1. (a) $\vec{F} = -k\vec{r} - mg\hat{y}$.

x component: $m\ddot{x} = -kx$

$$\Rightarrow x(t) = A\cos\omega t + B\sin\omega t$$

$$\omega = \sqrt{\frac{k}{m}}.$$

y component: $m\ddot{y} = -ky - mg$

Let $y' \equiv y + \frac{mg}{k}$.

$$\Rightarrow m\ddot{y}' = -ky'.$$

$$\Rightarrow y(t) = C\cos\omega t + D\sin\omega t - \frac{mg}{k}.$$

Initial conditions $x(0) = 0$ and $\dot{x}(0) = v_0\cos\theta$,

$$\Rightarrow x(t) = \left(\frac{v_0\cos\theta}{\omega}\right)\sin\omega t.$$

Similarly, initial conditions $y(0) = 0$ and $\dot{y}(0) = v_0\sin\theta$,

$$\Rightarrow y(t) = \left(\frac{mg}{k}\right)(\cos\omega t - 1) + \left(\frac{v_0\sin\theta}{\omega}\right)\sin\omega t.$$

(b) For small ωt, $\sin\omega t \approx \omega t$

$$\cos\omega t \approx 1 - \frac{(\omega t)^2}{2}.$$

$$\therefore x(t) \approx \left(\frac{v_0\cos\theta}{\omega}\right)\omega t = (v_0\cos\theta)t.$$

& using $\omega^2 = \frac{k}{m}$,

$$y(t) \approx \left(\frac{mg}{k}\right)\left(\frac{(-k/m)t^2}{2}\right) + \left(\frac{v_0\sin\theta}{\omega}\right)\omega t.$$

$$\therefore y(t) \approx -\frac{1}{2}gt^2 + (v_0\sin\theta)t.$$

(c.) For large ωt, the results are as follows.

For any t, the $x(t)$ motion is simple harmonic.

In order for the entire motion to be simple harmonic, we need it to be in a straight line.

i.e., $\frac{y}{x}$ must be a constant.

i.e., $\left(\frac{mg}{k}\right)(\cos\omega t - 1)$ term in $y(t)$ must be negligible.

We therefore need,

$$\frac{v_0 \sin\theta}{\omega} \gg \left(\frac{mg}{k}\right),$$

$$\text{i.e.,} \quad \frac{v_0 \sin\theta}{\omega} \gg \frac{g}{\omega^2}.$$

$$\text{i.e.,} \quad \omega \gg \frac{g}{v_0 \sin\theta}.$$

(d.) "Small ω"

When projectile hits the ground, $y \simeq 0$.

$$\Rightarrow \quad t = \left(\frac{2v_0 \sin\theta}{g}\right).$$

$$\therefore \quad \omega t \ll 1$$

$$\Rightarrow \quad \omega\left(\frac{2v_0 \sin\theta}{g}\right) \ll 1.$$

$$\text{i.e.,} \quad \omega \ll \left(\frac{g}{v_0 \sin\theta}\right) \Longleftrightarrow \text{"small } \omega\text{"}.$$

"Large ω"

As seen in (c) above, $\omega \gg \frac{g}{v_0 \sin\theta} \Longleftrightarrow$ "large ω".

(e.) We demand $y = 0$ when $\dot{x} = 0$.

$$\dot{x} = (v_0 \cos\theta) \cos\omega t = 0 \quad \text{when } \omega t = \frac{\pi}{2}.$$

The value of $y(t)$ at $\omega t = \frac{\pi}{2}$ is,

$$y\left(t = \frac{\pi}{2\omega}\right) = \frac{mg}{k}(0 - 1) + \left(\frac{v_0 \sin\theta}{\omega}\right)(1).$$

Setting this to zero,

$$\Rightarrow \frac{mg}{k} = \frac{v_0 \sin\theta}{\omega}. \quad \Rightarrow \frac{g}{\omega^2} = \frac{v_0 \sin\theta}{\omega}.$$

$$\Rightarrow \omega = \frac{g}{v_0 \sin\theta}.$$

(Exactly between the two limits in (d.) above).

