MA 101

Partial Derivatives

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Partial Derivative with Respect to x

The partial derivative of $f(x, y)$ with respect to x at the point (x_0, y_0) is

$$\lim_{h \to 0} \frac{f(x_0 + h, y_0) - f(x_0, y_0)}{h}$$
$$= \frac{\partial f}{\partial x}\Big|_{(x_0, y_0)}$$

provided the limit on the left exists.

Partial Derivative with Respect to y

The partial derivative of $f(x, y)$ with respect to y at the point (x_0, y_0) is

$$\lim_{k \to 0} \frac{f(x_0, y_0 + k) - f(x_0, y_0)}{k} = \left.\frac{\partial f}{\partial y}\right|_{(x_0, y_0)}$$

provided the limit on the left exists.

Finding the Slope of a Surface in the $y-$Direction

The plane $\mathbf{x = 1}$ intersects the paraboloid $\mathbf{z = x^2 + y^2}$ in a parabola. Find the slope of the tangent to the parabola at $(\mathbf{1}, \mathbf{2}, \mathbf{5})$

Find f_x and f_y.

- $\mathbf{f(x,y) = x^y}$.
- $\mathbf{f(x,y) = \int_x^y g(t)dt}$ (g is continuous for all t.)
- $\mathbf{f(x,y) = \sin^2(x - 3y)}$.
- $\mathbf{f(x,y) = \sum_{n=0}^{\infty} (xy)^n}$ $\quad (|xy| < 1)$.

Partial Derivatives Exist, But f Discontinuous

Let

$$\mathbf{f}(\mathbf{x},\mathbf{y}) = \begin{cases} \mathbf{0}, & ; \quad \mathbf{xy} \neq \mathbf{0} \\ \mathbf{1}, & ; \quad \mathbf{xy} = \mathbf{0}. \end{cases}$$

(a) Find the limit as $(\mathbf{x},\mathbf{y})$ approaches $(\mathbf{0},\mathbf{0})$ along the line $\mathbf{y} = \mathbf{x}$.

(b) Prove that $\mathbf{f}(\mathbf{x},\mathbf{y})$ is not continuous at the origin.

(c) Show that both partial derivatives $\frac{\partial \mathbf{f}}{\partial \mathbf{x}}$ and $\frac{\partial \mathbf{f}}{\partial \mathbf{y}}$ exist at the origin.

Remark

Mere existence of partial derivatives at any point does not imply that function is continuous at that point.

Differentiability and Continuity

1. **Example where function is continuous but partial derivatives do not exist:**

$$f(x,y) = \sqrt{x^2 + y^2} \text{ at } (0,0)$$

2. **It is still true in higher dimensions that differentiability at a point implies continuity.**
3. **What the previous example suggests is that we need a stronger requirement for differentiability in higher dimensions than the mere existence of the partial derivatives.**
4. **We define differentiability for functions of two variables at the end of this section and revisit the connection to continuity.**

Notations

$$\frac{\partial f}{\partial x} = f_x; \quad \frac{\partial f}{\partial y} = f_y.$$

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = f_{xx}; \quad \frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = f_{yy}.$$

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = (f_y)_x = f_{yx}; \quad \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = (f_x)_y = f_{xy}.$$

$$f_{xy} = f_{yx} \quad \text{is not always true.}$$

Under what condition $f_{xy} = f_{yx}$ is true.???

Example when $f_{xy} \neq f_{yx}$

$$\text{Let} \quad f(x,y) = \begin{cases} \frac{xy(x^2-y^2)}{x^2+y^2}, & ; \quad (x,y) \neq (0,0) \\ 0, & ; \quad (x,y) = (0,0). \end{cases}$$

Show that

$$f_{xy}(0,0) \neq f_{yx}(0,0).$$

$$f_{xy}(0,0) = (f_x)_y\Big|_{(x,y)=(0,0)} = \lim_{k\to 0} \frac{f_x(0,k) - f_x(0,0)}{k} = \lim_{k\to 0} \frac{-k-0}{k} = -1$$

$$f_{yx}(0,0) = (f_y)_x\Big|_{(x,y)=(0,0)} = \lim_{h\to 0} \frac{f_y(h,0) - f_y(0,0)}{h} = \lim_{h\to 0} \frac{h-0}{h} = 1$$

Therefore

$$f_{xy}(0,0) \neq f_{yx}(0,0).$$

Try yourself

Let $f(x, y)$ be defined by

$$f(x,y) = \begin{cases} x^2 \tan^{-1}\left(\frac{y}{x}\right) - y^2 \tan^{-1}\left(\frac{x}{y}\right), & ; \quad xy \neq 0 \\ 0, & ; \quad xy = 0. \end{cases}$$

Evaluate

$$f_{xy}(0,0) \text{ and } f_{yx}(0,0).$$

The Mixed Derivative Theorem or Clairaut's Theorem

Theorem

If $f(x, y)$ ***and its partial derivatives*** f_x, f_y, f_{xy} ***and*** f_{yx} ***are defined throughout an open region containing a point*** (a, b) ***and are all continuous at*** (a, b)***, then***

$$f_{xy}(a, b) = f_{yx}(a, b).$$

Implicit Differentiation

Example

Find the partial derivative $\frac{\partial z}{\partial x}$ if the equation $xz - \ln z = xy$ defines z as a function of two independent varibles x and y and partial derivatives exists.

Solution:

$$\frac{\partial z}{\partial x} = \frac{(y - z)z}{zx - 1}$$

Partial Derivatives of Still Higher Order:

$$f_{yyx} = (f_{yy})_x = \frac{\partial}{\partial x}(f_{yy}) = \frac{\partial}{\partial x}\frac{\partial^2 f}{\partial y^2} = \frac{\partial^3 f}{\partial x \partial y^2}.$$

Functions of Three Variables

$$\lim_{h\to 0}\frac{f(x_0+h,y_0,z_0)-f(x_0,y_0,z_0)}{h}=\left.\frac{\partial f}{\partial x}\right|_{(x_0,y_0,z_0)}=f_x(x_0,y_0,z_0)$$

if the limit on the left exists.

Similarly other derivatives with respect to y and z can be defined.

Example

If $f(x,y,z)=e^{3x+4y}\cos 5z$, show that

$$f_{xx}+f_{yy}+f_{zz}=0.$$

The Increment Theorem for Functions of Two Variables

Theorem

Suppose that the first partial derivatives of $\mathbf{f(x, y)}$ are defined throughout an open region $\mathbf{R}$ containing the point $(\mathbf{x_0}, \mathbf{y_0})$ and that $\mathbf{f_x}$ and $\mathbf{f_y}$ are continuous at $(\mathbf{x_0}, \mathbf{y_0})$. Then the change

$$\Delta z = f(x_0 + \Delta x, y_0 + \Delta y) - f(x_0, y_0)$$

in the value of $\mathbf{f}$ that results from moving from $(\mathbf{x_0}, \mathbf{y_0})$ to another point $(x_0 + \Delta x, y_0 + \Delta y)$ in $\mathbf{R}$ satisfies an equation of the form

$$\Delta z = f_x(x_0, y_0)\Delta x + f_y(x_0, y_0)\Delta y + \epsilon_1 \Delta x + \epsilon_2 \Delta y,$$

in which each of $\epsilon_1, \epsilon_2 \to \mathbf{0}$ as both $\Delta x, \Delta y \to 0$.

