HS301 - Financial Economics (Midsem)

Name : Chandravanshi Mangesh Shivaji
Roll No : 1801CS16
Date : 05/10/2020
Email : 1801cs16@iitp.ac.in

Q.1) ⟹

Given,
Utility function for an investor is
$$U = \ln(X)$$
where,
X is income
U is the Utility Level.

Gamble: Gain 5 ₹ or 7 ₹ with equal probabilities.

∴ Mean return (Expected) = E(X)

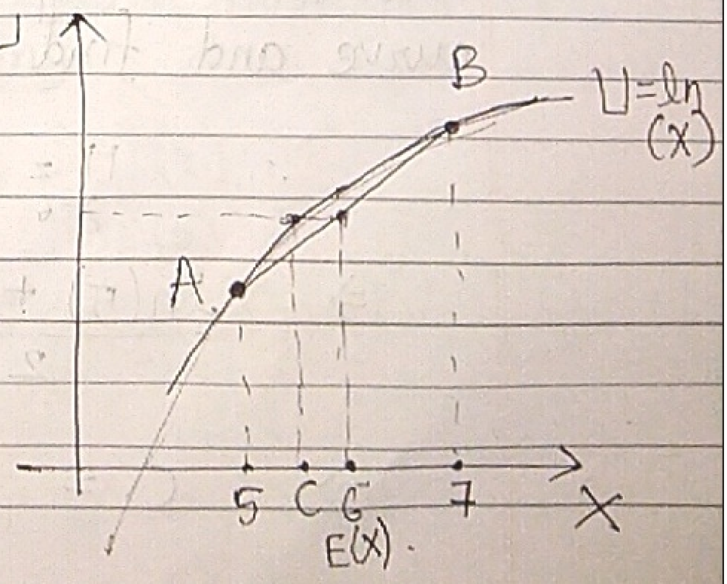
$$= (5)\left(\frac{1}{2}\right) + 7\left(\frac{1}{2}\right)$$

$$= 12/2 = 6 ₹$$

∴ $\boxed{E(X) = 6}$

From given information we can plot the corres-ponding graph,

where AB line is risk-neutral. (corresponds to)



i.e. Investor does not take part in gamble.

$A(5, \ln(5))$ and $B(7, \ln(7))$
from $U = \ln(x)$

Eq^n of line AB will be,

$$U - \ln(5) = \left(\frac{\ln(7) - \ln(5)}{7-5}\right)(x-5)$$

When $X = E(x) = 6$; the value of U will be

$$U_{X=6} = \ln(5) + \left(\frac{\ln(7) - \ln(5)}{2}\right)(6-5)$$

$$\Rightarrow \boxed{U_{X=6} = \frac{\ln(5) + \ln(7)}{2}}$$

For certainty equivalence (i.e. minimum amount of money for a person when it does not matter he/she takes part in the gamble or not),

we have to equate $U_{X=6}$ value to Utility curve and find corresponding x (Income)

$$\therefore U_{X=6} = \ln(C)$$

$$\Rightarrow \frac{\ln(5) + \ln(7)}{2} = \ln(C).$$

$$\Rightarrow C = e^{(\ln(5) + \ln(7))/2}$$

$\Rightarrow C = 5.916079$

$\boxed{C \simeq 5.916 \text{ ₹}}$

$\therefore$ The certainty equivalence is equal to 5.916 ₹.

$\therefore$ As $C < E(x)$;
we can say that investor is risk-averse.

$\Rightarrow$ $U = \ln(x)$

$$\frac{dU}{dx} = \frac{1}{x}$$

$$\frac{d^2U}{dx^2} = \frac{-1}{x^2} < 0$$

$\Rightarrow$ U concave $\Rightarrow$ risk averse investor.

$\Rightarrow$ Investor's risk aversion does not change with income X (By Arrow-Pratt measure of relative risk aversion)

$\Rightarrow$
$$A' = xA = x \cdot \frac{-U''(x)}{U'(x)}$$

$$= x \cdot \frac{-\left(-1/x^2\right)}{1/x} = 1.$$

A' is constant.

Q.2) ⇒ The risk-return frontier for an investor who buys two assets with same return but different risks (standard deviations).

