SERIES →

- An infinite series is the sum of an infinite sequence of numbers

$$a_1 + a_2 + a_3 + \cdots + a_n + \cdots$$

- Aim: To understand meaning of such an ∞ sum & to develop methods to calculate.

- Th sum of first n terms

$$s_n = a_1 + a_2 + \cdots + a_n$$

is an ordinary finite sum & can be calculated by normal addition. It is called n^{th} partial sum.

→ COMBINING SERIES

THEOREM 1 If Σa_n & Σb_n are convergent, then

i) Sum rule: $\Sigma (a_n + b_n) = \Sigma a_n + \Sigma b_n = A + B$

ii) Difference rule: $\Sigma (a_n - b_n) = \Sigma a_n - \Sigma b_n = A - B$

iii) Constant multiple Rule: $\Sigma k a_n = k \Sigma a_n = kA$

- ## Adding or Deleting Terms

We can add a finite no. of terms to a series or delete a finite no of terms without altering the series' convergence or divergence although in the case of convergence this will usually change the sum.

- ## Reindexing

As long as we preserve the order of its terms, we can reindex any series without altering its convergence. To raise the starting value of the index h units, replace the n in formula for a_n by $n-h$

$$\sum_{n=1}^{\infty} a_n = \sum_{n=2}^{\infty} a_{n-1}$$

NOTE: $1+\frac{1}{2}+\left(\frac{1}{4}+\frac{1}{4}\right)+\left(\frac{1}{8}+\frac{1}{8}+\frac{1}{8}+\frac{1}{8}\right)+\dots < 1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\frac{1}{6}+\dots$

$\Rightarrow 1+\frac{1}{2}+\frac{1}{2}+\frac{1}{2}\dots\dots < 1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\dots$

THEOREM 2:

- If $\sum_{n=1}^{\infty} a_n$ converges, then $\underbrace{\lim_{n\to\infty} a_n = 0}_{\text{necessary condition}}$

- converse is not true, ie if $\lim_{n\to\infty} a_n = 0$ then $\sum_{n=1}^{\infty}$ is not necessarily convergent.

  eg = $\sum 1/n$

- If $\lim_{n\to\infty} a_n \neq 0$, then we can directly conclude that $\sum_{n=1}^{\infty} a_n$ is divergent.

THEOREM 3:

- To show if $\sum a_n$ converges, we have to prove only that the partial sums of series ~~a~~ is bounded above.

- Corollary: A series $\sum_{n=1}^{\infty} a_n$ of non-negative terms is convergent if simply it is bounded above.

  Just prove that the sum is bounded & that is it for it being convergent.

→ To Find if $\sum a_n$ is Convergent

Given $\sum a_n$

$\Downarrow$

compute $\lim_{n\to\infty} a_n$

If $\lim_{n\to\infty} a_n \neq 0$ or doesn't exist

$\Downarrow$

$\sum a_n$ is divergent

If $\lim_{n\to\infty} a_n = 0$

$\Downarrow$

No conclusion can be drawn & further test required.

$\Downarrow$

For that we assume $a_n \geq 0$ $\therefore$ $\{s_n\}$ is non decreasing & as soon as $\{s_n\}$ is bounded above, we are done

$\Downarrow$

As in that case $\lim_{n\to\infty} s_n$ = Finite = Sum of series

$\Rightarrow \sum a_n$ is convergent

imp Improper Integration

$\int_a^\infty f(x)\,dx$ = Finite then integral = convergent (for $a \geq 0$)

$\int_0^\infty f(x)\,dx = \infty$ or $-\infty$ or d.n.e. then integral = divergent

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Date ______

Page ______

TUTORIAL 2

Q.

i) $\lim (2^n + 3^n)^{1/n} = 3$

$\Rightarrow (3^n)^{1/n} \leq (2^n + 3^n)^{1/n} \leq (3^n + 3^n)^{1/n}$

$\Rightarrow 3 \leq (2^n + 3^n)^{1/n} \leq (2 \cdot 3^n)^{1/n}$

$\Rightarrow 3 \leq (2^n + 3^n)^{1/n} \leq 2^{1/n} \cdot 3$

$\Rightarrow 3 \leq \lim (2^n + 3^n)^{1/n} \leq \lim (2^{1/n} 3)$

By Sandwich Theorem

$\lim (2^n + 3^n)^{1/n} = 3$

iii) $\lim \left(\frac{1}{n^2+1} + \frac{2}{n^2+2} + \cdots + \frac{n}{n^2+n} \right) = 1/2$

