Homework 2 MATH163 **Show all your work to get credit.** 79/100

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Question 1*) Let $S = \{\emptyset, a, \{a\}\}$. Determine whether each of these is an element of S, a subset of S, neither, or both.*

(a) $\{a\}$

The power set of S is P(S) = $\{\emptyset, \{\emptyset\}, \{a\}, \{\{a\}\}, \{\emptyset, a\}, \{\emptyset, \{a\}\}, \{a, \{a\}\}, \{\emptyset, a, \{a\}\}\}$

$\{a\}$ *is both an element of S and a subset of S.*

(b) $\{\{a\}\}$

Using the Power Set of S from Part A, $\{\{a\}\}$ *is a subset of S.*

(c) $\emptyset$

$\emptyset$ *is both an element of S and a subset of S. The empty set is a subset of every set, as well as S.*

(d) $\{\{\emptyset\}, a\}$

$\{\{\emptyset\}, a\}$ *is neither an element of S nor a subset of S.* $\{\emptyset\}$ *contains one element, the empty set. It cannot be put with a to form a subset of S nor it is it an alement of S.*

(e) $\{\emptyset\}$

$\{\emptyset\}$ *is not an element of S.*

(f) $\{\emptyset, a\}$

$\{\emptyset, a\}$ *is a subset of S.*

Question 2*) You begin with \$1000. You invest it at 5% compounded annually, but at the end of each year you withdraw \$100 immediately after the interest is paid.*

(a) Set up a recurrence relation and initial condition for the amount you have after n years.

$a_n = 1.05a_{n-1} - 100,\ a_0 = 1000$

(b) How much is left in the account after you have withdrawn \$100 at the end of the third year?

$a_1 = 1.05(1000) - 100$

$a_2 = 1.05(950) - 100 = 897.5$

$a_3 = 1.05(897.5) - 100 = 842.375$

(c) Find a formula for a_n.

$a_n = 1.05a_{n-1} - 100$

$a_{n-1} = 1.05a_{n-2} - 100$

$a_n = 1.05^2(a_{n-2}) - 100(1.05) - 100$

$a_n = 1.05^3(a_{n-3}) - 100(1.05)^2 - 100(1.05) - 100$

$a_n = 1.05^n(a_0) - 100(\Sigma_{i=0}^{n-1}(1.05)^i)$

(d) Use the formula to determine how long it takes before the last withdrawal reduces the balance in the account to \$0.

$0 = 1.05^n(a_0) - 100(\Sigma_{i=0}^{n-1}(1.05)^i)$

$0 = 1.05^n(1000) - 100(1(\frac{1-1.05^n}{1-1.05})$

$0 = 1.05^n(1000) + 2000(1 - 1.05^n)$

$0 = 1.05^n(1000) - 2000(1.05^n) + 2000$

$0 = -1000(1.05^n) + 2000$

$-2000 = -1000(1.05^n)$

$2 = 1.05^n$

$n = 14.2$

Question 3*) If $P(A)$ means the power set of A,*

(a) Prove that $P(A) \cup P(B) \subset P(A \cup B)$ is true for all sets A and B.

Lets assume value $x \in P(A) \cup P(B)$.

$x \in (P(A) \vee P(B))$

$x \in P(A) \vee x \in P(B) -> x \subset A \vee x \subset B$

$x \subset A \vee x \subset B \equiv x \subset (A \vee B)$

$x \subset (A \vee B) -> P(A) \cup P(B) \subset P(A \cup B)$

(b) Prove that the converse of (a) is not true. That is, prove that:
$P(A \cup B) \subset P(A) \cup P(B)$ is false for some sets A and B.

