Unit - 5 (1)

Mathematical Expectation & Probability Distribution

(1) A density function of a random variable X is

$$f(x) = \begin{cases} 2e^{-2x} & x \geq 0 \\ 0, & \text{otherwise} \end{cases}$$

find (i) $E(X)$ (ii) $V(X)$ (iii) σ_x (iv) $E[(x-1)^2]$

Soln: $E(X) = \int_{-\infty}^{\infty} x f(x)dx = 2\int_0^{\infty} x e^{-2x} dx$

$$= 2\left[\frac{xe^{-2x}}{-2} - \int \frac{e^{-2x}}{-2} dx\right]$$

$$= 2\left[\frac{xe^{-2x}}{-2} - \frac{1}{4} e^{-2x}\right]_0^{\infty}$$

$$= 2\left[0 - \left(0 - \frac{1}{4}\right)\right] = \frac{1}{2}$$

$$\boxed{E(X) = \frac{1}{2}}$$

(ii) $V(X) = E(X^2) - [E(X)]^2$

$$= \int_{-\infty}^{\infty} x^2 f(x)dx - \left[\frac{1}{2}\right]^2$$

$$= \int_0^{\infty} x^2 2e^{-2x} dx - \frac{1}{4}$$

$$= 2\left[\frac{x^2 e^{-2x}}{-2} - \int \frac{2xe^{-2x}}{-2} dx\right] - \frac{1}{4} = 2\left[\frac{x^2 e^{-2x}}{-2} + \int x e^{-2x} dx\right] - \frac{1}{4}$$

$$= 2\left[\frac{x^2 e^{-2x}}{-2} + \left(\frac{xe^{-2x}}{-2} - \int \frac{e^{-2x}}{-2} dx\right)\right] - \frac{1}{4}$$

$$= 2\left[\frac{x^2 e^{-2x}}{-2} - \frac{xe^{-2x}}{2} - \frac{e^{-2x}}{4}\right]_0^{\infty} - \frac{1}{4} = 2\left[0 - \left(-\frac{1}{4}\right)\right] - \frac{1}{4} = \frac{1}{2} - \frac{1}{4} = \frac{1}{4}$$

1

(iii) $\sigma_x = \sqrt{Var X} = \sqrt{1/4} = 0.5$

(iv)

$$E(X-1)^2 = E(X^2-2X+1) = E(X^2) - 2E(X) + E(1)$$

$$= \int_{-\infty}^{\infty} x^2 f(x)\,dx - 2\times\frac{1}{2} + 1$$

$$= \int_{-\infty}^{\infty} x^2 f(x)\,dx - 1 + 1$$

$$= \int_0^{\infty} x^2 f(x)\,dx = 2\int_0^{\infty} x^2 e^{-2x}\,dx$$

$$= 2\left[x^2 \cdot \frac{e^{-2x}}{-2} - \int 2x\frac{e^{-2x}}{-2}\,dx\right]$$

$$= 2\left[\frac{x^2 e^{-2x}}{-2} + \left(x\frac{e^{-2x}}{-2} - \int \frac{e^{-2x}}{-2}\,dx\right)\right]$$

$$= 2\left[\frac{x^2 e^{-2x}}{-2} - \frac{x e^{-2x}}{2} - \frac{e^{-2x}}{4}\right]_0^{\infty}$$

$$= 2\left[0 - \left(-\frac{1}{4}\right)\right] = \frac{1}{2}$$

(2) If a random variable such that $E[(x-1)^2] = 10$ & $E[(x-2)^2] = 6$ find (i) $E(X^2)$ (ii) $Var(X)$ (iii) σ_x

Sol: $E[(x-2)^2] = 6 \Rightarrow E[x^2 - 2x + 4] = 6$

$E(x^2) - 4E(x) + E(4) = 6$

$E(x^2) - 4E(x) + 4 = 6$ —— (1)

$E(x-1)^2 = E(x^2 - 2x + 1) = E(x^2) - 2E(x) + E(1)$

$= E(x^2) - 2E(x) + 1$

$E(x^2) - 2E(x) + 1 = 10$ —— (2)