Let

$$f(n) = \frac{1}{n^2+1} + \frac{2}{2+n^2} + \cdots + \frac{n}{n+n^2}$$

$$\Rightarrow \frac{1}{n^2+n} + \cdots + \frac{n}{n^2+n} \leq f(n) \leq \frac{1}{n^2+1} + \frac{2}{n^2+1} + \cdots + \frac{n}{n^2+1}$$

$\left(\because n^2+n > n^2+1 \right.$

$\left. \Rightarrow \frac{1}{n^2+n} < \frac{1}{n^2+1} \right)$

$$\Rightarrow \frac{n(n+1)}{2(n^2+n)} \leq f(n) \leq \frac{n(n+1)}{2(n^2+1)}$$

$$\Rightarrow \frac{1}{2} \leq f(n) \leq \frac{n^2+n}{\ldots}$$

Taking $\lim_{n\to\infty}$ and using sandwich theorem

$$\lim_{n\to\infty} f(n) = 1/2$$

ii) $$\lim \frac{1\cdot 3\cdot 5 \cdots\cdots (2n-1)}{2\cdot 4\cdot 6 \cdots 2n} = 0$$

Here $$x_n = \frac{1\cdot 3 \cdots\cdots (2n-1)}{2^n(1\cdot 2 \cdots n)} \frac{\times 2\cdot 4\cdot 6 \cdots 2n}{\times 2\cdot 4\cdot 6 \cdots 2n} = \frac{(2n)!}{2^{2n}(n!)^2}$$

Then we will prove by induction that

$$0 < x_n < \frac{1}{\sqrt{n+1}}$$ iv)

For $n=1$,

$$x_1 = \frac{2!}{2^2(1!)} = \frac{1}{2} \quad \& \quad 0 < 1/2 < \frac{1}{\sqrt{2}}$$

Let this be true for $n = m$ ie

$$\frac{(2m)!}{2^{2m}(m!)^2} < \frac{1}{\sqrt{m+1}}$$

now for ~~m~~ $n = m+1$, we have

$$x_{m+1} = \frac{(2(m+1)!)}{2^{2(m+1)}((m+1)!)^2}$$

$$= \frac{(2m+1)(m+1)}{2(m+1)^2} x_m < \frac{2m+1}{2(m+1)^2} \frac{1}{\sqrt{m+1}}$$

...ich theorem

$$=\frac{(2n)!}{2^{2n}(n!)^2}$$

... that

$$\frac{n+1}{\ldots 1)^2} \quad \frac{1}{\sqrt{n+1}}$$

$$\frac{2n+1}{2(n+1)^2} < \sqrt{\frac{n+1}{n+2}}$$

$$x_{n+1} < \frac{1}{\sqrt{n+1}} \sqrt{\frac{n+1}{n+2}} = \sqrt{\frac{1}{n+2}}$$

Hence it is true for all n

$$0 < x_{n+1} < \frac{1}{\sqrt{n+2}}$$

So, taking limit, $x_n = 0$

iv) $\lim (\sqrt{2} - 2^{1/3})(\sqrt{2} - 2^{1/5}) \ldots\ldots (\sqrt{2} - 2^{1/(2n+1)}) = 0$

Let $x_n = (\sqrt{2} - 2^{1/3}) \ldots\ldots (\sqrt{2} - 2^{1/(2n+1)})$

Since $2^{1/(2n+1)} > 1 \Rightarrow -2^{1/(2n+1)} < -1$

$$0 < x_n < (\sqrt{2} - 1)^n$$

$$0 < x_n < \frac{(\sqrt{2}-1)^n (\sqrt{2}+1)^n}{(\sqrt{2}+1)^n} = \frac{1}{(\sqrt{2}+1)^n}$$

Taking limit $x_n = 0$

10) b) $x_1 = 1$ & $x_{n+1} = \sqrt{2 + x_n}$

Note that $x_1 < 2$ now, assume that $x_n < 2$. Then $x_{n+1} = \sqrt{2 + x_n}$

$$< \sqrt{2+2} < 2$$

Hence by induction $x_n < 2$ for all $n \in \mathbb{N}$

now we will verify that $x_{n+1} \geq x_n$

$\Rightarrow \sqrt{2 + x_n} > x_n$

squaring, $2 + x_n > x_n^2$

$\Rightarrow x_n^2 - x_n < 2$

$\Rightarrow x_n(x_n - 1) < 2$

This is true since $(x_n - 1) < 1$ & $x_n < 2$

$\therefore$ x_n is increasing sequence &

$1 \leq x_n < 2$. So, its also bounded

By monotone convergent theorem, it also converges.

