Saturday

Unit 5 :
Mathematical expectation, (14 Marks)

Page No. 5
Date 9/2/2019

- Mathematical expectation :-

The mathematical expectation or expected value of D.R.V. 'X' having values $x_1, x_2 \ldots\ldots x_n$ with probabilities $p(X = x_i) = f(x_i)$

$i = 1, 2, \ldots\ldots n$

$$E(X) = \sum_{i=1}^{n} x_i f(x_i)$$

OR

$$E(X) = \sum x f(x)$$

If all probabilities are equal, then

$$E(X) = \frac{x_1 + x_2 + \ldots\ldots\ldots + x_n}{n}$$

i.e $E(X)$ is arithmetic mean of $x_1, x_2 \ldots\ldots x_n$

The mathematical expectation of Continuous random variable is

$$E(X) = \int_{-\infty}^{\infty} x f(x)\, dx$$

Ex: 1 In a lottery there are hundred prices of Rs. 25

50 prices of Rs. 50

20 prices of Rs. 100

10 prices of Rs. 200

4 prices of Rs. 500 &

1 price of Rs. 1000

{different prices for different values discrete

Assuming that 10,000 are to be sold. What is a $\overset{\text{expected}}{}$ fair price to pay for a ticket

ution:→ Let X be a random Variable denotes a amount of money clm on a ticket

$$X = (0, 25, 50, 100, 200, 500, 1000)$$

$$p(x=25) = \frac{100}{10{,}000} = 0.01$$

$$p(x=50) = \frac{50}{10{,}000} = 0.005$$

$$p(x=100) = \frac{20}{10{,}000} = 0.002$$

$$p(x=200) = \frac{10}{10{,}000} = 0.001$$

$$p(x=500) = \frac{4}{10{,}000} = 0.0004$$

$$p(x=1000) = \frac{1}{10{,}000} = 0.0001$$

$$P(X=0) = \frac{\overset{10{,}000-}{(100+50+20+10+4+1)}}{10{,}000} = \frac{9815}{10{,}000} = 0.9815$$

X	0	25	50	100	200	500	1000
f(x)	0.9815	0.01	0.005	0.002	0.001	0.0004	0.0001

∴ Fair price to pay for a ticket is,

$$E(X) = \Sigma\, x.f(x)$$

$$= \left((0\times 0.9815) + (25\times 0.01) + (50\times 0.005) + (100\times 0.002) + (200\times 0.001) + (500\times 0.0004) + (1000\times 0.0001)\right]$$

$$= 0 + 0.25 + 0.25 + 0.2 + 0.2 + 0.2 + 0.1$$

$$E(X) = 1.2$$

S-18

Ex:2 A random variable X has density function

$$f(x) = \begin{cases} e^{-x} & x \geq 0 \\ 0 & x < 0 \end{cases}$$

Find ⓘ $E(X)$

(ii) $E(X^2)$

(iii) $E(e^{2X/3})$

Solution:→ ⓘ $E(X) = \int_{-\infty}^{\infty} x\, f(x)\, dx$

$$= \int_{-\infty}^{0} 0 + \int_{0}^{\infty} x.e^{-x}\, dx$$

$$= \left[-x.e^{-x}\right]_0^{\infty} + \int e^{-x}\, dx$$

$$= \left.-e^{-x}\right]_0^{\infty}$$

$$= e^0$$

$$\boxed{E(X) = 1}$$

ⓘⓘ $E(x^2) = \int_{-\infty}^{\infty} x^2 f(x)\, dx$

$= \int_{0}^{\infty} x^2 e^{-x} dx$

$= -x^2 e^{-x}\Big]_0^{\infty} + \int 2x e^{-x} dx$

$= 0 + 2\int x e^{-x} dx$

$= 2 \cdot 1$

$\boxed{E(x^2) = 2}$

ⓘⓘⓘ $E[e^{2x/3}] = \int_{-\infty}^{\infty} e^{2x/3} f(x) \cdot dx$

$= \int_{0}^{\infty} e^{2x/3} e^{-x} dx$

$= \int_{0}^{\infty} e^{-1/3} dx$

$= \frac{e^{-1/3 x}}{-1/3}\Big]_0^{\infty}$

$\boxed{E[e^{2x/3}] = 3}$

Monday

Ex: 1 Let x be the random variable giving the no. of heads in three tosses of fair coin

