Date ___/___/______

- 8021
- nktomar@iitp.ac.in
- N.K.TOMAR

Page No.: ______

MA102 — Linear Algebra; Ordinary differential Equations (ODE).

- Linear Operators. $T(ax+by) = aT(x) + bT(y)$.

- Physical Phenomenon $\xrightarrow[\text{Modelling}]{\text{Mathematical}}$ Differential equations.

$\downarrow$ Solution

behaviour of variables involved in physical model.

eg. Newton's Second Law.

$$\boxed{\frac{d^2y}{dx^2} = \frac{f}{m}}$$

* Differential Equation :- An equation involving derivatives of one or more dependent variables w.r.t one or more independent variables is called a diff. eq^n (DE).

eg (1) $\frac{dy}{dx} = ay$; (1) $\frac{dy}{dx} = k(a-y)$, (2) $\frac{d^2y}{dx^2} = \frac{f}{m}$.

(4) $\frac{d^4y}{dx^4} + a\frac{d^2y}{dx^2} + by = \sin x$,

(x), where x is order of DE.

(2) $\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}$

★ Types
- ODE (Ordinary) { Only one independent variable.
- PDE (Partial).

Linear ODE — Non linear ODE.

★ Mathematical form of ODE :- $f(x, y, y', y'', \ldots) = 0$ where, x is independent variable.

Scanned by CamScanner

DE ≅ ODE
for MA102

☆ Order of DE :-

It is the highest order of derative involved in the DE.

☆ First Order ODE (DE) :-

Mathematical form of Ist order ODE :- $f(x,y,y')=0$

Standard form of Ist order ODE :-

(I) $\frac{dy}{dx} = f(x,y)$ (Derivative form)

(II) $M(x,y)\,dx + N(x,y)\,dy = 0$. (Differential form)

Unless specified; y is treated dependent & x as independent

☆ S. L. Ross: Differential Equations. (Textbook)

eg. Population Model. Convert into DE.

$x(t) \Rightarrow$ Population size at t
$b \Rightarrow$ birth rate
$d \Rightarrow$ death rate / individual / time.

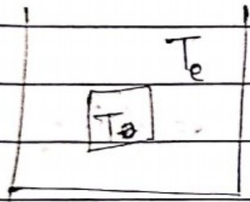
$$\Delta x = x(t+\Delta t) - x(t) = \big(bx(t) - dx(t)\big)\Delta t$$

$$\Rightarrow \frac{\Delta x}{\Delta t} = \underbrace{(b-d)}_{a\,=\,\text{const}}x(t).$$

(if rates are considered constant throughout)

Scanned by CamScanner

★ Newton's law of Cooling.



$$\frac{dT}{dt} = k(T_e - T).$$

★ Method I to solve Ist order DE (Exact DEs):-

Given:- $Mdx + Ndy = 0$. —— ①.

M, N are functions of x and y both,

DE ① is called exact DE if $\exists\ f(x,y)$ s.t.

$$df \equiv \left(\frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy\right)$$
$$\parallel$$
$$Mdx + Ndy.$$

$\Rightarrow df = 0$, Solution $\boxed{f(x,y) = c}$

Solution of DE is family of curves (not unique) which depends on a paramater which is nothing but constant of integration.

★ eg. $xdx + ydx = 0$

$\Rightarrow d\left(\frac{x^2}{2} + \frac{y^2}{2}\right) = 0$

Sol^n :- $\frac{x^2}{2} + \frac{y^2}{2} = c$.

$\Rightarrow \boxed{x^2 + y^2 = c}$

Scanned by CamScanner

Date ___/___/______ Page No.: ______

Discussion → To determine exactness of given DE, and corresponding solutions.

If ① is exact ⇒ $\frac{\partial f}{\partial x} = M$ and $\frac{\partial f}{\partial y} = N$ —(i).

$$\Rightarrow \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial M}{\partial y} \quad \& \quad \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial N}{\partial x}$$

(I) ⇒ $\boxed{\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}}$ ★ condition :- {partial derivatives of M & N are continuous}

So, ① is exact ⟺ $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ {& continuous}

⇑

Test to check exactness of DE

from (i) ; $\boxed{f(x,y) = \int M(x,y)\partial x + \phi(y)}$

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}\int M(x,y)\partial x + \frac{d\phi(y)}{dy}$$

from (i) ; $\boxed{N(x,y) = \frac{\partial}{\partial y}\int M(x,y)\partial x + \frac{d\phi}{dy}}$

★ Algorithm.

0. Write equation as $Mdx + Ndy = 0$.
1. Check if equation is exact —— (I).
2. If exact, find f
3. Sol $\boxed{f(x,y) = c}$

Maharaza

Scanned by CamScanner

Q. $(3x^2+4xy)\,dx + (2x^2+2y)\,dy = 0$.

$\Rightarrow$ Check for exactness.

$M = 3x^2+4xy$; $N = 2x^2+2y$.

$\frac{\partial M}{\partial y} = 4x$; $\frac{\partial N}{\partial x} = 4x$

$\Rightarrow \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \Rightarrow$ D.eq^n is exact.

Since, equation is exact, $\exists\, f(x,y)$ s.t.

$$\left\{ \frac{\partial f}{\partial x} = 3x^2+4xy \;;\; \frac{\partial f}{\partial y} = 2x^2+2y \right\}$$

$$\Rightarrow f(x,y) = 3.\frac{x^3}{3} + 4y.\frac{x^2}{2} + \phi(y)$$

$$\Rightarrow f(x,y) = x^3 + 2x^2y + \phi(y).$$

$$\Rightarrow \frac{\partial f}{\partial y} = 2x^2 + \frac{d\phi}{dy}.$$

$$\Rightarrow 2x^2+2y = 2x^2 + \frac{d\phi}{dy}$$

$$\Rightarrow \frac{d\phi}{dy} = 2y \Rightarrow \phi = y^2$$

$$\Rightarrow f(x,y) = x^3 + 2x^2y + y^2$$

$\therefore$ $\boxed{\text{Solution is } x^3 + 2x^2y + y^2 = C}$

Scanned by CamScanner

★ Method II :- Separation of Variable.

Form – $f(x)\,dx + g(y)\,dy = 0$.

$\frac{dF}{\partial x} = f(x)$ $\qquad \frac{\partial F}{\partial y} = g(y)$.

$F = \int f(x)dx + \phi(y) = \int f(x)dx + \int g(y)dy$

$\frac{\partial F}{dy} = \frac{d\phi}{dy} = g(y)$

Its solution is $\boxed{\int f(x)dx + \int g(y)dy = c}$

Generalized form for separation of Variables

$f(x).F(y)\,dx + g(y)\,G(x)\,dy = 0$. —— ①

By $\frac{1}{F(y)\,G(x)} \times$ ① we obtain,

$\frac{f(x)}{G(x)}\,dx + \frac{g(y)}{F(y)}\,dy = 0$

Soln :- $\boxed{\int \frac{f(x)}{G(x)}\,dx + \int \frac{g(y)}{F(y)}\,dy = c}$

★ Solve. $(x-4)y^4\,dx - x^3(y^2-3)\,dy = 0$ —— ①

⇒ By $\frac{1}{y^4 \times (-x^3)} \times$ ① we obtain.

Scanned by CamScanner

Date ___/___/______ Page No.: ______

$$(-1)\left(\frac{x-4}{x^3}\right).dx + \left(\frac{y^3-3}{y^4}\right) dy = 0.$$

$$(-1)\left(\frac{1}{x^2}.dx - \frac{4}{x^3}.dx\right) + \left(\frac{1}{y}.dy - \frac{3}{y^4}.dy\right) = 0$$

Integrating ;

$$(-1)\left[\int\frac{1}{x^2}.dx - \int\frac{4}{x^3}.dx\right] + \int\frac{1}{y}dy - \int\frac{3}{y^4}.dy = 0.$$

$$\Rightarrow (-1)\left[-\frac{1}{x} - \frac{4}{(-2)x^2}\right] + \left(\ln y - \frac{3}{-3y^3}\right) = 0$$

$$\Rightarrow \boxed{\frac{1}{x} - \frac{2}{x^2} + \ln y + \frac{1}{y^3} = C}$$

★ Tutorial-5 Linear Algebra

1.

Scanned by CamScanner

3. V is inner product space : then
$|\langle x, y\rangle| = \|x\| \cdot \|y\|$ iff
$x = cy$ or $y = cx$.

$\Rightarrow$

4. T is linear operator on V, $\|T(x)\| = \|x\|$ for all x Prove that T is one-one.

$\Rightarrow$

Let $x_1, x_2 \in V$

$\Rightarrow T(x_1) = T(x_2)$
$\Rightarrow T(x_1) - T(x_2) = 0$
$\Rightarrow T(x_1 - x_2) = 0$
$\Rightarrow \|T(x_1 - x_2)\| = \|0\| = 0.$
$\Rightarrow \|(x_1 - x_2\| = 0.$
$\Rightarrow x_1 - x_2 = 0$
$\Rightarrow x_1 = x_2$

$\therefore$ T is one one.

Scanned by CamScanner

★ Gram-Schmidt process to obtain orthonormal set.

★ v_1 is given vector then $w_1 = \dfrac{v_1}{|v_1|}$ is called orthonormal to v_1 vector

Let $\{v_1, \ldots, v_n\}$ is basis of V.
We can construct orthogonal basis of V $\{w_1, \ldots, w_n\}$ as follows.

Take $w_1 = v_1 /$

$$w_2 = v_2 - \frac{\langle v_2, w_1\rangle}{\langle w_1, w_1\rangle} w_1 \qquad (\because \langle w_1, w_2\rangle = 0$$

$$w_3 = v_3 - \frac{\langle v_3, w_1\rangle}{\langle w_1, w_1\rangle} w_1 - \frac{\langle v_3, w_2\rangle}{\langle w_2, w_2\rangle} w_2 .$$

Q.5) (b) $\{\underset{v_1}{1}\,;\ \underset{v_2}{p_1(t) = t}\,,\ \underset{v_3}{p_2(t) = t^2}\}$ of P_2 with the inner product $\langle p, q\rangle = \int_0^1 p(t)q(t)\,dt$.

Let, orthogonal set is $\{w_1, w_2, w_3\}$.

$w_1 = 1$.

$w_3 = t^3 + \frac{1}{6} - t$

$$w_2 = v_2 - \frac{\langle v_2, w_1\rangle}{\langle w_1, w_1\rangle} w_1$$

$$\langle v_2, w_1\rangle = \int_0^1 t \cdot 1\, dt = 1/2 .$$

$$w_2 = t - \frac{1/2}{1} \times 1 \rightarrow w_2 = t - \frac{1}{2} .$$

$$w_3 = t^2 - \int_0^1 t^2 dt - \int_0^1 t^2\left(t - \tfrac{1}{2}\right) dt \times \left(t - \tfrac{1}{2}\right) .$$

$$w_3 = t^2 - \frac{1}{3} + \left(\frac{1}{4} + \frac{1}{6}\right) \times \left(t - \frac{1}{2}\right) .$$

Assignment – 1. Sem II (Differential Equations)

Q.1)

		Linear	Nonlinear	Ordinary	Partial	Order
(i)	$y'' + 3y' + 20y = e^x$	✓	✗	✓	✗	2
(ii)	$\sqrt{1+y'^3} = x^2$ $\Rightarrow 1+y'^3 = x^4$	✗	✓	✓	✗	1
(iii)	$y'' + y^2 = \cos x$	✗	✓	✓	✗	2
(iv)	$y' + xy = \cos y'$	✗	✓	✓	✗	1
(v)	$(xy')' = xy$ $\Rightarrow y' + xy'' = xy$	✓	✗	✓	✗	2
(vi)	$u_x + u_y = 0$	✓	✗	✗	✓	1

$u(x,y)$

$\Rightarrow u_x = \dfrac{\partial u}{\partial x}$

$\&\ u_y = \dfrac{\partial u}{\partial y}$.

$u(x,y,t)$

$u_{xx} = \dfrac{\partial^2 u}{\partial x^2}$ $\quad u_t = \dfrac{\partial u}{\partial t}$.

$u_{yy} = \dfrac{\partial^2 u}{\partial y^2}$

		Linear	Nonlinear	Ordinary	Partial	Order
(vii)	$u_{xx} + u_{yy} = u_t$	✓	✗	✗	✓	2

Q.8) Show that following eqns are exact and hence find their general solution.

$$M(x,y)\cdot dx + N(x,y)\cdot dy = 0.$$

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \quad \text{(for exactness)}.$$

Maharaza

Scanned by CamScanner

Date ___/___/______ Page No.: ______

(i) $(\cos x \cos y - \cot x) \cdot dx - (\sin x \sin y)\, dy = 0$.

$\Rightarrow$

$$\frac{\partial M}{\partial y} = -\cos x \sin y \; ; \; \frac{\partial N}{\partial x} = -\cos x \cdot \sin y .$$

$\Rightarrow \int M dx = c_1$

$\Rightarrow \int \{\cos x \cdot \cos y - \cot x\} \cdot dx = c_1$.

$\Rightarrow \int \cos x \cdot \cos y \, dx - \int \cot x \cdot dx = c_1$

$\Rightarrow \sin x \cos y - \int \frac{\cos x}{\sin x} \cdot dx = c_1$.

$\Rightarrow \sin x \cos y - \ln(t) = c_1 \quad (t = \sin x)$

$\Rightarrow \sin x \cos y - \log_e(\sin x) + \phi(y) = c_1$.

$\rightarrow \frac{\partial f}{\partial y} = N$.

$\Rightarrow \sin x \sin y - \frac{d\phi}{dy} = 0$.

$\Rightarrow d\phi = \sin x \sin y \, dy$

$\rightarrow \phi(y) = \int \sin x \sin y \cdot dy$

$= -\sin x \cos y$.

$\rightarrow \boxed{\sin x \cos y - \log_e(\sin x) - \sin x \cos y = c_1}$.

Scanned by CamScanner

Linear Algebra continued.

★ **Diagonalization** $PAP^{-1} = D$

Let A be a square matrix of order n over the field F then if there exists an (non-singular) invertible matrix P such that $PAP^{-1} = D$ then we call matrix P diagonalize the matrix A, where D is a diagonal matrix

eg. $A_{3\times3}$.

λ_1 $x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ λ_2 $y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$ λ_3 $z = \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix}$

$$P = [x \ y \ z].$$

$$PAP^{-1} = D = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}.$$

★ If eigen vectors are linearly independent then only the given matrix can be diagonal-ized.

★ **Fixed point of function** :-

$f : A \to A$ $x \in A$, $f(x) = x$;
then x is called fixed point of f.

★ **Banach fixed Point Theorem** :-

Scanned by CamScanner

$< \alpha x + \beta y, z > = \alpha < x, z > + \beta < y, z >$ the left...

Page No.:

Date / /

* Bilinear form :- / Bilinear functional.

Defn :- let U and V be two vector spaces over the same field F. Let $W = U \times V$ (or $U \oplus V$). $(W, +, \cdot)$ is a vector space where addition and scalar multiplication W is defined as usual i.e.

$(x_1, x_2)\ (y_1, y_2) \in W$

$\Rightarrow$

$$(x_1, x_2) + (y_1, y_2) = (x_1 + y_1, x_2 + y_2).$$

$$\alpha (x_1, x_2) = (\alpha x_1, \alpha x_2)$$

A map $f : W \times W \to F$ is said to be a bilinear form if (i) $f(\alpha x + \beta y, z) = \alpha f(x, z) + \beta f(y, z)$ and (ii) $f(x, \gamma y + \delta z) = \gamma f(x, y) + \delta f(x, z)$ where, $x, y, z \in W$ ~~$\times W$~~ ; $\alpha, \beta, \gamma, \delta \in F$.

(i) linearity from left (ii) linearity from right.

Here, f is bilinear form from U to V.
If $U = V$; f is bilinear form on U.

eg. $V = \mathbb{R}^2 = \{ (x, y) \mid x, y \in \mathbb{R} \}$.

$f : V \times V \to F$ defined by.

$f(\alpha, \beta) = x_1 y_2 - x_2 y_1$.

where $\alpha = (x_1, x_2)$; $\beta = (y_1, y_2)$.

Whether f is bilinear form or not ?

Ans $\Rightarrow$

$$f(a\alpha + b\beta, \gamma) = f(a(x_1, x_2) + b(y_1, y_2), (z_1, z_2))$$
$$= f((ax_1 + by_1, ax_2 + by_2), (z_1, z_2))$$
$$= (ax_1 + by_1) z_2 - (ax_2 + by_2)(z_1). \quad \text{--- (1)}$$

$$f(\alpha, c\beta + d\gamma) = f((x_1, x_2), c(y_1, y_2) + d(z_1, z_2))$$
$$= f((x_1, x_2), (cy_1 + dz_1, cy_2 + dz_2))$$
$$= x_1 (cy_2 + dz_2) - x_2 (cy_1 + dz_1) \quad \text{--- (2)}$$

$a, b, c, d \in \mathbb{R}$; $\alpha = (x_1, x_2)$

$\gamma = (z_1, z_2)$; $\beta = (y_1, y_2)$

Scanned by CamScanner

Also check for $f_2(\alpha,\beta) = (x_1-y_1)^2 + 2x_2y_2$.

Date ___/___/______ Page No.: ______

$af(\alpha,\gamma) + bf(\beta,\gamma)$.

$= a(x_1z_2 - x_2z_1) + b(y_1z_2 - y_2z_1)$

$= (ax_1+by_1)z_2 - (ax_2+by_2)z_1$ — (3)

$cf(\alpha,\beta) + df(\alpha,\gamma)$.

$= c(x_1y_2 - x_2y_1) + d(x_1z_2 - x_2z_1)$

$= x_1(cy_2+dz_2) - x_2(cy_1+dz_1)$ — (4)

$\therefore$ from (1) & (3) & from (2) & (4) $\Rightarrow f$ is bilinear

* $f_2(\alpha,\beta) = (x_1-y_1)^2 + 2x_2y_2$

Maharaza

Scanned by CamScanner

$A^T = A$ Symmetrix
$A^T = -A$ Skew Symmetrix.

