① Investigate the linear dependence of the vectors $X_1 = (1,0,2,1)$, $X_2 = (3,1,2,1)$, $X_3 = (4,6,2,-4)$, $X_4 = (-6,0,-3,-4)$ and if possible find relation between them.

Sol^n: Given vectors,

$X_1 = (1,0,2,1)$, $X_2 = (3,1,2,1)$, $X_3 = (4,6,2,-4)$

$X_4 = (-6,0,-3,-4)$

$$k_1X_1 + k_2X_2 + k_3X_3 + k_4X_4 = 0 \quad \text{— ①}$$

$$k_1[1,0,2,1] + k_2[3,1,2,1] + k_3[4,6,2,-4] + k_4[-6,0,-3,-4] = [0,0,0,0]$$

$$k_1 + 3k_2 + 4k_3 - 6k_4 = 0 \quad \text{— ②}$$

$$0k_1 + k_2 + 6k_3 + 0k_4 = 0 \quad \text{— ③}$$

$$2k_1 + 2k_2 + 2k_3 - 3k_4 = 0 \quad \text{— ④}$$

$$k_1 + k_2 - 4k_3 - 4k_4 = 0 \quad \text{— ⑤}$$

By Cramers Rule:

$$\frac{k_1}{\begin{vmatrix} 3 & 4 & -6 \\ 1 & 6 & 0 \\ 2 & 2 & -3 \end{vmatrix}} = \frac{-k_2}{\begin{vmatrix} 1 & 4 & -6 \\ 0 & 6 & 0 \\ 2 & 2 & -3 \end{vmatrix}} = \frac{k_3}{\begin{vmatrix} 1 & 3 & -6 \\ 0 & 1 & 0 \\ 2 & 2 & -3 \end{vmatrix}} = \frac{-k_4}{\begin{vmatrix} 1 & 3 & 4 \\ 0 & 1 & 6 \\ 2 & 2 & 2 \end{vmatrix}}$$

$$\frac{k_1}{18} = -\frac{k_2}{54} = \frac{k_3}{9} = \frac{-k_4}{18}$$

$\Rightarrow \boxed{k_1 = 2,\ k_2 = -6,\ k_3 = 1,\ k_4 = -2}$ not all zero

& satisfy all eq^n ②, ③, ④, ⑤.

$\therefore$ Given vectors are linearly dependent.

The relation bet^n them is:

$$\boxed{2X_1 - 6X_2 + X_3 - 2X_4 = 0}$$

③ Investigate the linearly dependence of the vectors $X_1=(1,2,-1,3)$, $X_2=(2,-1,3,2)$, $X_3=(-1,8,-9,5)$ and if possible find relation betn them.

Sol: $k_1X_1+k_2X_2+k_3X_3=0$ —— ①

$k_1[1,2,-1,3]+k_2[2,-1,3,2]+k_3[-1,8,-9,5]=[0,0,0,0]$

$k_1+2k_2-k_3=0$ —— ②

$2k_1-k_2+8k_3=0$ —— ③

$-k_1+3k_2-9k_3=0$ —— ④

$3k_1+2k_2+5k_3=0$ —— ⑤

$$\frac{k_1}{\begin{vmatrix}2 & -1\\ -1 & 8\end{vmatrix}}=\frac{-k_2}{\begin{vmatrix}1 & -1\\ 2 & 8\end{vmatrix}}=\frac{k_3}{\begin{vmatrix}1 & 2\\ 2 & -1\end{vmatrix}}$$

$$\frac{k_1}{15}=\frac{-k_2}{10}=\frac{k_3}{-5}$$

$\Rightarrow \boxed{k_1=3\ ,\ k_2=-2\ ,\ k_3=-1}$ not all zero & satisfy all equations ②, ③ ④ & ⑤.

∴ given vectors are linearly dependent.

Reln betn them is: $\boxed{3X_1-2X_2-X_3=0}$

④

(3) Are the following vectors are linearly dependent? If so, find the relation between them.
$X_1 = [1, 1, 1, 3]$, $X_2 = [1, 2, 3, 4]$, $X_3 = [2, 3, 4, 7]$

Solⁿ:

$$k_1X_1 + k_2X_2 + k_3X_3 = 0 \quad \text{--- (1)}$$

$$k_1[1\ 1\ 1\ 3] + k_2[1\ 2\ 3\ 4] + k_3[2\ 3\ 4\ 7] = [0\ 0\ 0\ 0]$$

$$k_1 + k_2 + 2k_3 = 0 \quad \text{--- (2)}$$
$$k_1 + 2k_2 + 3k_3 = 0 \quad \text{--- (3)}$$
$$k_1 + 3k_2 + 4k_3 = 0 \quad \text{--- (4)}$$
$$3k_1 + 4k_2 + 7k_3 = 0 \quad \text{--- (5)}$$

By Cramer's Rule

$$\frac{k_1}{\begin{vmatrix} 1 & 2 \\ 2 & 3 \end{vmatrix}} = \frac{-k_2}{\begin{vmatrix} 1 & 2 \\ 1 & 3 \end{vmatrix}} = \frac{k_3}{\begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix}}$$

$$\frac{k_1}{-1} = \frac{-k_2}{1} = \frac{k_3}{1}$$

$\Rightarrow \boxed{k_1 = -1, \quad k_2 = -1, \quad k_3 = 1}$ & satisfy eqⁿ (2), (3), (4) & (5)

k_1, k_2, k_3 not all zero,

∴ Given vectors are linearly independent.

