Chapter -
Multiple Integrals

* → MULTI-VARIABLE FUNCTIONS →

- $f(x,y)$ defined as

$$f : D \longrightarrow R$$

where $D \subseteq R \times R$ (ie cartesian plane)

- consider $f(x,y) = x^2 + y^2$

$$f : R \times R \longrightarrow [0, \infty)$$

- consider $f(x,y) = \sqrt{1 - x^2 - y^2}$

$$D = \{ (x,y) : x^2 + y^2 \leq 1 \}$$

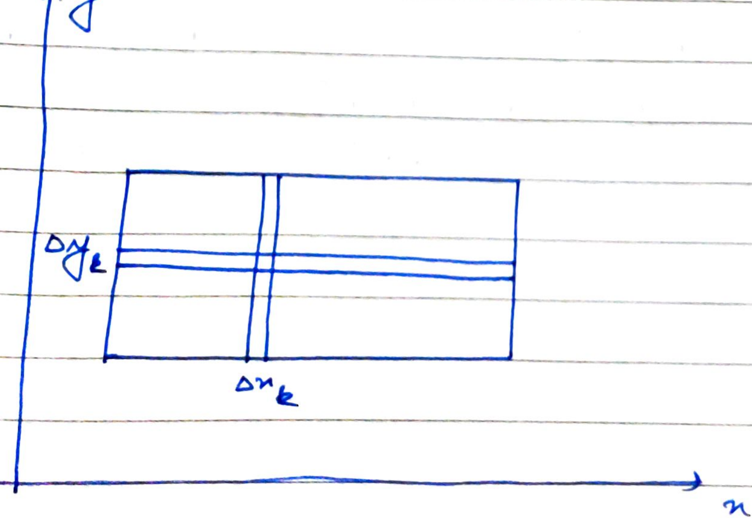
Range $= [0, 1]$

- Level Curves : Consider $f(x,y) = x^2 + y^2$

If we take $x^2 + y^2 = c$ where c is any constant, then the family of curves it represents ~~is~~ are called level c

MULTIPLE INTEGRALS

- Definite integ. of a cont. function $f(x)$ over an ~~integral~~ interval $[a, b]$ is defined as a limit of Riemann Sums
- now we will extend this to cont. func. of 2 variables $f(x, y)$ over a bounded region R in the plane.
- In both cases the integrals are limits of approximating Riemann sums.
- We can use multiple integrals to calculate quantities varying over 2 or 3 dimensions.
- Our main concern is now to identify region of integration.
- Considering ~~the~~ integral over a rectangle and a function defined on it $f(x, y)$ (say $f(x,y)$ is mass per unit area)



$\Delta A_k = \Delta x_k \Delta y_k$

$\therefore$ we can say, sum of elemental masses

~~mass of rectangle~~ $= \sum_{k=1}^{n} f(x_k, y_k)\, \Delta A_k$

For mass of rectangle $= \lim_{n\to\infty} \sum_{k=1}^{n} f(x_k, y_k)\, \Delta A_k$

$\therefore$ we write

$$\iint_R f(x,y)\, dA = \lim_{n\to\infty} \sum_{k=1}^{n} f(x_k, y_k)\, \Delta A_k$$

R → rectangle pts.

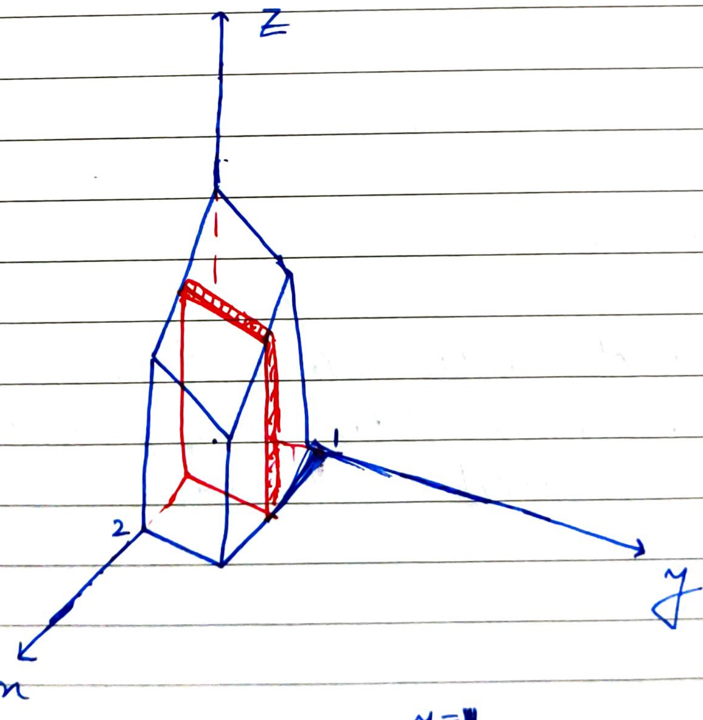
- $$\iint_R f(x,y)\, dA = \iint_R f(x,y)\, dx\, dy$$

- It can be shown that if $f(x,y)$ is continuous over R, ~~as~~ ~~to~~ then its double integral is calculatable. Also, even if the $f(x,y)$ is discontinuous at finite points, we can calculate the integral by including those points in the boundaries. Also, its calculatable if discont. occurs over smooth curves.

• We can develop a method to calculate double integrals (for volume calculation) using Riemann method. In single integrals, we had sum of rectangles, here we will get cuboids. And sum of volumes of such cuboids will give us Riemann sum.

Q: Calculate the volume under the plane $z = 4 - x - y$ over the rectangular region $R: 0 \le x \le 2,\ 0 \le y \le 1$ in xy plane.

Ans:



Area of red slice (area) $= \int_{y=0}^{y=1} (4 - x - y)\, dy$

Volume of slice element $= \left(\int_{y=0}^{1} (4 - x - y)\, dy \right) dx$

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$\therefore$ Volume of figure $= \displaystyle\int_{x=0}^{x=2} \left(\int_{y=0}^{y=1} (4-x-y)\,dy \right) dx$

$\Rightarrow$ Volume $= \displaystyle\int_{x=0}^{x=2} \left(4y - xy - \frac{y^2}{2} \right)\Big|_0^1 dx$

$= \displaystyle\int_{x=0}^{2} \left(4 - x - \frac{1}{2} \right) dx$

$= \left(4x - \frac{x^2}{2} - \frac{x}{2} \right)\Big|_0^2$

$= 7-2$

$\Rightarrow$ Volume $= 5$

We could also have taken ~~strip~~ area slice of area $\parallel$ to x-z plane. Even then, we would get volume $=5$.

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→ THEOREM: FUBINI'S THEOREM

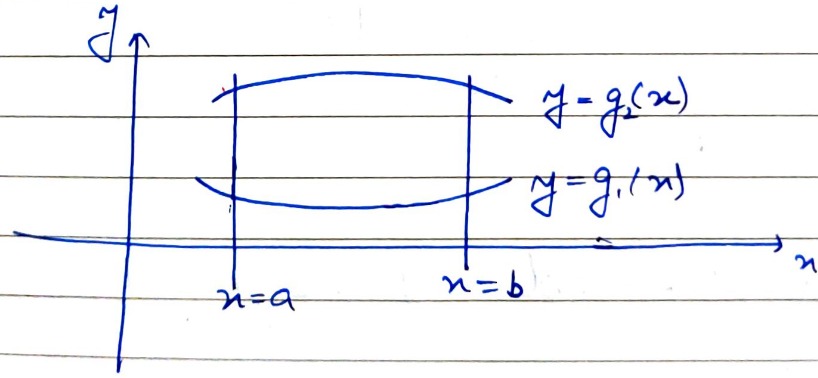
- If $f(x,y)$ is continuous throughout the rectangular region, then,

$$\iint_R f(x,y)\,dA = \int_c^d \int_a^b f(x,y)\,dx\,dy$$

$$= \int_a^b \int_c^d f(x,y)\,dy\,dx$$

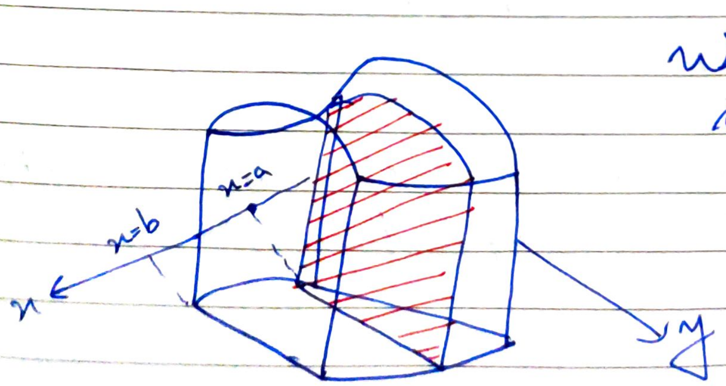
→ INTEGRAL OVER NON-RECTANGULAR REGIONS →

- consider, we need to calculate integral over



- Say, we need volume $(z = f(x,y))$

we take a strip and find its area



$$A = \int_{g_1(x)}^{g_2(x)} f(x,y)\, dy$$

$$\therefore V = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\, dy\, dx$$

→ THEOREM: STRONGER FUBINI'S THEOREM

- Let $f(x,y)$ be ~~a~~ continuous on a region R

1. If R is defined by $a \leq x \leq b$, $g_1(x) \leq y \leq g_2(x)$ with g_1 and g_2 continuous on $[a,b]$, then

$$\iint_R f(x,y)\, dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\, dy\, dx$$

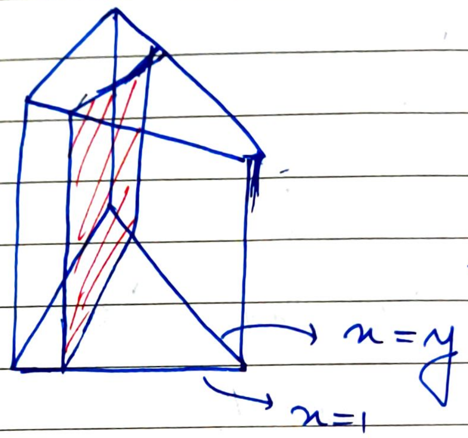
2. If R is defined by $c \leq y \leq d$, $h_1(y) \leq x \leq h_2(y)$ with h_1 and h_2 continuous on $[c,d]$ then

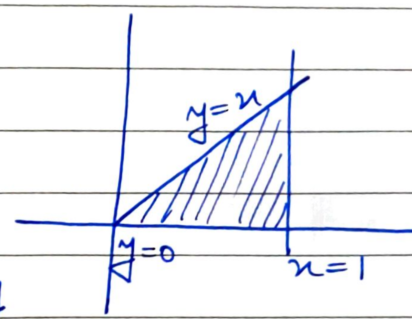
$$\iint_R f(x,y)\, dA = \int_c^d \int_{h_1(x)}^{h_2(x)} f(x,y)\, dx\, dy$$