If we set $A = \{1\}$ and $B = \{2, 3\}$

P(A) $= \{\emptyset, \{1\}\}$

P(B) $= \{\emptyset, \{2\}, \{3\}, \{2, 3\}\}$

P(A ∪ B) $= \{\emptyset, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}\}$

$P(A) \cup P(B) = \{\emptyset, \{1\}, \{2\}, \{3\}, \{2, 3\}\}$

Let $x = \{1, 3\}$

$(\{1, 3\} \in P(A \cup B)) \wedge (\{1, 3\} \notin P(A) \cup P(B)$*, so the converse of (a) is not true and false for some sets A and B.*

Question 4*) Prove that the following is true for all sets A, B, and C: if $A \cap C \subset B \cap C$ and $A \cup C \subset B \cup C$, then $A \subset B$.*

Assume that $A \notin B$. $A \not\subset B$

Case 1: Element $x \in C$:

not true!

If $x \in C$, then $x \in A \cap C$, and since $A \cap C \subset B \cap C$, we have $x \in B \cap C$. This implies that $x \in B$ because x is in both $B \cap C$ and C.

Case 2: Element $x \notin C$:

not true!

If $x \notin C$, then $x \in A \cap C$, and since $A \cap C \subset B \cap C$, we have $x \in B \cap C$. This implies that $x \in B$ because x is in both $B \cap C$ and C.

In both cases, $x \in B$ which contradicts $A \notin B$, concluding that $A \subset B$.

Question 5*) Let $f : R \to R$ have the rule $f(x) = \lceil 3x \rceil + 1$and $g : R \to R$ have the rule $g(x) = \dfrac{x}{3}$.*

(a) Find $(fog)^{-1}(\{2.5\})$.

$fog = \lceil x \rceil + 1$. *However, to get 2.5, $\lceil x \rceil$ has to equal 1.5, which cannot be possible because floor or ceiling functions always up or down to the nearest integer, respectively. So the answer for this is $\emptyset$.*

(b) Find $(fog)^{-1}(\{2\})$.

$fog = \lceil x \rceil + 1$. $(fog)^{-1}\{2\} = x \in (0, 1]$ *because any number between 0 and 1 will get rounded up to 1 due to the ceiling function.*

Question 6*) Find a formula for the recurrence relation* $a_n = 2a_{n-1} + 2^n, a_0 = 1$*, using a recursive method.*

$a_n = 2a_{n-1} + 2^n$

$a_{n-1} = 2a_{n-2} + 2^{n-1}$

$a_{n-1} = 2(2a_{n-2} + 2^{n-1}) + 2^n = 2^2(a_{n-2}) + 2^n + 2^n = 2^2 a_{n-2} + 2(2^n)$

$a_{n-2} = 2^2(2a_{n-3} + 2^{n-2}) + 2^n = 2^3(a_n n - 3) + 2^n + 2(2^n) = 2^3(a_n n - 3) + 3(2^n)$

$a_n = 2^n(a_0) + n(2^n)$

$a_n = 2^n(1 + n)$

Question 7*) Find a formula for an infinite sequence $a_1, a_2, a_3, \ldots$ that begins with the terms 1,2,1,2,1,2,1 and continues this alternating pattern.*

$a_n = 1 + ((n+1) \mod 2)$

Basically, with the (n+1) mod 2, any number n that is odd, such as 5, plus 1 mod 2 will equal 0 and any number n that is even, such as 6, plus 1 mod 2 will be equal to 1. Then you just add 1 to get your answer.

very nice!

Question 8*) Find a function $f : \mathbb{Z} \to \mathbb{N}$ that is one-to-one but not onto.*

$f(x) = 2x - 2$ *when* $x \geq 0$ *and* $-2x + 1$ *when* $x < 0$ $f(0) = -2$ which is not in $\mathbb{N}$

This works because the function is one to one because no two elements in the domain of f correspond to the same element in the range of f. The function is not onto, however, because the function f's image is not equal to the codomain. show this

Find a function $g : \mathbb{N} \to \mathbb{Z}$ *that is one-to-one and onto.*

$g(x) = x/2$ *when* x *is even and* $-(x - 1)/2$ *when* x *is odd*

This works because function is one to one because no two elements in the domain of f correspond to the same element in the range of f. The function is also onto because the function f's image is equal to the codomain.