2

$$E(x^2) - 4E(x) + 4 = 6 \quad \text{---} ①$$
$$E(x^2) - 2E(x) + 1 = 10 \quad \text{---} ②$$
$$-\ \ +\ \ -\ \ -$$

$$-2E(x) + 3 = -4$$
$$-2E(x) = -7$$

$$\boxed{E(x) = 7/2}$$

$$E(x^2) - 4E(x) + 4 = 6$$
$$E(x^2) - 2E(x) + 1 = 10$$

$$2E(x^2) - 6E(x) + 5 = 16$$
$$2E(x^2) - 6 \times 7/2 + 5 = 16$$
$$\Rightarrow 2E(x^2) - \frac{42}{2} = 11$$
$$\Rightarrow 2E(x^2) = 11 + 21$$
$$\Rightarrow 2E(x^2) = 32$$

(i) $\boxed{E(x^2) = 16}$

(ii) $Var(X) = E(x^2) - [E(x)]^2$

$$= 16 - (7/2)^2$$
$$= 16 - 49/4$$

$$\boxed{Var(x) = 15/4}$$

(iii) $\sigma x = \sqrt{var(X)} = \sqrt{15/2}$

(4) Find the mathematical expectation of discrete random variable X whose probability function is $p(x) = (1/2)^x$, $x = 1, 2, 3 \ldots$

Sol: We have, $E(X) = \sum_{x=1}^{\infty} x\, p(x)$

3

$\therefore E(x) = 1(1/2) + 2(1/2)^2 + 3(1/2)^3 + 4(1/2)^4 \cdots$

Let $S = 1/2 + 2(1/4) + 3(1/8) + 4(1/16) + \cdots$ —— ①

$\therefore \frac{1}{2}S = \frac{1}{4} + 2(1/8) + 3(1/16) + 4(1/32) + \cdots$

$\therefore S - \frac{1}{2}S = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots = 1$

i.e $\frac{1}{2}S = 1$ or $S = 2$

$\therefore$ from ①, $E(X) = 2$.

(5) A random variable X is defined by $f(x) = \begin{cases} -2, & \text{prob } 1/3 \\ 3, & \text{prob } 1/2 \\ 1, & \text{prob } 1/6 \end{cases}$

Find (i) $E(x)$ (ii) $E(2x+3)$ (iii) $E(x^2)$ (iv) $E(x^2+5x)$ (v) $Var(x)$

Sol:

(i) $E(X) = \Sigma x f(x) = -2\times 1/3 + 3\times\frac{1}{2} + 1\times\frac{1}{6}$

$= -2/3 + 3/2 + 1/6$

$E(X) = \frac{-4+9+1}{6} = \frac{6}{6} = 1$

$\boxed{E(X) = 1}$

(ii) $E(2x+3) = 2E(x) + E(3)$

$= 2(1) + 3$

$\boxed{E(2x+3) = 5}$

(iii) $E(x^2) = \Sigma x^2 f(x) = (-2)^2\frac{1}{3} + (3)^2\times\frac{1}{2} + (1)^2\times\frac{1}{6}$

$= \frac{4}{3} + \frac{9}{2} + \frac{1}{6}$

$E(x^2) = \frac{8+27+1}{6} = \frac{36}{6}$

$\boxed{E(x^2) = 6}$

(iv) $E(x^2+5x) = E(x^2) + 5E(X) = 6 + 5\cdot 1$

$\boxed{E(x^2+5x) = 11}$

4

(v) $Var(X) = E(X^2) - [E(X)]^2 = 6 - 1 = 5$

(7) Find the moment generating function for random variable X having density function $f(x) = \begin{cases} e^{-x}, & x \geq 0 \\ 0, & x < 0 \end{cases}$ and determine first four moment about origin and mean.

Sol^n:

$M_X(t) = E[e^{tX}] = \int_{-\infty}^{\infty} e^{tx} f(x)dx$

$\therefore M_X(t) = \int_{-\infty}^{0} e^{tx} f(x)dx + \int_{0}^{\infty} e^{tx} f(x)dx$

$= \int_0^{\infty} e^{tx} e^{-x} dx = \int_0^{\infty} e^{-(1-t)x} dx = \left[\frac{e^{-(1-t)x}}{-(1-t)}\right]_0^{\infty}$

$M_X(t) = \frac{e^{-\infty} - e^{0}}{-(1-t)} = \frac{-1}{-(1-t)} = \frac{1}{(1-t)}$

$\boxed{M_X(t) = \frac{1}{1-t}}$, for $t<1$

We have,

$M_X(t) = (1-t)^{-1} = 1 + t + t^2 + t^3 + t^4 + \cdots$

but $M_X(t) = 1 + \mu_1' t + \mu_2' \frac{t^2}{2!} + \mu_3' \frac{t^3}{3!} + \mu_4' \frac{t^4}{4!} + \cdots$

$\therefore \mu_1' = 1,\ \mu_2' = 2! = 2,\ \mu_3' = 3! = 6,\ \mu_4' = 4! = 24$

are the first four moments about the origin.