Date ___/___/_____ Page No.: _____

* Hermitian Matrix $A = (\bar{A})^T$

Skew Hermitian Matrix. $A = -(\bar{A})^T$.

* Jordan Decomposition Theorem.

* Any Linear transformation T is said to be symmetric / skew-symmetric if its matrix representation is symmetric / skew-symmetric.

* A bilinear form f on vector space V(F) is said to be symmetric if $f(x,y) = f(y,x)$; $x,y \in V$.

- skew symmetric if $f(x,y) = -f(y,x)$; $x,y \in V$.

- $f: V \times V \to F$; $B = \{x_1, x_2, \ldots, x_n\}$ is basis of V

- $f(x,y) = f\left(\sum_{i=1}^{n} a_i x_i, \sum_{j=1}^{n} b_j x_j\right)$.

$$= \sum_{j=1}^{n}\sum_{i=1}^{n} a_i b_j f(x_i, x_j).$$

$f(x_i, x_j) = a_{ij}$, $1 \le i,j \le n$

Matrix representation of bilinear form

$$[f]_B = (a_{ij})_{n \times n}$$

* Quadratic form :-

Let f be a bilinear form on a vector space V(F). Then a map $q: V \to F$ as

$$q(x) = f(x,x) \; ; \; \forall x \in V$$

is called quadratic form. $(\because f(x,y) = f(y,x))$

Maharaza

- It is symmetric bilinear form as $f(x,x) = f(x,x)$

Scanned by CamScanner

Date Qudratic form. Page No.:

Def^n using Inner product space.

let T be a symmetric linear operator on a vector space $V(F)$. Then a map $q : V \to F$ is called a quadratic form if

$$q(x) = \langle T(x), x \rangle ; \quad \forall x \in V.$$

Standard Inner Product
$x = (x_1, \dots, x_n)$; $y = (y_1, \dots, y_n)$.
$\langle x, y \rangle = x_1 y_1 + \dots + x_n y_n$

g $f(x,y) = ax^2 + 2hxy + by^2$

$$= \underset{1\times 2}{[x \;\; y]} \underset{2\times 2}{\begin{bmatrix} a & h \\ h & b \end{bmatrix}} \underset{2\times 1}{\begin{bmatrix} x \\ y \end{bmatrix}}$$

$$= \underset{1\times 2}{[ax+hy \quad hx+by]} \underset{2\times 1}{\begin{bmatrix} x \\ y \end{bmatrix}}$$

$$= [ax^2 + hxy + hxy + by^2]_{1\times 1}$$

$$= ax^2 + 2hxy + by^2.$$

let, $u = [x, y]$; $q(u) = \langle T(u), u \rangle$.

$$g(x,y) = ax^2 + by^2 + cz^2 + 2hxy + 2fyz + 2gxz.$$

$$\underset{1\times 3}{[x \;\; y \;\; z]} \underset{3\times 3}{\begin{bmatrix} a & h & f \\ h & b & g \\ f & g & c \end{bmatrix}} \underset{3\times 3}{\begin{bmatrix} x \\ y \\ z \end{bmatrix}}$$

Maharaza

Scanned by CamScanner

$$ds^2 = dx^2 + dy^2 + dz^2$$
$$ds^2 = dr^2 + r^2 d\theta^2 + dz^2.$$

Not in syllabus for now (IV sem)

positive definite	negative definite	indefinite.
All principal minors >0	All principal minors <0	Random.

principal minor $\Rightarrow$ determinants (all possible)

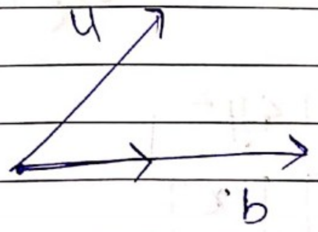
for 3×3 $\Rightarrow$ 1×1 ; 2×2 ; 3×3

Orthogonal $\langle x, y\rangle = 0$.

Orthonormal. $\langle x, y\rangle = 0$; $\langle x, x\rangle = 1$ & $\langle y, y\rangle = 1$

Tut-5.

Q.6).



$$P_b(u) = \frac{\langle u.b\rangle}{\|b\|^2}\vec{b}.$$

(a) $V = \mathbb{R}^2$; $u = (2,6)$ and $W = \{(x,y) \mid y = 4x\}$

$u = (2,6)$; $b = (1,4)$.

$$P_b(u) = \frac{26}{17}(1,4)$$

Orthogonal projection $=$ $\cancel{E}$ $P_b(u) = \frac{26}{17}(1,4)$

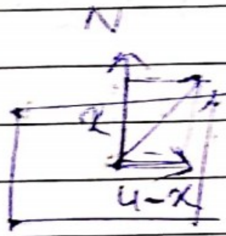
Scanned by CamScanner

Date __/__/____ Page No.: ______

(b) $V = \mathbb{R}^3$, $u = (2,1,3)$ and $W = \{(x,y,z) \mid x+3y-2z = 0\}$

$\Rightarrow$ $u = (2,1,3)$; $b = (1,3,-2)$.

$$\Rightarrow \quad u - \left[\frac{\langle u . b \rangle}{\|b\|^2} \vec{b}\right] (x)$$



$$(2,1,3) - \frac{(-1)}{14}(1,3,-2).$$

$$= \left(\frac{29}{14}, \frac{17}{14}, \frac{40}{14}\right).$$

(c). $V = P(\mathbb{R})$ $\quad \langle f,g \rangle = \int_0^1 f(t)\,g(t)\,dt.$

Q.9) Associated matrix of $f(X) = x^2 - 3xy + 4y^2$.

$$f(x) = x^T A x = x^2 - 3xy + 4y^2$$

$$= (x,y) \begin{bmatrix} 1 & -3/2 \\ -3/2 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

10) bilinear form from quadratic form.

(a) $x_1 x_2$.

Bilinear form

$f(x,y) = x^T A y$.

$$A = \begin{bmatrix} 0 & 1/2 & 0 & 0 \\ 1/2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Quadratic form

$f(x) = x^T A x$.

$x = (x_1\ x_2\ x_3\ x_4)$

$y = (y_1\ y_2\ y_3\ y_4)$.

Scanned by CamScanner

$$f(x,y) = x^T A y$$

$$= \begin{pmatrix} x_1 & x_2 & x_3 & x_4 \end{pmatrix} \begin{bmatrix} 0 & 1/2 & 0 & 0 \\ 1/2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix}$$

$$f(x,y) = \begin{bmatrix} \frac{x_2}{2} & \frac{x_1}{2} & 0 & 0 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix} \Rightarrow \left[\frac{x_2 y_1}{2} + \frac{x_1 y_2}{2} \right.$$

$$f(x,y) = (x_2 y_1 + x_1 y_2)/2$$

(b) $x_1 x_3 + x_4^2 = f(x) = x^T A x$

$$f(x,y) = x^T A y$$

$\Rightarrow$ $$A = \begin{bmatrix} 0 & 0 & 1/2 & 0 \\ 0 & 0 & 0 & 0 \\ 1/2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \underset{1\times 4}{(x_1 x_2 x_3 x_4)} \underset{4\times 4}{\begin{bmatrix} 0 & 0 & 1/2 & 0 \\ 0 & 0 & 0 & 0 \\ 1/2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}} \underset{4\times 1}{\begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix}}$$

$$f(x,y) = \begin{bmatrix} \frac{x_3}{2} & 0 & \frac{x_1}{2} & x_4 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix}$$

$$= \frac{x_3 y_1 + x_1 y_3}{2} + x_4 y_4.$$

Q.7) $V(F)$; $W \leq V$

$$W^\perp = \{ v \in V \mid \langle v, w \rangle = 0 \ \forall w \in W \}.$$

P.T. $z \in W^\perp \Leftrightarrow \langle z, v \rangle = 0 \ \forall v \in \beta$ is the basis of W.

Scanned by CamScanner

Differential Equations Continued

(Revision).

Mathematical form of ODE $f(x, y, y', y'', \ldots) = 0$.

Order of DE $\Rightarrow$ highest order of derivative involved.

Mathematical form of 1^{st} order ODE $\{f(x, y, y') = 0$

Standard form of 1^{st} order ODE

$\frac{dy}{dx} = f(x, y)$ —① $M(x, y)\,dx + N(x, y)\,dy = 0$ —②

(derivative form)   (differential form)

Exact ODE of first order: ② is called exact if $\exists$ $F(x, y)$ s.t. $dF = M\,dx + N\,dy$; moreover solution in the case is $\boxed{F(x, y) = c}$.

$\boxed{\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}}$ condition to determine exactness

Separation of variables $\Rightarrow$ $f(x)\,dx + g(y)\,dy = 0$.

Soln :- $\int f(x)\,dx + \int g(y)\,dy = C$.

General form for separable method

$f(x)F(y)\,dx + g(y).G(x).dy = 0$ — ③.

By $\frac{1}{F(y)\,G(x)} \times$ ③ we obtain

$f(x)/G(x) \cdot dx + g(y)/F(y) \cdot dy = 0$. — ③'

Soln :- $\boxed{\int \frac{f(x)}{G(x)}\,dx + \int \frac{g(y)}{F(y)}\,dy = C.}$

Maharaza

The solution is not exactly the solution for eq^n (3) as there may be some more or less solutions for the eq^n (3).

eq^n (3) need not be exact always but (3') is always exact

So, to make the eq^n (3) exact we have multiplifd it with a certain function. This function is called as Integrating factor (IF)

IF is not unique. It can be also completely different along with constant multiples of obtaine IF.

★ Linear first Order ODE's :-

* linear ODE :- An ODE is linear if dependent variable and its derivatives appear to the power 1. (Product of dependent variable and its derivatives are not allowed) otherwise ODE is non-linear.

eg. $\frac{dy}{dx} = ay$; $m\frac{d^2y}{dt^2} = f$; $\frac{d^4y}{dx^2} + a\frac{d^2y}{dx^2} + by = \sin x$.

Some Non-linear e.g. $\frac{d^4y}{dx^2} + a\frac{d^2y}{dx^2} + by = \sin y$;

$\frac{d^4y}{dx^2} + \left(\frac{dy}{dx}\right) \cdot \frac{d^2y}{dx^2} + by = \sin x$.

* Mathematical form of n^{th} order of linear ODE;

$$a_0 \cdot \frac{d^ny}{dx^n} + a_1 \frac{d^{n-1}y}{dx^{n-1}} + a_2 \cdot \frac{d^{n-2}y}{dx^{n-2}} + \ldots + a_{n-1}\frac{dy}{dx} + a_ny = b$$

Maharaza

where, a_i and b are either constants or functions of x only.

★ Linear First order ODE'S :-

$$\boxed{a_0 \cdot \frac{dy}{dx} + a_1 y = b}$$

General Mathematical form.

* Standard form of Ist order linear ODE :

$$\boxed{\frac{dy}{dx} + p(x) \cdot y = q(x)} \quad \text{--- (4)}$$

let u be a function s.t. u × L.H.S. of (4) is complete derivate of a function (i.e. Given ODE becomes exact). such u is called Integrating factor (IF) for (4).

$$u \cdot \frac{dy}{dx} + u \cdot p(x) \cdot y = u \cdot q(x)$$

gives $\frac{d}{dx}(uy) = u \cdot q(x)$ --- (5)

Thus for solution (4) integrate (5) w.r.t x;

for u; $u' = u \cdot p(x)$

$$\frac{du}{dx} = u \cdot p(x)$$

$$\Rightarrow \frac{du}{u} = p(x) \cdot dx$$

$$\Rightarrow \ln u = \int p(x) \cdot dx$$

$$\Rightarrow u = e^{\int p(x) \cdot dx}$$

$$\Rightarrow \boxed{u(x) = e^{\int p(x) \cdot dx}}$$

* Algorithm to solve I^{st} order linear ODE :-

① Write the equation in following form

$$\frac{dy}{dx} + p(x).y = q(x) \quad — ①$$

② Find IF ; $u(x) = e^{\int p(x).dx}$.

③ Multiply u in expression ① and obtain

$$(uy)' = uq(x)$$

④ Solution of ① is $\boxed{uy = \int uq(x).dx + c}$.

e.g. (I) $x.\frac{dy}{dx} - y = x^3$.

$\rightarrow \quad \frac{dy}{dx} - \frac{1}{x} y = x^2$.

IF ; $u(x) = e^{\int -(1/x).dx} = \frac{1}{x}$.

$$\frac{1}{x}.\frac{dy}{dx} - \frac{1}{x^2}.y = x$$

$$\frac{d}{dx}\left(\frac{1}{x}.y\right) = x.$$

$\Rightarrow \quad \frac{1}{x}.y = \frac{x^2}{2} + c.$

$\Rightarrow \quad \boxed{\frac{y}{x} = \frac{x^2}{2} + c} \Rightarrow$ Solution

(II) $(1+\cos x)y' - (\sin x).y = 2x$.

$\Rightarrow$ $$y' - \left(\frac{\sin x}{1+\cos x}\right).y = 2\left(\frac{x}{1+\cos x}\right)$$

IF. $$u(x) = e^{-\int \frac{\sin x}{1+\cos x}.dx}$$

$$= e^{\ln(1+\cos x)}$$

$$u(x) = 1+\cos x$$

$$(1+\cos x).y' - (\sin x)y = 2x.$$

$$\frac{d}{dx}\left((1+\cos x).y\right) = 2x.$$

$$(1+\cos x).y = \cancel{2}.\frac{x^2}{\cancel{2}} + c$$

$\Rightarrow$ $\boxed{y(1+\cos x) = x^2+c}$ is the solution.

★ First order ODE's :-

1. Exact method $Mdx + Ndy = 0$ is exact iff $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$.
2. Separation method $f(x).dx + g(y).dy = 0$.
   Soln :- $\int f(x).dx + \int g(y).dy = C$.
3. Linear ODE's method $\frac{dy}{dx} + p(x)y = q(x)$.
   Soln :- $(uy)' = uq$; $u = e^{\int p(x).dx}$

④ Make exact, if possible.

⑤ Substitution.

★ Bernoulli's Equation :-

$$\frac{dy}{dx} + p(x).y = q(x).y^a \quad (a.\text{constant}).$$

$$\Rightarrow y^{-a}.\frac{dy}{dx} + p(x).y^{1-a} = q(x).$$

Substitution $u = y^{1-a}$ $\quad \frac{du}{dx} = (1-a).y^{-a}.\frac{dy}{dx}$

$\Rightarrow$ By subs. $u = y^{1-a}$; we obtain

$$\frac{1}{1-a}.\frac{du}{dx} + p(x).u = q(x).$$

$$\boxed{\frac{du}{dx} + (1-a)p(x)u = (1-a)q(x)}.$$

Linear ; first find u and then (y).

e.g. $\frac{dy}{dx} - Ay = -By^2$ (A & B are constant).

$$\Rightarrow y^{-2}.\frac{dy}{dx} - Ay^{-1} = -B.$$

Let, $u = y^{-1}$; $\quad \frac{du}{dx} = \frac{-1}{y^2}.\frac{dy}{dx}.$

$$-\frac{du}{dx} - Au = -B. \Rightarrow \frac{du}{dx} + Au = B.$$

$$\frac{du}{dx} + Au = B.$$

Its solution

$$I.F. = e^{\int A.dx} = e^{Ax}.$$

$$e^{Ax}.\frac{du}{dx} + e^{Ax}.Au = Be^{Ax}$$

$$\Rightarrow \frac{d}{dx}(u.e^{Ax}) = Be^{Ax}$$

$$\Rightarrow \int (u.e^{Ax})' dx = \int Be^{Ax}.dx$$

$$\Rightarrow u.e^{Ax} = \frac{B}{A}.e^{Ax} + c.$$

$$u = y^{-1}$$

$$\Rightarrow \frac{e^{Ax}}{y} = \frac{B}{A}e^{Ax} + C'$$

$$\Rightarrow y = \frac{Ae^{Ax}}{Be^{Ax} + C}$$

* $\frac{dy}{dx} = \frac{y}{x} \Rightarrow \frac{dy}{dx} - \frac{1}{x}.y = 0.$

$$IF = e^{-\int \frac{1}{x}.dx} = 1/x.$$

$$\Rightarrow \frac{1}{x}.\frac{dy}{dx} - \frac{1}{x^2}.y = 0$$

$$\Rightarrow \left(\frac{1}{x}.y\right)' = 0$$

Maharaza

$$\Rightarrow \boxed{\frac{y}{x} = \text{Const.}}$$

Notation:- $M_y = \frac{\partial M}{\partial y}$; $N_x = \frac{\partial N}{\partial x}$.

Date ___/___/______ Page No.: ______

$\Rightarrow$ $x dy - y dx = 0$. $\Rightarrow$ $y.dx - x.dy = 0$. ——— ①

$M_y = \frac{\partial M}{\partial y} = +1$; $N_x = \frac{\partial N}{\partial x} = -1$ not exact.

★ Result I :- If $\frac{M_y - N_x}{N} = R(x)$; function of x alone

then I.F. $= e^{\int R(x).dx}$.

★ Result II :- If $\frac{N_x - M_y}{M} = R(y)$; function of y alone.

then I.F $= e^{\int R(y).dy}$.

Applying Result (I) ; $\frac{M_y - N_x}{N} = \frac{1+1}{-x} = \frac{-2}{x}$.

thus I.F. $= e^{-\int \frac{2}{x}.dx} = \cancel{2/x} . \frac{1}{x^2}$.

