MA 202 Project

Precession Of Mercury

Anushk Bhana
21110031

Ayush Modi
21110039

Kaushal Kothiya
211100107

Ameya Tajne
21110220

Vedant Kumbhar
21110234

April 26, 2023

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1 Preface

This report presents a detailed analysis of the phenomenon of precession on the planet Mercury, which is a fascinating topic in planetary science. Mercury, the innermost planet of our solar system, has a highly elliptical orbit around the sun, and is subject to a range of gravitational forces from other planets in the solar system.

One of the most intriguing aspects of Mercury's orbit is its precession, which refers to the slow rotation of the planet's orbit around the sun. This precession is caused by the gravitational forces of other planets in the solar system, as well as the curvature of spacetime near the massive sun. The precession of Mercury has been a subject of study for over a century, and was famously used as a key test of Einstein's theory of general relativity.

This report will explore the mathematical models used to describe the precession of Mercury, including the Runge-Lenz Vector, The Leapfrog Method and The simulation of differential precession. The report will also examine the latest observational data on the precession of Mercury, and the implications of these findings for our understanding of the fundamental laws of physics.

Overall, this report will provide a comprehensive overview of the precession of Mercury, and its significance for our understanding of the dynamics of the solar system and the nature of spacetime itself.

2 Motivation

The study of planetary motion and celestial mechanics has been a subject of fascination for scientists and astronomers for centuries. The precession of Mercury, in particular, has captivated the imagination of scientists and inspired them to develop new mathematical models and theories to explain the phenomenon.

Understanding the precession of Mercury has far-reaching implications, not only for our understanding of the solar system, but also for our understanding of the fundamental laws of physics. The precession of Mercury has been a key test of Einstein's theory of general relativity, and its precise measurement has helped to confirm this fundamental theory.

Moreover, the study of the precession of Mercury has broader implications for our understanding of the nature of spacetime and the dynamics of the universe as a whole. By studying the precession of Mercury, we can gain insights into the nature of gravity, the curvature of spacetime, and the behavior of other celestial bodies in the solar system.

Therefore, the study of the precession of Mercury is not only a fascinating topic in its own right, but also has important implications for our understanding of the fundamental laws of nature. This report aims to provide a comprehensive analysis of the precession of Mercury, from its historical context to the latest developments in mathematical modeling and observational data, in order to advance our understanding of this intriguing phenomenon.

3 Theory

Astronomers have known for a long time that the orbit of planets changes as time proceeds. Since the Earth's axis of rotation slightly wobbles over time in the process called the precession of the equinoxes. This phenomenon causes the position of other astronomical objects in space like stars and planets to shift gradually over time. The kinematic effect which is caused by the Earth's rotation and precession, should be separated from the dynamic effect which is the actual motion of the celestial body to obtain the accurate measurement of the motion, position and other properties of the object they are studying.

By the early 1800s, it was already suspected that there was something strange about the orbit of Mercury. Mercury has a very eccentric orbit. Eccentricity is a generally used astrodynamic term that is dimensionless and refers to how much the astronomical object which is orbiting around another body defers or deviates from the perfect circle. The orbit eccentricity of mercury is 0.2056 (eccentricity of the Earth is about 0.0167, just for reference)[1].

The one way to separate the kinematics of Earth from the dynamic effect is to take the fixed space of the frame of reference, which is the Sun (as it can be assumed to be at rest). From this perspective, the motion of Mercury can be analyzed more accurately. From this, the astronomers were able to approximate the precession rate of Mercury's perihelion is 574 seconds of arc/century. This means that the perihelion slowly rotates through an angle of $0.16°$.

It is obvious that other than the Sun, other planets such as Venus (this planet has the highest contribution due to the fact that it is closest to Mercury), Jupiter (this planet has the second highest contribution because of its substantial mass), and beyond Saturn the effect is negligible plays a pivotal role in influencing the orbit of mercury.