Let $\lim x_n = l$

Passing limit both sides

d)

$$l = \sqrt{2+l}$$

$$\Rightarrow l^2 = l+2$$

$$\Rightarrow l^2 - l - 2 = 0$$

$$\Rightarrow (l+1)(l-2) = 0$$

$$\Rightarrow l = -1, 2 \ \cancel{\text{sin}}$$

since $x_n \geq 1$

$$\Rightarrow \underline{\underline{l = 2}}$$

d) $x_n = 1 + \frac{1}{1!} + \frac{1}{2!} + \dots + \frac{1}{n!}$

$$x_{n+1} - x_n = \frac{1}{(n+1)!} > 0$$

$$\Rightarrow x_{n+1} > x_n \quad \forall n \in \mathbb{N}$$

$\therefore$ its monotonic & increasing

from given sequence, we see that

$$x_n \geq 1 \quad \forall n \in \mathbb{N}$$

$$x_n = 1 + \frac{1}{1!} + \frac{1}{2!} + \dots + \frac{1}{n!} \leq 1 + 1 + \frac{1}{2} + \frac{1}{2^2} + \dots + \frac{1}{2^{n-1}}$$

(Prove it by induction)

$\Rightarrow 1 < \lim_{n\to\infty} x_n < 3$ (bounded)

So,
If a sequence is increasing & has an upper bound then it converges to its supremum.

Just prove $\sup x_n = 3$

15. $x_n = \sqrt{n}$

$\lim_{n\to\infty} |x_{n+1} - x_n|$

$\Rightarrow \lim_{n\to\infty} |\sqrt{n+1} - \sqrt{n}|$

$= \lim_{n\to\infty} \frac{(\sqrt{n+1}-\sqrt{n})(\sqrt{n+1}+\sqrt{n})}{(\sqrt{n}+\sqrt{n+1})}$

$= \lim_{n\to\infty} \frac{1}{\sqrt{n}+\sqrt{n+1}}$

$=0$

Hence proved

Now

to prove $x_n = \sqrt{n}$ is not Cauchy

Let $|x_m - x_n| < \varepsilon \ \forall \ m, n \geq n_0$

18.

Let

$m := (k+x_0)^2 \Rightarrow x_m = \sqrt{(k+x_0)^2} = k+x_0$

$\& \ n = x_0^2 \Rightarrow x_n = x_0$

Then $|x_m - x_n| = k$ $\quad \& \ k \in \mathbb{Z}$

18. $x^3 - 5x + 1 = 0$

has root r where $0 < r < 1$

$r^3 - 5r + 1 = 0$

$\Rightarrow 5r = r^3 + 1$

$\Rightarrow r = \dfrac{r^3+1}{5}$

$\Rightarrow x_n = \dfrac{x_{n-1}^3 + 1}{5}$

Prove this is contractive.

Take $r_0 = 0.5$

Q17. If $x_1 = 2$ & $x_{n+1} = 2 + \frac{1}{x_n}$ for $n \geq 1$. Show $\{x_n\}$ is contractive sequence. Find limit

Ans: $x_1 = 2$

$$x_{n+1} = 2 + \frac{1}{x_n} \quad \forall n \geq 1$$

Claim: x_n is contractive sequence.

$$|x_{n+1} - x_n| \leq c\,|x_n - x_{n-1}|$$

where $0 < c \leq 1$

Now,

$$|x_{n+1} - x_n| = \left|2 + \frac{1}{x_n} - x_n\right|$$

$$= \left|2 + \frac{1}{x_n} - \left(2 + \frac{1}{x_{n-1}}\right)\right|$$

$$= \left|\frac{1}{x_n} - \frac{1}{x_{n-1}}\right|$$

Series (contd.) . . .

→ CAUCHY CRITERION

- $\sum a_n$ satisfies cauchy's criterion if its sequence $\{s_n\}$ of partial sums is a cauchy sequence.

ie $\quad |s_n - s_m| < \varepsilon \qquad \forall\ m, n > N$

now

$$|s_n - s_{m-1}| < \varepsilon \qquad \forall\ (m-1), n > N$$

$$\Rightarrow \left|\sum_{m}^{n} a_k\right| < \varepsilon$$

$$\therefore \boxed{\left|\sum_{m}^{n} a_k\right| < \varepsilon \qquad \forall\ (m, n) > N}$$

