MA101

MULTIPLE INTEGRALS

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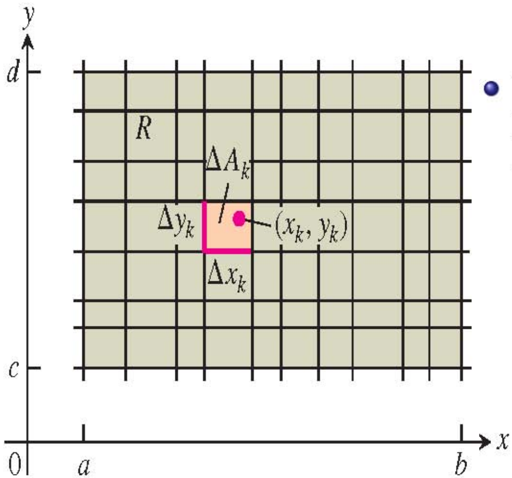
Figure: Rectangular grid partitioning the region R into small rectangles of area $A_k = \Delta x_k \Delta y_k$.



- There are many choices involved in a limit of this kind.

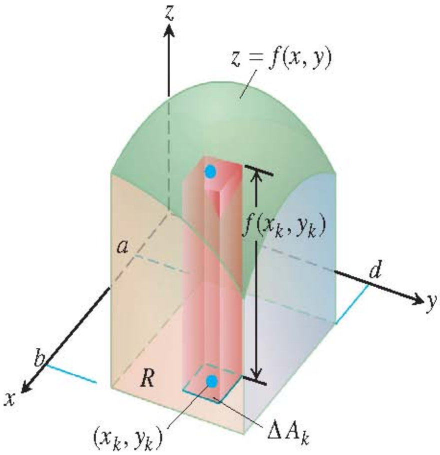
$$\lim_{n\to\infty} \sum_{k=1}^{n} f(x_k, y_k)\Delta A_k$$

- When a limit of the sums s_n exists, giving the same limiting value no matter what choices are made, then the function f is said to be integrable and the limit is called the double integral of f over R, written as

$$\iint_R f(x,y)\,dA \quad \text{or} \quad \iint_R f(x,y)\,dx\,dy.$$

- It can be shown that if $f(x,y)$ is a **continuous function** throughout R, then f is integrable, as in the single-variable case. Many **discontinuous functions** are also integrable, including functions which are discontinuous only on a finite number of points or smooth curves.

When $f(x,y)$ is a positive function over a rectangular region R in the xy−plane, we may interpret the double integral of f over R as the **volume** of the 3-dimensional solid region over the xy−plane bounded below by R and above by the surface $f(x,y)$.



Calculate the volume under the plane $z = 4 - x - y$ over the rectangular region $R : 0 \leq x \leq 2,\ 0 \leq y \leq 1$ in the xy-plane.

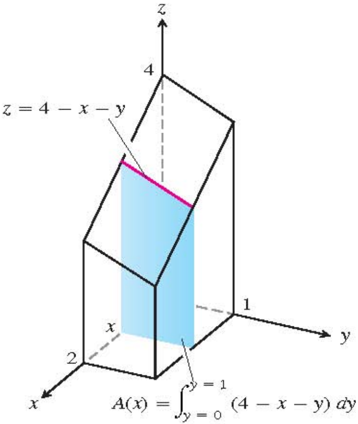


Figure:

$$\mathbf{V} = \int_{x=0}^{x=2} \int_{y=0}^{y=1} (4 - x - y)\,dy\,dx$$

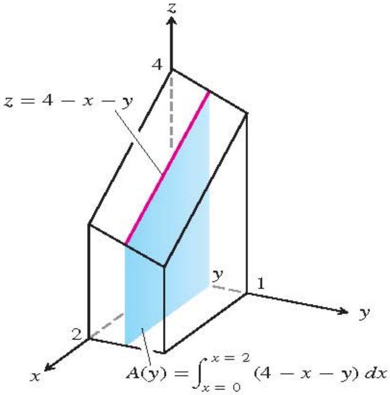


Figure:

$$\mathbf{V} = \int_{y=0}^{y=1} \int_{x=0}^{x=2} (4 - x - y)\,dx\,dy$$

THEOREM 1 Fubini's Theorem

Theorem

If $f(x, y)$ is continuous throughout the rectangular region then

$$\iint\limits_R \mathbf{f(x,y)dA} = \int_c^d \int_a^b \mathbf{f(x,y)dxdy} = \int_a^b \int_c^d \mathbf{f(x,y)dydx}.$$

Fubini's Theorem says that double integrals over rectangles can be calculated as ***iterated integrals***. Thus, we can evaluate a double integral by integrating with respect to one variable at a time.
Note: **Fubini proved his theorem in greater generality, but this is what it says in our setting.**

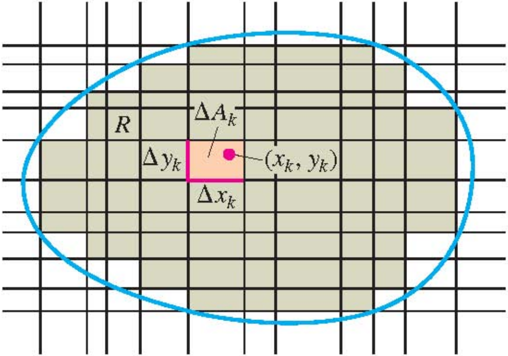
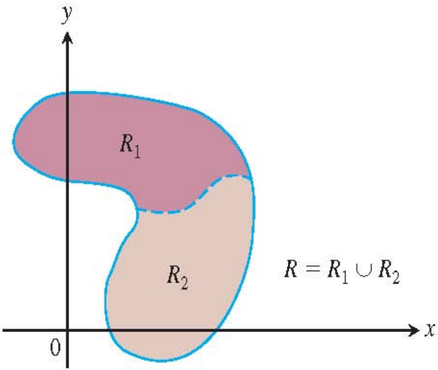


FIGURE 15.6 A rectangular grid partitioning a bounded nonrectangular region into rectangular cells.

Figure: The Additivity Property for rectangular regions holds for regions bounded by continuous curves.



$$\iint_R f(x, y)\, dA = \iint_{R_1} f(x, y)\, dA + \iint_{R_2} f(x, y)\, dA$$

Figure: The area of the vertical slice shown here is $\mathbf{A(x) = \int_{g_1(x)}^{g_2(x)} f(x,y)dy}$. To calculate the volume of the solid, we integrate this area from $x = a$ to $x = b$ and get $\mathbf{\int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)dydx}$.

