Q1. $\lim_{x \to \frac{\pi}{2}} \frac{\left[1-\tan\left(\frac{x}{2}\right)\right][1-\sin x]}{\left[1+\tan\left(\frac{x}{2}\right)\right][\pi-2x]^3}$ **is equal to**

A. $\frac{1}{8}$

B. 0

C. $\frac{1}{32}$

D. ∞

Ans: $\frac{1}{32}$

Solution: $\lim_{x \to \frac{\pi}{2}} \frac{\tan\left(\frac{\pi}{4}-\frac{x}{2}\right)(1-\sin x)}{(\pi-2x)^3}$

Let, $x = \frac{\pi}{2} + y$

$\Rightarrow \lim_{y \to 0} = \frac{\tan\left(\frac{-y}{2}\right)(1-\cos y)}{(-2y)^3}$

$= \lim_{y \to 0} \frac{-\tan\frac{y}{2} \cdot 2\sin^2\frac{y}{2}}{(-8)y^3} = \lim_{y \to 0} \frac{1}{32} \frac{\tan\frac{y}{2}}{\left(\frac{y}{2}\right)} \cdot \left[\frac{\sin\frac{y}{2}}{\frac{y}{2}}\right]^2$

$= \lim_{y \to 0} \frac{1}{32} \times \frac{\tan\frac{y}{2}}{\left(\frac{y}{2}\right)} \cdot \left(\lim_{y \to 0} \frac{\sin\frac{y}{2}}{\frac{y}{2}}\right)^2$

$\because \lim_{x \to 0} \frac{\sin x}{x} = \lim_{x \to 0} \frac{\tan x}{x} = 1$

$\Rightarrow \frac{1}{32} \times 1 \times 1^2$

$= \frac{1}{32}$

Q2. The value of $\lim_{x \to 0} \frac{1-\cos^3(\sin x)}{\sin x \sin(\sin x) \cos(\sin x)}$ **is equal to**

A. $\frac{3}{2}$

B. 1

C. 0

D. 2

Ans: $\frac{3}{2}$

Solution: Let, $\sin x = t$

$\Rightarrow \lim_{t \to 0} \frac{1-\cos^3 t}{t \sin t \cos t}$

$\Rightarrow \lim_{t \to 0} \frac{(1-\cos t)}{t^2} \times \left(\frac{t}{\sin t}\right) \times \frac{(1+\cos t+\cos^2 t)}{\cos t}$

$\Rightarrow \frac{1}{2} \times 1 \times \frac{(1+1+1)}{1} = \frac{3}{2}$

Q3. The value of $\lim_{x \to -\infty} \frac{x^2 \tan\left(\frac{1}{x}\right)}{\sqrt{4x^2-x+1}}$ **is equal to**

A. 1

B. $\frac{1}{2}$

C. -1

D. $-\frac{1}{2}$

Ans: $-\frac{1}{2}$

Solution: $\lim_{x \rightarrow-\infty} \frac{x^{\frac{\tan \left(\frac{1}{x}\right)}{\left(\frac{1}{x}\right)}}}{-x \sqrt{4-\frac{1}{x}+\frac{1}{x^2}}}=\frac{1}{-\sqrt{4}}=-\frac{1}{2}$

Q4.

$\lim\limits_{x \rightarrow 0} \frac{\tan x \sqrt{\tan x}-\sin x \sqrt{\sin x}}{x^3 \sqrt{x}}$ **equals**

A. $\frac{1}{4}$

B. $\frac{3}{4}$

C. $\frac{1}{2}$

D. 1

Ans: $\frac{3}{4}$

Solution:

Let, $L=\lim\limits_{x \rightarrow 0} \frac{\tan x \sqrt{\tan x}-\sin x \sqrt{\sin x}}{x^3 \sqrt{x}}$

$=\lim\limits_{x \rightarrow 0} \frac{\left(\tan x\right)^{\frac{3}{2}}\left[1-\left(\cos x\right)^{\frac{3}{2}}\right]}{x^{3 / 2} \cdot x^2}$

$=1^{\frac{3}{2}} \cdot \lim\limits_{x \rightarrow 0} \frac{1-\cos ^3 x}{x^2} \cdot \frac{1}{1+\left(\cos x\right)^{\frac{3}{2}}}$ (Rationalizing)

$=\lim\limits_{x \rightarrow 0} \frac{1-\cos x}{x^2} \cdot\left(1+\cos x+\cos ^2 x\right) \cdot \frac{1}{1+(\cos x)^{\frac{3}{2}}}$

$=\frac{1}{2} \cdot \frac{1}{2}(1+1+1)=\frac{3}{4}$.