By $\frac{1}{x} \times$ ① ; we obtain

$$\frac{y}{x^2} dx - \frac{1}{x} dy = 0 .$$

$\Rightarrow$ $d\left(\frac{-y}{x}\right) = 0$ $\Rightarrow$ soln: $\frac{-y}{x} = \text{const.}$

$$\Rightarrow \boxed{\frac{y}{x} = \text{const}}$$

Date __/__/____ Page No.: ____

Applying Result (II); $\frac{N_x - M_y}{M} = \frac{-1-1}{y} = \frac{-2}{y}$.

$$IF = e^{-\int \frac{2}{y} dy} = \frac{1}{y^2}.$$

$\frac{1}{y^2} \times (1)$; we obtain.

$$\Rightarrow \frac{1}{y^2} \times y dx - \frac{1}{y^2} x dy = 0.$$

$$\Rightarrow \frac{1}{y} \cdot dx - \frac{1}{y^2} \cdot x dy = 0.$$

$$\Rightarrow d\left(x \cdot \frac{1}{y}\right) = 0 \Rightarrow \frac{x}{y} = \text{const}$$

$$\Rightarrow \boxed{\frac{y}{x} = \text{const}}$$

Take $IF = \frac{1}{xy}$; Then by $\frac{1}{xy} \times (1)$; we obtain

$$\frac{1}{x} \cdot dx - \frac{1}{y} \cdot dy = 0$$

$$\Rightarrow \ln x - \ln y = C \Rightarrow \frac{x}{y} = \text{const}.$$

Take $IF = \frac{1}{x^2+y^2}$; $\frac{y\,dx - x\,dy}{x^2+y^2} = 0$.

Maharaza

Scanned by CamScanner

* More About Substitution Method :-

Type of Given ODE	Substitution	Given ODE reduces to
$\frac{dy}{dx} + p(x)y = q(x).y^a \; (a \neq 0,1)$	$z = y^{1-a}$	Linear form.
$f'(y)\frac{dy}{dx} + p(x).f(y) = q(x)$	$z = f(y)$	Linear form.

e.g. $\frac{dy}{dx} = e^{2x}e^{-y} - e^{x}$

$\Rightarrow e^{y}.\frac{dy}{dx} + e^{x}e^{y} = e^{2x}$ Sub $z = e^{y}$.

$\Rightarrow \frac{dz}{dx} + e^{x}z = e^{2x}$.

$IF = e^{\int e^{x}.dx} = e^{e^{x}}$.

$\Rightarrow e^{e^{x}}.\frac{dz}{dx} + e^{e^{x}}.e^{x}.z = e^{2x}.e^{e^{x}}$

$\Rightarrow \left(e^{e^{x}}.z\right)' = \int e^{2x}.e^{e^{x}}.dx + C.$

Take $t = e^{x}$, $\quad = \int t^{2}.e^{t}.\frac{dt}{t} + c.$

$dt = e^{x}dx.$

$= \int te^{t}dt + c$

$= te^{t} - \int e^{t}dt + c$

$= te^{t} - e^{t} + c.$

$e^{e^{x}}y = e^{x}.e^{e^{x}} - e^{e^{x}} + C.$

Maharaza

$\Rightarrow e^{y} = e^{x} - 1 + ce^{-e^{x}}.$

Date __/__/____ Page No.: ____

Type of Given ODE	substitution	Given ODE reduces to.
$\frac{dy}{dx} = f\left(\frac{y}{x}\right)$ Homogeneous equation of first order.	$y = zx$ $\cancel{(z = y/x)}$	separable form.

$y = zx \Rightarrow \frac{dy}{dx} = z + x.\frac{dz}{dx} = f(z)$.

$$\Rightarrow x.\frac{dz}{dx} = f(z) - z$$

$$\Rightarrow \frac{dz}{f(z) - z} = \frac{dx}{x}.$$

eg. $xy.\frac{dy}{dx} + 4x^2 + y^2 = 0$.

$$\Rightarrow \frac{dy}{dx} = \frac{-(4x^2 + y^2)}{xy} = -\frac{4 + \left(\frac{y}{x}\right)^2}{\left(\frac{y}{x}\right)}.$$

Sub :$\Rightarrow$ $y = zx$.

$$\Rightarrow z + x.\frac{dz}{dx} = -\left(\frac{4 + z^2}{z}\right)$$

$$\Rightarrow x.\frac{dz}{dx} = -\left[\frac{4 + z^2}{z} + z\right]$$

$$\Rightarrow \frac{x\,dz}{dx} = -\frac{4 + 2z^2}{z} = -2\left(\frac{2 + z^2}{z}\right)$$

$$\Rightarrow \frac{1}{(-2)}\frac{z}{2 + z^2}.dz = \frac{2.dx}{x}.$$

Maharaza

२१।२।२१

Date ___/___/______ Page No.: ______

Sol^n :- $-\frac{1}{4}\ln(2+z^2) = 2\ln x + c$.

Type of Given ODE	Substitution	Given ODE reduces to.
$\frac{dy}{dx} = f(ax+by+c)$; a, b, c are const.	$z = ax+by+c$	Separable form.

eg. $\frac{dy}{dx} = e^{9y-x}$ Sub. $z = 9y - x$

$\frac{dz}{dx} = 9\cdot\frac{dy}{dx} - 1$.

$\Rightarrow \frac{1}{9}\left(\frac{dz}{dx}+1\right) = e^z \Rightarrow \frac{dz}{dx} = 9e^z - 1$.

$\Rightarrow \frac{dz}{9e^z - 1} = dx$.

$\Rightarrow \frac{e^{-z}\cdot dz}{9 - e^{-z}} = dx$.

$\Rightarrow \ln(9 - e^{-z}) = x + c$.

$\Rightarrow \boxed{9 - e^{-z} = ce^x}$

☆ $(a_1x + b_1y + c_1)\cdot dx + (a_2x + b_2y + c_2)\, dy = 0$

$a_1 a_2 b_1 b_2 c_1 c_2$ are consts

$+ \quad \frac{a_1}{a_2} = \frac{b_1}{b_2}$.

Maharaza

$z = a_1 x + b_1 y$ Separable form.

Date ___/___/______ Page No.: ______

eg. $(x+2y+3).dx + (2x+4y-1)dy = 0$

Sub :- $z = x+2y \Rightarrow dz = dx + 2dy$

$(z+3)dx + (2z-1)\left(\frac{dz-dx}{2}\right) = 0$

$\Rightarrow \cancel{zdx} + 3dx + zdz - \cancel{zdx} - \frac{dz}{2} + \frac{dx}{2} = 0$

$\Rightarrow \left(3+\frac{1}{2}\right)dx + \left(z-\frac{1}{2}\right)dz = 0.$

$\Rightarrow 7dx + (2z-1)dz = 0$

$\Rightarrow \int 7dx = \int (1-2z)dz$

$\Rightarrow 7x = z - z^2 + C$

$\Rightarrow 7x = x+2y - (x^2+4y^2+4xy)$

$\Rightarrow 6x - 2y = -x - 4y^2 + 4xy$

$\Rightarrow x^2 + 4y^2 + 4xy + 6x - 2y = C.$

* $(a_1 x + b_1 y + c_1).dx + (a_2 x + b_2 y + c_2)dy = 0$

$a_1\ b_1\ c_1\ a_2\ b_2\ c_2$ are constants

But $\frac{a_1}{a_2} \neq \frac{b_1}{b_2}$

$z_1 = x - h$

$z_2 = y - k$

homogeneous eq^n in variables z_1 & z_2

where h & k are solution of

$a_1 h + b_1 k + c_1 = 0$

$a_2 h + b_2 k + c_2 = 0.$

Maharaza

Scanned by CamScanner

Date ___/___/______ Page No.: ______

eg. $(x-2y+1).dx + (4x-3y-6).dy = 0$.

$\Rightarrow \frac{1}{4} \neq \frac{-2}{-3}$ $z_1 = x-h$; $z_2 = y-k$;

$h-2k+1 = 0$
$4h-3k-6 = 0$
$h = 3$
$k = 2$
$\Rightarrow z_1 = x-3 \Rightarrow x = z_1+3$
$z_2 = y-2 \Rightarrow y = z_2+2$

$((z_1+3) - 2(z_2+2)+1).dz_1 + (4(z_1+3)-3(z_2+2)-6)dz_2 = 0$

$\Rightarrow (z_1 - 2z_2).dz_1 + (4z_1 - 3z_2)dz_2 = 0$.

$$\Rightarrow \frac{dz_2}{dz_1} = \frac{2z_2 - z_1}{4z_1 - 3z_2}$$

$$\frac{dz_2}{dz_1} = \frac{2-(z_1/z_2)}{4(z_1/z_2)-3} = \frac{2(z_2/z_1)-1}{4-3(z_2/z_1)}$$

Homogeneous form in z_1 & z_2

Sub :- $z_2 = z z_1$

Scanned by CamScanner

More about IF :-

Let $M(x,y)dx + N(x,y)dy$ be not exact.

Assignment-1 ; Sem II ; MA102.

Q.6) A. Reduce $y' = f\left(\frac{ax+by+c}{dx+ey+f}\right)$, $ae - bd \neq 0$ to a separable variable form. What if $ae = bd$?

$\Rightarrow$ Put $X = x-h$; $Y = y-k$.

$\Rightarrow dX = dx$; $dY = dy$.

h & k are the solution of $ax+by+c=0$
$dx+ey+f=0$.

$$\frac{dY}{dX} = \frac{aX+bY}{dX+eY} = \frac{a+b(Y/X)}{d+e(Y/X)}.$$

Now; Put $\frac{Y}{X} = V \Rightarrow Y = VX$; $Y' = V + XV'$

$$V + XV' = f\left(\frac{a+bV}{d+eV}\right) \rightarrow XV' = \frac{a+bV-dV-eV^2}{d+eV}.$$

$$\Rightarrow XV' = f(\,) - V$$

$$\Rightarrow \int \frac{dV}{f(\,)-V} = \int \frac{dX}{X} + c.$$

$$\Rightarrow \int \frac{dV}{G(V)} = \int \frac{dX}{X} + c.$$

If $ae = bd \Rightarrow \frac{a}{d} = \frac{b}{e} = \lambda$.

$\Rightarrow a = d\lambda$; $b = e\lambda$.

Maharaza

Date ___/___/______ Page No.. ______

$$\Rightarrow \frac{dy}{dx} = \frac{d\lambda x + e\lambda y + c}{dx + ey + f} = \frac{\lambda(dx+ey)+c}{dx+ey+f}.$$

Put ; $dx+ey = u \Rightarrow d + ey' = u'$.

$$\Rightarrow d+e\left(\frac{\lambda u+c}{u+c}\right) = u'$$

$$\Rightarrow \int \frac{dy}{d+e\left(\frac{\lambda u+c}{u+c}\right)} = \int dx + c.$$

(B) (i) $(x+2y+1)dx - (2x+y-1).dy = 0$.

$\rightarrow$ $z_1 = x-h$; $z_2 = y-k$.

$h+2k+1=0$

$-(2h+k-1)=0$.

$h=1$; $k=-1$.

$z_1 = x-1$; $z_2 = y+1$.

$dz_1 = dx$; $dz_2 = dy$.

$x = z_1+1$; $y = z_2-1$.

$$((z_1+1)+2(z_2-1)+1).dz_1 - (2(z_1+1)+(z_2-1)-1)dz_2 = 0$$

$$\Rightarrow \frac{dz_2}{dz_1} = \frac{z_1+2z_2}{2z_1+z_2}.$$

$$\frac{dz_2}{dz_1} = \frac{1+2(z_2/z_1)}{2+(z_2/z_1)}$$

Date __/__/____

Q.3) Let V be linear space of all twice differentiable functions with usual operations. Show that solⁿ of diff. eqⁿ $y'' + \alpha y' + \beta y = 0$ form a linear space. — ①

→ S be Solution set.

Let, $y_1, y_2 \in S$

$$y_1'' + \alpha y_1' + \beta y_1 = 0. \quad — ②$$
$$y_2'' + \alpha y_2' + \beta y_2 = 0. \quad — ③$$

To show linearity, Claim $y_1 + y_2$ is solⁿ of eqⁿ ①.

$(y_1+y_2)'' + \alpha(y_1+y_2)' + \beta(y_1+y_2)$

Proof :-

eqⁿ ② + eqⁿ ③

$$y_1'' + y_2'' + \alpha(y_1' + y_2') + \beta(y_1 + y_2) = 0$$

$$\Rightarrow (y_1+y_2)'' + \alpha(y_1+y_2)' + \beta(y_1+y_2) = 0$$

$$\left(\because \frac{d}{dx}(y+z) = \frac{dy}{dx} + \frac{dz}{dx} \right).$$

Q.4) $y' = \alpha y$, $x > 0$, α is constant. Show.

(i) If ϕ is any solⁿ and $\psi(x) = \phi(x) e^{-\alpha x}$, then $\psi(x)$ is a constant.

$\frac{dy}{y} = \alpha x$

$\ln y = \frac{\alpha x^2}{2} + \beta$.

$= Ce^{\frac{\alpha x^2}{2} + \beta}$

$\psi(x) = Ce^{\frac{\alpha x^2}{2} - \alpha x}$

$= Ce^{\frac{\alpha}{2}(x)(x-1)}$

Maharaza

Date __/__/__ Page No.: ____

Theory Continued

More about IF for $M(x,y).dx + N(x,y)dy = 0$ —— ①.

Suppose $F \equiv F(x,y)$ is IF for ① ;

$$\Rightarrow \quad \frac{\partial}{\partial y}(FM) = \frac{\partial}{\partial x}(FN)$$

$$FM dx + FN dy = 0 .$$

As this eqⁿ is exact;

$$\frac{\partial}{\partial y}(FM) = \frac{\partial}{\partial x}(FN)$$

$$\Rightarrow \boxed{\frac{\partial F}{\partial y} M + F . \frac{\partial M}{\partial y} = \frac{\partial F}{\partial x} N + F . \frac{\partial N}{\partial x}} \quad \text{—— ②}$$

Assume; $F \equiv F(x)$ Then $\frac{\partial F}{\partial y} = 0$ & $\frac{\partial F}{\partial x} = \frac{dF}{dx}$

—— ③

from ② & ③, we now obtain

$$F . \frac{\partial M}{\partial y} = \frac{dF}{dx} N + F . \frac{\partial N}{\partial x}$$

$$\Rightarrow \frac{dF}{dx} = \frac{FM_y - FN_x}{N} .$$

$$\Rightarrow \frac{dF}{F} = \frac{M_y - N_x}{N} . dx .$$

Maharaza

$$\Rightarrow F = e^{\int \frac{M_y - N_x}{N} . dx} .$$

Result I :-

$\Rightarrow$ If $\dfrac{M_y - N_x}{N} = R(x)$ then.

$$\Rightarrow \quad F = e^{\int R(x).dx}$$

Assume, $F = F(y)$; Then $\dfrac{\partial F}{\partial x} = 0$ & $\dfrac{dF}{dy} = \dfrac{\partial F}{\partial y}$ (4)

from (2) & (4).

$$\frac{dF}{dy}.M + F.\frac{\partial M}{\partial y} = F.\frac{\partial N}{\partial x}$$

$$\frac{dF}{dy} = \frac{FN_x - FM_y}{M}.$$

$$\Rightarrow \frac{dF}{F} = \frac{N_x - M_y}{M}.dy.$$

Result II :-

$$\Rightarrow \boxed{F = e^{\int R(y).dy}}$$

If $R(y) = \left(\dfrac{N_x - M_y}{M}\right)$.

★ Some more useful results.

(III) Eqn (1) $\Rightarrow (M_1 y)dx + (N_1 x)dy = 0$

& $Mx - Ny \neq 0$. $\quad M = M_1 y$

then $\quad N = N_1 x$,

$$IF = \frac{1}{Mx - Ny}.$$

Maharaza

Scanned by CamScanner

Date __/__/____ Page No.: ____

(IV) If M and N are homogeneous functions $\equiv$ $f\left(\frac{y}{x}\right)$ in (1) ; $Mx + Ny \neq 0$. Then

$$IF = \frac{1}{Mx + Ny}$$

Q $(xy\sin xy + \cos xy) \cdot y\,dx + (xy\sin xy - \cos xy)\,x\,dy = 0$

$\Rightarrow$ $Mx - Ny$

$= xy(\cancel{xy\sin xy} + \cos xy - \cancel{xy\sin xy} + \cos xy)$.

$= 2xy\cos xy \neq 0$.

$$\Rightarrow \quad IF = \frac{1}{2xy\cos xy}$$

$$\Rightarrow \frac{(xy\sin xy + \cos xy)y\,dx}{2xy\cos xy} + \frac{(xy\sin xy - \cos xy)\,x\,dy}{2xy\cos xy} = 0$$

$$\Rightarrow \left(\frac{y\tan xy}{2} + \frac{1}{2x}\right)dx + \left(\frac{x\tan xy}{2} - \frac{1}{2y}\right)dy = 0$$

Scanned by CamScanner

Date __/__/____ Page No.: ______

$y = f(x)$; $f(x,y) = c$; $\boxed{f(x,y,c) = 0}$ ⇒ prefeared form of solⁿ

D.E. from given curve.

e.g. $x^2 + y^2 = 2cx$ — ①

where c is a paramater.

$2x + 2y \cdot \frac{dy}{dx} = 2c$ — ②

not required ✗ $2x \cdot \frac{dx}{dy} + 2y = 2c \cdot \frac{dx}{dy}$ — ③.

from ① ⇒ we obtain $c = \frac{x^2+y^2}{2x}$ put it into ② ;

$$2x + 2y \cdot \frac{dy}{dx} = \frac{x^2+y^2}{x}$$

$$\Rightarrow 2y \cdot \frac{dy}{dx} = \frac{y^2-x^2}{x}$$

$$\Rightarrow \boxed{\frac{dy}{dx} = \frac{y^2-x^2}{2xy}}$$

Maharaza

Date __/__/____ Page No.: ____

e.g. $x^2 + y^2 = c^2$ ——① ; c is parameter.

$$2x + 2y \cdot \frac{dy}{dx} = 0 \quad \text{——②}$$

$$\Rightarrow \boxed{\frac{dy}{dx} = \frac{-x}{y}}$$

★ Algorithm :-

① Write eqⁿ as it is

② Differentiate with independent variable (generally x).

③ Eliminate c from both of above equations.

★ Geometrical view of first order ODE :-

derivative of solution curve at (x,y).

$$\frac{dy}{dx} = f(x,y) \Rightarrow y = g(x).$$

★ Topic to learn :-

IVP / General solⁿ of DE / Particular solⁿ / Interval of validity.