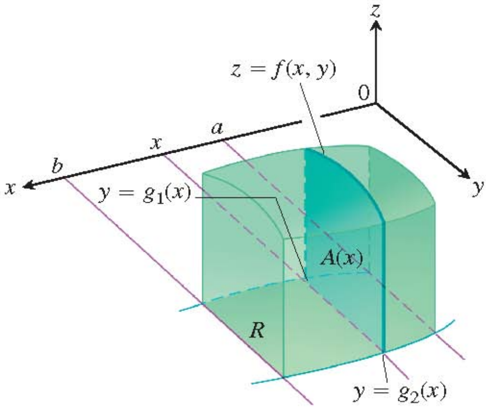
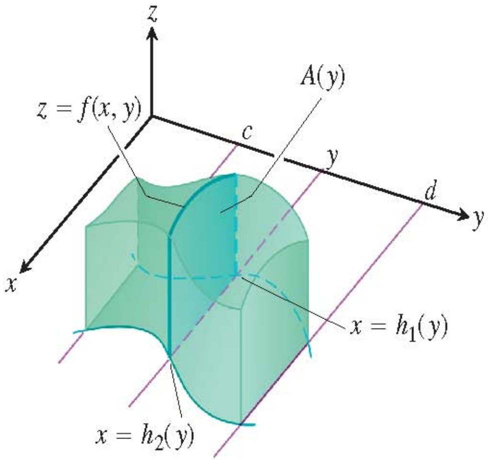


Figure: The volume of the solid shown here is
$\int_c^d A(y)dy = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x,y)dxdy.$



THEOREM 2 Fubini's Theorem Stronger Form

Theorem

Let $f(x, y)$ be continuous on a region R.

1. *If R is defined by $a \leq x \leq b$, $g_1(x) \leq y \leq g_2(x)$ with g_1 and g_2 continuous on $[a, b]$, then*

$$\iint_R \mathbf{f(x, y)dA} = \int_{\mathbf{a}}^{\mathbf{b}} \int_{\mathbf{g_1(x)}}^{\mathbf{g_2(x)}} \mathbf{f(x, y)dydx}.$$

2. *If R is defined by $c \leq y \leq d$, $h_1(y) \leq x \leq h_2(y)$ with h_1 and h_2 continuous on $[c, d]$, then*

$$\iint_R \mathbf{f(x, y)dA} = \int_{\mathbf{c}}^{\mathbf{d}} \int_{\mathbf{h_1(y)}}^{\mathbf{h_2(y)}} \mathbf{f(x, y)dxdy}.$$

Problem

Find the volume of the prism whose base is the triangle in the $xy-$plane bounded by the $x-$axis and the lines $y = x$ and $x = 1$ and whose top lies in the plane $z = f(x, y) = 3 - x - y$.

Important Remark

Note:

Although Fubini's Theorem assures us that a double integral may be calculated as an iterated integral in either order of integration, the value of one integral may be easier to find than the value of the other.

Problem

Evaluate the following double integral in two ways (both iterated integrals)

$$\iint_R \frac{\sin x}{x} dA,$$

where R is the triangle in the xy-plane bounded by the x-axis, the line $y = x$ and the line $x = 1$.

Like single integrals, double integrals of continuous functions have algebraic properties that are useful in computations and applications. If $f(x, y)$ and $g(x, y)$ are continuous, then

1. Constant Multiple:

$$\iint\limits_{\mathbf{R}} \mathbf{cf(x, y)dA} = \mathbf{c} \iint\limits_{\mathbf{R}} \mathbf{f(x, y)dA}. \ (\textit{any number } c)$$

2. Sum and Difference:

$$\iint\limits_{\mathbf{R}} (\mathbf{f(x, y)} \pm \mathbf{g(x, y)})\mathbf{dA} = \iint\limits_{\mathbf{R}} \mathbf{f(x, y)dA} \pm \int\int_{\mathbf{R}} \mathbf{g(x, y)dA}.$$

3. Domination:

(a)

$$\iint_R f(x,y)dA \geq 0 \quad \textit{if} \quad f(x,y) \geq 0$$

on R.

(b)

$$\iint_R f(x,y)dA \geq \iint_R g(x,y)dA$$

if

$$f(x,y) \geq g(x,y)$$

on R.

4. Additivity:

$$\iint_R f(x,y)dA = \iint_{R_1} f(x,y)dA + \iint_{R_2} f(x,y)dA$$

if R is the union of non overlapping regions R_1 and R_2.

Ex. 15.1 Problem 61

What region R in the $xy-$plane maximizes the value of

$$\iint_R (4 - x^2 - 2y^2)\, dA.$$

Give reasons for your answer.

Ex. 15.1 Problem 48

Find the volume of the solid cut from the square column $|x| + |y| \leq 1$ by the planes $z = 0$ and $3x + z = 3$.

Remark

Can we use symmetry of the region $R : \; |x| + |y| \leq 1$?

Area, Moments and Center of Mass

Definition

The area of a closed, bounded plane region R is

$$Area = \iint\limits_{\mathbf{R}} \mathbf{dA}.$$

Finding Area

Find the area of the region R enclosed by the parabola $y = x^2$ and the line $y = x + 2$.

Must to do!

Changing order of Integration

Change the order of the integration of the following integration

$$\int_0^{2a}\int_{\sqrt{2ax-x^2}}^{\sqrt{2ax}} f(x,y)\,dy\,dx,$$

where $f(x,y)$ is defined over the shaded region. What if area of shaded region is asked?

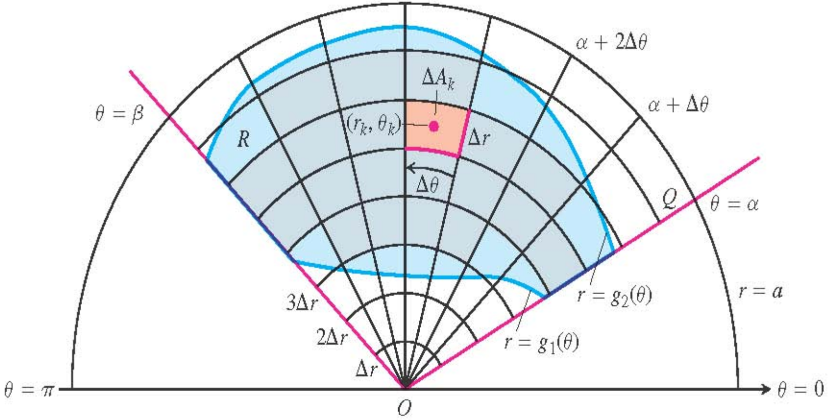
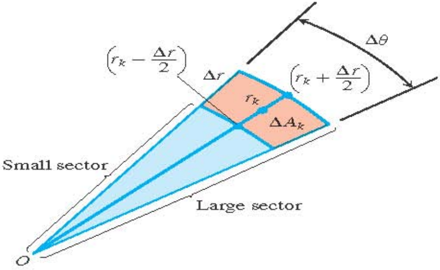


FIGURE 15.21 The region R: $g_1(\theta) \le r \le g_2(\theta)$, $\alpha \le \theta \le \beta$, is contained in the fan-shaped region Q: $0 \le r \le a$, $\alpha \le \theta \le \beta$. The partition of Q by circular arcs and rays induces a partition of R.