Q5. $\lim\limits_{x \rightarrow 0} \frac{(1-\cos x)(3+\cos 2 x)}{x \cdot \tan 2 x}=$

A. 0

B. 1

C. $\frac{1}{2}$

D. -1

Ans: 1

Solution:

$\lim_{x \to 0} \frac{1-\cos x}{x^2} \cdot \frac{(3+\cos 2x)}{1} \cdot \frac{2x}{\tan 2x} \cdot \frac{1}{2}$

$= \frac{1}{2} \cdot \left(4\right) \cdot \frac{1}{2} = 1$

Q6. The value of $\lim_{x \to 0} \frac{1-\cos^3 x}{x \sin x \cos x}$ is

A. $\frac{2}{5}$

B. $\frac{3}{5}$

C. $\frac{3}{2}$

D. $\frac{3}{4}$

Ans: $\frac{3}{2}$

Solution:

$\lim_{x \to 0} \frac{1-\cos^3 x}{x \sin x \cos x}$

$= \lim_{x \to 0} \frac{(1-\cos x)\left(1+\cos x+\cos^2 x\right)}{x \sin x \cos x}$

$= \lim_{x \to 0} \frac{2 \sin^2 \left(\frac{x}{2}\right)}{x \cdot 2 \sin \left(\frac{x}{2}\right) \cos \left(\frac{x}{2}\right)} \times \frac{\left(1+\cos x+\cos^2 x\right)}{\cos x}$

$= \lim_{x \to 0} \frac{\sin \left(\frac{x}{2}\right)}{2\left(\frac{x}{2}\right)} \times \frac{1+\cos x+\cos^2 x}{\cos \left(\frac{x}{2}\right) \cos x} = \frac{1}{2} \times 3 = \frac{3}{2}$

Q7. If $\lim_{x \to 0}\left(x^{-3} \sin 3x + ax^{-2} + b\right)$ exists and is equal to 0, then

A. $\mathrm{a} = -3$ and $\mathrm{b} = 9/2$

B. $a = 3$ and $b = 9/2$

C. $a = -3$ and $b = -9/2$

D. $a = 3$ and $b = -9/2$

Ans: $\mathrm{a} = -3$ and $\mathrm{b} = 9/2$

Solution:

$\lim_{x \to 0}\left(\frac{\sin 3x}{x^3} + \frac{a}{x^2} + b\right) = \lim_{x \to 0} \frac{\sin 3x + ax + bx^3}{x^3}$

$= \lim_{x \to 0} \frac{3\frac{\sin 3x}{3x} + a + bx^2}{x^2}$

limit exists if

$3 + a = 0$

or $a = -3$

$\therefore \quad L = \lim_{x \to 0} \frac{\sin 3x - 3x + bx^3}{x^3} = 27\left(\lim_{t \to 0} \frac{\sin t - t}{t^3} + b\right) = 0 \quad \left(\text{Putting } 3x = t\right) = -\frac{27}{6} + b = 0$

or $b=\frac{9}{2}$

Q8. If $f(x)$ is a differentiable function such that $f'(1)=4$ and $f'(4)=\frac{1}{2}$,, then value of $\lim_{x\to 0}\frac{f\left(x^2+x+1\right)-f(1)}{f\left(x^4-x^2+2x+4\right)-f(4)}$ is :-

A. 8

B. 16

C. 4

D. Does not exist

Ans: 4

Solution:

$\lim_{x\to 0}\frac{f\left(x^2+x+1\right)-f(1)}{f\left(x^4-x^2+2x+4\right)-f(4)}$ $\left(\frac{0}{0}\text{ form}\right)$

$\lim_{x\to 0}\frac{(2x+1)f^1\left(x^2+x+1\right)}{\left(4x^3-2x+2\right)f^1\left(x^4-x^2+2x+4\right)}$

$=\frac{f'(1)}{2f'(4)}=4$ (Applying L' Hospital's Rule)

Q9. If $\underset{x\to\infty}{lim}\left(\sqrt{x^2+x+2}-ax-b\right)=2$, then equation of circle whose centre is $(a,\ 2b)$ and radius 1 unit is