Differentiable Function

Definition

A function $\mathbf{z = f(x, y)}$ is **differentiable** at $(\mathbf{x_0}, \mathbf{y_0})$ if $\mathbf{f_x(x_0, y_0)}$ and $\mathbf{f_y(x_0, y_0)}$ exist and Δz satisfies an equation of the form

$$\Delta z = f_x(x_0, y_0)\Delta x + f_y(x_0, y_0)\Delta y + \epsilon_1 \Delta x + \epsilon_2 \Delta y,$$

in which each of $\epsilon_1, \epsilon_2 \to \mathbf{0}$ as both $\Delta x, \Delta y \to 0$. We call $\mathbf{f}$ **differentiable** if it is differentiable at every point in its domain.

Corollary

If the partial derivatives $\mathbf{f_x}$ and $\mathbf{f_y}$ of a function $\mathbf{f(x, y)}$ are continuous throughout an open region $\mathbf{R}$, then $\mathbf{f}$ is differentiable at every point of $\mathbf{R}$.

Differentiability Implies Continuity

Theorem

If a function $f(x, y)$ *is differentiable at* (x_0, y_0), *then* f *is continuous at* (x_0, y_0).

1. Let $\mathbf{w} = \mathbf{f}(\mathbf{x})$ be a differentiable function of $\mathbf{x}$ and $\mathbf{x}$ be a differentiable function of $\mathbf{t}$, then $\mathbf{w}$ becomes a differentiable function of $\mathbf{t}$ and
$$\frac{dw}{dt} = \frac{dw}{dx}\frac{dx}{dt}.$$
2. For functions of two or more variables the Chain Rule has several forms.
3. The form depends on how many variables are involved.
4. **Functions of Two Variables:**

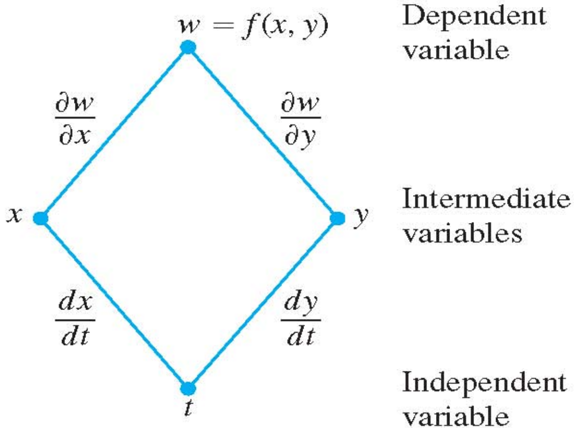
Theorem

If $\mathbf{w} = \mathbf{f}(\mathbf{x}, \mathbf{y})$ *has continuous partial derivatives* $\mathbf{f_x}$ *and* $\mathbf{f_y}$ *and if* $\mathbf{x} = \mathbf{x}(\mathbf{t}),\ \mathbf{y} = \mathbf{y}(\mathbf{t})$ *are differentiable functions of* $\mathbf{t}$, *then the composite* $\mathbf{f}(\mathbf{x}(\mathbf{t}), \mathbf{y}(\mathbf{t}))$ *is a differentiable function of* $\mathbf{t}$ *and*

$$\frac{dw}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}.$$

Tree Diagram

Chain Rule



$$\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt}$$

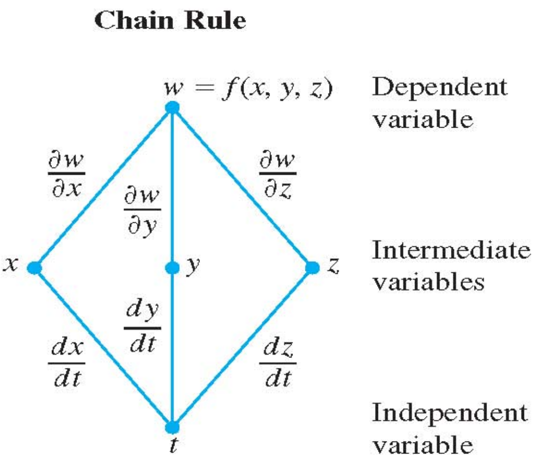
Chain Rule for Functions of Three Independent Variables

Theorem

If $\mathbf{w} = \mathbf{f}(\mathbf{x}, \mathbf{y}, \mathbf{z})$ *is differentiable and* $\mathbf{x}$, $\mathbf{y}$ & $\mathbf{z}$ *are differentiable functions of* $\mathbf{t}$, *then the composite* $\mathbf{w} = \mathbf{f}(\mathbf{x}(\mathbf{t}), \mathbf{y}(\mathbf{t}), \mathbf{z}(\mathbf{t}))$ *is a differentiable function of* $\mathbf{t}$ *and*

$$\frac{\mathbf{dw}}{\mathbf{dt}} = \frac{\partial \mathbf{f}}{\partial \mathbf{x}}\frac{\mathbf{dx}}{\mathbf{dt}} + \frac{\partial \mathbf{f}}{\partial \mathbf{y}}\frac{\mathbf{dy}}{\mathbf{dt}} + \frac{\partial \mathbf{f}}{\partial \mathbf{z}}\frac{\mathbf{dz}}{\mathbf{dt}}.$$

Tree Diagram



$$\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt} + \frac{\partial w}{\partial z}\frac{dz}{dt}$$

Functions Defined on Surfaces

Chain Rule for 2 Independent Variables & 3 Intermediate Variables

Theorem

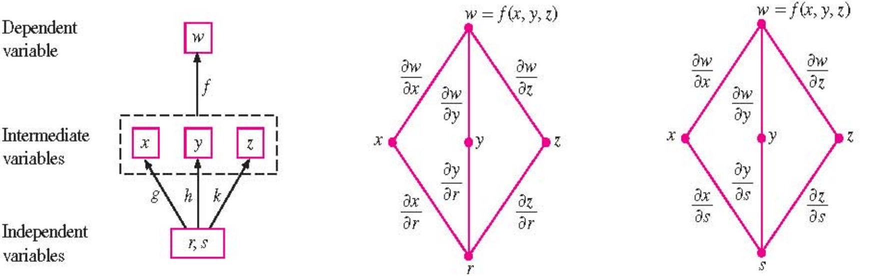
Suppose $\mathbf{w} = \mathbf{f}(\mathbf{x}, \mathbf{y}, \mathbf{z})$, $\mathbf{x} = \mathbf{g}(\mathbf{r}, \mathbf{s})$, $\mathbf{y} = \mathbf{h}(\mathbf{r}, \mathbf{s})$ *and* $\mathbf{z} = \mathbf{k}(\mathbf{r}, \mathbf{s})$. *If all four functions are differentiable, then* $\mathbf{w}$ *has partial derivatives with respect to* $\mathbf{r}$ *and* $\mathbf{s}$, *given by the formulas*

$$\frac{\partial w}{\partial r} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial r} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial r},$$

$$\frac{\partial w}{\partial s} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial s} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial s}.$$

Remark: **The first of these equations can be derived from the Chain Rule in previous theorem by holding $\mathbf{s}$ fixed and treating $\mathbf{r}$ as $\mathbf{t}$. The second can be derived in the same way, holding $\mathbf{r}$ fixed and treating $\mathbf{s}$ as $\mathbf{t}$.**

Tree Diagram



$$w = f\left(g(r,s), h(r,s), k(r,s)\right)$$

(a)

$$\frac{\partial w}{\partial r} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial r} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial r}$$

(b)

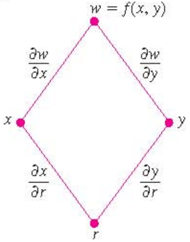
$$\frac{\partial w}{\partial s} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial s} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial s}$$