A version of Fubini's Theorem says that the limit approached by these sums can be evaluated by repeated single integrations with respect to r and θ as

$$\Delta \mathbf{A_k} = \mathbf{r_k}\Delta\theta\Delta\mathbf{r}.$$

$$S_n = \sum_{k=1}^{n} f(r_k, \theta_k) r_k \Delta r \Delta\theta.$$

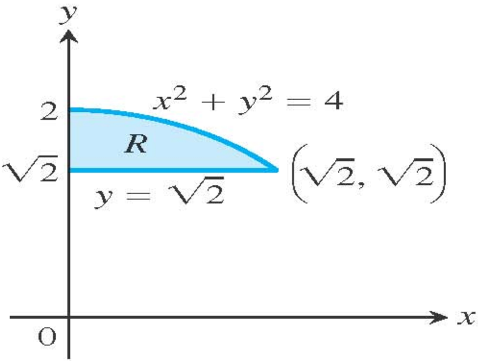
$$\lim_{n\to\infty} S_n = \iint_R f(r,\theta) r \, dr \, d\theta.$$

$$\iint_R f(r,\theta) dA$$

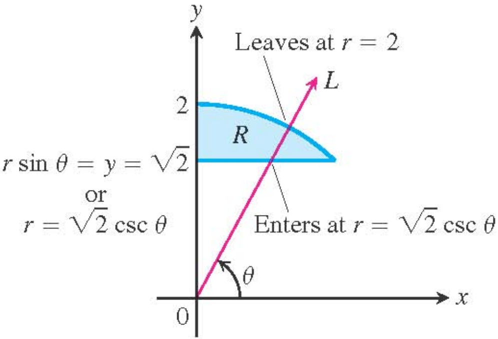
$$= \int_{\theta=\alpha}^{\theta=\beta} \int_{r=g_1(\theta)}^{r=g_2(\theta)} f(r,\theta) r \, dr \, d\theta.$$

To evaluate $\iint\limits_R f(x,y)\,dA$ over a region R in polar coordinates, integrating first with respect to r and then with respect to θ, take the following steps.

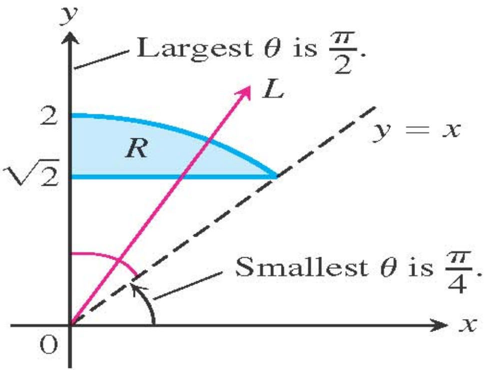
1. Sketch: Sketch the region and label the bounding curves.



2. Find the $r-$limits of integration:



3. Find the $\theta-$limit of integration: Find the smallest and largest θ values that bound R. These are the θ limits of integration.



$$\iint_R f(x,y)\,dA = \int_{\theta=\pi/4}^{\theta=\pi/2} \int_{r=\sqrt{2}\csc\theta}^{r=2} f(r,\theta) r\, dr\, d\theta.$$

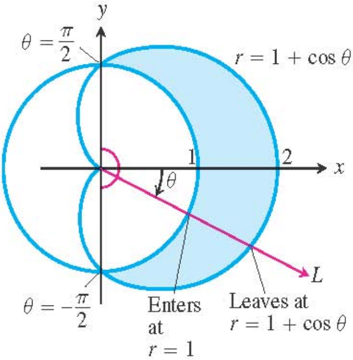
15.3, Problem 37 (a)

Evaluate

$$I = \int_0^\infty e^{-x^2}\,dx.$$

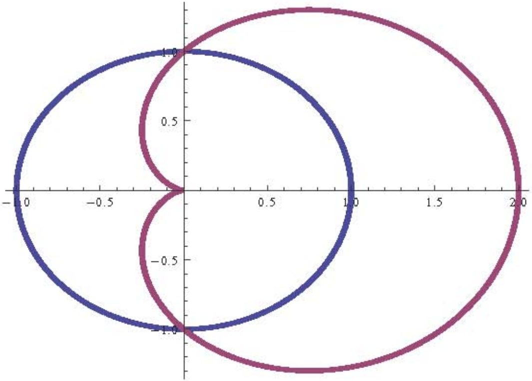
Cardioid overlapping a circle Ex. 15.3 (18)

Find the area of the region that lies inside the cardioid $r = 1 + cos\theta$ and outside the circle $r = 1$.



Volume of **noncircular right cylinder**

The region that lies inside the cardioid $r = 1 + \cos\theta$ and outside the circle $r = 1$ is the base of a solid right cylinder. The top of the cylinder lies in the plane $z = x$. Find the cylinder's volume.



- **What can be computed? :** Triple integrals calculate the volumes of three-dimensional shapes, the masses and moments of solids of varying density, and the average value of a function over a three dimensional region.
- **Origin: ?** Triple integrals arise in the study of vector fields and fluid flow in three dimensions. (Next topic of this course).

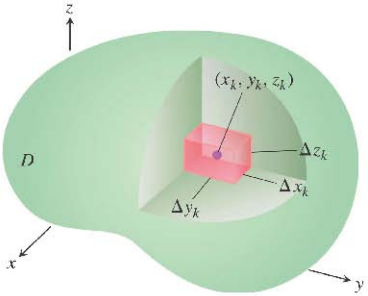


FIGURE 15.27 Partitioning a solid with rectangular cells of volume ΔV_k.