$\frac{dy}{dx} = 6y^2x \Rightarrow$ differential equation

$y(1) = \frac{1}{25} \Rightarrow$ Initial ~~value problem~~ condition

} Initial value problem

$$\Rightarrow \frac{dy}{y^2} = 6x\,dx. \Rightarrow -\frac{1}{y} = 3x^2 + c.$$

general solution of DE.

put $(y(1) = 1/25)$ in

$$c = -3 - \frac{1}{1/25} = -28.$$

Thus, particular solution of IVP.

$$-1/y = 3x^2 - 28 \Rightarrow \boxed{y = \frac{1}{28 - 3x^2}}$$

Maharaza

Date ___/___/_______ Page No.: ______

$y = \frac{1}{28-3x^2}$ I_1 I_2 I_3

$-\sqrt{\frac{28}{3}}$ $\sqrt{\frac{28}{3}}$

$\Downarrow$

Interval of Validity

$\left(-\sqrt{\frac{28}{3}}, \sqrt{\frac{28}{3}}\right)$.

The above function y is valid in I_2 interval only. Because if we include I_1 & I_3 the function will be discontinuous at some points i.e. not differentiable.

* $(\cos x).\frac{dy}{dx} + (\sin x)y = 2\cos^3 x \sin x - 1$;

$y\left(\frac{\pi}{4}\right) = 3\sqrt{2}$; $0 \le x < \frac{\pi}{2}$

$\Rightarrow$ $\frac{dy}{dx} + y.\tan x = \frac{2\cos^3 x \sin x - 1}{\cos x}$.

$IF \Rightarrow e^{\int \tan x dx} = e^{\ln \sec x} = \sec x$

$\sec x.\frac{dy}{dx} + y.\sec x.\tan x = 2\sin x\cos x - \sec^2 x$

$(\sec x.y)' = 2\sin x \cos x - \sec^2 x$.

$\Rightarrow y\sec x = \int \sin 2x dx - \int \sec^2 x dx$

$\Rightarrow y\sec x = -\frac{\cos 2x}{2} - \tan x + C$

$3\sqrt{2}.(\sqrt{2}) = -1 + C$

$\Rightarrow \boxed{C = 7}$.

Maharaza

$\Rightarrow y\sec x = -\frac{\cos 2x}{2} - \tan x + 7.$

* $2\frac{dy}{dx} - y = 4\sin(3x).$; $y(0) = y_0.$

~~$dy/dx - y/2 = 2\sin(3x).$~~

$\Rightarrow$ IF $\Rightarrow e^{-\frac{1}{2}x}$ $\Rightarrow$ ~~$y(x) =$~~

$\Rightarrow e^{-x/2}.\frac{dy}{dx} - \frac{y}{2}.e^{-x/2} = 2e^{-x/2}\sin(3x).$

$\Rightarrow \frac{d}{dx}(y.e^{-x/2}) = 2e^{-x/2}\sin(3x)$

$\Rightarrow y.e^{-x/2} = 2\int e^{-x/2}\sin(3x).dx.$

$I = \int \underset{v}{e^{-x/2}} \underset{u}{\sin(3x)}.dx$

$= \sin(3x)\frac{1}{\left(\frac{-1}{2}\right)}.e^{-x/2} - \int \frac{1}{\frac{-1}{2}}.e^{-x/2}3\cos 3x.dx$

$= -2\sin(3x)\,e^{-x/2} + 6\left[\cos 3x.\frac{1}{\left(\frac{-1}{2}\right)}.e^{-x/2} - 6\int 2\,e^{-x/2}\sin 3x\,.dx\right]$

$= -2e^{-x/2}\sin(3x) - 12\cos 3x.e^{-x/2} - 36 I.$

$37 I = -2(\sin(3x).e^{-x/2} + 6\cos 3x\, e^{-x/2})$

$\Rightarrow y.e^{-x/2} = \frac{-4}{37}\sin(3x).e^{-x/2} - \frac{24}{37}\cos 3x\, e^{-x/2}$

Maharaza

Scanned by CamScanner

$$y(x) = \frac{-24}{37}\cos 3x - \frac{4}{37}\sin 3x + \left(y_0 + \frac{24}{37}\right)e^{x/2}.$$

Date __/__/__ Page No.: ______

Now, consider if we were given

$y_0 < -24/37$	, $x \to \infty$	$y \to \infty$
$y_0 > -24/37$	, $x \to \infty$	$y \to \infty$
$y_0 = -24/37$	, $x \to \infty$	$y(x) = \frac{-24}{37}\cos 3x - \frac{4}{37}\sin 3x$

Existance Theorem :-

★ Consider $\left.\begin{array}{l}\frac{dy}{dx} = f(x,y) \\ y(x_0) = y_0\end{array}\right)$ IVP $+ x \in [x_0-\delta, x+\delta] = I$

If f is continuous on I then IVP has a solution in some interval $\begin{bmatrix} x_0-\epsilon, \\ x_0+\epsilon \end{bmatrix}$

★ Uniqueness Theorem :-

Consider $\left.\begin{array}{l}\frac{dy}{dx} = f(x,y) \\ y(x_0) = y_0\end{array}\right)$ IVP $+$ $x \in [x_0-\delta, x_0+\delta] = I$

If f is continuous on I + $\frac{\partial f}{\partial y}$ is bounded on I then IVP has a solution y in same interval $[x_0-\epsilon, x_0+\epsilon]$.

Recall: Consider $\frac{dy}{dx} = f(x,y)$; $y(x_0) = y_0$

Only sufficient tests

existence	uniqueness.
f is continuous near (x_0,y_0)	f is cont. + $\frac{\partial f}{\partial y}$ is bounded } near (x_0,y_0)

Maharaza

bounded $\Rightarrow \left|\frac{\partial f}{\partial y}\right| \le k$ (finite)

Date ___/___/_______ Page No.: _______

Q. $\frac{dy}{dx} = y^{1/3}$; $y(0) = 0$

$\Rightarrow \frac{dy}{y^{1/3}} = dx$

$\Rightarrow \frac{3}{2}\frac{1}{y^{-2/3}} = x + C.$

$\Rightarrow y^{2/3} = \frac{2}{3}(x+c)$

$\Rightarrow y^{2/3} = \frac{2}{3}x$

using $y(0)=0$.

$\Rightarrow y = \pm\left(\frac{2x}{3}\right)^{3/2}$

$+ \; y = 0,$

Now check

$\frac{\partial f}{\partial y} = \frac{1}{3}\frac{1}{y^{2/3}}$

Not bounded.

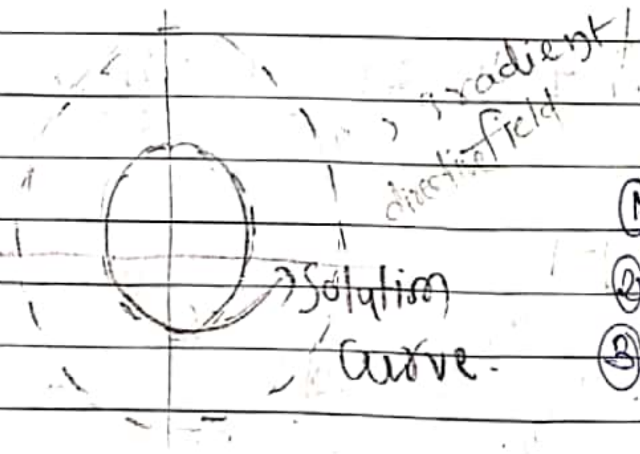
(but this does not imply non-unique solution.)

On the other hand, existance of multiple sol^{ns} implies f is non-conts. or $\frac{\partial f}{\partial y}$ is not bounded.

☆ Geometrical view :-

gradient/directive field $\longleftrightarrow$ $\frac{dy}{dx} = f(x,y)$.

Solution → Solution curve / Integral curve / Repres. curve



☆ Algorithm :- (to draw).

1. Take any slope c.
2. Plot $f(x,y)=c$ (isocline)
3. Draw line elements on above isocline with slope c.

Maharaza

Scanned by CamScanner

Date ___/___/_______ Page No.: _______

* $\frac{dy}{dx} = 1 + x - y$

(I) Let, $c = -1$

(II) plot $1 + x - y = -1$

$\Rightarrow y - x = 2$

(I) let $c = 0$

(II) plot $1+x-y=0$

$\Rightarrow y-x=1$

(I) let $c=1$

(II) plot $1+x-y=1$

$\Rightarrow x-y=0$

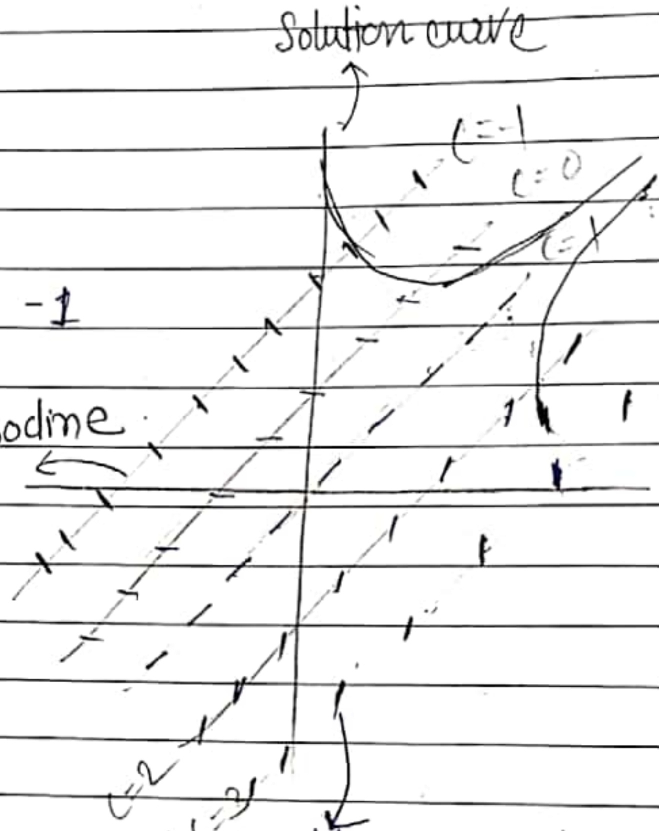


Fig. Gradient field corresponding to differential eqn

Picard's Iterative method

* $\frac{dy}{dx} = f(x,y) \rightarrow \int_{x_0}^{x} \frac{dy}{ds}\,ds = \int_{x_0}^{x} f(s,y)\,ds$

$y(x_0) = y_0$

Integral equation $\rightarrow$ $\boxed{y(x) = y(x_0) + \int_{x_0}^{x} f(s,y)\,ds}$

Start iteration

$y_1(x) = y(x_0) + y(x_0) + \int_{x_0}^{x} f(s, y_0)\,ds$

eg. $dy/dx = y \; ; \; y(0) = 1$ $\qquad y(x) = e^x$

$y(x) = y(x_0) + \int_{x_0}^{x} y(s)\,ds$

Maharaza

$y_1(x) = 1 + \int_{0}^{x} y_0(s)\,ds = 1 + \int_{0}^{x} ds = 1 + x$

$$y_2(x) = y(x_0) + \int_{x_0}^{x} f(s, y_1)\,ds.$$

$$y_2(x) = 1 + \int_0^x (1+s)\,ds.$$

$$y_2(x) = 1 + x + \frac{x^2}{2}.$$

$$y_3(x) = 1 + \int_0^x \left(1 + s + s^2/2\right) ds.$$

$$y_3(x) = 1 + x + \frac{x^2}{2} + \frac{x^3}{6}$$

☆ Tut. Ass. I . Sem II.

Q.7) $$y' = \frac{1 - xy^2}{2x^2y}$$

$y = vx^n$

$y' = vnx^{n-1} + x^n v'$

$$vnx^{n-1} + x^n v' = \frac{1 - x^{2n+1}v^2}{2x\;vx^{n+2}}. \qquad n = -1/2.$$

Ass II.

Q.5) $\Rightarrow$ Wronskian $\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = W$

Date / / Page No.:

☆ $y'' + P(x)y' + Q(x)y = 0$

y_1 & y_2 are solution then

$\rightarrow$

$y_1'' + P(x)y_1' + Q(x)y_1 = 0$ — ① $\quad W = y_1y_2' - y_1'y_2$

$y_2'' + P(x)y_2' + Q(x)y_2 = 0$ — ② $\quad W' = y_1y_2'' - y_1''y_2$

Q.4) $\Rightarrow$ ① $\times y_2$ − ② y_1 $\Rightarrow$ $(y_2y_1'' - y_1y_2'') + P(x)(y_2y_1' - y_1y_2') = 0$

$\Rightarrow \quad -W' + P(x)(-W) = 0$

$\Rightarrow \quad \frac{dW}{W} = -P(x).dx.$

$\Rightarrow \quad W = c.\underbrace{e^{-\int p(x).dx}}_{\neq 0}$

Thus; $\quad W = 0$ iff $c = 0$

$W \neq 0$ if $c \neq 0$. Hence, proved.

Q.6) (a) $\quad y'' + 3y' + 2y = 0. \quad (y = e^{mx})$

$\rightarrow$ find m.

$\Rightarrow \quad m^2e^{mx} + 3me^{mx} + 2e^{mx} = 0$

$\Rightarrow \quad e^{mx}(m^2 + 3m + 2) = 0$

$m = -1 ; -2$

$\therefore \quad y = e^{-x} ; e^{-2x}.$

Q.5) $\Rightarrow$ if y_1 & y_2 are L.D. $\quad y_2 = cy_1$

$W = \begin{vmatrix} y_1 & cy_1 \\ y_1' & cy_1' \end{vmatrix} = 0 \Rightarrow \underline{W = 0}.$

only if $\quad W = 0$

$\frac{y_1y_2' - y_2y_1' = 0}{y_1^2}$

$\Rightarrow \left(\frac{y_2}{y_1}\right)' = 0 \Rightarrow \underline{y_2 = cy_1}.$

Maharaza

Q.6) (b) $y = x^m$

(i) $x^2y'' - 4xy' + 4y = 0$.

$\Rightarrow x^2(m(m-1)x^{m-2}) - 4x(m(x^{m-1})) + 4x^m = 0$.

$\Rightarrow x^m(m^2 - m - 4m + 4) = 0$

$\Rightarrow$ As $x>0 \Rightarrow x^m \neq 0$.

$\Rightarrow m^2 - 5m + 4 = 0 \Rightarrow m = 1; 4$

$\Rightarrow y = x^4, x$.

Q.4) $\Rightarrow$ $W(y_1, y_2) = y_1y_2' - y_1'y_2$ (Wronskian of $y'' + P(x)y' + Q(x) = 0$)

$$\frac{dw}{dx} = y_1y_2'' + \cancel{y_1'y_2'} - \cancel{y_1'y_2'} - y_1''y_2.$$

$$\frac{dw}{dx} = y_1y_2'' - y_1''y_2$$

$y_1'' + P(x)y_1' + Q(x)y_1 = 0$

$y_2'' + P(x)y_2' + Q(x)y_2 = 0$.

$\Rightarrow y_1'' = -P(x) \cdot y_1' - Q(x)y_1$

$\Rightarrow y_2'' = -P(x)y_2' - Q(x)y_2$

$$\therefore \frac{dw}{dx} = P(x)(y_1'y_2 - y_1y_2').$$

$$\therefore \frac{dw}{dx} = P(x) \cdot (-w(x)).$$

$$\Rightarrow w(x) = \underbrace{c_1 e^{-\int P(x) \cdot dx}}_{\neq 0}.$$

If $c_1 = 0$; $W = 0$

$c_1 \neq 0$; $W \neq 0$.

Date ___/___/______

$M(x,y).dx + N(x,y).dy = 0$ — ①
be a non-exact DE.
let $\phi(x,y)$ be IF of ①.

$\phi M dx + \phi N dy = 0$ — ② will be exact.

So. ∃ a function s.t.
$$\phi[f(x,y)] = \phi M dx + \phi N dy.$$

let; G be any arbitrary ; $G = G(f)$.
Now, $G(f).\phi(x,y)$ will be IF also.

e.g $x dy - y dx = 0$
$M = -y$; $N = x$
$M_y = -1$; $N_x = 1$.
IF. $\phi(x,y) = \frac{1}{y^2}$
$\frac{1}{y^2}(x dy - y dx) = 0$
$\Rightarrow \frac{x.dy}{y^2} - \frac{dx}{y} = 0 \Rightarrow$ Sol^n $\Rightarrow \frac{x}{y} = c$.

$M = \frac{-1}{y}$; $N = \frac{x}{y^2}$
$\Rightarrow M_y = \frac{1}{y^2}$, $N_x = \frac{1}{y^2}$.

then. $G\left(\frac{x}{y}\right).\frac{1}{y^2}$ will be I.F.

e.g. $\frac{x}{y}\left(\frac{1}{y^2}\right)$, $\left(\frac{1}{y^2}\right).\sin\left(\frac{x}{y}\right)$.

