Q1. The value of x for which $\sin\left(\cot^{-1}(1+x)\right)=\cos\left(\tan^{-1}x\right)$ is

A. $\frac{1}{2}$

B. 1

C. 0

D. $-\frac{1}{2}$

Ans: $-\frac{1}{2}$

Solution: Let, $\cot^{-1}(x+1)=\theta$ and $\tan^{-1}x=\phi$

Now, given equation becomes $\sin\theta=\cos\phi$

$\Rightarrow \frac{1}{\sqrt{1+(x+1)^2}}=\frac{1}{\sqrt{1+x^2}}$

$\Rightarrow (x+1)^2+1=x^2+1 \Rightarrow x=-\frac{1}{2}$

Q2. The value of $\cos\left(\frac{1}{2}\cos^{-1}\left(\cos\left(\sin^{-1}\frac{\sqrt{63}}{8}\right)\right)\right)$ is

A. $\frac{3}{16}$

B. $\frac{3}{8}$

C. $\frac{3}{4}$

D. $\frac{3}{2}$

Ans: $\frac{3}{4}$

Solution:

The given trigonometric ratio

$= \cos\left(\frac{1}{2}\cos^{-1}\left(\cos\left(\cos^{-1}\frac{1}{8}\right)\right)\right)$

$= c \ \cos\left(\frac{1}{2}\cos^{-1}\frac{1}{8}\right)$

$= \sqrt{\frac{1+\cos\left(\cos^{-1}\frac{1}{8}\right)}{2}} = \frac{3}{4}$

Note: One may also proceed by writing the

ratio as $\cos\left(\frac{1}{2}\sin^{-1}\frac{\sqrt{63}}{8}\right)$.

Q3. The value of $\sin^{-1}\sin 17 + \cos^{-1}\cos 10$ is equal to

A. 27

B. -27

C. $17 - 5\pi$

D. $9\pi - 27$

Ans: $9\pi - 27$

Solution: $\sin^{-1}\sin 17 = \sin^{-1}\sin(17 - 5\pi + 5\pi)$

$= 5\pi - 17$

$\cos^{-1}(\cos 10) = \cos^{-1}\cos(10 - 3\pi + 3\pi)$

$= \cos^{-1}\cos\{3\pi + (10 - 3\pi)\}$

$= \cos^{-1}\{-\cos(10 - 3\pi)\}$

$= \pi - \cos^{-1}\cos(10 - 3\pi)$

$= \pi - (10 - 3\pi) = 4\pi - 10$

Hence, $\sin^{-1}\sin 17 + \cos^{-1}(\cos 10) = 9\pi - 27$

Q4. The value of $\sin\left\{\cot^{-1}\left[\cos\left(\cot^{-1}\left(\frac{1}{x}\right)\right)\right]\right\}$ is equal to $(x > 0)$

A. $\sqrt{\frac{1+x^2}{2+x^2}}$

B. $\sqrt{\frac{1-x^2}{2+x^2}}$

C. $\sqrt{\frac{1+x^2}{2-x^2}}$

D. $\sqrt{\frac{2+x^2}{1+x^2}}$

Ans: $\sqrt{\frac{1+x^2}{2+x^2}}$

Solution: Let $\cot^{-1}\left(\frac{1}{x}\right)=\alpha \Rightarrow \tan\alpha=x$

So, $\cos\alpha=\frac{1}{\sqrt{1+x^2}}$

$\therefore \sin\{\cot^{-1}[\cos(\tan^{-1}x)]\}$

$=\sin\left(\cot^{-1}\frac{1}{\sqrt{1+x^2}}\right)$

Let $\cot^{-1}\frac{1}{\sqrt{1+x^2}}=\beta \Rightarrow \cot\beta=\frac{1}{\sqrt{1+x^2}}$

$\therefore \sin\beta=\frac{\sqrt{(1+x^2)}}{\sqrt{(2+x^2)}}$

$\therefore \sin\beta=\sin\left\{\cot^{-1}\left[\cos\left(\cot^{-1}\left(\frac{1}{x}\right)\right)\right]\right\}=\sqrt{\frac{1+x^2}{2+x^2}}$

Q5. If the value of the expression $\tan\left(\frac{1}{2}\cos^{-1}\frac{2}{\sqrt{5}}\right)$ is in the form of $a+\sqrt{b}$ where $a,\ b\in Z$, then the value of $\frac{a+b}{b}$ is

Ans: 0.6

Solution: We know that,

$\tan\left(\frac{x}{2}\right)=\frac{1-\cos x}{\sin x}$

$\therefore \quad \tan\left(\frac{1}{2}\cos^{-1}\frac{2}{\sqrt{5}}\right)=\frac{1-\cos\left(\cos^{-1}\left(\frac{2}{\sqrt{5}}\right)\right)}{\sin\left(\cos^{-1}\left(\frac{2}{\sqrt{5}}\right)\right)}$