NOTE: Here we can't exchange limits as we did in simple Fubini's theorem

Q. Find the volume of the prism whose base is the triangle in the xy-plane bounded by the x-axis and the lines $y=x$ and $x=1$ and whose top lies in the plane $z=f(x,y)=3-x-y$

Ans:





We firstly write general expression for area of slice

$$A=\int_{x=y}^{x=1}(3-x-y)\,dx$$

Now integrating the slices over $y=0$ to $y=1$ we get volume

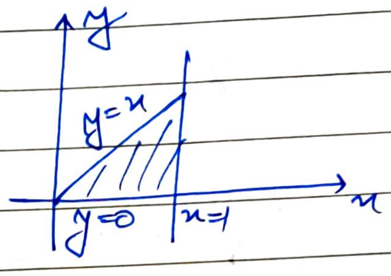
$$\boxed{V=\int_{y=0}^{y=1}\int_{x=y}^{x=1}(3-x-y)\,dx\,dy}$$

We could also have started from y to get

$$V=\int_{x=0}^{1}\int_{y=0}^{x}(3-x-y)\,dy\,dx$$

NOTE! It might not always be easy or feasible to calculate the integral over a particular variable (say x or y)

eg- Say we have $f(x,y) = \frac{\sin x}{x}$ over the region



$\therefore$ we can write

$$V = \int_{y=0}^{y=1} \int_{x=y}^{x=1} \frac{\sin x}{x} \, dx \, dy$$

But this is very difficult $\therefore$ we can write

$$V = \int_{x=0}^{x=1} \int_{y=0}^{y=x} \frac{\sin x}{x} \, dy \, dx$$

$$\Rightarrow V = \int_0^1 \frac{\sin x}{\cancel{x}} (\cancel{x}) \, dx$$

$$\Rightarrow V = -\cos x \Big|_0^1$$

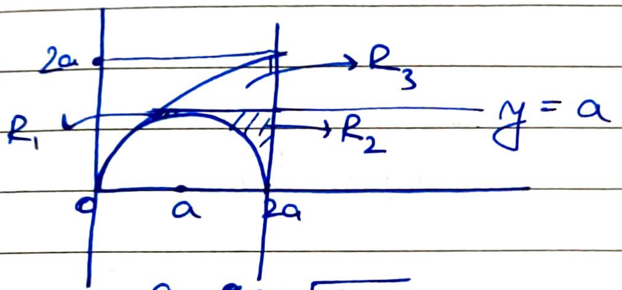
$$\Rightarrow V = 1 - \cos 1$$

Q→ change the order of the integration of:

$$\int_0^{2a}\int_{\sqrt{2ax-x^2}}^{\sqrt{2ax}} f(x,y)\,dy\,dx$$

where $f(x,y)$ is defened over the shaded region. what if area of shaded region is asked.

Ans:



$$\boxed{\begin{aligned} a^2 &= 2ax \\ x &= a/2 \end{aligned}}$$

R_1 ~~area~~ $\equiv \int_0^a \int_{y^2/2a}^{a-\sqrt{a^2-y^2}} f(x,y)\,dx\,dy$

$y^2 = 2ax$

$y^2 = 2ax - x^2$

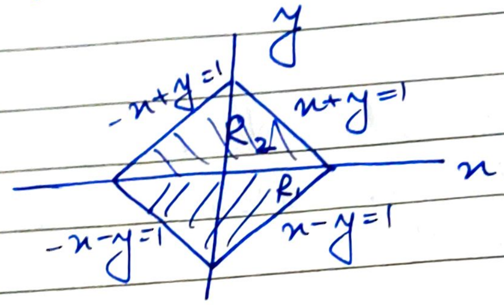
$x = a \pm \sqrt{a^2-y^2}$

$$R_2 \equiv \int_0^a \int_{a+\sqrt{a^2-y^2}}^{2a} f(x,y)\,dx\,dy$$

$$R_3 \equiv \int_a^{2a} \int_{y^2/2a}^{2a} f(x,y)\,dx\,dy$$

Q- Find volume of the solid cut from the square column $|x|+|y|=1$ and $z=3-3x$ plane and $z=0$ plane.

Ans:



$R = R_1 \cup R_2$

$\therefore$ volume $= \iint_{R_1} z \, dx \, dy + \iint_{R_2} z \, dx \, dy$

$\Rightarrow$ volume $= \int_{-1}^{0} \int_{-1-y}^{1+y} (3-3x) \, dx \, dy + \int_{0}^{1} \int_{-1+y}^{1-y} (3-3x) \, dx \, dy$

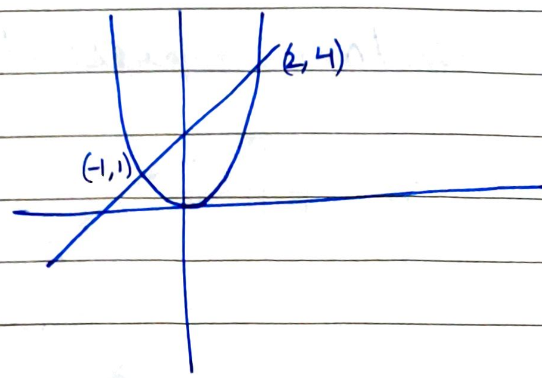
→ Area, moments and COM →

- The area of a closed, bounded plane region R is

$$\text{Area} = \iint_R dA$$

Q) Find area of region R enclosed by parabola $y = x^2$ and line $y = x + 2$.

Ans:



$$x^2 = x + 2$$

$$x^2 - x - 2 = 0$$

$$\therefore x = -1, 2$$

$$\int_{-1}^{2}\int_{x^2}^{x+2} dy\, dx = \int_{-1}^{2} (x + 2 - x^2)\, dx$$

$$= \left. \frac{x^2}{2} + 2x - \frac{x^3}{3} \right|_{-1}^{2}$$

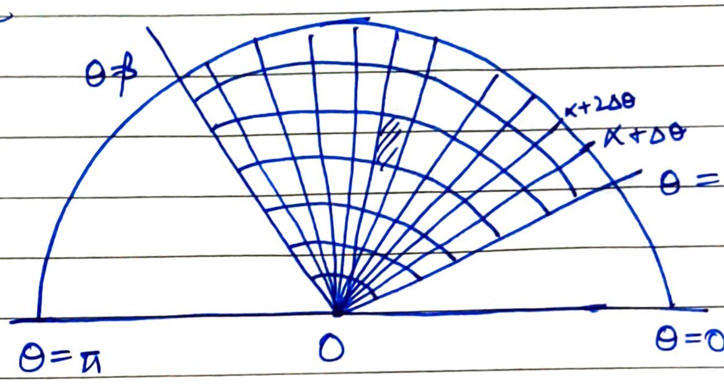
$$= \left(2 + 4 - \frac{8}{3} - \frac{1}{2} + 2 - \frac{1}{3}\right)$$

$$= 8 - 3 - 1/2 = 5 - 1/2 = 9/2$$

→ DOUBLE INTEGRALS IN POLAR FORM

- Double integrals are sometimes easier to compute in polar form.

- For elemental representation, we have

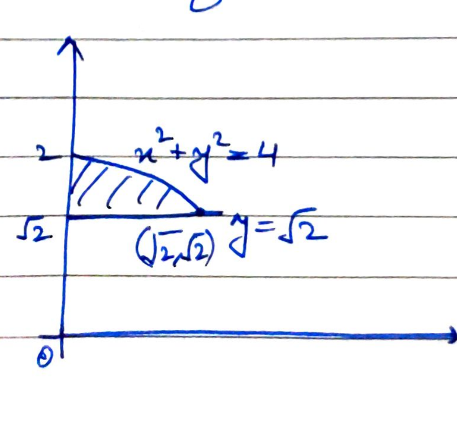


- Elemental Area = $r\,d\theta\,dr$

- ∴ we write double integration as

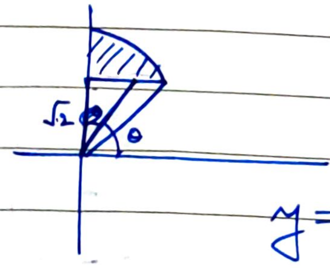
$$A = \iint_R f(r,\theta)\, r\, d\theta\, dr$$

Q→ Find area of



Ans:

$$\int_{\theta=\pi/4}^{\theta=\pi/2}\int_{r=\sqrt{2}\,\mathrm{cosec}\,\theta}^{r=2} r\,dr\,d\theta$$



$y = r\sin\theta$

$r\,\cancel{\cos\theta}\ \sin = \sqrt{2}$

$r = \sqrt{2}\ \cancel{\sec\theta}\ \mathrm{cosec}\,\theta$

Q→ Integrate $I = \int_0^\infty e^{-x^2}dx$ using double integrals.

Ans:

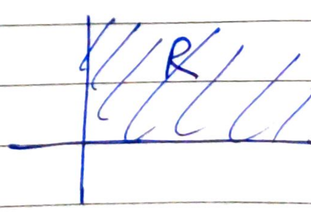
$$I = \int_0^\infty e^{-x^2}dx \qquad I = \int_0^\infty e^{-y^2}dy$$

$$I^2 = \int_0^\infty e^{-x^2}dx \int_0^\infty e^{-y^2}dy$$

$$\Rightarrow I^2 = \int_0^\infty\int_0^\infty e^{-(x^2+y^2)}\,dx\,dy$$

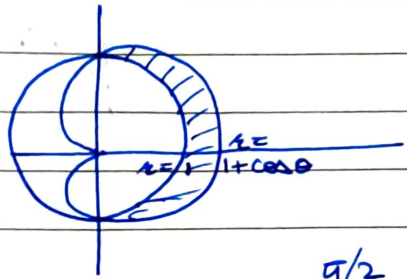
We can carry this out since limits are const. and e^{-x^2} is indep of y & vice versa

Converting to polar form



$$\Rightarrow I^2 \int_0^{\pi/2}\int_0^\infty e^{-r^2}\,r\,dr\,d\theta$$

Q→ Cardoid overlapping a circle



Ans: Required area = $\int_{-\pi/2}^{\pi/2}\int_{1}^{1+\cos\theta} r\,dr\,d\theta$

Q→ If a right cylinder has base as darkened in previous question and top lies in $z=x$ plane. Find volume

Ans.. $V = \iint_R z\,r\,dr\,d\theta$

$z = x = r\cos\theta$

$\Rightarrow V = \int_{-\pi/2}^{\pi/2}\int_{1}^{1+\cos\theta} r^2\cos\theta\,d\theta$