Let $F(x, y, z)$ be a function defined on a closed bounded region D in space. Partition a rectangular boxlike region containing D into rectangular cells by planes parallel to the coordinate axis.

$$S_n = \sum_{k=1}^{n} F(x_k, y_k, z_k)\Delta V_k.$$

When F is continuous and the bounding surface of D is formed from finitely many smooth surfaces joined together along finitely many smooth curves, then F is integrable (Check the Validity of this statement). **The following limit is called the triple integral of F over D**

$$\lim_{n\to\infty} S_n = \iiint_D F(x, y, z)dV.$$

If $F(x, y, z) = 1$ then $\iiint\limits_D 1 \cdot dV$ represents volume of the space occupied by the domain D.

Definition

The volume of a closed, bounded region D in space is

$$\iiint\limits_{\mathbf{D}} \mathbf{dV}.$$

Remark: **This definition is in agreement with our previous definitions of volume (???)**

1. We evaluate a triple integral by applying a **three-dimensional version of Fubini's Theorem** to evaluate it by three repeated single integrations.
2. How many iterated integrals can be framed? **Answer : Six**

a. $\iiint\limits_{D} F(x, y, z)\,dz\,dy\,dx.$

b. $\iiint\limits_{D} F(x, y, z)\,dz\,dx\,dy.$

c. $\iiint\limits_{D} F(x, y, z)\,dy\,dz\,dx.$

d. $\iiint\limits_{D} F(x, y, z)\,dy\,dx\,dz.$

e. $\iiint\limits_{D} F(x, y, z)\,dx\,dy\,dz.$

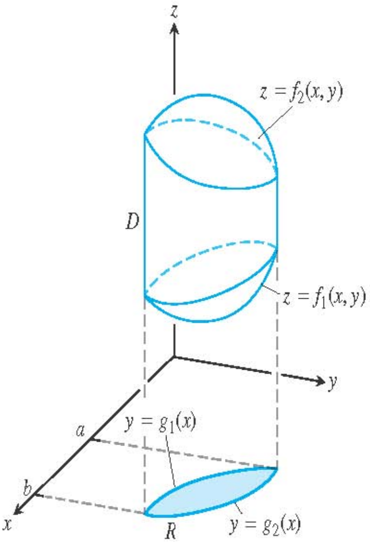
f. $\iiint\limits_{D} F(x, y, z)\,dx\,dz\,dy.$

Step 1: Sketch

Consider

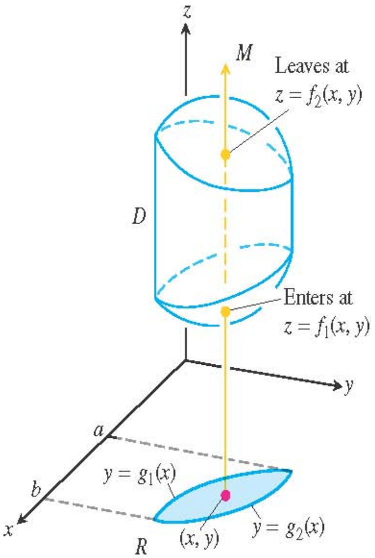
$$\iiint_D \mathbf{F(x, y, z)dzdydx}.$$

Sketch the region D along with its "shadow" R (vertical projection) in the xy−plane. Label the upper and lower bounding surfaces of D and the upper and lower bounding curves of R.



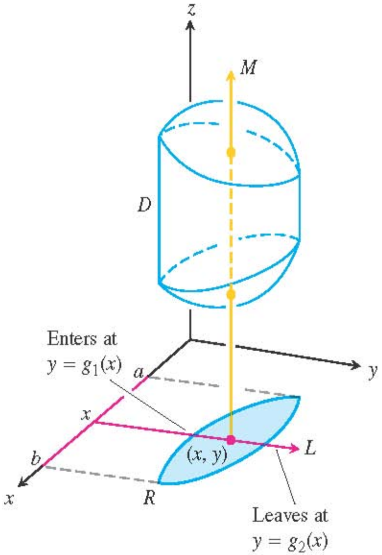
Step 2: Find the z-limits of integration

Draw a line M passing through a typical point (x, y) in R parallel to the z−axis. As z increases, M enters D at $z = f_1(x, y)$ and leaves at $z = f_2(x, y)$. These are the z−limits of integration.



Step 3: Find the y-limits of integration

Draw a line L through (x, y) parallel to the $y-$axis. As y increases, L enters R at $g_1(x)$ and leaves at $g_2(x)$. These are the $y-$limits of integration.

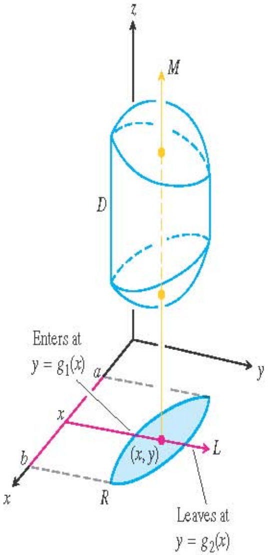


Step 4: Find the x-limits of integration

Choose $x-$limits that include all lines through R parallel to the $y-$axis ($x = a$ and $x = b$ in the adjacent figure). These are the $x-$limits of integration. The integral is

$$\int_{x=a}^{x=b}\int_{y=g_1(x)}^{y=g_2(x)}\int_{z=f_1(x,y)}^{z=f_2(x,y)} F(x,y,z)\,dz\,dy\,dx.$$

Remark: Follow a similar procedures for other combinations of dx, dy, dz. The "shadow" of region D lies in the plane of the last two variables.



If $F(x, y, z)$ and $G(x, y, z)$ are continuous, then

1. Constant Multiple:

$$\iiint\limits_{D} cF(x,y,z)dV = c \iiint\limits_{D} F(x,y,z)dV. \ (\textit{any number } c)$$

2. Sum and Difference:

$$\iiint\limits_{D} (F(x,y,z) \pm G(x,y,z))dV = \iiint\limits_{D} F\,dV \pm \iiint\limits_{D} G\,dV.$$

3. Domination:

(a)
$$\iiint\limits_{D} F(x,y,z)\,dV \geq 0 \quad \textit{if} \quad F(x,y,z) \geq 0$$
on D.

(b)
$$\iiint\limits_{D} F(x,y,z)\,dV \geq \iiint\limits_{D} G(x,y,z)\,dV$$
if
$$F(x,y,z) \geq G(x,y,z)$$
on D.

4. Additivity:
$$\iiint\limits_{D} F(x,y,z)\,dV = \iiint\limits_{D_1} F(x,y,z)\,dV + \iiint\limits_{D_2} F(x,y,z)\,dV$$
if D is the union of non overlapping regions D_1 and D_2.