$=\frac{1-2/\sqrt{5}}{\sin\left(\sin^{-1}\left(1/\sqrt{5}\right)\right)}$

$= \frac{1-2/\sqrt{5}}{1/\sqrt{5}} = \sqrt{5} - 2$

Now, $a + \sqrt{b} = -2 + \sqrt{5}$

$\Rightarrow a = -2$ and $b = 5$

Hence, $\frac{a+b}{b} = \frac{3}{5} = 0.6$

Q6. If $\sin^{-1}\frac{1}{3} + \sin^{-1}\frac{2}{3} = \sin^{-1} x$, then the value of x is

A. 0

B. $\frac{\left(\sqrt{5}-4\sqrt{2}\right)}{9}$

C. $\frac{\left(\sqrt{5}+4\sqrt{2}\right)}{9}$

D. $\frac{\pi}{2}$

Ans: $\frac{\left(\sqrt{5}+4\sqrt{2}\right)}{9}$

Solution: Given that, $\sin^{-1}\frac{1}{3} + \sin^{-1}\frac{2}{3} = \sin^{1} x$

Taking sine on both sides

$\Rightarrow \left(\frac{1}{3}\sqrt{1-\frac{4}{9}} + \frac{2}{3}\sqrt{1-\frac{1}{9}}\right) = x$

$\Rightarrow \left(\frac{1}{3} \cdot \frac{\sqrt{5}}{3} + \frac{2}{3} \cdot \frac{\sqrt{8}}{3}\right) = x$

$\Rightarrow \left(\frac{\sqrt{5}+4\sqrt{2}}{9}\right) = x$

$\therefore \quad x = \left(\frac{\sqrt{5}+4\sqrt{2}}{9}\right)$

Q7. The complete solution set of the inequality $\cos^{-1}(\cos 4) > 3x^2 - 4x$ is

A. $\left(0, \frac{2+\sqrt{6\pi-8}}{3}\right)$

B. $\left(\frac{2-\sqrt{6\pi-8}}{3}, 0\right)$

C. $(-2, 2)$

D. $\left(\frac{2-\sqrt{6\pi-8}}{3}, \frac{2+\sqrt{6\pi-8}}{3}\right)$

Ans: $\left(\frac{2-\sqrt{6\pi-8}}{3}, \frac{2+\sqrt{6\pi-8}}{3}\right)$

Solution: As, $\cos^{-1}(\cos 4) = \cos^{-1}\{\cos(2\pi - 4)\} = 2\pi - 4$

$\Rightarrow \quad 2\pi - 4 > 3x^2 - 4x$

$\Rightarrow \quad 3x^2 - 4x - (2\pi - 4) < 0$

$\Rightarrow \quad \frac{2-\sqrt{6\pi-8}}{3} < x < \frac{2+\sqrt{6\pi-8}}{3}$

Q8. If $x = \sin(2\tan^{-1}3)$ and $y = \sin\left(\frac{1}{2}\tan^{-1}\frac{4}{3}\right)$, then

A. $2x = 1 - y$

B. $x^2 = 1 - 2y$

C. $x^2 = 1 + y$

D. $y^2 = 2x - 1$

Ans: $y^2 = 2x - 1$

Solution:

Let, $x=\sin 2\theta$ (where $\tan\theta=3$)

$\Rightarrow x=\frac{2\tan\theta}{1+\tan^2\theta}=\frac{6}{1+9}=\frac{3}{5}$

If $\alpha=\tan^{-1}\frac{4}{3}$

$\Rightarrow y=\sin\left(\frac{\alpha}{2}\right)=\frac{1}{\sqrt{2}}\sqrt{1-\cos\alpha}$

$=\frac{1}{\sqrt{2}}\sqrt{1-\frac{3}{5}}=\frac{1}{\sqrt{5}}$

$\therefore y^2=\frac{1}{5}\Rightarrow y^2=2x-1$

Q9. $\tan\left(2\tan^{-1}\frac{1}{5}+\sec^{-1}\frac{\sqrt{5}}{2}+2\tan^{-1}\frac{1}{8}\right)$ **is equal to:**

A. 1

B. 2

C. $\frac{1}{4}$

D. $\frac{5}{4}$

Ans: 2

Solution:

Given,

$\tan\left(2\tan^{-1}\frac{1}{5}+\sec^{-1}\frac{\sqrt{5}}{2}+2\tan^{-1}\frac{1}{8}\right)$

$=\tan\left(2\left(\tan^{-1}\frac{1}{5}+\tan^{-1}\frac{1}{8}\right)+\tan^{-1}\left(\frac{1}{2}\right)\right)$

$=\tan\left(2\left(\tan^{-1}\frac{\frac{1}{5}+\frac{1}{8}}{1-\frac{1}{40}}\right)+\tan^{-1}\left(\frac{1}{2}\right)\right)$

$=\tan\left[2\tan^{-1}\left(\frac{1}{3}\right)+\tan^{-1}\left(\frac{1}{2}\right)\right]$

$=\tan\left[\tan^{-1}\left(\frac{2\times\frac{1}{3}}{1-\left(\frac{1}{3}\right)^2}\right)+\tan^{-1}\left(\frac{1}{2}\right)\right]$