A. $x^2+y^2+2x+6y+9=0$

B. $x^2+y^2-2x+6y+1=0$

C. $x^2+y^2-2x+6y+9=0$

D. none of these

Ans: $x^2+y^2-2x+6y+9=0$

Solution:

$\underset{x\to\infty}{lim}\left(\sqrt{x^2+x+2}-ax-b\right)=2$

$a=1;b=\frac{-3}{2}$ required equation is $(x-1)^2+(y+3)^2=1$

Q10. For a positive integer m, if $\lim_{x\to\infty}\left(x^3\ \ln\left(\frac{x+1}{x}\right)+\frac{x}{2}-x^2\right)=\frac{1}{m}$. Then the value of m is

A. 1

B. 2

C. 3

D. 4

Ans: 3

Solution:

$$\lim_{x\to\infty} x^3 \ \ln\left(1+\frac{1}{x}\right)+\frac{x}{2}-x^2$$

$$x=\frac{1}{t}$$

$$\lim_{t\to0}\left(\frac{\ln(1+t)}{t^3}+\frac{1}{2t}-\frac{1}{t^2}\right)=\lim_{t\to0}\frac{2\ \ln(1+t)+t^2-2t}{2t^3}$$

$$=\lim_{t\to0}\frac{2\left(t-\frac{t^2}{2}+\frac{t^3}{3}-\frac{t^4}{4}+\ldots\right)+t^2-2t}{2t^3}$$

$$=\lim_{t\to0}\left(\frac{1}{3}-\frac{t}{4}+\frac{t^2}{5}\ldots\ldots\right)=\frac{1}{3}=\frac{1}{m}\Rightarrow m=3.$$

Q11. The value of $\lim_{n\to\infty}\frac{3.2^{n+1}-4.5^{n+1}}{5.2^n+7.5^n}$ is equal to

A. $\frac{3}{5}$

B. $-\frac{4}{7}$

C. $-\frac{20}{7}$

D. 0

Ans: $-\frac{20}{7}$

Solution: $\lim_{n\to\infty}\frac{3.2^{n+1}-4.5^{n+1}}{5.2^n+7.5^n}$

$$=\lim_{n\to\infty}\frac{5^n\left(6.\left(\frac{2}{5}\right)^n-20\right)}{5^n\left(5.\left(\frac{2}{5}\right)^n+7\right)}=-\frac{20}{7}\left(\because\lim_{n\to\infty}\left(\frac{2}{5}\right)^n=0\right)$$

Q12. The value of $\lim_{x\to0}\frac{\ln(2-\cos 15x)}{\ln^2(\sin 3x+1)}$ is equal to

Ans: 12.5

Solution:

$$\lim_{x\to0}\frac{\ln(2-\cos 15x)}{\ln^2(\sin 3x+1)}$$

$$=\lim_{x\to0}\frac{\ln\{1+(1-\cos 15x)\}}{\ln^2(1+\sin 3x)}$$

$$=\lim_{x\to0}\frac{1-\cos 15x}{(\sin 3x)^2}\ (\text{Applying}\lim_{x\to0}\frac{\ln(1+x)}{x}=1)$$

$$=\lim_{x\to0}\frac{(15x)^2}{2(3x)^2}\ \ (\text{using standard limit})$$

$$=\frac{1(225)}{2\times9}=12.5$$

Q13. Let $f:\mathbb{R}\rightarrow\mathbb{R}$ be a function defined as $f(x)=a\sin\left(\frac{\pi[x]}{2}\right)+[2-x], a\in\mathbb{R}$, where $[t]$ is the greatest integer less than or equal to t. If $\lim\limits_{x\rightarrow-1} f(x)$ exists, then the value of $\int_0^4 f(x)dx$ is equal to

A. -1

B. -2

C. 1

D. 2

Ans: -2

Solution:

Given,

$f(x)=a\sin\left(\frac{\pi[x]}{2}\right)+[2-x], a\in\mathbb{R}$

Now given $\lim\limits_{x\rightarrow-1} f(x)$ exists,

So $\lim\limits_{x\rightarrow-1^{+}} a\sin\left(\pi\frac{[x]}{2}\right)+[2-x]=-a+2$

And $\lim\limits_{x\rightarrow-1^{-}} \text{asin}\left(\pi\frac{[x]}{2}\right)+[2-x]=0+3=3$

