Inverse Z-transform by Power Series method:

In this method we find inverse Z-transform by expanding the func^n. $F(z)$ in Convergent series in Powers of z either by actual division or by binomial expansion. Then the sequence $f(n)$ Can be determined by inspection. This method is useful when the region of Convergence is specified. This method requires following information:

(i) Partial fraction method

(ii) Binomial Series expansion

$$\cancel{(1 \pm z^n)}\ (1+z)^n = 1 + nz + \frac{n(n-1)}{2!}z^2 + \frac{n(n-1)(n-2)}{3!}z^3 + \cdots$$

is valid for $|z| < 1$

iii) Defination of Two sided Z-transform

$$Z\{f(n)\} = \sum_{n=-\infty}^{\infty} f(n) \cdot z^{-n}$$

Ex: Using Power series method, find

(i) $Z^{-1}\left[\frac{1}{z^2-3z+2}\right]$ in the regions

(a) $|z|<1$ (b) $1<|z|<2$ (c) $|z|>2$

Soln: Let $F(z) = \frac{1}{z^2-3z+2} = \frac{1}{(z-1)(z-2)}$

By Partial fraction method,

$$\frac{1}{(z-1)(z-2)} = \frac{A}{z-1} + \frac{B}{z-2}$$

$\therefore A = -1$ & $B = 1$

$$\therefore \frac{1}{(z-1)(z-2)} = \frac{1}{z-2} - \frac{1}{z-1}$$

$$\therefore F(z) = \frac{1}{z-2} - \frac{1}{z-1}$$

(a) $|z|<1$

we have

$$\frac{1}{z-1} = \frac{1}{-1+z} = \frac{-1}{1-z}$$

& Let $|z|<2$

$\Rightarrow \frac{|z|}{2} < 1$

we have $\frac{1}{z-2} = \frac{1}{2\left(\frac{z}{2}-1\right)} = \frac{-1}{2\left(1-\frac{z}{2}\right)}$

$$\therefore F(z) = \frac{-1}{2\left(1-\frac{z}{2}\right)} - \frac{-1}{1-z}$$

$$= -\frac{1}{2}\left(1-\frac{z}{2}\right)^{-1} + (1-z)^{-1}$$

$$= -\frac{1}{2}\left[1+\frac{z}{2}+\left(\frac{z}{2}\right)^2+\cdots+\left(\frac{z}{2}\right)^n+\cdots\right] + \left[1+z+z^2+\cdots+z^n+\cdots\right]$$

$$= -\left[\frac{1}{2} + \frac{z}{2^2} + \frac{z^2}{2^3} + \cdots + \frac{z^n}{2^{n+1}} + \cdots\right]$$

$$+ \left[1 + z + z^2 + \cdots + z^n + \cdots\right]$$

$$= -\sum_{n=0}^{\infty} \frac{1}{2^{n+1}} z^n + \sum_{n=0}^{\infty} z^n$$

$$= \sum_{n=0}^{\infty} \left[1 - \frac{1}{2^{n+1}}\right] z^n \quad \cdots$$

$$= \sum_{n \leq 0}^{0} \left[1 - \frac{1}{2^{-n+1}}\right] z^{-n}$$

Taking inverse z-transform

$$z^{-1}\{F(z)\} = f(n) = 1^n - \cancel{2}\, 1/2^{-n+1}$$

$$= 1^n - 2^{n-1}$$

(b) $1 < |z| < 2$

$1 < |z|$ & $|z| < 2$

$\Rightarrow \frac{1}{|z|} < 1$; $\Rightarrow \frac{|z|}{2} < 1$

$\therefore \frac{1}{z-1} = \frac{1}{z(1-\frac{1}{z})}$; $\therefore \frac{1}{z-2} = \frac{1}{2(\frac{z}{2}-1)} = \frac{-1}{2(1-\frac{z}{2})}$

$$F(z) = -\frac{1}{2} \cdot \frac{1}{1-\frac{z}{2}} - \frac{1}{z} \cdot \frac{1}{1-1/z}$$

$$= -\frac{1}{2}\left(1-\frac{z}{2}\right)^{-1} - \frac{1}{z}\left(1-\frac{1}{z}\right)^{-1}$$