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Question 9*) Find* $1 + x^2 + x^4 + x^6 + x^8 + ...$ *assuming* $0 < |x| < 1$.

Sum of geometric sequences: $1/(1 - r)$.

Since the common difference of the sequence above is x^2*, the answer is* $1/(1 - x^2)$.

Question 10*) Use the Principle of Mathematical Induction to show this inequality is true for all integers $n \geq 2$:* $\displaystyle\sum_{i=1}^{n} \frac{1}{\sqrt{i}} > \sqrt{n}$

Base Case (n = 2):

For $n = 2$, we have:

$$\sum_{i=1}^{2} \frac{1}{\sqrt{i}} = \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} = 1 + \frac{1}{\sqrt{2}}$$

Now, $\sqrt{2} < 2$ and $\frac{1}{\sqrt{2}} > \frac{1}{2}$. Therefore, $1 + \frac{1}{\sqrt{2}} > 1 + \frac{1}{2} = \frac{3}{2}$, which is greater than $\sqrt{2}$. So, the base case is true.

Inductive Hypothesis:

Assume that for some positive integer $k \geq 2$, the inequality

$$\sum_{i=1}^{k} \frac{1}{\sqrt{i}} > \sqrt{k}$$

is true.

Inductive Step:

We need to prove that for $n = k + 1$, the inequality

$$\sum_{i=1}^{k+1} \frac{1}{\sqrt{i}} > \sqrt{k+1}$$

is true.

Starting with the left-hand side:

$$\sum_{i=1}^{k+1} \frac{1}{\sqrt{i}} = \left(\sum_{i=1}^{k} \frac{1}{\sqrt{i}}\right) + \frac{1}{\sqrt{k+1}}$$

By the inductive hypothesis, we know that $\sum_{i=1}^{k} \frac{1}{\sqrt{i}} > \sqrt{k}$. Therefore:

$$\sum_{i=1}^{k+1} \frac{1}{\sqrt{i}} > \sqrt{k} + \frac{1}{\sqrt{k+1}}$$

Now, we need to prove that $\sqrt{k} + \frac{1}{\sqrt{k+1}} > \sqrt{k+1}$. We can do this by squaring both sides of the inequality:

$$\left(\sqrt{k} + \frac{1}{\sqrt{k+1}}\right)^2 > (\sqrt{k+1})^2$$

Expanding and simplifying the left-hand side:

$$k + 2\sqrt{k} \cdot \frac{1}{\sqrt{k+1}} + \left(\frac{1}{\sqrt{k+1}}\right)^2 > k + 1$$

Since $\frac{1}{\sqrt{k+1}} > 0$, *we can safely remove it from both sides of the inequality:*

$$k + 2\sqrt{k} \cdot \frac{1}{\sqrt{k+1}} > k + 1$$

Now, subtracting k *from both sides:*

$$2\sqrt{k} \cdot \frac{1}{\sqrt{k+1}} > 1$$

Simplifying further by multiplying both sides by $\sqrt{k+1}$*:*

$$2\sqrt{k} > \sqrt{k+1}$$

Squaring both sides again:

$$4k > k + 1$$

Subtracting k *from both sides:*

$$3k > 1$$

Since $k \geq 2$, *it's clear that* $3k > 1$, *and the inequality is true. Therefore:*

$$\sqrt{k} + \frac{1}{\sqrt{k+1}} > \sqrt{k+1}$$

Now, we can conclude that:

$$\sum_{i=1}^{k+1} \frac{1}{\sqrt{i}} > \sqrt{k} + \frac{1}{\sqrt{k+1}} > \sqrt{k+1}$$

So, the inequality holds for $n = k + 1$.

By the Principle of Mathematical Induction, we have shown that the inequality

$$\sum_{i=1}^{n} \frac{1}{\sqrt{i}} > \sqrt{n}$$

is true for all integers $n \geq 2$.