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TUTORIAL

1. Let $f(x) = [x]$, $x \in [0,3]$

$$f(x) = \begin{cases} 0 & ; 0 \le x < 1 \\ 1 & ; 1 \le x < 2 \\ 2 & ; 2 \le x < 3 \\ 3 & ; x = 3 \end{cases}$$

Since f is bounded on $[0,3]$, $|f(x)| \le 3$ $\forall x \in [0,3]$. f is continuous on $[0,3]$ except at $x = 1, 2, 3$. Since f has finite no. of discont. points, $\Rightarrow f$ is integrable on $[0,3]$.

now, $$\int_0^2 [x]\,dx = \int_0^1 [x]\,dx + \int_1^2 [x]\,dx = 1$$

4. Here $$g(x) = \frac{1}{\alpha}\int_0^x f(t)\,\{\sin(\alpha x)\cos(\alpha t) - \cos(\alpha(x))\sin(\alpha t)\}\,dt$$

$$= \frac{1}{\alpha}\left[\sin(\alpha x)\int_0^x f(t)\cos(\alpha t)\,dt - \cos(\alpha x)\int_0^x f(t)\sin(\alpha t)\,dt\right]$$

$$\Rightarrow g'(x) = \frac{1}{\alpha}\left[\alpha\cos(\alpha x)\int_0^x f(t)\cos(\alpha t)\,dt + \sin(\alpha x)f(x)\cos(\alpha x)\right.$$

$$\left. + \alpha\sin(\alpha(x))\int_0^x f(t)\sin(\alpha t)\,dt - \cos(\alpha(x))f(x)\sin(\alpha x)\right]$$

$$\Rightarrow g''(x) = f(x) + \alpha\cos(\alpha x)\int_0^x f(t)\sin(\alpha t)\,dt - \alpha\sin(\alpha x)\int_0^x f(t)\cos(\alpha t)\,dt$$

$$g''(x) = f(x) - \alpha^2 \frac{1}{\alpha}\left[\sin(\alpha x)\int_0^x f(t)\cos(\alpha t)\,dt - \cos(\alpha x)\int_0^x f(t)\sin(\alpha t)\,dt\right]$$

$$\Rightarrow g''(x) = f(x) - \alpha^2 g(x)$$

$$\Rightarrow f(x) = g''(x) + \alpha^2 g(x)$$

By Fundamental Theorem of Calculus

Let f is integrable function on $[a,b]$. For $x \in [a,b]$

Let $F(x) = \int_a^x f(t)\,dt$

Then f is continuous on $[a,b]$. If f is continuous at x_0 in (a,b) then f is diff at x_0 in (a,b) at $F'(x_0) = f(x_0)$

5. Let $F(x) = \int_0^x f(t)\,dt - x^3$ on $[0,1]$

Here $F(1) = \int_0^1 f(t)\,dt - 1$

$= 1 - 1 = 0$

Also $F(0) = 0$

Here F is continuous on $[0,1]$ & F is differentiable on $(0,1)$

Also, $F(0) = F(1)$

Applying ~~*~~ Rolle's Theorem

$\exists$ c in $(0,1)$ such that

$F'(c) = 0$ where $c \in (0,1)$

$f(c) - 3c^2 = 0$

$\Rightarrow f(c) = 3c^2$

7. Let $F(x) = \int_0^x f(t)\,dt$, $G(x) = \sin 2x$

Applying Cauchy mean value theorem on $[0, \pi/4]$

$\frac{F(\pi/4) - F(0)}{G(\pi/4) - G(0)} = \frac{F'(c)}{G'(c)}$ for $c \in (0, \pi/4)$

$\Rightarrow \frac{\int_0^{\pi/4} f(t)\,dt - 0}{\sin \pi/2} = \frac{f(c)}{2\cos 2c}$

$\Rightarrow f(c) = 2\cos 2c \int_0^{\pi/4} f(t)\,dt$

Let $F(x) = \int_a^x f(t)\,dt \Rightarrow F'(x) = f(x)$

8. $\int_a^x f(t)\,dt = \int_a^x f(t) + \int_x^b f(t)\,dt - \int_x^b f(t)\,d(t)$

$= \int_a^b f(t)\,dt - \int_x^b f(t)\,dt$

$= \int_a^b f(t)\,dt - \int_a^x f(t)\,dt$ (given in que)

$F(x) = F(b) - F(x)$

$\Rightarrow 2F(x) = F(b)$

$\Rightarrow 2F'(x) = 0$

$\Rightarrow F'(x) = 0$

$\Rightarrow f(x) = 0$

9. Since f, g are diff. on $[a,b]$ $\therefore$ fg is diff on $[a,b]$

$\Rightarrow f, g$ continuous on $[a,b]$

f, g Integrable on $[a,b]$

$\therefore fg' + f'g$ is Integrable on $[a,b]$

ie $(fg)'$ is Integrable on $[a,b]$

So by Fundamental theorem

$$\int_a^b (fg)'\,dx = (fg)\Big|_a^b = f(b)g(b) - f(a)g(a)$$

Also

$$\int_a^b (fg)'\,dx = \int_a^b (f'g + g'f)\,dx = \int_a^b f'g\,dx + \int_a^b fg'\,dx$$

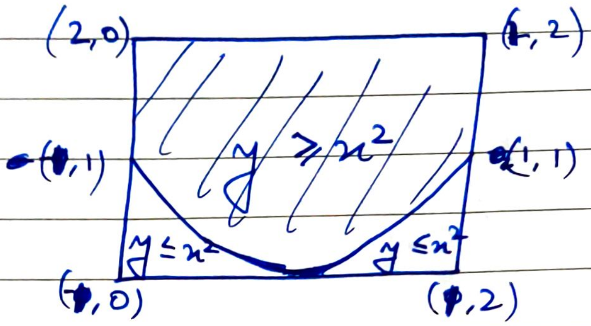
$$\Rightarrow \int_a^b (fg)'\,dx = f(b)g(b) - f(a)g(a)$$

$$\Rightarrow \int_a^b f(x)\,g'(x)\,dx = f(b)g(b) + f(a)g(a) - \int_a^b f'(x)g(x)\,dx$$

Q→ Solve:

$$I = \int_{-1}^{1}\int_{0}^{2} |y - x^2| \, dy \, dx$$

Ans:



$$\therefore I = \int_{-1}^{1}\int_{0}^{x^2} -(y - x^2)\, dy\, dx + \int_{-1}^{1}\int_{x^2}^{2} (y - x^2)\, dy\, dx$$

$$= \int_{-1}^{1} \left(-\frac{y^2}{2} + x^2 y\right)\Big|_{0}^{x^2} dx + \int_{-1}^{1} \left(\frac{y^2}{2} - x^2 y\right)\Big|_{x^2}^{2}$$

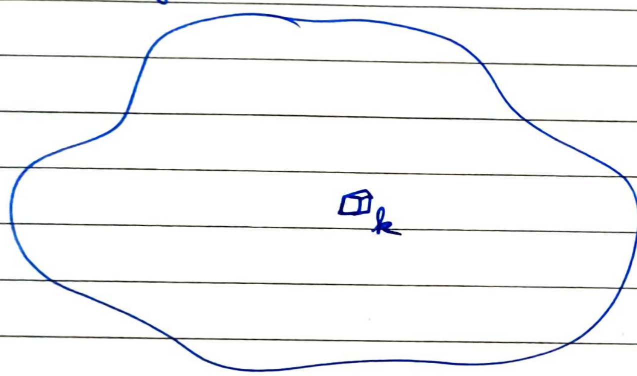
$$= \int_{-1}^{1} \left(-\frac{x^4}{2} + x^4\right) dx + \int_{-1}^{1} \left[2 - 2x^2 - \left(\frac{x^4}{2} - x^4\right)\right] dx$$

$$= \int_{-1}^{1} x^4 \, dx$$

* → Triple Integrals →

- A triple integral is basically defined with respect to small volume elements.

- For a region D



We can create a triple integral by writing summation

$$\therefore \quad I = \sum_{k=1}^{n} F(x,y,z)\, \Delta x\, \Delta y\, \Delta z$$

$$\therefore \quad I = \iiint_D F(x,y,z)\, dx\, dy\, dz$$

- If $F(x,y,z) = 1$, then we get volume of the specific region D.

$$\therefore \quad V = \iiint_D dx\, dy\, dz$$

NOTE: To solve, first find limits of z, then obtain projection of the volume on x-y plane & then accordingly find x & y limits.

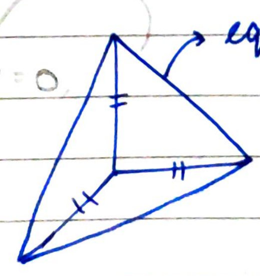
→ SOME PROPERTIES OF TRIPLE INTEGRALS

→ SOME SIMPLE PROBLEMS →

a) **Volume of a Cuboid:** (bounded by planes $x=\pm a$; $y=\pm b$; $z=\pm c$)

$$\therefore \quad \boxed{V=\int_{-c}^{c}\int_{-b}^{b}\int_{-a}^{a} dx\,dy\,dz}$$

b) **Volume of shape formed by planes $x=0$; $y=0$; $z=0$ and $ax+by+cz=d$**

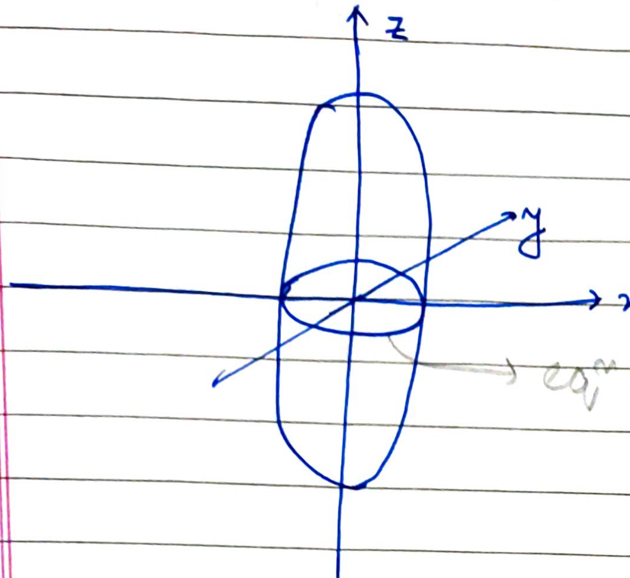


→ eq^n of this line is $ax+by=d$ (by putting $z=0$)

putting $z=0$ we get equation of line in x-y plane

$$\boxed{V=\int_{x=0}^{d/a}\int_{y=0}^{(d-ax)/b}\int_{z=0}^{(d-by-ax)/c} dz\,dy\,dx}$$

c) <u>Volume of ellipsoid</u> : $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$



eq^n of this circle is obtained by putting $z=0$ in given equation (Think!!)