Let r_1 and r_2 be correspondings returns for two assets.

σ_1 and σ_2 be their std. deviations resp.

Portfolio's return $(\bar{r}) = w_1 r_1 + w_2 r_2$

Portfolio's risk $(\sigma) = \sqrt{w_1^2 \sigma_1^2 + 2 w_1 w_2 \sigma_{12} + w_2^2 \sigma_2^2}$ — (I)

where w_1, w_2 are the corresponding weights.

$r_1 = r_2$ (Given).

$\Rightarrow$ let $r_1 = r_2 = R$.

$\therefore \bar{r} = (w_1 + w_2) R$

$\Rightarrow \bar{r} = R$ (As $w_1 + w_2 = 1$).

So, whatever the correlation or risk between the assets return will be same and equal to R.

$$\rho_{12} = \frac{\sigma_{12}}{\sigma_1 \sigma_2}$$

From (I), we can write

$$\sigma^2 = w_1^2\sigma_1^2 + 2w_1w_2\sigma_{12} + w_2^2\sigma_2^2$$

$$\Rightarrow \sigma^2 = (1-w_2)^2\sigma_1^2 + 2\underset{(1-w_2)}{w_1}w_2\sigma_{12} + w_2^2\sigma_2^2$$

$$\sigma^2 = \sigma_1^2 - 2w_2\sigma_1^2 + w_2^2\sigma_1^2 + 2\sigma_{12}w_2 - 2\sigma_{12}w_2^2 + w_2^2\sigma_2^2$$

$$\frac{\partial\sigma^2}{\partial w_2} = 0 - 2\sigma_1^2 + 2w_2\sigma_1^2 + 2\sigma_{12} - 4\sigma_{12}w_2 + 2w_2\sigma_2^2$$

For minimum σ;

$$\frac{\partial\sigma^2}{\partial w_2} = 0$$

$$\Rightarrow 0 = -2\sigma_1^2 + 2\sigma_{12} + w_2(2\sigma_1^2 + 2\sigma_2^2 - 4\sigma_{12})$$

$$\Rightarrow \boxed{w_2 = \frac{\sigma_1^2 - \sigma_{12}}{\sigma_1^2 + \sigma_2^2 - 2\sigma_{12}}}$$

$$\Rightarrow \boxed{w_1 = \frac{\sigma_2^2 - \sigma_{12}}{\sigma_1^2 + \sigma_2^2 - 2\sigma_{12}}}$$

These are proportions for each stock for minimum amount of σ (risk).

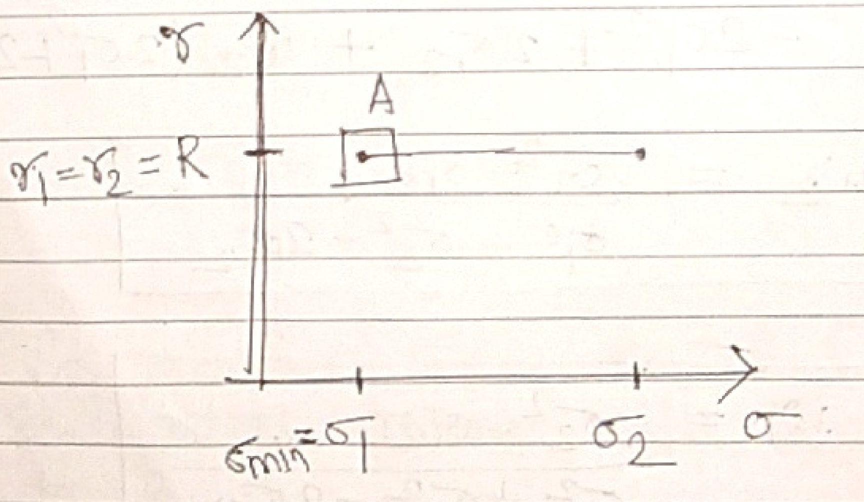
On putting values of w_1 and w_2 in σ,

$$\sigma_{min} = \sqrt{\sigma_1^6 + \sigma_2^6 - 2\sigma_1^4\sigma_{12} - 2\sigma_2^4\sigma_{12} + 2\sigma_1^2\sigma_2^2\sigma_{12} - \sigma_1^2\sigma_{12}^2 - \sigma_2^2\sigma_{12}^2 + 2\sigma_{12}^3}$$