Find i) Expectation of x] i) E(x)

(ii) var(x)

(iii) 6X

Sol^n →

3 2 2 1 0 1 1 2

$S = \{HHH, HHT, HTH, HTT, TTT, THT, TTH, THH\}$

$X = \{0, 1, 2, 3\}$

X	0	1	2	3
f(x)	1/8	3/8	3/8	1/8

(i) E(x) :-

$E(x) = \Sigma x f(x) = \mu$

$= 0 \cdot 1/8 + 1 \cdot 3/8 + 2 \cdot 3/8 + 3 \cdot 1/8$

$= 12/8 = 3/2$

(ii) Var(x) :-

$V(x) = E[(x-\mu)^2]$

$= E[(x - 3/2)^2]$

$= E[x^2 - 3x + 9/2]$

$= E(x^2) - E(3x) + E(9/4)$

$= \Sigma x^2 f(x) - 3F(x) + 9/4 E(1)$

$= (0)^2 \cdot 1/8 + 1 \times 3/8 + (2)^2 \times 3/8 + (3)^2 \times 1/8$

$- 3 \times 3/2 + 9/4$

$= 24/8 - 9/2 + 9/4$

$6x^2 =$ $V(x) = 3/4$

(iii) $\sigma x = s.d$

$= \sqrt{3/4} = \sqrt{3}/2$

Ex:2 A random variable x is defined by

$$x = \begin{cases} -2 & \text{prob } 1/3 \\ 3 & \text{prob } 1/2 \\ 1 & \text{prob } 1/6 \end{cases}$$

Find (i) $E(X)$ (iii) $E(X^2)$

(ii) $E(2X+3)$ (iv) $E(X^2+5X)$ &

(v) $Var(x)$

Ans→

x	-2	3	1
$f(x)$	$1/3$	$1/2$	$1/6$

(i) $E(X)$:-

$$E(x) = \Sigma x f(x)$$

$$= -2\left(1/3\right) + 3\left(1/2\right) + 1\left(1/6\right) = 1$$

(ii) $E(2x+3)$:-

step 1:- $E(2x+3) = \Sigma(2x+3) f(x)$

$$= (2(-2)+3)\,1/3 + (2(3)+3)\,1/2 + (2(1)+3)\,1/6$$

$$= 5$$

OR

step 2:- $E(2x+3) = 2E(x) + E(3)$

$$= 2 + 3E(1)$$

$$= 2+3$$

$$= 5$$

(iii) $E(x^2)$:-

$F(x^2) = \Sigma x^2 f(x)$

$= (-2)^2(1/3) + (3)^2(1/2) + 1^2 \times 1/6$

$= 4/3 + 9/2 + 1/6 = 6$.

(iv) $E(x^2+5x)$:-

$E(x^2+5x) = E(x^2) + 5E(x)$

$= 6 + 5(1)$

$= 11$.

(v) $Var(x)$:- $E[(x-\mu)^2]$

$= E[(x-1)^2] = E[x^2 - 2x + 1]$

$= E(x^2) - 2E(x) + E(1)$

$= 6 - 2(1) + 1 \Rightarrow 5$.

Ex:3 Let x be the random variable with density function, $F(x) = \begin{cases} 3x^2 & , 0 \le x \le 1 \\ 0 & , \text{otherwise} \end{cases}$

Find $E(x)$ & $Var(x)$

Ans→ (i) $E(x)$:-

$$E(x) = \int_{-\infty}^{\infty} x f(x) dx$$

$$= \int_0^1 x \cdot 3x^2 dx$$

$$= \int 3x^3 dx = \left.\frac{3x^4}{4}\right]_0^1$$

$$= 3/4$$

$Var(x) =$

$$Var\ X = E[(X-\mu)^2]$$

$$= E[(x-3/4)^2]$$

$$= E[x^2 - 3/2\, x + 9/16]$$

$$= E(x^2) - 3/2\, E(X) + 9/16\, E(1)$$

$$= \int_0^1 x^2 . 3x^2 dx - 3/2\,(3/4) + 9/16$$

$$= 3/80$$

↳ density ⇒ Continuous ⇒ Integration

Ex: 4 A random Variable x has density funition given by $f(x) = \begin{cases} 2e^{-2x} & x \geq 0 \ (0 \text{ to } \infty) \\ 0 & x < 0 \ (-\infty \text{ to } 0) \end{cases}$

