12 Marks.

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Unit 1 : Laplace Transform

- Defination :-

If $f(t)$ is a function defined for $t \geq 0$ then laplace transform of $f(t)$ is given by,

$$L[f(t)] = \int_0^\infty e^{-st} f(t)\, dt = \bar{f}(s)$$

- Laplace transform of some standard funition :-

① $L[1] = \frac{1}{s}$

② $L[e^{at}] = \frac{1}{s-a}$

③ $L[e^{-at}] = \frac{1}{s+a}$

④ $L[\sin at] = \frac{a}{s^2+a^2}$ अ - a

⑤ $L[\cos at] = \frac{s}{s^2+a^2}$ क - s

⑥ $L[\sinh at] = \frac{a}{s^2-a^2}$

(7) $L[\cosh at] = \frac{s}{s^2 - a^2}$

(8) $L[t^n] = \frac{n!}{s^{n+1}} = \frac{\Gamma_{n+1}}{s^{n+1}}$

Ex:- Find $L[1]$

→ By defination

$$L[f(t)] = \int_0^\infty e^{-st} f(t)\,dt$$

$$L[1] = \int_0^\infty e^{-st} \cdot 1\,dt$$

$$= \left[\frac{e^{-st}}{-s}\right]_0^\infty$$

$$= 0 + \frac{e^0}{s}$$

$$\therefore \boxed{L[1] = \frac{1}{s}}$$

Ex:- $L[e^{at}]$

→ By defination

$$L[f(t)] = \int_0^\infty e^{-st} f(t)\,dt$$

$$L[e^{at}] = \int_0^\infty e^{-st} e^{at}\,dt$$

$$= \int_0^\infty e^{-t(s-a)}\,dt$$

$$= \left[\frac{e^{-t(s-a)}}{-(s-a)}\right]_0^\infty \Rightarrow 0 + \frac{e^0}{s-a} \Rightarrow \frac{1}{s-a}$$

$\int e^{ax}\sin bx\,dx = \frac{e^{ax}}{a^2+b^2}[a\sin bx - b\cos bx]$

(3) $L[e^{-at}]$

→ By defination

$$L[f(t)] = \int_0^\infty e^{-st}[f(t)]\,dt$$

$$L[e^{-at}] = \int_0^\infty e^{-st}.(e^{-at})\,dt$$

$$= \int_0^\infty e^{-t(s+a)}\,dt$$

$$= \left[\frac{e^{-t(s+a)}}{-(s+a)}\right]_0^\infty$$

$$= 0 + \frac{e^0}{s+a}$$

$$\boxed{L[e^{-at}] = \frac{1}{s+a}}$$

(4) $L[\sin at]$

→ By defination

$$L(f(t)) = \int_0^\infty e^{-st}(f(t)].dt$$

$$L[\sin at] = \int_0^\infty e^{-st}[\sin at].dt$$

$$= \left[\frac{e^{-st}}{s^2+a^2}[-s\sin at - b\cos at]\right]_0^\infty$$

$$= 0 - \frac{e^0}{s^2+a^2}(-s\sin 0 - a\cos 0)$$

$$= \frac{-1}{s^2+a^2}(-a)$$

$$L[\sin at] = \frac{a}{s^2+a^2}$$

(5) $L(\cos at)$ =

→ By defination

$$L(\cos at) = \int_0^\infty e^{-st} \cos at \, dt$$

$$= \frac{e^{-st}}{s^2+a^2}\left[-s\cos at + a\sin at\right]_0^\infty$$

$$= 0 - \frac{1}{s^2+a^2}[-s]$$

$$L(\cos at) = \frac{s}{s^2+a^2}$$

property 1 :-

- Linearity Polarity :-

let $f(t)$, $g(t)$ and $h(t)$ be functions of 't' then laplace transform of linear combination of this functions is

$$L[af(t) + bg(t) - ch(t)]$$

$$= aL[f(t)] + bL[g(t)] - cL[h(t)]$$

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Ex: $(t^2+1)^3 + e^{-5t} + \cosh 3t$

$\rightarrow$ $L[(t^2+1)^3 + e^{-5t} + \cosh 3t] =$

$L[(t^2+1)^3] + L[e^{-5t}] + L[\cosh 3t]$

3!