Case (I) : Perfectly positive correlated assets $\rho_{12} = 1$.

$$\Rightarrow \rho_{12} = \sigma_{12}/\sigma_1\sigma_2$$
$$\Rightarrow \sigma_{12} = \sigma_1\sigma_2$$

$$\bar{r} = R$$
$$\sigma = w_1\sigma_1 + w_2\sigma_2$$
$$\Rightarrow \sigma = \sigma_1 + w_2(\sigma_2 - \sigma_1).$$

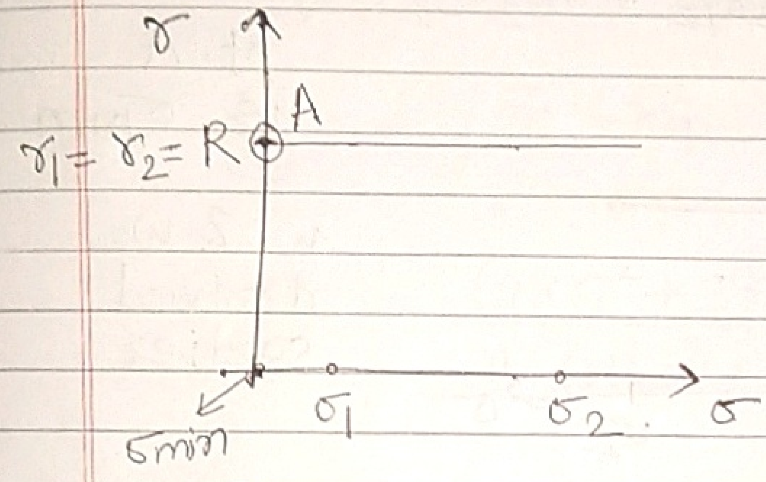


Without loss of generality, $\sigma_1 < \sigma_2$

A is pt. of min risk $(w_2 = 0)$.

Case (II) : Perfectly Negatively Correlated assets $\rho_{12} = -1$

$$\bar{r} = R$$
$$\sigma = |w_1\sigma_1 - w_2\sigma_2|.$$



For pt. A $\sigma = 0$

$$\Rightarrow |w_1\sigma_1 - w_2\sigma_2| = 0$$
$$\Rightarrow w_1\sigma_1 = w_2\sigma_2$$
$$\Rightarrow \boxed{\frac{w_1}{w_2} = \frac{\sigma_2}{\sigma_1}}$$

∴ This the perfect case for diversification.

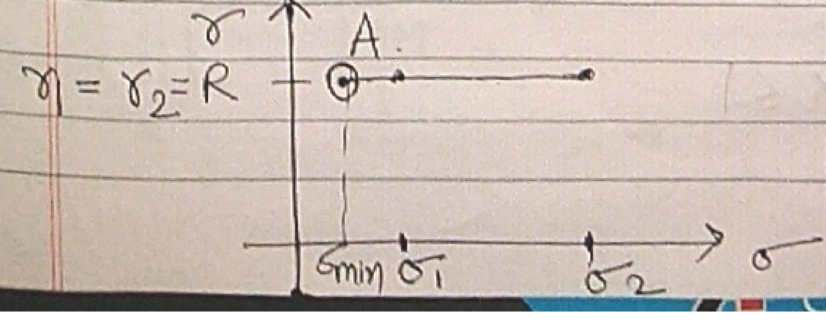
Case (III) $\rho_{12} = 0$; Uncorrelated assets $\Rightarrow \sigma_{12} = 0$.

$$\bar{r} = R$$
$$\sigma = \sqrt{\sigma_1^2 w_1^2 + \sigma_2^2 w_2^2}$$

for min σ ; pt. A.

$$w_2 = \frac{\sigma_1^2 - \sigma_{12}}{\sigma_1^2 + \sigma_2^2 - 2\sigma_{12}}$$ (Derived earlier for general case.