Moments about the mean are:

$\mu_1 = E(X-\mu) = \mu - \mu = 0$

$\mu_2 = \mu_2' - \mu = 2 - 1 = 1$

$\mu_3 = \mu_3' - 3\mu_2'\mu + 2\mu^3 = 6 - 3(2)(1) + 2(1) = 2$

$\mu_4 = \mu_4' - 4\mu_3'\mu + 6\mu_2'\mu^2 - 3\mu^4 = 24 - 4(6)(1) + 6(2)(1)^2 - 3(1)^4 = 9$

(8) A density function of random variable X is

$f(x) = \begin{cases} 2e^{-2x}, & x \geq 0 \\ 0, & x < 0 \end{cases}$ Find moment generating function and first four moments about origin.

Sol:
$$M_X(t) = E[e^{tX}] = \int_{-\infty}^{\infty} e^{tX} f(x)dx = \int_0^{\infty} e^{tX} \cdot 2e^{-2X} dx$$

$$= 2\int_0^{\infty} e^{-(2-t)x} dx = \left. \frac{2e^{-(2-t)x}}{-(2-t)} \right]_0^{\infty}$$

$$= \cancel{\frac{-2}{(2-t)}\left[e^{-(2-t)}\right.}$$

$$= \frac{2}{-(2-t)}\left[e^{-\infty} - e^{0}\right] = \frac{2}{(2-t)}, \text{ for } t<2$$

We have $M_X(t) = 2(2-t)^{-1} = 2 \cdot 2^{-1}(1-t/2)^{-1}$

$$M_X(t) = 1 + t/2 + t^2/4 + t^3/8 + \frac{t^4}{16} + \cdots$$

but $M_X(t) = 1 + \mu_1' t + \mu_2' t^2/2! + \mu_3' t^3/3! + \mu_4' \frac{t^4}{4!} + \cdots$

$\Rightarrow \mu_1' = \frac{1}{2}, \mu_2' = \frac{1}{2}, \mu_3' = \frac{3}{4}, \mu_4' = \frac{3}{2}$ moments about origin.

(10) Out of 800 families with 5 children each, how many would you expect to have (i) 3 boys (ii) 5 girls (iii) Either 2 or 3 boys? Assume equal probabilities for boys and girls.

Sol: Let p & q be the probabilities of success and failure respectively. Let n be the number of trials.

$p = 1/2, q = 1/2, n = 5$

(i) $P(3 \text{ boys}) = P(X=3) = {}^5C_3 \left(\frac{1}{2}\right)^3 \left(\frac{1}{2}\right)^2$

$= 5/16$

6

∴ the no. of families having exactly 3 boys is $\frac{5}{16} \times 800 = 250$

(ii) $P(5 \text{ girls}) = P(X=5) = {}^5C_5 \left(\frac{1}{2}\right)^5 \left(\frac{1}{2}\right)^0 = \left(\frac{1}{2}\right)^5 = \frac{1}{32}$

∴ the number of families having 5 girls is $\frac{1}{32} \times 800 = 25.$

(iii) $P(2 \text{ or } 3 \text{ boys}) = P(X=2) + P(X=3) = {}^5C_2 \left(\frac{1}{2}\right)^2 \left(\frac{1}{2}\right)^3 + {}^5C_3 \left(\frac{1}{2}\right)^3 \left(\frac{1}{2}\right)^2$

$$= 10\left(\frac{1}{2}\right)^5 + 10\left(\frac{1}{2}\right)^5$$

$$= 20\left(\frac{1}{2}\right)^5 = \frac{5}{8}$$

∴ the number of families having 2 or 3 boys is $5/8 \times 800 = 500.$

~~In a normal distribution 31% of items are under~~

Q.5. Find moment generating function and first four moments about origin for r.v. X is given by

$$X = \begin{cases} 1/2, & \text{prob } 1/2 \\ -1/2, & \text{prob } 1/2 \end{cases}$$

