HS-801 Financial Economics (ENDSEM).

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Q1) ⇒ Given, European Call Option

Stock Price (S_0) = 49 \$
Strike Price (k) = 50 \$
risk free rate (r) = 5% = 0.05
volatility (σ) = 20% = 0.2
time to maturity (τ) = 20 weeks = $\frac{20 \times 7}{365}$ yrs.

we know, Call Option Price (C)

$$= S_0 N(d_1) - k e^{-r\tau} N(d_2)$$

From BSH equation, call option price

where, N is commulative normal distribution

$$d_1 = \frac{\ln(S_0/K) + (r + \sigma^2/2)\tau}{\sigma\sqrt{\tau}}$$

$$d_2 = d_1 - \sigma\sqrt{\tau}$$

$\tau = T - t$ (T: strike date, t: current date)

⇒ Theta(θ) = $\frac{-dC}{d\tau}$ = rate of change of option price wrt. passage of time

$= \frac{dC}{dt}$

$$= \frac{-d}{d\tau}\left(S_0 N(d_1) - k e^{-r\tau} N(d_2)\right)$$

$$= -S_0 \cdot \frac{\partial N(d_1)}{\partial \tau} + (-r) \cdot k \cdot e^{-r\tau} N(d_2) + k e^{-r\tau} \frac{\partial N(d_2)}{\partial \tau}$$

$$= -S_0 \frac{\partial N(d_1)}{\partial d_1} \cdot \frac{\partial d_1}{\partial T} - rke^{-rT} \cdot N(d_2) + ke^{-rT} \frac{\partial N(d_2)}{\partial d_2} \cdot \frac{\partial d_2}{\partial T}$$

$$= -\frac{S_0}{\sqrt{2\pi}} e^{-d_1^2/2} \left(\frac{-\ln(S_0/K)}{2\sigma T^{3/2}} + \frac{r+\sigma^2/2}{2\sigma\sqrt{T}} \right) - rke^{-rT} N(d_2)$$

$$+ \cancel{Ke^{-rT}} \left(\frac{1}{\sqrt{2\pi}} \cdot e^{-d_1^2/2} \frac{S_0}{\cancel{K}} \cancel{e^{rT}} \right) \cdot \left(\frac{-\ln(S_0/K)}{2\sigma T^{3/2}} + \frac{r-\sigma^2/2}{2\sigma\sqrt{T}} \right)$$

$$\left(\because d_2 = d_1 - \sigma\sqrt{T} \text{ and } \frac{\partial N(d_2)}{\partial d_2} = \frac{1}{\sqrt{2\pi}} e^{-d_2^2/2} \right)$$

$$= \frac{-S_0}{\sqrt{2\pi}} e^{-d_1^2/2} \left(\frac{-\ln(S_0/K)}{2\sigma T^{3/2}} + \frac{r+\sigma^2/2}{2\sigma\sqrt{T}} \right) - rke^{-rT} N(d_2)$$

$$+ \frac{S_0 e^{-d_1^2/2}}{\sqrt{2\pi}} \left(\frac{-\ln(S_0/K)}{2\sigma T^{3/2}} + \frac{r-\sigma^2/2}{2\sigma\sqrt{T}} \right)$$

$$= \frac{-S_0}{\sqrt{2\pi}} e^{-d_1^2/2} \left(\cancel{\frac{-\ln(S_0/K)}{2\sigma T^{3/2}}} + \frac{\cancel{r}+\sigma^2/2}{2\sigma\sqrt{T}} + \cancel{\frac{\ln(S_0/K)}{2\sigma T^{3/2}}} - \frac{(\cancel{r}-\sigma^2/2)}{2\sigma\sqrt{T}} \right)$$

$$- rke^{-rT} N(d_2)$$

$$= \frac{-S_0}{\sqrt{2\pi}} e^{-d_1^2/2} \left(\frac{2\sigma^2/\cancel{2}}{2\sigma\sqrt{T}} \right) - rke^{-rT} N(d_2)$$

$$\boxed{\text{Theta}(\theta) = \frac{-S_0 \sigma N'(d_1)}{2\sqrt{T}} - rke^{-rT} N(d_2)}$$

Option Price (C) = $S_0 N(d_1) - Ke^{-r\tau} N(d_2) \approx 2.396$ \$

$$\text{Theta}(\theta) = \frac{-(49)(0.2)N'(d_1)}{2\sqrt{\frac{140}{365}}} - (0.05)(50)e^{-(0.05)\left(\frac{140}{365}\right)} N(d_2)$$

$$d_1 = \frac{\ln\left(\frac{S_0}{K}\right) + \left(r + \sigma^2/2\right)\tau}{\sigma\sqrt{\tau}} = \frac{\ln\left(\frac{49}{50}\right) + \left(0.05 + \frac{(0.2)^2}{2}\right)\frac{140}{365}}{0.2 \times \sqrt{\frac{140}{365}}}$$