$$\boxed{V = \int_{x=-a}^{a} \int_{y=-b\sqrt{1-\frac{x^2}{a^2}}}^{b\sqrt{1-\frac{x^2}{c^2}}} \int_{z=-c\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}}^{c\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}} dz\,dy\,dx}$$

d) <u>Volume of Right circular Cylinder with base $x^2+y^2=r^2$ and bounded by $z=\pm h$</u>

$$\boxed{V = \int_{-r}^{r} \int_{-\sqrt{r^2-x^2}}^{\sqrt{r^2-x^2}} \int_{-h}^{h} dz\,dy\,dx}$$

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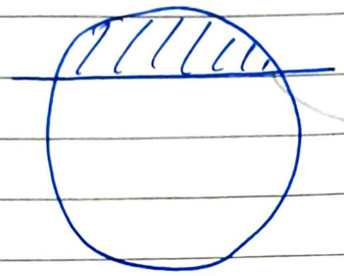
c) Volume bounded above by the sphere $x^2+y^2+z^2=25$ and bounded by plane $z=4$

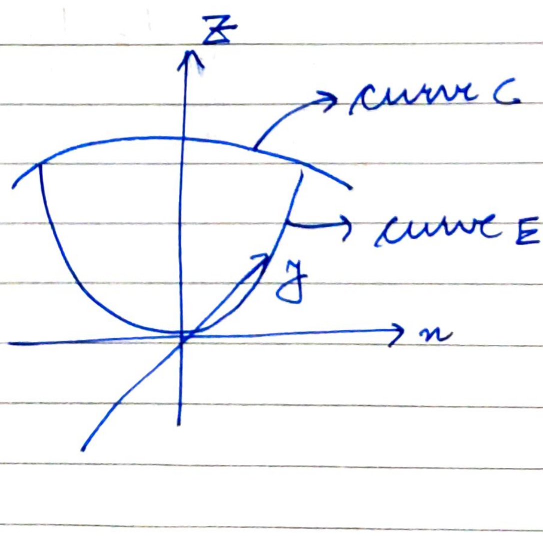


→ eqn of this circle is obtained by putting z=4 in eqn to get $x^2+y^2=9$

$$V=\int_{-3}^{3}\int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}}\int_{4}^{\sqrt{25-x^2-y^2}} dz\,dy\,dz$$

Q→ Find volume of region D enclosed by surfaces $\underbrace{z=x^2+3y^2}_{\text{elliptical paraboloid (E)}}$ and $\underbrace{z=8-x^2-y^2}_{\text{circular paraboloid (inverted) (C)}}$

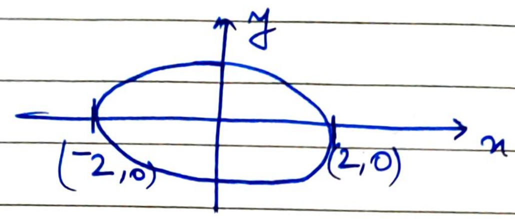
Ans. We will have a figure like



To find projection on x-y plane, eliminate z. ∴ we get

$$2x^2 + 4y^2 = 8$$

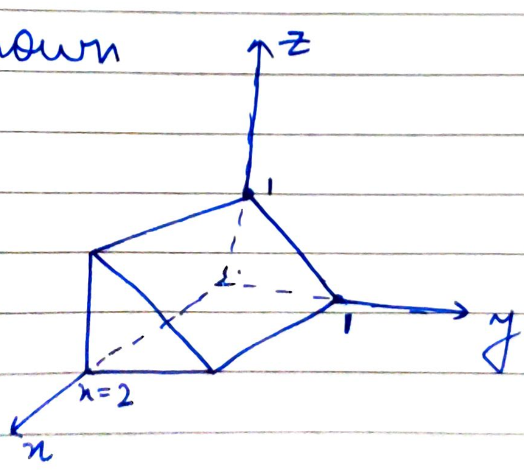
$$\Rightarrow \boxed{x^2 + 2y^2 = 4}$$



∴ we can now write volume

$$V = \int_{-2}^{2} \int_{-\frac{1}{\sqrt{2}}\sqrt{4-x^2}}^{\frac{1}{\sqrt{2}}\sqrt{4-x^2}} \int_{x^2+3y^2}^{8-x^2-y^2} dz\, dy\, dx$$

Q→ Find volume of region bounded by $x=2$; $x=0$; $y=0$; $z=0$; and plane $y+z=1$ as shown in 6 ways.



Ans.

$$V = \int_{y=0}^{1}\int_{x=0}^{2}\int_{z=0}^{1-y} dz\,dx\,dy$$

$$V = \int_{x=0}^{2}\int_{y=0}^{1}\int_{z=0}^{1-y} dz\,dy\,dx$$

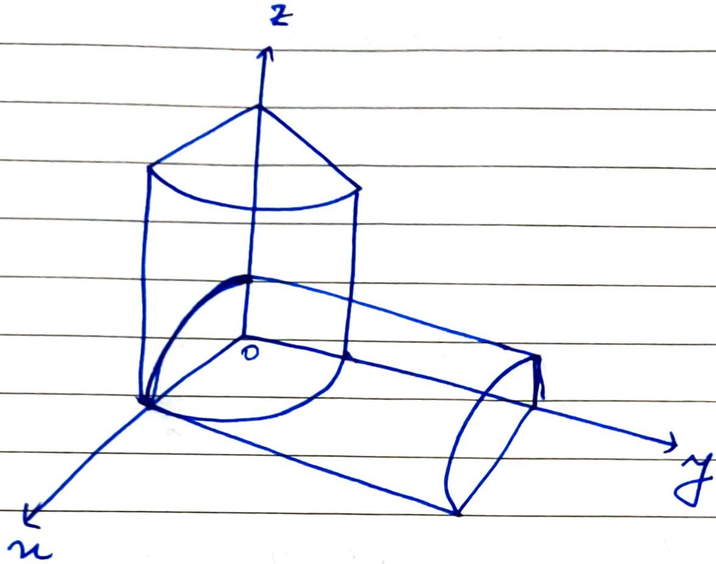
$$V = \int_{z=0}^{1}\int_{x=0}^{2}\int_{y=0}^{1-z} dy\,dx\,dz$$

$$V = \int_{x=0}^{2}\int_{z=0}^{1}\int_{y=0}^{1-z} dy\,dz\,dx$$

$$V = \int_{z=0}^{1}\int_{y=0}^{1-z}\int_{x=0}^{2} dx\,dy\,dz$$

$$V = \int_{y=0}^{1}\int_{z=0}^{1-y}\int_{x=0}^{2} dx\,dz\,dy$$

Q→ The region common to interior of cylinders $x^2+y^2=1$ and $x^2+z^2=1$ one eighth of which is shown



Ans:

$$V = 8\int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^{\sqrt{1-x^2}} dz\, dy\, dx$$

Or $$V = \int_{-1}^{1} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} dz\, dy\, dx$$

$$\Rightarrow V = \int_{-1}^{1} 4(1-x^2)\, dx$$

$$\Rightarrow V = 4\left(x - \frac{x^3}{3}\right)\Bigg|_{-1}^{1} \Rightarrow V = \frac{16}{3}$$

classmate

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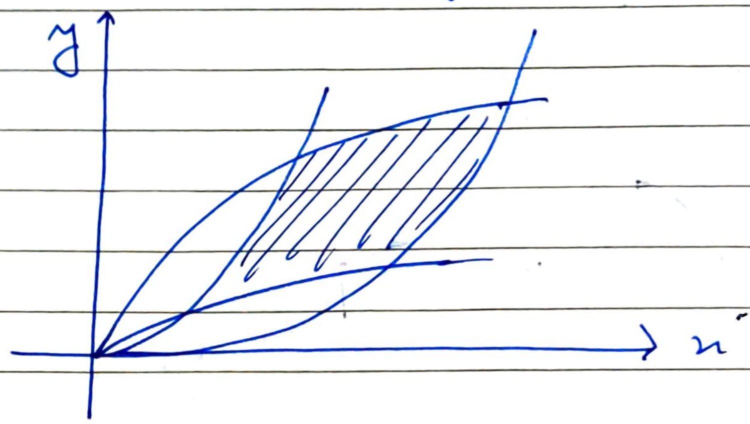
→ Substitutions in Multiple Integrals →

Q→ Find area bounded b/w four parabolas
$y^2 = 16x$, $y^2 = 256x$, $y = x^2$, $y = 8x^2$

Ans:

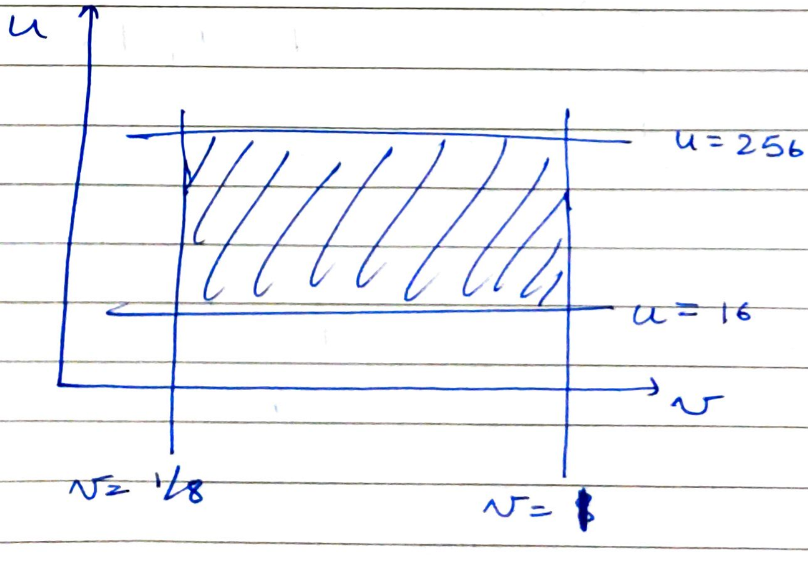


~~To~~ To find this area, we need at least 3 integrals & heavy solving.

now if we assume,

$\frac{y^2}{x} = u$ and $\frac{x^2}{y} = v$

and take u and v on axis



We can now easily compute area of this rectangle as

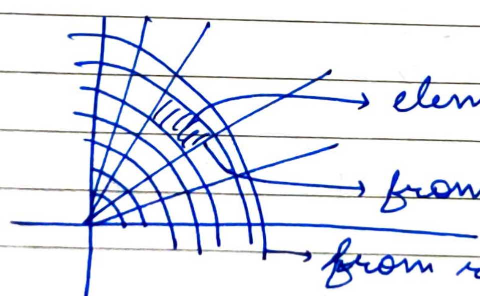
$$A = \int_{1/8}^{1} \int_{16}^{256} du\, dv$$

But this isn't equal to area of our required region. For that we have

$$\boxed{A = \int_{1/8}^{1} \int_{16}^{256} |J|\, du\, dv} = \iint_R dx\, dy$$

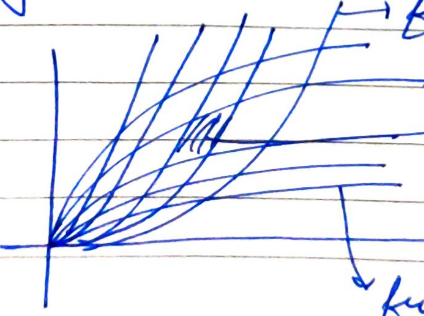
$|J| \rightarrow$ Jacobian factor.

