Probability and Statistics (MA2001)
(Conditional probability, total probability, Bayes theorem, and independence)

Lectures-4 and 5

by

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Conditional probability

Example

Let us toss two fair coins. Let A denote that both coins show same face and B denote at least one coin shows head. Obtain the probability of happening of A given that B has already occured.

Solution

Listen to my lecture.

Definition

Let $(\Omega, \mathcal{F}, P)$ be a probability space and $B \in \mathcal{F}$ be a fixed event such that $P(B) > 0$. Then, the conditional probability of event A given that B has already occured is defined as

$$P(A|B) = \frac{P(A \cap B)}{P(B)}.$$

Example

Six cards are dealt at random (without replacement) from a deck of 52 cards. Find the probability of getting all cards of heart in a hand (event A) given that there are at least 5 cards of heart in the hand (event B).

Solution

Clearly,

$$P(A \cap B) = P(A) = \frac{\binom{13}{6}}{\binom{52}{6}} \quad \text{and} \quad P(B) = \frac{\binom{13}{5}\binom{39}{1} + \binom{13}{6}}{\binom{52}{6}}.$$

Thus, $P(A|B) = \frac{\binom{13}{6}}{\binom{13}{5}\binom{39}{1}+\binom{13}{6}}$.

Conditional probability (cont...)

Example

An urn contains four red and six black balls. Two balls are drawn successively, at random and without replacement, from the urn. Find the probability that the first draw resulted in a red ball and the second draw resulted in a black ball.

Solution

Let A denote the event that the first draw results in a red ball and B that the second ball results in a black ball. Then,

$$P(A \cap B) = P(A)P(B|A) = \frac{4}{10} \times \frac{6}{9} = \frac{12}{45}.$$

Total probability

Theorem of total probability

Let $(\Omega, \mathcal{F}, P)$ be a probability space and let $\{E_i; i \in \mathcal{A}\}$ be a countable collection of mutually exclusive and exhaustive events (that is, $E_i \cap E_j = \phi$ for $i \neq j$ and $P(\cup_{i\in\mathcal{A}} E_i) = P(\Omega) = 1$) such that $P(E_i) > 0$ for all $i \in \mathcal{A}$. Then, for any event $E \in \mathcal{A}$,

$$P(E) = \sum_{i\in\mathcal{A}} P(E \cap E_i) = \sum_{i\in\mathcal{A}} P(E|E_i)P(E_i).$$

Proof

Let $F = \cup_{i\in\mathcal{A}} E_i$. Then, $P(F) = P(\Omega) = 1$ and $P(F^c) = 1 - P(F) = 0$. Again, $E \cap F^c \subset F^c \Rightarrow 0 \leq P(E \cap F^c) \leq P(F^c) = 0$.

Total probability (cont...)

Proof (cont...)

Thus,

$$\begin{aligned} P(E) &= P(E \cap F) + P(E \cap F^c) \\ &= P(E \cap F) \\ &= P(\cup_{i \in \mathcal{A}}(E \cap E_i)) \\ &= \sum_{i \in \mathcal{A}} P(E \cap E_i) \\ &= \sum_{i \in \mathcal{A}} P(E|E_i)P(E_i), \end{aligned}$$

since E_i's are disjoint implies that $E_i \cap E$'s are disjoint.

Example

Urn U_1 contains four white and six black balls and urn U_2 contains six white and four black balls. A fair die is cast and urn U_1 is selected if the upper face of die shows 5 or 6 dots, otherwise urn U_2 is selected. If a ball is drawn at random from the selected urn find the probability that the drawn ball is white.

Solution

W $\rightarrow$ drawn ball is white; $E_1 \rightarrow$ Urn U_1 is selected; $E_2 \rightarrow$ Urn U_2 is selected.

Here, $\{E_1, E_2\}$ is a collection of mutually exclusive and exhaustive events. Thus,

$$\begin{aligned} P(W) &= P(E_1)P(W|E_1) + P(E_2)P(W|E_2) \\ &= \frac{2}{6} \times \frac{4}{10} + \frac{4}{6} \times \frac{6}{10} = \frac{8}{15}. \end{aligned}$$

Bayes theorem

Theorem

Let $(\Omega, \mathcal{F}, P)$ be a probability space and let $\{E_i; i \in \mathcal{A}\}$ be a countable collection of mutually exclusive and exhaustive events with $P(E_i) > 0$ for $i \in \mathcal{A}$. Then, for any event $E \in \mathcal{F}$, with $P(E) > 0$, we have

$$P(E_j|E) = \frac{P(E|E_j)P(E_j)}{\sum_{i\in\mathcal{A}} P(E|E_i)P(E_i)},\ j \in \mathcal{A}.$$

Proof

For $j \in \mathcal{A}$, $P(E_j|E) = \frac{P(E_j \cap E)}{P(E)} = \frac{P(E|E_j)P(E_j)}{P(E)} = \frac{P(E|E_j)P(E_j)}{\sum_{i\in\mathcal{A}} P(E|E_i)P(E_i)}$ from the theorem of total probability.