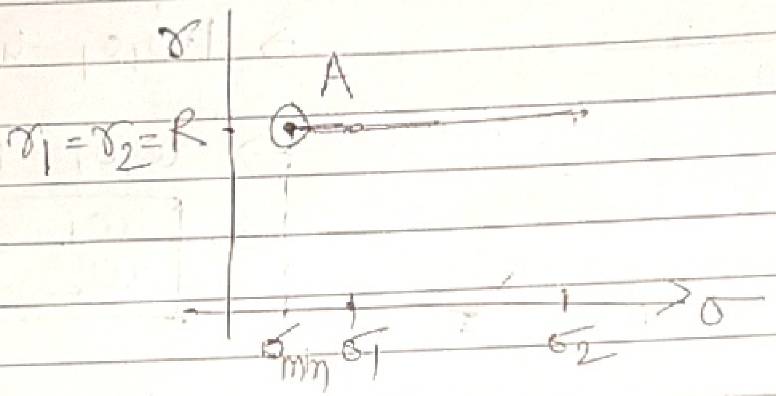
$$\Rightarrow \boxed{w_2 = \frac{\sigma_1^2}{\sigma_1^2 + \sigma_2^2}} \quad \boxed{w_1 = \frac{\sigma_2^2}{\sigma_1^2 + \sigma_2^2}}$$



Case IV : $-1 < \rho_{12} < 1$; ~~Unknown~~
Imperfect correlation.

$$\bar{r} = R$$
$$\sigma = \sqrt{w_1^2\sigma_1^2 + 2w_1w_2\sigma_{12} + w_2^2\sigma_2^2}$$

pt. A is σ_{min}.



w_1 & w_2 derived earlier.

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Q.3) ⇒ CAPM Equation :- Capital Asset Price Model.

If the market portfolio M is efficient then the expected return $\bar{r}_i$ of any asset 'i' satisfies

$$\bar{r}_i - r_f = \beta_i(\bar{r}_M - r_f).$$
$$\text{where } \beta_i = \sigma_{iM}/\sigma_M^2.$$

Proof :- Assume 'α' is invested in asset 'i'
'$1-\alpha$' is invested in market 'M'

$$0 \leq \alpha \leq 1$$

Portfolio's return $(\bar{r}_\alpha) = \alpha \bar{r}_i + (1-\alpha)\bar{r}_M$

Portfolio's risk (SD) $= \sigma_\alpha$

$$= \left(\alpha^2 \sigma_i^2 + 2\alpha(1-\alpha)\sigma_{iM} + (1-\alpha)^2 \sigma_M^2\right)^{1/2}$$

$$\frac{d\bar{r}_\alpha}{d\alpha} = \bar{r}_i - \bar{r}_M$$

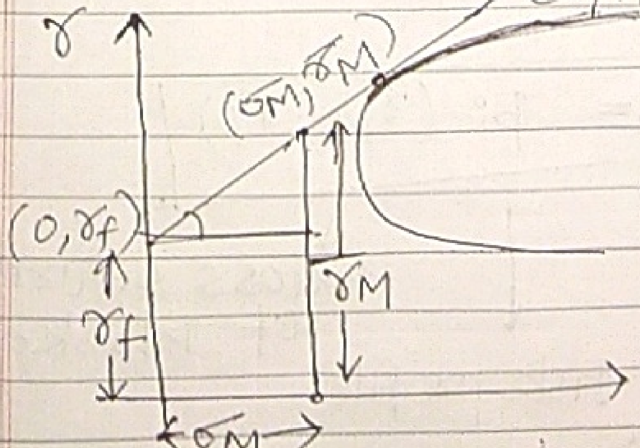
$$\frac{d\sigma_\alpha}{d\alpha} = \frac{\left(2\alpha\sigma_i^2 + 2\sigma_{iM} - 4\alpha\sigma_{iM} + 2\alpha\sigma_M^2 - 2\sigma_M^2\right)}{2\left(\alpha^2\sigma_i^2 + 2\alpha(1-\alpha)\sigma_{iM} + (1-\alpha^2)\sigma_M^2\right)^{1/2}}$$