NOTE: Jacobian factor for conversion to spherical polar system is r. Geometrically we can find it as



Similarly we can find elementary area for any system.

eg- for u & v above



→ JACOBIAN FACTOR

- We ~~define~~ can find jacobian factor as:

- $$\boxed{J(u,v) = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial y}{\partial u}\frac{\partial x}{\partial v} = \frac{\partial(x,y)}{\partial(u,v)}}$$

- $$J(u,v) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$$

- eg - Jacobian for spherical polar system.

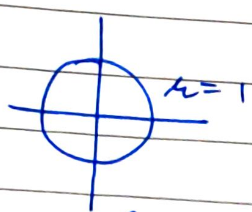
$$J(r,\theta) = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix}$$

$$x = r\cos\theta \quad , \quad y = r\sin\theta$$

$$u = r \qquad v = \theta$$

$$\therefore \quad J = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r$$

- eg- computing area of circle



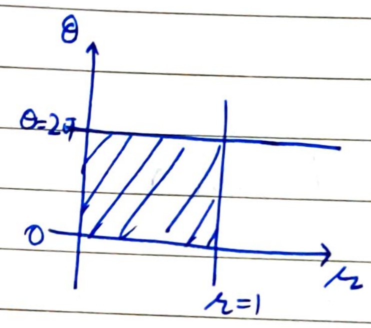
$x^2+y^2=1$

In cartesian form

$$\int_{-1}^{1}\int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} dx\,dy$$

In polar form

$$\int_{\theta=0}^{2\pi}\int_{r=0}^{1} r\,dr\,d\theta$$



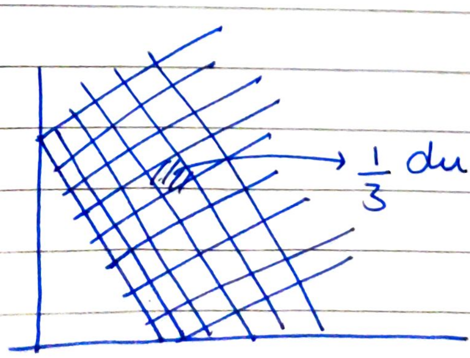
Q→ Evaluate area bounded by lines

$y=-2x+4$, $y=-2x+7$, $y=x-2$, $y=x+1$

Ans. Let

$u = y+2x$ $\Rightarrow$ $y=\dfrac{u+2v}{3}$

$v = y-x$ $\quad x=\dfrac{u-v}{3}$



$\to \frac{1}{3}\,du\,dv$ (Since $|J|=1/3$)

$$|J| = \begin{vmatrix} 1/3 & -1/3 \\ 1/3 & 2/3 \end{vmatrix} = 1/3$$

$$\iint_R dx\,dy = \int_{v=-2}^{1}\int_{u=4}^{7} |J|\,du\,dv$$

$$= \int_{v=-2}^{1}\int_{u=4}^{7} \frac{1}{3}\,du\,dv$$

Q→ Find area bounded in 1^{st} que.

Ans: $\frac{y^2}{x} = u$, $\frac{y}{x^2} = v$

$\frac{y^3}{x^3} = uv$ | $xy = \frac{u}{v}$

We need x & y in terms of u & v

$\frac{y^3}{x^3}\,x^3y^3 = uv\,\frac{u^3}{v^3} \Rightarrow y^6 = \frac{u^4}{v^2} \Rightarrow \boxed{y = \frac{u^{2/3}}{v^{1/3}}}$

Similarly

$x = u^{1/3}v^{-2/3}$

$$|J| = \begin{vmatrix} \frac{1}{3}u^{-2/3}v^{-2/3} & -\frac{2}{3}u^{1/3}v^{-5/3} \\ \frac{2}{3}u^{-1/3}v^{-1/3} & -\frac{1}{3}u^{2/3}v^{-4/3} \end{vmatrix}$$

$$\therefore J = -\frac{1}{9} v^{-2} + \frac{4}{9} v^{-2} = \frac{1}{3v^2}$$

$$\Rightarrow \iint dx\, dy = \int_{v=1/8}^{1} \int_{u=16}^{256} \frac{1}{3v^2}\, du\, dv$$

- Generalisation in 3-D

$$\iiint_D F(x,y,z)\, dx\, dy\, dz = \iiint_G H(u,v,w) \left| J(u,v,w) \right| du\, dv\, dw$$

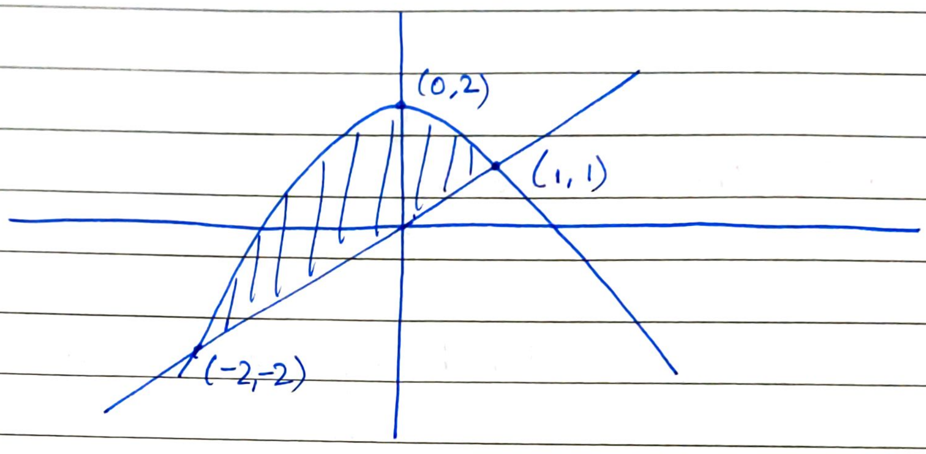
where

$$\left| J(u,v,w) \right| = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix}$$

TUTORIAL (MULTIPLE INTEGRALS)

5.

d. $y = 2 - x^2$ and $y = x$



$y - 2 = -x^2$

$x^2 = 2 - y$

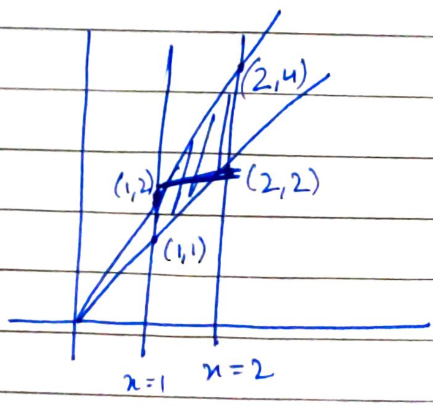
$\therefore$ we can write volume as

$$V = \int_{-2}^{1} \int_{y=x}^{2-x^2} x^2 \, dy \, dx$$ (taking vert. strip elements)

6. $f(x,y) = x/y$

$y = x$ to $y = 2x$

$x = 1$ to $x = 2$



$$\therefore \; I = \int_1^2 \int_x^{2x} \frac{x}{y} \, dy \, dx$$ (by taking vertical strip elements)

or in horizontal strips,

$$I = \int_1^2 \int_1^y \frac{x}{y} \, dx \, dy + \int_2^4 \int_{y/2}^2 \frac{x}{y} \, dx \, dy$$

Varimp

7.
$$\iint_R (x^2+y^2-9) \, dA$$

We have to minimize the integral. In order to minimize the integral, we want the domain to include all the points where the integrand is -ve and exclude all the points where integrand is +ve. So

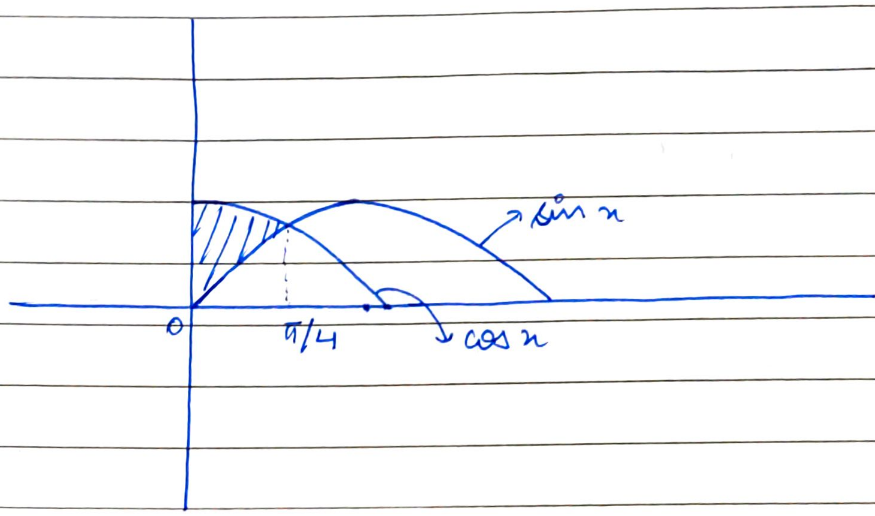
$$x^2+y^2-9 \leq 0$$

$$\text{or } x^2+y^2 \leq 9$$

Which is a closed disc of radius 3 centered at origin.

8.

a) $$\int_0^{\pi/4}\int_{\sin x}^{\cos x} dy\, dx$$



imp NOTE: At least draw one diagram when you solve question.
Prefer drawing a 3-D view and a 2-D projection both.

Q→ calculate volume of ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

Ans: Consider

$$\frac{x}{a} = u \ ; \ \frac{y}{b} = v \ ; \ \frac{z}{c} = w$$

$\therefore$ $u^2+v^2+w^2=1$ (sphere)

$$\therefore \iiint\limits_{\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2} \le 1} dx\, dy\, dz = \iiint\limits_{u^2+v^2+w^2 \le 1} abc\ du\, dv\, dw$$

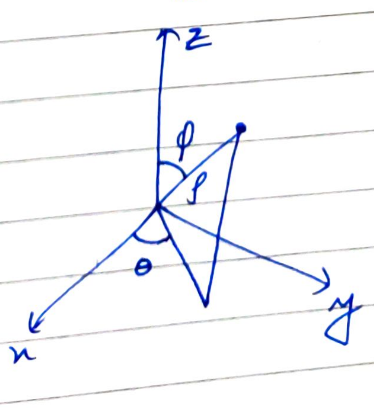
$\therefore J = abc$

$\therefore$ we can directly write

$$V = abc \underbrace{\iiint\limits_{u^2+v^2+w^2\le 1} du\, dv\, dw}_{\text{volume of sphere}}$$

$\therefore V = \frac{4}{3}\pi abc$ clearly.