Bayes theorem (cont...)

Note

- $P(E_j),\ j \in \mathcal{A}$ are known as the prior probabilities.
- $P(E_j|E)$ are known as the posterior probabilities.

Bayes theorem (cont...)

Example

Urn U_1 contains four white and six black balls and urn U_2 contains six white and four black balls. A fair die is cast and urn U_1 is selected if the upper face of die shows 5 or 6 dots, otherwise urn U_2 is selected. A ball is drawn at random from the selected urn.

- Given that the drawn ball is white, what is the conditional probability that it came from U_1.
- Given that the ball is white, find the conditional probability that it came from urn U_2.

Solution

W $\rightarrow$ drawn ball is white;

E_1 $\rightarrow$ Urn U_1 is selected;

E_2 $\rightarrow$ Urn U_2 is selected.

Bayes theorem (cont...)

Solution (contd...)

E_1 and E_2 are mutually exclusive and exhaustive events.

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\begin{aligned}
(i)\ P(E_1|W) &= \frac{P(W|E_1)P(E_1)}{P(W|E_1)P(E_1) + P(W|E_2)P(E_2)} \\
&= \frac{\frac{4}{10} \times \frac{2}{6}}{\frac{4}{10} \times \frac{2}{6} + \frac{6}{10} \times \frac{4}{6}} = \frac{1}{4}.
\end{aligned}
$$

(ii) Since E_1 and E_2 are mutually exclusive and $P(E_1 \cup E_2|W) = P(\Omega|W) = 1$, we have

$$P(E_2|W) = 1 - P(E_1|W) = \frac{3}{4}.$$

Definition

Let $(\Omega, \mathcal{F}, P)$ be a probability space and A and B be two events. Events A and B are said to be

- negatively associated (correlated) if $P(A \cap B) < P(A)P(B)$
- positively associated (correlated) if $P(A \cap B) > P(A)P(B)$
- independent if $P(A \cap B) = P(A)P(B)$
- dependent if they are not independent.

Note

- If $P(B) = 0$, then $P(A \cap B) = 0 = P(A)P(B)$ for all $A \in \mathcal{F}$. That is, if $P(B) = 0$, then any event $A \in \mathcal{F}$ and B are independent.
- If $P(B) > 0$, then A and B are said to be independent if and only if $P(A|B) = P(A)$.

Independence

Let $(\Omega, \mathcal{F}, P)$ be a probability space. Let $\mathcal{A} \subset \mathbb{R}$ be an index set and let $\{E_\alpha : \alpha \in \mathcal{A}\}$ be a collection of events in $\mathcal{F}$.

- Events $\{E_\alpha : \alpha \in \mathcal{A}\}$ are said to be pairwise independent if any pair of events E_α and E_β, $\alpha \neq \beta$ in the collection $\{E_j : j \in \mathcal{A}\}$ are independent, that is, if $P(E_\alpha \cap E_\beta) = P(E_\alpha)P(E_\beta)$, $\alpha, \beta \in \mathcal{A}$ and $\alpha \neq \beta$.
- Let $\mathcal{A} = \{1, 2, \cdots, n\}$ for some $n \in \mathcal{N}$. The events $E_1, \cdots, E_n$ are said to be independent if for any sub collection $\{E_{\alpha_1}, \cdots, E_{\alpha_k}\}$ of $\{E_1, \cdots, E_n\}$ $(k = 2, 3, \cdots, n)$

$$P(\cap_{j=1}^{n} E_{\alpha_j}) = \prod_{j=1}^{n} P(E_{\alpha_j}).$$

Independence

- Independence $\Rightarrow$ pairwise independence
- pairwise independence $\nRightarrow$ Independence (always!)

Example of independent events

- Rolling two dice, x_1 and x_2. Let A be the event $x_1 = 3$ and B be the event $x_2 = 4$. Then, A and B are independent.

Example

Take four identical marbles. On the first write symbols $A_1A_2A_3$. On each of the other three, write A_1, A_2 and A_3, respectively. Put the four marbles in an urn and draw one at random. Let E_i denote the event that the symbol A_i appears on the drawn marble. Then, show that E_1, E_2 and E_3 are not independent though they are pairwise independent.

Solution

See during the lecture!

Example

Consider the experiment of tossing of coin three times. Let H_i denote the event that the ith toss is head. Are H_1, H_2, H_3 independent?

Solution

See during the lecture!

References

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Thank You