By Binomial expansion

$$= -\frac{1}{2}\left[1 + \frac{z}{2} + \left(\frac{z}{2}\right)^2 + \cdots + \left(\frac{z}{2}\right)^n + \cdots\right] - \frac{1}{z}\left[1 + \frac{1}{z} + \left(\frac{1}{z}\right)^2 + \cdots + \left(\frac{1}{z}\right)^n + \cdots\right]$$

$$= -\left[\frac{1}{2} + \frac{z}{2^2} + \frac{z^2}{2^3} + \cdots + \frac{z^n}{2^{n+1}} + \cdots\right] - \left[z^{-1} + z^{-2} + \cdots + z^{-(n+1)} + \cdots\right]$$

$$= -\sum_{n=0}^{\infty} \frac{1}{2^{n+1}} z^{n} - \sum_{n=1} 1^{n} \cdot z^{-n}$$

$$= -\sum_{n\le 0} \frac{1}{2^{-n+1}} z^{-n} - \sum_{n=1}^{\infty} 1^{n} \cdot z^{-n}$$

$$\therefore Z^{-1}\{F(z)\} = f(n) = \begin{cases} \frac{1}{2^{-n+1}}, & n \le 0 \\ 1^{n}, & n \ge 1 \end{cases}$$

(c) $|z| > 2$

$2 < |z|$

$\frac{2}{|z|} < 1$

$\therefore \frac{1}{z-2} = \frac{1}{z(1-\frac{2}{z})}$

Let $|z| > 1$

& $1 < |z|$

$\cancel{\Rightarrow 1 \cdot |z| > 1}$

$\Rightarrow \frac{1}{|z|} < 1$

$\therefore \frac{1}{z-1} = \frac{1}{z(1-\frac{1}{z})}$

$$\therefore F(z) = \frac{1}{z} \cdot \frac{1}{1-\frac{2}{z}} - \frac{1}{z} \cdot \frac{1}{1-\frac{1}{z}}$$

$$= \frac{1}{z}\left(1-\frac{2}{z}\right)^{-1} - \frac{1}{z}\left(1-\frac{1}{z}\right)^{-1}$$

$$= \frac{1}{z}\left[1 + \frac{2}{z} + \left(\frac{2}{z}\right)^{2} + \dots + \left(\frac{2}{z}\right)^{n} + \dots\right]$$

$$- \frac{1}{z}\left[1 + \frac{1}{z} + \left(\frac{1}{z}\right)^{2} + \dots + \left(\frac{1}{z}\right)^{n} + \dots\right]$$

$$= \left[z^{-1} + 2z^{-2} + 2^{2}z^{-3} + \dots + 2^{n-1} \cdot z^{-n} + \dots\right]$$

$$- \left[z^{-1} + z^{-2} + \dots + z^{-(n)} + \dots\right]$$

$$= \cancel{\sum_{n=1}^{\infty} 2^{-(n+1)} z^{-n}}$$

$$= \sum_{n=1}^{\infty} 2^{n-1} \cdot z^{-n} - \sum_{n=1}^{\infty} z^{-n}$$

$$= \left[\sum_{n=1}^{\infty} 2^{n-1} - 1\right] z^{-n}$$

$$\Rightarrow Z^{-1}\{F(z)\} = f(n) = 2^{n-1} - 1, \quad k \ge 1$$
$$= 0, \quad k \le 0$$

Z-transform

Convolution Theorem:

Statement: If $Z^{-1}\{F(z)\} = f(n)$ and $Z^{-1}\{G(z)\} = g(n)$

then $Z^{-1}\{F(z).G(z)\} = \sum_{m=0}^{n} f(m).g(n-m)$

Ex: Using Convolution theorem, evaluate

① $Z^{-1}\left\{\frac{z^2}{(z-a)^2}\right\}$ ② $Z^{-1}\left\{\frac{z^2}{(z-1)(z-3)}\right\}$.

Soln: ① $\frac{z^2}{(z-a)^2} = \frac{z}{z-a} \cdot \frac{z}{z-a}$

$\therefore$ $F(z) = \frac{z}{z-a}$ & $G(z) = \frac{z}{z-a}$

$Z^{-1}\{F(z)\} = Z^{-1}\left\{\frac{z}{z-a}\right\}$ & $Z^{-1}\{G(z)\} = Z^{-1}\left\{\frac{z}{z-a}\right\}$

$\Rightarrow f(n) = a^n$ $\Rightarrow g(n) = a^n$

$\Rightarrow f(m) = a^m$ But, $g(n-m) = a^{n-m}$

By Convolution theorem

$Z^{-1}\{F(z).G(z)\} = \sum_{m=0}^{n} f(m).g(n-m)$

$$\Rightarrow Z^{-1}\left\{\frac{z}{z-a} \cdot \frac{z}{z-a}\right\} = \sum_{m=0}^{n} a^m . a^{n-m}$$

$$= a^n \sum_{m=0}^{n} a^{m-m}$$

$$= a^n \sum_{m=0}^{n} 1$$

$$= a^n \left[1 + \sum_{m=1}^{n} 1\right]$$

$= a^n (m+1) \quad \because \sum_{m=1}^{n} 1 = 1+1+1+1+\cdots n \text{ term}$

$= n \cdot 1 = n$

(2) Soln: We have $\frac{z^2}{(z-1)(z-3)} = \frac{z}{z-1} \cdot \frac{z}{z-3}$

$\because F(z) = \frac{z}{z-1}$ & $G(z) = \frac{z}{z-3}$

$z^{-1}\{F(z)\} = z^{-1}\{\frac{z}{z-1}\}$ & $z^{-1}\{G(z)\} = z^{-1}\{\frac{z}{z-3}\}$