The relation betⁿ them is:

$$-X_1 - X_2 + X_3 = 0 \Rightarrow \boxed{X_1 + X_2 - X_3 = 0}$$

(4) Check whether following set of vectors are lineary dependent, if dependent find the relⁿ betⁿ them,
$X_1 = [2, -1, 3, 2]$ $X_2 = [1, 3, 4, 2]$, $X_3 = [3, -5, 2, 2]$

Solⁿ: $k_1X_1 + k_2X_2 + k_3X_3 = 0$ --- (1)

$$k_1[2, -1, 3, 2] + k_2[1, 3, 4, 2] + k_3[3, -5, 2, 2] = [0, 0, 0, 0]$$

$2k_1 + k_2 + 3k_3 = 0$ --- (2) $\quad -k_1 + 3k_2 - 5k_3 = 0$ --- (3)

$3k_1 + 4k_2 + 2k_3 = 0$ --- (4) $\quad 2k_1 + 2k_2 + 2k_3 = 0$ --- (5)

$$\frac{k_1}{\begin{vmatrix}1 & 3\\3 & -5\end{vmatrix}} = \frac{-k_2}{\begin{vmatrix}2 & 3\\-1 & -5\end{vmatrix}} = \frac{k_3}{\begin{vmatrix}2 & 1\\-1 & 3\end{vmatrix}}$$

$$\frac{k_1}{-14} = \frac{-k_2}{-7} = \frac{k_3}{7}$$

$\Rightarrow \boxed{k_1 = -2,\ k_2 = 1,\ k_3 = 1}$ not all zero satisfy all eq^n (3) (3) (4) + (5).

$\therefore$ Given vectors are l.d.

Rel^n bet^n them is:

$\boxed{-2X_1 + X_2 + X_3 = 0}$.

(5) Find the Modal Matrix for the matrix

$$A = \begin{bmatrix}1 & 0 & -1\\1 & 2 & 1\\2 & 2 & 3\end{bmatrix}$$

Sol^n: Characteristic Eq^n: $|A - \lambda I| = 0$

$$\begin{vmatrix}1-\lambda & 0 & -1\\1 & 2-\lambda & 1\\2 & 2 & 3-\lambda\end{vmatrix} = 0$$

$\lambda^3 - S_1\lambda^2 + S_2\lambda - |A| = 0$

$\lambda^3 - 6\lambda^2 + [4+5+2]\lambda - 6 = 0$ chara. eq^n.

Eigen Values:

$\lambda_1 = 1,\ \lambda_2 = 3,\ \lambda_3 = 2$

Let $X = \begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}$ be the eigen vector corresponding

to eigen value λ s.t. $[A - \lambda I]X = 0$

$$\begin{bmatrix}1-\lambda & 0 & -1\\1 & 2-\lambda & 1\\2 & 2 & 3-\lambda\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$$

$(1-\lambda)x_1 + 0x_2 - x_3 = 0$ —— (1)

$x_1 + (2-\lambda)x_2 + x_3 = 0$ —— (2)

$2x_1 + 2x_2 + (3-\lambda)x_3 = 0$ —— (3)

For $\lambda_1 = 1$

(1) $\Rightarrow 0x_1 + 0x_2 - x_3 = 0$

(2) $\Rightarrow x_1 + x_2 + x_3 = 0$

(3) $\Rightarrow 2x_1 + 2x_2 + 2x_3 = 0$

$\rightarrow x_1 + x_2 + x_3 = 0$

(1) $\Rightarrow \boxed{x_3 = 0}$

$x_1 + x_2 + x_3 = 0 \Rightarrow x_1 + x_2 + 0 = 0$

$\Rightarrow x_1 + x_2 = 0$

Let $x_1 = 1 \Rightarrow x_2 = -1$

$\therefore X_1 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$

For $\lambda_2 = 3$

(1) $\Rightarrow -2x_1 + 0x_2 - x_3 = 0$

(2) $\Rightarrow x_1 - x_2 + x_3 = 0$

(3) $\Rightarrow 2x_1 + 2x_2 + 0x_3 = 0$

$$\frac{x_1}{\begin{vmatrix} 0 & -1 \\ -1 & 1 \end{vmatrix}} = \frac{-x_2}{\begin{vmatrix} -2 & -1 \\ 1 & 1 \end{vmatrix}} = \frac{x_3}{\begin{vmatrix} -2 & 0 \\ 1 & -1 \end{vmatrix}}$$

$$\frac{x_1}{-1} = \frac{-x_2}{-1} = \frac{x_3}{2}$$

$\boxed{x_1 = -1, \; x_2 = 1, \; x_3 = 2}$

$X_2 = \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix}$

For $\lambda_3 = 2$

$$\begin{aligned} ① &\Rightarrow -x_1 + 0x_2 - x_3 = 0 \\ ② &\Rightarrow x_1 + 0x_2 + x_3 = 0 \end{aligned} \Bigg] \rightarrow x_1 + x_3 = 0$$

$$③ \Rightarrow 2x_1 + 2x_2 + x_3 = 0 \rightarrow 2x_1 + 2x_2 + x_3 = 0$$

Let $x_3 = 2 \Rightarrow x_1 = -2$

$$\therefore 2x_1 + 2x_2 + x_3 \Rightarrow -4 + 2x_2 + 2 = 0$$

$$\Rightarrow -2 + 2x_2 = 0$$

$$\Rightarrow \boxed{x_2 = 1}$$

$$\therefore X_3 = \begin{bmatrix} -2 \\ 1 \\ 2 \end{bmatrix}$$

$$\therefore \text{Modal matrix } B = \begin{bmatrix} 1 & -1 & -2 \\ -1 & 1 & 1 \\ 0 & 2 & 2 \end{bmatrix}$$