$$\Rightarrow \left.\frac{d\sigma_\alpha}{d\alpha}\right|_{\alpha=0} = \frac{2\sigma_{iM} - 2\sigma_M^2}{2\sigma_M} = \frac{\sigma_{iM} - \sigma_M^2}{\sigma_M}.$$

$$\frac{d\bar{r}_\alpha}{d\sigma_\alpha} = \frac{d\bar{r}_\alpha/d\alpha}{d\sigma_\alpha/d\alpha}$$

$$\left.\frac{d\bar{r}_\alpha}{d\sigma_\alpha}\right|_{\alpha=0} = \frac{(\bar{r}_i - \bar{r}_M)\sigma_M}{\sigma_{iM} - \sigma_M^2}$$

capital Market line (M)



$\alpha = 0$; All money is in the market

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When both curves touch each other; their Slopes should be equal.

eqn of Capital Market Line;

$$(\bar{r} - r_f) = \left(\frac{\bar{r}_M - r_f}{\sigma_M - 0}\right)(\sigma - 0)$$

$$\Rightarrow \bar{r} = \left(\frac{\bar{r}_M - r_f}{\sigma_M}\right)\sigma + r_f$$

$$\text{Slope} = \frac{\bar{r}_M - r_f}{\sigma_M}$$

$$\Rightarrow \frac{(\bar{r}_i - \bar{r}_M)\,\sigma_M}{\sigma_{iM} - \sigma_M^2} = \frac{\bar{r}_M - r_f}{\sigma_M}$$

$$\Rightarrow \bar{r}_i \sigma_M^2 - \cancel{\bar{r}_M \sigma_M^2} = \bar{r}_M \sigma_{iM} - \cancel{\bar{r}_M \sigma_{iM}^2} - r_f \sigma_{iM} + r_f \sigma_M^2$$

$$\Rightarrow (\bar{r}_i - r_f)\sigma_M^2 = (\bar{r}_M - r_f)\sigma_{iM}$$

$$\Rightarrow \bar{r}_i - r_f = \frac{\sigma_{iM}}{\sigma_M^2}(\bar{r}_M - r_f)$$

$$\Rightarrow \boxed{\bar{r}_i - r_f = \beta_i(\bar{r}_M - r_f)}$$

excess return for 'i' stock

excess return of Market

risk profile of 'i' stock; $\beta_i = \sigma_{iM}/\sigma_M^2$.

$\Rightarrow$ Excess Return of 'i' stock $\propto$ Excess return of Market,

where β_i is the proportionality constant.

★ Idiosyncratic and Market risk

from CAPM,

$$\bar{r}_i = r_f + \beta_i(\bar{r}_M - r_f) + \varepsilon_i \quad \text{—— (1)}$$

Here, ε_i is idiosyncratic/random error.

Assumptions $\left\{ \begin{array}{l} E(\varepsilon_i) = 0 \\ Cov(\varepsilon_i, \bar{r}_M) = 0 \end{array} \right\}$

from (1) we can write

$$E(\bar{r}_i) = E(r_f) + \beta_i E(\bar{r}_M) - \beta_i E(r_f) + 0 \quad \text{—— (2)}$$

$$E[\bar{r}_i - E(\bar{r}_i)]^2 = E[\beta_i\{\bar{r}_M - E(\bar{r}_M)\}]^2 + Var(\varepsilon_i)$$

$$\Rightarrow \sigma_i^2 = \beta_i^2 \sigma_M^2 + Var(\varepsilon_i)$$

$\beta_i^2\sigma_M^2$: Market/Systematic Risk

$Var(\varepsilon_i)$: Idiosyncratic/Non-systematic risk

Market Risk	Idiosyncratic Risk.
The risk that is inherent into the overall market and can not generally resolved by simple diversification of the portfolio.	The risk that is inherent into that specific asset(s) totally uncorrelated with market, can be reduced using diversification.
eg. Company's dependence on the CEO. Apple and Steve Jobs. Valuation relative to price multiples fail for apple when he was absent (due to illness)	eg. The Great Recession of 2008, Impact of Covid-19. (These are examples of market risk).

eg. of Idiosyncratic Risk