Find (i) E(x) (ii) var(x) (iii) $E[(x-1)^2]$

Ans →

(i) E(x) :-

$$E(x) = \int_{-\infty}^{\infty} x f(x) dx$$

$$= \int_0^{\infty} x . 2e^{-2x} dx$$

$$= 2\left[\frac{xe^{-2x}}{-2} - \int \frac{e^{-2x}}{-2} dx\right.$$

$$= 2\left[\frac{x.e^{-2x}}{-2} - \frac{e^{-2x}}{4}\right]_0^{\infty}$$

$$= 0 - 2[0 - 1/4] \Rightarrow E(x) = 1/2$$

$\therefore \quad E(X) = 1/2$

(ii) Var X :-

$$Var\ X = E[(x-\mu)^2]$$
$$= E[(X-1/2)^2]$$
$$= E[X^2 - \cancel{X} + 1/4]$$

$$= E(X^2) - E(X) + E(1/4)$$

$$= \int_0^{\infty} x^2 f(x)\,dx - 1/2 + 1/4$$

$$= 2\int x^2 e^{-2x} dx - 1/4$$

$$= 2\left[\frac{x^2 e^{-2x}}{-2} - \int 2 \frac{x e^{-2x}}{-2} dx - 1/4 \right.$$

$$= 2\left[\frac{x^2 e^{-2x}}{-2} + \int x e^{-2x} dx \right] - \frac{1}{4}$$

$$= 2\left\{ \left[\frac{x^2 e^{-2x}}{-2} \right]_0^{\infty} + \frac{1}{2} \right] - \frac{1}{4}$$

$$= 2[1/2] - 1/4$$

$$= 1 - 1/4$$
$$= 3/4$$

(iii) $E[(X-1)^2]$:- $= E(X^2 - 2X + 1)$

$$= E(X^2) - 2E(X) + E(1)$$
$$= 1 - \cancel{2} \times 1/\cancel{2} + 1$$
$$= 1$$

• Moments and moment generating function:-

Moments about Mean:-

Let x be the random variable with mean 'μ' then rth moment OR rth Central moment of random variable 'x' about the Mean is defined as $\mu_r = E[(x-\mu)^r]$, $r = 0,1,2,3 \dots$

Clearly,

$$\mu_0 = E(1) = 1,$$

$$\mu_1 = E(X-\mu) = E(X) - \mu$$
$$= \mu - \mu$$
$$= 0$$

$$\boxed{\mu_0 = 1} \quad \& \quad \boxed{\mu_1 = 0}$$

put r = 2 for μ_2

$$\mu_2 = E[(X-\mu)^2] = \text{Var} X = \sigma^2$$

• If 'x' is discrete random variable with probability function f(x) then

$$\mu_r = \Sigma (x-\mu)^r f(x)$$

& If x is C.R.V with probability f(x) then

$$\mu_r = \int_{-\infty}^{\infty} (x-\mu)^r f(x)\, dx$$

- $\mu_1' = \left[\frac{d}{dt} M_x(t)\right]_{t=0}$
- $\mu_2' = \left[\frac{d^2}{dt^2} M_x(t)\right]_{t=0}$
- $\mu_3' = \left[\frac{d^3}{dt^3} M_x(t)\right]_{t=0}$
- $\mu_4' = \left[\frac{d^4}{dt^4} M_x(t)\right]_{t=0}$

[x:1] $f(x) = \begin{cases} e^{-x} & x \geq 0 \\ 0 & x < 0 \end{cases}$

S.1ⁿ →

$$M_x(t) = E[e^{tx}] = \int_{-\infty}^{\infty} e^{tx} f(x)\,dx$$

$$= \int_0^{\infty} e^{tx} e^{-x}\,dx$$

$$= \int_0^{\infty} e^{-x(1-t)}\,dx$$

$$= \left.\frac{e^{-x(1-t)}}{-(1-t)}\right]_0^{\infty}$$

$$= \frac{0 + e^0}{1-t}$$

$$= + \frac{1}{1-t}$$

$$M_x(t) = \frac{1}{(1-t)} = (1-t)^{-1} = 1 + t + t^2 + t^3 + t^4 + \cdots$$

$$M_x(t) = 1 + \mu_1' t + \mu_2 t^2/2! + \mu_3' t^3/3! + \mu_4' t^4/4! + \cdots$$