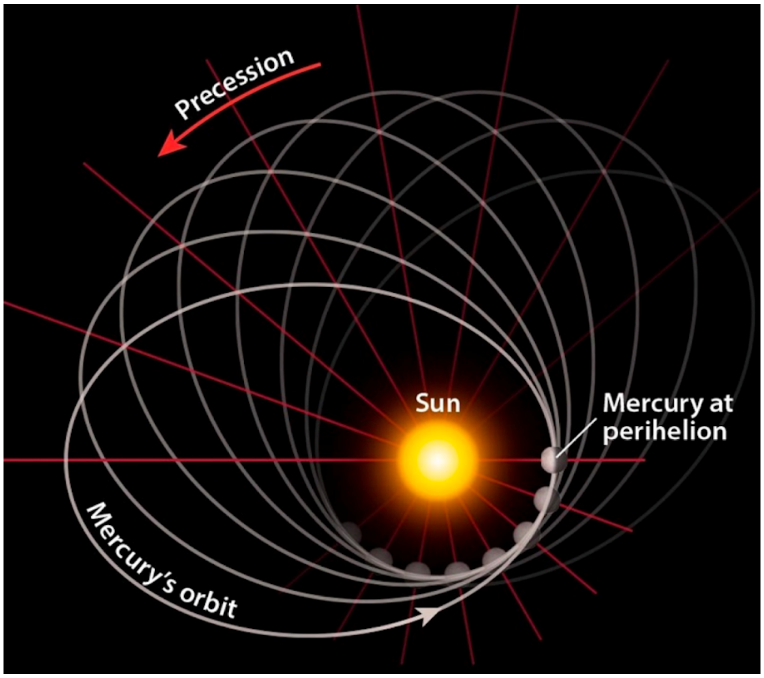

Figure 1 : Orbit of Mercury [2]

So taking into consideration all the known effects of these planets on mercury we can use Newton's theory of gravity, the expected precession was found out to be 531" leaving the 43" (574-531) omitted. This error though small cannot be ignored. It was after Einstein published the General Theory of Relativity in 1916 that this problem could be solved. It turns out that the General Theory of Relativity predicted the 43 arc second shift.

Using the General Theory of Relativity, the gravitational field can be modified as

$$F(r) = -\frac{GMm}{r^2}(1 + \frac{\lambda}{r^2})$$

Where $\lambda = 1.096 \times 10^{-8} AU^2$ for Mercury.

3.1 Runge-Lenz Vector

The Runge-Lenz vector is a conserved vector quantity that describes the shape and orientation of an orbit in space. It was first introduced by Carl Runge and Wilhelm Lenz in the late 19th century, and has since been widely used in the study of celestial mechanics. The Runge-Lenz vector is defined as follows:

$$\vec{A} = \vec{p} \times \vec{L} - mk\hat{r}$$

where $\vec{p} = m\vec{v}$ is the momentum and $\vec{L}$ is the angular momentum. Mass of the orbiting body is denoted by m and the force constant is denoted by k.

This $\vec{A}$ is seen to be constantly moving purely in the Inverse-square law of Force. Which can be shown as

$$\frac{d\vec{A}}{dt} = 0$$

For elliptical paths, $\vec{A}$ always points in the direction of perihelion from the origin. For circular paths, $\vec{A} = 0$.

Sometimes $\vec{A}$ can also be seen to be rotating with the perihelion. This will occur when the force is not absolute inverse-square in r. This rotation with the perihelion can greatly assist us, as we can follow $\vec{A}$ and the angle that it covers will give the precession. Let the two components of $\vec{A}$ be A_x and A_y. then the angle of precession can be found by