2. (a.) $F = ma$ in the tangential direction yields,

$$-mg \sin\theta = m v \frac{dv}{dx}.$$

write $dx = l\,d\theta$.

$$\Rightarrow -\int_{\theta_0}^{\theta} mgl \sin\theta \, d\theta = \int_0^v m v \, dv.$$

$$\Rightarrow v = \pm\sqrt{2gl(\cos\theta - \cos\theta_0)}.$$

$$\therefore \int dt = \int \frac{dx}{v}$$

$$\Rightarrow T = 4\int_0^{\theta_0} \frac{l\,d\theta}{v} = 4\int_0^{\theta_0} \frac{l\,d\theta}{\sqrt{2gl(\cos\theta - \cos\theta_0)}}$$

$$\therefore T = \sqrt{\frac{8l}{g}} \int_0^{\theta_0} \frac{d\theta}{\sqrt{\cos\theta - \cos\theta_0}}.$$

(b.) Using $\cos\phi = 1 - 2\sin^2(\frac{\phi}{2})$,

$$\& \quad \sin x = \frac{\sin(\theta/2)}{\sin(\theta_0/2)} \Rightarrow \cos x\, dx = \frac{\frac{1}{2}\cos(\frac{\theta}{2})\, d\theta}{\sin(\frac{\theta_0}{2})}.$$

$$\& \quad \cos\frac{\theta}{2} = \sqrt{1 - \sin^2\frac{\theta_0}{2}\sin^2 x}.$$

$$T = \sqrt{\frac{8l}{g}} \int_0^{\theta_0} \frac{d\theta}{\sqrt{\cos\theta - \cos\theta_0}}.$$

$$= \sqrt{\frac{4l}{g}} \int_0^{\theta_0} \frac{d\theta}{\sqrt{\sin^2(\frac{\theta_0}{2}) - \sin^2(\frac{\theta}{2})}}.$$

$$= 2\sqrt{\frac{l}{g}} \int_0^{\theta_0} \frac{d\theta}{\sin(\frac{\theta_0}{2})\sqrt{1 - \sin^2 x}}$$

$$= 2\sqrt{\frac{l}{g}} \int_0^{\frac{\pi}{2}} \left(\frac{2\sin(\frac{\theta_0}{2})\cos x\, dx}{\cos(\frac{\theta}{2})}\right) \frac{1}{\sin(\frac{\theta_0}{2})\cos x}.$$

$$= 4\sqrt{\frac{l}{g}} \int_0^{\frac{\pi}{2}} \frac{dx}{\sqrt{1 - \sin^2(\frac{\theta_0}{2})\sin^2 x}}.$$

$$\approx 4\sqrt{\frac{l}{g}} \int_0^{\frac{\pi}{2}} \left(1 + \frac{1}{2}\sin^2(\frac{\theta_0}{2})\sin^2 x + \cdots\right) dx.$$

$$\approx 4\sqrt{\frac{l}{g}} \int_0^{\pi/2} \left(1 + \frac{1}{2}(\frac{\theta_0}{2})^2 \sin^2 x + \cdots\right) dx.$$

$$= 2\pi\sqrt{\frac{l}{g}}\left(1 + \frac{\theta_0^2}{16} + \cdots\right)$$

$$\left\{ \int_0^{\pi/2} \sin^2 x\, dx = \frac{1}{2}\int_0^{\pi/2} (1 - \cos 2x)\, dx = \frac{\pi}{4}. \right.$$

(3) We have (as derived in class):

$$t = \frac{1}{K} \ln\left(\frac{g + Ku}{g + Kv}\right).$$

We have to invert the expression first.

$$\Rightarrow \left(\frac{g + Kv}{g + Ku}\right) = e^{-Kt}.$$

$$\Rightarrow g + Kv = (g + Ku)e^{-Kt}$$

$$= Kue^{-Kt} + ge^{-Kt}.$$

$$\Rightarrow Kv = Kue^{-Kt} - g(1 - e^{-Kt})$$

$$\therefore v = ue^{-Kt} - \frac{g}{K}(1 - e^{-Kt}).$$

P.T.O.