$\mu_1' = 1$

$\frac{\mu_2'}{2!} = 1 \Rightarrow \mu_2' = 2$

$\frac{\mu_3'}{3!} = 1 \Rightarrow \mu_3' = 6$

$\frac{\mu_4'}{4!} = 1 \Rightarrow \mu_4' = 24$

Moment about mean:

$\mu_0 = 1$

$\mu_1 = 0$

$\mu_2 = \mu_2' - \mu^2 = 2 - 1 = 1$

$\mu_3 = \mu_3' - 3\mu_2'\mu + 2\mu^3 = 6 - 3(2)(1) + 2(1)$

$= 2$

$\mu_4 = \mu_4' - 4\mu_3'\mu + 6\mu_2'\mu^2 - 4\mu_1'\mu^3 + \mu_0'\mu^4$

$= 3$

Ex: 2 Find the moment generating function of the random variable x.
Also find first four moments about the origin.

$$x = \begin{cases} 1/2 & \text{prob } 1/2 \\ -1/2 & \text{prob } 1/2 \end{cases}$$

Soln → Moment generating function Mxt,

$$Mx(t) = E[e^{tx}] = \Sigma e^{tx} f(x)$$

$$= e^{t/2}(1/2) + e^{-t/2} \cdot 1/2$$

$$= \frac{e^{t/2} + e^{-t/2}}{2} \leftarrow \text{prob}(1/2)$$

trignometry = cos

hyperbolic = cosh

$$= \cosh(t/2)$$

no coefficient ∴ $\mu_3' = 0$

$$Mx(t) = 1 + \mu_1' t + \mu_2' t^2/2! + \mu_3' t^3/3! + \mu_4' t^4/4! + \cdots$$

$$Mx(t) = \cosh(t/2) = 1 + \frac{(t/2)^2}{2!} + \frac{(t/2)^4}{4!} + \frac{(t/2)^6}{6!} + \frac{(t/2)^8}{8!}$$

equate the same power

$\mu_1' = 0$

$\mu_2' = 1/4$

$\mu_3' = 0$

$\mu_4' = 1/16$

Ex:3 Find moments generating function & ~~of~~ first four moments about the origin for random Variable X :

$$X = \begin{cases} 1 & \text{prob } 1/2 \\ -1 & \text{prob } 1/2 \end{cases}$$

Sol^n → Moment generating function

$$Mx(t) = E[e^{tx}] = \Sigma e^{tx} \cdot f(x)$$

$$= \frac{e^t + e^{-t}}{2}$$

$$= \cos t$$

$$\cancel{\mu_1'} = \frac{d}{dt}(\cos t)$$

$$\mu_1' = \frac{1}{2} \frac{d}{dt}\left[e^t + e^{-t}\right]_{t=0}$$

$$= \frac{1}{2}\left[e^t - e^{-t}\right]_{t=0} = 0$$

$$\mu_2' = \frac{1}{2} \frac{d^2}{dt^2}\left[e^t + e^{-t}\right]$$

$$= \frac{1}{2}\left[e^t + e^{-t}\right]_{t=0}$$

$$= 1$$

Tuesday

Page No. 18
Date : 12/2/2019

$$\mu_3' = \frac{1}{2}\frac{d^3}{dt^3}\left[e^t + e^{-t}\right]$$

$$= \frac{1}{2}\left[e^t - e^{-t}\right] = 0$$

$$\mu_4' = \frac{1}{2}\frac{d^4}{dt^4}\left[e^t + e^{-t}\right]$$

$$= 1$$

Wednesday

GM

Ex:2) A random variable X has density function given by $f(x) = 2e^{-2x} \quad x \geq 0$

$= 0 \quad x < 0$

Find (i) moment generating function

(ii) First four moments about origin.