$\Rightarrow f(n) = 1^n$ & $g(n) = 3^n$

But $f(m) = 1^m$ & $g(n-m) = 3^{n-m}$

By Convolution thm.

$$z^{-1}\{F(z) \cdot G(z)\} = \sum_{m=0}^{n} f(m) \cdot g(n-m)$$

$$z^{-1}\left\{\frac{z}{z-1} \cdot \frac{z}{z-3}\right\} = \sum_{m=0}^{n} 1^m \cdot 3^{n-m}$$

$$= 3^n \sum_{m=0}^{n} 1^m \cdot 3^{-m}$$

$$= 3^n \sum_{m=0}^{n} \left(\frac{1}{3}\right)^m$$

$$= 3^n \left[1 + \sum_{m=1}^{n} \left(\frac{1}{3}\right)^m\right] \quad \text{—} (1)$$

Now, $\sum_{m=1}^{n} \left(\frac{1}{3}\right)^m$ is finite geometric series whose first term is $\frac{1}{3}$ and Common ratio $\frac{1}{3}$

since Common ratio $\frac{1}{3} < 1$,

$$\therefore \sum_{m=1}^{n} \left(\frac{1}{3}\right)^m = \frac{(1^{st} \text{ term}) \left[1 - (\text{Common ratio})^n\right]}{1 - \text{Common ratio}}$$

$$\sum_{m=1}^{n}\left(\frac{1}{3}\right)^m = \frac{\frac{1}{3}\left[1-\left(\frac{1}{3}\right)^n\right]}{1-\frac{1}{3}}$$

$$= \frac{\frac{1}{3}\left[1-\frac{1}{3^n}\right]}{\frac{3-1}{3}} = \frac{1}{2\cdot 3^n}\left[3^n-1\right]$$

$\therefore$ equation (1) becomes

$$Z^{-1}\left\{\frac{z^2}{(z-1)(z-3)}\right\} = \left[3^n\left[1+\frac{(3^n-1)}{2\cdot 3^n}\right]\right]$$

$$= 3^n\cdot\left[\frac{2\cdot 3^n+3^n-1}{2\cdot 3^n}\right]$$

$$= \frac{1}{2}\left[2\cdot 3^n+3^n-1\right]$$

$$= \frac{1}{2}\left[3^n(2+1)-1\right]$$

$$= \frac{1}{2}\left[3^n\cdot 3-1\right] = \frac{1}{2}\left[3^{n+1}-1\right]$$

Ex: Using Convolution thm. find $Z^{-1}\left\{\frac{z^2}{(z-a)(z-b)}\right\}$

Soln: -

Ex: Find $Z\{\sin(3k+5)\}$

Sol^n. $Z\{\sin(3k+5)\} = Z\{\sin 3k \cos 5 + \cos 3k \sin 5\}$

$= \cos 5 . Z\{\sin 3k\} + \sin 5\, Z\{\cos 3k\}$

$= \cos 5 . \dfrac{z \sin 3}{z^2 - 2z\cos 3 + 1} +$

$\sin 5 . \dfrac{z(z - \cos 3)}{z^2 - 2z\cos 3 + 1}$

$$= \frac{z^2 \sin 5 + z(\sin 3 \cos 5 - \cos 3 \sin 5)}{z^2 - 2z\cos 3 + 1}$$

$$= \frac{z^2 \sin 5 - z \sin 2}{z^2 - 2z\cos 3 + 1}$$

simillarly, find $Z\{\cos(3k+5)\}$

c) $|z|>2 \Rightarrow \frac{2}{|z|}<1$, $\therefore$ we write $\frac{1}{z-2}=\frac{1}{z\left(1-\frac{2}{z}\right)}$

Since $|z|>2$, we have $|z|>1$, *i.e.* $\frac{1}{|z|}<1$, $\therefore$ we write $\frac{1}{z-1}=\frac{1}{z\left(1-\frac{1}{z}\right)}$