3 contd

$$v = u e^{-Kt} - \frac{g}{K}(1 - e^{-Kt}).$$

Integrating once

$$\Rightarrow \quad z = -\frac{u}{K} e^{-Kt} - \frac{gt}{K} - \frac{g}{K^2} e^{-Kt} + C$$

$$\text{At } t = 0, \ z = 0 \quad \Rightarrow C = \frac{u}{K} + \frac{g}{K^2}.$$

$$\therefore \quad z = \frac{u}{K}(1 - e^{-Kt}) + \frac{g}{K^2}(1 - e^{-Kt}) - \frac{gt}{K}.$$

$$\Rightarrow \quad z = \left(\frac{u}{K} + \frac{g}{K^2}\right)(1 - e^{-Kt}) - \frac{gt}{K}.$$

$$\text{At } t = t_{max} = \frac{1}{K}\ln\left(1 + \frac{Ku}{g}\right), \ z = z_{max}.$$

$$\Rightarrow \quad z_{max} = \left(\frac{u}{K} + \frac{g}{K^2}\right)\left(1 - \frac{g}{g + Ku}\right) - \frac{g}{K^2}\ln\left(1 + \frac{Ku}{g}\right).$$

on rearrangement

$$z_{max} = \frac{u}{K} - \frac{g}{K^2}\ln\left(1 + \frac{Ku}{g}\right). \qquad \text{Q.E.D.}$$

4 Assume that the motion starts from the origin & takes place along the z-axis which is pointing vertically downwards. Following notation used in class, eqn. of motion is,

$$m\frac{dv}{dt} = mg - mKv^2.$$

When the body attains terminal velocity, $\frac{dv}{dt} = 0$.

$$\Rightarrow \quad mg - mKv^2 \Big|_{v = v_T} = 0.$$

$$\therefore \quad v_T^2 = \frac{g}{K}. \qquad \text{i.e., } K = \frac{g}{v_T^2}.$$

$$\Rightarrow \boxed{\frac{dv}{dt} = g\left(1 - \frac{v^2}{V_T^2}\right)} .$$

Thus, $\displaystyle\int \frac{dv}{V_T^2 - v^2} = \frac{g}{V_T^2}\int dt .$

Hence, $\displaystyle\frac{g}{V_T^2} t = \frac{1}{2V_T}\int\left(\frac{1}{(V_T - v)} + \frac{1}{(V_T + v)}\right) dv.$

$$= \frac{1}{2V_T} \ln\left(\frac{V_T + v}{V_T - v}\right) + C.$$

At $t = 0$ we have, $v = 0$. $\Rightarrow C = 0$.

$$\therefore \boxed{t = \frac{V_T}{2g} \ln\left(\frac{V_T + v}{V_T - v}\right)}$$

$\Downarrow$ inverted

$$\boxed{v = V_T \tanh\left(\frac{gt}{V_T}\right) .}$$

$$\therefore \quad z = V_T \int \tanh\left(\frac{gt}{V_T}\right) dt .$$

$$= \frac{V_T^2}{g} \ln\left(\cosh\left(\frac{gt}{V_T}\right)\right) + D.$$

At $t = 0$, $z = 0$ $\Rightarrow D = 0$.

$$\therefore \quad z = \frac{V_T^2}{g} \ln\left(\cosh\left(\frac{gt}{V_T}\right)\right) .$$

If the body falls through a height $z = H$ in time $t = T$, then,

$$T = \left(\frac{V_T}{g}\right) \cosh^{-1}\left(e^{gH/V_T^2}\right).$$

5 (a) $\bar{F} = m\bar{a}$

$$\Rightarrow \begin{cases} \ddot{x} = -\alpha \dot{x} \\ \ddot{y} = -g - \alpha \dot{y} \end{cases}$$

Let $\dot{x} = u_x$

$\Rightarrow \dot{u}_x = -\alpha u_x$

$\therefore u_x = Ae^{-\alpha t} = \dot{x}$.