$= \tan\left[\tan^{-1}\left(\frac{3}{4}\right)+\tan^{-1}\left(\frac{1}{2}\right)\right]$

$= \tan\left[\tan^{-1}\frac{\frac{3}{4}+\frac{1}{2}}{1-\frac{3}{8}}\right]$

$= \tan\tan^{-1}\frac{\frac{5}{4}}{\frac{5}{8}}$

$= \tan\tan^{-1}2$

$= 2$

Q10. The value of $2\sin^{-1}\frac{4}{5}+2\sin^{-1}\frac{5}{13}+2\sin^{-1}\frac{16}{65}$ is equal to

A. $\frac{3\pi}{2}$

B. $\frac{\pi}{2}$

C. π

D. 2π

Ans: π

Solution: $2\left[\sin^{-1}\frac{4}{5}+\sin^{-1}\frac{5}{13}+\sin^{-1}\frac{16}{65}\right]$
$= 2\sin^{-1}\left(\frac{4}{5}\sqrt{1-\left(\frac{5}{13}\right)^2}+\frac{5}{13}\sqrt{1-\left(\frac{4}{5}\right)^2}\right)+2\sin^{-1}\frac{16}{65}$
$= 2\sin^{-1}\left(\frac{48}{65}+\frac{15}{65}\right)+2\sin^{-1}\left(\frac{16}{65}\right)$
$= 2\left[\sin^{-1}\left(\frac{63}{65}\right)+\sin^{-1}\left(\frac{16}{65}\right)\right]$
$= 2\left[\cos^{-1}\left(\frac{16}{65}\right)+\sin^{-1}\left(\frac{16}{65}\right)\right]\left(\because \sin^{-1}x=\cos^{-1}\sqrt{1-x^2}\right)$
$= \pi$

Q11. The number of solution of the equation $2\tan^{-1}x+\cot^{-1}x=\frac{7\pi}{6}$ is

A. 0

B. 1

C. 2

D. 3

Ans: 0

Solution:

$2\tan^{-1}x+\cot^{-1}x=\frac{7\pi}{6}$

$\Rightarrow \tan^{-1}x+\frac{\pi}{2}=\frac{7\pi}{6}\Rightarrow \tan^{-1}x=\frac{2\pi}{3}$

$\Rightarrow \tan^{-1}x\in\left(\frac{-\pi}{2},\ \frac{\pi}{2}\right)(x\in\phi)$

Q12. The value of $\tan^{-1}\left[\frac{\sqrt{1-\sin x}+\sqrt{1+\sin x}}{\sqrt{1-\sin x}-\sqrt{1+\sin x}}\right]\left(\forall x\in\left[0,\frac{\pi}{2}\right]\right)$ **is equal to**

A. $\frac{x}{2}-\frac{\pi}{2}$

B. $\frac{x}{2}+\frac{\pi}{2}$

C. $\frac{x}{2}-\pi$

D. $\frac{\pi}{2}-\frac{x}{2}$

Ans: $\frac{x}{2}-\frac{\pi}{2}$

Solution: $\tan^{-1}\left[\frac{\sqrt{1-\sin x}+\sqrt{1+\sin x}}{\sqrt{1-\sin x}-\sqrt{1+\sin x}}\right]$

$=\tan^{-1}\left[\frac{\left(\sqrt{1-\sin x}+\sqrt{1+\sin x}\right)}{\left(\sqrt{1-\sin x}-\sqrt{1+\sin x}\right)}\frac{\left(\sqrt{1-\sin x}+\sqrt{1+\sin x}\right)}{\left(\sqrt{1-\sin x}+\sqrt{1+\sin x}\right)}\right]$

$=\tan^{-1}\left[\frac{(1-\sin x)+(1+\sin x)+2\sqrt{1-\sin^2 x}}{(1-\sin x)-(1+\sin x)}\right]=\tan^{-1}\left[\frac{2(1+\cos x)}{-2\sin x}\right]$

$=\tan^{-1}\left[\frac{-2\cos^2\left(\frac{x}{2}\right)}{2\sin\left(\frac{x}{2}\right)\cos\left(\frac{x}{2}\right)}\right]=\tan^{-1}\left(-\cot\frac{x}{2}\right)=\tan^{-1}\left[\cot\left(\pi-\frac{x}{2}\right)\right]$

$= \frac{\pi}{2} - \cot^{-1}\left[\cot\left(\pi - \frac{x}{2}\right)\right] = \frac{\pi}{2} - \left(\pi - \frac{x}{2}\right) = \frac{x}{2} - \frac{\pi}{2}$

Q13. If the equation $\sin^{-1}(4x^2 - 12x + 10) + \cos^{-1}(12x - 4x^2 - 10) + \lambda x = 0$ has a real solution, then λ is equal to

A. $\frac{\pi}{4}$

B. $-\pi$

C. $\frac{\pi}{2}$

D. $\frac{-\pi}{2}$

Ans: $-\pi$

Solution:

$\sin^{-1}\left(1 + (2x-3)^2\right) + \cos^{-1}\left(-1 - (2x-3)^2\right) + \lambda x = 0$

$x = \frac{3}{2}$

$\Rightarrow \frac{\pi}{2} + \pi + \frac{3\lambda}{2} = 0 \Rightarrow \lambda = -\pi.$

Q14. If $\cos^{-1} x + \cos^{-1} y + \cos^{-1} z = \pi$, then

A. $x^2 + y^2 + z^2 + xyz = 0$

B. $x^2 + y^2 + z^2 + 2xyz = 0$

C. $x^2 + y^2 + z^2 + xyz = 1$

D. $x^2 + y^2 + z^2 + 2xyz = 1$

Ans: $x^2 + y^2 + z^2 + 2xyz = 1$

Solution: Given that $\cos^{-1} x + \cos^{-1} y + \cos^{-1} z = \pi$

$\Rightarrow \cos^{-1}(x)+\cos^{-1}(y)+\cos^{-1}(z)=\cos^{-1}(-1)$

$\Rightarrow \cos^{-1}(x)+\cos^{-1}(y)=\cos^{-1}(-1)-\cos^{-1}(z)$

$\Rightarrow \cos^{-1}\left(xy-\sqrt{1-x^2}\sqrt{1-y^2}\right)=\cos^{-1}\{(-1)(z)\}$

$\Rightarrow xy-\sqrt{\left(1-x^2\right)\left(1-y^2\right)}=-2$

$\Rightarrow (xy+z)=\sqrt{\left(1-x^2\right)\left(1-y^2\right)}$

Squaring both sides we get $x^2+y^2+z^2+2xyz=1$ Trick: Put $x=y=z=\frac{1}{2}$, so that

$\cos^{-1}\frac{1}{2}+\cos^{-1}\frac{1}{2}+\cos^{-1}\frac{1}{2}=\pi$

Obviously (d) holds for these values of x, y, z.

Q15. If $\left(\cot^{-1}x\right)^2-7\left(\cot^{-1}x\right)+10>0$, then the range of x will be

A. $(-\infty\ ,\ \cot 2)$

B. $(-\infty\ ,\ \cot 5)$

C. $(\cot 2\ ,\ \cot 5)$

D. $(\cot 2,\ \infty)$

Ans: $(\cot 2,\ \infty)$

Solution: $\left(\cot^{-1}x-5\right)\left(\cot^{-1}x-2\right)>0$

$\Rightarrow \cot^{-1}x \in(-\infty,\ 2)\cup(5,\ \infty)$

$\Rightarrow \cot^{-1}x \in(0,\ 2)$ (Taking intersection with range of $\cot^{-1}x$)

$\Rightarrow n\in(\cot 2,\ \infty)$

Q16. The value of a for which $ax^2+\sin^{-1}\left(x^2-2x+2\right)+\cos^{-1}\left(x^2-2x+2\right)=0$ has a real solution, is

A. $-\frac{2}{\pi}$

B. $\frac{2}{\pi}$

C. $-\frac{\pi}{2}$

D. $\frac{\pi}{2}$

Ans: $-\frac{\pi}{2}$

Solution: Here, $x^2-2x+2=(x-1)^2+1\geq 1$

But, $-1\leq\left(x^2-2x+2\right)\leq 1$

Which is possible only when

$x^2-2x+2=1$

$\Rightarrow x=1$

Then, $a(1)^2+\sin^{-1}(1)+\cos^{-1}(1)=0$

$\Rightarrow a+\frac{\pi}{2}+0=0$

$\Rightarrow a=-\frac{\pi}{2}$

Q17. Number of real roots of the equation $\sin^{-1}\sin x=\cos^{-1}\cos\ 4$ **in** $[0,2\pi]$ **is**

A. 0

B. 1

C. 2

D. more than 2

Ans: 0

Solution:

$\cos^{-1}\cos 4 = 2\pi - 4 > \frac{\pi}{2}$ and $\sin^{-1}\sin x \leq \frac{\pi}{2}$

$\Rightarrow \sin^{-1}\sin x = \cos^{-1}\cos 4$has no real root.

Q18. The greatest and the least value of $\left(\sin^{-1}x\right)^3 + \left(\cos^{-1}x\right)^3$ are

A. $-\frac{\pi}{2}, \frac{\pi}{2}$

B. $-\frac{\pi^3}{8}, \frac{\pi^3}{8}$

C. $\frac{7\pi^3}{8}, \frac{\pi^3}{32}$

D. None of these

Ans: $\frac{7\pi^3}{8}, \frac{\pi^3}{32}$

Solution: We have $\left(\sin^{-1}x\right)^3 + \left(\cos^{-1}x\right)^3$

$= \left(\sin^{-1}x + \cos^{-1}x\right)^3 - 3\sin^{-1}x\cos^{-1}x\left(\sin^{-1}x + \cos^{-1}x\right)$