So, $\lim\limits_{x\rightarrow-1} f(x)$ exist when $-a+2=3\Rightarrow a=-1$

Now,

$\int_0^4 f(x)dx=\int_0^1 f(x)dx+\int_1^2 f(x)dx+\int_2^3 f(x)dx+\int_3^4 f(x)dx$

$\Rightarrow \int_0^4 f(x)dx=\int_0^1 -\sin\left(\frac{\pi[x]}{2}\right)+[2-x]dx+\int_1^2 -\sin\left(\frac{\pi[x]}{2}\right)+[2-x]dx+\int_2^3 -\sin\left(\frac{\pi[x]}{2}\right)+[2-x]dx+\int_3^4 -\sin\left(\frac{\pi[x]}{2}\right)+[2-x]dx$

$=\int_0^1(0+1)dx+\int_1^2(-1+0)dx+\int_2^3(0-1)dx+\int_3^4(1-2)dx$

$=1-1-1-1=-2$

Q14. The integer n for which $\lim_{x\rightarrow0}\frac{(\cos x-1)(\cos x-e^x)}{x^n}$ is a finite nonzero number is

A. 1

B. 2

C. 3

D. 4

Ans: 3

Solution: $\frac{(\cos x-1)(\cos x-e^x)}{x^n}$

$$= \frac{1}{x^n}\left(\left(1-\frac{x^2}{2!}+\frac{x^4}{4!}-\ldots\right)-1\right)\left(\left(1-\frac{x^2}{2!}+\frac{x^4}{4!}-\ldots\right)-\left(1+x+\frac{x^2}{2!}+\frac{x^3}{3!}\ldots\right)\right)$$

$$= \frac{1}{x^n}\left[\left(-\frac{x^2}{2!}+\frac{x^4}{4!}-\ldots\right)\left(-x-x^2-\frac{x^3}{3!}\ldots\right)\right]$$

$$= \frac{-1}{x^{n-3}}\left[\left(-\frac{1}{2!}+\frac{x^2}{4!}-\ldots\right)\left(1+x+\frac{x^2}{3!}\ldots\right)\right]$$

$\therefore$ For $\lim_{x\to 0}\frac{(\cos x-1)(\cos x-e^x)}{x^n}$ to exist as a nonzero number we must have $n-3=0 \Rightarrow n=3$.

Q15. If the largest value of the $\lim\limits_{x\to\infty}\left(1+\frac{a}{x}\right)^{\frac{x}{b}}$ where $a,\, b$ lies in the interval $\left[\frac{1}{5},\, 403\right]$ is e^λ, then λ equals

A. 2015

B. 2016

C. 2017

D. 2018

Ans: 2015

Solution:

$\lim\limits_{x\to\infty}\left(1+\frac{a}{x}\right)^{\frac{x}{b}} = e^{\lim_{x\to\infty}\frac{x}{b}\left(1+\frac{a}{x}-1\right)}$

$[\because 1^\infty \text{ form}]$

$= e^{\frac{a}{b}} = e^{\frac{403\times 5}{1}} = e^{2015} \equiv e^\lambda \Rightarrow \lambda = 2015.$

Q16. $\operatorname{Lim}\limits_{n\to\infty}\left(\frac{2n^2-3}{2n^2-n+1}\right)^{\frac{n^2-1}{n}}$ is equal to

A. $\frac{1}{\sqrt{e}}$

B. $\sqrt{e}$

C. e

D. $\frac{1}{e}$

Ans: $\sqrt{e}$

Solution:

$\mathrm{L} = e^l$

$l = \operatorname{Lim}\limits_{n\to\infty}\frac{n^2-1}{n}\left[\frac{2n^2-3-2n^2+n-1]}{2n^2-n+1}\right]$

$= \operatorname{Lim}\limits_{n\to\infty}\frac{n^2\left(1-\frac{1}{n^2}\right)-n\left(1-\frac{4}{n}\right)}{n\cdot n^2\left(2-\frac{1}{n}+\frac{1}{n^2}\right)} = \frac{1}{2}$

$\therefore \mathrm{L}=\sqrt{e}$.