Ex: Find laplace transform of $2t^{3/2} + 5e^{-3t} - 3\cos^2 2t$

$\Rightarrow$ Apply laplace,

$L[2t^{3/2} + 5e^{-3t} - 3\cos^2 2t]$

Laplace polarity $\rightarrow$ $= 2L[t^{3/2}] + 5L[e^{-3t}] - 3L[\cos^2 2t]$

$= 2\dfrac{\Gamma\frac{3}{2}+1}{s^{\frac{3}{2}+1}} + 5\dfrac{1}{s+3} - 3L\left[\dfrac{1+\cos 4t}{2}\right]$

$= 2\dfrac{\frac{3}{2}\Gamma\frac{3}{2}}{s^{5/2}} + \dfrac{5}{s+3} - \dfrac{3}{2}\{L(1) + L(\cos 4t)\}$

$= \dfrac{3\,\frac{1}{2}\Gamma\frac{1}{2}}{s^{5/2}} + \dfrac{5}{s+3} - \dfrac{3}{2}\left[\dfrac{1}{s} + \dfrac{s}{s^2+16}\right]$

$= \dfrac{3/2\sqrt{\pi}}{s^{5/2}} + \dfrac{5}{s+3} - \dfrac{3}{2}\left[\dfrac{1}{s} + \dfrac{s}{s^2+16}\right]$

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Change Scale Property

Change Scale prop

Property 2 :- ~~First Shifting Property~~.

If $L[f(t)] = \bar{f}(s)$ then
$$L[f(at)] = \frac{1}{a}\bar{f}(s/a)$$

Proof :-
$$L[f(t)] = \bar{f}(s) = \int_0^\infty e^{-st} f(t)\,dt$$

$$L[f(at)] = \int_0^\infty e^{-st} f(at)\,dt$$

put $at = p \Rightarrow t = p/a \Rightarrow dt = \frac{dp}{a}$

$a\,dt = dp$ ($p = at$)

As $t = 0 \Rightarrow p = 0$
$t = \infty \Rightarrow p = \infty$

$$= \int_0^\infty e^{-\left(\frac{p}{a}\right)s} f(p)\,\frac{dp}{a}$$

$$= \frac{1}{a}\int_0^\infty e^{-s\left(p/a\right)} f(p)\,dp$$

$$= \frac{1}{a}\int_0^\infty e^{-\left(s/a\right)p} f(p)\,dp$$

$$= \frac{1}{a}\bar{f}\left(\frac{s}{a}\right)$$

Ex: If $L[\cos t] = \frac{s}{s^2+1}$ then find $L[\cos 3t]$

by (change scale ~~*~~) property). first shift property

Ans →

By Change Scale property

$$L[f(at)] = \frac{1}{a}\bar{f}(s/a)$$

$$L[\cos 3t] = \frac{1}{3}\bar{f}(s/3)$$

$$= \frac{1}{3}\frac{s/3}{(s/3)^2-1}$$

Ex: Given $L[f(t)] = \frac{1}{\sqrt{s^2+1}}$ find $L[f(3t)]$

Sol^n ⇒ By change Scale property

$$L[f(at)] = \frac{1}{a}\bar{f}(s/a)$$

$$L[f(3t)] = \frac{1}{3}\bar{f}\left(s/3\right)$$

$$= \frac{1}{3}\frac{1}{\sqrt{(s/3)^2+1}}$$

$$= \frac{1}{3}\frac{1}{\sqrt{\frac{s^2+9}{9}}} = \frac{1}{\cancel{3}}\frac{\cancel{3}}{\sqrt{s^2+9}}$$

$$= \frac{1}{\sqrt{s^2+9}}$$

Property 3:

Ex: First Shifting Property :-

If $L[f(t)] = \bar{f}(s)$ then

$L[e^{-at} f(t)] = \bar{f}(s+a)$ &

$L[e^{at} f(t)] = \bar{f}(s-a)$

→ By defination

$$L[f(t)] = \int_0^\infty e^{-st} f(t)\,dt$$

$$L[e^{-at} f(t)] = \int_0^\infty e^{-st} e^{-at} f(t)\,dt$$

$$= \int_0^\infty e^{-t(s+a)} f(t)\,dt$$

$$= \bar{f}(s+a)$$

for 2nd

$$L[e^{at} f(t)] = \int_0^\infty e^{-st} e^{+at} f(t)\,dt$$

$$= \int_0^\infty e^{-t(s-a)} f(t)\,dt$$

$$= \bar{f}(s-a)$$

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Ex:- find laplace transform of e^{at}

$\Rightarrow$ $L[e^{at}] = L[e^{at} \cdot 1]$

$$= \frac{1}{s-a}$$

Ex :- find laplace transform of $e^{4t}\sin 3t$

$\Rightarrow$

$$L[e^{4t}\sin 3t] = \frac{3}{(s-4)^2+9}$$

$\sin 3t = \frac{3}{s^2+(3)^2}$

$= \text{for } e^{4t}$

$= (s-4)^2$

$$= \frac{3}{s^2-8s+25}$$

Ex:- find laplace transform of $\{\sin hat \cos at\}$

$\Rightarrow$ $L[\sin hat + \cos at] = L\left[\frac{e^{at}-e^{-at}}{2}\cos at\right]$

$$= \frac{1}{2}\left\{L[e^{at}\cos at] - L[e^{-at}\cos at]\right\}$$

$$= \frac{1}{2}\left\{\frac{(s-a)}{(s-a)^2+a^2} - \frac{s}{(s+a)^2+a^2}\right\}$$

$$= \frac{1}{2}\left\{\frac{s-a}{s^2-2sa+2a^2} - \frac{s+a}{s^2+2as+2a^2}\right\}$$