$= \frac{\pi^3}{8} - 3\left(\sin^{-1}x\cos^{-1}x\right)\frac{\pi}{2}$

$= \frac{\pi^3}{8} - \frac{3\pi}{2}\sin^{-1}x\left(\frac{\pi}{2} - \sin^{-1}x\right)$

$= \frac{\pi^3}{8} - \frac{3\pi^2}{4}\sin^{-1}x + \frac{3\pi}{2}\left(\sin^{-1}x\right)^2$

$= \frac{\pi^3}{8} + \frac{3\pi}{2}\left[\left(\sin^{-1}x\right)^2 - \frac{\pi}{2}\sin^{-1}x\right]$

$= \frac{\pi^3}{8} + \frac{3\pi}{2}\left[\left(\sin^{-1}x - \frac{\pi}{4}\right)^2\right] - \frac{3\pi^3}{32}$

$= \frac{\pi^3}{32} + \frac{3\pi}{2}\left(\sin^{-1}x - \frac{\pi}{4}\right)^2$

$\therefore$ The least value is $\frac{\pi^3}{32}$

and since $\left(\sin^{-1}x - \frac{\pi}{4}\right)^2 \leq \left(\frac{3\pi}{4}\right)^2$

$\therefore$ The greatest value is $\frac{\pi^3}{32} + \frac{9\pi^2}{16} \times \frac{3\pi}{2} = \frac{7\pi^3}{8}$.

Q19. The real solutions of the equation $\tan^{-1}\sqrt{x(x+1)}+\sin^{-1}\sqrt{x^2+x+1}=\frac{\pi}{2}$ are

A. $-1,\ 0$

B. $0,\ 1$

C. $-1,\ 1$

D. $-1,\ 2$

Ans: $-1,\ 0$

Solution: $\tan^{-1}\sqrt{x(x+1)}+\sin^{-1}\sqrt{x^2+x+1}=\frac{\pi}{2}$

$\Rightarrow \tan^{-1}\sqrt{x^2+x}+\sin^{-1}\sqrt{x^2+x+1}=\frac{\pi}{2}$

This equation holds true, if

$$x^2+x\geq 0 \text{ and } 0\leq x^2+x+1\leq 1$$

Now, $x^2+x\geq 0$ and $0\leq x^2+x+1\leq 1$

$\Rightarrow \quad x^2+x\geq 0$ and $x^2+x+1\leq 1 \quad [\because x^2+x+1>0 \text{ for all } x]$

$\Rightarrow \quad x^2+x\geq 0$ and $x^2+x\leq 0$

$\Rightarrow \quad x^2+x=0 \quad \Rightarrow \quad x=0,-1.$

Clearly, these two values satisfy the given equation. Hence, $x=-1,\ 0$ are the solutions of the given equation.

Q20. The number of integers for which the equation $\sin^{-1} x+\cos^{-1} x+\tan^{-1} x=n$ has real solution(s) is

A. 0

B. 1

C. 2

D. 3

Ans: 2

Solution:

Given equation is $\frac{\pi}{2}+\tan^{-1} x=n, \forall x \in[-1,1]$

Now LHS $\in\left[\frac{\pi}{4}, \frac{3 \pi}{4}\right]$

Integers in this interval are 1 & 2

Hence, there are 2 integers for which the equation has real solutions

Q21. Let $a=\left(\sin^{-1} x\right)^{\sin^{-1} x}, b=\left(\sin^{-1} x\right)^{\cos^{-1} x}, c=\left(\cos^{-1} x\right)^{\sin^{-1} x}, d=\left(\cos^{-1} x\right)^{\cos^{-1} x}$ and if $x \in(0,1)$, then

A. $a>b>d>c$

B. $d>c>a>b$

C. $b>a>d>c$

D. $a<b<d<c$

Ans: $d>c>a>b$

Solution:

It is given that $a=\left(\sin^{-1} x\right)^{\sin^{-1} x}$,

$b=\left(\sin^{-1}x\right)^{\cos^{-1}x}, c=\left(\cos^{-1}x\right)^{\sin^{-1}x}$

$d=\left(\cos^{-1}x\right)^{\cos^{-1}x}$

Also, here $x \in(0,1)$

$\Rightarrow \cos^{-1}x<\sin^{-1}x$

Also, $\cos^{-1}x>1$

and $\sin^{-1}x<1$

$\therefore \left(\cos^{-1}x\right)^{\cos^{-1}x}$ is greatest and $\left(\sin^{-1}x\right)^{\cos^{-1}x}$ is least.

$\Rightarrow \left(\sin^{-1}x\right)^{\sin^{-1}x}<\left(\cos^{-1}x\right)^{\sin^{-1}x}$

$\Rightarrow d>c>a>b$

Q22.