Q17. $\lim_{x \rightarrow 2} \frac{3^x+3^{3-x}-12}{3^{-\frac{x}{2}}-3^{1-x}}$ **is equal to**

Ans: 36

Solution: Let, $3^{\frac{x}{2}}=t, x \rightarrow 2 \Rightarrow t \rightarrow 3$

$\lim_{t \rightarrow 3} \frac{t^{2^{\frac{2}{t^2}}}-12}{\frac{1}{t}-\frac{3}{t^2}}=\lim_{t \rightarrow 3} \frac{t^4+27-12 t^2}{t-3}$

$\lim_{t \rightarrow 3} \frac{\left(t^2-3\right)(t+3)(t-3)}{(t-3)}=6 \times 6=36$

Q18. The value of $\lim_{x \rightarrow 0}\left(\frac{1+\tan x}{1+\sin x}\right)^{\frac{2}{\sin x}}$ **is equal to**

A. 0

B. 1

C. -1

D. None of these

Ans: 1

Solution: Let, $f(x)=\frac{1+\tan x}{1+\sin x}$

and $g(x)=\frac{2}{\sin x}$

Clearly, $f(x) \rightarrow 1$ and $g(x) \rightarrow \infty$ as $x \rightarrow 0$

$\therefore \lim_{x \rightarrow 0}\left(\frac{1+\tan x}{1+\sin x}\right)^{\frac{2}{\sin x}}=e^{\lim_{x \rightarrow 0} \frac{2}{\sin x}\left(\frac{1+\tan x}{1+\sin x}-1\right)}$

$\left\{\text{ using } \lim_{x \rightarrow a}[f(x)]^{g(x)}=e^{\lim_{x \rightarrow a} g(x)[f(x)-1]} \text{ for } 1^{\infty} \text{ form }\right\}$

$=e^{\lim_{x \rightarrow 0} \frac{2}{\sin x}\left(\frac{\tan x-\sin x}{1+\sin x}\right)}=e^{\lim_{x \rightarrow 0} \frac{2(1-\cos x)}{\cos x(1+\sin x)}}$

$=e^0=1$

Q19. The value of $\lim\limits_{x \rightarrow 0^{+}}((x \cot x)+(x \ln x))$ **is equal to**

A. 1

B. 2

C. 3

D. 0

Ans: 1

Solution: $\lim\limits_{x \rightarrow 0^{+}} x \cot x+\lim\limits_{x \rightarrow 0^{+}} x \ln x$

$=\lim\limits_{x \rightarrow 0^{+}} \frac{x}{\tan x}+\lim\limits_{x \rightarrow 0^{+}} \frac{\ln x}{\left(\frac{1}{x}\right)}$

$= \lim_{x \to 0^+} \frac{x}{\tan x} + \lim_{x \to 0^+} \frac{\left(\frac{1}{x}\right)}{\left(-\frac{1}{x^2}\right)} = 1 + \lim_{x \to 0^+} (-x)$

$= 1 + 0 = 1$

Q20. The value of $\lim_{x \to \frac{\pi}{2}} \frac{\left[\frac{x}{3}\right]}{\ln(\sin x)}$ (where, $[.]$ denotes the greaest integer function)

A. does not exist

B. is equal to 1

C. is equal to 0

D. is equal to -1

Ans: is equal to 0

Solution: $\because \frac{\pi}{6} < 1$,

$\therefore \left[\frac{\pi}{6}\right] = 0$

$\therefore \lim_{x \to \frac{\pi}{2}} \frac{\left[\frac{x}{2}\right]}{\ln(\sin x)} = 0$

Q21. The $\lim_{x \to 0} x^8 \left[\frac{1}{x^3}\right]$ (where $[x]$ is greatest integer function) is (Mark incorrect option)

A. a nonzero real number

B. a rational number

C. an integer

D. zero

Ans: a nonzero real number

Solution: Since $x - 1 \leq [x] \leq x$ for all $x \in \mathbf{R}$ so

$$\frac{1}{x^3} - 1 \leq \left[\frac{1}{x^3}\right] \leq \frac{1}{x^3}$$

$\Rightarrow \quad x^8 \left(\frac{1}{x^3} - 1\right) \leq x^5 \left[\frac{1}{x^3}\right] \leq x^5$ for all x

But $\lim_{x \to 0} x^5 = 0 = \lim_{x \to 0} \left(x^5 - x^8\right)$ so

$\lim_{x \to 0} x^8 \left[\frac{1}{x^3}\right] = 0 \in \mathbf{I} \subseteq \mathbf{Q}$

Q22. If $\lim_{x \to 0} \frac{\sin 2x - a \sin x}{x^3}$ exists finitely, then the value of a is

A. 0

B. 2

C. 1

D. 4

Ans: 2

Solution: $\lim_{x \to 0} \frac{\sin x(2\cos x - a)}{x \cdot x^2} = \lim_{x \to 0}\left(\frac{\sin x}{x}\right)\left(\frac{2\cos x - a}{x^2}\right)$

For this limit to exists finitely,

$\lim_{x \to 0} \frac{2\cos x - a}{x^2} = \text{finite}$

$\therefore$ It must be $\frac{0}{0}$ form

$\therefore 2\cos(0) - a = 0 \Rightarrow a = 2$

Q23.