Thus, we express $\bar{f}(z)$ as

$$\bar{f}(z)=\frac{1}{z}\cdot\frac{1}{1-\frac{2}{z}}-\frac{1}{z}\cdot\frac{1}{1-\frac{1}{z}}=\frac{1}{z}\left(1-\frac{2}{z}\right)^{-1}-\frac{1}{z}\left(1-\frac{1}{z}\right)^{-1}$$

$$=\frac{1}{z}\left[1+\frac{2}{z}+\left(\frac{2}{z}\right)^2+\ldots+\left(\frac{2}{z}\right)^k+\ldots\right]-\frac{1}{z}\left[1+\frac{1}{z}+\left(\frac{1}{z}\right)^2+\ldots+\left(\frac{1}{z}\right)^k+\ldots\right]$$

$$=\left[z^{-1}+2z^{-2}+2^2z^{-3}+\ldots+2^kz^{-(k+1)}+\ldots\right]-\left[z^{-1}+z^{-2}+z^{-3}+\ldots+z^{-(k+1)}+\ldots\right]$$

$$=\sum_{k=1}^{\infty}2^{k-1}z^{-k}-\sum_{k=1}^{\infty}z^{-k}=\sum_{k=1}^{\infty}\left(2^{k-1}-1\right)z^{-k}$$

$$\therefore \quad Z^{-1}\left[\bar{f}(z)\right]=\{f(k)\}=\left\{2^{k-1}-1\right\},\ k\ge 1$$

$$=0 \quad ,\ k\le 0$$

(Ans.)

ii) We write $\bar{f}(z)=\frac{15z}{(4-z)(4z-1)}=\frac{A}{4-z}+\frac{B}{4z-1}$ (resolving into partial fractions)

$\Rightarrow \quad 15z=A(4z-1)+B(4-z)$

Putting $4-z=0$, *i.e.* $z=4$ we get $60=15A$, $\therefore A=4$

Putting $4z-1=0$, *i.e.* $z=1/4$ we get $15/4=(15/4)B$, $\therefore B=1$

$$\therefore \quad \bar{f}(z)=\frac{4}{4-z}+\frac{1}{4z-1}$$

$\frac{1}{4}<|z|<4 \Rightarrow \frac{1}{4}<|z| \Rightarrow \frac{1}{|4z|}<1$, $\therefore$ we write $\frac{1}{4z-1}=\frac{1}{4z\left(1-\frac{1}{4z}\right)}$

and $|z|<4 \Rightarrow \frac{|z|}{4}<1$, $\therefore$ we write $\frac{4}{4-z}=\frac{4}{4\left(1-\frac{z}{4}\right)}=\frac{1}{1-\frac{z}{4}}$

$$\therefore \quad \bar{f}(z)=\frac{1}{\left(1-\frac{z}{4}\right)}+\frac{1}{4z\left(1-\frac{1}{4z}\right)}=\left(1-\frac{z}{4}\right)^{-1}+\frac{1}{4z}\left(1-\frac{1}{4z}\right)^{-1}$$

$$=\left[1+\frac{z}{4}+\left(\frac{z}{4}\right)^2+.....+\left(\frac{z}{4}\right)^k+...\right]+\frac{1}{4z}\left[1+\frac{1}{4z}+\left(\frac{1}{4z}\right)^2+...+\left(\frac{1}{4z}\right)^k+...\right]$$

$$=\left[1+\frac{z}{4}+\left(\frac{z}{4}\right)^2+...+\left(\frac{z}{4}\right)^k+...\right]+\left[\frac{1}{4z}+\left(\frac{1}{4z}\right)^2+\left(\frac{1}{4z}\right)^3+...+\left(\frac{1}{4z}\right)^{k+1}+...\right]$$

$$=\sum_{k=0}^{\infty}\left(\frac{1}{4}\right)^k z^k+\sum_{k=1}^{\infty}\left(\frac{1}{4z}\right)^k$$

$$=\sum_{k\le 0} 4^k z^{-k}+\sum_{k=1}^{\infty}\left(\frac{1}{4}\right)^k z^{-k}$$

$$\therefore \quad Z^{-1}\left[\bar{f}(z)\right]=\{f(k)\}=\begin{cases}\{4^k\} & , k\le 0\\ \{(1/4)^k\}, & k\ge 1\end{cases}$$ **(Ans.)**

iii) We write

$$\bar{f}(z)=\frac{2z^2-10z+13}{(z-2)(z-3)^2}=\frac{A}{z-2}+\frac{B}{z-3}+\frac{C}{(z-3)^2}$$