(c)

f is a function of two variables instead of three

If **f** is a function of two variables instead of three, each equation in the previous theorem becomes correspondingly one term shorter.
Suppose $\mathbf{w} = \mathbf{f}(\mathbf{x}, \mathbf{y})$, $\mathbf{x} = \mathbf{g}(\mathbf{r}, \mathbf{s})$, $\mathbf{y} = \mathbf{h}(\mathbf{r}, \mathbf{s})$. If all three functions are differentiable, then **w** has partial derivatives with respect to **r** and **s**, given by the formulas

$$\frac{\partial \mathbf{w}}{\partial \mathbf{r}} = \frac{\partial \mathbf{f}}{\partial \mathbf{x}}\frac{\partial \mathbf{x}}{\partial \mathbf{r}} + \frac{\partial \mathbf{f}}{\partial \mathbf{y}}\frac{\partial \mathbf{y}}{\partial \mathbf{r}}, \qquad \frac{\partial \mathbf{w}}{\partial \mathbf{s}} = \frac{\partial \mathbf{f}}{\partial \mathbf{x}}\frac{\partial \mathbf{x}}{\partial \mathbf{s}} + \frac{\partial \mathbf{f}}{\partial \mathbf{y}}\frac{\partial \mathbf{y}}{\partial \mathbf{s}}.$$

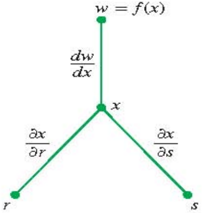


$$\frac{\partial w}{\partial r} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial r}$$

f is a function of one variable only

If **f** is a function of **x** alone, our equations become even simpler. Suppose $\mathbf{w} = \mathbf{f}(\mathbf{x})$ and $\mathbf{x} = \mathbf{g}(\mathbf{r}, \mathbf{s})$ then we have

$$\frac{\partial \mathbf{w}}{\partial \mathbf{r}} = \frac{\mathbf{dw}}{\mathbf{dx}}\frac{\partial \mathbf{x}}{\partial \mathbf{r}}, \qquad \frac{\partial \mathbf{w}}{\partial \mathbf{s}} = \frac{\mathbf{dw}}{\mathbf{dx}}\frac{\partial \mathbf{x}}{\partial \mathbf{s}}.$$



$$\frac{\partial w}{\partial r} = \frac{dw}{dx}\frac{\partial x}{\partial r}$$

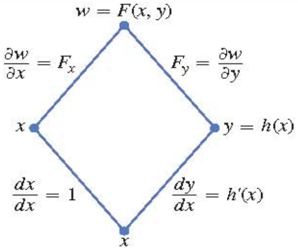
$$\frac{\partial w}{\partial s} = \frac{dw}{dx}\frac{\partial x}{\partial s}$$

Implicit Differentiation

Theorem

Suppose that $\mathbf{F}(\mathbf{x},\mathbf{y})$ *is differentiable and that the equation* $\mathbf{F}(\mathbf{x},\mathbf{y}) = \mathbf{0}$ *defines* $\mathbf{y}$ *as a differentiable function of* $\mathbf{x}$*. Then at any point where* $\mathbf{F_y} \neq \mathbf{0}$

$$\frac{\mathbf{dy}}{\mathbf{dx}} = -\frac{\mathbf{F_x}}{\mathbf{F_y}}.$$



$$\frac{dw}{dx} = F_x \cdot 1 + F_y \cdot \frac{dy}{dx}$$

Implicit Differentiation: Problem

Problem

Assuming that the following equation define $\mathbf{y}$ *as a differentiable function of* $\mathbf{x}$, *find the value of* $\frac{dy}{dx}$ *at the given point.*

$$xe^{y} + \sin xy + y - \ln 2 = 0, \qquad (0, \ln 2).$$

Let

$$F = xe^{y} + \sin xy + y - \ln 2$$

$$\frac{dy}{dx} = -\frac{F_x}{F_y} = -\frac{e^{y} + y\cos xy}{xe^{y} + x\cos xy + 1}.$$

$$\left.\frac{dy}{dx}\right|_{(0,\ln 2)} = -2 - \ln 2.$$

Chain Rule: Problem

Problem

Draw tree diagram and write chain rule for

$$\frac{\partial \mathbf{w}}{\partial \mathbf{u}} \quad \& \quad \frac{\partial \mathbf{w}}{\partial \mathbf{v}}$$

where $\mathbf{w} = \mathbf{g}(\mathbf{x}, \mathbf{y})$ *and* $\mathbf{x} = \mathbf{h}(\mathbf{u}, \mathbf{v})$ *and* $\mathbf{y} = \mathbf{k}(\mathbf{u}, \mathbf{v})$.

Problem

Draw tree diagram and write chain rule for

$$\frac{\partial \mathbf{w}}{\partial \mathbf{s}} \quad \& \quad \frac{\partial \mathbf{w}}{\partial \mathbf{t}}$$

where $\mathbf{w} = \mathbf{g}(\mathbf{u})$ *and* $\mathbf{u} = \mathbf{h}(\mathbf{s}, \mathbf{t})$.

Three Variable Implicit Differentiation

Suppose that $\mathbf{z}$ is given implicitly as a function $\mathbf{z} = \mathbf{f}(\mathbf{x}, \mathbf{y})$ by an equation of the form $\mathbf{F}(\mathbf{x}, \mathbf{y}, \mathbf{z}) = \mathbf{0}$, i.e., $\mathbf{F}(\mathbf{x}, \mathbf{y}, \mathbf{f}(\mathbf{x}, \mathbf{y})) = \mathbf{0}$ for all $(\mathbf{x}, \mathbf{y})$ in the domain of $\mathbf{f}$. If $\mathbf{F}$ and $\mathbf{f}$ are differentiable, then one can use the Chain rule to differentiate the equation $\mathbf{F}(\mathbf{x}, \mathbf{y}, \mathbf{z}) = \mathbf{0}$ as follows:

$$\frac{\partial \mathbf{F}}{\partial \mathbf{x}}\frac{\partial \mathbf{x}}{\partial \mathbf{x}} + \frac{\partial \mathbf{F}}{\partial \mathbf{y}}\frac{\partial \mathbf{y}}{\partial \mathbf{x}} + \frac{\partial \mathbf{F}}{\partial \mathbf{z}}\frac{\partial \mathbf{z}}{\partial \mathbf{x}} = \mathbf{0}.$$

Hence

$$\frac{\partial \mathbf{F}}{\partial \mathbf{x}} + \frac{\partial \mathbf{F}}{\partial \mathbf{z}}\frac{\partial \mathbf{z}}{\partial \mathbf{x}} = \mathbf{0}, \quad \left(\because \frac{\partial \mathbf{y}}{\partial \mathbf{x}} = \mathbf{0}\right)$$

thus

$$\frac{\partial \mathbf{z}}{\partial \mathbf{x}} = -\frac{\mathbf{F_x}}{\mathbf{F_z}}, \quad \text{provided} \quad \mathbf{F_z} \neq \mathbf{0}.$$

Similarly

$$\frac{\partial \mathbf{z}}{\partial \mathbf{y}} = -\frac{\mathbf{F_y}}{\mathbf{F_z}}, \quad \text{provided} \quad \mathbf{F_z} \neq \mathbf{0}.$$

Laplace equations

Problem

Show that if $w = f(u, v)$ *satisfies the Laplace equation*

$$f_{uu} + f_{vv} = 0$$

and if $u = \frac{x^2 - y^2}{2}$ *and* $v = xy$ *then* w *satisfies the Laplace equation*

$$w_{xx} + w_{yy} = 0.$$

Problem

Let

$$
\begin{aligned}
w &= f(x, y, z),\\
x &= g(r, s),\\
y &= h(r, s),\\
z &= k(r, s),\\
r &= p(t),\\
s &= q(t).
\end{aligned}
$$

Then find

$$\frac{dw}{dt}.$$

Exercise 14.4

(a) express dw/dt as a function of t, both by using the Chain Rule and by expressing w in terms of t and differentiating directly with respect to t. Then (b) evaluate dw/dt at the given value of t.