The value of $tan^{-1}\left(\frac{9}{19}\right)+tan^{-1}\left(\frac{9}{49}\right)+tan^{-1}\left(\frac{9}{97}\right)+tan^{-1}\left(\frac{9}{163}\right)+\ \ldots\ \infty$ equals

A. $\tan^{-1}(3)$
B. $\tan^{-1}\left(\frac{1}{3}\right)$
C. $\tan^{-1}\left(\frac{2}{3}\right)$
D. $\tan^{-1}\left(\frac{3}{2}\right)$

Ans: $\tan^{-1}\left(\frac{3}{2}\right)$

Solution:

$S=\lim\limits_{m\rightarrow\infty}\sum_{n=1}^{m}tan^{-1}\left(\frac{9}{9n^2+3n+7}\right)=\lim\limits_{m\rightarrow\infty}\sum_{n=1}^{m}tan^{-1}\left(\frac{1}{1+n^2+\frac{n}{3}-\frac{2}{9}}\right)$

$= \lim_{m\to\infty} \sum_{n=1}^{m} tan^{-1}\left(\frac{\left(n+\frac{2}{3}\right)-\left(n-\frac{1}{3}\right)}{1+\left(n+\frac{2}{3}\right)\left(n-\frac{1}{3}\right)}\right) = \lim_{m\to\infty} \sum_{n=1}^{m}\left[tan^{-1}\left(n+\frac{2}{3}\right) - tan^{-1}\left(n-\frac{1}{3}\right)\right]$

Q23. If $y = \tan^{-1}\frac{1}{1+x+x^2} + \tan^{-1}\frac{1}{x^2+3x+3} + \tan^{-1}\frac{1}{x^2+5x+7} + \ldots. +$ upto $2n$ terms $(\forall x \geq 0)$, then $y(0)$ is

A. $\tan^{-1}(n)$

B. $\tan^{-1}(2n)$

C. $2\tan^{-1}(n)$

D. 0

Ans: $\tan^{-1}(2n)$

Solution: $y = \tan^{-1}\frac{1}{1+x+x^2} + \tan^{-1}\frac{1}{x^2+3x+3} + \ldots + 2n$ terms

$= \tan^{-1}\frac{(x+1)-x}{1+x(1+x)} + \tan^{-1}\frac{(x+2)-(x+1)}{1+(x+1)(x+2)} + \ldots\ldots$(2n terms)

$= \tan^{-1}(x+1) - \tan^{-1}x + \tan^{-1}(x+2) - \tan^{-1}(x+1) + \ldots + \tan^{-1}(x+2n) - \tan^{-1}(x+(2n-1))$

$= \tan^{-1}(x+2n) - \tan^{-1}x$

$y(0) = \tan^{-1}(2n)$

Q24. The value of the expression $\cot^{-1}\frac{1}{2} + \cot^{-1}\frac{9}{2} + \cot^{-1}\frac{25}{2} + \cot^{-1}\frac{49}{2} +$ .... upto n terms is

A. $\tan^{-1}2n$

B. $\tan^{-1}(2n-1)$

C. $\tan^{-1}n$

D. $\tan^{-1}2n - \tan^{-1}1$

Ans: $\tan^{-1}2n$

Solution:

Given expression $= \tan^{-1} 2 + \tan^{-1} \frac{2}{9} + \tan^{-1} \frac{2}{25} + \tan^{-1} \frac{2}{49} + \ldots\ldots$

General term $= \frac{2}{(2n-1)^2} = \frac{2}{4n^2-4n+1} = \frac{2}{1+4n(n-1)} = \frac{2n-(2n-2)}{1+2n(2n-2)}$

$T_n = \tan^{-1} 2n - \tan^{-1}(2n-2)$

$\therefore$ Sum of the series

$= \tan^{-1} 2 - \tan^{-1} 0 + \tan^{-1} 4 - \tan^{-1} 2 + \tan^{-1} 6 - \tan^{-1} 4 + \ldots \tan^{-1} 2\text{n} - \tan^{-1}(2\text{n} - 2)$

$= \tan^{-1} 2n - \tan^{-1} 0 = \tan^{-1} 2n$

Q25. The value(s) of x satisfying the equation $\sin^{-1}(1-x) - 2\sin^{-1} x = \frac{\pi}{2}$ is/are

A. 0

B. $\frac{1}{2}$

C. $0, \frac{1}{2}$

D. $-\frac{1}{2}$

Ans: 0

Solution: We have,

$\sin^{-1}(1-x) - 2\sin^{-1} x = \frac{\pi}{2}$

$\Rightarrow \sin^{-1}(1-x) = \frac{\pi}{2} + 2\sin^{-1} x$

$\Rightarrow 1 - x = \sin\left(\frac{\pi}{2} + 2\sin^{-1} x\right)$

$\Rightarrow 1 - x = \cos\left(2\sin^{-1} x\right)$

$\Rightarrow 1 - x = \cos\left\{\cos^{-1}\left(1 - 2x^2\right)\right\} \left[\because 2\sin^{-1} x = \cos^{-1}\left(1 - 2x^2\right)\right]$

$\Rightarrow 1 - x = \left(1 - 2x^2\right)$

$\Rightarrow x = 2x^2 \Rightarrow x(2x-1) = 0 \Rightarrow x = 0, \frac{1}{2}$

For, $x = \frac{1}{2}$, we have

LHS $=\sin^{-1}(1-x)-2\sin^{-1}x$

$=\sin^{-1}\frac{1}{2}-2\sin^{-1}\frac{1}{2}=-\sin^{-1}\frac{1}{2}=\frac{-\pi}{6}\neq \text{R.H.S.}$

So, $x=\frac{1}{2}$ is not a root of the given equation.