Sol^n →

(i) $M_x(t) = E[e^{tx}]$

$$= \int_{-\infty}^{\infty} e^{tx} f(x)\, dx$$

$$= 2\int_{0}^{\infty} e^{tx} e^{-2x}\, dx$$

$$= 2\int_{0}^{\infty} e^{-x(2-t)}\, dx$$

$$= 2\left[\frac{e^{-x(2-t)}}{-(2-t)}\right]_0^{\infty}$$

$$\therefore M_x(t) = 2\left[0 + \frac{1}{2-t}\right]$$

Ans → $\frac{-2}{(t-2)}$

Moment generating function $(M_x(t)) = \frac{-2}{(t-2)}$

(ii) Moment generating function about origin:-

$$\mu_1' = \frac{d}{dt}[M_x(t)]_{t=0} = \frac{+2}{(t-2)^2}\bigg|_{t=0} = \frac{1}{2}$$

$$\mu_2' = \frac{d^2}{dt^2}\left[\frac{-2}{(t-2)}\right] = \frac{2(-2)}{(t-2)^3}\bigg|_{t=0} = \frac{1}{2}$$

$$\mu_3' = \frac{d^3}{dt^3}\left[\frac{-2}{(t-2)}\right] = \frac{-4(-3)}{(t-2)^4}\bigg|_{t=0} = \frac{12}{16} = \frac{3}{4}$$

$$\mu_4' = \frac{d^4}{dt^4}\left[\frac{-2}{(t-2)}\right] = \frac{(-4)(-3)(-4)}{(t-2)^5}\bigg|_{t=0} = \frac{3}{2}$$

- Characteristic function :—

The characteristic function for random variable x can be obtained by substitution $t = i\omega$

where i = imaginary unit,

In the moment generating function of random variable X it is denoted by

$$\phi_X(\omega) = E\left[e^{i\omega x}\right]$$

If x is discrete random variable then

$$\phi_X(\omega) = \Sigma \cdot e^{i\omega x} f(x)$$

If ϕ is continuous random variable then

$$\phi_X(\omega) = \int_{-\infty}^{\infty} e^{i\omega x} f(x)\, dx.$$

Ex: 1 Find the characteristic function of random variable x with density function f(x)

$$f(x) = \begin{cases} x/2 & 0 \leq x \leq 2 \\ 0 & \text{otherwise} \end{cases}$$

Soln → By defination of characteristic function

$$\phi_x(\omega) = E[e^{i\omega x}]$$

$$= \int_{-\infty}^{\infty} e^{i\omega x} f(x)\, dx$$

$$= \int_0^2 e^{i\omega x} \cdot x/2 \, dx$$

$$= \frac{1}{2}\left[\frac{x e^{i\omega x}}{i\omega} - \int \frac{e^{i\omega x}}{i\omega} dx\right]$$

$$= \frac{1}{2}\left[\frac{x e^{i\omega x}}{i\omega} - \frac{e^{i\omega x}}{i^2\omega^2}\right]_0^2$$

$$= \frac{1}{2}\left[\frac{2e^{2i\omega}}{i\omega} + \frac{\cancel{2}e^{2i\omega}}{\omega} - \frac{1}{\omega^2}\right]$$

Ex: 2 Find (i) Mean, (ii) The Variance (iii) The moment generating function (iv) characteristic function for the random Variable x, having density function

c

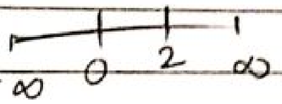
$$f(x) = \begin{cases} cx, & 0 \le x \le 2 \\ 0, & \text{otherwise} \end{cases}$$

Also find Value of c for random Variable X with given density function

Sol^n →

By defination of density function

$$\int_{-\infty}^{\infty} f(x)\,dx = 1$$



$$\int_0^2 cx\,dx = 1$$

$$\left.\frac{cx^2}{2}\right]_0^2 = 1$$

$$2c = 1$$

$$\boxed{c = 1/2}$$

① $$\text{Mean} = \mu = E(X) = \int_{-\infty}^{\infty} x \cdot f(x)\,dx$$

$$= \int_0^2 cx^2\,dx = \frac{1}{2}\left[\frac{x^3}{3}\right]_0^2$$

$$= \boxed{\mu = 4/3}$$

(ii) Variance :-

$$\text{Variance} = V(x) = E\left[(x-\mu)^2\right]$$

$$= \int_{-\infty}^{\infty} (x-\mu)^2 f(x)\,dx$$

$$= \cancel{\frac{1}{2}} = \frac{1}{2}\int_0^2 x\left(x-\frac{4}{3}\right)^2 dx$$

$$= \frac{1}{2}\int x\left[x^2 - \frac{8}{3}x + \frac{16}{9}\right] dx$$

$$= \frac{1}{2}\int\left[x^3 - \frac{8}{3}x^2 + \frac{16}{9}x\right] dx$$

$$= \frac{1}{2}\left[\frac{x^4}{4} - \frac{8}{3}\frac{x^3}{3} + \frac{16}{9}\frac{x^2}{2}\right]_0^2$$