(3) $\mathbf{w} = \frac{\mathbf{x}}{\mathbf{z}} + \frac{\mathbf{y}}{\mathbf{z}}$, $\mathbf{x} = \cos^2 \mathbf{t}$, $\mathbf{y} = \sin^2 \mathbf{t}$ and $\mathbf{z} = \mathbf{1}/\mathbf{t}$ at $\mathbf{t} = \mathbf{3}$.

(6) $\mathbf{w} = \mathbf{z} - \sin \mathbf{xy}$, $\mathbf{x} = \mathbf{t}$, $\mathbf{y} = \ln \mathbf{t}$ and $\mathbf{z} = \mathbf{e}^{\mathbf{t}-\mathbf{1}}$ at $\mathbf{t} = \mathbf{1}$.

Exercise 14.4

(a) express $\partial z/\partial u$ and $\partial z/\partial v$ as functions of u and v both by using the Chain Rule and by expressing z directly in terms of u and v before differentiating. Then (b) evaluate $\partial z/\partial u$ and $\partial z/\partial v$ at the given point (u, v).

(8) $\mathbf{z} = \tan^{-1}(\mathbf{x}/\mathbf{y})$, $\mathbf{x} = \mathbf{u} \cos \mathbf{v}$, $\mathbf{y} = \mathbf{u} \sin \mathbf{v}$ and $(u, v) = (1.3, \pi/6)$.

Exercise 14.4

(a) express $\partial w/\partial u$ and $\partial w/\partial v$ as functions of u and v both by using the Chain Rule and by expressing w directly in terms of u and v before differentiating. Then (b) evaluate $\partial w/\partial u$ and $\partial w/\partial v$ at the given point (u, v).

(9) $\mathbf{w = xy + yz + zx}$, $\mathbf{x = u + v}$, $\mathbf{y = u - v}$ and $\mathbf{z = uv}$ where $(u, v) = (1/2, 1)$.

Exercise 14.4

(a) express $\partial u/\partial x$, $\partial u/\partial y$ and $\partial u/\partial z$ as functions of x, y and z both by using the Chain Rule and by expressing u directly in terms of x, y and z before differentiating. Then (b) evaluate $\partial u/\partial x$, $\partial u/\partial y$ and $\partial u/\partial z$ at the given point (x, y, z).

(11) $\mathbf{u = \frac{p-q}{q-r}}$, $\mathbf{p = x + y + z}$, $\mathbf{q = x - y + z}$ and $\mathbf{r = x + y - z}$ where $(x, y, z) = (\sqrt{3}, 2, 1)$.

Suppose that the function $\mathbf{f}(\mathbf{x}, \mathbf{y})$ is defined throughout a region $\mathbf{R}$ in the $\mathbf{xy}$−plane, that $\mathbf{P_0}(\mathbf{x_0}, \mathbf{y_0})$ is a point in $\mathbf{R}$, and that $\widehat{\mathbf{u}} = \mathbf{u_1}\widehat{\mathbf{i}} + \mathbf{u_2}\widehat{\mathbf{j}}$ is a unit vector. Then the equations

$$\mathbf{x} = \mathbf{x_0} + \mathbf{s}\,\mathbf{u_1}, \qquad \mathbf{y} = \mathbf{y_0} + \mathbf{s}\,\mathbf{u_2}$$

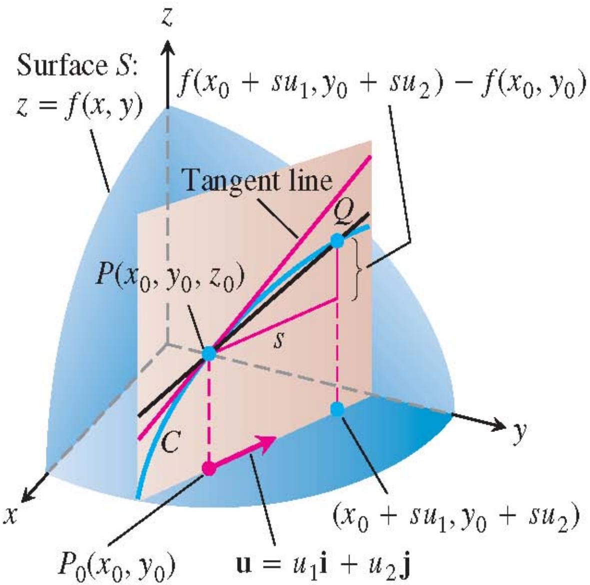
parametrize the line through $\mathbf{P_0}$ parallel to $\widehat{\mathbf{u}}$. If the parameter $\mathbf{s}$ measures arc length from $\mathbf{P_0}$ in the direction of $\widehat{\mathbf{u}}$, the rate of change of $\mathbf{f}$ at $\mathbf{P_0}$ in the direction of $\widehat{\mathbf{u}}$ is calculated by $\frac{\mathbf{df}}{\mathbf{ds}}$ at $\mathbf{P_0}$.

Definition

The derivative of $\mathbf{f}$ at $\mathbf{P_0}(\mathbf{x_0}, \mathbf{y_0})$ in the direction of the unit vector $\mathbf{u} = \mathbf{u_1}\widehat{\mathbf{i}} + \mathbf{u_2}\widehat{\mathbf{j}}$ is the number

$$(\mathbf{D_u f})_{\mathbf{P_0}} = \left(\frac{\mathbf{df}}{\mathbf{ds}}\right)_{\mathbf{u},\mathbf{P_0}} = \lim_{\mathbf{s}\to 0} \frac{\mathbf{f}(\mathbf{x_0} + \mathbf{su_1}, \mathbf{y_0} + \mathbf{su_2}) - \mathbf{f}(\mathbf{x_0}, \mathbf{y_0})}{\mathbf{s}},$$

provided the limit exists. (See Solved Example)


z
Surface S:
z = f(x, y)
f(x0 + su1, y0 + su2) − f(x0, y0)
Tangent line
Q
P(x0, y0, z0)
s
C
y
x
(x0 + su1, y0 + su2)
P0(x0, y0)
u = u1i + u2j

Gradient Vector

The gradient vector (gradient) of $\mathbf{f}(\mathbf{x},\mathbf{y})$ at a point $\mathbf{P_0}(\mathbf{x_0},\mathbf{y_0})$ is the vector

$$\textit{grad } f = \nabla f = \frac{\partial \mathbf{f}}{\partial \mathbf{x}}\widehat{\mathbf{i}} + \frac{\partial \mathbf{f}}{\partial \mathbf{y}}\widehat{\mathbf{j}}$$

obtained by evaluating the partial derivatives of $\mathbf{f}$ at $\mathbf{P_0}$.