The value of $\lim_{x \to 0} \frac{\sin x}{3}\left[\frac{5}{x}\right]$ is equal to

(where, $[.]$ represents the greatest integer function)

A. $\frac{1}{3}$

B. 0

C. $\frac{5}{3}$

D. 1

Ans: $\frac{5}{3}$

Solution: $\frac{5}{x} - 1 < \left[\frac{5}{x}\right] \le \frac{5}{x}$

$\underbrace{\frac{\sin x}{3}\left(\frac{5}{x} - 1\right)}_{h(x)} < \underbrace{\frac{\sin x}{3}\left[\frac{5}{x}\right]}_{f(x)} \le \underbrace{\frac{\sin x}{3}\left(\frac{5}{x}\right)}_{g(x)}$

by sandwich theorem

$\because \lim_{x \to 0} g(x) = \lim_{x \to 0} h(x) = \frac{5}{3}$

$\therefore \lim_{x \to 0} f(x) = \frac{5}{3}$

Q24.

The value of the limit

$$\lim_{n \to \infty} n^2 \left\{\sqrt{\left(1 - \cos\frac{1}{n}\right)\sqrt{\left(1 - \cos\frac{1}{n}\right)\sqrt{\left(1 - \cos\frac{1}{n}\right) \ldots\ldots \infty}}}\right\} \text{ is}$$

A. $\frac{1}{2}$

B. -2

C. 2

D. $-\frac{1}{2}$

Ans: $\frac{1}{2}$

Solution:

Let the given expression be y.

Then, $y=\lim_{n\to\infty} n^2$

$$\left\{\sqrt{\left(1-\cos\frac{1}{n}\right)\sqrt{\left(1-\cos\frac{1}{n}\right)\sqrt{\left(1-\cos\frac{1}{n}\right)}}}\right.$$

On putting $\frac{1}{n}=\theta\ldots\ldots\infty$

So that, $n\to\infty\Rightarrow\theta\to 0$

Thus,

$$\mathrm{y}=\lim_{\theta\to 0}\frac{1}{\theta^2}\left(1-\cos\theta\right)^{\frac{1}{2}+\frac{1}{2^2}+\frac{1}{2^3}+\ldots\text{to }\infty}$$

$$=\lim_{\theta\to 0}\left(\frac{1-\cos\theta}{\theta^2}\right)$$

$$\left\{\because\ \frac{1}{2}+\frac{1}{2^2}+\ldots\infty=1\right\}$$

$$=\lim_{\theta\to\infty}\frac{2\sin^2\ \theta/2}{\theta^2}$$

$$=\lim_{\theta\to 0}2\left(\frac{\sin\ \theta/2}{\theta/2}\right)^2\times\frac{1}{4}$$

$$=2\cdot 1^2\cdot\frac{1}{4}=\frac{1}{2}$$

Q25. The value of $\lim_{n\to\infty}\frac{[x]+[2^2x]+[3^2x]+\ldots..+[n^2x]}{1^2+2^2+3^2+\ldots.+n^2}$ is equal to (where $[x]$ represents the greatest integer part of x)

A. x

B. $2x$

C. $\frac{x}{2}$

D. $\frac{x}{6}$

Ans: x

Solution: Let $f(x)=\frac{[x]+[2^2x]+[3^2x]+\ldots+[n^2x]}{1^2+2^2+3^2+\ldots+n^2}$

Now, we have,

$f(x)\leq\frac{x+2^2x+3^2x+\ldots+n^2x}{1^2+2^2+3^2+\ldots+n^2}=x$

and, $f(x) > \frac{(x-1)+(2^2x-1)+(3^2x-1)+\ldots+(n^2x-1)}{1^2+2^2+3^2+\ldots+n^2}$

$= \frac{x\Sigma n^2 - n}{\Sigma n^2} = x - \frac{6}{(n+1)(2n+1)}$ $(\because x-1 \leq [x] < x,\ \forall x \in R)$