1. Volume of a Cuboid[1] bounded by planes $x = \pm a$, $y = \pm b$, $z = \pm c$.
2. Volume of shape formed by planes $x = 0$, $y = 0$, $z = 0$ and $ax + by + cz = d$.
3. Volume of Ellipsoid

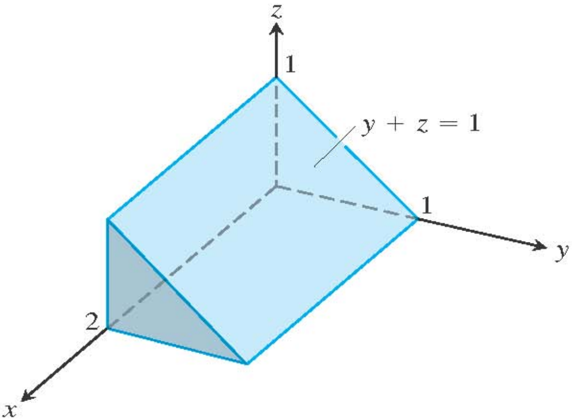
$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1.$$

4. Volume of Right Circular Cylinder whose base is $x^2 + y^2 = r^2$ and bounded above and below by planes $z = \pm h$.
5. Volume bounded above by the sphere $x^2 + y^2 + z^2 = 25$ and bounded below by the plane $z = 4$.

[1] Read exact definitions of Cylinder, Prism, Pyramid, Cone, Ellipsoid, Hyperboloid, Paraboloid, etc.

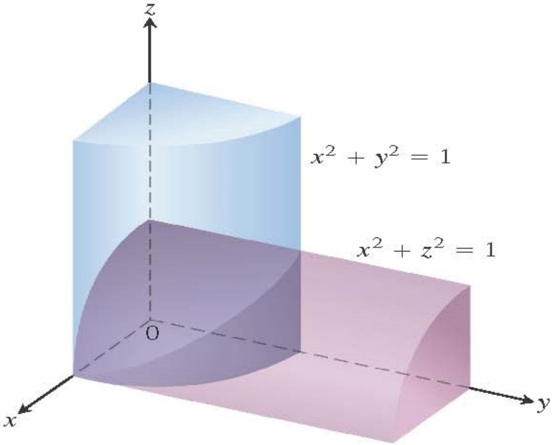
Example 4

Find the volume the region shown in the figure given below by evaluating $\iiint\limits_D dV$ in six ways.



Example 4

The region common to the interiors of the cylinders $x^2 + y^2 = 1$ and $x^2 + z^2 = 1$ one-eighth of which is shown in the following figure.



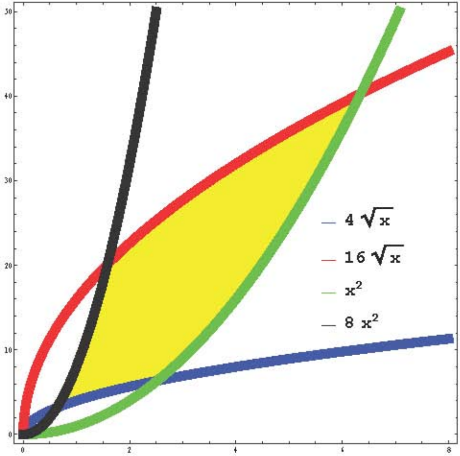
Suggestions

Suggestions

Solve Exercise 15.4 Problem 21 to 36. (Try to do in by all six iterated integrals).

Motivation

Find the area bounded between four parabolas $y^2 = 16x$, $y^2 = 256x$, $y = x^2$, $y = 8x^2$.



Substitutions known to us

- Cartesian to Polar

$$\iint_R f(x,y)\,dx\,dy = \iint_R f(r\cos\theta, r\sin\theta)\, r\,dr\,d\theta.$$

- Cartesian to Cylindrical

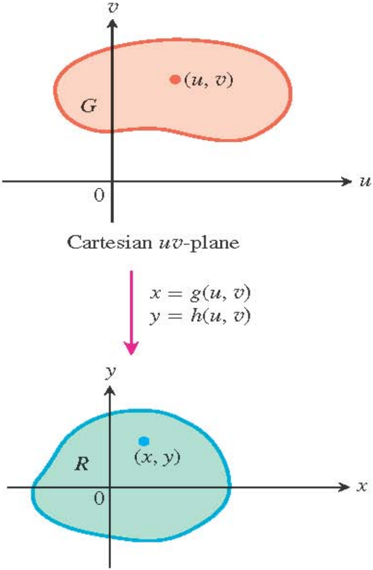
$$\iiint_D F(x,y,z)\,dx\,dy\,dz = \iiint_D f(r\cos\theta, r\sin\theta, z)\, r\,dz\,dr\,d\theta.$$

- Cartesian to Spherical

$$\iiint_D F(x,y,z)\,dx\,dy\,dz$$

$$= \iiint_D f(\rho\sin\phi\cos\theta, \rho\sin\phi\sin\theta, \rho\cos\phi)\, \rho^2\sin\phi\,d\rho\,d\phi\,d\theta.$$

Substitutions in Double Integrals



Suppose that a region G in the uv−plane is transformed one-to-one into the region R in the xy−plane by equations of the form

$$\mathbf{x} = \mathbf{g}(\mathbf{u}, \mathbf{v}), \quad \mathbf{y} = \mathbf{h}(\mathbf{u}, \mathbf{v}).$$

R the image of G under the transformation, and G the preimage of R. $\mathbf{f}(\mathbf{x}, \mathbf{y})$ defined on R can be thought of as a function

$$\mathbf{f}(\mathbf{g}(\mathbf{u}, \mathbf{v}), \mathbf{h}(\mathbf{u}, \mathbf{v}))$$

over G.

1. How is the integral of $\mathbf{f}(\mathbf{x}, \mathbf{y})$ over $\mathbf{R}$ related to the integral of $\mathbf{f}(\mathbf{g}(\mathbf{u}, \mathbf{v}), \mathbf{h}(\mathbf{u}, \mathbf{v}))$ over $\mathbf{G}$?
2. The answer is: If $\mathbf{g}$, $\mathbf{h}$, have continuous partial derivatives and $\mathbf{J}(\mathbf{u}, \mathbf{v})$ is zero only at isolated points, if at all, then

$$\iint_{\mathbf{R}} \mathbf{f}(\mathbf{x}, \mathbf{y})\, \mathbf{dx}\, \mathbf{dy} = \iint_{\mathbf{G}} \mathbf{f}(\mathbf{g}(\mathbf{u}, \mathbf{v}), \mathbf{h}(\mathbf{u}, \mathbf{v}))\, |\mathbf{J}(\mathbf{u}, \mathbf{v})|\, \mathbf{du}\, \mathbf{dv}.$$