$$= \frac{1}{2}\left[\frac{16}{4} - \frac{8}{9}(8) + \frac{16}{9}\frac{(4)}{2}\right]$$

$$\boxed{V(x) = \frac{2}{9}}$$

(iii) The moment generating function :-

$$M_x(t) = E\left[e^{tx}\right]$$

$$= \int_{-\infty}^{\infty} e^{tx} f(x)\,dx$$

$$= \frac{1}{2}\int_0^2 e^{-tx} x\,dx$$

$$= \frac{1}{2}\left[x\frac{e^{tx}}{t} - \int \frac{e^{tx}}{t}dx\right]$$

$$= \frac{1}{2}\left[\frac{xe^{tx}}{t} - \frac{e^{tx}}{t^2}\right]_0^2$$

$$= \frac{1}{2}\left[\frac{2e^{2t}}{t} - \frac{e^{2t}}{t^2} - 0 + \frac{1}{t^2}\right]$$

$$= \frac{1}{2}\left[\frac{2e^{2t}}{t} + \frac{1}{t^2}(1-e^{2t})\right]$$

$$=$$

(iv) Characteristic function :-

$$\phi_X(\omega) = E\left[e^{i\omega X}\right]$$

$$= \int_{-\infty}^{\infty} e^{i\omega x} f(x)\,dx$$

$$= \frac{1}{2}\int_0^2 x\,e^{i\omega x}\,dx$$

$$= \frac{1}{2}\left[\frac{xe^{i\omega x}}{i\omega} - \int \frac{e^{i\omega x}}{i\omega}dx\right]$$

$$= \frac{1}{2}\left[\frac{xe^{i\omega x}}{i\omega} - \frac{e^{i\omega x}}{i^2\omega^2}\right]_0^2$$

$$= \frac{1}{2}\left[\frac{2e^{2i\omega}}{i\omega} + \frac{e^{2i\omega}}{\omega^2} - 0 - \frac{1}{\omega^2}\right]$$

$$= \frac{1}{2}\left[\frac{2e^{2i\omega}}{i\omega} + \frac{(e^{2i\omega}-1)}{\omega^2}\right]$$

Ex:3 Find (i) Mean (ii) Variance (iii) The moment generating function for the uniform distribution

$$f(x) = \begin{cases} 1/b-a & a<x<b \\ 0 & ,\text{otherwise} \end{cases}$$

Also find first two moments about origin

Sol^n →

Expectation of X

(i) Mean :-

$$\text{Mean} = \mu = E(X) = \int_{-\infty}^{\infty} x f(x)\,dx$$

$$= \int_a^b x \frac{1}{b-a}\,dx$$

$$= \frac{1}{b-a}\left[\frac{x^2}{2}\right]_a^b$$

$$= \frac{1}{b-a}\left[\frac{b^2-a^2}{2}\right]$$

$$= \frac{(b-a)(b+a)}{2(b-a)}$$

$$\boxed{E(X) = \frac{a+b}{2}}$$

(ii) Variance :-

$$\text{Var}\, X = E\left[(X-\mu)^2\right]$$

$$= E\left[\left(X-\frac{a+b}{2}\right)^2\right]$$

$$= E\left[x-c\right]^2$$

$$= E\left[x^2-2xc+c^2\right]$$

$$= E(x^2) - 2cE(X) + c^2 E[1]$$

$$= \int_a^b x^2 f(x)\,dx - 2c$$

$$= \int_a^b x^2 \frac{1}{b-a}\,dx - 2 = \left(\frac{a+b}{2}\right) + c^2$$

$$= \left.\frac{x^3}{3}\right]_a^b \frac{1}{b-a} - 2\left(\frac{a+b}{2}\right)\left(\frac{a+b}{2}\right) + \left(\frac{a+b}{2}\right)^2$$

$$= \frac{b^3-a^3}{3}\,\frac{1}{b-a} - \left(\frac{a+b}{2}\right)^2$$

$$= \frac{\cancel{(b-a)}(b^2+ab+a^2)}{3\,\cancel{(b-a)}} - \left(\frac{a+b}{2}\right)^2$$