Theorem

If $\mathbf{f}(\mathbf{x},\mathbf{y})$ is differentiable in an open region containing $\mathbf{P_0}(\mathbf{x_0},\mathbf{y_0})$ then

$$(\mathbf{D}_{\widehat{\mathbf{u}}}\mathbf{f})_{\mathbf{P_0}} = \left(\frac{\mathbf{df}}{\mathbf{ds}}\right)_{\mathbf{u},\mathbf{P_0}} = \left.\frac{\partial \mathbf{f}}{\partial \mathbf{x}}\right|_{\mathbf{P_0}}\mathbf{u_1} + \left.\frac{\partial \mathbf{f}}{\partial \mathbf{y}}\right|_{\mathbf{P_0}}\mathbf{u_2} = (\nabla \mathbf{f})_{\mathbf{P_0}} \cdot \widehat{\mathbf{u}}.$$

is the dot product of the gradient $\mathbf{f}$ at $\mathbf{P_0}$ and $\widehat{\mathbf{u}}$.

Problem

Find out derivative of the function $f(x, y)$ at P_0 in the direction of $\vec{\mathbf{A}}$.

$$f(x, y) = 2x^2 + y^2,$$
$$P_0 = (-1, 1),$$
$$\vec{\mathbf{A}} = 3\hat{i} - 4\hat{j}.$$

Solution: $-\mathbf{4}$.

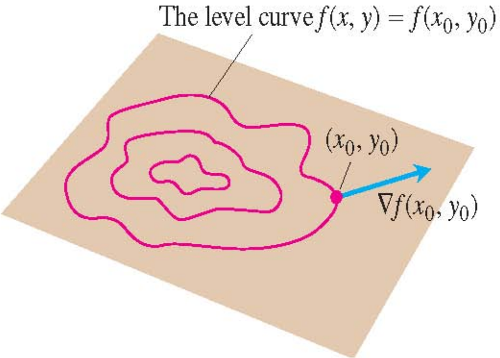
Gradients and Tangents to Level Curves & Level Surfaces

If a differentiable function $f(x, y)$ has a constant value c along a smooth curve (making the curve a level curve $r(t) = g(t)\hat{i} + h(t)\hat{j}$ of f), then $f(g(t), h(t)) = c$. Differentiating both sides of this equation with respect to t leads to the following equations

$$\frac{\partial f}{\partial x}\frac{dg}{dt} + \frac{\partial f}{\partial y}\frac{dh}{dt} = 0 \quad \Rightarrow \quad \left(\frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j}\right) \cdot \left(\frac{dg}{\partial t}\hat{i} + \frac{dh}{dt}\hat{j}\right) = 0$$

$$\nabla f \cdot \frac{dr}{dt} = 0$$

The above equation says that ∇f is normal to the tangent vector $\frac{dr}{dt}$, so it is normal to the curve.



Remark

At every point in the domain of a differentiable function $\mathbf{f(x, y)}$*, the gradient of* $\mathbf{f}$ *is normal to the level curve through* $\mathbf{(x_0, y_0)}$

Problem

Find an equation for the tangent to the ellipse at (-2,1)

$$\frac{x^2}{4} + y^2 = 2.$$

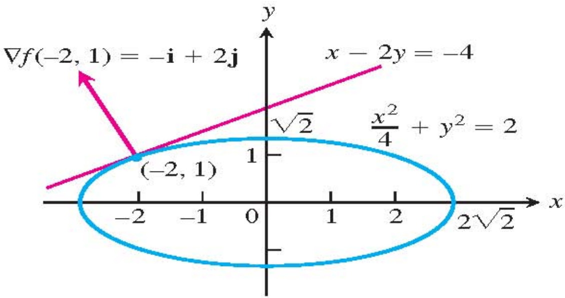


Figure: Equation of Tangent Line : $x - 2y = -4$.

Finding Directions of Maximal, Minimal, and Zero Change

Find the directions in which $f(x, y) = (x^2/2) + (y^2/2)$.

(a) Increases most rapidly at the point $(1, 1)$.

(b) Decreases most rapidly at $(1, 1)$.

(c) What are the directions of zero change in f at $(1, 1)$?

Directional derivatives for function of three variables

Let $\mathbf{f}(\mathbf{x}, \mathbf{y}, \mathbf{z})$ be the function and the point at which we are looking for directional derivative is $\mathbf{P_0}(\mathbf{x_0}, \mathbf{y_0}, \mathbf{z_0})$ along the direction of the unit vector $\widehat{\mathbf{u}} = \mathbf{u_1}\widehat{\mathbf{i}} + \mathbf{u_2}\widehat{\mathbf{j}} + \mathbf{u_3}\widehat{\mathbf{k}}$.

$$(D_{\widehat{u}}f)_{P_0} = \left(\frac{df}{ds}\right)_{u,P_0} = (\nabla f)_{P_0} \cdot \widehat{u} = \left.\frac{\partial f}{\partial x}\right|_{P_0} u_1 + \left.\frac{\partial f}{\partial y}\right|_{P_0} u_2 + \left.\frac{\partial f}{\partial z}\right|_{P_0} u_3.$$

Problem

Find the gradient of the function $\mathbf{f(x, y) = y - x}$ *at the point* $(\mathbf{2}, \mathbf{1})$. *Then sketch the gradient together with the level curve that passes through* $(\mathbf{2}, \mathbf{1})$.

Problem

Find the derivative of the function $\mathbf{f(x, y, z) = xy + yz + zx}$ *at* $\mathbf{P_0(1, -1, 2)}$ *in the direction of* $\mathbf{A} = \mathbf{3\widehat{i} + 6\widehat{j} - 2\widehat{k}}$.

Problem

Find the directions in which the function $\mathbf{f(x, y, z) = xe^y + z^2}$ *increase and decrease most rapidly at* $\mathbf{P_0(1, \ln 2, 1/2)}$ *Then find the derivatives of the function in these directions..*

Definitions and equations

- The tangent plane at the point $P_0(x_0, y_0, z_0)$ on the level surface $f(x, y, z) = c$ of a differentiable function f is the plane through P_0 normal to ∇f_{P_0}. The equation is given by

$$f_x(P_0)(x - x_0) + f_y(P_0)(y - y_0) + f_z(P_0)(z - z_0) = 0.$$

- The normal line of the surface at P_0 is the line through P_0 parallel to ∇f_{P_0}. The equation is given by

$$\frac{x - x_0}{f_x(P_0)} = \frac{y - y_0}{f_y(P_0)} = \frac{z - z_0}{f_z(P_0)} = t.$$

- The plane tangent to the surface $z = f(x, y)$ of a differentiable function f at the point $P_0(x_0, y_0, z_0)$ is

$$f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0) - (z - z_0) = 0.$$

Problem

Find the tangent plane and normal line of the surface $\mathbf{f(x,y,z) = x^2 + y^2 + z - 9 = 0}$ ***at*** $(1,2,4)$.

- $\nabla \mathbf{f}|_{\mathbf{P_0}} = \mathbf{2\widehat{i} + 4\widehat{j} + \widehat{k}}$
- Equation of Tangent Plane: $\mathbf{2x + 4y + z = 14}$
- Equation of Normal Line: $\mathbf{\frac{x-1}{2} = \frac{y-2}{4} = \frac{z-4}{1} = t}$

Problem

Tangent Line to the Curve of Intersection: *The cylinder* $\mathbf{f(x,y,z) = x^2 + y^2 - 2 = 0}$ *and the plane* $\mathbf{g(x,y,z) = x + z - 4 = 0}$ *meet in an ellipse* $\mathbf{E}$*. Find parametric equations for the line tangent to* $\mathbf{E}$ *at the point* $\mathbf{P_0(1,1,3)}$*.*

- $\nabla f|_{(1,1,3)} = 2\widehat{i} + 2\widehat{j}$, $\nabla g|_{(1,1,3)} = \widehat{i} + \widehat{j}$.
- $\nabla f|_{(1,1,3)} \times \nabla g|_{(1,1,3)} = 2\widehat{i} - 2\widehat{j} - 2\widehat{k}$.
- Tangent Line at $(1,1,3)$, $\frac{x-1}{2} = \frac{y-1}{-2} = \frac{z-3}{-2}$.

