_{it}) & if\, y_{it} = 1 \end{cases}
\tag{10}
$$

.