We can also do this by converting further to spherical form.



$\therefore u = \rho \sin\phi \cos\theta$

$v = \rho \sin\phi \sin\theta$

$w = \rho \cos\phi$

$$\therefore \Rightarrow V = \int_0^{2\pi}\int_0^{\pi}\int_0^{1} abc\, \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$$

$$= \frac{abc}{3} \times 2 \times 2\pi = \frac{4\pi abc}{3}$$

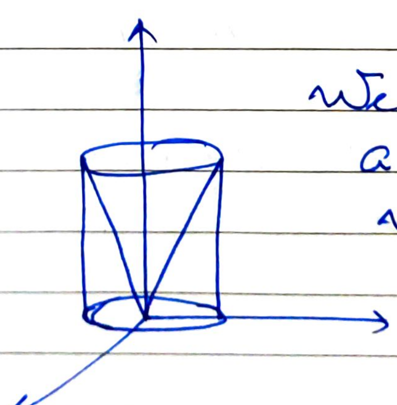
HW

Q Find volume of cylinder with radius 1 and height 1 in spherical system using following order

$$\iiint_D \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$$

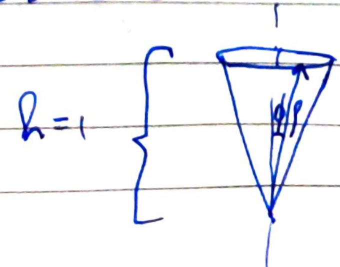
Also try to change the order by plotting the transformation of volume in $\rho - \theta - \phi$ cartesian system.

Ans:



We divide it into 2 parts. a cone & rest of the volume

For cone



we ~~too~~ can write $\rho\cos\phi = 1$

$\Rightarrow \rho = \sec\phi$

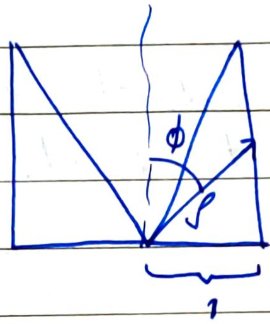
for every elemental volume

& ϕ varies from $[0, \pi/4]$.

& θ varies from $[0, 2\pi)$

$$\therefore \quad V_{cone} = \int_{\theta=0}^{2\pi} \int_{\phi=0}^{\pi/4} \int_{\rho=0}^{\sec\phi} \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$$

For rest of part



we can write

$\rho \sin\phi = 1$

$\Rightarrow \rho = \operatorname{cosec}\phi$

For any elemental volume.

also ϕ varies from $[\pi/4, \pi/2]$

also θ varies from $[0, 2\pi]$

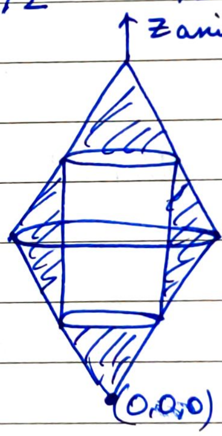
$$\therefore V_{rest} = \int_{\theta=0}^{2\pi} \int_{\phi=\pi/4}^{\pi/2} \int_{\rho=0}^{\operatorname{cosec}\phi} \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$$

$$\therefore V_{total} = V_{cone} + V_{rest}$$

Ans:

Q: Find volume inside cones $z = 2 - \sqrt{x^2+y^2}$ & $z = \sqrt{x^2+y^2}$ but outside cylinder $x^2+y^2 = 1/2$ using ~~spherical~~ cylindrical coordinates

Ans:



In cartesian

z varies from $\sqrt{x^2+y^2}$ to $2 - \sqrt{x^2+y^2}$

& $\sqrt{x^2+y^2} = r$ in cylindrical system

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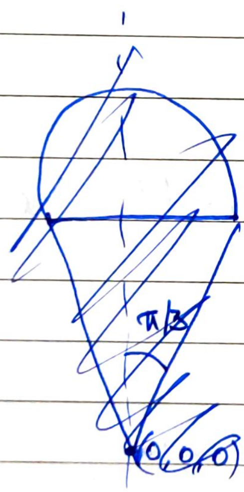
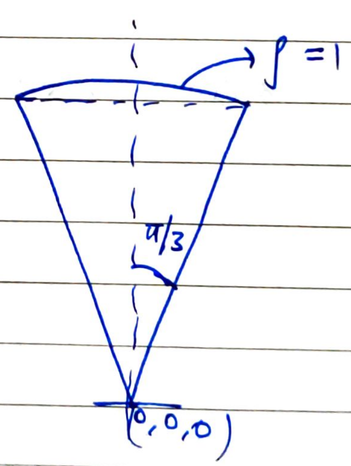
Q: Find volume of ice cream cone in spherical system

Ans:





$$\therefore \quad V = \int_{\theta=0}^{2\pi} \int_{\phi=0}^{\pi/3} \int_{\rho=0}^{1} \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$$

Q.→ Solve using spherical system :

The solid bounded below by the sphere $\rho = 2\cos\phi$ and above by the cone

$z = \sqrt{x^2 + y^2}$

Ans.

$x = \rho \sin\phi \cos\theta$

$y = \rho \sin\phi \sin\theta$

$z = \rho \cos\phi$

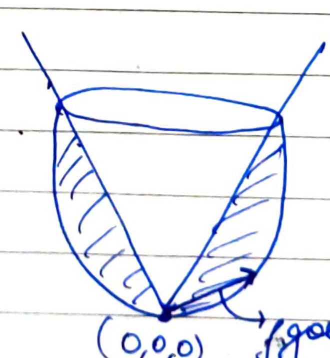
$x^2 + y^2 + z^2 = \rho^2$

$\Rightarrow x^2 + y^2 + z^2 = 4\cos^2\phi$

$\Rightarrow x^2 + y^2 + z^2 = 2z$

ie we have a sphere with centre at $(0,0,1)$

$$\therefore V = \int_{\theta=0}^{2\pi} \int_{\phi=\pi/4}^{\pi/2} \int_{\rho=0}^{2\cos\phi} \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$$



ρ goes from zero to sphere (ie $\rho = 0$ to $\rho = 2\cos\phi$)

Q→ The solid bounded below by the xy-plane, on the sides by the sphere $\rho=2$ and above by cone $\phi = \pi/3$

Ans:

$$V = \int_{\theta=0}^{2\pi} \int_{\phi=\pi/4}^{\pi/2} \int_{\rho=0}^{2} \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$$

Q→ Find volume of region bounded ~~below~~ below and above by cones

$z = \sqrt{x^2+y^2}$ & $z = 2\sqrt{x^2+y^2}$ and laterally by cylinders $x^2+y^2=1$ and $x^2+y^2=16$

Ans: Using cylindrical system

$z = \sqrt{x^2+y^2} = r$

to

$z = 2\sqrt{x^2+y^2} = 2r$

$x^2+y^2 = 1 \Rightarrow r=1$

$x^2+y^2 = 16 \Rightarrow r=4$

$$\therefore V = \int_{\theta=0}^{2\pi} \int_{r=1}^{4} \int_{z=r}^{2r} r \, dz \, dr \, d\theta$$

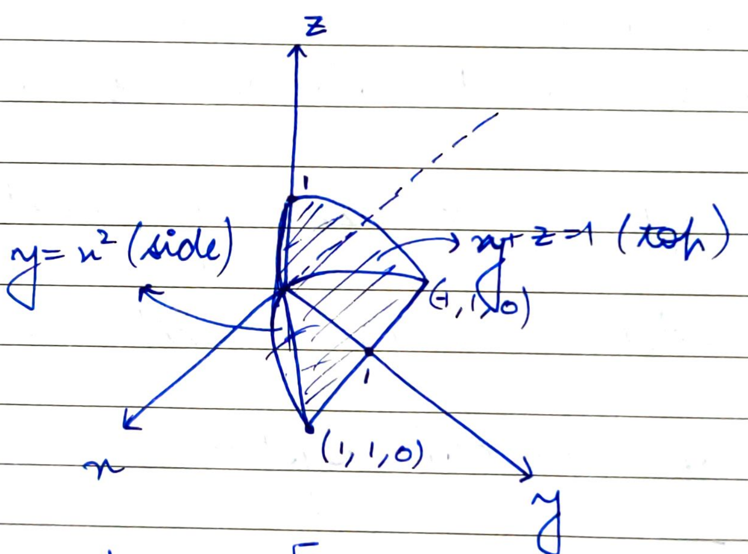
Q→ Find the volume which lies b/w 2 spheres $x^2+y^2+z^2=4$ & $x^2+y^2+z^2=16$ & above x-y plane

Ans.

$$V=\int_{\theta=0}^{2\pi}\int_{\phi=0}^{\pi/2}\int_{\rho=2}^{4}\rho^2\sin\phi\,d\rho\,d\phi\,d\theta$$

Q→ Find volume for region bounded by $y=x^2$, xy plane and $y+z=1$

Ans:



$$V=\int_{0}^{1}\int_{y=0}^{1-z}\int_{x=-\sqrt{y}}^{\sqrt{y}}dx\,dy\,dz$$

$$V=\int_{z=0}^{1}\int_{x=-\sqrt{(z-1)(-1)}}^{\sqrt{(z-1)}}\int_{y=x^2}^{1-z}dy\,dx\,dz$$

$$V = \int_{-1}^{1} \int_{x^2}^{1} \int_{0}^{1-y} dz\, dy\, dx$$

Q. Find volume in cylindrical system for a cylinder (right circular) bounded below by xy plane & above by $4-y=z$ and with eqn $r=2\sin\theta$

Ans:

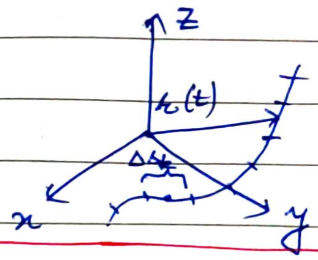
$y = r\sin\theta$

$$V = \int_{\theta=0}^{\pi} \int_{r=0}^{2\sin\theta} \int_{z=0}^{4-r\sin\theta} dz\, r\,dr\, d\theta$$

Line Integrals →

- Suppose that $f(x, y, z)$ is a real valued f^n we wish to integrate over the curve

$C: \vec{r}(t) = g(t)\hat{i} + h(t)\hat{j} + k(t)\hat{k}$; $a \le t \le b$
lying within domain of f.