Upon integration,

$\dot{x} = Ae^{-\alpha t}$. At $t=0$, $\dot{x} = v_0 \cos\theta$.

$\therefore \dot{x} = v_0 \cos\theta \, e^{-\alpha t}$.

Integrating once more,

$$x = -\left(\frac{v_0 \cos\theta}{\alpha}\right) e^{-\alpha t} + B.$$

At $t = 0$, $x = 0$ $\cancel{\Rightarrow B = 0 + 1 = 1}$.

$$\Rightarrow B = \frac{v_0 \cos\theta}{\alpha}.$$

$$\boxed{\therefore x(t) = \left(\frac{v_0 \cos\theta}{\alpha}\right)\left(1 - e^{-\alpha t}\right).}$$

$$\ddot{y} = -g - \alpha\dot{y} = -\alpha\left(\dot{y} + \frac{g}{\alpha}\right)$$

$$\therefore \frac{d}{dt}\left(\dot{y} + \frac{g}{\alpha}\right) = -\alpha\left(\dot{y} + \frac{g}{\alpha}\right).$$

$$\Rightarrow \dot{y} + \frac{g}{\alpha} = Ce^{-\alpha t}.$$

At $t = 0$, $\dot{y} = v_0 \sin\theta$.

$$\Rightarrow C = v_0 \sin\theta + \frac{g}{\alpha}.$$

$$\therefore \boxed{\dot{y} = \left(v_0 \sin\theta + \frac{g}{\alpha}\right) e^{-\alpha t} - \frac{g}{\alpha}}$$

Integrating once more,

$$y = -\frac{1}{\alpha}\left(v_0 \sin\theta + \frac{g}{\alpha}\right) e^{-\alpha t} - \frac{gt}{\alpha} + D.$$

At $t = 0$, $y = 0 \Rightarrow D = \frac{1}{\alpha}\left(v_0 \sin\theta + \frac{g}{\alpha}\right)$.

$$\therefore \quad \boxed{y(t) = \frac{1}{\alpha}\left(v_0 \sin\theta + \frac{g}{\alpha}\right)\left(1 - e^{-\alpha t}\right) - \frac{gt}{\alpha}}$$

(b:) Given $m\alpha v_0 = mg$

$$\Rightarrow \frac{g}{\alpha} = v_0.$$

$$\therefore \quad \dot{y} = (v_0 \sin\theta + v_0)e^{-\alpha t} - v_0.$$

At the highest point $\dot{y} = 0$

Let this happen at $t = t_p$

$$\Rightarrow e^{-\alpha t_p} = \frac{1}{(1+\sin\theta)}.$$

The value of x at time $t = t_p$ is,

$$x_p = \frac{v_0 \cos\theta}{g/v_0}\left(1 - \frac{1}{(1+\sin\theta)}\right)$$

$$\therefore \quad x_p = \frac{v_0^2 \cos\theta \sin\theta}{g(1+\sin\theta)}.$$

$$\frac{dx_p}{d\theta} = 0 \quad \Rightarrow \quad \sin^3\theta + 2\sin^2\theta - 1 = 0.$$

$$\therefore \quad (\sin\theta + 1)(\sin^2\theta + \sin\theta - 1) = 0.$$

Roots,

$$\left.\begin{aligned} \sin\theta &= -1 \\ \sin\theta &= -\frac{(\sqrt{5}+1)}{2} \end{aligned}\right\} \text{unphysical for present case.}$$

$$\sin\theta = \frac{\sqrt{5}-1}{2}.$$

$$\therefore \quad \theta = \sin^{-1}\left(\frac{\sqrt{5}-1}{2}\right) \approx 38.2^\circ.$$