Date ___/___/______

Page No.: ______

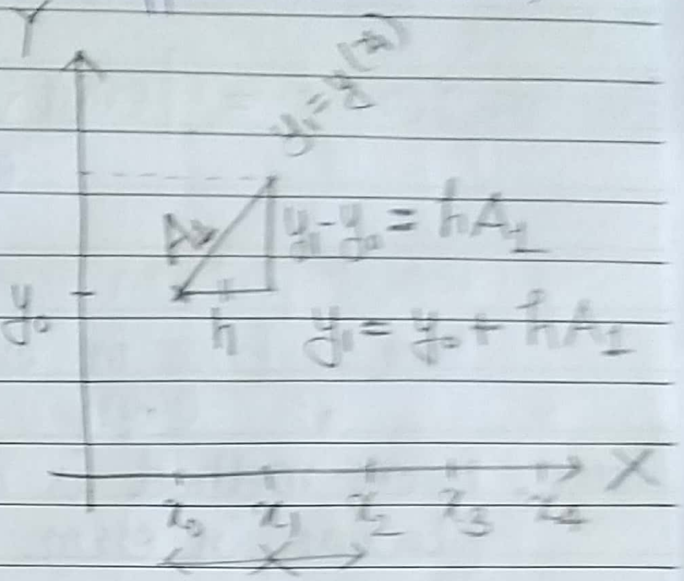
* $\frac{dy}{dx} = f(x,y)\ ;\ y(x_0) = y_0$ $\frac{dy}{dx} = x^2 - y^2\ ;\ y(0) = 1$

* Euler's Method :- (Numerical Approximation Method)

$$x_{n+1} = x_n + h$$

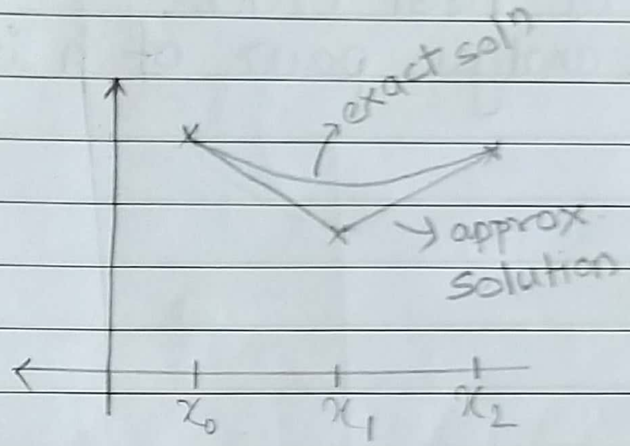
$$y_{n+1} = y_n + hA_n$$

$$A_n = f(x_n, y_n)$$



h = stepsize

$$\frac{y_1 - y_0}{h} \simeq f(x_0, y_0)$$



* We can find an approximate solution through this meth-od.

$\frac{dy}{dx} = x^2 - y^2;$

$y(0) = 1$

$h = 0.1$

n	x_n	y_n	$A_n = f(x_n, y_n)$	hA_n
0	0	1	-1	-0.1
1	0.1	0.9	$(0.1)^2 - (0.9)^2$	
↓	↓	↓	↓	↓

plot curve using values in above table

RK4

Date ___/___/______ Page No.: ______

* $\frac{dy}{dx} + 2y = 2 - e^{-4x}$ | Take step size $h = 0.1$

$y(0) = 1$ | $y(0.1)$; $y(0.2)$; $y(0.3)$; $y(0.4)$; $y(0.5)$

Calculate. ⇒ $y(x)$ (analytic)

$$\text{Absolute \% error} = \left|\frac{\text{exact value} - \text{approx. value}}{\text{exact value}}\right| \times 100$$

Repeat problem for $h = 0.05$.

Claim :- Error in euler's method is of order h

i.e. $e \sim h$.

hence, Euler's method is of first order.

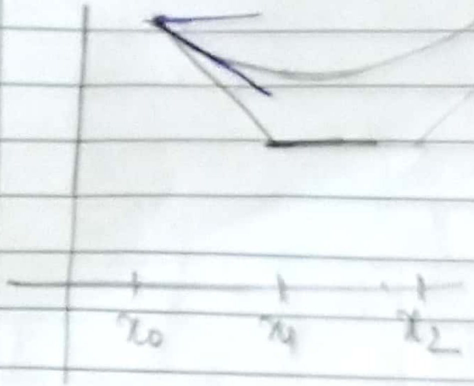
(because in error analysis power of h is 1)

Maharaza

Date ___/___/______ Page No.: ______

RK-2 {Runge Kutta}

★



$x_{n+1} = x_n + h$

$k_1 = f(x_n, y_n)$

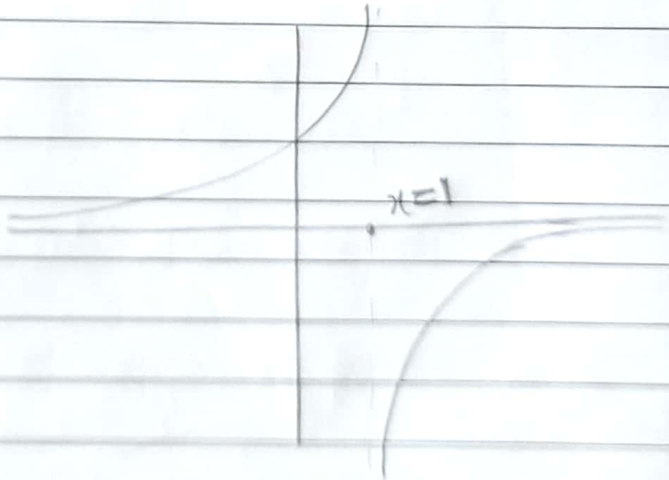
$k_2 = f\left(x_n + \frac{h}{2}, y_n + \frac{1}{2}k_1h\right)$

$y_{n+1} = y_n + hk_2$

$e \sim h^2$

* $\frac{dy}{dx} = y^2$; $0 \le x \le 2$; $y(0) = 1$

Ans $y = \frac{1}{1-x}$



Singularity in curve at $x=1$.

$a_0(x)\frac{d^ny}{dx^n} + a_1(x)\frac{d^{n-1}y}{dx^{n-1}} + \ldots + a_n(x)\,y = f(x)$

Date: / /

Page No.:

★ Linear Differential Equations of second order :-

St. form $\boxed{\frac{d^2y}{dx^2} + p(x).\frac{dy}{dx} + q(x).y = r(x)}$ — ⊛

If $r(x) \equiv 0$; then ⊛ is called homogeneous differential equation.

If p, q and r are constant then ⊛ is called diff. equation of constant coefficients.

Consider $\frac{d^2y}{dx^2} + p(x)\frac{dy}{dx} + q(x).y = 0$ —①

General Solution $\boxed{y = c_1y_1 + c_2y_2}$

where y_1 and y_2 are two LI solutions of ①
c_1 & c_2 are arb. constants

Existence / Uniqueness :-

Equation ⊛ with initial data $y(x_0) = y_0$; $y'(x_0) = y_1$ has unique solution if p, q, r are continuous functions on $[a,b]$ {Here $x_0 \in [a,b]$}

Theorem :- If y_1 and y_2 are any two solutions of ① on $[a,b]$. Then their Wronskian $W(y_1,y_2)$ is either identically zero or never zero. Further y_1 & y_2 are LD iff $W(y_1,y_2)$ is identically zero.

Proof :

$$W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = y_1 y_2' - y_2 y_1'$$

$$\Rightarrow \quad W' = \cancel{y_1' y_2'} + y_1 y_2'' - \cancel{y_2' y_1'} - y_2 y_1''$$

$$W' = y_1 y_2'' - y_2 y_1'' \quad \text{--- (2)}$$

Since y_1 & y_2 are sol^n of (1) ie.

$$y_1'' + P y_1' + q y_1 = 0 \quad \text{--- (3)}$$
$$y_2'' + P y_2' + q y_2 = 0 \quad \text{--- (4)}$$

from $y_2 \times (3) - y_1 \times (4)$; we obtain.

$$y_2 y_1'' + y_2 P y_1' + y_2 q y_1 = 0$$
$$- y_1 y_2'' - y_1 P y_2' - y_1 q y_2 = 0$$

$$(y_2 y_1'' - y_1 y_2'') + p\,[y_2 y_1' - y_1 y_2'] = 0.$$

$$\Rightarrow \quad \frac{dW}{dx} + pW = 0 \quad \text{--- (5)}$$

from (5), we obtain

$$W = C.\underbrace{e^{-\int p(x).dx}}_{\neq 0}$$

Thus,

$W \equiv 0$ if $C = 0$.

OR W is never zero if $C \neq 0$.

Maharaza

Date __/__/____ Page No.: ____

→ if y_1 & y_2 are LD ⇒ $y_2 = cy_1$

Then, $W = \begin{vmatrix} y_1 & cy_1 \\ y_1' & c.y_1' \end{vmatrix} = 0$ → $\boxed{W = 0}$

only if

$$W = 0$$

$$\frac{y_1 y_2' - y_2 y_1'}{y_1^2} = 0$$

$$\Rightarrow \left(\frac{y_2}{y_1}\right)' = 0 \Rightarrow \boxed{y_2 = cy_1}$$

Consider $y'' + py' + qy = 0$ —① (p & q are real constant function)
$(p, q \in \mathbb{R})$

Theorem:- If $(u+iv)$ is a complex solution of ①. Then u and v are two solutions of ①.

Proof:- $(u+iv)'' + p(u+iv)' + q(u+iv) = 0$

$\Rightarrow (u'' + pu' + qu) + i(v'' + pv' + qv) = 0$

$\Rightarrow u'' + pu' + qu = 0$
& $v'' + pv' + qv = 0$

Hence, proved.

Theorem:- If y_1 is a solution of ① then $\exists$ $u(x)$ s.t $y_2 = y_1 u$ is also a solution of ①.

Maharaza

Date __/__/__ Page No.: ____

Proof: Put, y_2 in ①

$$y_1''u + 2y_1'u' + y_1u'' + p(y_1'u + y_1u') + q(y_1u) = 0$$

$$\Rightarrow \underbrace{(y_1'' + py_1' + qy_1)}_{=0 \text{ as } y_1 \text{ is a solution}} u + (2y_1' + py_1)u' + y_1u'' = 0$$

$$\Rightarrow y_1A' + (2y_1' + py_1)A = 0 \quad (A = u')$$

On solving, $A = \frac{1}{y_1^2} \cdot e^{-\int p(x)\,dx}$

$$\Rightarrow \boxed{u = \int \frac{1}{y_1^2} \cdot e^{-\int p(x)\,dx}\,dx}$$

Assume $y = e^{mx}$ is a solution of $y'' + py' + qy = 0$.

$$m^2 e^{mx} + pm\,e^{mx} + qe^{mx} = 0$$

$$\Rightarrow (m^2 + pm + q)\underbrace{e^{mx}}_{\neq 0} = 0$$

$$\Rightarrow \boxed{m^2 + pm + q = 0} \qquad m = \frac{-p \pm \sqrt{p^2 - 4q}}{2}$$

Step ①. Write Auxilliary equation w.r.t. DE

$m^2 + pm + q = 0$.

Case ① : $p^2 > 4q$

we get two distinct real roots say m_1 and m_2.

Case (II) : $p^2 < 4q$.

$m = a \pm ib$.

$y = e^{(a+ib)x} = e^{ax} e^{ibx} = e^{ax}(\cos bx + i\sin bx)$.

$y_1; y_2 = e^{ax}\cos bx \,;\, e^{ax}\sin bx$ are solutions

Case (III) : $p^2 = 4q$

$m = m_1, m_1$

$y_1 = e^{m_1 x} = e^{-P/2(x)}$

$y_2 = \left[\int \frac{1}{y_1^2} \cdot e^{-\int p(x)\cdot dx} \cdot dx\right] y_1 \iff u\, y_1$.

$\Rightarrow \quad u = \int \frac{1}{e^{-px}} \cdot e^{-px} \cdot dx = x$.

$\Rightarrow \quad y_2 = x y_1 = x e^{m_1 x}$. $\Rightarrow \quad y_2 = x e^{m_1 x}$

Step (II) Solve Aux. eqn for m

Case (I)

$m = m_1, m_2$ (real)

$m_1 \neq m_2$.

G.soln $y = c_1 e^{m_1 x} + c_2 e^{m_2 x}$

Case (II)

$m = a \pm ib$.

$y = e^{ax}[c_1 \cos bx + c_2 \sin bx]$

Case (III)

$m = m_1, m_2$

$m_1 = m_2 = m$

$y = e^{mx}[c_1 + c_2 x]$.

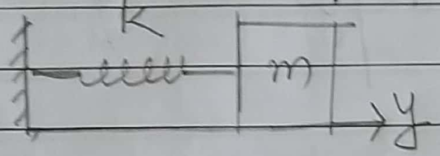
★ Bank Account Model :-

$x(t)$ = money in account at time t
$I(t)$ = Interest rate / money-time
$q(t)$ = deposit rate / time

$$x(t+\Delta t) - x(t) = I(t)\cdot x(t)\,\Delta t + q(t)\,\Delta t$$

$$\Rightarrow \frac{dx}{dt} - I(t)\cdot x(t) = q(t).$$

★ Undamped Oscillations :-



$$m\cdot\frac{d^2y}{dt^2} = -ky.$$

$$\Rightarrow \boxed{y'' + \underbrace{\frac{k}{m}}_{\omega_0^2} y = 0}$$

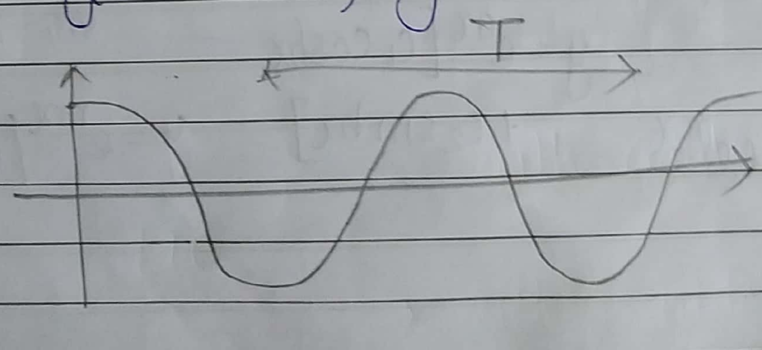
$$m^2 + \omega_0^2 = 0 \Rightarrow m = \pm i\omega_0.$$

$$a = 0 \;;\; b = \omega_0$$

$$\therefore\; y = e^{(0)x}\left[c_1\cos\omega_0 t + c_2\sin\omega_0 t\right].$$

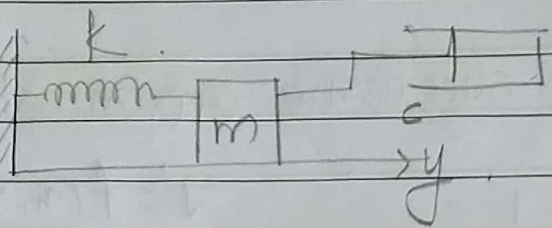
$$y(t) = c_1\cos\omega_0 t + c_2\sin\omega_0 t.$$

$$y(0) = 1 \;;\; y'(0) = 0.$$



$$T = \frac{2\pi}{\omega_0} \;;\; \text{freq.} = \frac{\omega_0}{2\pi} = \frac{1}{2\pi}\sqrt{\frac{k}{m}}$$

Date Overdamped



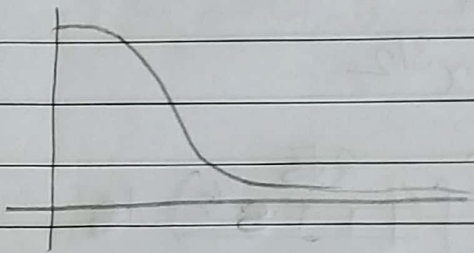
$$m\frac{d^2y}{dt^2} = -ky - cy'$$

$$\Rightarrow y'' + \frac{c}{m}y' + \frac{k}{m}y = 0.$$

$y'' + 4y' + 3y = 0.$

Aux. e. $m^2 + 4m + 3 = 0$ $\quad y = c_1e^{-t} + c_2e^{-3t}$

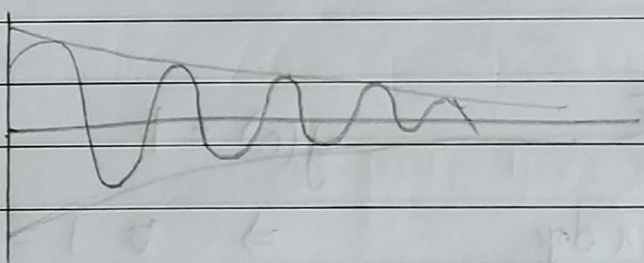
$m = -1, -3.$



$y'' + 4y' + 5y = 0.$ (Underdamped)

$m^2 + 4m + 5 = 0.$ $\quad y = e^{-2t}[c_1\cos t + c_2\sin t]$

$m = -2 \pm i$



$y'' + 4y' + 4y = 0$ (critically damped)

MA102 AssgnII SemII

Q7) Picard's method of successive approximation.

(i) $y' = 2\sqrt{x}$; $y(0) = 1$ $\qquad y' = f(x,y) = \sqrt{x}$

$\Rightarrow$ $y_{n+1} = y_0 + \int_0^x f(x.y_n).dx$

$y_1 = 1 + \int_0^x 2\sqrt{s}.ds$ $\qquad f(x_n, y_n) = \sqrt{x}$

$y_1 = 1 + \frac{4}{3}x^{3/2}$

$y_2 = 1 + \int_0^x 2\sqrt{s} \, \cancel{\left(1 + \frac{4}{3}s^{3/2}\right)} \, ds$

$= 1 + \frac{4}{3}x^{3/2}$

$\rightarrow$ $y_n = 1 + \frac{4}{3}x^{3/2}$

$\frac{dy}{dx} = 2\sqrt{x}$ $\qquad y(0) = 1$

$dy = 2\sqrt{x}\, dx$ $\qquad \Rightarrow 1 = c$

$y = \frac{4}{3}x^{3/2} + c$ $\qquad \Rightarrow y = 1 + \frac{4}{3}x^{3/2}$

LDE $\rightarrow$ with constant coefficients

LDE $\rightarrow$ with variable coefficients.

Date / /
Assign 3.

Page No.:

Q.1) (i) $y'' - 4y' + 3y = 0$.

$\Rightarrow$ $m^2 - 4m + 3 = 0$

$m^2 - 3m - m + 3 = 0$

$m(m-1) - 3(m-1) = 0$

$m = 1, 3$.

$\therefore y = c_1 e^x + c_2 e^{3x}$ is the solution

Q.2)

(ii) $(1-x^2) \cdot y'' - 2xy' + 2y = 0$; $y_1 = x$.