$$\Rightarrow \quad 2z^2-10z+13=A(z-3)^2+B(z-2)(z-3)+C(z-2)$$

Putting $z-2=0$, *i.e.* $z=2$ we get $1=A$, $\quad \therefore\ A=1$

Putting $z-3=0$, *i.e.* $z=3$ we get $1=C$, $\quad \therefore\ C=1$

Constants : $13=9A+6B-2, \quad \therefore\ B=1$

$$\therefore \quad \bar{f}(z)=\frac{1}{z-2}+\frac{1}{z-3}+\frac{1}{(z-3)^2}$$

$2<|z|<3 \Rightarrow \dfrac{2}{|z|}<1$, $\therefore$ we write $\dfrac{1}{z-2}=\dfrac{1}{z\left(1-\frac{2}{z}\right)}$,

and $|z|<3 \Rightarrow \dfrac{|z|}{3}<1$, $\therefore$ we write $\dfrac{1}{z-3}=-\dfrac{1}{3\left(1-\frac{z}{3}\right)}$ and $\dfrac{1}{(z-3)^2}=\dfrac{1}{9\left(1-\frac{z}{3}\right)^2}$

$(1+x)^{-n} = 1 - nx + \frac{n(n+1)}{2!}x^2 - \frac{n(n+1)(n+2)}{3!}x^3 + \dots$

$(1-x)^{-n} = 1 + nx + \frac{n(n+1)}{2!}x^2 + \frac{n(n+1)(n+2)}{3!}x^3 + \dots$

$$\therefore \quad \bar{f}(z) = \frac{1}{z}\cdot\frac{1}{1-\frac{2}{z}} - \frac{1}{3}\cdot\frac{1}{1-\frac{z}{3}} + \frac{1}{9}\cdot\frac{1}{\left(1-\frac{z}{3}\right)^2} = \frac{1}{z}\left(1-\frac{2}{z}\right)^{-1} - \frac{1}{3}\left(1-\frac{z}{3}\right)^{-1} + \frac{1}{9}\left(1-\frac{z}{3}\right)^{-2}$$

$$= \frac{1}{z}\left[1+\frac{2}{z}+\left(\frac{2}{z}\right)^2+\dots+\left(\frac{2}{z}\right)^k+\dots\right] - \frac{1}{3}\left[1+\frac{z}{3}+\left(\frac{z}{3}\right)^2+\dots+\left(\frac{z}{3}\right)^k+\dots\right]$$

$$+\frac{1}{9}\left[1+2\frac{z}{3}+3\left(\frac{z}{3}\right)^2+\dots+(k+1)\left(\frac{z}{3}\right)^k+\dots\right]$$

$$= \left[z^{-1}+2z^{-2}+2^2z^{-3}+\dots+2^kz^{-(k+1)}+\dots\right] - \frac{1}{3}\left[1+\frac{z}{3}+\left(\frac{z}{3}\right)^2+\dots+\left(\frac{z}{3}\right)^k+\dots\right]$$

$$+\frac{1}{9}\left[1+2\frac{z}{3}+3\left(\frac{z}{3}\right)^2+\dots+(k+1)\left(\frac{z}{3}\right)^k+\dots\right]$$

$$= \sum_{k=1}^{\infty} 2^{k-1}z^{-k} - \sum_{k=0}^{\infty}\frac{1}{3}\left(\frac{z}{3}\right)^k + \sum_{k=0}^{\infty}\frac{1}{9}(k+1)\left(\frac{z}{3}\right)^k$$

$$= \sum_{k=1}^{\infty} 2^{k-1}z^{-k} + \sum_{k=0}^{\infty}\left(\frac{k+1}{3^{k+2}} - \frac{1}{3^{k+1}}\right)z^k$$

$$= \sum_{k=1}^{\infty} 2^{k-1}z^{-k} + \sum_{k=0}^{\infty}\frac{k-2}{3^{k+2}}z^k = \sum_{k=1}^{\infty} 2^{k-1}z^{-k} + \sum_{k\le 0}\frac{-k-2}{3^{-k+2}}z^{-k}$$

$$\therefore \quad Z^{-1}\left[\bar{f}(z)\right] = \{f(k)\} = \begin{cases} \left\{2^{k-1}\right\}, & k \ge 1 \\ \left\{\frac{-(k+2)}{3^{-k+2}}\right\}, & k \le 0 \end{cases} \qquad \textbf{(Ans.)}$$

Example 2. Using power series method, find

$Z^{-1}\left[\frac{1}{(z-5)^3}\right]$ when $|z| > 5$. Determine the region of convergence.