Estimating the Change in f in a Direction $\widehat{\mathbf{u}}$.

- If f is a function of a single variable, we have

$$df = f'(P_0)ds \quad \text{(Ordinary Derivative} \times \text{Increament)}.$$

- For a function of two or more variables, we can write

$$df = (\nabla f_{P_0} \cdot \widehat{\mathbf{u}})ds \quad \text{(Directional Derivative} \times \text{Increament)}.$$

Problem

Estimate how much the value of $f(x, y, z) = y \sin x + 2yz$ will change if the point $P(x, y, z)$ moves 0.1 unit from $(0, 1, 0)$ straight toward $(2, 2, -2)$.

Solution: $\widehat{\mathbf{u}} = \frac{2\widehat{i}+\widehat{j}-2\widehat{k}}{3}$, $\nabla f|_{(0,1,0)} = \widehat{i} + 2\widehat{k}$.
$ds = 0.1$. Therefore $df = (\nabla f|_{(0,1,0)} \cdot \widehat{\mathbf{u}})0.1 = (-2/3)(0.1) \approx -0.067$ unit.

"Functions of two variables can be complicated, and we sometimes need to replace them with simpler ones that give the accuracy required for specific applications without being so difficult to work with."

Definition

The linearization of a function $f(x, y)$ at a point where f is differentiable is the function

$$L(x, y) = f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0).$$

The approximation $f(x, y) \approx L(x, y)$ is the standard linear approximation of f at (x_0, y_0).

Remark

The plane $z = L(x, y)$ is tangent to the surface $z = f(x, y)$ at the point (x_0, y_0) Thus, the linearization of a function of two variables is a tangent-plane approximation in the same way that the linearization of a function of a single variable is a tangent-line approximation.

Problem

Find the linearization of $f(x, y) = x^2 - xy + (1/2)y^2 + 3$ at $(3, 2)$.

Solution: $f(3, 2) = 8$, $f_x(3, 2) = 4$, $f_y(3, 2) = -1$. $L(x, y) = 4x - y - 2$.

Error

The Error in the Standard Linear Approximation

Total Differential

Total Differential

If we move from a point (x_0, y_0) to a point $(x_0 + dx, y_0 + dy)$ nearby the resulting change

$$\mathbf{df = f_x(x_0, y_0)dx + f_y(x_0, y_0)dy}$$

in the linearization of f is called **total differential** of f.

Problem

Suppose that a cylindrical can is designed to have a radius of 1 *inch*. and a height of 5 *inch*., but that the radius and height are off by the amounts $dr = +0.03$ and $dh = -0.1$. Estimate the resulting absolute change in the volume of the can.

Solution: $dV \approx 2\pi rhdr + \pi r^2 dh \approx 0.63\ inch^3$.

14.5 Problem 30: Changing temperature along a circle

Is there a direction $\widehat{u}$ in which the rate of change of the temperature function $T(x, y, z) = 2xy - yz$ (temperature in degrees Celsius, distance in feet) at $P(1, -1, 1)$ is $-3^\circ C/Ft$? Give reasons for your answer.

Hint: Compute extreme rate of changes, i.e., $\nabla f(1, -1, 1). \pm \frac{\nabla f(1,-1,1)}{|\nabla f(1,-1,1)|}$. (Answer: No such direction exists).

14.5 Problem 31

The derivative of $f(x, y)$ at $P_0(1, 2)$ in the direction of $\widehat{\mathbf{i}} + \widehat{\mathbf{j}}$ is $2\sqrt{2}$ and in the direction of $-2\widehat{\mathbf{j}}$ is -3. What is the derivative of f in the direction of $-\widehat{\mathbf{i}} - 2\widehat{\mathbf{j}}$? Give reasons for your answer.

Hint: Consider $\nabla f(1, 2) = f_x(1, 2)\widehat{\mathbf{i}} + f_y(1, 2)\widehat{\mathbf{j}}$. (Answer: $\frac{-7}{\sqrt{5}}$).

14.5 Problem 32

The derivative of $f(x, y, z)$ at a point P is greatest in the direction of $\widehat{\mathbf{i}} + \widehat{\mathbf{j}} - \widehat{\mathbf{k}}$. In this direction, the value of the derivative is $2\sqrt{3}$.
(a) What is ∇f at P? Give reasons for your answer?
(b) What is the derivative of f at P in the direction of $\widehat{\mathbf{i}} + \widehat{\mathbf{j}}$?

Hint: Consider $\nabla f|_P = \lambda(\widehat{\mathbf{i}} + \widehat{\mathbf{j}} - \widehat{\mathbf{k}})$. (Answer: (a) $2\widehat{\mathbf{i}} + 2\widehat{\mathbf{j}} - 2\widehat{\mathbf{k}}$ (b) $2\sqrt{2}$).

14.5 Problem 33, Directional derivatives and scalar components

How is the derivative of a differentiable function $f(x, y, z)$ at a point P_0 in the direction of a unit vector $\widehat{\mathbf{u}}$ related to the scalar component of $(\nabla f)_{P_0}$ in the direction of $\widehat{\mathbf{u}}$? Give reasons for your answer.

Hint: Geometrical interpretation of Dot Product.

14.6 Problem 5

Find equations for the (a) tangent plane and (b) normal line at the point P_0 on the given surface.

$$\cos \pi x - x^2 y + e^{xz} + yz = 4, \quad P_0(0, 1, 2).$$

Solution:

$\nabla f_{(0,1,2)} = 2\widehat{\mathbf{i}} + 2\widehat{\mathbf{j}} + \widehat{\mathbf{k}}.$

Tangent Plane: $2(x-0)+2(y-1)+1(z-2)=0$, i.e., $2x+2y+z=4$.

Normal Line : $\frac{x-0}{2} = \frac{y-1}{2} = \frac{z-2}{1} = t.$

Problem 14[1]

Find parametric equations for the line tangent to the curve of intersection of the surfaces at the given point.
Surface: $f = xyz - 1 = 0$, $g = x^2 + 2y^2 + 3z^2 - 6 = 0$.
Point: $(1, 1, 1)$.

Solution: $\nabla f(1,1,1) = \widehat{\mathbf{i}} + \widehat{\mathbf{j}} + \widehat{\mathbf{k}}$.
$\nabla g(1,1,1) = 2\widehat{\mathbf{i}} + 4\widehat{\mathbf{j}} + 6\widehat{\mathbf{k}}$.
$\nabla f(1,1,1) \times \nabla g(1,1,1) = 2\widehat{\mathbf{i}} - 4\widehat{\mathbf{j}} + 2\widehat{\mathbf{k}}$
Tangent Line : $\frac{x-1}{2} = \frac{y-1}{-4} = \frac{z-1}{2} = t$.

[1]Suggested Problems from 14.6 are 23, 24.

1. Continuous functions of two variables assume extreme values on closed and bounded domains.
2. How to search these extreme values?

   Answer: Through Partial Derivatives. But how?