$\Rightarrow$ $y'' - \left(\frac{2x}{1-x^2}\right) y' + \left(\frac{2}{1-x^2}\right) \cdot y = 0$

$y_2(x) = u\, y_1(x)$. $y(x) = c_1 y_1(x) + c_2 y_2(x)$

for $y'' + p(x) y' + q(x) \cdot y = 0$.

then $u(x) = \int \frac{1}{y_1^2} \cdot e^{-\int p(x)\, dx} \cdot dx$

$\rightarrow$ $u(x) = \int \frac{1}{x^2} \cdot e^{+\int \frac{2x}{1-x^2} \cdot dx} \cdot dx$

$= \int \frac{1}{x^2} \cdot e^{-\log(1-x^2)}\, dx$.

$u(x) = \int \frac{1}{x^2(1-x^2)} \cdot dx$

$\int \frac{1}{1-x^2} \cdot dx = \int \frac{1}{x^2} dx + \int \frac{1}{(1-x^2)} dx = \frac{-1}{x} + \frac{1}{2} \log\left(\frac{1+x}{1-x}\right)$

$= \int \frac{1}{2}\left(\frac{1}{1+x} + \frac{1}{1-x}\right) dx$

Scanned by CamScanner

$$y(x) = c_1 x + c_2 x\left\{\frac{-1}{x} + \frac{1}{2}\log\left(\frac{1+x}{1-x}\right)\right\}$$

Q.3) (iv) $y''' - 6y'' + 11y' - 6y = 0$

$m^3 - 6m^2 + 11m - 6 = 0$

$m_1 = 1$

$$\begin{array}{c|cccc} 1 & 1 & -6 & 11 & -6 \\ & & 1 & -5 & 6 \\ \hline & 1 & -5 & 6 & \boxed{0} \end{array}$$

$m^2 - 5m + 6 = 0 \quad m = 3, 2$

$m = 1, 2, 3$

$\therefore\ y = c_1 e^{x} + c_2 e^{2x} + c_3 e^{3x}$ is the solution.

Q.4)
(i) $x^2 y'' + 2xy' - 12y = 0$

$\Rightarrow$ $x^2 y'' + axy' + by = 0$ —— ①

$\boxed{\text{Put } x = e^{z} \Rightarrow z = \log x}$

diff. w.r.t. x.

$$\Rightarrow \frac{dz}{dx} = \frac{1}{x}$$

Now, $\frac{dy}{dx} = \frac{dy}{dz} \times \frac{dz}{dx} = \frac{1}{x} \cdot \frac{dy}{dz}$

$$\Rightarrow x\frac{dy}{dx} = \frac{dy}{dz} = xy'$$

$x^n y^n = D(D-1)(D-2)\ldots(D-(n-1))$

$x^2 y^2 = D(D-1)$

$xy = D$

Date __/__/__

$$\frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(\frac{1}{x}\cdot\frac{dy}{dz}\right)$$

$$= \frac{-1}{x^2}\cdot\frac{dy}{dz} + \frac{1}{x}\cdot\frac{d}{dx}\left(\frac{dy}{dz}\right)$$

$$\frac{d^2y}{dx^2} = \frac{-1}{x^2}\cdot\frac{dy}{dz} + \frac{1}{x}\cdot\frac{d^2y}{dz^2}\cdot\frac{dz}{dx}$$

$$\Rightarrow x^2\cdot\frac{d^2y}{dx^2} = -\frac{dy}{dz} + x\cdot\frac{d^2y}{dz^2}\cdot\frac{dz}{dx}$$

$$= -\frac{dy}{dz} + \cancel{x}\cdot\frac{d^2y}{dz^2}\left(\frac{1}{\cancel{x}}\right)$$

$$\Rightarrow x^2\frac{d^2y}{dx^2} = -\frac{dy}{dz} + \frac{d^2y}{dz^2} = x^2y''$$

① changes to

$$\Rightarrow \left(-\frac{dy}{dz} + \frac{d^2y}{dz^2}\right) + a\left(\frac{dy}{dz}\right) + by = 0$$

$$\Rightarrow \boxed{\frac{d^2y}{dz^2} + (a-1)\cdot\frac{dy}{dz} + by = 0}$$ General for ①

for given eq. $\frac{d^2y}{dz^2} + \frac{dy}{dz} - 12y = 0$

$\Rightarrow m^2 + m - 12 = 0$ $m = 3, -4$

$\Rightarrow y(z) = c_1 e^{3z} + c_2 e^{-4z}$

$y(x) = c_1 x^3 + c_2 x^{-4}$.

Maharaza

(ii) $x^2y'' + xy' + y = 0$

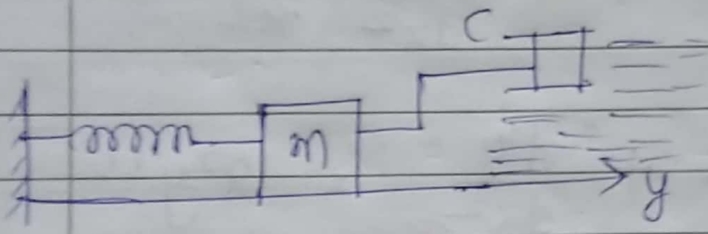
$$\Rightarrow \quad \frac{d^2y}{dz^2} + (1-1)\cdot\frac{dy}{dz} + y = 0$$

$$\Rightarrow \quad \frac{d^2y}{dz^2} + y = 0.$$

$$\Rightarrow \quad m^2 + 1 = 0 \qquad m = \pm 1$$

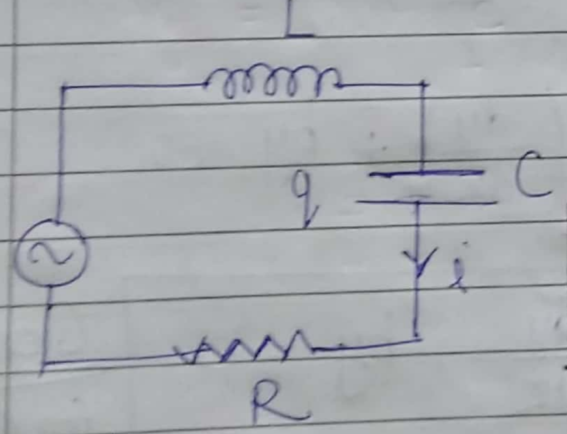
★ $y'' + p(x)y' + q(x)\cdot y = r(x)$.

(Inhomogeneous / Non-homogeneous)



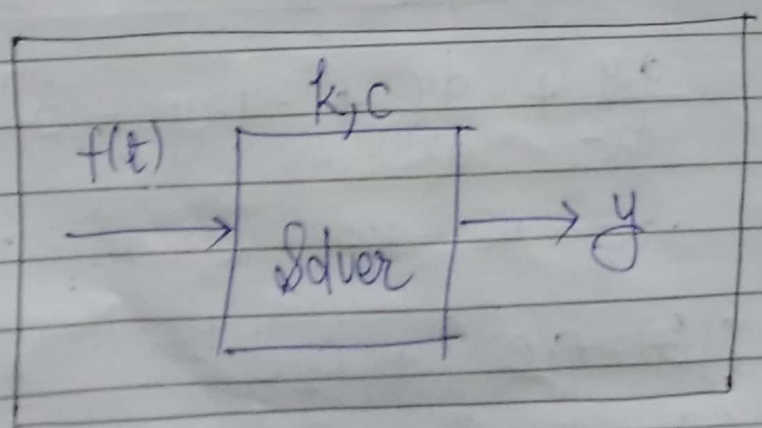
$y'' = -ky - cy' + f(t)$

$\Rightarrow \boxed{y'' + cy' + ky = f(t)}$.



$L\cdot\frac{di}{dt} + \frac{q}{c} + Ri = V_e$

$$\boxed{L\cdot\frac{d^2q}{dt^2} + R\cdot\frac{dq}{dt} + \frac{q}{C} = V_e}$$



Differential Operator is linear.

Date / /

How to find solution of

$$y'' + p(x).y' + q(x).y = r(x) \quad \text{--- (1)}$$

Gen. sol^n $y = y_c + y_p$.

Corresponding homogeneous equation of (1)

$$\boxed{y'' + p(x).y' + q(x).y = 0} \quad \text{--- (2)}$$

where, y_c = sol. of (2) $\Rightarrow$ complementary sol

y_p = sol. of (1) $\Rightarrow$ Particular solution

$y'' + py' + qy = r$ --- (1)

$y'' + py' + qy = 0$ --- (2)

$y(x_0) = y_0$; $y'(x_0) = y_1$

$D^2y + pDy + qy = 0$.

$\rightarrow (D^2 + pD + q)y = 0$.

$\rightarrow Ly = 0$.

$L = (D^2 + pD + q)$.

Gen. sol. $y = c_1y_1 + c_2y_2$

$$L(c_1y_1 + c_2y_2) = c_1L(y_1) + c_2L(y_2) = 0.$$

$\{c_1y_1 + c_2y_2\} \geq u$

$\{\tilde{c}_1\tilde{y}_1 + \tilde{c}_2\tilde{y}_2\}$

$c_1y_1(x_0) + c_2y_2(x_0) = y_0$

$c_1y_1'(x_0) + c_2y_2'(x_0) = y_1$

$$\begin{bmatrix} y_1 & y_2 \\ y_1' & y_2' \end{bmatrix}_{x_0} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} y_0 \\ y_1 \end{bmatrix}$$

If $L = D^2 + PD + q$ then (1) becomes

$Ly = r$.

Given, $Ly_p = r$; $Ly = r$

$\therefore$ $L(y - y_p) = 0$

solution of homogeneous eq^n

$\therefore$ $y - y_p = y_c$

$\therefore$ $\boxed{y = y_c + y_p}$ is the general solution

{ $c_1y_1 + c_2y_2$ is only (unique) solution for DE }

★ Variation of Parameters (VOP)

Date __/__/__ Page No.:

$$y'' + py' + qy = r(x) \quad \text{—} (1)$$

$$y'' + py' + qy = 0 \quad \text{—} (2)$$

(I) Solve (2) $y_c = c_1 y_1 + c_2 y_2$ $\left\{ c_1 \,\&\, c_2 \text{ are consts} \right\}$

(II) $y_p = f_1 y_1 + f_2 y_2$ $\left\{ f_1 \,\&\, f_2 \text{ are functions of independent variable} \right\}$

$$f_1 = \int \frac{W_1}{W} \cdot r \, dx = -\int \frac{y_2 \, r \, dx}{W}$$

$$f_2 = \int \frac{W_2}{W} \cdot r \, dx = \int \frac{y_1 \, r \, dx}{W}$$

where,

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$$

W_j = replace jth col of W by last column of I (order of DE) $\left\{ \text{Here, } \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$

$$W_1 = \begin{vmatrix} 0 & y_2 \\ 1 & y_2' \end{vmatrix} = -y_2 .$$

$$W_2 = \begin{vmatrix} y_1 & 0 \\ y_1' & 1 \end{vmatrix} = y_1 .$$

(III) $y = y_c + y_p .$

Maharaza

Date ___/___/______ Page No.: ______

Q. $y'' + y = \operatorname{cosec} x$

$$W = \begin{vmatrix} \sin x & \cos x \\ \cos x & -\sin x \end{vmatrix}$$

$\Rightarrow$ Aux eqⁿ $m^2 + 1 = 0$.

$$= \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix}$$

$y = y_c + y_p$.

$y_c = C_1 \cos x + C_2 \sin x$ $\qquad W = 1$

$y_p = f_1 \cos x + f_2 \sin x$.

where, $f_1 = \int \frac{W_1}{W} r \, dx = -\int \frac{y_2}{W} r \, dx$

$$= -\int \frac{\sin x \cdot \operatorname{cosec} x}{1} \, dx$$

$\Rightarrow \quad f_1 = -x$.

$f_2 = \int \frac{W_2}{W} r \, dx = +\int \frac{y_1}{W} r \, dx$

$\Rightarrow f_2 = \int \frac{\cos x}{\sin x} \, dx = \ln(\sin x)$.

Hence,

$\therefore \quad y = C_1 \cos x + C_2 \sin x - x \cos x + \ln(\sin x) \sin x$

Date __/__/____ Page No.: ____

The: Hom. Eqn. Known → y_1 ; $y_2 = uy_1$

where, $u = \int \frac{1}{y_1^2} \cdot e^{-\int p(x)\,dx}$

★ $y'' + py' + qy = 0$

⇒ $y = x^m$

$m(m-1)x^{m-2} + m.p.x^{m-1} + qx^m = 0$

⇒ $[m(m-1) + mpx + qx^2] \cdot x^{m-2} = 0$

⇒ $\boxed{m(m-1) + mpx + qx^2 = 0}$

If $px + qx^2 = 0$ then $y_1 = x$

$2 + 2px + qx^2 = 0$ then $y_1 = x^2$

★ $x^2 y'' - 2x(1+x) \cdot y' + 2(1+x) \cdot y = x^3$

⇒ $y'' - \underbrace{\frac{2x(1+x)}{x^2}}_{p} y' + \underbrace{\frac{2(1+x)}{x^2}}_{q} y = x$

$px + qx^2 = 0$ ∴ $y_1 = x$ is one solution

$y_2 = uy_1$;

$u = \int \frac{1}{y_1^2} \cdot e^{-\int p(x)\,dx} \cdot dx$

$= \int \frac{1}{x^2} \cdot e^{+\int \frac{2x(1+x)}{x^2} \cdot dx} \, dx$

Maharaza

Date __/__/____ Page No.: ____

$$\therefore \quad I = e^{+\int \frac{2(1+x)}{x}.dx} = e^{+\int\left(\frac{2}{x}+2\right).dx}$$

$$= e^{+2.\ln x + 2x}$$

$$= \frac{e^{\ln(x^{+2})}}{e^{-2x}} = \frac{1}{x^2} e^{+2x} = x^2.e^{2x}$$

$$u = \int \frac{1}{\cancel{x^2}}.\cancel{x^2}.e^{2x}.dx = \frac{e^{2x}}{2}$$

$$\therefore \quad y_c = C_1 x + \frac{1}{2} C_2 x.e^{2x}.$$

$$W = \begin{vmatrix} x & \frac{x.e^{2x}}{2} \\ 1 & \frac{e^{2x}}{2}+xe^{2x} \end{vmatrix}$$

$$y_p = f_1 x + f_2 . \frac{xe^{2x}}{2}.$$

$$W = \cancel{\frac{xe^{2x}}{2}} + x^2 e^{2x} - \cancel{\frac{x.e^{2x}}{2}}$$

$$W = x^2.e^{2x}$$

$$f_1 = \int \frac{W_1}{W} r.dx = \int \frac{-y_2}{W} r.dx.$$

$$= -\int \frac{x.e^{2x}}{2(x^2 e^{2x})} x.dx$$

$$= \frac{-x}{2}$$

$$f_2 = \int \frac{W_2}{W} r dx = \int \frac{y_1 r dx}{W}$$

$$= \int \frac{x.x.dx}{x^2 e^{2x}} = \int e^{-2x} dx$$

$$f_2 = -\frac{e^{-2x}}{2}$$

$$y_p = \frac{-x^2}{2} + \frac{-e^{-2x}}{2}.\frac{xe^{2x}}{2} = \frac{-x^2}{2} - \frac{x}{4}$$

$$y = C_1 x + \frac{C_2 x e^{2x}}{2} - \frac{x^2}{2} - \frac{x}{4}$$

$$= x\left(C_1 - \frac{1}{4}\right) - \frac{x^2}{2} + \frac{C_2 x e^{2x}}{2}$$

Page No.:

Date __/__/____

* $y^{(4)} - 5y^{(2)} + 4y = 0$

$\Rightarrow$ Aux. eq^n $m^4 - 5m^2 + 4 = 0$

$\cancel{m=1}$

$m^4 - m^2 - 4m^2 + 4 = 0$

$\Rightarrow m^2(m^2-1) - 4(m^2-1) = 0$

$\Rightarrow m^2 = 4 \; ; \; m^2 = 1$

$\Rightarrow m = \pm 2 \; ; \pm 1$.

$y = c_1e^{-x} + c_2e^{x} + c_3e^{-2x} + c_4e^{+2x}$.

* $y^{(4)} - 8y^{(2)} + 16y = 0$.

$\Rightarrow$ Aux. eq^n $m^4 - 8m^2 + 16 = 0$.

$m^4 - 4m^2 - 4m^2 + 16 = 0$.

$\Rightarrow m^2(m^2-4) - 4(m^2-4) = 0$

$\Rightarrow m = \pm 2 \; ; \pm 2$.

$y = (c_1 + c_2x)e^{-2x} + (c_3 + c_4x)e^{2x}$.

(If $m = -2, -2, -2, 2$, then $y = (c_1 + c_2x + c_3x^2)e^{-2x} + c_4e^{2x}$.)

* $x^3 y''' - 3x^2 y'' + 6xy' - 6y = x^4 \ln x$.

$\Rightarrow$ (I) $x^3 y''' - 3x^2 y'' + 6xy' - 6y = 0$.

Let, $y = x^m$.

$x^3 \cdot (m)(m-1)(m-2) \cdot x^{m-3} - 3x^2 \cdot (m)(m-1) \cdot x^{m-2} + 6x \cdot m \cdot x^{m-1} - 6x^m = 0$.

$\Rightarrow m(m-1)(m-2) \cdot x^m - 3m(m-1) \cdot x^m + 6m \cdot x^m - 6x^m = 0$.

Maharaza

9.

$\Rightarrow x^m \{ m(m-1)(m-2) - 3m(m-1) + 6m - 6 \} = 0$

$m=1, \; Q=0, \;$ then $y_1 = x$.

$m=2, \; Q=0, \;$ then $y_2 = x^2$.

$m=3, \; Q=0, \;$ then $y_3 = x^3$.

$\therefore y_c = c_1 x + c_2 x^2 + c_3 x^3$.