Sol:

$$M_X(t) = E(e^{tX}) = \sum e^{tx} f(x)$$

$$= e^{t/2}\left(\tfrac{1}{2}\right) + e^{-t/2}\left(\tfrac{1}{2}\right) = \frac{e^{-t/2} + e^{t/2}}{2} = \cosh(t/2)$$

Using Taylor series expansion of $\cosh(t/2)$, we get

$$M_X(t) = 1 + \frac{(t/2)^2}{2!} + \frac{(t/2)^4}{4!} + \cdots$$

but $M_X(t) = 1 + \mu_1' \cdot t + \mu_2' \cdot t^2/2! + \mu_3' t^3/3! + \mu_4' \cdot t^4/4! + \cdots$

∴ $\mu_1' = 0$, $\mu_2' = 1/4$, $\mu_3' = 0$, $\mu_4' = 1/16$ are the first four moments about the origin.

7

(9) Find the moment generating function for the uniform distribution $f(x) = \begin{cases} \frac{1}{b-a}, & a<x<b \\ 0, & \text{otherwise} \end{cases}$ and also find the first ~~three~~ four moments about origin.

Sol^n:
$$M_x(t) = E[e^{tx}] = \int_{-\infty}^{\infty} e^{tx} f(x)\,dx$$
$$= \int_a^b e^{tx} \frac{1}{b-a}\,dx$$
$$= \left.\frac{e^{tx}}{t(b-a)}\right]_a^b = \frac{e^{bt}-e^{at}}{t(b-a)}$$

First two moments about origin:

Moment about origin, $\boxed{\mu_r' = E(X^r)}$ Formula

$$\mu_1' = E(X) = \int_a^b x\frac{1}{b-a}\,dx$$
$$= \left.\frac{x^2}{2(b-a)}\right]_a^b$$
$$= \frac{b^2-a^2}{2(b-a)} = \frac{b+a}{2}$$

$$\boxed{\mu_1' = \frac{b+a}{2}}$$

$$\mu_2' = E(X^2) = \int_a^b x^2 \cdot f(x)\,dx$$
$$= \int_a^b x^2 \cdot \frac{1}{b-a}\,dx = \left.\frac{x^3}{3(b-a)}\right]_a^b$$
$$= \frac{b^3-a^3}{3(b-a)} = \frac{\cancel{(b-a)}(b^2+ab+a^2)}{3\cancel{(b-a)}} = \frac{b^2+ab+a^2}{3}$$

$$\mu_3' = E(X^3) = \int_a^b x^3 \cdot f(x)\,dx = \int_a^b x^3 \frac{1}{b-a}\,dx$$
$$= \left.\frac{x^4}{4(b-a)}\right]_a^b = \frac{b^4-a^4}{4(b-a)}$$

$$\mu_4' = E(x^4) = \int_a^b x^4 f(x)\,dx = \int_a^b x^4 \cdot \frac{1}{b-a}\,dx$$

$$= \left.\frac{x^5}{b-a}\right]_a^b$$

$$\boxed{\mu_4' = \frac{b^5 - a^5}{b-a}}$$

(9) A R.V. X can assume the values $+1$ & -1 with probability $\frac{1}{2}$ each. Find (i) Moment generating funct. (ii) first four moment about origin.

Sol^n:

$$X = \begin{cases} 1 & 1/2 \\ -1 & 1/2 \end{cases}$$

$$M_X(t) = E[e^{tX}] = \sum e^{tx} f(x)$$

$$M_X(t) = e^{t} \cdot 1/2 + e^{-t} \cdot 1/2 = \frac{e^{t} + e^{-t}}{2} = \cosh t$$

Using Taylor's series expansion of $\cosh t$, we get

$$M_X(t) = 1 + t^2/2! + t^4/4! + \cdots$$

but $M_X(t) = 1 + \mu_1' t + \mu_2' t^2/2! + \mu_3' t^3/3! + \mu_4' t^4/4! + \cdots$

$\therefore \boxed{\mu_1' = 0,\ \mu_2' = 1,\ \mu_3' = 0,\ \mu_4' = 1}$ first four moments about origin.

9

(13) In a certain factory turning out razor blades there is a small chance of 0.002 for any blades to be defective. The blades are supplied in a packet of 10, use Poisson distribution to calculate the approximate numbe of packet containing no defective, one defective, two defective blade respectively in a consignment of 10000 packets.