3. The factor $\mathbf{J}(\mathbf{u}, \mathbf{v})$, whose absolute value appears in the above Equation, is the Jacobian of the coordinate transformation, named after German mathematician **Carl Jacobi**. It measures how much the transformation is expanding or contracting the area around a point in $\mathbf{G}$ as $\mathbf{G}$ is transformed into $\mathbf{R}$.

$$\mathbf{J}(\mathbf{u}, \mathbf{v}) = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial y}{\partial u}\frac{\partial x}{\partial v} = \frac{\partial(\mathbf{x}, \mathbf{y})}{\partial(\mathbf{u}, \mathbf{v})}.$$

Another perspective

1. For Cartesian to Polar instead of considering Polar system on the same plane with r as radial distance and θ angle, let us consider $r-\theta$ as mutually perpendicular axes on a different plane and try to understand these substitutions. Take example of a Circle $r=1$.

$$\iint_R f(x,y)\,dx\,dy = \iint_R f(r\cos\theta, r\sin\theta)\,|J(r,\theta)|\,dr\,d\theta.$$

$$J(r,\theta) = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = r$$

Exercise 15.7: Problem 6

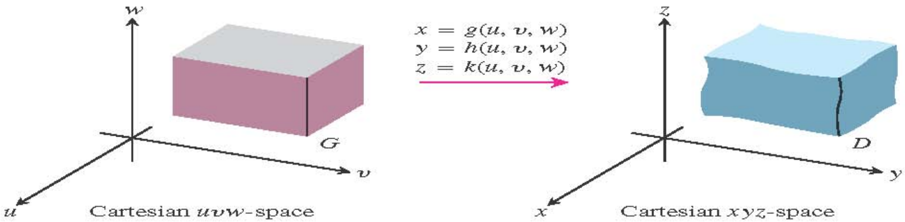
Evaluate the integral

$$\iint_R (2x^2 - xy - y^2)\, dx\, dy.$$

for the region R in the first quadrant bounded by the lines $y = -2x + 4$, $y = -2x + 7$, $y = x - 2$ and $y = x + 1$. [Explain meaning of $dxdy$ & $|J|dudv$, by drawing them on the same plane]

Four Parabolas

Find the area bounded between four parabolas $y^2 = 16x$, $y^2 = 256x$, $y = x^2$, $y = 8x^2$. [Explain meaning of **dxdy** & $|$**J**$|$**dudv**, by drawing them on the same plane]



Suppose that a region G in the uvw−space is transformed one-to-one into the region D in the xyz−space by equations of the form

$$\mathbf{x} = \mathbf{g}(\mathbf{u}, \mathbf{v}, \mathbf{w}), \quad \mathbf{y} = \mathbf{h}(\mathbf{u}, \mathbf{v}, \mathbf{w}), \quad \mathbf{z} = \mathbf{k}(\mathbf{u}, \mathbf{v}, \mathbf{w}).$$

D the image of G under the transformation, and G the preimage of D. F(x, y, z) defined on D can be thought of as a function

$$\mathbf{f}(\mathbf{g}(\mathbf{u}, \mathbf{v}, \mathbf{w}), \mathbf{h}(\mathbf{u}, \mathbf{v}, \mathbf{w}), \mathbf{k}(\mathbf{u}, \mathbf{v}, \mathbf{w})) = \mathbf{H}(\mathbf{u}, \mathbf{v}, \mathbf{w})$$

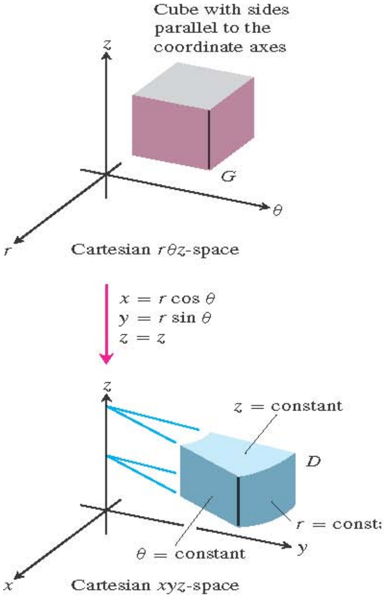
defined on G.

1. How is the integral of $\mathbf{F}(\mathbf{x}, \mathbf{y}, \mathbf{z})$ over $\mathbf{D}$ related to the integral of $\mathbf{H}(\mathbf{u}, \mathbf{v}, \mathbf{w})$ over $\mathbf{G}$?
2. The answer is: If $\mathbf{g}$, $\mathbf{h}$, $\mathbf{k}$ have continuous partial derivatives and $\mathbf{J}(\mathbf{u}, \mathbf{v}, \mathbf{w})$ is zero only at isolated points, if at all, then

$$\iiint_{\mathbf{D}} \mathbf{F}(\mathbf{x}, \mathbf{y}, \mathbf{z})\, \mathbf{dx}\, \mathbf{dy}\, \mathbf{dz} = \iiint_{\mathbf{G}} \mathbf{H}(\mathbf{u}, \mathbf{v}, \mathbf{w})\, |\mathbf{J}(\mathbf{u}, \mathbf{v}, \mathbf{w})|\, \mathbf{du}\, \mathbf{dv}\, \mathbf{dw}.$$

3. The factor $\mathbf{J}(\mathbf{u}, \mathbf{v}, \mathbf{w})$, whose absolute value appears in the above Equation, is the Jacobian of the coordinate transformation. It measures how much the transformation is expanding or contracting the volume around a point in $\mathbf{G}$ as $\mathbf{G}$ is transformed into $\mathbf{D}$.