(ii). $y_P = f_1 x + f_2 x^2 + f_3 x^3$.

$$W = \begin{vmatrix} x & x^2 & x^3 \\ 1 & 2x & 3x^2 \\ 0 & 2 & 6x \end{vmatrix} = 2x^3 \qquad W_2 = \begin{vmatrix} x & 0 & x^3 \\ 1 & 0 & 3x^2 \\ 0 & 1 & 6x \end{vmatrix}$$

$= -3x^3 + x^3 = -2x^3$

$$W_1 = \begin{vmatrix} 0 & x^2 & x^3 \\ 0 & 2x & 3x^2 \\ 1 & 2 & 6x \end{vmatrix} = x^4 \qquad W_3 = \begin{vmatrix} x & x^2 & 0 \\ 1 & 2x & 0 \\ 0 & 2 & 1 \end{vmatrix}$$

$= 2x^2 - x^2 = x^2$.

$f_1 = \int \frac{W_1}{W} x \, dx = \int \frac{x^4}{2x^3} . x . \ln x \, dx$.

$= \int \frac{x^2}{2} \ln x \, dx = \ln x . \frac{x^3}{6} - \int \frac{x^3}{6} . \frac{1}{x} dx$.

$f_1 = \ln x . \frac{x^3}{6} - \frac{x^3}{18} = \frac{x^3}{6}(\ln x - 1/3)$

$f_2 = \int \frac{W_2}{W} x \, dx = \frac{-2x^3}{2x^3} . x \ln x = -\int x . \ln x$.

$f_2 = -\ln x . \frac{x^2}{2} + \int \frac{x^2}{2} \frac{1}{x} . dx$

$f_2 = -\frac{x^2}{2} \ln x + \frac{x^2}{4} = \frac{x^2}{2}\left(-\ln x + \frac{1}{2}\right)$.

Scanned by CamScanner

Date ___/___/______ Page No.: ______

$$f_3 = \int \frac{W_3}{W} x \, dx \doteq \int \frac{x^2}{2x^3} x \ln x \, dx$$

$$= \int \frac{\ln x}{2} dx = \frac{1}{2} x(\ln x - 1)$$

$$y = y_c + y_p$$

$$y = c_1 x + c_2 x^2 + c_3 x^3 + \frac{x^3}{6}(\ln x - 1) x$$
$$+ \frac{x^2}{2}\left(\frac{1}{2} - \ln x\right) x^2 + \frac{x}{2}(\ln x - 1) x^3$$

* __Inhom__ $y'' + py' + qy = r$ — ①

Let, u be a sol. of $y'' + py' + qy = 0$.

Assume :- $\boxed{y = uv}$ is general sol. of ①.

Change $y \Rightarrow v$ in ①. — ide

$$\left.\begin{aligned} y &= uv \\ y' &= uv' + u'v \\ y'' &= 2u'v' + uv'' + u''v \end{aligned}\right\} \text{put in ①}$$

$$2u'v' + uv'' + u''v + puv' + pu'v + quv = r.$$

$$\Rightarrow uv'' + (2u' + pu)v' + \underbrace{(u'' + pu' + qu)}_{=0}v = r.$$

$=0$ as u is a solution.

$$\Rightarrow uv'' + (2u' + pu)v' = r.$$

Date __/__/____ Page No.: ____

Take $\boxed{v' = z}$; —— ②

$uz' + (2u' + pu)z = r$

On Solving for z;

use z to solve eqⁿ ②

we will get v.

Put v in $y = uv$ to obtain general solution.

$x^2 y'' - 2x(1+x).y' + 2(1+x).y = x^3$.

$\Rightarrow$ $y'' - \frac{2x(1+x)}{x^2}.y' + \frac{2(1+x)}{x^2}.y = x$.

$p x + q x^2 = 0$, $y_1 = x$ is the solution.

Here $y = (x)v$, $u = x$

Let, $y = uv$ be the solution

$\Rightarrow$ $y = xv$

$\Rightarrow$ $y' = xv' + v$

$\Rightarrow$ $y'' = v' + xv'' + v'$

$\Rightarrow$ $y'' = 2v' + xv''$

$(2v' + xv'') - \frac{2x(1+x)}{x^2}.(xv' + v) + \frac{2(1+x)}{x^2}.xv = x$

$\Rightarrow xv'' + (2 - 2(1+x))v' + \left(\frac{-2x(1+x)}{x^2} + \frac{2x(1+x)}{x^2}\right)v = x$

$\Rightarrow$ $xv'' + -2xv' = x$.

$\Rightarrow$ $v'' - 2v' = 1$

$\Rightarrow$ $v' = z$

$\Rightarrow$ $z' - 2z = 1$

$\Rightarrow$ $\frac{dz}{1+2z} = dx$ $\Rightarrow$ $\ln(1+2z) = 2x + C$

$z = e^{2x} - 1/2$

Maharaza

$$v' = \frac{c_1 e^{2x} - 1}{2}$$

$$\Rightarrow 2\,dv = (c_1 e^{2x} - 1)\cdot dx$$

$$\Rightarrow 2v = \frac{c_1 e^{2x}}{2} - x$$

$$\Rightarrow v = \frac{c_1 e^{2x}}{4} - \frac{x}{2} + c_2$$

$$y = (x)\left(\frac{c_1 e^{2x}}{4} - \frac{x}{2} + c_2\right)$$

$$y = \frac{c_1 x e^{2x}}{4} - \frac{x^2}{2} + c_2 x$$

$$\boxed{y = c_2 x - \frac{x^2}{2} + \frac{c_1 x e^{2x}}{4}}$$

★ Proof of VOP :-

$$y'' + p(x)\cdot y' + q(x)\cdot y = r(x) \quad \text{—— ①}$$

$$y'' + p(x)y' + q(x)y = 0 \quad \text{—— ②}$$

① $y_c = c_1 y_1 + c_2 y_2$ (y_c is gen. solⁿ of ②)

② $y_p = f_1 y_1 + f_2 y_2$

Assume :- $y = f_1 y_1 + f_2 y_2$

$$y' = f_1 y_1' + f_2 y_2' + \boxed{(f_1' y_1 + f_2' y_2)}$$

Assume 0

Date __/__/__ Page No.:

$$y'' = f_1'y_1' + f_1y_1'' + f_2'y_2' + f_2y_2''$$

subs y, y', y'' in ①

$\Rightarrow$
$$(f_1'y_1' + f_1y_1'' + f_2'y_2' + f_2y_2'') + p(f_1y_1' + f_2y_2')$$
$$+ q(f_1y_1 + f_2y_2) = r.$$

$$\Rightarrow f_1\underbrace{(y_1'' + py_1' + qy_1)}_{=0} + f_2\underbrace{(y_2'' + py_2' + qy_2)}_{=0}$$
$$+ f_1'y_1' + f_2'y_2' = r$$

(since, y_1 & y_2 are LI solutions of ①)

$$\Rightarrow \boxed{f_1'y_1' + f_2'y_2' = r}$$

$$\left.\begin{aligned} f_1'y_1 + f_2'y_2 &= 0 \\ f_1'y_1' + f_2'y_2' &= r. \end{aligned}\right\} \text{obtained equations.} \quad \begin{aligned} &— ③ \\ &— ④ \end{aligned}$$

eqⁿ ③ $\times y_1'$ − eqⁿ ④ $\times y_1$.

$$f_1'y_1y_1' + f_2'y_2y_1' = 0$$
$$- \; f_1'y_1'y_1 \mp f_2'y_2'y_1 = \mp r y_1$$

$$f_2'(y_2y_1' - y_2'y_1) = -ry_1$$

$$\Rightarrow f_2' = \frac{-y_1r}{y_2y_1' - y_2'y_1}$$

$$\Rightarrow f_2 = \int \frac{y_1r}{y_1y_2' - y_2y_1'}\, dx$$

Date __/__/____ Page No.: ____

Similarly,

$eq^n ③ \times y_2' - eq^n ④ \times y_2$

$f_1'(y_1y_2' - y_2y_1') = -y_2 r$

$$\Rightarrow f_1 = -\int \frac{y_2}{y_1y_2' - y_2y_1'} \cdot r\, dx$$

$$\therefore f_1 = -\int \frac{y_2}{y_1y_2' - y_2y_1'} \cdot r\, dx \; ; \; f_2 = \int \frac{y_1}{y_1y_2' - y_2y_1'} \cdot r\, dx$$

$$\equiv \int \frac{W_1}{W} r\, dx \qquad \equiv \int \frac{W_2}{W} \cdot r\, dx$$

★ eg. ①

$$A = \begin{bmatrix} -2 & 2 \\ 2 & -5 \end{bmatrix}$$

$|A - \lambda I| = 0$

$$\begin{vmatrix} -2-\lambda & 2 \\ 2 & -5-\lambda \end{vmatrix} = 0$$

$\lambda^2 + 7\lambda + 6 = 0$

$\lambda = -1, -6$

eigen vector of A is element of null space of $(A - \lambda I)$.

$A = n \times n$

$(R^n \to R)$

$\boxed{Ax = b}$ $\boxed{Ax = \lambda x}$

① $(A - \lambda I)x = 0$ has non-zero x as solution

$\Updownarrow$

$|A - \lambda I| = 0$.

★ Algebraic Multiplicity (a.m)

The frequency of a specific eigen value is called its algebraic multiplicity.

-1 has a.m. 1 ; -6 has a.m. 1.

Maharaja

Dimension of Null space = n − Rank

Page No.:

Date / /

★ Geometric Multiplicity :- (g.m).

Number of LI eigen vectors corresponding to a specific λ, is the geometric multiplicity of that λ. = Nullity of [A − λI].

(1) eigen vector calculation

for λ = 1 ; $\begin{bmatrix} -1 & 2 \\ 2 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$-x_1 + 2x_2 = 0$

$2x_1 - 4x_2 = 0$.

REF RREF Eigen vectors

$[A - \lambda I] = \begin{bmatrix} \boxed{-1} & 2 \\ 2 & -4 \end{bmatrix} \sim \begin{bmatrix} \boxed{-1} & 2 \\ 0 & 0 \end{bmatrix} \sim \begin{bmatrix} \boxed{1} & -2 \\ 0 & 0 \end{bmatrix} \quad \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

for λ = −6 REF RREF

$[A - \lambda I] = \begin{bmatrix} 4 & 2 \\ 2 & 1 \end{bmatrix} \sim \begin{bmatrix} 4 & 2 \\ 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 1/2 \\ 0 & 0 \end{bmatrix} \quad \begin{bmatrix} -1/2 \\ 1 \end{bmatrix}$

$\boxed{g.m. \le a.m.}$

Date ___/___/_______ Page No.: _______

★ $\begin{bmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{bmatrix}$ $\quad |A-\lambda I| = 0$.

$$\begin{bmatrix} -2-\lambda & 1 & 1 \\ 1 & -2-\lambda & 1 \\ 1 & 1 & -2-\lambda \end{bmatrix} = 0$$

$$(-2-\lambda)\left[4+\lambda^2+4\lambda \underset{-1}{}\right] - 1\left[-3-\lambda\right] + 1\left[3+\lambda\right] = 0$$

$$-8-2\lambda^2-8\lambda-4\lambda-\lambda^3-4\lambda^2 + 6+2\lambda = 0.$$
$$+2+\lambda$$

$\Rightarrow \quad \lambda^3 + 9\lambda + 6\lambda^2 = 0$

$\Rightarrow \quad \lambda(\lambda^2+9+6\lambda) = 0$

$\Rightarrow \quad \lambda = 0, -3, -3$

Here, (-3 has a.m. 2).

for $\lambda = -3$, RREF.

$$[A-\lambda I] = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \sim \begin{bmatrix} \boxed{1} & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

eigen vectors of A for $\lambda=-3$ $\quad \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \Rightarrow$ Basis of Null Space of $[A-\lambda I]$.

Here, (-3 has g.m. 2)

for $\lambda = 0$ $\quad$ eigen vector

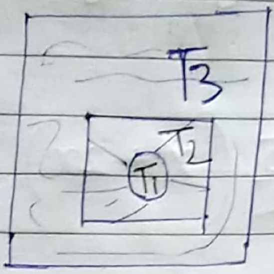
$$[A-\lambda I] = \begin{bmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{bmatrix} \approx \Big[\qquad \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

Maharaza

{ Good eigen value g.m. = a.m.
Bad eigen value g.m. < a.m }

★ System of Linear Differential Equations :-

① $$\frac{dT_1}{dt} = a(T_2 - T_1)$$

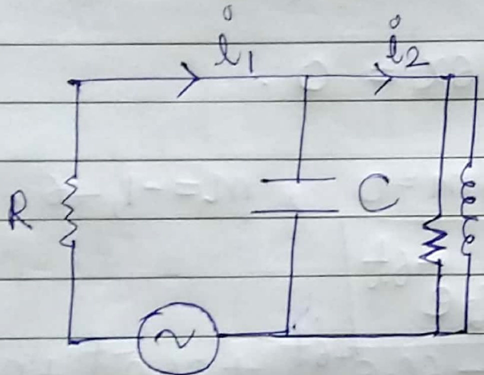


$$\frac{dT_2}{dt} = a(T_1 - T_2) + b(T_3 - T_2)$$

$$\frac{dT_1}{dt} = -aT_1 + aT_2$$

$$\frac{dT_2}{dt} = aT_1 - (a+b)T_2 + bT_3$$

②



$$Ri_1 + \frac{1}{C}q_1 = V$$

$$R\frac{dq_1}{dt} + \frac{q_1}{C} = V$$

const. coefficient homogeneous system
$$\left.\frac{dx}{dt} = ax + by \;;\; \frac{dy}{dt} = cx + dy\right\}$$

$$\frac{dx}{dt} = f(x,y,t) \;;\; \frac{dy}{dt} = g(x,y,t)$$

x, y are dependent variables
t is independent variable

(I) Elimination Method :-

Date ___/___/______ Page No.: ______

eg (1) $\frac{dx}{dt} = -2x + 2y$; $\frac{dy}{dt} = 2x - 5y$

$x(0) = x_0 = 5$

$y(0) = y_0 = 10$

$\Rightarrow y = \frac{x' + 2x}{2}$

$\Rightarrow \left(\frac{x'+2x}{2}\right)' = 2x - \frac{5}{2}\left(x' + 2x\right)$

$\Rightarrow \frac{x''}{2} + x' = 2x - \frac{5x'}{2} - \frac{10x}{2}$

$\Rightarrow \frac{x''}{2} + \frac{2x'}{2} = \frac{4x - 10x - 5x'}{2}$

$\Rightarrow x'' + 7x' + 6x = 0$

$m^2 + 7m + 6m = 0$; $m = -1, -6$

$\Rightarrow \boxed{x(t) = c_1 e^{-t} + c_2 e^{-6t}}$

$y(t) = \frac{1}{2}\left\{-c_1 e^{-t} - 6c_2 e^{-6t} + 2c_1 e^{-t} + 2c_2 e^{-6t}\right\}$

$= \frac{1}{2}\left\{c_1 e^{-t} - 4c_2 e^{-6t}\right\}$

$\Rightarrow \boxed{y(t) = \frac{c_1 e^{-t}}{2} - 2c_2 e^{-6t}}$

At $t=0$; $x = 5$; $2 \times 5 = 2c_1 + 2c_2$

At $t=0$; $y = 10$; $10 = \frac{c_1}{2} - 2c_2$

$\cancel{20}^{4} = \frac{\cancel{5}c_1}{2} \Rightarrow c_1 = 8$

$c_2 = -3$

Scanned by CamScanner

Method (II) (works only if no. of eigen values and no. of LI eigenvectors is equal)

Date __/__/____ Page No.: ____

equivalent form
$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} ; \begin{bmatrix} x(0) \\ y(0) \end{bmatrix} = \begin{bmatrix} x_0 \\ y_0 \end{bmatrix}$$

$$\boxed{X' = AX \; ; \; X(0) = X_0}$$ $x(t) = ze^{At}$

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} e_1 \\ e_2 \end{bmatrix} e^{\lambda t} ,$$ trial solution

$$\Rightarrow \lambda . \begin{bmatrix} e_1 \\ e_2 \end{bmatrix} . e^{\lambda t} = A \begin{bmatrix} e_1 \\ e_2 \end{bmatrix} e^{\lambda t}$$

$$\Rightarrow \underset{\neq 0}{e^{\lambda t}} \left(A \begin{bmatrix} e_1 \\ e_2 \end{bmatrix} - \lambda \begin{bmatrix} e_1 \\ e_2 \end{bmatrix} \right) = 0$$

$$\therefore \boxed{A \begin{bmatrix} e_1 \\ e_2 \end{bmatrix} = \lambda \begin{bmatrix} e_1 \\ e_2 \end{bmatrix}}$$

for eg. (1).
$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = c_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} e^{-t} + c_2 \begin{bmatrix} -1/2 \\ 1 \end{bmatrix} e^{-6t}$$

for (3) x (3) system;

$$\begin{bmatrix} x(t) \\ y(t) \\ z(t) \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} e^{0t} + c_2 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} e^{-3t} + c_3 \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} e^{-3t}$$

eg. $\begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}$ only one LI eigen vector.

Maharaza

Date ___/___/___

Page No.: ______

★ For Real symmetric Matrix, all eigen values are good.

★ Algorithm to solve system of d.e. :

$\dot{X} = AX$; $X = \begin{bmatrix} x \\ y \end{bmatrix}$; $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

Step I :- Find eigen values/ vectors of A.

Step II :- $\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = c_1 \vec{X_1} e^{\lambda_1 t} + c_2 \vec{X_2} e^{\lambda_2 t}$

where, $\vec{X_1}$, $\vec{X_2}$ are eigen vectors corresponding to λ_1, λ_2.

Works only if all eigen values are good.
i.e. For every eigen value, a.m. (algebraic multi-plicity) is equal to g.m. (geometric multiplicity).

Scanned by CamScanner

Date ___/___

(from here on t is considered as independent variable) generally domain for $t \in [0, \infty)$

★ Laplace Transformation :-

for a function $f(t)$,

$$\boxed{\mathcal{L}[f(t)] = F(s) = \int_0^\infty f(t)e^{-st}\,dt}$$ is laplace transform

where, s is parameter (variable constant).