(8) Reduce the given matrix to a diagonal form:

$$\begin{bmatrix} -1 & 1 & 2 \\ 0 & -2 & 1 \\ 0 & 0 & -3 \end{bmatrix}$$

Sol^n: Characteristic equation:

$$|A-\lambda I| = 0 \Rightarrow \begin{vmatrix} -1-\lambda & 1 & 2 \\ 0 & -2-\lambda & 1 \\ 0 & 0 & -3-\lambda \end{vmatrix} = 0$$

$$\lambda^3 + 6\lambda^2 + [6+3+2]\lambda + 6 = 0$$

$$\lambda^3 + 6\lambda^2 + 11\lambda + 6 = 0 \quad \text{characteristic eq}^n.$$

$$\boxed{\lambda_1 = -1,\ \lambda_2 = -2,\ \lambda_3 = -3} \text{ Eigen Values}$$

Let $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ be the eigen vector corresponding to eigen value λ such that $[A-\lambda I]X = 0$

$$\begin{bmatrix} -1-\lambda & 1 & 2 \\ 0 & -2-\lambda & 1 \\ 0 & 0 & -3-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$(-1-\lambda)x_1 + x_2 + 2x_3 = 0 \quad \text{——} (1)$$

$$0x_1 + (-2-\lambda)x_2 + x_3 = 0 \quad \text{——} (2)$$

$$0x_1 + 0x_2 + (-3-\lambda)x_3 = 0 \quad \text{——} (3)$$

For $\lambda_1 = -1$

$$(1) \Rightarrow 0x_1 + x_2 + 2x_3 = 0$$

$$(2) \Rightarrow 0x_1 - x_2 + x_3 = 0$$

$$(3) \Rightarrow 0x_1 + 0x_2 - 2x_3 = 0 \Rightarrow$$

$$\frac{x_1}{\begin{vmatrix} 1 & 2 \\ -1 & 1 \end{vmatrix}} = \frac{-x_2}{\begin{vmatrix} 0 & 2 \\ 0 & 1 \end{vmatrix}} = \frac{x_3}{\begin{vmatrix} 0 & 1 \\ 0 & -1 \end{vmatrix}} \Rightarrow \frac{x_1}{+3} = \frac{-x_2}{0} = \frac{x_3}{0}$$

$\therefore X_1 = \begin{bmatrix} +3 \\ 0 \\ 0 \end{bmatrix}$

For $\lambda_2 = -2$

$① \Rightarrow x_1 + x_2 + 2x_3 = 0$

$② \Rightarrow 0x_1 + 0x_2 + x_3 = 0$

$③ \Rightarrow 0x_1 + 0x_2 - x_3 = 0$

$$\frac{x_1}{\begin{vmatrix} 1 & 2 \\ 0 & 1 \end{vmatrix}} = \frac{-x_2}{\begin{vmatrix} 1 & 2 \\ 0 & 1 \end{vmatrix}} = \frac{x_3}{\begin{vmatrix} 1 & 1 \\ 0 & 0 \end{vmatrix}}$$

$$\frac{x_1}{+1} = \frac{-x_2}{1} = \frac{x_3}{0}$$

$$X_2 = \begin{bmatrix} +1 \\ -1 \\ 0 \end{bmatrix}$$

For $\lambda_3 = -3$

$① \Rightarrow 2x_1 + x_2 + 2x_3 = 0$

$② \Rightarrow 0x_1 + 1x_2 + x_3 = 0$

$③ \Rightarrow 0x_1 + 0x_2 + 0x_3 = 0$

$$\frac{x_1}{\begin{vmatrix} 1 & 2 \\ 1 & 1 \end{vmatrix}} = \frac{-x_2}{\begin{vmatrix} 2 & 2 \\ 0 & 1 \end{vmatrix}} = \frac{x_3}{\begin{vmatrix} 2 & 1 \\ 0 & 1 \end{vmatrix}}$$

$$\frac{x_1}{-1} = \frac{-x_2}{2} = \frac{x_3}{2}$$

$$X_3 = \begin{bmatrix} -1 \\ -2 \\ 2 \end{bmatrix}$$

Modal Matrix $B = \begin{bmatrix} 3 & 1 & -1 \\ 0 & -1 & -2 \\ 0 & 0 & 2 \end{bmatrix}$

$|B| \neq 0$

Diagonalization of Matrix:

$$B^{-1}AB = \begin{bmatrix} 3 & 1 & -1 \\ 0 & -1 & -2 \\ 0 & 0 & 2 \end{bmatrix}^{-1} \begin{bmatrix} -1 & 1 & 2 \\ 0 & -2 & 1 \\ 0 & 0 & -3 \end{bmatrix} \begin{bmatrix} 3 & 1 & -1 \\ 0 & -1 & -2 \\ 0 & 0 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 1/3 & 1/3 & 1/2 \\ 0 & -1 & -1 \\ 0 & 0 & 1/2 \end{bmatrix} \begin{bmatrix} -1 & 1 & 2 \\ 0 & -2 & 1 \\ 0 & 0 & -3 \end{bmatrix} \begin{bmatrix} 3 & 1 & -1 \\ 0 & -1 & -2 \\ 0 & 0 & 2 \end{bmatrix}$$

$$B^{-1}AB = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -3 \end{bmatrix}$$

(10) Use Sylvester's theorem to show that $\log_e e^A = A$ where $A = \begin{bmatrix} 3 & 2 \\ 2 & 3 \end{bmatrix}$

Sol:

Let $P(A) = \log_e e^A$

By Sylvester's theorem,

$$P(A) = P(\lambda_1) Z(\lambda_1) + P(\lambda_2) Z(\lambda_2) \quad \text{--- (1)}$$

$$[\lambda I - A] = \begin{bmatrix} \lambda-3 & -2 \\ -2 & \lambda-3 \end{bmatrix}$$

$$|\lambda I - A| = \lambda^2 - 6\lambda + 5 = \phi(\lambda)$$

$\lambda_1 = 1, \lambda_2 = 5$ eigen values

$$P(\lambda_1) = P(1) = \log_e e^1 = 1 \quad \text{--- (2)}$$

$$P(\lambda_2) = P(5) = \log_e e^5 = 5 \quad \text{--- (3)}$$

$$Z[\lambda_2] = \frac{\text{adj}|\lambda I - A|}{\phi'(\lambda)} = \frac{\text{adj}\begin{bmatrix} \lambda-3 & -2 \\ -2 & \lambda-3 \end{bmatrix}}{2\lambda - 6} = \frac{\begin{bmatrix} \lambda-3 & 2 \\ 2 & \lambda-3 \end{bmatrix}}{2\lambda-6}$$

$$Z[\lambda_1] = Z[1] = \frac{\begin{bmatrix} -2 & 2 \\ 2 & -2 \end{bmatrix}}{-4} \quad \text{--- (4)}$$

$$Z[\lambda_2] = Z[5] = \frac{\begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}}{4} \quad \text{--- (5)}$$

$\therefore (1) \Rightarrow$

$$P(A) = \log_e e^A = 1 \frac{\begin{bmatrix} -2 & 2 \\ 2 & -2 \end{bmatrix}}{-4} + \frac{5 \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}}{4}$$

$$\log_A e^A = \frac{1}{4}\left\{\begin{bmatrix} 2+10 & -2+10 \\ -2+10 & 2+10 \end{bmatrix}\right\} = \frac{1}{4}\begin{bmatrix} 12 & 8 \\ 8 & 12 \end{bmatrix}$$

$$\Rightarrow \log_A e^A = \begin{bmatrix} 3 & 2 \\ 2 & 3 \end{bmatrix}$$

$$\Rightarrow \boxed{\log_A e^A = A}$$

(1) Use sylvester's theorem to show that find A^{50}; where $A = \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}$

Sol: Let $P(A) = A^{50}$ —— (1)

By Sylvester's theorem,

$P(A) = P(\lambda_1)Z(\lambda_1) + P(\lambda_2)Z(\lambda_2)$ —— (2)

$[\lambda I - A] = \begin{bmatrix} \lambda-1 & 0 \\ 0 & \lambda-3 \end{bmatrix}$

$\phi(\lambda) = |\lambda I - A| = \begin{vmatrix} \lambda-1 & 0 \\ 0 & \lambda-3 \end{vmatrix}$

$\phi(\lambda) = \lambda^2 - 4\lambda + 3$, $\phi'(\lambda) = 2\lambda - 4$

$\lambda^2 - 4\lambda + 3 = 0 \Rightarrow \boxed{\lambda = 1, 3}$

$P(\lambda_1) = P(1) = 1^{50}$

$P(\lambda_2) = P(3) = 3^{50}$

$Z[\lambda_2] = \dfrac{\text{adj}\begin{bmatrix} \lambda-1 & 0 \\ 0 & \lambda-3 \end{bmatrix}}{2\lambda-4} = \dfrac{\begin{bmatrix} \lambda-3 & 0 \\ 0 & \lambda-1 \end{bmatrix}}{2\lambda-4}$

$Z[1] = \dfrac{\begin{bmatrix} -2 & 0 \\ 0 & 0 \end{bmatrix}}{-2}$, $Z[3] = \dfrac{\begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix}}{2}$

(2) $\Rightarrow$ $P(A) = A^{50} = 1^{50}\dfrac{\begin{bmatrix} -2 & 0 \\ 0 & 0 \end{bmatrix}}{-2} + 3^{50}\dfrac{\begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix}}{2}$

$A^{50} = 1^{50}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + 3^{50}\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$

$$\boxed{A^{50} = \begin{bmatrix} 1^{50} & 0 \\ 0 & 3^{50} \end{bmatrix}}$$

(12) If $A = \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}$ find A^{200} by sylvester's theorem.

Sol: Let $P(A) = A^{200}$ —— ①

By Sylvester's theorem:

$P(A) = P(\lambda_1) Z(\lambda_1) + P(\lambda_2) Z(\lambda_2)$ —— ①

$[\lambda I - A] = \begin{bmatrix} \lambda-1 & -2 \\ 0 & \lambda-3 \end{bmatrix}$

$\phi(\lambda) = |\lambda I - A| = \begin{vmatrix} \lambda-1 & -2 \\ 0 & \lambda-3 \end{vmatrix} = 0$

$\Rightarrow \phi(\lambda) = \lambda^2 - 4\lambda + 3$

$\lambda^2 - 4\lambda + 3 = 0 \Rightarrow \lambda_1 = 1, \lambda_2 = 3$

$P(\lambda_1) = P(1) = 1^{200}, \quad P(\lambda_2) = P(3) = 3^{200}$

$Z[\lambda_i] = \dfrac{adj[\lambda I - A]}{\phi'(\lambda)} = \dfrac{adj \begin{bmatrix} \lambda-1 & -2 \\ 0 & \lambda-3 \end{bmatrix}}{2\lambda - 4} = \dfrac{\begin{bmatrix} \lambda-3 & 2 \\ 0 & \lambda-1 \end{bmatrix}}{2\lambda - 4}$