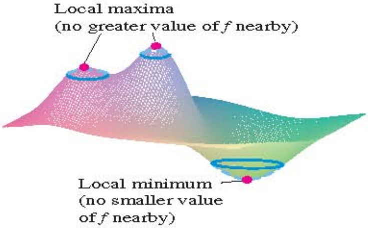
Local Maximum, Local Minimum: Let $f(x, y)$ be defined on a region R containing the point (a, b). Then

1. $f(a, b)$ is a local maximum value of f if for all domain points (x, y) in an open disk centered at (a, b).

$$f(a, b) \geq f(x, y)$$

2. $f(a, b)$ is a local minimum value of f if for all domain points (x, y) in an open disk centered at (a, b).

$$f(a, b) \leq f(x, y)$$



Absolute Maxima and Minima

3. If $f(a,b) \geq f(x,y)$ for ALL points (x,y) in the domain of f, then f has an ABSOLUTE MAXIMUM at (a,b).
4. If $f(a,b) \leq f(x,y)$ for ALL points (x,y) in the domain of f, then f has an ABSOLUTE MINIMUM at (a,b).
5. **How to search absolute extrema of a continuous function $f(x,y)$ on closed & bounded region R?**

(a) List the interior points of R where f may have local maxima and minima.

(b) List the boundary points of R where f has local maxima and minima.

(c) Since absolute maxima and minima are also local maxima and minima, the absolute maximum and minimum values of f appear somewhere in the lists.

1. A function of two variables can assume extreme values only at
   domain boundary points or at interior domain points
   where both first partial derivatives are ZERO or
   where one or both of the first partial derivatives FAIL to EXIST. (known as CRITICAL POINTS)
2. However, the vanishing of derivatives at an interior point (a, b) does not always signal the presence of an extreme value. The surface that is the graph of the function might be shaped like a SADDLE right above (a, b) and cross its tangent plane there.

Saddle point: Let (a, b) be a critical point of a differentiable function $f(x, y)$. If in EVERY open disc centered at (a, b), there are domain points (x, y) where

$$f(x, y) < f(a, b)$$

and domain points (x, y) where

$$f(x, y) > f(a, b),$$

then the point $(a, b, f(a, b))$ on the surface $z = f(x, y)$ is called saddle point of the surface.

SHOW 3D PLOTS

Theorem (First derivative test):
If $f(x, y)$ has a local maximum or minimum at an interior point (a, b) of its domain and the first order partial derivatives of f exist there, then $f_x(a, b) = 0$ and $f_y(a, b) = 0$.

Second Derivative Test for Local Extrema:

Let $f(x, y)$ and **its first and second partial derivatives are continuous throughout a disc centered at** (a, b) and $f_x(a, b) = 0 = f_y(a, b)$. Now find out

$$f_{xx}f_{yy} - f_{xy}^2, \text{ at } (a, b)$$

1. If $f_{xx}f_{yy} - f_{xy}^2 = 0$ at (a, b)**Test is Inconclusive**.
2. If $f_{xx}f_{yy} - f_{xy}^2 < 0$ at (a, b) then **Saddle Point at** (a, b).
3. If $f_{xx}f_{yy} - f_{xy}^2 > 0$ at (a, b) and $f_{xx}(a, b) > 0$ (or $f_{yy}(a, b) > 0$) then **Point of Minima at** (a, b).
4. If $f_{xx}f_{yy} - f_{xy}^2(a, b) > 0$ and $f_{xx}(a, b) < 0$ (or $f_{yy}(a, b) < 0$) then **Point of Maxima at** (a, b).

Proof for Maxima and Minima

$$\begin{aligned} f(x,y) &= f(x-a+a, y-b+b) \\ &= f(a,b) + (x-a)f_x(a,b) + (y-b)f_y(a,b) \\ &\quad + \frac{1}{2}\Big\{ f_{xx}(a,b)(x-a)^2 + 2f_{xy}(a,b)(x-a)(y-b) \\ &\quad + (y-b)^2 f_{yy}(a,b) \Big\} + \cdots \end{aligned}$$

Let $(x-a) = h$ and $(y-b) = k$. At Max & Min $f_x(a,b) = f_y(a,b) = 0$. Thus

$$f(x,y) - f(a,b) = \frac{1}{2}k^2 \left\{ A\left(\frac{h}{k}\right)^2 + 2B\left(\frac{h}{k}\right) + C. \right\}$$

$$f(x,y) - f(a,b) = \frac{1}{2}k^2 \left\{ A\xi^2 + 2B\xi + C. \right\}$$

Where $A = f_{xx}(a,b)$, $B = f_{xy}(a,b)$, $C = f_{yy}(a,b)$ and $\xi = \frac{h}{k}$.

Remark:

The extreme values of $f(x, y)$ can occur only at

1. boundary points of the domain of f,
2. critical points (interior points where $f_x = f_y = 0$ or points where f_x or f_y fail to exist).

Problem

Find all the local maxima, local minima, and saddle points of the function

$$f(x, y) = x^3 + y^3 + 3x^2 - 3y^2 - 8.$$

Solution: Critical Points are $(0,0)$, $(0,2)$, $(-2,0)$, $(-2,2)$.
$(0,0)$ is a SADDLE POINT.
$(0,2)$ is LOCAL MINIMUM and $f(0,2) = -12$.
$(-2,0)$ is LOCAL MAXIMUM and $f(-2,0) = -4$.
$(-2,2)$ is a SADDLE POINT.

14.7: Exercise 37

Problem: **Find the absolute maxima and minima of the following function on the given domain**

$$\mathbf{f(x,y)} = \left(\mathbf{4x} - \mathbf{x^2}\right)\cos\mathbf{y}, \quad \mathbf{1} \leq \mathbf{x} \leq \mathbf{3}, \quad -\frac{\pi}{\mathbf{4}} \leq \mathbf{y} \leq \frac{\pi}{\mathbf{4}}.$$

Ans: Absolute Maximum at $\mathbf{4}$ at $(\mathbf{2}, \mathbf{0})$ and Absolute Minimum $\frac{\mathbf{3}\sqrt{\mathbf{2}}}{\mathbf{2}}$ at $\left(3, \pm\frac{\pi}{4}\right)$ and $\left(1, \pm\frac{\pi}{4}\right)$.

Motivation: Article 14.8, Example 1

Finding a Minimum with Constraint

Find the point

$$P(x, y, z)$$

closest to the origin on the plane

$$2x + y - z = 5.$$

Formulation:

$$\text{Minimize} : f(x, y, z) = x^2 + y^2 + z^2$$

$$\text{Subject to} : 2x + y - z = 5$$

Motivation: Example 2 Article 14.8

Finding a Minimum with Constraint

Find the point

$$P(x, y, z)$$

closest to the origin on the hyperbolic cylinder

$$x^2 - z^2 - 1 = 0.$$

Formulation:

$$\text{Minimize}: \quad f(x, y, z) = x^2 + y^2 + z^2 \tag{1}$$

$$\text{Subject to}: \quad x^2 - z^2 - 1 = 0 \tag{2}$$

Alternate: **Another way to find the points on the cylinder $x^2 - z^2 - 1 = 0$ closest to the origin is to imagine a small sphere $x^2 + y^2 + z^2 - a^2 = 0$ centered at the origin expanding like a soap bubble until it just touches the cylinder.**

Motivation: Example 2 Article 14.8; Contd.