Soln - Given that,

Number of defective blades in a packets are,

$n = 10$ and $p = 0.002$

$\therefore \lambda = np = \cancel{10 \times 0.0} \; 10 \times 0.002$

$$\boxed{\lambda = 0.02}$$

By Poisson distribution,

$$P(X = x) = \frac{\lambda^x \cdot e^{-\lambda}}{x!}, \quad x = 0, 1, 2, \ldots$$

(i) no defective

$$P(X = 0) = \frac{(0.02)^0 \cdot e^{-0.02}}{0!} = 0.9801$$

(ii) one defective

$$P(X = 1) = \frac{(0.02)^1 e^{-0.02}}{1!} = 0.0196$$

(iii) Two defective

$$P(X = 2) = \frac{(0.02)^2 \cdot e^{-0.02}}{2!} = 0.0001960$$

(12) If 3% of electric bulb manufactured by a Company are defective, find the Probability that in a sample of 100 bulb (a) More than 5 (b) between 1 and 3 (c) at the most 2 (d) at least 2 bulb will be defective.

Sol^n - Let x denotes the number of defective bulbs.

We have $p = 3\% = \frac{3}{100} = 0.03$

number of trial, $n = 100$

By Poisson distribution,

$$P(X=x) = \frac{\lambda^x . e^{-\lambda}}{x!}, \quad x = 0, 1, 2, \ldots$$

$\because \lambda = np = 100(0.03) = 3$

(a) More than 5

$$P(X>5) = P(X=6) + P(X=7) + \cdots + P(X=100)$$

$$= 1 - [P(X \leq 5)]$$

$$= 1 - [P(X=0) + P(X=1) + P(X=2) + P(X=3) + P(X=4) + P(X=5)]$$

$$= 1 - \left[\frac{3^0 e^{-3}}{0!} + \frac{3^1 e^{-3}}{1!} + \frac{3^2 e^{-3}}{2!} + \frac{3^3 e^{-3}}{3!} + \frac{3^4 e^{-3}}{4!} + \frac{e^5 e^{-3}}{5!}\right]$$

$$= 1 - \left[\left(1 + 3 + \frac{9}{2} + \frac{27}{6} + \frac{81}{24} + \frac{243}{120}\right)(0.04979)\right]$$

$P(X>5) = \underline{0.08386}$

(b) between 1 and 3

$$P(1 \leq X \leq 3) = P(X=1) + P(X=2) + P(X=3)$$

$$= \frac{3^1 e^{-3}}{1!} + \frac{3^2 e^{-3}}{2!} + \frac{3^3 e^{-3}}{3!} = \underline{0.5975}$$

(c) at the most 2

$$P(X \le 2) = P(X=0) + P(X=1) + P(X=2)$$

$$= \frac{3^0 e^{-3}}{0!} + \frac{3^1 e^{-3}}{1!} + \frac{3^2 e^{-3}}{2!}$$

$$P(X \le 2) = 0.4232$$

(d) At least 2

$$P(X \ge 2) = P(X=2) + P(X=3) + \cdots + P(X=100)$$

$$= 1 - [P(X \le 2)]$$

$$= 1 - [P(X=0) + P(X=1)]$$

$$= 1 - \left[\frac{3^0 e^{-3}}{0!} + \frac{3^1 e^{-3}}{1!}\right]$$

$$= 1 - 4(0.04979)$$

$$\boxed{P(X \ge 2) = 0.8008}$$

(14) Find the moment generating function for random variable X having density function

$$f(x) = \begin{cases} e^{-x}, & x \geq 0 \\ 0, & x < 0 \end{cases}$$ and determine first four moment about origin and mean.

