Required fourier, is,

$$f(x) = \sum_{n=1}^{\infty} \frac{-2n(-1)^n}{\pi(a^2+n^2)} \sin nx$$

Fourier Transforms :- (6 Marks)

Let $f(x)$ be the function defined in interval $-\infty$ to ∞ then fourier transform of $f(x)$ is given by $F(f(x)) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{isx} f(x)\,dx = \bar{f}(s)$

where

F is fourier transform Operator

The inverse fourier transform of $\bar{f}(s)$ is given by,

$$F^{-1}[\bar{f}(s)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-isx} \bar{f}(s)\,ds = f(x)$$

where

F^{-1} = fourier inverse transform Operator

Fourier Cosine & sine transforms :-

1. $F_c\{f(x)\} = \bar{f}_c(s) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(x) \cos sx\,dx$
2. $f(x) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} \bar{F}_c(s) \cos sx\,ds$
3. $F_s\{f(x)\} = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(x) \sin sx\,dx = \bar{f}_s(s)$
4. $f(x) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} \bar{f}_s(s) \sin sx\,ds$

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Ex :- Find fourier transform of

$$f(x) = \begin{cases} 1 & , \text{ for } |x|<1 \\ 0 & , \text{ for } |x|>1 \end{cases}$$, hence

find $\int_0^\infty \frac{\sin x}{x} dx$.

Ans → Given function is even
Therefore we take fourier for cosine transform.

formula ⇒ $F_c\{f(x)\} = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \cos sx \, dx = \bar{f}(s)$

$= \sqrt{\frac{2}{\pi}} \left\{ \int_0^1 1 \cos sx \, dx + \int_0^\infty 0 \right\}$

$= \sqrt{\frac{2}{\pi}} \left[\frac{\sin sx}{s} \right]_0^1$

$= \sqrt{\frac{2}{\pi}} \left[\frac{\sin s}{s} \right]$

$= \bar{f}(s)$ cosine

Inverse fourier transform :-

$f(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty \bar{f}(s) \cos sx \, ds$

$e^{-\infty} = 0$

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$= \sqrt{\frac{2}{\pi}} \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{\sin s \cos sx}{s} ds$

$= \frac{2}{\pi} \int_0^\infty \frac{\sin s \cos sx}{s} ds$

put $x = 0$

$\frac{2}{\pi} \int_0^\infty \frac{\sin s}{s} ds = 1$

Parameter $s = x$

$\int_0^\infty \frac{\sin x}{x} dx = \frac{\pi}{2}$

Ex: find the fourier sine & cosine transform of e^{-ax}

Ans → fourier cosine transform is given by,

$F_c\{f(x)\} = \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-ax} \cos sx \, dx$

$= \sqrt{\frac{2}{\pi}} \left\{ \frac{e^{-ax}}{a^2+s^2} (-a \cos sx - s \sin sx) \right\}_0^\infty$

$= \sqrt{\frac{2}{\pi}} \left\{ 0 - \frac{1}{a^2+s^2} (-a \cdot 1) \right\}$

$= + \frac{a}{a^2+s^2} \sqrt{\frac{2}{\pi}}$

$\int e^{ax} \cos bx$

$\therefore \frac{e^{ax}}{a^2+b^2} [a \cos bx + b \sin bx]$

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Show that → Inverse (find also)

Fourier Sine transform:

$$F_s\{f(x)\} = \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-ax} \sin sx \, dx$$

$$= \sqrt{\frac{2}{\pi}} \left\{ \frac{e^{-ax}}{a^2+s^2} (-a\sin sx - s\cos sx) \right\}_0^\infty$$

$$= \sqrt{\frac{2}{\pi}} \left\{ 0 - \frac{1}{a^2+s^2} (s) \right\}$$

$$= \frac{-s}{s^2+a^2} \sqrt{\frac{2}{\pi}}$$

Ex: find fourier Sine transform of

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$e^{-|x|}$ and hence show that

$$\int_0^\infty \frac{x \sin mx}{1+x^2} dx = \frac{\pi}{2} e^{-m}, \; m>0$$

Sol^n ⇒ fourier Sine transform:-

$$F_s(f(x)) = \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-x} \sin sx \, dx$$