$$S_n = \sum_{k=1}^{n} f(x_k, y_k, z_k)\, \Delta s_k$$

$$\boxed{\lim_{n \to \infty} S_n = \int_C f(x, y, z)\, ds = \int_a^b f(x(t), y(t), z(t))\, |v(t)|\, dt}$$

where $|v(t)| = \dfrac{ds}{dt}$

- Line integrals have the useful property that if a curve C is made by joining a finite number of curves $C_1, C_2, \ldots, C_n$ end to end then the integral of a function over C is the sum of the integrals over the curves that make it up

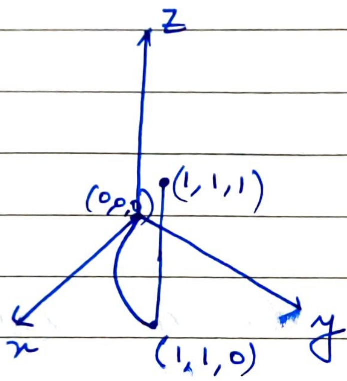
$$\int_C f\, ds = \int_{C_1} f\, ds + \int_{C_2} f\, ds + \ldots + \int_{C_n} f\, ds$$

Q→ Integrate $f(x,y,z) = x + \sqrt{y} - z^2$

over the path (as shown) from $(0,0,0)$ to $(1,1,1)$ given by

$$C_1 \equiv r(t) = t\hat{i} + t^2\hat{j} \quad : 0 \le t \le 1$$

$$C_2 \equiv r(t) = \hat{i} + \hat{j} + t\hat{k} \quad : 0 \le t \le 1$$



Ans $$\int_{C_1} f(x,y,z)\, ds = \int_0^1 (t + \sqrt{t^2} - 0)\sqrt{1+4t^2}\, dt$$

For C_1: $\vec{v} = \dfrac{d\vec{r}}{dt} = 1\hat{i} + 2\hat{j}$

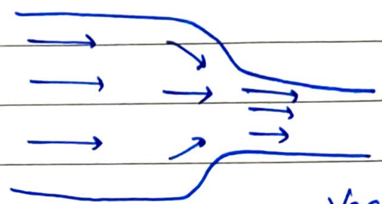
$\dfrac{ds}{dt} = |\vec{v}|$

$$+ \int_{C_2} \left(1 + \sqrt{1} - t^2\right)\sqrt{1}\, dt$$

For C_2: $\dfrac{d\vec{r}}{dt} = 1\hat{k}$

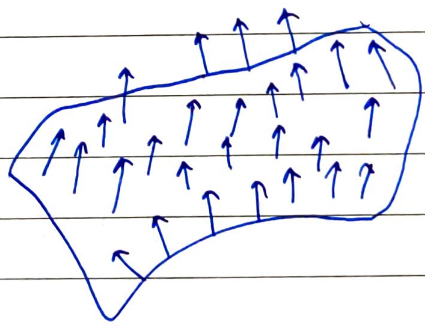
→ VECTOR FIELDS →

- It is a field representing a collection of vectors representing a quantity over a region.
- eg- fluid flow in a tube



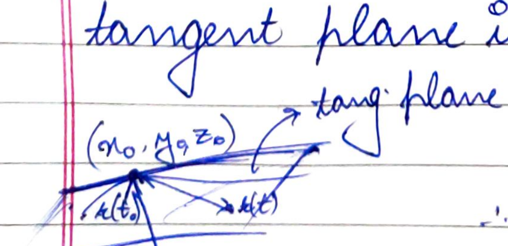
Vector field

- eg- gradient vectors along a surface



∇f or gradient f is always orthogonal to level surface of 'f'

We can use this to write equation of tangent plane in vector form



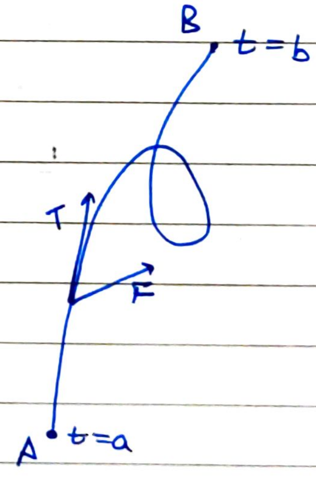
$\therefore$ tangent eq^n

$$\boxed{(r(t) - r(t_0)) \cdot \vec{\nabla f}\Big|_{t_0} = 0}$$

WORK DONE

- Suppose that the vector field

$F = M(x,y,z)\hat{i} + N(x,y,z)\hat{j} + P(x,y,z)\hat{k}$



represents a force throughout a region in space (it might be the force of gravity or an electromagnetic force of some kind) and that $r(t)$

$$\boxed{W = \int_{t=a}^{b} \vec{F}\cdot\vec{T}\,ds}$$

where $\vec{T}$ is unit tangent vector at the point.

$\vec{T} = \left(d\vec{r}/ds \right) / |d\vec{r}/ds|$

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- Other ways to write work integral

$$W = \int_a^b F \cdot T\, ds$$

$$= \int_a^b \vec{F} \cdot d\vec{r}$$

$$= \int_a^b F \cdot \frac{dr}{dt}\, dt$$

$$= \int_a^b \left(M \frac{dx}{dt} + N \frac{dy}{dt} + P \frac{dz}{dt}\right) dt$$

$$= \int_a^b (M\,dx + N\,dy + P\,dz)$$

$$\therefore \boxed{W = \int_a^b (M\,dx + N\,dy + P\,dz)}$$

Q: Find workdone by $F = xy\hat{i} + yz\hat{j} + zx\hat{k}$ from (0, 0, 0) to (1, 1, 1) over each of the paths.

a) Straight line path $C_1 : r(t) = t\hat{i} + t\hat{j} + t\hat{k}$, $0 \le t \le 1$

Ans:

$$W = \int \cancel{xy\hat{i}}\ F \cdot \left(\frac{dr}{dt}\right) dt$$

$$\Rightarrow W = \int_0^1 (t^2\hat{i} + t^2\hat{j} + t^2\hat{k}) \cdot (\hat{i} + \hat{j} + \hat{k})\ dt$$

$$\Rightarrow W = \int_0^1 3t^2\ dt$$

$$\Rightarrow W = 1N$$

b) Curved path: $C_2 : r(t) = t\hat{i} + t^2\hat{j} + t^4\hat{k} ; 0 \le t \le 1$

Ans: $$W = \int_0^1 (t^3\hat{i} + t^6\hat{j} + t^5\hat{k}) \cdot (\hat{i} + 2t\hat{j} + 4t^3\hat{k})\ dt$$

$$= \int_0^1 (t^3 + 2t^7 + 4t^8)\ dt$$

$$= \left(\frac{1}{4} + \frac{1}{4} + \frac{4}{9}\right)$$

c) The path $C_3 \cup C_4$ consisting of line

FLOW INTEGRALS

- Instead of being a force field, suppose that F represents the velocity field of a fluid flowing through a region in space (a tidal basin). Under these circumstances, the integral F.T along a curve in the region gives the fluids flow along the curve.

- If r(t) is a smooth curve in the domain of a continuous velocity field F, the flow along the curve from t=a to t= is

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$$\boxed{\text{Flow} = \int_a^b F \cdot T\, ds}$$

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The integral in this case is called flow integral. If the curve is a closed loop, the flow ~~case~~ is called the circulation around the curve.

Q→ Find the circulation of the field

$F = -y\hat{i} + x\hat{j}$ around and across the closed semicircular path that consists of the semicircular path that consists of the semicircular arch

$r_1(t) = (a\cos(t))\,\hat{i} + (a\sin t)\,\hat{j}$, $0 \le t \le \pi$
followed by line segment.
$r_2(t) = t\,\hat{i}$, $-a \le t \le a$

Ans: circulation $= \int_{C_1} \vec{F} \cdot \vec{T}\, ds$

$$= \int_{t=0}^{\pi} (-a\sin t\,\hat{i} + a\cos t\,\hat{j}) \cdot (-a\sin t\,\hat{i} + a\cos t\,\hat{j})\, dt$$

$$+ \int_{t=-a}^{a} (0\hat{i} + t\hat{j}) \cdot \hat{i}\, dt$$

$$= \pi a^2$$

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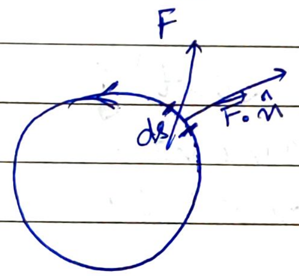
→ **FLUX** →

- ~~The~~ To find the rate at which a fluid is entering or leaving a region enclosed by a smooth curve C in the xy plane, we calculate the line integral over C of the scalar component of the fluid's velocity field in direction of curve's outward pointing unit normal

- C = smooth closed curve
  F = $M(x,y)\hat{i} + N(x,y)\hat{j}$ → continuous vector field
  n = outward pointing unit normal ~~field~~ vector on the curve

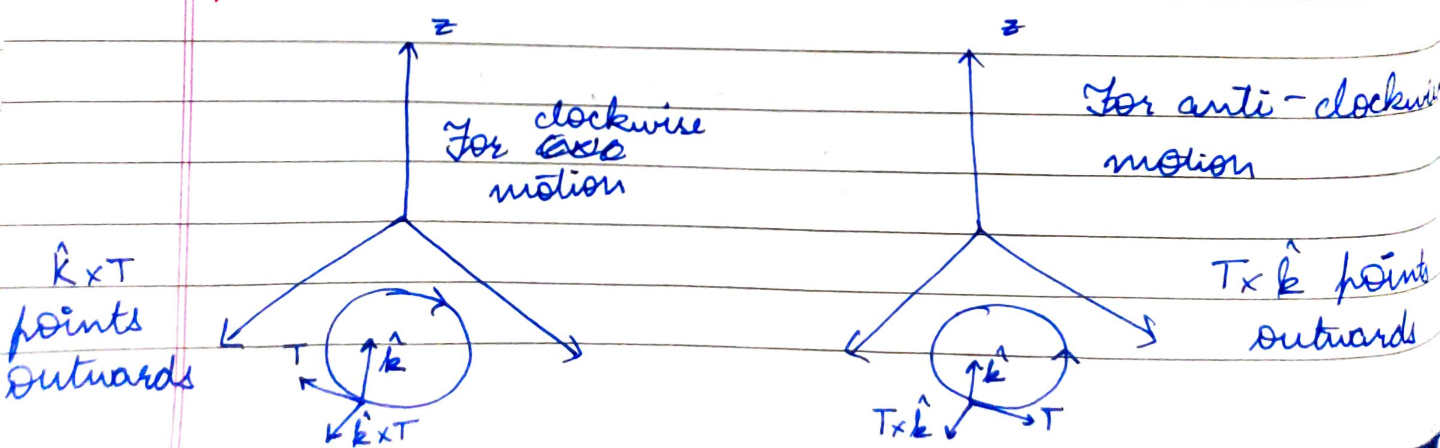
$\therefore$ flux of F across C $= \int_C \vec{F}\cdot\hat{n}\,ds$

- 

Flux $= \int_C \vec{F}\cdot\hat{n}\,ds$

NOTE: Many flux calculations involve no motion at all.