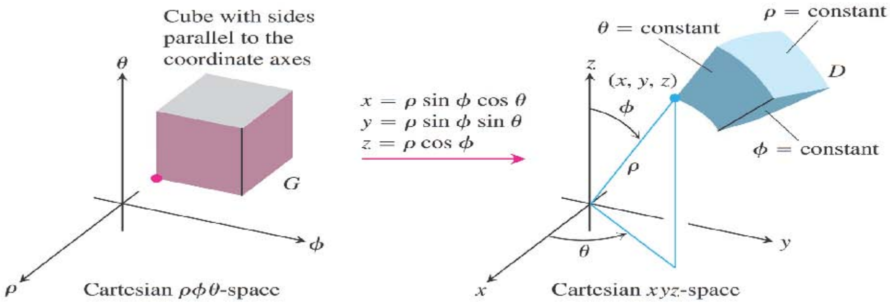
$$\mathbf{J}(\mathbf{u}, \mathbf{v}, \mathbf{v}) = \begin{vmatrix} \frac{\partial \mathbf{x}}{\partial \mathbf{u}} & \frac{\partial \mathbf{x}}{\partial \mathbf{v}} & \frac{\partial \mathbf{x}}{\partial \mathbf{w}} \\ \frac{\partial \mathbf{y}}{\partial \mathbf{u}} & \frac{\partial \mathbf{y}}{\partial \mathbf{v}} & \frac{\partial \mathbf{y}}{\partial \mathbf{w}} \\ \frac{\partial \mathbf{z}}{\partial \mathbf{u}} & \frac{\partial \mathbf{z}}{\partial \mathbf{v}} & \frac{\partial \mathbf{z}}{\partial \mathbf{w}} \end{vmatrix} = \frac{\partial(\mathbf{x}, \mathbf{y}, \mathbf{z})}{\partial(\mathbf{u}, \mathbf{v}, \mathbf{w})}.$$



$$\iiint_D \mathbf{F(x, y, z)\, dx\, dy\, dz}$$

$$= \iiint_G \mathbf{H(r, \theta, z)}\, |\mathbf{J(r, \theta, z)}|\, \mathbf{dr\, d\theta\, dz}.$$

$\mathbf{J(r, \theta, z)} = \mathbf{r}$. Verify



$$\iiint\limits_{D} \mathbf{F(x,y,z)\,dx\,dy\,dz} = \iiint\limits_{G} \mathbf{H}(\mathbf{r},\phi,\theta)\,|\mathbf{J}(\rho,\phi,\theta)|\,\mathbf{d}\rho\,\mathbf{d}\phi\,\mathbf{d}\theta.$$

$$\mathbf{J}(\rho,\phi,\theta) = \rho^{2}\sin\phi. \text{ Verify}$$

Volume of Ellipsoid

Find the volume of ellipsoid

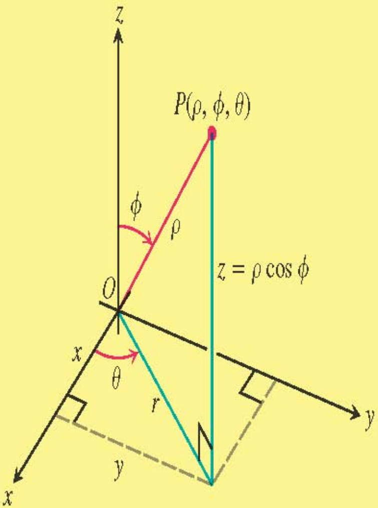
$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1.$$

Home Work

Volume of a Cylinder in Spherical System

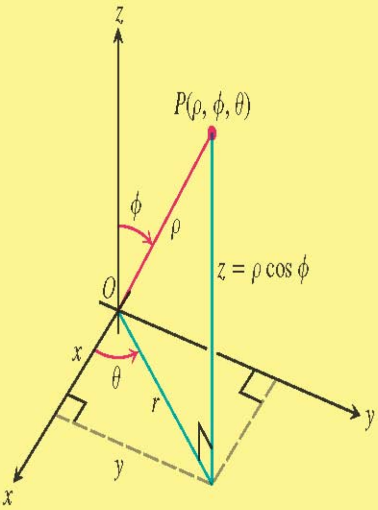
Find the volume of the region bounded between $\mathbf{z} = \mathbf{2} - \sqrt{\mathbf{x}^2 + \mathbf{y}^2}$ and $\mathbf{z} = \sqrt{\mathbf{x}^2 + \mathbf{y}^2}$ which lies outside the right circular cylinder $\mathbf{x}^2 + \mathbf{y}^2 = \mathbf{1}/\mathbf{2}$. First try Cartesian system and then try cylindrical system.

1. Spherical coordinates represent a point P in space by ordered triples (ρ, ϕ, θ) in which
2. ρ is the distance of P from origin and is always positive.
3. ϕ is the angle $\overrightarrow{OP}$ makes with the positive $z-$axis $(0 \leq \phi \leq \pi)$.
4. θ is the angle from cylindrical coordinates.

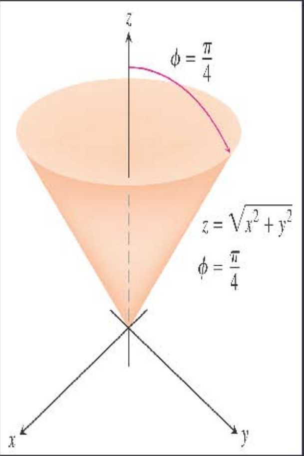


Equations Relating Spherical Coordinates to Cartesian and Cylindrical Coordinates

1. $\mathbf{r} = \rho \sin\phi,$
2. $\mathbf{x} = \mathbf{r}\cos\theta = \rho\sin\phi\cos\theta,$
3. $\mathbf{z} = \rho\cos\phi,$
4. $\mathbf{y} = \mathbf{r}\sin\theta = \rho\sin\phi\sin\theta,$
5. $$\rho = \sqrt{\mathbf{x}^2 + \mathbf{y}^2 + \mathbf{z}^2} = \sqrt{\mathbf{r}^2 + \mathbf{z}^2}.$$