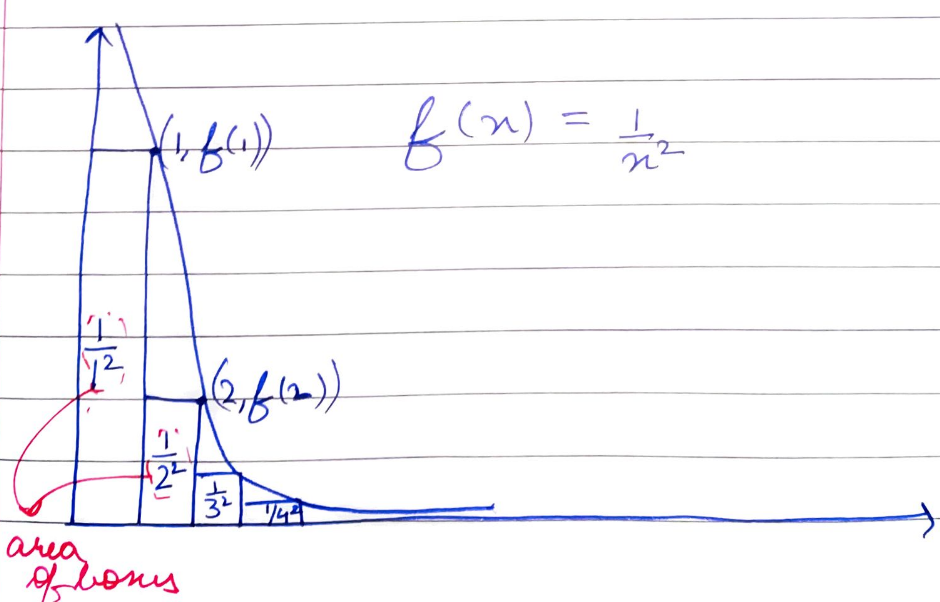
→ INTEGRAL TEST THEOREM →

Q→ Does the following series converge?

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \cdots\cdots + \frac{1}{n^2} + \cdots$$

Ans:

$f(x) = \frac{1}{x^2}$



classmate

Date

Page

now area of boxes is < area of integral

Carefully observe

$$\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\dots+\frac{1}{n^2}+\dots < 1+\int_1^\infty \frac{dx}{x^2}$$

$$\Rightarrow \sum_{n=1}^{\infty} \frac{1}{n^2} < 1+1$$

$$\Rightarrow \sum_{n=1}^{\infty} \frac{1}{n^2} < 2$$

$\therefore$ The series is bounded $\therefore$ It is clearly convergent.

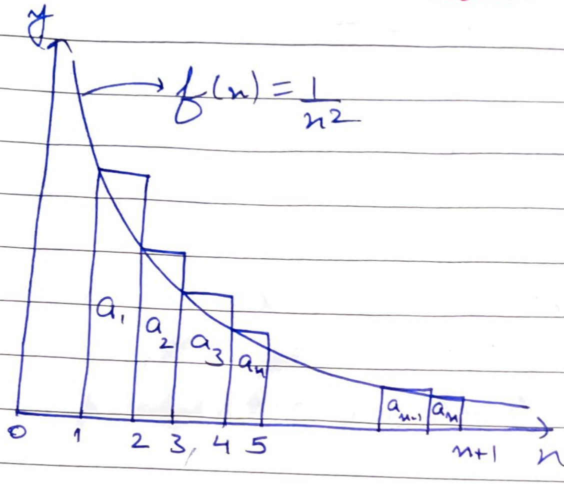
NOTE: However, the sum of $\frac{1}{n^2}$ is not 2. We just know its bounded. But this doesn't mean 2 is its sum.

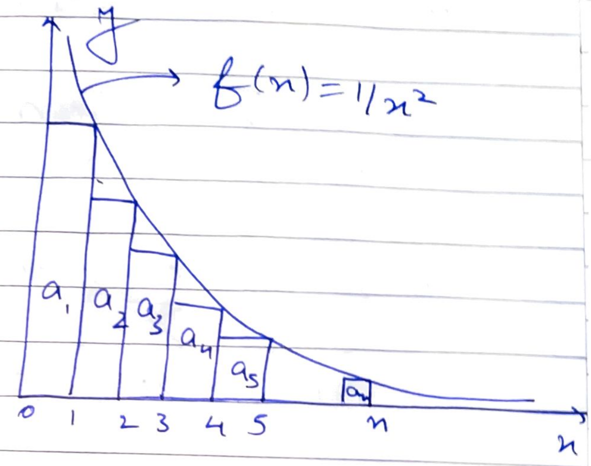
The series & integral need not have the same value in convergent case, eg we compute $\sum_{n=1}^{\infty} \frac{1}{n} < 2$ but $\sum_{n=1}^{\infty} 1/n^2 = \frac{\pi^2}{6}$

PTO

NOTE: If in the integral eqⁿ on right side, if $\int f(x)\,dx$ is not defined ⇒ sum not bounded above ~~cannot~~ But we can't say anything about it ∴ we move to inequal. of left. ~~If defined then it has lower bound~~

Geometrical Interpretation of Integral Test





∴ From above.