④ $L\left[\int_0^t f(u)\,du\right] = \frac{\bar{f}(s)}{s}$

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$= \frac{1}{2}$

Integration

6 Property 6 : L.S. of multiplication by t :

If $L[f(t)] = \bar{f}(s)$ then

$$L[t\,f(t)] = (-1)^{1}\frac{d}{ds}\bar{f}(s)$$

$$L[t^n f(t)] = (-1)^n \frac{d^n}{ds^n}\bar{f}(s)$$

→ Consider $\bar{f}(s) = \int_0^\infty e^{-st} f(t)\,dt$

$$\frac{d}{ds}\bar{f}(s) = \frac{d}{ds}\int_0^\infty e^{-st} f(t)\,dt$$

$$\frac{d}{ds}\bar{f}(s) = \int \frac{\partial}{\partial s} e^{-st} f(t)\,dt$$

$$= \int -t\,e^{-st} f(t)\,dt$$

$$(-1)^1\frac{d}{ds}f(s) = \int_0^\infty e^{-st}\,t\,f(t)\,dt$$

$$= L[t\,f(t)]$$

$$\Rightarrow L[t\,f(t)] = (-1)^1\frac{d}{ds}\bar{f}(s)$$

Now, we use method of induction

In method of induction we prove the statement for $n=1$ first
then assume that statement is true for $n=m$ and prove that it is true for $n=m+1$

$$P(n) = (-1)^n \frac{d^n}{ds^n} \bar{f}(s) = L[t^n f(t)]$$

$$\bar{f}(s) = \int_0^\infty e^{-st} f(t)\, dt$$

$$\frac{d}{ds}\bar{f}(s) = \int_0^\infty \frac{\partial}{\partial s} e^{-st} f(t)\, dt$$

$$= \int_0^\infty -t\, e^{-st} f(t)\, dt$$

$$(-1)^1 \frac{d}{ds}\bar{f}(s) = \int_0^\infty e^{-st} (t\, f(t))\, dt = L[t f(t)]$$

$$\therefore P(1) = (-1)^1 \frac{d}{ds} \bar{f}(s) = L[t f(t)]$$

Proved for $n=1$

Assume that $P(m) \Rightarrow n=m$

$$P(m) = (-1)^m \frac{d^m}{ds^m} \bar{f}(s) \text{ is true}$$

~~then~~, $L[t^m f(t)] = \int_0^\infty e^{-st}\, t^m f(t)\, dt$

~~then~~, $(-1)^m \frac{d^m}{ds^m} \bar{f}(s) = \int_0^\infty e^{-st}\, t^m f(t)\, dt$

Diff n^{th} side w.r.t. s

$$(-1)^n \frac{d^{m+1}}{ds^{m+1}} \bar{F}(s) = \int_0^\infty \frac{\partial}{\partial s} e^{-st} t^m f(t)\,dt$$

$$= \int_0^\infty -t e^{-st} t^m f(t)\,dt$$

$$(-1)^{m+1} \frac{d^{m+1}}{ds^{m+1}} \bar{F}(s) = -\int_0^\infty e^{-st} (t^{m+1} f(t))\,dt$$

$$P(m+1) = (-1)^{m+1} \frac{d^{m+1}}{ds^{m+1}} \bar{F}(s) = L[t^{m+1} f(t)]$$

Thus the statement is true for m+1

Thus,

$P(m) \Rightarrow P(m+1)$

$\therefore$ P(n) is true for all 'n'.

$$\therefore L[t^n f(t)] = (-1)^n \frac{d^n}{ds^n} \bar{F}(s)$$

differentiat I

Property 7 : Division by t :

If $L[f(t)] = \bar{f}(s)$ then $L\left[\frac{f(t)}{t}\right] = \int_s^\infty \bar{f}(s)\,ds$

Q.