$$= \left(\frac{b^2+ab+a^2}{3}\right) - \left(\frac{(a+b)}{2}\right)^2$$

(iii) Moment generating funition :-

$$M_x(t) = E[e^{tx}] = \int_a^b e^{tx} \left\{ \frac{1}{b-a} dx \right.$$

$$= \frac{e^{tx}}{t}\Big]_a^b \frac{1}{b-a}$$

$$M_x(t) = \frac{e^{bt} - e^{at}}{t} \frac{1}{b-a}$$

$$\therefore M_x(t) = \frac{e^{t(b-a)}}{t(b-a)}$$

& for,

Two moments about origin

$$\mu_1' = \frac{d}{dt}\left[M_x(t)\right]_{t=0}$$

$$= \frac{1}{b-a} \frac{d}{dt}\left[\frac{e^{bt} - e^{at}}{t}\right]$$

$$= \frac{1}{b-a}\left\{\frac{t(be^{bt} - ae^{at}) - (e^{bt} - e^{at})}{t^2}\right\}_{t=0}$$

$$0$$

$$\mu_1' = 0$$

$$\mu_2' = \frac{d^2}{dt^2}[M_x(t)]$$

$$= \frac{d}{dt}\left[\frac{bte^{bt} - ate^{at} - e^{bt} - e^{at}}{t^2}\right]$$

$$= \frac{t^2[be^{bt} + b^2te^{bt} - ae^{at} - a^2te^{at} - be^{bt} - ae^{at}] - 2t[bte^{bt} - ate^{at} - e^{bt} - e^{at}]}{t^4}$$

$$\mu_2' = 0$$

Ex: 1 Find Mathematical expectation of discrete Random Variable 'x' whose probability function is $f(x) = \left(\frac{1}{2}\right)^x, x = 1, 2, 3, ----$

Soln → ~~$E(X) =$~~

By defination

$$E(x) = \Sigma x f(x)$$

$$= \sum_{x=1}^{\infty} x\left(\frac{1}{2}\right)^x$$

$$E(x) = S = 1\cdot\left(\frac{1}{2}\right)^1 + 2\left(\frac{1}{2}\right)^2 + 3\left(\frac{1}{2}\right)^3 + ----$$

$$\frac{S}{2} = \frac{1}{2}\left[1\left(\frac{1}{2}\right)^1 + 2\left(\frac{1}{2}\right)^2 + 3\left(\frac{1}{2}\right)^3 + ----\right]$$

$$\frac{S}{2} = \frac{1}{4} + \left(\frac{1}{2}\right)^2 + \frac{3}{2}\left(\frac{1}{2}\right)^3 + -----$$

$$S - S/2 = 1/2 + 1/4 + 1/8 + 1/16 + \cdots$$

$$= 1/2 + 1/2\left(1/2\right) + 1/2\left(1/2\right)^2 + 1/2\left(1/2\right)^3 + \cdots$$

$$= \frac{a}{1-r} = \frac{1/2}{1-1/2} = 1$$

$$S/2 = 1 \rightarrow S = 2$$

$$\boxed{E(x) = 2}$$

Ex:2] If Random variable 'x' is such that $E[(x-1)^2] = 10$, $E(x-2)^2] = 6$ find (i) $E(x)$ (ii) Var X (iii) σx

Solⁿ→ (i) $E(x)$:

$$E(x-1)^2 = E(x^2 - 2x + 1)$$

$$= E(x^2) - 2E(x) + E(1)$$

$$E(x^2) - 2E(x) + 1 = 10 \quad -①$$

$$E[(x-2)^2] = E[x^2 - 4x + 4] = E(x^2) - 4E(x) + 4$$

$$E(x^2) - 2E(x) + 1 = 10 \quad -①$$

$$E(x^2) - 4E(x) + 4 = 6 \quad -②$$

$$\cancel{E(x^2)} - 2E(x) = 9$$

$$\cancel{E(x^2)} - 4E(x) = 2$$

$$- \qquad + \qquad -$$

$$2E(x) = 7$$

$$\boxed{E(x) = 7/2}$$

$$E(x^2) - 2E(x) = 9$$
$$E(x^2) = 9 + 2E(x)$$
$$= 9 + 2\left(7/2\right)$$

$$E(x^2) = 16$$

(ii) Var x :