$$\boxed{\int_1^{n+1} f(x)\,dx \le a_1 + a_2 + \dots + a_n \le a_1 + \int_1^{n} f(x)\,dx}$$

If this term is not defined, then series is not bounded above

- Theorem:
  - Let $\{a_n\}$ be a sequence of +ve terms
  - Suppose that $a_n = f(n)$ where
  - f is a continuous, positive, decreasing function of x for all $x \ge N$ (N is a +ve integer)
  - Then series

$$\sum_{n=N}^{\infty} a_n$$

classmate

Date ____
Page ____

THEOREM : COMPARISON TEST (DCT)

- Let Σa_n be a series of non-negative terms, then:

a) Σa_n converges if there is a convergent series Σc_n with $a_n \leq c_n$ for all $n > N$, for some integer N

b) Σa_n diverges if there is a div. series Σd_n with $a_n \geq d_n$ for all $n > N$, for some integer N.

THEOREM : LIMIT COMPARISON TEST (LCT)

~~Suppose~~

- This is a comparison test useful for series in which a_n is rational function of n.

- Suppose $a_n > 0$, $b_n > 0$ & $c_n > 0$ for $\forall\ n \geq N$ (N is an integer)

a) If $\lim_{n\to\infty} \frac{a_n}{b_n} = c > 0$ (finite) then Σa_n and Σb_n both converge or both diverge.

b) If $\lim_{n\to\infty} \frac{a_n}{b_n} = 0$ & Σb_n converges, then Σa_n converges

c) If $\lim_{n \to \infty} \frac{a_n}{b_n} = \infty$ & Σb_n diverges, then Σa_n also diverges.

→ THEOREM : RATIO TEST

- Let Σa_n be a series with +ve terms & suppose that

$$\lim_{n \to \infty} \frac{a_{n+1}}{a_n} = \rho$$

Then,

a) the series converges if $\rho < 1$
b) the series diverges if $\rho > 1$
c) the test fails if $\rho = 1$

- Proof based on geometric series.

→ THEOREM: ROOT TEST

- For $a_n \geq 0$ $\forall n \geq N$

suppose $\lim_{n \to \infty} \sqrt[n]{a_n} = \rho$

Then

a) convergent if $\rho < 1$
b) divergent if $\rho > 1$
c) failure if $\rho = 1$

- Again, proof based on geometric series.

THEOREM: INTEGRAL TEST

- Let $\{a_n\}$ be a sequence of +ve terms
- Suppose $a_n = f(n)$ where, f is a continuous, +ve, decreasing function of n for all $n \geq N$ (N a +ve integer)
- Then the series
$$\sum_{n=N}^{\infty} a_n$$
and the integral
$$\int_N^{\infty} f(x)\,dx$$
both converge or both diverge.

eg- $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^p}$ Applying integral test

$a_n = f(n)$

$$f(x) = \frac{1}{x(\ln x)^p}, \; p>0$$

for $x \geq 3$

$$\int_3^{\infty} \frac{dx}{x(\ln x)^p} = \int_{\ln 3}^{\infty} \frac{dt}{t^p} \Bigg] = \begin{cases} <\infty, & p>1 \\ =\infty, & p \leq 1 \end{cases}$$

...ve terms

...s a
...function
...integer)

...ge

...tegral

...$b > 1$
...$b \leq 1$

Q→ Testing conv. & divergence.

2. $$\sum_{n=1}^{\infty} \frac{3}{n+\sqrt{n}}$$

Limit comparison test with $b_n = \frac{1}{n}$

$$\therefore \lim_{n\to\infty} \frac{3n}{n+\sqrt{n}} = 3$$

Now since series $\frac{1}{n}$ is divergent

$\therefore$ a_n must also be divergent

3. $$\sum_{n=1}^{\infty} \frac{1}{n!}$$

Ratio test

$$\lim_{n\to\infty} \frac{n!}{(n+1)!} = \lim_{n\to\infty} \frac{1}{(n+1)} = 0$$

$\therefore$ the series is convergent.

4. $\sum_{n=1}^{\infty} \frac{2n+1}{n^2+3n-2}$

Ans: Ratio test

$$\lim_{n\to\infty} \frac{(2(n)+3)(n^2+3n-2)}{((n+1)^2+3(n+1)-2)(2n+1)}$$

$$= \frac{\cancel{2n^3}}{\cancel{2n^3}} = 1 \quad \text{(fails)}$$

Limit comparison test

$b_n = 1/n$

$$\therefore \lim_{n\to\infty} \frac{(2n+1)n}{n^2+3n-2} = 2$$

$\therefore$ finite limit

$\therefore$ since $1/n$ divergent $\therefore$ given a_n is divergent.