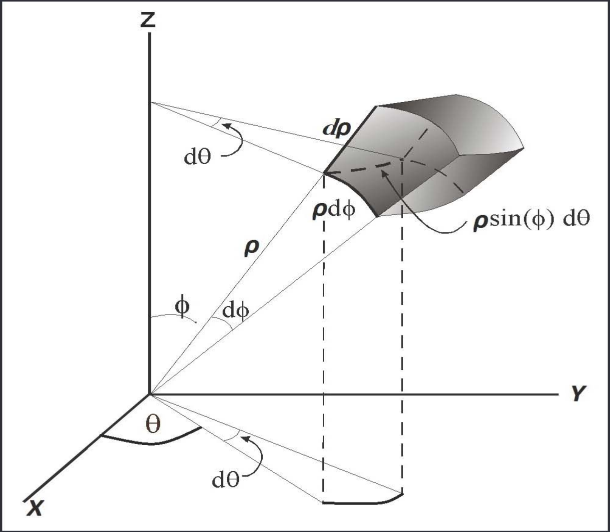


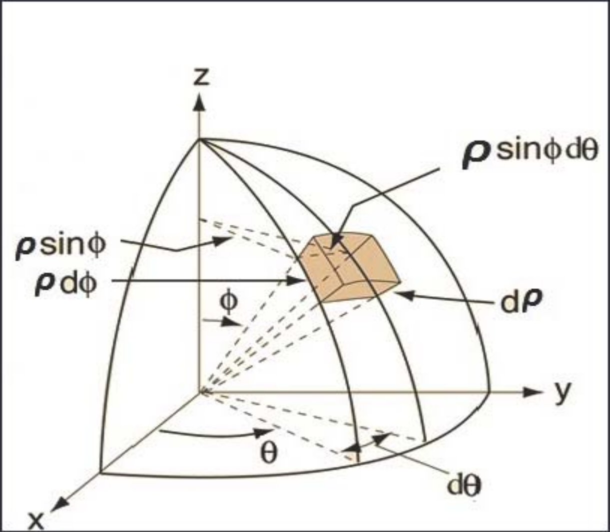
Converting Cartesian to Spherical

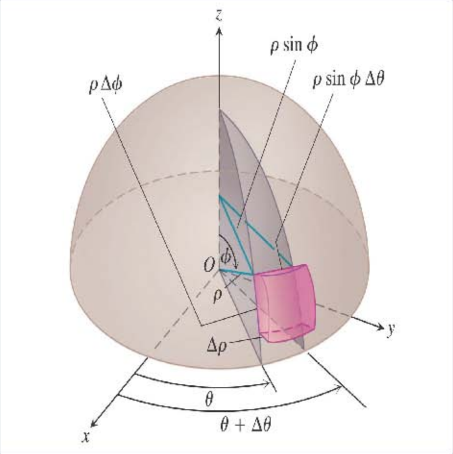


Find a spherical coordinate equation for the cone $z = \sqrt{x^2 + y^2}$.
Solution:
$\rho \cos \phi = \rho \sin \phi$.
So, equation of the cone $z = \sqrt{x^2 + y^2}$ in spherical system is

$$\phi = \frac{\pi}{4}.$$

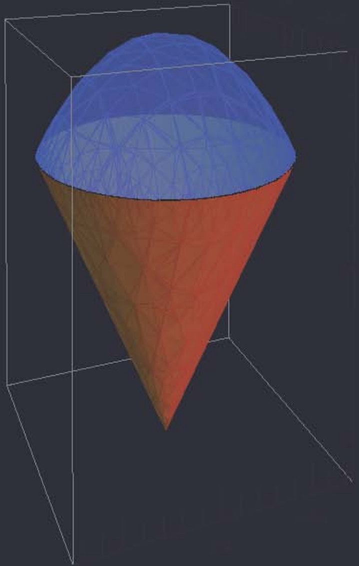

Z
dθ
dρ
ρdϕ
ρsin(ϕ) dθ
ρ
ϕ
dϕ
Y
θ
dθ
X


z
ρ sinϕ dθ
ρ sinϕ
ρ dϕ
ϕ
dρ
y
θ
dθ
x



$$dV = d\rho \cdot \rho\, d\phi \cdot \rho \sin\phi\, d\theta$$

$$dV = \rho^2 \sin\phi\, d\rho\, d\phi\, d\theta.$$

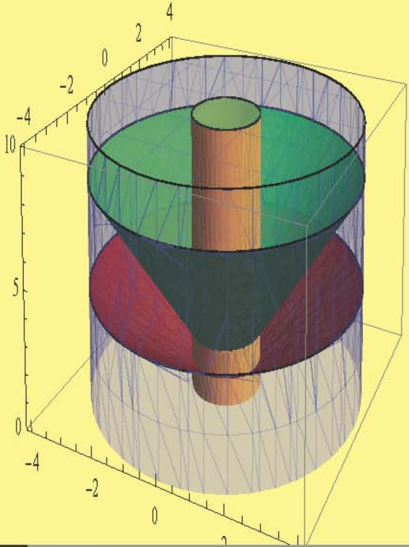


Find the volume of the "ice cream cone" D cut from the solid sphere $\rho \leq 1$ by the cone $\phi = \frac{\pi}{3}$.

Home Work

Find the volume of region bounded below and above by cones $z = \sqrt{x^2 + y^2}$, $z = 2\sqrt{x^2 + y^2}$ and laterally by cylinders $x^2 + y^2 = 1$ and $x^2 + y^2 = 16$.

$$V = \int_{\theta=0}^{\theta=2\pi} \int_{r=1}^{r=4} \int_{z=r}^{2r} dz\, r dr\, d\theta.$$



Problem

Find the volume of the cylinder $x^2 + y^2 = 16$ which lies out side cone $z = \sqrt{x^2 + y^2}$ and is above $z = 2$ plane. Sketch the required region in 3D and its projection in 2D.

Problem

Find the volume of the cone $z = \sqrt{x^2 + y^2}$ which lies outside the cylinder $x^2 + y^2 = 16$ and is below the plane $z = 6$. Sketch the required region in 3D and its projection in 2D.

Exercise 21

Set up the iterated integral for evaluating

$$\int_{-1}^{1}\int_{x^2}^{1}\int_{0}^{1-y} dz\, dy\, dx$$

over the given region D for remaining all different different orders of dx, dy & dz.

Cylindrical coordinates represent a point **P** in space by ordered triples (r, θ, z) in which
1. r and θ are polar coordinates for the vertical projection of P on the xy-plane.
2. z is the rectangular vertical coordinate.

Equations Relating Rectangular (x, y, z) and Cylindrical Coordinates (r, θ, z)

$$x = r\cos\theta, y = r\sin\theta, z = z$$

$$r^2 = x^2 + y^2, \tan\theta = \frac{y}{x}$$

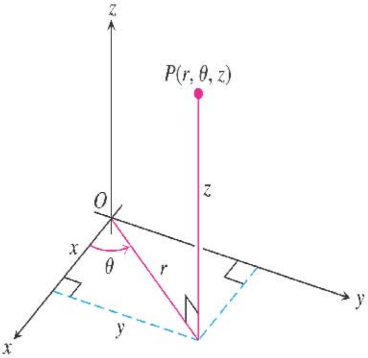


FIGURE 15.36 The cylindrical coordinates of a point in space are r, θ, and z.

Elementary Volume

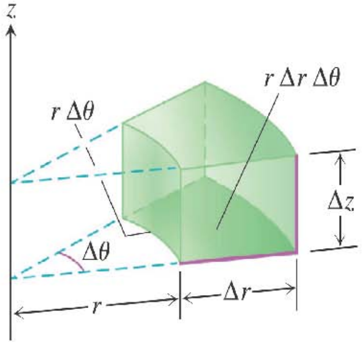


FIGURE 15.38 In cylindrical coordinates the volume of the wedge is approximated by the product $\Delta V = \Delta z\, r\,\Delta r\,\Delta\theta$.