Clearly, $x=0$ satisfies the equation

Here, $x=0$ is the root of the given equation.

Q26. Solution set of $\left[\sin^{-1}x\right]>\left[\cos^{-1}x\right]$, where [.] denotes the greatest integer function, is

A. $\left[\frac{1}{\sqrt{2}},1\right]$

B. $(\cos 1,\ \sin 1)$

C. $[\sin 1,\ 1]$

D. None of these

Ans: $[\sin 1,\ 1]$

Solution: $\because\ \left[\sin^{-1}x\right]>\left[\cos^{-1}x\right]$

$\Rightarrow\quad x>0$

Here, $\left[\cos^{-1}\ x\right]=\begin{cases}0, & x\in(\cos 1,\ 1]\\ 1, & x\in(0\ ,\ \cos 1]\end{cases}$

and $\left[\sin^{-1}\ x\right]=\begin{cases}0\ , & x\in(0,\ \sin 1\ )\\ 1, & x\in[\sin 1,\ 1]\end{cases}$

$\therefore\quad x\in[\sin 1,\ 1]$

Q27. If $\cot^{-1}(\alpha)=\cot^{-1}2+\cot^{-1}8+\cot^{-1}18+\cot^{-1}32+\ldots.$ upto 100 terms, then α is:

A. 1.01

B. 1.00

C. 1.02

D. 1.03

Ans: 1.01

Solution:

Given $\cot^{-1}(\alpha)=\cot^{-1}(2)+\cot^{-1}(8)+\cot^{-1}(18)+\ldots$

$=\sum_{n=1}^{100}\tan^{-1}\left(\frac{2}{4n^2}\right)$

$=\sum_{n=1}^{100}\tan^{-1}\left(\frac{(2n+1)-(2n-1)}{1+(2n+1)(2n-1)}\right)$

$=\sum_{n=1}^{100}\tan^{-1}(2n+1)-\tan^{-1}(2n-1)$

$=\tan^{-1}201-\tan^{-1}1$

$=\tan^{-1}\left(\frac{200}{202}\right)$

$\therefore \cot^{-1}\left(\alpha\right)=\cot^{-1}\left(\frac{202}{200}\right)$

$\alpha=1.01$

Q28. For $k\in\mathbb{R}$, let the solutions of the equation $\cos\left(\sin^{-1}\left(x\cot\left(\tan^{-1}\left(\cos\left(\sin^{-1}x\right)\right)\right)\right)\right)=k, 0<|x|<\frac{1}{\sqrt{2}}$ be α and β, where the inverse trigonometric functions take only principal values. If the solutions of the equation $x^2-bx-5=0$ are $\frac{1}{\alpha^2}+\frac{1}{\beta^2}$ and $\frac{\alpha}{\beta}$, then $\frac{b}{k^2}$ is equal to ______.

Ans: 12

Solution:

Given,

$\cos\left(\sin^{-1}\left(x\cot\left(\tan^{-1}\left(\cos\left(\sin^{-1}x\right)\right)\right)\right)\right)=k$

Now simplifying $\cos\left(\sin^{-1}x\right)=\cos\left(\cos^{-1}\sqrt{1-x^2}\right)=\sqrt{1-x^2}$

So, $\cos\left(\sin^{-1}\left(x\cot\left(\tan^{-1}\left(\cos\left(\sin^{-1}x\right)\right)\right)\right)\right)=k$

becomes $\cos\left(\sin^{-1}\left(x\cot\left(\tan^{-1}\sqrt{1-x^2}\right)\right)\right)=k$

And now solving $\cot\left(\tan^{-1}\sqrt{1-x^2}\right)=\cot\cot^{-1}\left(\sqrt{\frac{1}{\sqrt{1-x^2}}}\right)=\frac{1}{\sqrt{1-x^2}}$

So, $\cos\left(\sin^{-1}\left(x\cot\left(\tan^{-1}\sqrt{1-x^2}\right)\right)\right)=k$ becomes

$\cos\left(\sin^{-1}\left(\frac{x}{\sqrt{1-x^2}}\right)\right)=k$

Now solving $\cos\left(\sin^{-1}\left(\frac{x}{\sqrt{1-x^2}}\right)\right)=\frac{\sqrt{1-2x^2}}{\sqrt{1-x^2}}$

So, $\frac{\sqrt{1-2x^2}}{\sqrt{1-x^2}}=k$

$\Rightarrow 1-2x^2=k^2\left(1-x^2\right)$

$\Rightarrow\left(k^2-2\right)x^2=k^2-1$

$\Rightarrow x^2=\frac{k^2-1}{k^2-2}$

So, roots are $\alpha=\sqrt{\frac{k^2-1}{k^2-2}}\Rightarrow\alpha^2=\frac{k^2-1}{k^2-2}$