Thus, we have,

$x - \frac{6}{(n+1)(2n+1)} < f(x) \leq x$

Now, we have,

$\lim_{n\to\infty} x - \frac{6}{(n+1)(2n+1)} = x\ \ \&\ \lim_{n\to\infty} x = x$

Hence, by Sandwich Theorem, we have

$\lim_{n\to\infty} f(x) = x$

Q26. Let $\alpha,\ \beta \in R$ be such that $\lim_{x\to 0} \frac{x^2\tan(ax)}{\beta x - \tan(2x)} = 1$, then the value of $5\beta + 3\alpha$ is :

Ans: 2.00

Solution: $\lim_{x\to 0} \frac{x^2\tan(\alpha x)}{\beta x - \tan(2x)} = 1 \to \lim_{x\to 0} \frac{x^2\tan(\alpha x)}{\beta x - \left\{2x + \frac{(2x)^3}{3} + \frac{2(2x)^5}{15}\right\}} = 1$

$\to \beta = 2$ and $\frac{3\alpha}{8} = -1,$

So, $5\beta + 3\alpha = 2$

Q27. $\lim_{x\to 0} \frac{\cos(\sin x) - \cos x}{x^4}$ is equal to

A. $\frac{1}{3}$
B. $\frac{1}{6}$
C. $\frac{1}{4}$
D. $\frac{1}{12}$

Ans: $\frac{1}{6}$

Solution:

Consider $\lim_{x\to 0}\frac{\cos(\sin x)-\cos x}{x^4}\left(\frac{0}{0}\text{ form}\right)$

$$\lim_{x\to 0}\frac{2\sin\left(\frac{\sin x+x}{2}\right)\cdot\sin\left(\frac{x-\sin x}{2}\right)}{x^4}$$

$$=\lim_{x\to 0}2\left[\frac{\sin\left(\frac{\sin x+x}{2}\right)}{\left(\frac{\sin x+x}{2}\right)}\right]\left[\frac{\sin\left(\frac{x-\sin x}{2}\right)}{\left(\frac{x-\sin x}{2}\right)}\right]\times\left(\frac{\sin x+x}{2}\right)\times\left(\frac{x-\sin x}{2}\right)\times\frac{1}{x^4}$$

$$\lim_{x\to 0}2\left[\frac{\sin\left(\frac{\sin x+x}{2}\right)}{\left(\frac{\sin x+x}{2}\right)}\right]\left[\frac{\sin\left(\frac{x-\sin x}{2}\right)}{\left(\frac{x-\sin x}{2}\right)}\right]\times\left(\frac{x^2-\sin^2 x}{4x^4}\right)$$

$$\lim_{x\to 0}2\times\left(\frac{x^2-\sin^2 x}{4x^4}\right)\ \left(\frac{0}{0}\text{ form}\right)\quad\left(\because\ \lim_{t\to 0}\frac{\sin t}{t}=1\right)$$

Applying L'Hospital's Rule,

$$\lim_{x\to 0}2\times\left(\frac{2x-2\sin x\cos x}{4\cdot 4x^3}\right)=\lim_{x\to 0}\left(\frac{2x-\sin 2x}{8x^3}\right)\quad\left(\frac{0}{0}\text{ form}\right)$$

$$\lim_{x\to 0}\left(\frac{2-2\cos 2x}{24x^2}\right)\ \left(\frac{0}{0}\text{ form}\right)$$

$$\lim_{x\to 0}\left(\frac{4\sin 2x}{48x}\right)=\frac{1}{6}\lim_{x\to 0}\left(\frac{\sin 2x}{2x}\right)=\frac{1}{6}$$

Q28. The value of $\lim_{n\to\infty}\frac{[r]+[2r]+\ldots+[nr]}{n^2}$, where r is non-zero real number and $[r]$ denotes the greatest integer less than or equal to r, is equal to :

A. $\frac{r}{2}$
B. r
C. $2r$
D. 0

Ans: $\frac{r}{2}$

Solution:

We know that $r\leq[r]<r+1$

$2r\leq[2r]<2r+1$

$3r\leq[3r]<3r+1$

$\vdots\ \vdots\ \vdots$

$nr\leq[nr]<nr+1$

On adding all the above inequalities, we get

$r+2r+\ldots+nr \leq [r]+[2r]+\ldots+[nr] < r+2r+\ldots nr+n$

Using $1+2+3+\ldots+n=\frac{n(n+1)}{2},\ n \in N$,

$\frac{n(n+1)}{2}\cdot r \leq [r]+[2r]+\ldots+[nr] < \frac{n(n+1)}{2}\cdot r+n$

$\Rightarrow \frac{\frac{n(n+1)}{2}\cdot r}{n^2} \leq \frac{[r]+[2r]+\ldots+[nr]}{n^2} < \frac{\frac{n(n+1)}{2}\cdot r+n}{n^2}$