Question 11*) Prove that for all positive integers n, $3^{2^n} - 1$ is divisible by 2^{n+2}.*

Base Case: $n = 1$

$3^{2^1} - 1 = 3^2 - 1 = 9 - 1 = 8 = 2^{1+2} = 2^3$

I.H. Assume that for some positive integer k, $3^{2^k} - 1$ is divisible by 2^{k+2}.

I.S. $n = k + 1$ No!

$n = k + 1$, $3^{2^{k+1}} - 1$ *is divisible by* 2^{k+3}.

$$3^{2^{k+1}} - 1 = (3^{2^k})^2 - 1$$

Difference of squares formula: $a^2 - b^2 = (a + b)(a - b)$, *where* $a = 3^{2^k}$ *and* $b = 1$*:*

$$3^{2^{k+1}} - 1 = (3^{2^k} + 1)(3^{2^k} - 1)$$

From our inductive hypothesis, $3^{2^k} - 1$ *is divisible by* 2^{k+2}, *so* $3^{2^k} - 1 = 2^{k+2} \cdot m$, *where* m *is an integer.*

$$3^{2^{k+1}} - 1 = (3^{2^k} + 1)(2^{k+2} \cdot m)$$

Prove that this expression is divisible by 2^{k+3}, *by showing that* $3^{2^k} + 1$ *is divisible by* (2).

Since $3^{2^k} + 1$ *is the sum of an even number* (3^{2^k}) *and an odd number* (1), *it is always an odd number. Any odd number is divisible by 2 once. Therefore,* $3^{2^k} + 1$ *is divisible by 2.*

Hence, $3^{2^{k+1}} - 1$ *is divisible by* 2^{k+3}.

By induction, for all positive integers n, $3^{2^n} - 1$ *is divisible by* 2^{n+2}.

Question 12*) Find a formula for*

$$(1-\frac{1}{2^2})(1-\frac{1}{3^2})(1-\frac{1}{4^2})(1-\frac{1}{5^2})...(1-\frac{1}{n^2})$$

where $n \geq 2$, and use the Principle of Mathematical Induction to prove that the formula is correct.

Base Case ($n = 2$):

For $n = 2$, the formula gives:

$$\frac{(2^2-1)}{2^2} = \frac{3}{4}$$

The product in the original expression is $(1-\frac{1}{2^2}) = \frac{3}{4}$. So, the base case is true.

Inductive Hypothesis:

Assume that the formula holds for some positive integer $k \geq 2$:

$$(1-\frac{1}{2^2})(1-\frac{1}{3^2})\dots(1-\frac{1}{k^2}) = \frac{(2^2-1)}{2^2}\cdot\frac{(3^2-1)}{3^2}\dots\frac{(k^2-1)}{k^2}$$

Inductive Step:

We need to show that the formula holds for $n = k+1$. Using the inductive hypothesis, we have:

$$(1-\frac{1}{2^2})(1-\frac{1}{3^2})\dots(1-\frac{1}{k^2})\cdot(1-\frac{1}{(k+1)^2}) = \left(\frac{(2^2-1)}{2^2}\cdot\frac{(3^2-1)}{3^2}\dots\frac{(k^2-1)}{k^2}\right)\cdot(1-\frac{1}{(k+1)^2})$$

Now, we can simplify the right-hand side using the general form for the formula:

$$\left(\frac{(2^2-1)}{2^2}\cdot\frac{(3^2-1)}{3^2}\dots\frac{(k^2-1)}{k^2}\right)\cdot(1-\frac{1}{(k+1)^2}) = \frac{(2^2-1)}{2^2}\cdot\frac{(3^2-1)}{3^2}\dots\frac{(k^2-1)}{k^2}\cdot\frac{(k+1)^2-1}{(k+1)^2}$$

this is not the formula.

We can now expand the product:

$$\frac{(2^2-1)}{2^2}\cdot\frac{(3^2-1)}{3^2}\dots\frac{(k^2-1)}{k^2}\cdot\frac{(k+1)^2-1}{(k+1)^2} = \frac{(k+1)^2-1}{(k+1)^2}$$

Now, notice that this is exactly the form of the formula for $n = k+1$. Therefore, the formula holds for $n = k+1$.