Solution. We write $\bar{f}(z) = \frac{1}{(z-5)^3}$

$$|z| > 5 \quad \Rightarrow \quad \frac{5}{|z|} < 1, \quad \therefore \text{ we write } \frac{1}{(z-5)^3} = \frac{1}{z^3\left(1-\frac{5}{z}\right)^3}$$

ii) $\dfrac{z^2}{(2z+1)(z-1)}$ **Ans.** $\left\{\dfrac{1}{3}\,1^k + \dfrac{1}{6}\left(-\dfrac{1}{2}\right)^k\right\}$, $k \geq 0$

iii) $\dfrac{8z^2}{(2z-1)(4z-1)}$ **Ans.** $\left\{\left(\dfrac{1}{2}\right)^{k-1} - \left(\dfrac{1}{2}\right)^{2k}\right\}$

iv) $\left(\dfrac{z}{z-a}\right)^3$ **Ans.** $\left\{\dfrac{a^k}{2}\,(k+1)\,(k+2)\right\}$, $k \geq 0$

C. Partial fraction method

We frequently need to determine the inverse transform of a rational expression of the form $P(z)/Q(z)$ where $P(z)$ and $Q(z)$ are polynomials in z and degree of $P(z)$ < degree of $Q(z)$. In such cases the procedure, as for Laplace transforms, is first to resolve the expression into partial fractions and then to use the table of transforms. We shall now illustrate the approach through some examples.

ILLUSTRATIVE EXAMPLES

Example 1. Obtain the inverse Z-transforms of following functions :

i) $\dfrac{z}{(2z+1)(z-3)}$, ii) $\dfrac{2z^2-7z}{(z-1)^2(z-3)}$, iii) $\dfrac{3z^2+2z+1}{z^2-3z+2}$,

iv) $\dfrac{4z^2-2z}{z^3-5z^2+8z-4}$.

(R.T.M.N.U. CT/CE, IT W-2007)

Solution. i) We write $\bar{f}(z) = \dfrac{z}{(2z+1)(z-3)}$

We notice that in most of the standard results of I.Z.T. the factor z in the numerator, we therefore resolve $\bar{f}(z)/z$ into partial fractions [In any particular example, if there is no factor z available in the numerator, we must simply resolve $\bar{f}(z)$].

$$\frac{\bar{f}(z)}{z} = \frac{1}{(2z+1)(z-3)} = \frac{A}{2z+1} + \frac{B}{z-3} \qquad \ldots(1)$$

$$\Rightarrow \qquad 1 = A\,(z-3) + B\,(2z+1) \qquad \ldots(2)$$

Putting $2z+1=0$, *i.e.* $z=-1/2$ in equation (2), we get $1 = -\dfrac{7}{2}A$, $\therefore \quad A = -2/7$

Putting $z-3=0$, *i.e.* $z=3$ in equation(2), we get $1=7B$, $\therefore \quad B = 1/7$

Substituting the values of A and B in equation (1), we get

$$\bar{f}(z) = -\frac{2}{7}\frac{z}{2z+1} + \frac{1}{7}\frac{z}{z-3}$$

$$= -\frac{2}{7}\cdot\frac{1}{2}\frac{z}{[z+(1/2)]} + \frac{1}{7}\frac{z}{z-3}$$

$$\therefore \quad Z^{-1}\left[\bar{f}(z)\right] = -\frac{1}{7}\, Z^{-1}\left[\frac{z}{\{z+(1/2)\}}\right] + \frac{1}{7} Z^{-1}\left[\frac{z}{z-3}\right]$$

i.e. $$Z^{-1}\left[\frac{z}{(2z+1)(z-3)}\right] = \left\{\frac{1}{7}\left[3^k - \left(-\frac{1}{2}\right)^k\right]\right\}, \quad k \geq 0$$ **(Ans.)**

ii) We write $$\bar{f}(z) = \frac{2z^2 - 7z}{(z-1)^2\,(z-3)}$$

$$\Rightarrow \quad \frac{\bar{f}(z)}{z} = \frac{2z-7}{(z-1)^2\,(z-3)} \qquad \ldots(1)$$

Now $$\frac{2z-7}{(z-1)^2\,(z-3)} = \frac{A}{z-3} + \frac{B}{z-1} + \frac{C}{(z-1)^2}$$

$$\Rightarrow \quad 2z - 7 = A\,(z-1)^2 + B\,(z-1)\,(z-3) + C\,(z-3) \qquad \ldots(2)$$

Putting $z-3=0$, *i.e.* $z=3$ in equation (2) we get $-1 = 4A$, $\therefore\ A = -1/4$

Putting $z-1=0$, *i.e.* $z=1$ in equation (2) we get $-5 = -2C$, $\therefore\ C = 5/2$