At each point of contact, the cylinder and sphere have the same tangent plane and normal line. Therefore, if the sphere and cylinder are represented as the level surfaces obtained by setting

$$f(x,y,z) = x^2 + y^2 + z^2 - a^2; \quad g(x,y,z) = x^2 - z^2 - 1,$$

we get the following,

$$\nabla f = \lambda \nabla g$$

$$2x\widehat{i} + 2y\widehat{j} + 2z\widehat{k} = \lambda \left(2x\widehat{i} - 2z\widehat{k} \right).$$

$$2x = 2\lambda x, \quad 2y = 0, \quad 2z = -2\lambda z.$$

$$x = \pm 1, \quad y = 0, \quad z = 0. \quad ???$$

Note: This method is known as Method of Lagrange's Multipliers.

THEOREM 12 The Orthogonal Gradient Theorem

Suppose that $\mathbf{f}(\mathbf{x},\mathbf{y},\mathbf{z})$ is a differentiable region whose interior contains a smooth curve

$$\mathbf{C} : \mathbf{r}(t) = g(t)\hat{\mathbf{i}} + h(t)\hat{\mathbf{j}} + k(t)\hat{\mathbf{k}}.$$

If $\mathbf{P_0}$ is a point on $\mathbf{C}$ where $\mathbf{f}$ has a local maximum or minimum relative to its values on $\mathbf{C}$, then $\nabla\mathbf{f}$ is orthogonal to $\mathbf{C}$ at $\mathbf{P_0}$.

Corollary

At the points on a smooth curve $\mathbf{r}(t) = g(t)\hat{\mathbf{i}} + h(t)\hat{\mathbf{j}}$ *where a differentiable function* $\mathbf{f}(\mathbf{x},\mathbf{y})$ *takes on its local maxima and minima relative to its values on the curve,* $\nabla\mathbf{f}\cdot\mathbf{v} = 0$*, where* $\mathbf{v} = \frac{d\mathbf{r}}{dt}$*.*

The Method of Lagrange's Multipliers

Suppose that $\mathbf{f(x,y,z)}$ is a differentiable region whose interior contains a smooth curve (on the surface $\mathbf{g} = 0$)

$$\mathbf{C} : \mathbf{r}(t) = g(t)\hat{\mathbf{i}} + h(t)\hat{\mathbf{j}} + k(t)\hat{\mathbf{k}}.$$

If $\mathbf{P_0}$ is a point on $\mathbf{C}$ where $\mathbf{f}$ has a local maximum or minimum relative to its values on $\mathbf{C}$, then $\nabla\mathbf{f}$ is orthogonal to $\mathbf{C}$ at $\mathbf{P_0}$ (and hence on the surface $\mathbf{g} = 0$ at (P_0)).

Lagrange's Multiplier

- **Suppose that $f(x, y, z)$ and $g(x, y, z)$ are differentiable and that P_0 is a point on the surface $g(x, y, z) = 0$ where f has a local maximum or minimum value relative to its other values on the surface.**
- **Then f takes on a local maximum or minimum at P_0 relative to its values on every differentiable curve through P_0 on the surface $g(x, y, z) = 0$.**
- **Therefore, ∇f is orthogonal to the velocity vector of every such differentiable curve through P_0.** So is ∇**g** (because ∇**g** orthogonal to the level surface **g** = **0**).
- **Therefore, at P_0, ∇f is some scalar multiple λ of ∇g.**

How to use Lagrange's multiplier?

- Let $f(x, y, z)$ and $g(x, y, z)$ are differentiable functions.
- To find the Maximum and Minimum values of $f(x, y, z)$ subject to constraint $g(x, y, z) = 0$.
- STEP 1: Find the values of x, y, z and λ that simultaneously satisfy the equations

$$\nabla f = \lambda \nabla g \quad \text{and} \quad g(x, y, z) = 0.$$

- Evaluate $f(x, y, z)$ at all the points (x, y, z) that result from STEP 1.
- The largest of these values is Maximum and the smallest is the minimum.

Problem

Find point on $\mathbf{z^2 = xy + 4}$ *near to origin.*

Formulation:

$$\text{Minimze} : \mathbf{f(x, y, z) = x^2 + y^2 + z^2}$$

$$\text{Subject to} : \mathbf{g(x, y, z) = z^2 - xy - 4 = 0.}$$

$$\nabla \mathbf{f} = \lambda \nabla \mathbf{g}$$

$$\mathbf{2x = -\lambda y}, \quad \mathbf{2y = -\lambda x}, \quad \mathbf{2z = 2\lambda z}, \quad \mathbf{z^2 - xy - 4 = 0}$$

$$\text{Points} : \mathbf{(0, 0, \pm 2)} \quad \& \quad \mathbf{(2, -2, 0)} \quad \& \quad \mathbf{(-2, 2, 0)}.$$

Point of Minima is $\mathbf{(0, 0, \pm 2)}$

NECESSARY BUT NOT SUFFICIENT

Problem

Maximize $f(x, y) = x + y$ ***subject to*** $g(x, y) = xy - 16 = 0$.

Solution: (4,4) and (-4,-4) is NOT CORRECT.

Lagrange Multipliers with Two Constraints

Find the extreme values of a differentiable function $f(x, y, z)$ whose variables are subject to two constraints. If the constraints are

$$g_1(x, y, z) = 0, \quad g_2(x, y, z) = 0$$

and g_1, g_2 are differentiable with ∇g_1 not parallel to ∇g_2.

Many problems require us to find the extreme values of a differentiable function $f(x, y, z)$ whose variables are subject to two constraints. If the constraints are

$$g_1(x, y, z) = 0, \quad g_2(x, y, z) = 0.$$

and $g_1(x, y, z)$ and $g_2(x, y, z) = 0$ are differentiable, with ∇g_1 not parallel to ∇g_2.

To find the constrained local maxima and minima of f we introduce two Lagrange multipliers λ and μ. That is, we locate the points $P(x, y, z)$ where f takes on its constrained extreme values by finding the values of and x, y, z, λ, μ that simultaneously satisfy the equations

$$\nabla f = \lambda \nabla g_1 + \mu \nabla g_2, \quad g_1(x, y, z) = 0, \quad g_2(x, y, z) = 0.$$

Problem 37, Article 14.8

Find extreme values of the function $\mathbf{f(x,y,z) = x^2yz + 1}$ on the intersection of the plane $\mathbf{z = 1}$ with the sphere $\mathbf{x^2 + y^2 + z^2 = 10}$.

Formulation:

Minimze : $\mathbf{f(x,y,z) = x^2yz + 1}$

Subject to : $\mathbf{g_1 = z - 1 = 0}, \quad \mathbf{g_2 = x^2 + y^2 + z^2 - 10 = 0}.$

$\nabla \mathbf{f} = \lambda \nabla \mathbf{g_1} + \mu \nabla \mathbf{g_2}.$

To find the point of Maxima and Minima we need to solve:

$\mathbf{xyz = \mu x, \; x^2z = 2\mu y, \; x^2y = \lambda + 2\mu z,}$

$\mathbf{z = 1, \; x^2 + y^2 + z^2 = 10}.$

Which gives: $\mathbf{(0, \pm 3, 1)}$ and $\mathbf{(\pm\sqrt{6}, \pm\sqrt{3}, 1)}$

$f(0, \pm 3, 1) = 1$ and $f(\pm\sqrt{6}, \pm\sqrt{3}, 1) = 1 \pm 6\sqrt{3}$.

Maximum of f is $1 + 6\sqrt{3}$ at $(\pm\sqrt{6}, \sqrt{3}, 1)$ and Minimum is $1 - 6\sqrt{3}$ at $(\pm\sqrt{6}, -\sqrt{3}, 1)$.