And $\beta=\sqrt{\frac{k^2-1}{k^2-2}}\Rightarrow\beta^2=\frac{k^2-1}{k^2-2}$

Now finding $\frac{1}{\alpha^2}+\frac{1}{\beta^2}=2\left(\frac{k^2-2}{k^2-1}\right)$ and $\frac{\alpha}{\beta}=-1$

So, sum of roots of $x^2-bx-5=0$ will be $=\frac{1}{\alpha^2}+\frac{1}{\beta^2}+\frac{\alpha}{\beta}=b$

$\Rightarrow \frac{2\left(k^2-2\right)}{k^2-1}-1=b \quad \cdots(1)$

Product of roots of $x^2-bx-5=0$ will be $=\left(\frac{1}{\alpha^2}+\frac{1}{\beta^2}\right)\frac{\alpha}{\beta}=-5$

$\Rightarrow \frac{2\left(k^2-2\right)}{k^2-1}(-1)=-5$

$\Rightarrow 2k^2-4=5k^2-5$

$\Rightarrow 3k^2=1 \Rightarrow k^2=\frac{1}{3} \quad \cdots \text{Put in } (1)$

$\Rightarrow b=\frac{2\left(k^2-2\right)}{k^2-1}-1=5-1=4$

$\frac{b}{k^2}=\frac{4}{\frac{1}{3}}=12$

Q29. The number of solutions of the equation $\sin^{-1}x=(\sin x)^{-1}$ is/are

A. one

B. two

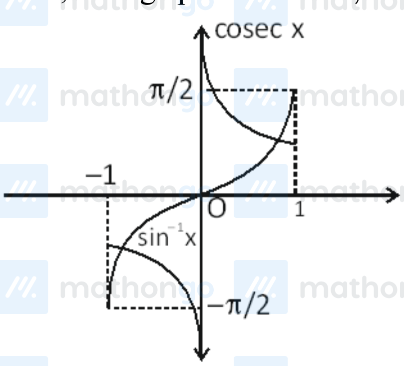
C. three

D. zero

Ans: two

Solution: $\because (\sin x)^{-1}=\frac{1}{\sin x}=\operatorname{cosec} x$

Now, from graphs of $\sin^{-1} x, \operatorname{cosec} x$



Clearly, both graph intersects at two points

$\therefore$ two solutions

Q30. The number of real solutions (x, y) where $|y| = \sin x, y = \cos^{-1}(\cos x), -2\pi \leq x \leq 2\pi$, is

A. 2

B. 1

C. 3

D. 4

E. 0

Ans: 2

Solution:

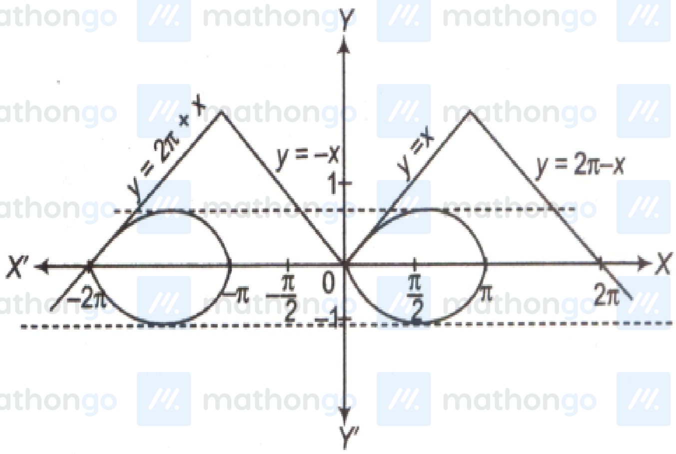
In $[0, \pi],\ |y| = \sin x,\ y = \cos^{-1}(\cos x) = x$

In $[\pi, 2\pi],\ |y| = \sin x,\ y = \cos^{-1}\{\cos(2\pi - x)\} = 2\pi - x$

In $[-\pi, 0],\ |y| = \sin x,\ y = \cos^{-1}\{\cos(-x)\} = -x$

In $[-2\pi, -\pi],\ |y| = \sin x,\ y = \cos^{-1}\{\cos(2\pi + x)\} = 2\pi + x$

Plotting the graphs, we have



There are 2 solutions, i.e., $(0,0)$ and $(-2\pi, 0)$.

Answer Key

Q1 (4)	**Q2** (3)	**Q3** (4)	**Q4** (1)
Q5 (0.6)	**Q6** (3)	**Q7** (4)	**Q8** (4)
Q9 (2)	**Q10** (3)	**Q11** (1)	**Q12** (1)
Q13 (2)	**Q14** (4)	**Q15** (4)	**Q16** (3)
Q17 (1)	**Q18** (3)	**Q19** (1)	**Q20** (3)
Q21 (2)	**Q22** (4)	**Q23** (2)	**Q24** (1)
Q25 (1)	**Q26** (3)	**Q27** (1)	**Q28** (12)
Q29 (2)	**Q30** (1)		