$$= \sqrt{\frac{2}{\pi}} \left\{ \frac{e^{-x}}{1+s^2} (-\sin sx - s\cos sx) \right\}_0^\infty$$

$$= \sqrt{\frac{2}{\pi}} \cdot \left\{ 0 - \frac{1}{1+s^2} (+s) \right\}$$

$$= \frac{s}{s^2+1}\sqrt{\frac{2}{\pi}}$$

Inverse Sine transform :-

$$f(x) = \sqrt{\frac{2}{\pi}}\int_0^\infty \bar{F}(s)\sin sx\, ds$$

$$= \sqrt{\frac{2}{\pi}}\int_0^\infty \sqrt{\frac{2}{\pi}}\frac{s}{s^2+1}\sin sx\, ds$$

$$e^{-x} = \frac{2}{\pi}\int_0^\infty \frac{s\sin sx}{s^2+1}\, ds$$

$$\frac{\pi}{2}e^{-x} = \int_0^\infty \frac{s\sin sx}{s^2+1}\, ds$$

Put $s = x \Rightarrow ds = dx$ & then s by m

$$\int_0^\infty \frac{x\sin mx}{x^2+1}\, ds = \frac{\pi}{2}e^{-m}$$

Q. 24 find fourier transform for

$$f(x) = \begin{cases} 1-x^2 & |x|<1 \\ 0 & |x|>1 \end{cases}$$ Hence show that

$$\int_0^\infty \left(\frac{\sin x - x\cos x}{x^3}\right)\cos\left(\frac{x}{2}\right) dx$$

Ans. Given function is even
∴ Fourier Cosine transform of given fn

$$F_c\{f(x)\} = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x)\cos sx\, dx = \bar{f}(s)$$

$$= \sqrt{\frac{2}{\pi}} \left\{ \int_0^1 (1-x^2)\cos sx\, dx + \int_0^\infty 0 \right\}$$

$$= \sqrt{\frac{2}{\pi}} \left\{ \frac{(1-x^2)\sin sx}{s} - \int \frac{-2x\sin sx}{s} dx \right.$$

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$$= \sqrt{\frac{2}{\pi}} \left\{ \frac{(1-x^2)\sin sx}{s} + \frac{2}{s} \left\{ \frac{-x\cos sx}{s} + \int \frac{\cos sx}{s} \right\} \right.$$

$$= \sqrt{\frac{2}{\pi}} \left\{ \frac{(1-x^2)\sin sx}{s} - \frac{2x\cos sx}{s^2} + \frac{2}{s^3}\sin \right.$$

$$= \sqrt{\frac{2}{\pi}} \left\{ 0 - \frac{2}{s^2}\cos s + \frac{2}{s^3}\sin s \right\}$$

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$$= \sqrt{\frac{2}{\pi}} \left\{ \frac{-2}{s^2} \cos s + \frac{2}{s^3} \sin s \right\}$$

$$= 2\sqrt{\frac{2}{\pi}} \left(\frac{-s\cos s + \sin s}{s^3} \right)$$

Inverse Cosine Transform

$$f(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty 2\sqrt{\frac{2}{\pi}} \left(\frac{-s\cos s + \sin s}{s^3} \right) \cos sx \, dx$$

$$(1-x^2) = \frac{4}{\pi} \int_0^\infty \left(\frac{\sin s - s\cos s}{s^3} \right) \cos sx \, ds$$

~~Replace~~ Put $s = x$

$$(1-s^2) = \frac{4}{\pi} \int_0^\infty \frac{(\sin x - x\cos x)}{x^3} \cos xs \, dx$$

Put $s = 1/2$

$$\left(1 - 1/4\right) = \frac{4}{\pi} \int_0^\infty \frac{\sin x - x\cos x}{x^3} \cos\left(\frac{x}{2}\right) dx$$

$$= \frac{3}{4} \times \frac{\pi}{4}$$

$$= \frac{3\pi}{16}$$

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