$Z[\lambda_1] = Z[1] = \dfrac{\begin{bmatrix} -2 & 2 \\ 0 & 0 \end{bmatrix}}{-2}$

$Z[\lambda_2] = Z[3] = \dfrac{\begin{bmatrix} 0 & 2 \\ 0 & 2 \end{bmatrix}}{2}$

① $\Rightarrow P(A) = A^{200} = \dfrac{1^{200} \begin{bmatrix} -2 & 2 \\ 0 & 0 \end{bmatrix}}{-2} + \dfrac{3^{200} \begin{bmatrix} 0 & 2 \\ 0 & 2 \end{bmatrix}}{2}$

$A^{200} = 1^{200} \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix} + 3^{200} \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$

$= \begin{bmatrix} 1^{200} & -1^{200} \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 3^{200} \\ 0 & 3^{200} \end{bmatrix}$

$\boxed{A^{200} = \begin{bmatrix} 1^{200} & -1^{200} + 3^{200} \\ 0 & 3^{200} \end{bmatrix}}$

(13) Use Sylvester's theorem to show that $\sin^2 A + \cos^2 A = I$ where $A = \begin{bmatrix} 1 & 2 \\ -1 & 4 \end{bmatrix}$

Sol: Let $P(A) = \sin^2 A + \cos^2 A$ —— (1)

By Sylvester's, $P(A) = P(\lambda_1) Z(\lambda_1) + P(\lambda_2) Z(\lambda_2)$ —— (2)

$$[\lambda I - A] = \begin{bmatrix} \lambda-1 & -2 \\ 1 & \lambda-4 \end{bmatrix}$$

$$\phi(\lambda) = \lambda^2 - 5\lambda + 6$$

$$\lambda^2 - 5\lambda + 6 = 0 \Rightarrow \lambda_1 = 2, \lambda_2 = 3$$

$$P(\lambda_1) = P(2) = \sin^2 2 + \cos^2 2$$

$$P(\lambda_2) = P(3) = \sin^2 3 + \cos^2 3$$

$$Z[\lambda_2] = \frac{\operatorname{adj}[\lambda I - A]}{\phi'(\lambda)} = \frac{\operatorname{adj}\begin{bmatrix} \lambda-1 & -2 \\ 1 & \lambda-4 \end{bmatrix}}{2\lambda - 5} = \frac{\begin{bmatrix} \lambda-4 & 2 \\ -1 & \lambda-1 \end{bmatrix}}{2\lambda-5}$$

$$Z[\lambda_1] = Z[2] = \frac{\begin{bmatrix} -2 & 2 \\ -1 & 1 \end{bmatrix}}{-1} = \begin{bmatrix} 2 & -2 \\ 1 & -1 \end{bmatrix}$$

$$Z[\lambda_2] = Z[3] = \frac{\begin{bmatrix} -1 & 2 \\ -1 & 2 \end{bmatrix}}{1} = \begin{bmatrix} -1 & 2 \\ -1 & 2 \end{bmatrix}$$

$\therefore (1) \Rightarrow$

$$\sin^2 A + \cos^2 A = (\sin^2 2 + \cos^2 2)\begin{bmatrix} 2 & -2 \\ 1 & -1 \end{bmatrix} + (\sin^2 3 + \cos^2 3)\begin{bmatrix} -1 & 2 \\ -1 & 2 \end{bmatrix}$$

$$= 1\begin{bmatrix} 2 & -2 \\ 1 & -1 \end{bmatrix} + \begin{bmatrix} -1 & 2 \\ -1 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

$$\Rightarrow \boxed{\sin^2 A + \cos^2 A = I}$$

(1) Find largest eigen value and corresponding eigen vector for the matrix $A = \begin{bmatrix} 1 & 6 & 1 \\ 1 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$

Sol: let $X^{(0)} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ be the initial approximation.

$$AX^{(0)} = \begin{bmatrix} 1 & 6 & 1 \\ 1 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = 1 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \lambda^{(1)} X^{(1)}$$

$$AX^{(1)} = \begin{bmatrix} 1 & 6 & 1 \\ 1 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 7 \\ 3 \\ 0 \end{bmatrix} = 7 \begin{bmatrix} 1 \\ 0.4285 \\ 0 \end{bmatrix} = \lambda^{(2)} X^{(2)}$$

$$AX^{(2)} = \begin{bmatrix} 1 & 6 & 1 \\ 1 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 0.4285 \\ 0 \end{bmatrix} = \begin{bmatrix} 3.571 \\ 1.857 \\ 0 \end{bmatrix} = 3.571 \begin{bmatrix} 1 \\ 0.5200 \\ 0 \end{bmatrix}$$

$$= \lambda^{(3)} X^{(3)}$$

$$AX^{(3)} = \begin{bmatrix} 1 & 6 & 1 \\ 1 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 0.5200 \\ 0 \end{bmatrix} = \begin{bmatrix} 4.12 \\ 2.04 \\ 0 \end{bmatrix} = 4.12 \begin{bmatrix} 1 \\ 0.4951 \\ 0 \end{bmatrix} = \lambda^{(4)} X^{(4)}$$

$$AX^{(4)} = \begin{bmatrix} 1 & 6 & 1 \\ 1 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 0.4951 \\ 0 \end{bmatrix} = \begin{bmatrix} 3.9706 \\ 1.9902 \\ 0 \end{bmatrix} = 3.970 \begin{bmatrix} 1 \\ 0.5012 \\ 0 \end{bmatrix} = \lambda^{(5)} X^{(5)}$$