$$\theta = tan^{-1}(A_y/A_x)$$

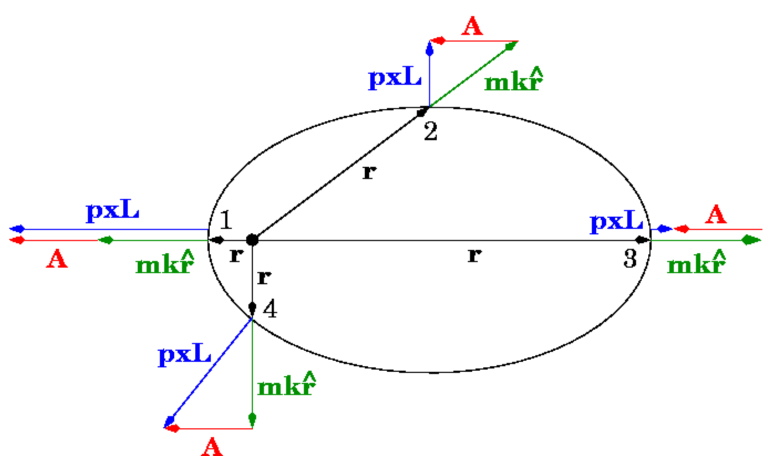

Figure 2 : A simple visualisation for Runge-Lenz Vector [3]

3.2 Leapfrog Method with Time Transformation

We can also use the Leapfrog method to calculate the Runge-Lenz vector but the results that are obtained are unsatisfactory for the practical application as they are quite random. The presumption that decreasing the 'h' (time interval) generally leads to more accurate results does not work here because the numerical algorithm generates more round-off errors. The main culprit to this problem is that there is a singularity at r = 0, which causes noteworthy variation in speed and force near the perihelion of the orbit. To obtain more accurate results, singularity needs to be smoothed out which is especially important when the orbit is highly elliptical.

The optimal way to counteract the problem is to use time transformation. This can be done by taking smaller steps near the perihelion of Mercury's orbit to obtain more accurate results.

Take vector equation of motion as

$$\frac{d\overline{r}}{dt} = \overline{v} \tag{1}$$

$$\frac{d\overline{r}}{dt} = \overline{a}(r) \tag{2}$$

We then introduce fictitious time s to a time transformation function

$$\frac{ds}{dt} = \Omega(r) \tag{3}$$

Using chain rule on the using the s term we modify the above two equation as

$$\frac{d\overline{r}}{dt} = \frac{\overline{v}}{\Omega(r)} \tag{4}$$

$$\frac{d\overline{v}}{dt} = \frac{\overline{a}(r)}{\Omega(r)} \tag{5}$$

We need to treat the $\Omega(r)$ as generalized velocity because the equation of motion involves both velocity and position. This poses a problem when using the Leapfrog method as it requires the equation of motion to be in a specific form in order to preserve area.

$$W = \Omega(r) \tag{6}$$

Using $\frac{dW}{ds} = \nabla\Omega(r).\frac{\overline{v}}{\Omega(r)}$ and (3). Also adding (4) and (5)

$$\frac{d\overline{r}}{dt} = \frac{\overline{v}(r)}{W} \tag{7}$$

$$\frac{dt}{ds} = \frac{1}{W} \tag{8}$$

$$\frac{d\overline{v}}{ds} = \frac{\overline{a}(r)}{\Omega(r)} \tag{9}$$

$$\frac{dW}{ds} = \nabla\Omega(r).\frac{\overline{v}}{\Omega(r)} \tag{10}$$

If we treat $\overline{r}$ and t as generalized coordinates and $\overline{v}$ and W as the generalized velocities above equation we say the above equation are in proper form for Leapfrog method. A good choice for planetary motion is $1/r^n$ taking the simple case where $n = 1$ we get

$$\Omega(r) = \frac{1}{r} \tag{11}$$

This ensures that $\Omega(r)$ is large, r and dt are small also one add on benefit is it is easy to calculate the gradient $\nabla\Omega(r) = \frac{-\overline{r}}{r^3}$.

Using (11) we can write the Leapfrog algorithm which is comparable to (1) and (3).

$$\overline{r}_{\frac{1}{2}} = \overline{r}_1 + \overline{v}_1\frac{h}{2W_1} \tag{12}$$

$$\overline{v}_1 = \overline{v}_0 + \overline{a}(\overline{r}_{\frac{1}{2}})r_{\frac{1}{2}}h \tag{13}$$

$$W_1 = W_0 - \overline{r}_{\frac{1}{2}}.\frac{v_1 + \overline{v}_0}{2}.\frac{h}{(r_{\frac{1}{2}})^2} \tag{14}$$

$$\overline{r}_1 = \overline{r}_{\frac{1}{2}} + \overline{r}_1 \frac{h}{2W_1} \tag{15}$$

Here, the subscript 0 and 1 denotes the beginning and the end of a time step h respectively. But since we have not used actual time, rather we have used transformed time (s). The actual time is returned as t_1 and the difference $t_1 - t_0$ is not constant thus regularizing the singularity.