5. $\sum_{n=2}^{\infty} \frac{1}{(\ln n)^2}$

Ans: LCT

$b_n = \frac{1}{n} \rightarrow$ (div. series)

$$\lim_{n\to\infty} \frac{n}{(\ln n)^2} = \lim_{n\to\infty} \frac{n}{2(\ln n)} = \frac{n}{2}$$

$= \infty$

$\therefore$ divergent

~~Root test~~

~~$\lim_{n\to\infty} (\ln n)^{2/n}$~~

6. $\sum_{n=1}^{\infty} \frac{(\ln n)^2}{n^3}$

LCT

$b_n = \frac{1}{n^2}$ (convergent series)

$$\lim_{n\to\infty} \frac{(\ln n)^2}{n} = 2\frac{\ln n}{n} = \frac{2}{n}$$

$= 0$

$\therefore$ convergent

7. $\sum_{n=1}^{\infty} \frac{a^n}{n!}$

Ans: Ratio test

$$\lim_{n\to\infty} \frac{a^{n+1}}{(n+1)!\, a^n} \cancel{n!} = \frac{a}{n+1} = 0$$

$\therefore$ convergent.

8. $\sum_{n=1}^{\infty} \frac{n^{10}}{n!}$

Ratio test

$$\lim_{n\to\infty} \frac{(n+1)^{10}}{(n+1)\; n^{10}} = 0$$

$\therefore$ convergent.

9. $\sum_{n=1}^{\infty} \frac{n^n}{(2^n)^2}$

Ans:

=0

→ ALTERNATING SERIES →

- A series in which terms are alternatively +ve and negative is an alternating series

- eg – a) $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} - \cdots + \frac{(-1)^{n+1}}{n} + \cdots$ Convergent (we'll see)

  b) $-2 + 1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \cdots +$

→ THEOREM: The Alternating Series (Leibnitz's Theorem)

- The series $\sum_{n=1}^{\infty} (-1)^{n+1} u_n = u_1 - u_2 + u_3 - u_4 - \cdots$

converges if all 3 of the following conditions are satisfies

AS1 The u_n's are all +ve

AS2 $u_n \geq u_{n+1}$ for all for $n \geq N$, some integer N

AS3 $u_n \to 0$ ie $\lim_{n \to \infty} u_n = 0$

eg – $\sum (-1)^{n+1} \frac{1}{n}$ is convergent Leibnitz's Theorem

imp NOTE: Important Result →

If odd terms of sequence ie a_{2k+1} & even terms ie a_{2k} converge to same limit, then a_n converges to same limit. PTO

classmate

Date
Page

ie $a_{2k+1} \to L$ $\quad a_{2k} \to L$ then $a_n \to L$

Proof: $|a_{2k} - L| < \varepsilon$

$|a_{2k+1} - L| < \varepsilon$

$\therefore |a_n - L| < \varepsilon$

- PROOF OF LIEBNITZ'S THEOREM

Consider that n is even (ie $n = 2m$)
lets write partial sum

$$s_{2m} = (u_1 - u_2) + (u_3 - u_4) + \text{—} + (u_{2m-1} - u_{2m})$$

using (AS2) we have $s_{2m+2} \geq s_{2m}$

($\{s_{2m}\}$ is non-decreasing sequence)

$$s_{2m} = u_1 - \{(u_2 - u_3) + (u_4 - u_5) + \text{—} + (u_{2m-2} - u_{2m-1})$$

THEOREM : ABSOLUTE CONVERGENCE

- A series $\sum a_n$ converges absolutely (is absolutely convergent) if the corresponding series of absolute values $\sum |a_n|$ converges. eg- $\sum (-1)^{n+1} \frac{1}{n^2}$ is absolutely convergent.

- A series that converges but does not converge absolutely is called conditionally convergent series.

NOTE: If absolute series is conv. then alternating will necessarily be convergent

PROOF:

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} (a_n + |a_n| - |a_n|)$$

$$= \sum_{n=1}^{\infty} (a_n + |a_n|) - \sum_{n=1}^{\infty} |a_n| = S_1 - S_2 \text{ (say)}$$

The series $S_2 = \sum_{n=1}^{\infty} |a_n|$ is given to be convergent. So if we can establish the convergence of $\sum_{n=1}^{\infty} a_n + |a_n|$, we are done.

$$-|a_n| \leq a_n \leq |a_n| \Rightarrow 0 \leq a_n + |a_n| \leq 2$$

Q.