$$AX^{(5)} = \begin{bmatrix} 1 & 6 & 1 \\ 1 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 0.5012 \\ 0 \end{bmatrix} = \begin{bmatrix} 4.0072 \\ 2.0024 \\ 0 \end{bmatrix} = 4.007 \begin{bmatrix} 1 \\ 0.4997 \\ 0 \end{bmatrix} = \lambda^{(6)} X^{(6)}$$

$$AX^{(6)} = \begin{bmatrix} 1 & 6 & 1 \\ 1 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 0.4997 \\ 0 \end{bmatrix} = \begin{bmatrix} 3.9982 \\ 1.9994 \\ 0 \end{bmatrix} = 3.99 \begin{bmatrix} 1 \\ 0.5 \\ 0 \end{bmatrix} = \lambda^{(7)} X^{(7)}$$

$$AX^{(7)} = \begin{bmatrix} 1 & 6 & 1 \\ 1 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 0.5 \\ 0 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \\ 0 \end{bmatrix} = 4 \begin{bmatrix} 1 \\ 0.5 \\ 0 \end{bmatrix} = \lambda^{(8)} x^{(8)}$$

$$AX^{(8)} = \begin{bmatrix} 1 & 6 & 1 \\ 1 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 0.5 \\ 0 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \\ 0 \end{bmatrix} = 4 \begin{bmatrix} 1 \\ 0.5 \\ 0 \end{bmatrix} = \lambda^{(9)} X^{(9)}$$

Largest eigen value is 4 and corresponding eigen vector $\begin{bmatrix} 1 \\ 0.5 \\ 0 \end{bmatrix}$

(15) Find the Largest eigen value and corresponding eigen vector ~~for~~ the matrix $A = \begin{bmatrix} 4 & 1 \\ 1 & 3 \end{bmatrix}$

Sol^n: Let $X^{(0)} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ be the initial approximation.

$$AX^{(0)} = \begin{bmatrix} 4 & 1 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \end{bmatrix} = 4 \begin{bmatrix} 1 \\ 0.25 \end{bmatrix} = \lambda^{(1)} X^{(1)}$$

$$AX^{(1)} = \begin{bmatrix} 4 & 1 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 0.25 \end{bmatrix} = \begin{bmatrix} 4.25 \\ 1.75 \end{bmatrix} = 4.75 \begin{bmatrix} 1 \\ 0.4117 \end{bmatrix} = \lambda^{(2)} X^{(2)}$$

$$AX^{(2)} = \begin{bmatrix} 4 & 1 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 0.4117 \end{bmatrix} = \begin{bmatrix} 4.4117 \\ 2.2351 \end{bmatrix} = \begin{bmatrix} 1 \\ 0.5066 \end{bmatrix} = \lambda^{(3)} X^{(3)}$$

$$AX^{(3)} = \begin{bmatrix} 4 & 1 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 0.5066 \end{bmatrix} = \begin{bmatrix} 4.5066 \\ 2.5198 \end{bmatrix} = 4.5066 \begin{bmatrix} 1 \\ 0.5591 \end{bmatrix} = \lambda^{(4)} X^{(4)}$$

$$AX^{(4)} = \begin{bmatrix} 4 & 1 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 0.5591 \end{bmatrix} = \begin{bmatrix} 4.5591 \\ 2.6773 \end{bmatrix} = 4.5591 = \begin{bmatrix} 1 \\ 0.5872 \end{bmatrix} = \lambda^{(5)} X^{(5)}$$

$$AX^{(5)} = \begin{bmatrix} 4 & 1 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 0.5872 \end{bmatrix} = \begin{bmatrix} 4.5872 \\ 2.7616 \end{bmatrix} = 4.5872 \begin{bmatrix} 1 \\ 0.602 \end{bmatrix} = \lambda^{(6)} X^{(6)}$$

$$AX^{(6)} = \begin{bmatrix} 4 & 1 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 0.602 \end{bmatrix} = \begin{bmatrix} 4.602 \\ 2.806 \end{bmatrix} = 4.602 \begin{bmatrix} 1 \\ 0.6097 \end{bmatrix} = \lambda^{(7)} X^{(7)}$$

$$AX^{(7)} = \begin{bmatrix} 4 & 1 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ 0.6097 \end{bmatrix} = \begin{bmatrix} 4.6097 \\ 2.8291 \end{bmatrix} = 4.6097\begin{bmatrix} 1 \\ 0.6187 \end{bmatrix} = \lambda^{(8)}X^{(8)}$$

$$AX^{(8)} = \begin{bmatrix} 4 & 1 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ 0.6137 \end{bmatrix} = \begin{bmatrix} 4.6137 \\ 2.8411 \end{bmatrix} = 4.6137\begin{bmatrix} 1 \\ 0.6157 \end{bmatrix} = \lambda^{(9)}X^{(9)}$$

$$AX^{(9)} = \begin{bmatrix} 4 & 1 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ 0.6157 \end{bmatrix} = \begin{bmatrix} 4.6157 \\ 2.8471 \end{bmatrix} = 4.6157\begin{bmatrix} 1 \\ 0.6168 \end{bmatrix} = \lambda^{(10)}X^{(10)}$$

$\therefore$ Largest eigen value is 4.6157 and corresponding eigen vector is $\begin{bmatrix} 1 \\ 0.6168 \end{bmatrix}$

(16) Find largest eigen value and corresponding eigen vector for the matrix $A = \begin{bmatrix} -4 & -5 \\ 1 & 2 \end{bmatrix}$