Now, $\lim_{n\to\infty}\frac{n(n+1)\cdot r}{2\cdot n^2}=\lim_{n\to\infty}\frac{n^2\left(1+\frac{1}{n}\right)\cdot r}{2\cdot n^2}=\frac{r}{2}$

and $\lim_{n\to\infty}\frac{\frac{n(n+1)}{2}\cdot r+n}{n^2}=\lim_{n\to\infty}\frac{n^2\left\{\left(1+\frac{1}{n}\right)\cdot r+\frac{1}{n}\right\}}{2\cdot n^2}=\frac{r}{2}$

So, by Sandwich Theorem, we can conclude that

$\lim_{n\to\infty}\frac{[r]+[2r]+\ldots+[nr]}{n^2}=\frac{r}{2}$.

Q29. The value of $\lim_{x\to 0}\frac{e^{-\frac{x^2}{2}}-\cos x}{x^3\tan x}$ is equal to

A. $\frac{1}{4}$

B. $\frac{1}{8}$

C. $\frac{1}{12}$

D. $\frac{1}{16}$

Ans: $\frac{1}{12}$

Solution: Using expansion,

$$\lim_{x\to 0}\frac{\left(1+\left(-\frac{x^2}{2}\right)+\frac{\left(\frac{x^4}{4}\right)}{2!}+\ldots\right)-\left(1-\frac{x^2}{2!}+\frac{x^2}{4!}-\ldots\right)}{x^3\left(x+\frac{x^3}{3}+\ldots\right)}$$

$$\lim_{x\to 0}\frac{(1-1)+\left(-\frac{x^2}{2}+\frac{x^2}{2}\right)+\left(\frac{x^4}{8}-\frac{x^4}{24}\right)+\ldots}{x^3\left(x+\frac{x^3}{3}+\ldots\right)}$$

$\Rightarrow \frac{1}{8}-\frac{1}{24}=\frac{1}{12}$

Q30. If $\lim_{x\to\infty}\frac{ae^x+b\cos x+c+dx}{x\sin^2 x}=3$, then the value of $272\frac{abd}{c^3}$ is equal to

Ans: 34

Solution:

Using expansions, we get,

$$\lim_{x\to 0}\frac{a\left(1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\ldots\right)+b\left(1-\frac{x^2}{2!}+\ldots\right)+c+dx}{x\left(x-\frac{x^3}{3!}+\ldots\right)^2}=3$$

$$\lim_{x\to 0}\frac{(a+b+c)+(a+d)x+\left(\frac{a-b}{2}\right)x^2+\frac{a}{6}x^3+\ldots}{x^3\left(1-\frac{x^2}{3!}+\ldots\right)^2}=3$$

$\because$ in the denominator lowest power of x is 3

For the limit to be finite, the numerator should also have the least power of x as 3

$\therefore a+b+c=0 \quad \ldots(1)$

$a+d=0\ldots(2)$

$\frac{a-b}{2}=0\ldots(3)$

Now, $\frac{\left(\frac{a}{6}\right)}{1}=3\Rightarrow a=18$

From $(1),(2),(3)$, we get,

$a=18,\ b=18,\ c=-36,\ d=-18$

$\frac{abd}{c^3}=\frac{-(18)^3}{-8(18)^3}=\frac{1}{8}$

Answer Key

Q1 (3)	**Q2** (1)	**Q3** (4)	**Q4** (2)
Q5 (2)	**Q6** (3)	**Q7** (1)	**Q8** (3)
Q9 (3)	**Q10** (3)	**Q11** (3)	**Q12** (12.5)
Q13 (2)	**Q14** (3)	**Q15** (1)	**Q16** (2)
Q17 (36)	**Q18** (2)	**Q19** (1)	**Q20** (3)
Q21 (1)	**Q22** (2)	**Q23** (3)	**Q24** (1)
Q25 (1)	**Q26** (2.00)	**Q27** (2)	**Q28** (1)
Q29 (3)	**Q30** (34)		