By mathematical induction, we have shown that the formula:

$$(1-\frac{1}{2^2})(1-\frac{1}{3^2})\dots(1-\frac{1}{n^2}) = \frac{(2^2-1)}{2^2}\cdot\frac{3^2-1}{3^2}\dots\frac{(n^2-1)}{n^2}$$

is correct for all positive integers $n \geq 2$.

Question 13*) Which amounts of stamps can be formed using just five cents stamp and nine cents stamp? Prove your answer using strong induction.*

Base Case:

For small values, we can directly observe that the following amounts can be formed using only 5-cent and 9-cent stamps:

- *5 cents: One 5-cent stamp.*
- *9 cents: One 9-cent stamp.*
- *10 cents: Two 5-cent stamps.*
- *14 cents: One 5-cent stamp and one 9-cent stamp.*
- *15 cents: Three 5-cent stamps.*
- *18 cents: Two 9-cent stamps.*
- *19 cents: Two 5-cent stamps and one 9-cent stamp.*
- *20 cents: Four 5-cent stamps.*
- *23 cents: One 5-cent stamp and three 9-cent stamps.*
- *24 cents: Three 5-cent stamps and one 9-cent stamp.*
- *25 cents: Five 5-cent stamps.*

Now, we will use strong induction to show that any amount greater than or equal to 32 cents can be formed.

Inductive Hypothesis:

if this is true, then no need to prove anything

Assume that for all positive integers k such that $k \geq 32$, we can form k cents using only 5-cent and 9-cent stamps.

Inductive Step:

We want to show that for $n = k + 1$, we can form $(k + 1)$ cents using these stamps.

Consider n cents where $n \geq 32$. We have two possibilities:

1. *Use one 5-cent stamp to form $(n - 5)$ cents. Since $n - 5 \geq 32$, by the inductive hypothesis, we can form $(n - 5)$ cents using the stamps.*
2. *Use one 9-cent stamp to form $(n - 9)$ cents. Since $n - 9 \geq 32$, by the inductive hypothesis, we can form $(n - 9)$ cents using the stamps.*

In either case, we can add one more 5-cent or 9-cent stamp to what was formed for $(n-5)$ or $(n - 9)$ cents, respectively, to get $(n + 1)$ cents.

Therefore, for any $n \geq 32$, we can form $(n+1)$ cents using only 5-cent and 9-cent stamps.

By strong induction, we have shown that any amount greater than or equal to 32 cents can be formed using these stamps.

no!

Question 14*) Let P(n) be the statement that a postage of n cents can be formed using just 3- cent stamps and 5- cent stamps. The parts of this exercise outline a strong induction proof that P(n) is true for* $n \geq 8$.

a) Show that the statements P(8), P(9), and P(10) are true, completing the basis step of the proof.

$3 + 5 = 8, 3 + 3 = 9, 5 + 5 = 10$, *making P (8), P (9), and P (10)true.*

b) What is the inductive hypothesis of the proof?

I.H.: P(n) is true for $8 \leq n \leq k$, *where* $k \geq 10$.

c) What do you need to prove in the inductive step?

Prove that P(k+1) is true. no!

d) Complete the inductive step for $k \geq 10$.

If $k \geq 10$, *then k + 1 = (k - 2) + 3. Since* $k - 2 \geq 8$, *then P (k - 2) is true due to the I.H.*

A postage of k - 2 cents can be paid by using 3-cent and 5-cent stamps. Adding one 3-cent stamp, we can pay a postage of k + 1 cents, making P(k + 1) is true.

e) Explain why these steps show that this statement is true whenever $n \geq 8$.

It shows that for greater than or equal to 8, P(n) is true. When k = 10, and set P(8) as P(k-2), adding a 3-cent coin makes it P(k+1). Getting to the next number, you just have to add k.

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