1. $\sum_{n=1}^{\infty} (-1)^{n+1} \left(\frac{n}{10}\right)^{n}$

Ans: Divergent by n^{th} term test ie n^{th} term's limit $\neq 0$

2. $\sum_{n=1}^{\infty} (-1)^{n+1} \left(\frac{10^n}{n^{10}}\right)$

Ans: Divergent by n^{th} term test ie n^{th} term's limit $\neq 0$

3. $\sum_{n=2}^{\infty} (-1)^{n+1} \left(\frac{1}{\ln n}\right)$

Ans: $\lim_{n\to\infty} \frac{1}{\ln n} = 0$

~~limit could~~ $\therefore$ further test required

$u_n \geq 0$

$u_n \geq u_{n+1}$

$\lim_{n\to\infty} u_n \to 0$

$\therefore$ convergent by Alternating ~~co~~ series test.

4. $\sum_{n=1}^{\infty} (-1)^{n+1} \left(\frac{\ln n}{n}\right)$

5. $\sum_{n=1}^{\infty} (-1)^{n} \frac{\sin n}{n^2}$

6. $\sum_{n=1}^{\infty} (-1)^{n} \frac{1}{\ln n^3}$

HW Q → Discuss convergence

a) $\sum a_n$ where $a_{n+1} = \frac{n}{n+1} a_n$, $a_1 = 3$

b) $\sum a_n$ where $a_{n+1} = \frac{n + \ln n}{n+10} a_n$, $a_1 = 1/2$

c) $\sum e^{-\alpha n} n^{\beta}$

Power Series

- A power series about $x=0$ is a series of form

$$\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + \dots + c_n x^n + \dots$$

- A power series about $x=a$ is a series of form

$$\sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1(x-a) + \dots + c_n(x-a)^n + \dots$$

in which the centre a and $c_0 \dots c_n$ are constants.

imp
NOTE: At its centre, power series is always convergent.

- Power series is convergent if its sum is finite

eg- Testing for convergence using ratio test.

1. $$\sum_{n=1}^{\infty} \frac{(-1)^{n-1} x^n}{n} = x - \frac{x^2}{2} + \frac{x^3}{3} - \dots$$

Sol

It is convergent if we prove abs. convergence. We do this using ratio test

$$\left|\frac{u_{n+1}}{u_n}\right| = \left|\frac{x^{n+1}(n)}{(n+1)x^n}\right| = \left|\frac{x}{\frac{n+1}{n}}\right|$$

$$\lim_{n\to\infty}\left|\frac{x}{\frac{n+1}{n}}\right| = |x|$$

$\therefore$ for $|x| < 1$
we have convergence

$\therefore$ $-1 < x < 1$
but we have to check at 1 & -1

At $x=1$ series is

$$1 - \frac{1}{2} + \frac{1}{3} \cdots\cdots$$

using leibnitz rule, since u_n is +ve $u_n \geq 0$
$\lim_{n\to\infty} u_n = 0$ $\therefore$ we can say it
is convergent

At $x = -1$

2. $\sum_{n=1}^{\infty} \frac{(-1)^{n-1} x^{2n-1}}{2n-1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \dots$

$\cancel{\frac{x^{2n-1}}{2n-1}}$

$$\lim_{n\to\infty} \left| \frac{x^{2n+1}}{2n+1} \times \frac{2n-1}{x^{2n-1}} \right| = |x^2|$$

$$= x^2$$

convergence at

$x^2 < 1$

$-1 < x < 1$

HW
3. $\sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \ldots$

*

HW 4. $\sum_{n=0}^{\infty} n!\, x^n = 1 + x + \ldots$

Theorem: Convergence theorem for power series.

- Consider the power series

$$\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \dots$$

i) If it converges for $x = c \neq 0$, then it converges absolutely for all x with $|x| < |c|$

ii) If the series diverges for $x = d$ then it diverges for all $|x| > |d|$

- Thus for power series $\sum c_n (x-a)^n$ we have 3 possibilities.

a) It might converge at only $x = a$
b) converges everywhere
c) converge on some interval of radius R centred at $x = a$.

- COROLLARY

The convergence of $\sum c_n (x-a)^n$ has 3 possibilities.

a) There is a +ve no. R such that the series diverges for x with $|x-a| > R$ but converges absolutely for x with $|x-a| < R$. The series may or may not converge at end points $a-R$ & $a+R$

b) The series converges absolutely for every x $(R=\infty)$

c) The series converges at $x=a$ & diverges elsewhere $(R=0)$

RADIUS OF CONVERGENCE →

- R is called the radius of convergence of power series.
- and interval of radius R centered at $x=a$ is called interval of convergence.
- The interval of convergeince may be open, closed, or half-open, depending on the particular series.
- At pts $|x-a|<R$ the series conv. absolutely.
- If series converges for all x, we say its radius of convergence is ∞.
- If it converges only at $x=a$, we say rad. of convergence is zero

classmate
Date
Page

NOTE: Power series always converges for the center. So, our aim is to determine for what non-zero value of $x-a$ or values of $x \neq a$ it can converges.