From equation (2), coefficients of z : $2 = -2A - 4B + C$

$$= \frac{1}{2} - 4B + \frac{5}{2} = 3 - 4B\,, \quad \therefore\ B = 1/4$$

Substituting the values of A, B, C in equation (1), we obtain

$$\bar{f}(z) = -\frac{1}{4}\frac{z}{z-3} + \frac{1}{4}\frac{z}{z-1} + \frac{5}{2}\frac{z}{(z-1)^2}$$

$$\therefore \quad Z^{-1}\left[\bar{f}(z)\right] = -\frac{1}{4} Z^{-1}\left[\frac{z}{z-3}\right] + \frac{1}{4} Z^{-1}\left[\frac{z}{z-1}\right] + \frac{5}{2} Z^{-1}\left[\frac{z}{(z-1)^2}\right]$$

$$= \left\{-\frac{1}{4}3^k + \frac{1}{4}1^k + \frac{5}{2}k\ 1^k\right\}$$

i.e. $$Z^{-1}\left[\frac{2z^2-7z}{(z-1)^2\,(z-3)}\right] = \left\{\frac{5}{2}k + \frac{1}{4}\left(1^k - 3^k\right)\right\}, \quad k \geq 0$$ **(Ans.)**

iii) We write $\bar{f}(z)=\dfrac{3z^2+2z+1}{z^2-3z+2}=3+\dfrac{11z-5}{z^2-3z+2}=3+\dfrac{11z-5}{(z-1)(z-2)}=3-\dfrac{6}{z-1}+\dfrac{17}{z-2}$

$$\therefore \quad Z^{-1}\left[\frac{3z^2+2z+1}{z^2-3z+2}\right]=3\,Z^{-1}[1]-6\,Z^{-1}\left[\frac{1}{z-1}\right]+17\,Z^{-1}\left[\frac{1}{z-2}\right]$$

$$=\left\{3\,\delta(k)-6\cdot 1^{k-1}+17\cdot 2^{k-1}\right\},\ k\ge 1 \qquad \textbf{(Ans.)}$$

iv) We write $\bar{f}(z)=\dfrac{4z^2-2z}{z^3-5z^2+8z-4}$

$$\Rightarrow \quad \frac{\bar{f}(z)}{z}=\frac{4z-2}{z^3-5z^2+8z-4}=\frac{4z-2}{(z-1)(z-2)^2}=\frac{A}{z-1}+\frac{B}{z-2}+\frac{C}{(z-2)^2} \qquad \ldots(1)$$

$$\therefore \quad 4z-2=A(z-2)^2+B(z-1)\,(z-2)+C(z-1) \qquad \ldots(2)$$

Putting $z-1=0$, *i.e.* $z=1$ in equation (2) we get $2=A$, $\therefore$ $A=2$

Putting $z-2=0$, *i.e.* $z=2$ in equation (2) we get $6=C$, $\therefore$ $C=6$

From equation (2), coefficients of z^2 : $0=A+B$, $\therefore$ $B=-2$

$\therefore$ from equation (1), we have

$$\bar{f}(z)=2\frac{z}{z-1}-2\frac{z}{z-2}+6\frac{z}{(z-2)^2}$$

$$\Rightarrow \quad Z^{-1}\left[\bar{f}(z)\right]=2\,Z^{-1}\left[\frac{z}{z-1}\right]-2\,Z^{-1}\left[\frac{z}{z-2}\right]+3\,Z^{-1}\left[\frac{2z}{(z-2)^2}\right]$$

$$\Rightarrow \quad Z^{-1}\left[\frac{4z^2-2z}{z^3-5z^2+8z-4}\right]=\left\{2\cdot 1^k-2\cdot 2^k+3k\cdot 2^k\right\},\ k\ge 0$$

$$=\left\{2\left(1^k-2^k\right)+3k\cdot 2^k\right\},\ k\ge 0 \qquad \textbf{(Ans.)}$$

Example 2. Obtain the inverse Z-transforms of

i) $\dfrac{z}{z^2+a^2}$, ii) $\dfrac{z}{z^2-2\sqrt{3}\,z+4}$, iii) $\dfrac{z^2}{(z-1)^2\left(z^2-z+1\right)}$.