Soln: - $M_X(t) = E(e^{tX}) = \int_{-\infty}^{\infty} e^{tx} f(x)\,dx$

$$M_X(t) = \int_{-\infty}^{0} e^{tx}(0)\,dx + \int_{0}^{\infty} e^{tx} e^{-x}\,dx$$

$$= \int_{0}^{\infty} e^{-(1-t)x}\,dx = \left[\frac{e^{-(1-t)x}}{-(1-t)}\right]_0^{\infty}$$

$$\boxed{M_X(t) = \frac{1}{1-t}}$$

We have, Moment about origin:

$$\cancel{M_X(t) = \frac{d}{dt}} \quad \mu_r' = \left[\frac{d}{dt} M_X(t)\right]_{t=0}$$

$$\mu_1' = \left[\frac{d}{dt} M_X(t)\right]_{t=0} = \left[\frac{d}{dt}\left(\frac{1}{1-t}\right)\right]_{t=0}$$

$$\mu_1' = \frac{1}{(1-t)^2}\bigg|_{t=0} = 1 = \mu$$

$$\mu_2' = \left[\frac{d^2}{dt^2} M_X(t)\right]_{t=0} = \left[\frac{d^2}{dt^2}\left(\frac{1}{1-t}\right)\right]_{t=0} = \left[\frac{d}{dt}\left(\frac{1}{(1-t)^2}\right)\right]_{t=0}$$

$$\boxed{\mu_2' = \frac{2}{(1-t)^3}\bigg|_{t=0} = 2}$$

$$\mu_3' = \left[\frac{d^3}{dt^3} M_x(t)\right]_{t=0} = \left[\frac{d^3}{dt^3}\left(\frac{1}{(1-t)}\right)\right]_{t=0}$$

$$\mu_3' = \left[\frac{d}{dt}\frac{2}{(1-t)^2}\right]_{t=0} = \frac{6}{(1-t)^4}\bigg|_{t=0}$$

$$\boxed{\mu_3' = 6}$$

$$\mu_4' = \left[\frac{d^4}{dt^4} M_x(t)\right]_{t=0} = \left[\frac{d^4}{dt^4}\left(\frac{1}{1-t}\right)\right]_{t=0}$$

$$\mu_4' = \left[\frac{d}{dt}\frac{6}{(1-t)^4}\right]_{t=0} = \frac{24}{(1-t)^5}\bigg|_{t=0}$$

$$\boxed{\mu_4' = 24}$$

<u>Moment</u> <u>about</u> <u>mean</u> <u>are</u>,

$$\mu_1 = E(X-\mu) = \mu - \mu = 0$$

$$\boxed{\mu_1 = 0}$$

$$\mu_2 = \mu_2' - \mu^2 = 2 - 1$$

$$\boxed{\mu_2 = 1}$$

$$\mu_3 = \mu_3' - 3\mu_2'\mu + 2\mu^3$$
$$= 6 - 3(2)(1) + 2(1)^3$$

$$\boxed{\mu_3 = 2}$$

$$\mu_4 = \mu_4' - 4\mu_3'\mu + 6\mu_2'\mu^2 - 3\mu^4$$
$$= 24 - 4(6)(1) + 6(2)(1^2) - 3(1^4)$$

$$\boxed{\mu_4 = 9}$$

(15) Ten coin are tossed simultaneously. Find the probability of getting (i) at least seven heads (ii) exactly seven heads (iii) at most seven heads.

Sol^n^ - Here $n=10$, $p=\frac{1}{2}$ & $q=\frac{1}{2}$

By Binomial distribution,

$$P(X=x) = {}^{n}C_{x}\, p^{x} q^{n-x}, \quad x = 0, 1, 2, \cdots n$$

(i) at least seven heads

$$P(X \geq 7) = P(X=7) + P(X=8) + P(X=9) + P(X=10)$$

$$= {}^{10}C_7 \left(\tfrac{1}{2}\right)^7 \left(\tfrac{1}{2}\right)^3 + {}^{10}C_8 \left(\tfrac{1}{2}\right)^8 \left(\tfrac{1}{2}\right)^2 + {}^{10}C_9 \left(\tfrac{1}{2}\right)^9 \left(\tfrac{1}{2}\right)^1$$

$$+ {}^{10}C_{10} \left(\tfrac{1}{2}\right)^{10} \left(\tfrac{1}{2}\right)^0 = \frac{120}{1024} + \frac{45}{1024} + \frac{10}{1024} + \frac{1}{1024}$$

$$= 0.117 + 0.043 + 0.00097 + 0.000097 = \underline{0.1610}$$

(ii) exactly seven heads

$$P(X=7) = {}^{10}C_7 \left(\tfrac{1}{2}\right)^7 \left(\tfrac{1}{2}\right)^{10-7} = 120 \times \frac{1}{1024} = \frac{15}{128} = \underline{0.117}$$