Q. 3

→ Consider

$$\int_s^\infty \bar{f}(s)\,ds = \int_s^\infty \int_0^\infty e^{-st} f(t)\,dt\,ds$$

Ans →

$$= \int_0^\infty \left[\int_s^\infty e^{-st}\,ds\right] f(t)\,dt$$

$$= \int_0^\infty \left[\frac{e^{-st}}{-t}\right]_s^\infty f(t)\,dt$$

$$= -\int_0^\infty 0 + \frac{e^{-st}}{t}\,\frac{f(t)}{\cancel{t}}\,dt$$

$$= \int_0^\infty e^{-st}\left[\frac{f(t)}{t}\right] dt$$

$$= L\left[\frac{f(t)}{t}\right]$$

Q.9

Q. Find L.T of $\frac{e^{-at}-e^{-bt}}{t}$, hence evaluate

$$\int_0^\infty \frac{e^{-at}-e^{-bt}}{t}\,dt$$

Ans→ By laplace of,

$$L\left[\frac{f(t)}{t}\right]=\int_s^\infty \bar{f}(s)\,ds$$

$$=\int_s^\infty L[e^{-at}-e^{-bt}]\,ds$$

$$=\int_s^\infty \left[\frac{1}{s+a}-\frac{1}{s+b}\right]ds$$

$$=\int_s^\infty \frac{1}{s+a}\,ds-\int_s^\infty \frac{1}{s+b}\,ds$$

$$=\log(s+a)\Big]_s^\infty-\log(s+b)\Big]_s^\infty$$

$$=\log\frac{s+a}{s+b}\Big]_s^\infty$$

$$=\log\left[\frac{\cancel{s}(1+a/s)}{\cancel{s}(1+b/s)}\right]_s^\infty$$

$$=\log\left(\frac{1+a/\infty}{1+b/\infty}\right)-\log\left(\frac{1+a/s}{1+b/s}\right)$$

$$=\log\left(\frac{1+0}{1+0}\right)-\log\left(\frac{s+a}{s+b}\right)$$

$L \cos at = \frac{s}{s^2+a^2}$

$L \cos bt = \frac{s}{s^2+b^2}$

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$$L\left[\frac{f(t)}{t}\right] = \int_0^\infty e^{-st}\frac{f(t)}{t}\,dt = \log\left(\frac{s+b}{s+a}\right)$$

$$\int_0^\infty e^{-st}\left(\frac{e^{-at}-e^{-bt}}{t}\right)dt = \log\left(\frac{s+b}{s+a}\right)$$

i=1

put $s=0$

$$\int_0^\infty \frac{e^{-at}-e^{-bt}}{t}\,dt = \log\frac{b}{a}$$

Ex: Find laplace transform of $\frac{\cos at - \cos bt}{t}$,

Q.21

hence evaluate $\int_0^\infty \frac{\cos at - \cos bt}{t}\,dt$.

Ans→ $$L\left[\frac{\cos at - \cos bt}{t}\right] = \int_s^\infty \left[\frac{s}{s^2+a^2} - \frac{s}{s^2+b^2}\right]ds$$

$$= \frac{1}{2}\int_s^\infty \frac{2s}{s^2+a^2}\,ds - \frac{1}{2}\int_s^\infty \frac{2s}{s^2+b^2}\,ds$$

$$= \frac{1}{2}\log(s^2+a^2) - \frac{1}{2}\log(s^2+b^2)$$

$$= \frac{1}{2}\left[\log(s^2+a^2)^{1/2} - \log(s^2+b^2)^{1/2}\right]$$

$$= \frac{1}{2}\log\left[\frac{(s^2+a^2)}{(s^2+a^2)}\right]_s^\infty$$

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w.r.t s karna nl try to put s^2 so down. $\frac{1}{s}$

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$$= \frac{1}{2}\log\left[\frac{s^2(1+a^2/s^2)}{s^2(1+b^2/s^2)}\right]_s^\infty$$

$$= \frac{1}{2}\left[\log\frac{(1+a^2/\infty)}{(1+b^2/\infty)} - \log\left(\frac{1+a^2/s^2}{1+b^2/s^2}\right)\right]$$

$$= \frac{1}{2}\left[\log 1 - \log\left(\frac{s^2+a^2}{s^2+b^2}\right)\right]$$

$$= \frac{1}{2}\left[0+\log\left(\frac{s^2+b^2}{s^2+a^2}\right)\right]$$

$$\therefore L[\cos at - \cos bt] = \log\left(\frac{s^2+b^2}{s^2+a}\right)^{1/2}$$

put $s=0$

$$\int_0^\infty \frac{(\cos at-\cos bt)}{t}\,dt = \log\left(\frac{b}{a}\right)^{2\cdot 1/2}$$

$$= \log\frac{b}{a}$$

$$\int_0^\infty \frac{(\cos at-\cos bt)}{t}\,dt = \log\frac{b}{a}$$

Ex:- Find laplace transform of $\frac{\sin^2 t}{t}$

$$\int_0^\infty e^{-t}\left[\frac{\sin^2 t}{t}\right] dt$$

$$\rightarrow \quad L\left[\frac{\sin^2 t}{t}\right] = \int_s^\infty L\left[\sin^2 t\right] ds$$

$$= \int_s^\infty L\left[1 - \frac{\cos 2t}{2}\right] ds$$

$$= \frac{1}{2}\int_s^\infty \left[\frac{1}{s} - \frac{s}{s^2+4}\right] ds$$

$$= \frac{1}{2}\left\{\frac{2}{2}\log s - \frac{1}{2}\log(s^2+4)\right]_s^\infty$$