Soln: $X^{(0)} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$$AX^{(0)} = \begin{bmatrix} -4 & -5 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -4 \\ 1 \end{bmatrix} = -4\begin{bmatrix} 1 \\ -0.25 \end{bmatrix} = \lambda^{(1)}X^{(1)}$$

$$AX^{(1)} = \begin{bmatrix} -4 & -5 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ -0.25 \end{bmatrix} = \begin{bmatrix} -2.75 \\ 0.5 \end{bmatrix} = -2.75\begin{bmatrix} 1 \\ -0.1818 \end{bmatrix} = \lambda^{(2)}X^{(2)}$$

$$AX^{(2)} = \begin{bmatrix} -4 & -5 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ -0.1818 \end{bmatrix} = \begin{bmatrix} -3.091 \\ 0.6364 \end{bmatrix} = -3.091\begin{bmatrix} 1 \\ -0.2058 \end{bmatrix} = \lambda^{(3)}X^{(3)}$$

$$AX^{(3)} = \begin{bmatrix} -4 & -5 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ -0.2058 \end{bmatrix} = \begin{bmatrix} -2.971 \\ 0.5884 \end{bmatrix} = -2.971\begin{bmatrix} 1 \\ -0.1980 \end{bmatrix} = \lambda^{(4)}X^{(4)}$$

$$AX^{(4)} = \begin{bmatrix} -4 & -5 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ -0.1980 \end{bmatrix} = \begin{bmatrix} -3.01 \\ 0.604 \end{bmatrix} = -3.01\begin{bmatrix} 1 \\ -0.2006 \end{bmatrix} = \lambda^{(5)}X^{(5)}$$

$$AX^{(5)} = \begin{bmatrix} -4 & -5 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ -0.2006 \end{bmatrix} = \begin{bmatrix} -2.997 \\ 0.5988 \end{bmatrix} = -2.997\begin{bmatrix} 1 \\ -0.1997 \end{bmatrix} = \lambda^{(6)}X^{(6)}$$

$$AX^{(6)} = \begin{bmatrix} -4 & -5 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ -0.1997 \end{bmatrix} = \begin{bmatrix} -3.0015 \\ 0.6006 \end{bmatrix} = -3.0015 \begin{bmatrix} 1 \\ -0.2000 \end{bmatrix} = \lambda^{(7)} X^{(7)}$$

$$AX^{(7)} = \begin{bmatrix} -4 & -5 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ -0.2000 \end{bmatrix} = \begin{bmatrix} -3 \\ 0.6 \end{bmatrix} = -3 \begin{bmatrix} 1 \\ -0.2 \end{bmatrix} = \lambda^{(8)} X^{(8)}$$

Largest eigen value is -3 and corresponding eigen vector is $\begin{bmatrix} 1 \\ -0.2 \end{bmatrix}$

Q7) Find the eigen values and eigen vectors and modal matrix for the matrix $A = \begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$

Soln: Characteristic eqn is $|A - \lambda I| = 0$

$$\begin{vmatrix} 8-\lambda & -6 & 2 \\ -6 & 7-\lambda & -4 \\ 2 & -4 & 3-\lambda \end{vmatrix} = 0$$

$\lambda^3 - 18\lambda^2 + 45\lambda - 0 = 0$ or $\lambda(\lambda-3)(\lambda-15) = 0$

$\boxed{\lambda = 0, 3, 15}$ eigen values

Let $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ be the eigen vector corresponding to eigen value λ such that $[A - \lambda I]X = 0$

$$\therefore \begin{bmatrix} 8-\lambda & -6 & 2 \\ -6 & 7-\lambda & -4 \\ 2 & -4 & 3-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$(8-\lambda)x_1 - 6x_2 + 2x_3 = 0$ —— (1)

$-6x_1 + (7-\lambda)x_2 - 4x_3 = 0$ —— (2)

$2x_1 - 4x_2 + (3-\lambda)x_3 = 0$ —— (3)

For $\lambda = 0$

① $\Rightarrow 8x_1 - 6x_2 + 2x_3 = 0$

② $\Rightarrow -6x_1 + 7x_2 - 4x_3 = 0$

③ $\Rightarrow 2x_1 - 4x_2 + 3x_3 = 0$

$$\frac{x_1}{\begin{vmatrix} -6 & 2 \\ 7 & -4 \end{vmatrix}} = \frac{-x_2}{\begin{vmatrix} 8 & 2 \\ -6 & -4 \end{vmatrix}} = \frac{x_3}{\begin{vmatrix} 8 & -6 \\ -6 & 7 \end{vmatrix}}$$

$$\frac{x_1}{10} = \frac{-x_2}{-20} = \frac{x_3}{20}$$

$\Rightarrow x_1 = 1, \quad x_2 = 2, \quad x_3 = 2$

$$X_1 = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$$

For $\lambda = 3$

① $\Rightarrow 5x_1 - 6x_2 + 2x_3 = 0$

② $\Rightarrow -6x_1 + 4x_2 - 4x_3 = 0$

③ $\Rightarrow 2x_1 - 4x_2 + 0x_3 = 0$

$$\frac{x_1}{\begin{vmatrix} -6 & 2 \\ 4 & -4 \end{vmatrix}} = \frac{-x_2}{\begin{vmatrix} 5 & 2 \\ -6 & -4 \end{vmatrix}} = \frac{x_3}{\begin{vmatrix} 5 & -6 \\ -6 & 4 \end{vmatrix}}$$