(iii) at most seven heads

$$P(X \leq 7) = P(X=0) + P(X=1) + P(X=2) + P(X=3)$$

$$+ P(X=4) + P(X=5) + P(X=6) + P(X=7)$$

$$= {}^{10}C_0 \left(\tfrac{1}{2}\right)^0 \left(\tfrac{1}{2}\right)^{10-0} + {}^{10}C_1 \left(\tfrac{1}{2}\right)^1 \left(\tfrac{1}{2}\right)^9 + {}^{10}C_2 \left(\tfrac{1}{2}\right)^2 \left(\tfrac{1}{2}\right)^8 +$$

$${}^{10}C_3 \left(\tfrac{1}{2}\right)^3 \left(\tfrac{1}{2}\right)^7 + {}^{10}C_4 \left(\tfrac{1}{2}\right)^4 \left(\tfrac{1}{2}\right)^6 + {}^{10}C_5 \left(\tfrac{1}{2}\right)^5 \left(\tfrac{1}{2}\right)^5$$

$$+ {}^{10}C_6 \left(\tfrac{1}{2}\right)^6 \left(\tfrac{1}{2}\right)^4 + {}^{10}C_7 \left(\tfrac{1}{2}\right)^7 \left(\tfrac{1}{2}\right)$$

$$= \frac{1}{1024} + \frac{45}{1024} + \frac{120}{1024} + \frac{210}{1024} + \frac{252}{1024} + \frac{210}{1024} + \frac{120}{1024}$$

$$= \frac{121}{128} = \underline{0.9453}$$

(16) If the probability that an individual suffers a bad reaction from injection of a given serum is 0.001, determines the probability that out of 2,000 individuals (a) exactly 3 (b) more than 2 individuals will suffer a bad reaction.

Soln. Given: $n = 2,000$ and $p = 0.001$

$\therefore \lambda = np = 2000 \times 0.001 = 2$

By Poissons distribution.

$$P(X=x) = \frac{\lambda^x . e^{-\lambda}}{x!}$$

(a) exactly 3

$$P(X=3) = \frac{2^3 e^{-2}}{3!} = \frac{4}{3e^2} = \underline{\underline{0.1804}}$$

(b) more than 2

$$P(X>2) = 1 - P(\leq 2)$$

$$= 1 - [P(X=0) + P(X=1) + P(X=2)]$$

$$= 1 - \left[\frac{2^0 e^{-2}}{0!} + \frac{2^1 e^{-2}}{1!} + \frac{2^2 e^{-2}}{2!}\right]$$

$$= 1 - 0.6766$$

$$= \underline{\underline{0.3234}}$$

(17) The marks obtained in a certain exam follow normal distribution with mean 45 and SD 10. If 1,300 students appeare at the examination, calculate the number of students scoring (i) less than 35 marks and (ii) more than 65 marks

Sol^n: - Given: Mean $= \mu = 45$ and SD $= 10 = \sigma$

Let X be denotes the number of students score in examination.

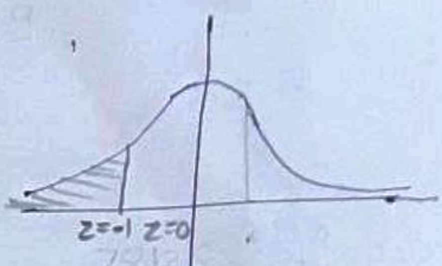
Therefore the standardized variable,

$$Z = \frac{X - \mu}{\sigma} = \frac{X - 45}{10}$$

(i) less than 35 marks:

Standard unit

$$Z = \frac{35 - 45}{10} = -1$$



$P(X < 35) = P(Z < -1)$

$= 0.5 - (\text{area between } z=0 \text{ to } z=1)$

$= 0.5 - 0.3413$

$= 0.1587$

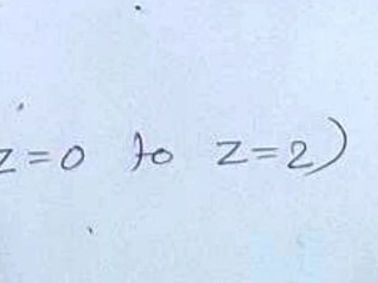
Expected number of students scoring less than 35 marks are $0.1587 \times 1300 = 206$

(ii) more than 65 marks

~~Standard uni~~

Standard unit

$$Z = \frac{X - \mu}{\sigma} = \frac{65 - 45}{10} = 2.0$$



$P(X > 65) = P(Z > 2)$

$= 0.5 - (\text{area between } z=0 \text{ to } z=2)$

$= 0.5 - 0.4772$

$= 0.0228$

Expected number of students scoring more than 65 marks are $0.0228 \times 1300 = 30$

(18) Suppose that the customers arriving at ticket Counter according to Poisson Process with a mean rate of 2 per minutes. Then in arrival of 5 minutes find the Probability that the number of Customers is (i) exactly 5 (ii) less than 4 (iii) greater than 3.