4 Results

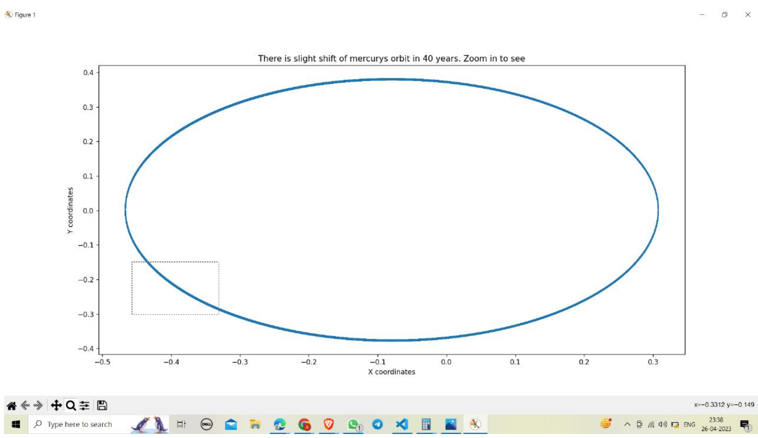

Figure 1 : Shift in Orbit of Mercury

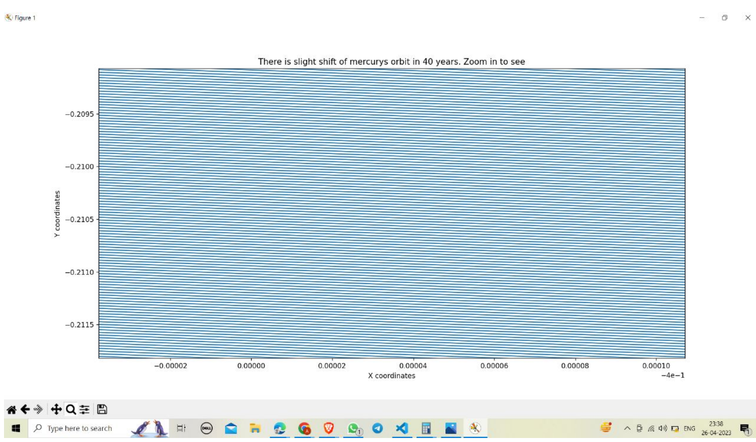

Figure 2 : Shift in Orbit of Mercury (zoomed in)

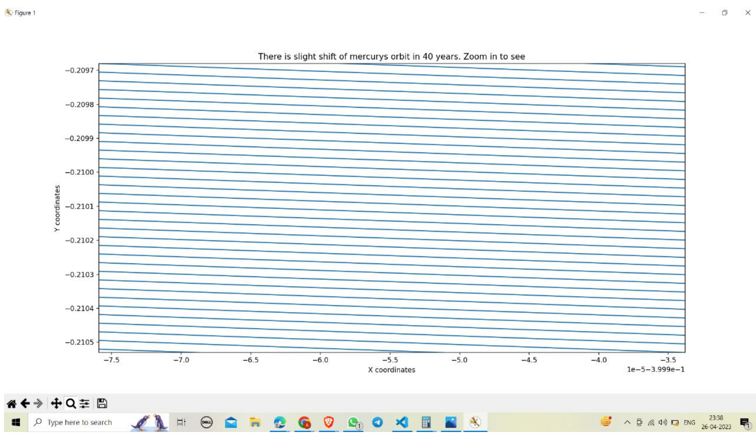

Figure 3 : Shift in Orbit of Mercury (more zoomed in)

