Probability and Statistics (MA2001)
(Random variable and distribution function)

Lectures-6, 7 & 8

by

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Random variable

The sample space (Ω), event space ($\mathscr{F}$) and probability measure ($\mathcal{P}$) provide the underlying structure for a random experiment.

Defining a random variable is crucial because it allows us to quantify and analyze the outcomes of a random experiment in a more systematic and mathematically rigorous way.

The random variable bridges the gap between the abstract concepts and the numerical outcomes that we can work with.

Specifically, a random variable enables us to:

- Assign numerical values to the outcomes of a random experiment. This allows us to perform mathematical operations and statistical analysis on the outcomes, which is not possible with just the sample space and event space.
- Compute the probability distribution of the random variable, which describes the likelihood of different numerical outcomes occurring. This probability distribution can then be used to calculate expected values, variances, and other statistical properties.
- Develop and apply a wide range of probability models and statistical techniques, such as hypothesis testing, parameter estimation, and regression analysis. These methods rely on the mathematical properties of random variables and their associated probability distributions.

Random variable (cont...)

Definition

Let $(\Omega, \mathcal{F}, P)$ be a probability space and let $X : \Omega \to \mathbb{R}$ be a given function. We say that X is a random variable if

$$X^{-1}(\mathcal{B}) \in \mathcal{F} \quad \text{for all} \quad \mathcal{B} \in \mathcal{B}_1,$$

where $\mathcal{B}_1$ is the Borel sigma-field.

Alternative

Let $(\Omega, \mathcal{F}, P)$ be a probability space. Then, a real valued measurable function defined on the sample space is known as the random variable.

Random variable (cont...)

Theorem

Let $(\Omega, \mathcal{F}, P)$ be a probability space and let $X : \Omega \to \mathbb{R}$ be a given function. Then, X is a random variable if and only if

$$X^{-1}((-\infty, a]) = \{w \in \Omega : X(\omega) \leq a\} \in \mathcal{F}$$

for all $a \in \mathbb{R}$.

Random variable (cont...)

Example

Consider the experiment of tossing of a coin. Then, the sample space is $\Omega = \{H, T\}$. Define X as the number of heads. Then, $X(H) = 1$ and $X(T) = 0$. Consider

$$\mathcal{F} = \text{Power set}(\Omega) = \{\phi, \{H\}, \{T\}, \Omega\}.$$

Our goal is to show that $X : \Omega \to \mathbb{R}$ is a random variable. Here,

$$\{w \in \Omega : X(\omega) \leq a\} = \begin{cases} \phi, & a < 0 \\ \{T\}, & 0 \leq a < 1 \\ \Omega, & a \geq 1 \end{cases}$$

which belongs to $\mathcal{F}$. Thus, X is a random variable.

Distribution function

Definition

A function $F : \mathcal{R} \to \mathcal{R}$ defined by

$$F_X(x) = P(X \leq x) = P((-\infty, x]), \ x \in \mathcal{R}$$

is called the distribution function of the random variable X. It is also denoted by $F_X(x)$.

Theorem

Let F_X be the distribution function of a random variable X. Then,

- F_X is non-decreasing.
- F_X is right continuous.
- $F_X(\infty) = 1$ and $F_X(-\infty) = 0$.

Distribution function (cont...)

Example

Suppose that a fair coin is independently flipped thrice. Then, the sample space is

$$\Omega = \{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT\}.$$

Let X be a random variable, which denotes the number of heads. Then,

$$P(X = 0) = \frac{1}{8} = P(X = 3), \; P(X = 1) = \frac{3}{8} = P(X = 2).$$

Distribution function (cont...)

Example (cont...)

The distribution function of X is

$$F_X(x) = \begin{cases} 0, & x < 0 \\ \frac{1}{8}, & 0 \leq x < 1 \\ \frac{1}{2}, & 1 \leq x < 2 \\ \frac{7}{8}, & 2 \leq x < 3 \\ 1, & x \geq 3. \end{cases}$$

- $F_X(x)$ is non-decreasing, right continuous, $F_X(+\infty) = 1$ and $F_X(-\infty) = 0$. Moreover, $F_X(x)$ is a step function having discontinuities at $0, 1, 2$ and 3.
- Sum of the sizes of the jumps is equal to 1.

Distribution function (cont...)

Note

Let $-\infty < a < b < \infty$. Then,

- $P(a < X \leq b) = P(X \leq b) - P(X \leq a)$
- $P(a < X < b) = P(X < b) - P(X \leq a)$
- $P(a \leq X < b) = P(X < b) - P(X < a)$
- $P(a \leq X \leq b) = P(X \leq b) - P(X < a)$
- $P(X \geq a) = 1 - P(X < a)$
- $P(X > a) = 1 - P(X \leq a)$

Example

Find the value of k such that

$$F_X(x) = \begin{cases} 0, & x < 2 \\ \frac{2}{3}, & 2 \leq x < 5 \\ \frac{7-6k}{6}, & 5 \leq x < 9 \\ \frac{3k^2-6k+7}{6}, & 9 \leq x < 14 \\ \frac{16k^2-16k+19}{16}, & 14 \leq x \leq 20 \\ 1, & x > 20. \end{cases}$$

will be a distribution function.

For solution see the lecture!!

Example

Let

$$F_X(x) = \begin{cases} 0, & x < 0 \\ \frac{x}{4}, & 0 \leq x < 1 \\ \frac{x}{3}, & 1 \leq x < 2 \\ \frac{3x}{8}, & 2 \leq x < \frac{5}{2} \\ 1, & x \geq \frac{5}{2}. \end{cases}$$

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Check if this is a distribution function. Find $P(1 < X \leq 5/2)$, $P(1 \leq X < 5/2)$, and $P(X = 1)$.

For solution see the lecture!!

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Distribution function (cont...)

Example

Consider a function $G : \mathcal{R} \to \mathcal{R}$ defined by

$$G(x) = \begin{cases} 0, & x < 0 \\ 1 - e^{-x}, & x \geq 0 \end{cases}$$

Observations

- Clearly, G is nondecreasing, continuous and satisfies $G(-\infty) = 0$ and $G(\infty) = 1$. Thus, G is a distribution function for a random variable X.
- Since G is continuous, we have $P(X = x) = G(x) - G(x^-) = 0$ for all $x \in \mathcal{R}$, where $G(x^-)$ is the left hand limit of G at the point x.

Distribution function (cont...)

Example (cont...)

- For $-\infty < a < b < \infty$, $P(a < X < b) = P(a \leq X < b) = P(a \leq X \leq b) = P(a < X \leq b) = G(b) - G(a)$.
- $P(X \geq a) = P(X > a) = 1 - G(a)$ and $P(X < a) = P(X \leq a) = G(a)$.
- $P(2 < X \leq 3) = G(3) - G(2) = e^{-2} - e^{-3}$
- $P(-2 < X \leq 3) = G(3) - G(-2) = 1 - e^{-3}$
- $P(X \geq 2) = 1 - G(2) = e^{-2}$
- $P(X > 5) = 1 - G(5) = e^{-5}$.
- Note that the sum of sizes of jumps of G is 0.

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Thank You