$$= \frac{1}{2}\left\{\frac{1}{2}\log s^2 - \frac{1}{2}\log(s^2+4)\right]_s^\infty$$

$$= \frac{1}{4}\left[\log \frac{s^2}{s^2+4}\right]_s^\infty$$

$$= \frac{1}{4}\left\{\log\left[\frac{s^2}{s^2(1+4/s^2)}\right]\right]_s^\infty$$

$$= \frac{1}{4}\left[\log\left(\frac{1}{1+4/\infty}\right) - \log\left(\frac{1}{1+4/s^2}\right)\right]$$

$$= \frac{1}{4}\left[\log 1 - \log \frac{s^2}{s^2+4}\right]$$

$$= \frac{1}{4}\left[\log \frac{s^2+4}{s^2}\right]$$

$$L\left[\frac{\sin^2 t}{t}\right] = \frac{1}{4}\left[\log \frac{s^2+4}{s^2}\right]$$

$$\int_0^\infty e^{-st} \frac{\sin^2 t \, dt}{t} = \frac{1}{4}\log \frac{s^2+4}{s^2}$$

Put $s = 1$

$$\int_0^\infty e^{-t} \frac{\sin^2 t}{t} dt = \frac{1}{4}\log 5.$$

Ex :- Find L.T. of $t^2 \cos 3t \, e^{-t}$

$\rightarrow$ $L[f(t)] = \int_0^\infty e^{-st} f(t) \, dt$

$$L[e^{-t}(t^2 \cos 3t)] = L[e^{at} f(t)]$$

$$= \bar{f}(s+a)$$

$$L[t^2 \cos 3t] = (-1)^2 \frac{d^2}{ds^2}\left(\frac{s}{s^2+9}\right)$$

$$= \frac{d}{ds}\left(\frac{(s^2+9) - s(2s)}{(s^2+9)^2}\right)$$

$\frac{u}{v}$

$$= \frac{d}{ds}\left(\frac{-s^2+9}{(s^2+9)^2}\right)$$

multiply $(-1)^1$

$1t = (-1)^1$ 1 time diff

$2t = (-1)^2$. 2 time diff

$$= \frac{(s^2+9)^2(-2s) - 2(-s^2+9)(s^2+9)2s}{(s^2+9)^4}$$

$$= \frac{-2s(s^2+9) - 4s(-s^2+9)}{(s^2+9)^3}$$

$$= \frac{-2s^3 - 18s + 4s^2 - 36s}{(s^2+9)^3}$$

$$= \frac{2s^3 - 54s}{(s^2+9)^3}$$

$$= \frac{2s(s^2-27)}{(s^2+9)^3}$$

$$L[e^{-t}(t^2\cos 3t)] = \frac{2(s+1)((s+1)^2-27)}{((s+1)^2+9)^3}$$

Ex :- $\int_0^\infty t e^{-2t} \cos t \, dt$

$\rightarrow \int_0^\infty e^{-2t}(t\cos t)dt = L[t\cos t]_{s=2}$

$$= (-1)^1 \frac{d}{ds} \left. \frac{s}{s^2+1} \right]_{s=2}$$

$$= -1 \left. \frac{(s^2+1)1 - s(2s)}{(s^2+1)^2} \right|_{s=2}$$

$= -1\left[\frac{5-8}{25}\right]$

$= +\frac{3}{25}$

Ex :- Find L.T. of $\frac{\sin t}{t}$ & hence evaluate $\int_0^{\infty} \frac{\sin t}{t}\,dt$

$\rightarrow$ $L\left[\frac{\sin t}{t}\right] = \int_s^{\infty} \bar{f}(s)\,ds$

$= \int_s^{\infty} L[\sin t]\,ds$

$= \int_s^{\infty} \frac{1}{s^2+1}\,ds$

$= \left[\tan^{-1}s\right]_s^{\infty}$

$= \tan^{-1}\infty - \tan^{-1}s$

$= \pi/2 - \tan^{-1}s$

$= \cancel{\cot^{-1}s}$

$L\left[\frac{\sin t}{t}\right] = \cancel{\cot^{-1}s}\ \pi - \tan^{-1}s$

$\int_0^{\infty} e^{-st}\frac{\sin t}{t}\,dt = \frac{\pi}{2} - \tan^{-1}s$

Put $s=0$

$\int_0^{\infty} \frac{\sin t}{t}\,dt = \pi/2 - \tan^{-1}0$

$= \pi/2 - 0$

$= \pi/2$

Tuesday

- Inverse laplace transform :-

If $L[f(t)] = f(s)$ then $L^{-1}[f(s)] = f(t)$

Their are different methods to find inverse laplace transform by (i) partial fraction (ii) Convolution theorem (iii) Properties / By using properties