$$\frac{x_1}{16} = \frac{-x_2}{-8} = \frac{x_3}{-16}$$

$x_1 = 2, \quad x_2 = 1, \quad x_3 = 2$

$$X_2 = \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}$$

For $\lambda = 15$

① $\Rightarrow -7x_1 - 6x_2 + 2x_3 = 0$

② $\Rightarrow -6x_1 - 8x_2 - 4x_3 = 0 \Rightarrow 3x_1 + 4x_2 + 2x_3 = 0$

③ $\Rightarrow 2x_1 - 4x_2 - 12x_3 = 0 \Rightarrow x_1 - 2x_2 - 6x_3 = 0$

$$\frac{x_1}{\begin{vmatrix} 6 & 2 \\ 4 & 2 \end{vmatrix}} = \frac{-x_2}{\begin{vmatrix} -7 & 2 \\ 3 & 2 \end{vmatrix}} = \frac{x_3}{\begin{vmatrix} -7 & -6 \\ 3 & 4 \end{vmatrix}}$$

$$\frac{x_1}{-20} = \frac{-x_2}{-20} = \frac{x_3}{-10}$$

$x_1 = -2$, $x_2 = 2$, $x_3 = -1$

$$X_3 = \begin{bmatrix} -2 \\ 2 \\ -1 \end{bmatrix}$$

Modal Matrix $B = \begin{bmatrix} 1 & 2 & -2 \\ 2 & 1 & 2 \\ 2 & -2 & -1 \end{bmatrix}$

(9) Reduce the given matrix to a diagonal form

$$A = \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$$

Sol^n Chara. eq^n is $|A - \lambda I| = 0$

$$\begin{vmatrix} 6-\lambda & -2 & 2 \\ -2 & 3-\lambda & -1 \\ 2 & -1 & 3-\lambda \end{vmatrix} = 0$$

$$\lambda^3 - 12\lambda^2 + 36\lambda - 32 = 0$$

$$\lambda = 2, 2, 8$$

Let $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ be the eigen value vector corresponding to eigen value λ such that

$[A - \lambda I] = 0$

$(6-\lambda)x_1 - 2x_2 + 2x_3 = 0$ —— (1)

$-2x_1 + (3-\lambda)x_2 - x_3 = 0$ —— (2)

$2x_1 - x_2 + (3-\lambda)x_3 = 0$ —— (3)

For $\lambda=2$

$$\begin{rcases} (1) \Rightarrow 4x_1-2x_2+2x_3=0 \\ (2) \Rightarrow -2x_1+x_2-x_3=0 \\ (3) \Rightarrow 2x_1-x_2+x_3=0 \end{rcases} \rightarrow 2x_1-x_2+x_3=0$$

Let $x_1=1$, $x_2=0 \Rightarrow x_3=-2$

$$X_1=\begin{bmatrix}1\\0\\-2\end{bmatrix}$$

For $\lambda=2$

$2x_1-x_2+x_3=0$

Let $x_1=0$, $x_2=1 \Rightarrow x_3=1$

$$X_2=\begin{bmatrix}0\\1\\1\end{bmatrix}$$

For $\lambda=8$

$$-2x_1-2x_2+2x_3=0 \Rightarrow x_1+x_2-x_3=0$$
$$-2x_1-5x_2-x_3=0 \Rightarrow 2x_1+5x_2+x_3=0$$
$$2x_1-x_2-5x_3=0 \Rightarrow 2x_1-x_2-5x_3=0$$

$$\frac{x_1}{\begin{vmatrix}1&-1\\5&1\end{vmatrix}}=\frac{-x_2}{\begin{vmatrix}1&-1\\2&1\end{vmatrix}}=\frac{x_3}{\begin{vmatrix}1&1\\2&5\end{vmatrix}}$$

$$\frac{x_1}{6}=\frac{-x_2}{3}=\frac{x_3}{3}$$

$\Rightarrow x_1=2$, $x_2=-1$, $x_3=1$

$$X_3=\begin{bmatrix}2\\-1\\1\end{bmatrix}$$

Modal matrix $B=\begin{bmatrix}1&0&2\\0&1&-1\\-2&1&1\end{bmatrix}$

Now $|B| \neq 0$

$$B^{-1}AB = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -1 \\ -2 & 1 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix} \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -1 \\ -2 & 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1/3 & 1/3 & -1/3 \\ 1/3 & 5/6 & 1/6 \\ 1/3 & -1/6 & 1/6 \end{bmatrix} \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix} \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -1 \\ -2 & 1 & 1 \end{bmatrix}$$

$$B^{-1}AB = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 8 \end{bmatrix}$$ is the required diagonal form

Q5. Show that the vectors $[0, 1, -2], [1, -1, 1], [1, 2, 1]$ form a linearly independent set.

Sol: $k_1X_1 + k_2X_2 + k_3X_3 = 0$ — (1)

$k_1[0, 1, -2] + k_2[1, -1, 1] + k_3[1, 2, 1] = [0, 0, 0]$

$0k_1 + k_2 + k_3 = 0$ — (2)

$k_1 - k_2 + 2k_3 = 0$ — (3)

$-2k_1 + k_2 + k_3 = 0$ — (4)

$$\frac{k_1}{\begin{vmatrix} 1 & 1 \\ -1 & 2 \end{vmatrix}} = \frac{-k_2}{\begin{vmatrix} 0 & 1 \\ 1 & 2 \end{vmatrix}} = \frac{k_3}{\begin{vmatrix} 0 & 1 \\ 1 & -1 \end{vmatrix}}$$

$$\frac{k_1}{3} = \frac{-k_2}{-1} = \frac{k_3}{-1}$$

$\Rightarrow k_1 = 3, \quad k_2 = 1, \quad k_3 = -1$ not all zeros, but not satisfy eq^n (4)

$\therefore$ given vectors are linearly independent.