- Calculating flux across a smooth closed plane curve



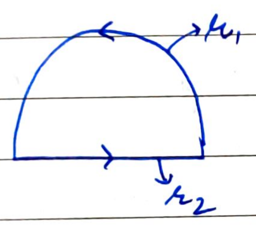
$$\hat{n} = \hat{k} \times \hat{T} = \hat{k} \times \left(\frac{dx}{ds}\hat{i} + \frac{dy}{ds}\hat{j}\right)$$

$$\Rightarrow \hat{n} = \frac{dx}{ds}\hat{j} - \frac{dy}{ds}\hat{i}$$

$$\text{Flux} = \int_C (M\hat{i} + N\hat{j}) \cdot \left(\frac{dx}{ds}\hat{j} - \frac{dy}{ds}\hat{i}\right) ds$$

$$= -\int_C (M\,dy - N\,dx)$$ For clockwise

Q→ Calculate flux around



$r_1(t) = a(\cos t)\hat{i} + a(\sin t)\hat{j}$, $t \in [0, \pi]$

$r_2(t) = t\hat{i}$, $t \in [0, 1]$

$\vec{F} = -y\hat{i} + x\hat{j}$

Ans. For r_1,

$\hat{n} = \hat{T} \times \hat{k}$

$$\hat{n} = \frac{dr_1}{dt} \Big/ \left|\frac{dr_1}{dt}\right| \times \hat{k} = (\sin t\,\hat{i} + \cos t\,\hat{j}) \times \hat{k}$$

$$= (\sin t)\hat{j} + (\cos t)\hat{i}$$

For r_2

$$\hat{n} = \hat{T} \times \hat{k} = \hat{i} \times \hat{k} = -\hat{j}$$

$$\therefore \text{Flux} = \int_0^{\pi} (\sin t\,\hat{j} + \cos t\,\hat{i}) \cdot (-y\hat{i} + x\hat{j})\, d$$

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Ans: $\hat{n} = \hat{T} \times \hat{k}$

$= \dfrac{dr_1}{ds} \Big/ \left|\dfrac{dr_1}{ds}\right|$

$= (-\sin t\,\hat{i} + \cos t\,\hat{j})\dfrac{dt}{ds}$

$\hat{n}$ for $r_2 = \hat{T} \times \hat{k} = -\hat{j}$

$\therefore$ Flux $= \displaystyle\int_0^{\pi} (-\sin t\,\hat{i} + \cos t\,\hat{j})\frac{dt}{\cancel{ds}} \cdot (-y\hat{i} + x\hat{j})\,\cancel{ds}$

$+ \displaystyle\int_0^{1} (-\hat{j}) \cdot (\cancel{-} a\sin t\,\cancel{\hat{i}} + a\cos t\,\hat{j})\,ds$

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Path Independence →

- It denotes independence of an integral b/w two points irrespective of the path taken b/w them.

→ Conservative Field

- If a field is defined over an open region D and we have path independence over any 2 pts. A and B in D, then, we say that the field is a conservative field.

→ Potential Function

- If we have a conservative field in region D which can be written as

$$\boxed{F = \nabla f}$$

Then in that case, 'f' is called the potential f^n of the field F.

→ Analogue of Fundamental Theorem of Calculus in Higher Dimension

- In 1-D, we have $\int_a^b f'(x)\,dx = f(b) - f(a)$

- If f is potential f^n of F then we can evaluate work integral over any path b/w A and B by

$$\int_A^B \vec{F}\cdot dr = \int_A^B \nabla f \cdot dr = f(B) - f(A)$$

THEOREM 1 : Fundamental Theorem of Line Integrals

- Let $F = M(x,y,z)\hat{i} + N(x,y,z)\hat{j} + P(x,y,z)\hat{k}$

be a vector field whose components are continuous throughout an open connected region D in space. Then there exist a diff. f^n 'f' such that

$$\boxed{F = \nabla f = \frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j} + \frac{\partial f}{\partial z}\hat{k}}$$

if and only if for all points A and B in D the value of $\int_A^B F \cdot dr$ is independent of the path joining A to B in D

- If the integral is indep. of the path from A to B, its value is

$$\int_A^B F \cdot dr = f(B) - f(A)$$

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→ THEOREM 2:

- The following statements are equivalent

a) $\int F \cdot dr = 0$ in a closed loop.
b) F is a conservative field.

From Theorem 1 and 2

- $$F = \nabla f \text{ on } D$$

$$\Updownarrow$$

$$F \text{ is conservative on } D$$

$$\Updownarrow$$

$$\oint F \cdot dr = 0 \text{ on any closed path in } D$$

→ TEST FOR CONSERVATIVE FIELD

- We keep in mind the assumption that domain of F is connected & simply connected.

- Let $F = M(x,y,z)\,\hat{i} + N(x,y,z)\,\hat{j} + P(x,y,z)\,\hat{k}$ be a field whose component functions have continuous first partial derivatives. Then F is conservative if and only if.

$$\frac{\partial P}{\partial y} = \frac{\partial N}{\partial z}, \quad \frac{\partial M}{\partial z} = \frac{\partial P}{\partial x}, \quad \frac{\partial N}{\partial x} = \frac{\partial M}{\partial y}$$

PROOF: F is conservative

$$\Rightarrow \vec{F} = \nabla f$$

$$\Rightarrow M\hat{i} + N\hat{j} + P\hat{k} = \frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j} + \frac{\partial f}{\partial z}\hat{k}$$

$$\Rightarrow M = \frac{\partial f}{\partial x}, \quad N = \frac{\partial f}{\partial y}, \quad P = \frac{\partial f}{\partial z}$$

now, $\frac{\partial M}{\partial y} = \frac{\partial^2 f}{\partial y \partial x}$ & $\frac{\partial N}{\partial x} = \frac{\partial^2 f}{\partial x \partial y}$

$\therefore \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ & similarly for other 2 relations.

Hence proved.

\# (We have a second part of proof where we need domain of F connected. We will do it later)

→ FINDING A POTENTIAL FUNCTION →

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~~imp~~ Q→ Verify if pot. f^n exists & then find it for

$$F = (2xy - z^2)\hat{i} + (x^2 + 2yz)\hat{j} + (y^2 - 2zx)\hat{k}$$

Ans: $M = (2xy - z^2)$
$N = (x^2 + 2yz)$
$P = (y^2 - 2zx)$

$\frac{\partial M}{\partial y} = 2x$; $\frac{\partial N}{\partial x} = 2x$; $\frac{\partial M}{\partial z} = -2z$

$\frac{\partial P}{\partial x} = -2z$; $\frac{\partial N}{\partial x} = 2x$; $\frac{\partial M}{\partial y} = 2x$

Hence F is conservative and potential f^n exists.

$\nabla f = \vec{F}$

$$\frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j} + \frac{\partial f}{\partial z}\hat{k} = M\hat{i} + N\hat{j} + P\hat{k}$$

$\frac{\partial f}{\partial x} = M$; $\frac{\partial f}{\partial y} = N$; $\frac{\partial f}{\partial z} = P$

$\frac{\partial f}{\partial x} = 2xy - z^2$

Integrating wrt. x

$$f(x,y,z) = x^2y - z^2x + h(y,z)$$

partial
on, diff this
wrt x,
we get
zero

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$\cancel{\frac{\partial f}{\partial y}}$ Taking f from previous step

$$\frac{\partial f}{\partial y} = \cancel{x^2} + \frac{\partial h}{\partial y} = \cancel{x^2} + 2yz$$

$$\Rightarrow \frac{\partial h}{\partial y} = 2yz$$

Integrating w.r.t. y

$$\Rightarrow h(y,z) = y^2 z + g(z)$$

Taking h from previous step

$$f(x,y,z) = x^2 y - z^2 x + y^2 z + g(z)$$

Par. Diff w.r.t. z

$$\frac{\partial f}{\partial z} = -2zx + y^2 + g'(z) = P$$

$$\Rightarrow \cancel{-2zx} + \cancel{y^2} + g'(z) = \cancel{y^2} - \cancel{2xz}$$

$$g'(z) = 0$$

$$\Rightarrow g(z) = c$$

$$\therefore f(x,y,z) = x^2 y - z^2 x + y^2 z + c$$

Exact Differential Forms →

- **Definition:**

Any expression $M(x,y,z)\,dx + N(x,y,z)\,dy + P(x,y,z)\,dz$ is a differential form.
A differential form is exact on a domain D in space if $M(x,y,z)\,dx + N(x,y,z)\,dy + P(x,y,z)\,dz = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz$

$$= df$$

for some scalar f^n 'f' throughout D.

- **Test for exactness:**

The differential form is exact if and only if

$$M = \frac{\partial f}{\partial x} \; ; \; N = \frac{\partial f}{\partial y} \; ; \; P = \frac{\partial f}{\partial z}$$

This is equivalent to saying that field

$F = M\hat{i} + N\hat{j} + P\hat{k}$ is conservative

- This is helpful in calculating integrals like

$$\int_A^B \left(M(x,y,z)\,dx + N(x,y,z)\,dy + P(x,y,z)\,dz\right)$$

$$= \int_A^B df = f(B) - f(A)$$

simply.

Q→ Find a potential f^n and evaluate the integral

$$\int_{(0,2,1)}^{(1,\pi/2,2)} \left(2\cos y\, dx + \left(\frac{1}{y} - 2x\sin y\right) dy + \frac{1}{z} dz\right)$$

Ans: $M = 2\cos y$; $P = \frac{1}{z}$

$N = \frac{1}{y} - 2x\sin y$

$\frac{\partial f}{\partial x} = 2\cos y$

$\Rightarrow f(x,y,z) = 2x\cos y + h(y,z)$

$\frac{\partial f}{\partial y} = \cancel{-2x\sin y} + \frac{\partial h}{\partial y} = \frac{1}{y} \cancel{- 2x\sin y}$

$\Rightarrow \frac{\partial h}{\partial y} = \frac{1}{y}$

$\Rightarrow h(y,z) = \ln(y) + g(z)$

$\therefore f = 2x\cos y + \ln y + g(z)$

$\frac{\partial f}{\partial z} = g'(z) = \frac{1}{z}$

$\Rightarrow g(z) = \ln z + c$

$\therefore f(x,y,z) = 2x\cos y + \ln y + \ln z + c$

$$\therefore \quad I = \int_{(0,2,1)}^{(1,\frac{\pi}{2},2)} d(2x\cos y + \ln y + \ln z + c)$$

$\Rightarrow I = \ln \pi/2$