× Find Inverse laplace transform of $\frac{s^2-3}{(s+2)(s-3)(s^2-2s+5)}$

~~Solution~~ :-

Standard formulae :-

1. $L^{-1}\left[\frac{1}{s}\right] = 1$

2. $L^{-1}\left[\frac{1}{s-a}\right] = e^{at}$

3. $L^{-1}\left[\frac{1}{s+a}\right] = e^{-at}$

4. $L^{-1}\left[\frac{1}{s^2+a^2}\right] = \frac{1}{a}\sin at$

5. $L^{-1}\left[\frac{s}{s^2+a^2}\right] = \cos at$

(c) $L^{-1}\left[\frac{1}{s^n}\right] = \frac{t^{n-1}}{n!}$

Ex:- Find $L^{-1}\left[\frac{2s^2-6s+5}{s^3-6s^2+11s-6}\right]$

Ans→

$$\frac{2s^2-6s+5}{s^3-6s^2+11s-6} = \frac{2s^2-6s+5}{(s-1)(s-2)(s-3)}$$

$s^2\cancel{-2s}+s)$

$$= \frac{A}{(s-1)} + \frac{B}{(s-2)} + \frac{C}{(s-3)}$$

$$\Rightarrow 2s^2-6s+5 = A(s-2)(s-3) + B(s-1)(s-3) + C(s-1)(s-2)$$

$s=2$

$8-12+5 = -B \Rightarrow \boxed{B=-1}$

$s=3$

$18-18+5 = 2C \Rightarrow \boxed{C = 5/2}$

$s=1$

$1 = 2A \Rightarrow \boxed{A = 1/2}$

$$L^{-1}\left[\frac{2s^2-6s+5}{s^3-6s^2+11s-6}\right] = \frac{1}{2}L^{-1}\left[\frac{1}{s-1}\right] - 1L^{-1}\left[\frac{1}{s-2}\right] + \frac{5}{2}L^{-1}\left[\frac{1}{s-3}\right]$$

$$= \frac{1}{2}e^{t} - 1e^{2t} + \frac{5}{2}e^{3t}$$

Ex :- Find $L^{-1}\left[\frac{s^2+2s+3}{(s^2+2s+2)(s^2+2s+5)}\right]$

$\Rightarrow$ put $s^2+2s=t$

$$L^{-1}\left[\frac{s^2+2s+3}{(s^2+2s+2)(s^2+2s+5)}\right]$$

$$= L^{-1}\left[\frac{t+3}{(t+2)\,(t+5)}\right]$$

$$\Rightarrow \frac{t+3}{(t+2)(t+5)} = \frac{A}{t+2} + \frac{B}{t+5}$$

$$t+3 = A(t+5) + B(t+2)$$

$\Rightarrow t=-5$

$-2 = -3B \Rightarrow \boxed{B = 2/3}$

$\Rightarrow t=-2$

$1 = 3A \Rightarrow \boxed{A = \frac{1}{3}}$

$$L^{-1}\left[\frac{s^2+2s+3}{(s^2+2s+2)(s^2+2s+5)}\right] = \frac{1}{3}L^{-1}\left[\frac{1}{s^2+2s+2}\right] + \frac{2}{3}L^{-1}\left[\frac{1}{s^2+2s+5}\right]$$

$(s+1)^2+1^2 \Rightarrow s^2+2s+1+1 \Rightarrow s^2+2s+2$

$$L^{-1}\left[\frac{s^2+2s+3}{(s^2+2s+2)(s^2+2s+5)}\right] = \frac{1}{3}L^{-1}\left[\frac{1}{(s+1)^2+1^2}\right] + \frac{2}{3}L^{-1}\left[\frac{1}{(s+1)^2+2^2}\right]$$

$$= \frac{1}{3}e^{-t}L^{-1}\left[\frac{1}{s^2+1^2}\right] + \frac{2}{3}e^{-t}L^{-1}\left[\frac{1}{s^2+2^2}\right]$$

$$= \frac{1}{3}e^{-t}\sin t + \frac{2}{3}e^{-t}\frac{1}{2}\sin 2t$$

Ex: Find $L^{-1}\left[\frac{s^2-3}{(s+2)(s-3)(s^2+2s+5)}\right]$

$$\Rightarrow \frac{s^2-3}{(s+2)(s-3)(s^2+2s+5)} = \frac{A}{(s+2)} + \frac{B}{s-3} + \frac{Cs+D}{s^2+2s+5}$$