$$= E\left[(x-\mu)^2\right]$$
$$= E\left(x - 7/4\right)^2$$

$$= E\left[x^2 - 7x + 49/4\right]$$

$$= E(x^2) - 7E(x) + 49/4$$
$$= 16 - 7\left(7/2\right) + 49/4$$

$$= 16 - 49/2 + 49/4$$

+4

$$= 16 - 49/4$$

$$\boxed{\text{Var } x = 15/4}$$

(iii) σx :

$$\sigma x = \sqrt{\text{Var } x}$$
$$= \sqrt{15/4}$$

$$\boxed{\sigma x = \frac{\sqrt{15}}{2}}$$

[-x:3] $Var\,X = E(X^2) - [E(X)]^2$

i.e. $\sigma^2 = E(X^2) - \mu$

Soln →

$$Var\,X = E[(X-\mu)^2]$$
$$= E(X^2 - 2X\mu + \mu^2)$$
$$= E(X^2) - 2\mu E(X) + E(\mu^2)$$
$$= E(X^2) - 2\mu^2 + \mu^2$$
$$= E(X^2) - \mu^2$$

$$\boxed{Var(X) = E(X^2) - [E(X)]^2}$$

$$\boxed{\therefore \sigma^2 = E(X^2) - \mu^2}$$

Ex: 4] If X and Y are independent Random Variable then ~~Variance~~. $Var(X \pm Y) = Var(X) + Var(Y)$

i.e. $\sigma^2_{x \pm y} = \sigma x^2 + \sigma y^2$

Soln → Let

$$E(X) = \mu_1 \,,\; E(Y) = \mu_2$$
$$E(X+Y) = \mu_1 + \mu_2$$
$$Var(X+Y) = E[(X+Y) - (\mu_1 + \mu_2)^2]$$
$$= E[(X-\mu_1) + (Y-\mu_2)]^2]$$
$$= E[(X-\mu_1)^2 + 2(X-\mu_1)(Y-\mu_2) + (Y-\mu_2)^2]$$
$$= E(X-\mu_1)^2 + 2E(X-\mu_1)(Y-\mu_2) + E(Y-\mu_2)^2]$$
$$= E(X-\mu_1)^2 + 2E[XY - X\mu_2 - \mu_1 Y + \mu_1\mu_2] + E(Y-\mu_2)^2$$

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$$= E(x^2) - [E(X)]^2 + 2[E(xy) - \mu_2 E(X) - \mu_1 E(Y) + E(\mu_1 \mu_2)] + E(y^2) - [E(y)]^2$$

$$= E(y^2) - \mu_1^2 + 2[\cancel{\mu_1 \mu_2} - \cancel{\mu_1 \mu_2} - \cancel{\mu_1 \mu_2} + \cancel{\mu_1 \mu_2}] + E(y^2) - \mu_2^2$$

$$Var(x+y) = (E(x^2) - [E(x)]^2) + ([E(y^2) - E(y)^2])$$

$$= Var\,x + Var\,y$$

similarly

$$Var(x-y) = Var\,x - Var\,y$$

$$\boxed{\therefore Var(x \pm y) = Var\,x \pm Var\,y}$$

Ex. 1 The quantity Expectation of $E[(x-\alpha)^2]$ is minimum when $\alpha = \mu$

Sol^n $\rightarrow$ $E[(x-\alpha)^2] = E[(x - \mu + \mu - \alpha)^2]$

$$= E[(x-\mu) + (\mu - \alpha)]^2 \quad (a+b)^2$$

$$= E[(x-\mu)^2 + 2(x-\mu)(\mu-\alpha) + (\mu-\alpha)^2]$$

$$= E[(x-\mu)^2] + 2E[(x-\mu)(\mu-\alpha) + E(\mu-\alpha)^2$$

$$= E(x-\mu)^2 + 2E(x-\mu)E(\mu-\alpha) + E(\mu-\alpha)^2$$

$$= E(x-\mu)^2 + 2\{(E(x) - E(\mu)]\} + E(\mu-\alpha)^2$$

$$= E(x-\mu)^2 + 2[(\mu-\mu)E(\mu-\alpha)\} + E(\mu-\alpha)^2$$

$$E[(x-\alpha)^2] = E(x-\mu)^2 + E(\mu-\alpha)^2$$

expectation of $E(x-\alpha)^2$ occurs,

If ~~μ~~= $\mu-\alpha=0$ i.e $\alpha=\mu$

==X==