Soln: ~~Let~~ Given: $\lambda = 2$

By Poisson distribution

$$P(X=x) = \frac{\lambda^x \cdot e^{-\lambda}}{x!}, \quad x = 0, 1, 2, \cdots$$

(i) exactly 5

$$P(X=5) = \frac{2^5 e^{-2}}{5!} = 0.03608$$

(ii) less than 4

$$P(X<4) = P(X=0) + P(X=1) + P(X=2) + P(X=3)$$

$$= \frac{2^0 e^{-2}}{0!} + \frac{2^1 e^{-2}}{1!} + \frac{2^2 e^{-2}}{2!} + \frac{2^3 e^{-2}}{3!}$$

$$= 0.8571$$

(iii) greater than 3

$$P(X>3) = P(X=4) + P(X=5)$$

$$= \frac{2^4 e^{-2}}{4!} + \frac{2^5 e^{-2}}{5!}$$

$$= 0.1263$$

(19) A machine producest bolts which are 10% defective. Find the probability that in a random sample of 400 bolts produced by this machine.

(i) between 30 and 50. (ii) at the most 30

(iii) 55 or more of the bolts will be defective.

Soln :- Let X denotes the number of bolts produced by the machine.

Since no. of trials $n = 400$ is large and the probability of defective bolt $p = 10\% = \frac{10}{100} = 0.1$ is close to 0.

Here we use normal distribution.

$\therefore$ Standardized variable,

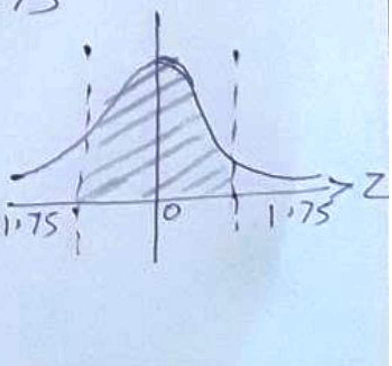
$$Z = \frac{X - \mu}{\sigma} = \frac{X - np}{\sqrt{npq}} = \frac{X - 40}{6}$$

Treating the data continuous, it follows,

(i) between 30 and 50

$$(29.5 \text{ standard unit}) = \frac{29.5 - 40}{6} = -1.75$$

$$\text{and } (50.5 \text{ in standard unit}) = \frac{50.5 - 40}{6} = 1.75$$



$\therefore P(30 \leq X \leq 50) = P(-1.75 \leq Z \leq 1.75)$

$= (\text{area between } Z = -1.75 \text{ and } Z = 1.75)$

$= 2(\text{area between } Z = 0 \text{ and } Z = 1.75)$

$= 2(0.4599) = \underline{0.9198}$

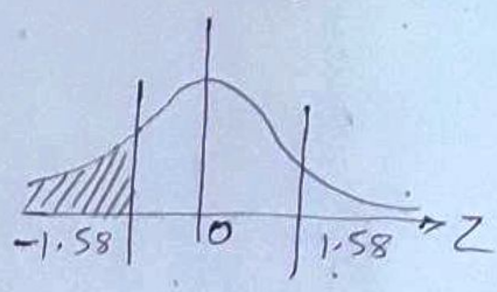
(ii) (30.5 in standard units) $= \frac{30.5-40}{6} = -1.58$

$\therefore P(X \leq 30) = P(Z \leq -1.58)$

$= 0.5 - (\text{area between } z=-1.58 \text{ and } z=0)$

$= 0.5 - 0.4429$

$= \underline{\underline{0.0571}}$



(iii) (54.5 in standard units) $= \frac{54.5-40}{6} = 2.42$

$\therefore P(X \geq 55) = P(Z \geq 2.42)$

$= 0.5 - (\text{area between } z=0 \text{ and } z=2.42)$

$= 0.5 - 0.4922$

$= \underline{0.0078}$