Q→ Discuss convergence of

$$\sum_{n=1}^{\infty} \frac{(-1)^n (x+2)^n}{n}$$

Ans.

$$\lim_{n\to\infty} \left|\frac{u_{n+1}}{u_n}\right| < 1 \Rightarrow \left| \frac{(x+2)^{n+1}}{\cancel{n+1}} \cdot \frac{\cancel{n}}{(x+2)^n} \right|$$

$\Rightarrow |x+2| < 1$

$\Rightarrow -3 < x < -1$

∴ series is absolutely convergent here.

now at $x=-3$, we have $1+\frac{1}{2}+\frac{1}{3}+\dots$ (harmonic series)

∴ it is divergent

At $x=-1$, we have alternating harmonic series. Using alternating series test, it is convergent but not absolutely convergent.

∴ Abs. conv. for $(-3,-1)$

conv. for $[-3,-1]$

⇒ rad. of conv. $=2$

Q→ $\sum_{n=1}^{\infty} a_n$ where $a_{n+1} = \frac{n}{n+1} a_n$, $a_1 = 3$

Ans. $a_{n+1} = \frac{n}{n+1} \frac{n-1}{n} \frac{n-2}{n-1} \cdots \cdots \frac{2}{3} \frac{1}{2} a_1$

$\Rightarrow a_{n+1} = \frac{3}{n+1}$

$\therefore$ Harmonic series. Hence, divergent.

Q→ $a_{n+1} = \frac{n + \ln n}{n+10} a_n$ & $a_1 = 1/2$

Ans. Ratio test

$\frac{a_{n+1}}{a_n} = \frac{n + \ln n}{n+10}$

Now

$\ln n > 10$ for $n \geq [e^{10}] + 1$

$\Rightarrow n + \ln n > n + 10$

$\Rightarrow \frac{n + \ln n}{n+10} > 1$

$\Rightarrow \frac{a_{n+1}}{a_n} > 1$ for $n \geq [e^{10}] + 1$

& similarly

$\frac{a_{n+1}}{a_n} < 1$ for $n \leq [e^{10}]$

We have established that

$\cancel{a_{n+1}/a_n}$ $a_{n+1} > a_n$ for $n \geq [e^{10}]+1$

Now since ~~a~~ at $n = [e^{10}]+1$,

$a_n > 0$

$\therefore a_{n+1} > 0$

$\therefore \lim_{n\to\infty} a_{n+1} > 0$

$\therefore$ The sequence is divergent.

(Overall sequence is divergent since at ∞ $\lim_{n\to\infty} a_{n+1} > 0$)

→ DIRICHLET TEST

- For $\sum_{n=1}^{\infty} a_n \times b_n$

$$\left.\begin{array}{l} \text{If } a_n \geq 0, \quad a_{n+1} \leq a_n \\ \lim_{n\to\infty} a_n = 0 \text{ and} \\ \left|\sum_{n=1}^{N} b_n\right| \leq M \quad \forall n \in \mathbb{N} \end{array}\right] \Rightarrow \text{then } \sum_{n=1}^{\infty} a_n b_n \text{ is converge}$$

$e^{10}] + 1$

Q→ $\sum_{n=1}^{\infty} (-1)^n \frac{\sin^2 n}{n} = \cancel{\Sigma}$

Ans. $= \underbrace{\sum_{n=1}^{\infty} \frac{(-1)^n}{2n}}_{\text{conv. by Liebnitz rule}} - \sum_{n=1}^{\infty} (-1)^n \frac{\cos 2n}{2n}$

$a_n = \frac{1}{2n}$ & $b_n = (-1)^n \cos 2n$ }

~~∴ using Dirichlet test, the given series is convergent.~~

$$\sum_{k=1}^{N} \cos ka = \frac{\sin \frac{Na}{2} \cos \frac{(N+1)a}{2}}{\sin a/2} \qquad a \neq 2\pi l, \; l \in \mathbb{Z}$$

At $a = 2$

$$= \left| \frac{\sin \frac{N(\pi-2)}{2} \cos \frac{(N+1)(\pi-2)}{2}}{\sin\left(\frac{\pi-2}{2}\right)} \right|$$

ce

gent.

Q. $\sum_{n=1}^{\infty} e^{-\alpha n} n^{\beta}$

Ans: $\alpha < 0, \forall \beta$ Divergent (n^{th} Term test)

$\alpha = 0,$ $\begin{cases} \beta < -1 & \text{Convergent} \\ \beta \geq -1 & \text{Divergent} \end{cases}$

$\alpha > 0, \forall \beta$ Convergent.