$$s^2-3 = A(s-3)(s^2+2s+5) + B(s+2)(s^2+2s+5) + (Cs+D)(s+2)(s-3)$$

$\Rightarrow s=3$

$6 = 100B \Rightarrow B = 3/50$

$\Rightarrow s=-2$

$1 = -25A \Rightarrow A = -1/25$

$\Rightarrow s=0$

$-3 = -15A + B(10) - 6D = 0$

$-3 = -15\left(\frac{-1}{25}\right) + \frac{3}{50}(10) - 6D = 0$

$-3 = \frac{3}{5} + \frac{3}{5} - 6D$

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$$CD = \frac{6}{5} + \frac{3}{1}$$

$$D = \frac{21}{30} = \frac{7}{10}$$

$\Rightarrow s = 1$

$$-2 = \frac{-1}{25}(-2)(8) + \frac{3}{50}(24) + \left(C + \frac{7}{10}\right)(-$$

$$-2 = \frac{16}{25} + \frac{36}{25} - 6C - \frac{21}{5}$$

$$6C = \frac{52}{25} - \frac{21}{5} + 2$$

$$C = \frac{1}{6}\left[\frac{52}{25} - \frac{21}{5} + 2\right]$$

$$\boxed{C = -\frac{1}{50}}$$

Q10 $\Rightarrow L^{-1}\left[\dfrac{s^2-3}{(s+2)(s-3)(s^2+2s+5)}\right] =$

$$L^{-1}\left[\frac{-1/25}{s+2}\right] + L^{-1}\left[\frac{3/50}{s-3}\right] + L^{-1}\left[\frac{-\frac{1}{50}s + 7/10}{s^2+2s+5}\right]$$

$$= \frac{-1}{25}e^{-2t} + \frac{3}{50}e^{3t} - \frac{1}{50}L^{-1}\left[\frac{s}{s^2+2s+5}\right] +$$

$$\frac{7}{10}L^{-1}\left[\frac{1}{s^2+2s+5}\right]$$

$$= \frac{-1}{25} e^{-2t} + \frac{3}{50} e^{3t} - \frac{1}{50} L^{-1}\left[\frac{(s+1)-1}{(s+1)^2+2^2}\right]$$

$$+ \frac{7}{10} L^{-1}\left[\frac{1}{(s+1)^2+2^2}\right]$$

$$= \frac{-1}{25} e^{-2t} + \frac{3}{50} e^{3t} - \frac{1}{50} \left\{ \cancel{e^{t}} L^{-1}\left[\frac{s+1}{(s+1)^2+(2)^2}\right] \right\}$$

$$+ \frac{7}{10} L^{-1}\left[\frac{1}{(s+1)^2+2^2}\right]$$

$$= \frac{-1}{25} e^{-2t} + \frac{3}{50} e^{3t} - \frac{1}{50} \left\{ e^{-t} L^{-1}\left[\frac{s}{s^2+2^2}\right] - e^{-t} \right.$$

$$\left. L^{-1}\left[\frac{1}{s^2+2^2}\right] \right\} + \frac{7}{10} e^{-t} L^{-1}\left[\frac{1}{s^2+2^2}\right]$$

$$= \frac{-1}{25} e^{-2t} + \frac{3}{50} e^{3t} - \frac{1}{50} e^{-t} \left\{ \cos 2t - \frac{1}{2} \sin 2t \right\}$$

$$+ \frac{7}{10} e^{-t} \frac{1}{2} \sin 2t$$

$$\left.\frac{+7/10}{2s+5}\right]$$

(2) $L^{-1}\left[\frac{1}{(s-2)(s+2)^2}\right]$

$$\Rightarrow \frac{1}{(s-2)(s+2)^2} = \frac{A}{s-2} + \frac{B}{s+2} + \frac{C}{(s+2)^2}$$

$$1 = A(s+2)^2 + B(s-2)(s+2) + C(s-2)$$

$s = -2$

$$1 = -4C \Rightarrow \boxed{C = \frac{-1}{4}}$$

$s = 2$

$$1 = 16A \Rightarrow \boxed{A = \frac{1}{16}}$$

$s = 0$

$$1 = \frac{1}{16}(2) + B(-4) - \frac{1}{4}(-2)$$

$$1 = \frac{1}{8} - 4B + \frac{1}{2}$$

$$4B = \frac{1}{8} + \frac{1}{2} - 1$$

$$4B = \frac{5}{8} - 1$$

$$B = -\frac{3}{8} \times \frac{1}{4}$$

$$B = \frac{-3}{32}$$

$$L^{-1}\left[\frac{1}{(s-2)(s+2)^2}\right] = L^{-1}\left[\frac{1/16}{s-2}\right] + L^{-1}\left[\frac{-3/32}{s+2}\right] + L^{-1}\left[\frac{-1/4}{(s+2)^2}\right]$$

$$= \frac{1}{16}e^{2t} - \frac{3}{32}e^{-2t} - \frac{1}{4}e^{-2t}L^{-1}\left[\frac{1}{s^2}\right]$$