Solution. i) We write

$$\bar{f}(z)=\frac{z}{z^2+a^2}$$

ii) We have $\dfrac{z^2}{(z-1)(z-3)} = \dfrac{z}{z-1} \cdot \dfrac{z}{z-3}$

$\therefore \quad Z^{-1}\left[\dfrac{z}{z-1}\right] = \{1^k\}$ and $Z^{-1}\left[\dfrac{z}{z-3}\right] = \{3^k\}$

By convolution theorem, we obtain

$$Z^{-1}\left[\frac{z^2}{(z-1)(z-3)}\right] = \left\{\sum_{m=0}^{k} 1^m \cdot 3^{k-m}\right\}$$

$$= \left\{3^k \sum_{m=0}^{k} \left(\frac{1}{3}\right)^m\right\}$$

$$= \left\{3^k \left[1 + \sum_{m=1}^{k} \left(\frac{1}{3}\right)^m\right]\right\} \quad \ldots (1)$$

Now, $\sum_{m=1}^{k} \left(\frac{1}{3}\right)^m$ is a finite geometric series whose first term is $\frac{1}{3}$ and common ra

Since common ratio $\frac{1}{3} < 1$,

$$\therefore \quad \sum_{m=1}^{k} \left(\frac{1}{3}\right)^m = \frac{(\text{first term})\left[1-(\text{common ratio})^k\right]}{1-\text{common ratio}}$$

$$= \frac{\frac{1}{3}\left(1-\frac{1}{3^k}\right)}{1-(1/3)} = \frac{1}{2 \cdot 3^k}\left(3^k - 1\right)$$

$\therefore$ equation (1) becomes

$$Z^{-1}\left[\frac{z^2}{(z-1)(z-3)}\right] = \left\{3^k\left[1 + \frac{\left(3^k-1\right)}{2 \cdot 3^k}\right]\right\}$$

$$= \left\{3^k \cdot \frac{\left(3 \cdot 3^k - 1\right)}{2 \cdot 3^k}\right\} = \left\{\frac{1}{2}\left(3^{k+1}-1\right)\right\}$$

iii) We have

$$e^{2/z} = e^{1/z} \cdot e^{1/z}$$

Let $\quad Z^{-1}\left[e^{1/z}\right] = \{f(k)\}$

$$\therefore \quad Z\{f(k)\} = e^{1/z}$$

$$= 1 + \frac{1}{1!}\left(\frac{1}{z}\right) + \frac{1}{2!}\left(\frac{1}{z}\right)^2 + \ldots.. + \frac{1}{k!}\left(\frac{1}{z}\right)^k + \ldots..$$

$$= \sum_{k=0}^{\infty} \frac{1}{k!} z^{-k} = Z\left\{\frac{1}{k!}\right\} \qquad \Rightarrow f(k) = \frac{1}{k!}$$

$$\therefore \quad Z^{-1}\left[e^{1/z}\right] = \left\{\frac{1}{k!}\right\} \text{ and } Z^{-1}\left[e^{1/z}\right] = \left\{\frac{1}{k!}\right\}$$

By convolution theorem, we obtain

$$Z^{-1}\left[e^{2/z}\right] = \left\{\sum_{m=0}^{k} \frac{1}{m!} \cdot \frac{1}{(k-m)!}\right\}$$

$$= \left\{\frac{1}{k!} + \frac{1}{(k-1)!} + \frac{1}{2!} \cdot \frac{1}{(k-2)!} + \frac{1}{3!} \cdot \frac{1}{(k-3)!} + \ldots.. + \frac{1}{k!}\right\}$$

$$= \left\{\frac{1}{k!} + \frac{k}{k!} + \frac{k(k-1)}{2!\ k!} + \frac{k(k-1)(k-2)}{3!\ k!} + \ldots.. + \frac{1}{k!}\right\}$$

$$= \left\{\frac{1}{k!}\left[1 + k + \frac{1}{2!} k(k-1) + \frac{1}{3!} k(k-1)(k-2) + \ldots.. + 1\right]\right\}$$

$$= \left\{\frac{1}{k!}(1+1)^k\right\} \text{ , by binomial theorem.}$$

$$= \left\{\frac{2^k}{k!}\right\} \qquad \textbf{(Ans.)}$$

Example 2. Verify convolution theorem for sequences

$$\{f(k)\} = \{k\} \quad \text{and} \quad \{g(k)\} = \{k\}$$

Solution. By convolution theorem, we have

$$Z^{-1}\left[\bar{f}(z)\, \bar{g}(z)\right] = \left\{\sum_{m=0}^{k} f(m)\, g(k-m)\right\}$$

or

$$\bar{f}(z)\, \bar{g}(z) = Z\left\{\sum_{m=0}^{k} f(m)\, g(k-m)\right\} \qquad \ldots(1)$$

$$\because \{f(k)\} = \{k\}, \quad \therefore \quad Z\{f(k)\} = Z\{k\} = \frac{z}{(z-1)^2} \qquad \left[\because Z\left\{k\, a^k\right\} = \frac{az}{(z-a)^2}\right]$$