$$\Rightarrow \boxed{d_1 = 0.05366}$$

$$d_2 = d_1 - \sigma\sqrt{T} = (0.05366) - (0.2)\sqrt{\frac{140}{365}}$$

$$= 0.05366 - 0.1239$$

$$\Rightarrow \boxed{d_2 = -0.07024}$$

$$N'(d_1) = \frac{1}{\sqrt{2\pi}} e^{-(0.05366)^2/2} = \underline{0.3984}$$

$N(d_2) = N(-0.07024) = \underline{0.4720}$ (from CDF Table)

$$\text{Theta}(\theta) = \frac{-(49)(0.2)(0.3984)}{2\sqrt{140/365}} - (0.05)(50)e^{-(0.05)(140/365)}(0.4720)$$

$$= -3.15208 - 1.15758$$
$$= -4.30966$$

(only valid for short time) $\therefore$ $\boxed{\text{Theta}(\theta) \cong -4.31}$ \$/yr.

So, by amount 4.31 option's value will decrease every ~~day~~ year till maturity.

As this rate is dependent on stock price and other volatile factors. It is not necessarily fixed for a year

$\Rightarrow$ Gamma(γ) $= \frac{\partial^2 \pi}{\partial S^2} \doteq$ double derivative of option price with respect to stock price.

i.e. $\frac{\partial^2 C}{\partial S^2}$ C is call option price

$= \frac{\partial}{\partial S}\left(\frac{\partial C}{\partial S}\right)$

$C = S_0 N(d_1) - ke^{-rT} N(d_2)$

$$\boxed{\frac{\partial C}{\partial S} = N(d_1)}$$

$= \frac{\partial}{\partial S} N(d_1)$

$= \frac{\partial N(d_1)}{\partial d_1} \cdot \frac{\partial d_1}{\partial S}$

$= N'(d_1) \cdot \frac{\partial}{\partial S}\left(\frac{\ln(S_0/K) + (r + \sigma^2/2)T}{\sigma\sqrt{T}}\right)$

$= N'(d_1) \cdot \frac{1}{\sigma\sqrt{T}} \cdot \left(\frac{1}{S_0}\right)$

$$\Rightarrow \boxed{\gamma = \frac{N'(d_1)}{\sigma S_0 \sqrt{T}}}$$

$$\text{Gamma}(\gamma) = \frac{0.3984}{(0.2)(49)\sqrt{140/365}} = 0.06564.$$

$$\boxed{\text{Gamma}(\gamma) = 0.06564}$$

γ tells the variation/volatility of option price wrt. assets price (here, stock price).

Q.2) ⇒ Put-Call Parity

Relationship between the price of European Put Option and European Call option of the same class means which have same underly-ing asset price, strike price and expiration date is given by the principle of Put-Call Parity.

If the prices of the put and call options diverge so that this relationship does not hold, an arbitrage opportunity exists, meaning sophisticated traders earn a risk free profit. Such opportunities are rare and short-lived in liquid markets.

According to BSM equation;

Call Option Price (C) $= S_0 N(d_1) - ke^{-rT} N(d_2)$ —①

Put Option Price (P) $= ke^{-rT} N(-d_2) - S_0 N(-d_1)$ —②

$$d_1 = \frac{\ln(S_0/k) + (r + \sigma^2/2)T}{\sigma\sqrt{T}}$$

$$d_2 = d_1 - \sigma\sqrt{T}$$

where; r is risk free interest rate

T is time to maturity

k is strike price

S_0 is stock price (asset's price)

σ is volatility

$N(x)$ represents cummulative normal distribution for x.

Put-Call Parity equation

$$\boxed{C + ke^{-rT} = S_0 + P} \quad \text{--- (3)}$$

$$\text{L.H.S.} = C + ke^{-rT}$$

$$= \{S_0 N(d_1) - ke^{-rT} N(d_2)\} + ke^{-rT}$$

(from ①)

$$= \{S_0(1 - N(-d_1)) - ke^{-rT}(\cancel{1} - N(-d_2))\} + \cancel{ke^{-rT}}$$

$$= \{ke^{-rT} N(-d_2) - S_0 N(-d_1)\} + S_0$$

$$= P + S_0 \quad \text{(from ②)}$$

$$= \text{R.H.S.}$$

Hence, the Put-Call Parity equation is established.

Q.3) ⇒ If S (Stock Price) follows a geometric Brow-nian motion represented by $dS = \mu S dt + \sigma S dz$, then using Itô's lemma derive the process followed by $\ln(S)$. Notations have standard meanings.

Soln :-

Itô's Process :- It is a type of stochastic process described by Kiyoshi Ito, a Japanese mathematician, which can be written as the sum of the integral of a process over time and of another process over a Brownian Motion.