$$= \frac{1}{16}e^{2t} - \frac{3}{32}e^{-2t} - \frac{1}{4}e^{-2t}\frac{t}{2}$$

Convolution Theorem

Statement :- If $L[f(t)] = \bar{f}(s)$, $L[g(t)] = \bar{g}(s)$ then

$$L^{-1}[\bar{f}(s)\cdot\bar{g}(s)] = \int_0^t f(u)\,g(t-u)\,du$$

Ex : Find $L^{-1}\left[\frac{1}{s(s^2+1)}\right]$ by Convolution theorem

Q.U

By Convolution Thm

$$L^{-1}[\bar{f}(s)\cdot\bar{g}(s)] = \int_0^t f(u)\,g(t-u)\,du$$

$$L^{-1}\left[\frac{1}{s^2+1}\cdot\frac{1}{s}\right] = \int_0^t \sin u \cdot 1\,du$$

$$= -\cos u\Big]_0^t$$

$$= -\cos t + \cos 0$$

$$= 1 - \cos t$$

② Find $L^{-1}\left[\frac{1}{(s+1)(s^2+1)}\right]$ by Convolution thm.

$\Rightarrow$ By Convolution thm,

$L^{-1}[f(s).g(s)] = \int_0^t f(u).g(t-u)du$

$L^{-1}\left[\frac{1}{s^2+1} \cdot \frac{1}{s+1}\right] = \int_0^t \sin u \, e^{-t+u} du$

$= \int_0^t e^{u} \sin u du$

$= e^{-t}\left[\frac{e^u}{1+1}[\sin u - \cos u]\right]_0^t$

$= e^{-t}\left\{\frac{e^t}{2}(\sin t - \cos t) - \frac{e^0}{2}(\sin 0 - \cos 0)\right\}$

$= \frac{e^{-t}}{2}\left\{e^t(\sin t - \cos t) + \frac{1}{2}\right\}$.

③ Find $L^{-1}\left[\frac{s^2}{(s^2+4)(s^2+9)}\right]$ by Convolution theorem

Q.14

$\Rightarrow L^{-1}\left[\frac{s^2}{(s^2+4)(s^2+9)}\right] = L^{-1}\left[\left(\frac{s}{s^2+4}\right)\left(\frac{s}{s^2+9}\right)\right]$

$= \int_0^t \cos 2u \cdot \cos 3u(t-u) du$

$= \int_0^t \cos 2u \cos(3t-3u) du$

$= \int_0^t \left[\frac{1}{2}(\cos(2u+3t-3u) + \cos(2u-3t+3u)\right.$

$$= \frac{1}{2}\int_0^t [\cos(-u+3t) + \cos(5u-3t)]du$$

$$= \frac{1}{2}\left[\frac{\sin(-u+3t)}{-1} + \frac{\sin(5u-3t)}{5}\right]_0^t$$

$$= \frac{1}{2}\left[\frac{\sin(-t+3t)}{-1} + \frac{\sin(5t-3t)}{5} - \frac{\sin(3t)}{-1} - \frac{\sin(-3t)}{5}\right]$$

$$= \frac{1}{2}\left[-\sin 2t + \frac{1}{5}\sin 2t + \sin 3t + \frac{\sin 3t}{5}\right]$$

$$= \frac{1}{2}\left[\frac{-4}{5}\sin 2t + \frac{6}{5}\sin 3t\right]$$

* Unit Step function :- OR Heaviside unit step function is given by

$$u(t) = \begin{cases} 1 & t > 0 \\ 0 & t \leq 0 \end{cases}$$

NOTE :

1. $L[f(t)\,u(t)] = \bar{f}(s)$
2. $L[f(t-a)\,u(t-a)] = e^{-as} L[f(t)\,u(t)]$
$= e^{-as}\bar{f}(s)$

Ex. Express the given function in terms of unit step function $f(t) = \begin{cases} t-1 & 1<t<2 \\ 3-t & 2<t<3 \end{cases}$

and hence find its Laplace transform

→ Unit step function representation of given function is,

$$[f(t)] = (t-1)[u(t-1) - u(t-2)] + (3-t)[u(t-2) - u(t-3)]$$

$$= (t-1)u(t-1) + [(3-t) - (t-1)]u(t-2) - (3-t)u(t-3)$$

$$f(t) = (t-1)u(t-1) + (4-2t)u(t-2) + (t-3)u(t-3)$$

$$L[f(t)] = L[(t-1)u(t-1)] - 2L[(t-2)u(t-2)] + L[(t-3)u(t-3)]$$

$$= e^{-s}L[t] - 2e^{-2s}L[t] + e^{-3s}L[t]$$

$$= e^{-s}\frac{1}{s^2} - 2e^{-2s}\cdot\frac{1}{s^2} + e^{-3s}\frac{1}{s^2}$$