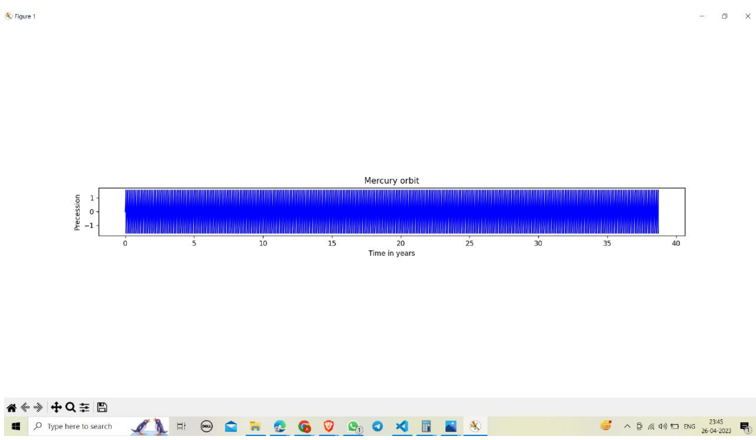

Figure 4 : Precession of Mercury's orbit

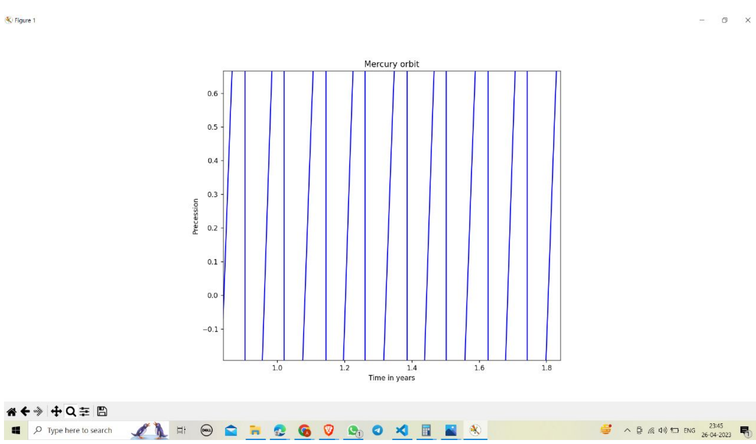

Figure 5 : Precession of Mercury's Orbit (Sinusoidal pattern is visible when zoomed in)

5 Conclusion

In conclusion, the precession of Mercury is a fascinating phenomenon that has been the subject of study and research for centuries. Through our report, we have explored some of the key concepts and methods used to understand this phenomenon, including the Runge-Lenz vector and the Leapfrog Method with Time Transformation. By applying these techniques and running simulations with the codes we provided, we were able to gain a deeper understanding of the precession of Mercury and the underlying physical principles at work. Overall, our report highlights the importance of continued research and exploration in this field, as we seek to unlock the mysteries of the universe and the laws that govern it.

In addition, our findings highlight how numerical simulations and computational approaches can enhance our comprehension of intricate physical systems, such as the precession of Mercury. Through the utilization of modern computational tools and methods, we can simulate and model complex systems with exceptional accuracy and precision. This not only deepens our comprehension of the fundamental physical principles involved but also has practical implications in various fields, such as Astronomy, Engineering, and Physics.

In conclusion, our report provides a valuable contribution to the ongoing study of the precession of Mercury and serves as a reminder of the importance of continued scientific inquiry and exploration. We hope that our findings and insights will inspire further research and discovery in this fascinating field, and we look forward to seeing what new insights and breakthroughs emerge in the years to come.

6 References

[1] Mercury fact sheet NASA. NASA. Available at: https://nssdc.gsfc.nasa.gov/planetary/factsheet/mercuryfac
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[2] The mystery of Mercury's missing arcseconds (2020) Astronomical Returns. Available at: https://www.astronomicalreturns.com/2020/05/the-mystery-of-mercurys-missing.html (Accessed: April 25, 2023).

[3] File:Laplace Runge Lenz vector.svg (no date) Wikimedia Commons. Available at: https://commons.wikimedia.org/wiki/File:$Laplace_Runge_Lenz_vector.svg$ (Accessed: April 25, 2023).