This is generalized form for weiner process where 'a' and 'b' parameters are not constant but they are dependent on time and underlying asets value.

i.e.; $dx = a(x,t)dt + b(x,t)dz$

$$\Delta x = a(x,t)\, dt + b(x,t)\, \varepsilon \sqrt{\Delta t}$$

$a(x,t) \Rightarrow$ drift term

$b(x,t) \Rightarrow$ variance term

Itô's Lemma

Stock price behaves stochaistically and its changes depends on time. Hence an option price depends on time and the underlying stock. Behaviour of stochaistic process for an option was prepared by Ito

Itô's lemma says that a function is dependent on x, t variables where ~~h~~ follows itô's process, then h satisfies following PDE

$$dh = \left[\frac{\partial h}{\partial x} \cdot a + \frac{\partial h}{\partial t} + \frac{1}{2} \frac{\partial^2 h}{\partial x^2} b^2\right] dt$$

$$+ \frac{\partial h}{\partial x} b \, dz$$

where x follows generalized weiner's process

$$dx = a(x,t)\, dt + b(x,t)\, dz.$$

Derivation of Itô's Lemma :-

If x (underlying asset or stock) changes by small unit Δx then

$$\Delta h \approx \frac{dh}{dx} \Delta x + \frac{\partial h}{\partial t} \Delta t.$$

from Taylor's Expansion : $\Delta h = \frac{\partial h}{\partial x} \Delta x + \frac{1}{2} \frac{\partial^2 h}{\partial x^2} \Delta x^2 + \ldots$

$$+ \frac{\partial h}{\partial t} \cdot \Delta t + \frac{1}{2} \frac{\partial^2 h}{\partial t^2} \Delta t^2 + \ldots$$

$$+ \frac{\partial^2 h}{\partial x \partial t} \Delta x \Delta t + \ldots \quad \text{--- (1)}$$

For smaller values of Δx & Δt ;

$$\Delta h = \frac{\partial h}{\partial x} \Delta x + \frac{\partial h}{\partial t} \Delta t + \frac{1}{2} \frac{\partial^2 h}{\partial x^2} \Delta x^2 + \frac{1}{2} \frac{\partial^2 h}{\partial t^2} \Delta t^2 + \frac{\partial^2 h}{\partial t \partial x} \Delta t \Delta x \quad \text{--- (2)}$$

we know,

$$\Delta x = a\Delta t + b\varepsilon\sqrt{\Delta t}$$

$$\Delta x^2 = (a\Delta t + b\varepsilon\sqrt{\Delta t})^2$$

$$= a^2(\Delta t)^2 + b^2\varepsilon^2\Delta t + 2ab\,\Delta t^{3/2}$$

Ignoring higher order terms for Δt (as $\Delta t \to 0$)

$$\Delta x^2 = b^2\varepsilon^2\Delta t + \text{negligible terms}$$

from ② $\Rightarrow$ $dh = \frac{\partial h}{\partial x}dx + \frac{\partial h}{\partial t}dt + \frac{1}{2}\frac{\partial^2 h}{\partial x^2}b^2\varepsilon^2 dt$

now we know, $Var(\varepsilon) = 1$ (as it follows Std. Norm. dist.)

$\Rightarrow 1 = E(\varepsilon^2) - [E(\varepsilon)]^2$

$\Rightarrow 1 = E(\varepsilon^2)$ (As $E(\varepsilon) = 0$)

$\Rightarrow \underline{\underline{\varepsilon^2 = 1}}$ ($E(\varepsilon^2)$ is constant & $Var(\varepsilon)$ is constant).

$$\Rightarrow dh = \frac{\partial h}{\partial x}\left[a\,dt + b\,dz\right] + \frac{\partial h}{\partial t}dt + \frac{1}{2}\frac{\partial^2 h}{\partial x^2}b^2 dt$$

$$\Rightarrow dh = \underbrace{\left(a\frac{\partial h}{\partial x} + \frac{\partial h}{\partial t} + \frac{1}{2}\frac{\partial^2 h}{\partial x^2}b^2\right)}_{\text{new drift term}} dt + \underbrace{b\frac{\partial h}{\partial x}}_{\text{new variance factor}} dz$$

<u>Hence, Proved</u>

For given question,

$$\boxed{h(S,t) = \ln(S)}$$

$$\Rightarrow dh = \left[a \cdot \frac{\partial h}{\partial S} + \frac{\partial h}{\partial t} + \frac{1}{2}\frac{\partial^2 h}{\partial S^2} b^2 \right] \cdot dt$$

$$+ b \cdot \frac{\partial h}{\partial S} \cdot dz.$$

we know, $dS = \mu S\, dt + \sigma S\, dz$

$$\Rightarrow a = \mu S \ \& \ b = \sigma S.$$

$$dh = \left[\mu \cancel{S} \cdot \left(\frac{1}{\cancel{S}}\right) + 0 + \frac{1}{2}\left(\frac{-1}{\cancel{S^2}}\right)(\sigma \cancel{S})^2 \right] dt$$

$$+ \sigma \cancel{S} \left(\frac{1}{\cancel{S}}\right) \cdot dz.$$

$$dh = \left(\mu - \frac{\sigma^2}{2}\right) dt + \sigma dz$$

$$\boxed{h(S,t) = \int_0^t \left(\mu - \frac{\sigma^2}{2}\right) dt + \int_0^t \sigma dz}$$

So, $h(S,t) = \ln(S)$ follows a ~~geometric~~ ito's process ~~brownian motion~~ with parameters $\left(\mu - \sigma^2/2, \sigma\right)$.