(provided integral converges).

eg. laplace transform of $f(t) = 1$

$$\mathcal{L}[1] = \int_0^\infty e^{-st}\,.dt.$$

$$= \lim_{R\to\infty} \int_0^R e^{-st}\,.dt.$$

$$= \lim_{R\to\infty} \left.\frac{e^{-st}}{-s}\right|_0^R$$

$$= \lim_{R\to\infty} \left.\frac{e^{-sR}}{-s}\right|_0^R + \frac{1}{s}$$

If s is positive, $\lim_{R\to\infty} \frac{e^{-sR}}{-s} \to 0$

$$\boxed{\mathcal{L}[1] = \frac{1}{s} \quad (s>0)}$$

★ $f(t) \xrightarrow{\mathcal{L}} F(s)$ Laplace transform $= \mathcal{L}$

★ Condition of existance for laplace transform of $f(t)$.

Date __/__/____ Page No.:______

For a laplace transform to exist, $f(t)$ is of 'exponential order'.

Sufficient only condition

Not necessary

i.e. $\boxed{|f(t)| < me^{kt}}$ for some $m>0$; $k>0$.

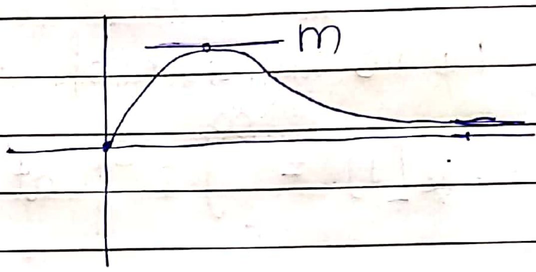
let, $f(t) = e^{t^2}$

laplace transform for e^{t^2} does not exist

$f(t) = t^n$; $t^n \leq me^{kt}$

$$\frac{t^n}{e^{kt}} \leq m.$$

$f(t) = \frac{1}{t}$



for t^n, LT exists. (laplace transform)

★ Basically, the growth of $f(t)$ should not exceed the growth of e^{kt} i.e. $f(t)$ should not become ∞ before e^{kt}.

Because in this case; for the product $f(t) \cdot e^{-st}$, e^{-st} will be able to counter the growth of $f(t)$.

e.g. Calculate $L[e^{at}]$ $\qquad L[f(t)] = \int_0^\infty f(t) \cdot e^{-st}\, dt$.

$$\Rightarrow L[e^{at}] = \int_0^\infty e^{at} \cdot \cancel{f(t)}\, e^{-st} \cdot dt$$

$$= \int_0^\infty e^{-(s-a)t} \cdot dt.$$

Maharaja

$$= \lim_{R\to\infty} \left. \frac{e^{-(s-a)t}}{-(s-a)} \right|_0^R$$

$$= 0 + \frac{1}{s-a} \quad (\text{If } s>a).$$

$$\therefore \boxed{L[e^{at}] = \frac{1}{s-a}.} \quad s>a.$$

e.g. Calculate $L[e^{at} f(t)] \Rightarrow F(s-a)$ ~~first shifting thm~~ exp. shift formula

$$\Rightarrow = \int_0^\infty e^{at} . f(t) e^{-st} dt.$$

$$= \int_0^\infty f(t) . e^{-(s-a)t} dt.$$

$$= F(s-a).$$

Here, $s-a>0 \Rightarrow s>a$ for integral to converge.

$$\therefore \boxed{L[e^{at} f(t)] = F(s-a)}$$

★ Calculate $L[t^n]$.

$\Rightarrow$

$$L[t^n] = \int_0^\infty t^n e^{-st}\, dt$$

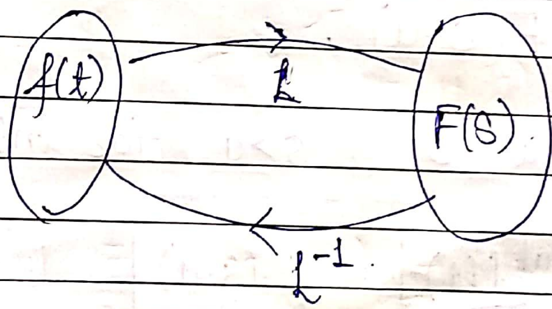
$$= \lim_{R\to\infty} \cancel{\left. t^n \cdot \frac{e^{-st}}{-s} \right|_0^R} \rightarrow (0) + \frac{n}{s} \int_0^\infty t^{n-1} e^{-st}\, dt$$

$$= \frac{n}{s} L[t^{n-1}]$$

$$= \frac{n(n-1)}{s^2} L[t^{n-2}]$$

$$\therefore L[t^n] = \frac{n!}{s^n} L[1] = \frac{n!}{s^n} \cdot \frac{1}{s} = \frac{n!}{s^{n+1}}$$

$$\therefore \boxed{L[t^n] = \frac{n!}{s^{n+1}}}$$



★ <u>Inverse Laplace</u> :-

$L[t^n] = \frac{n!}{s^{n+1}}$; t^n is inverse laplace of $\frac{n!}{s^{n+1}}$

$L[1] = \frac{1}{s}$; 1 is inverse laplace of $\frac{1}{s}$
$(s>0)$

time domain $\longrightarrow$ frequency domain. $\Big\}$ Laplace Transformation

Date ___/___/___ Page No.: ______

* **Algorithm :-**

$$\boxed{\text{Differential Equation in } y} \;/\; \text{IVP.}$$

$$\Big\downarrow L$$

$$\boxed{\text{algebraic equation}} \xrightarrow[\text{for } y]{\text{Solve it}} \boxed{y(s) = \frac{p(s)}{q(s)}} \xrightarrow{L^{-1}} \boxed{\text{Sol. of D.E. } y(t).}$$

* $y = y(t) \; ; \; y' = \dfrac{dy}{dt}$

$$L[y'(t)] = \int_0^\infty \underset{\text{II}}{y'(t)} . \underset{\text{I}}{e^{-st}} \, dt$$

$$= \lim_{R\to\infty} e^{-st}. \, y(t)\Big|_0^R + s \int_0^\infty e^{-st} y(t).dt.$$

$$L[y'(t)] = s.L[y(t)] - y(0).$$

$$\therefore \; L[y'(t)] = s.L[y(t)] - y(0).$$

$$\cancel{= s.y(s) - y(0)}$$

$$\boxed{L[y'(t)] = s.Y(s) - y(0)}$$

$$L[y''(t)] = s.L[y'] - y'(0).$$

$$\boxed{L[y''(t)] = s^2.Y(s) - s\,y(0) - y'(0)}$$

$\cos at = \frac{e^{iat} + e^{-iat}}{2}$; $\sin at = \frac{e^{iat} - e^{-iat}}{2}$

Date __/__/____ Page No.:____

☆ $L[1] = \frac{1}{s}$ $(s > 0)$.

☆ $L[e^{at}] = \frac{1}{s-a}$ $(s > a)$

☆ $L[e^{at} f(t)] = F(s-a)$ $(s > a)$

☆ $L[t^n] = \frac{n!}{s^{n+1}}$

☆ $L[f(t)] = \int_0^\infty f(t) \cdot e^{-st} \cdot dt = F(s)$.

☆ $L(\cos at) = \frac{s}{s^2+a^2}$ $(s > 0)$

☆ $L(\sin at) = \frac{a}{s^2+a^2}$ $(s > 0)$.

☆ $L[y'(t)] = s \cdot Y(s) - y(0)$

☆ $L[y''(t)] = s^2 \cdot Y(s) - s\,y(0) - y'(0)$.

☆ $L[f+g] = L(f) + L(g)$; $L[cf] = c\,L[f]$

Date ___/___/______

Page No.: ______

eg. $y'' + y = t$; $y(0) = 1$; $y'(0) = -2$.

$$\Rightarrow (s^2 Y - s + 2) + Y = \frac{1}{s^2}$$

$$\Rightarrow Y = \frac{1}{s^2(s^2+1)} + \frac{s}{s^2+1} - \frac{2}{s^2+1}$$

Take Inverse Laplace;

$$y(t) = \cos t - 2\sin t + \mathcal{L}^{-1}\left[\frac{1}{s^2(s^2+1)}\right]$$

$$\mathcal{L}^{-1}\left[\frac{1}{s^2(s^2+1)}\right] = \mathcal{L}^{-1}\left[\frac{1}{s^2} - \frac{1}{s^2+1}\right]$$

$$= t - \sin t.$$

$$y(t) = \underbrace{\cos t - 3\sin t}_{y_c} + \underbrace{t}_{y_p}.$$

* Laplace transform & Inverse laplace both are linear transformations.

* ESE | Sat 4/5 | Max. marks = 50 | Time = 3H.

	Marks
ODE (I/II order)	$\Rightarrow$ 18–15
Laplace / Sys. of DE / Numerical Methods	$\Rightarrow$ 12–15. → 4 Post MSE
Linear Algebra	$\Rightarrow$ 10 → 6 Pre MSE
Small marks questions	$\Rightarrow$ 10 (like quiz)

MA102 $\Rightarrow$ 50.

*

$$f^{-1}(y) = \{x \in D : f(x) = y\}.$$

Inverse of a Laplace Transform.

⇓

Unique ⇒ if functions are considered continuous

★ May not be unique when discontinuous functions are also considered.

e.g.

$$g(t) = \begin{cases} 1 ; & 0 \le t < 3 \\ 2 ; & t = 3 \\ 1 ; & t > 3. \end{cases}$$

$$L[g(t)] = \int_0^3 g(t)\, e^{-st}\, dt + \int_3^\infty g(t) . e^{-st} . dt.$$

$$= \left(\frac{e^{-3s}}{(-s)} + \frac{1}{s} \right) + \left. \frac{e^{-st}}{(-s)} \right|_3^\infty$$

$$L[g(t)] = \frac{1}{s}$$

★

$$L[f(t)] \rightsquigarrow F(s) \qquad L[f(t).g(t)] = \int_0^\infty f(t).g(t)\, e^{-st} . dt.$$

$$L[g(t)] \rightsquigarrow G(s)$$

$$g(t) * f(t).$$

$$|||$$

$$F(s) . G(s) \xrightarrow{L^{-1}} f(t) * g(t) . \{\text{convolution}\}$$

$$= \int_0^t f(u) . g(t-u) . du.$$

Integration w.r.t dummy variable (u)

Date ___/___/___ Page No.: ______

★ Inverse laplace of $\left\{ \frac{1}{s(s^2+1)} \right\}$.

(I)

$\Rightarrow \quad \mathcal{L}^{-1}\left[\frac{1}{s(s^2+1)}\right] = \mathcal{L}^{-1}\left[\frac{1}{s} - \frac{s}{s^2+1}\right]$

$$\frac{1}{s(s^2+1)} = \frac{A}{s} + \frac{Bs+C}{s^2+1} = \frac{1}{s} - \frac{s}{s^2+1}$$

$$\frac{1}{s(s^2+1)} = \frac{As^2 + Bs^2 + (A) + (Cs)}{s(s^2+1)}$$

$A + B = 0$

$A = 1 \, ; \, B = -1 \, ; \quad C = 0$.

$$= \mathcal{L}^{-1}\left[\frac{1}{s}\right] - \mathcal{L}^{-1}\left[\frac{s}{s^2+1}\right]$$

$$\boxed{\mathcal{L}^{-1}\left[\frac{1}{s(s^2+1)}\right] = 1 - \cos t}$$

(II)

$$1 * \sin t = \sin t * 1 = \int_0^t (1).(\sin u)\, du.$$

$$= -\cos u \Big|_0^t$$

$$= -\cos t + 1 = 1 - \cos t.$$

$$\mathcal{L}^{-1}\left[\frac{1}{s(s^2+1)}\right] = 1 - \cos t.$$

Maharaza

Q. $y'' - 3y' + 2y = 4e^{2t}$;

$y(0) = -3$; $y'(0) = 5$.

$$\Rightarrow (s^2Y + 3s - 5) - 3[sY + 3] + 2Y = 4 \cdot \frac{1}{s-2}$$

$$\Rightarrow s^2Y + 3s - 5 - 3sY - 9 + 2Y = \frac{4}{s-2}$$

$$\cancel{\Rightarrow Y(s^2 - 4s + 4) + Ys - 2Y + 3s - 14 = \frac{4}{s-2}}$$

$$\cancel{\Rightarrow Y(s-2)^2 + Y}$$

$$Y(s^2 - 3s + 2) + 3s - 14 = \frac{4}{s-2}$$

$$\Rightarrow Y(s-1)(s-2) + 3s - 14 = \frac{4}{s-2}$$

$$\Rightarrow Y(s-1)(s-2) = \frac{4}{s-2} + 14 - 3s$$

$$= \frac{4 + 14s - 28 - 3s^2 + 6s}{s-2}$$

$$\Rightarrow Y = \frac{-3s^2 + 20s - 24}{(s-1)(s-2)^2}$$

172.16.1.252/~nktomar.

Date ___/___/_____ Page No.: _____

$$\Rightarrow Y = \frac{-7}{s-1} + \frac{4}{s-2} + \frac{4}{(s-2)^2}.$$

$$y(t) = -7e^{t} + 4e^{2t} + 4te^{2t}.$$

★ Assign.

1. Find the Laplace Transform of following equations

(1) sin hat. $\sinh at = \frac{1}{2}\left(e^{at} - e^{-at}\right)$.

$$\Rightarrow \mathcal{L}[\sinh at] = \mathcal{L}\left[\frac{1}{2}e^{at} - e^{-at}\right]$$

$$= \frac{1}{2}\cdot\mathcal{L}[e^{at}] - \frac{1}{2}\mathcal{L}[e^{-at}]$$

$$= \frac{1}{2}\left[\left(\frac{1}{s-a}\right) - \left(\frac{1}{s+a}\right)\right]$$

$$\Rightarrow \frac{1}{2}\left[\frac{\cancel{s}+a - \cancel{s}+a}{s^2-a^2}\right] = \frac{a}{s^2-a^2}$$

Scanned by CamScanner

Date __/__/____

Page No.:

(2) $\cos(\omega t + \theta)$. $\cos(\omega t+\theta) = \cos\omega t.\cos\theta - \sin\omega t.\sin\theta$

$\Rightarrow L[\cos\omega t \cos\theta - \sin\omega t.\sin\theta]$

$= \cos\theta.\ L[\cos\omega t] - \sin\theta.\ L[\sin\omega t]$.

$= \cos\theta.\left(\frac{s}{s^2+\omega^2}\right) - \sin\theta\left(\frac{a}{s^2+\omega^2}\right)$.

$= \frac{s\cos\theta - a\sin\theta}{s^2+\omega^2}$

2. Find Inverse Laplace transform of following.

(1) $\frac{1}{s^2(s^2+a^2)} = F(s)$

$\Rightarrow L^{-1}[F(s)] = L^{-1}\left(\frac{1}{s^2(s^2+a^2)}\right)$

$= L^{-1}\left(\frac{1}{a^2}\left(\frac{1}{s^2} - \frac{1}{s^2+a^2}\right)\right)$

$= \frac{1}{a^2}.L^{-1}\left(\frac{1}{s^2}\right) - \frac{1}{a.a^2}.L^{-1}\left(\frac{a}{s^2+a^2}\right)$

Scanned by CamScanner

Ans. $e^{-t} + \frac{1}{3}\sin 3t - \cos 3t$

$$= \frac{1}{a^2}\cdot t - \frac{1}{a^3}\sin at.$$

(Actual question in test $y'' + 9y = 10e^{-t}$

(3) $y'' + 9y' = 10e^{-t}$, $y(0) = 0$; $y'(0) = 0$.

$$\Rightarrow \mathcal{L}[y''] + 9\mathcal{L}[y'] = 10\mathcal{L}[e^{-t}]$$

$$\Rightarrow [s^2Y - 9s(0) - (0)] + 9[sY - (0)] = 10\cdot\frac{1}{(s+1)}$$

$$\Rightarrow s^2Y + 9sY = \frac{10}{s+1}$$

$$\Rightarrow (s^2+9s)Y = \frac{10}{(s+1)}$$

$$\Rightarrow Y = \frac{10}{(s+1)(s)(s+9)}$$

Take Inverse laplace,

$$\Rightarrow \cancel{y(t)}\ Y = \frac{A}{s+1} + \frac{B}{s} + \frac{C}{s+9}$$

$$\Rightarrow \frac{10}{(s+1)(s)(s+9)} = \frac{A(s^2+9) + B(s^2+10s+9) + C(s^2+s)}{s(s+1)(s+9)}$$

$$\Rightarrow 10 = (A+B+C)s^2 + (10B+C)s + 9(A+B)$$

$A+B+C=0$ $\quad A+B = 10/9 \quad B = 1/9$; $A = 1$.

$10B+C=0$ $\quad 10B = 10/9 \quad C = -10/9$

$\Rightarrow 9B-A=0$

$$Y = \frac{1}{s+1} + \frac{1}{9s} + \frac{-10}{9(s+9)}$$

$$y(t) = \mathcal{L}^{-1}\left[\frac{1}{s+1}\right] + \frac{1}{9}\mathcal{L}^{-1}\left[\frac{1}{s}\right] - \frac{10}{9}\mathcal{L}^{-1}\left[\frac{1}{s+9}\right]$$

$$y(t) = e^{-t} + \frac{1}{9} - \frac{10}{9}e^{-9t}.$$

★ (Convolution Theorem)

Let $F(s)$ and $G(s)$ denote the laplace transform of functions $f(t)$ and $g(t)$ resp. Then the product $F(s).G(s)$ is the laplace transform of the convolution of functions f and g, defined by

$$f * g(t) = \int_0^t f(u).g(t-u).du \quad \text{ie}$$

$$\mathcal{L}(f*g(t)) = F(s).G(s).$$

a. Prove above thm & b. Show $f*g = g*f$.

Proof:- a. $F(s) = \mathcal{L}(f(t))$; $G(s) = \mathcal{L}(g(t))$

$$\mathcal{L}[f*g(t)] = F(s).G(s).$$

$$F(s).G(s) = \int_0^\infty f(\sigma)e^{-s\sigma}.d\sigma \int_0^\infty g(\tau).e^{-s\tau}d\tau.$$

$$= \int_0^\infty \left[\int_0^\infty f(\sigma).e^{-s(\sigma+\tau)}d\sigma\right] g(\tau).d\tau$$

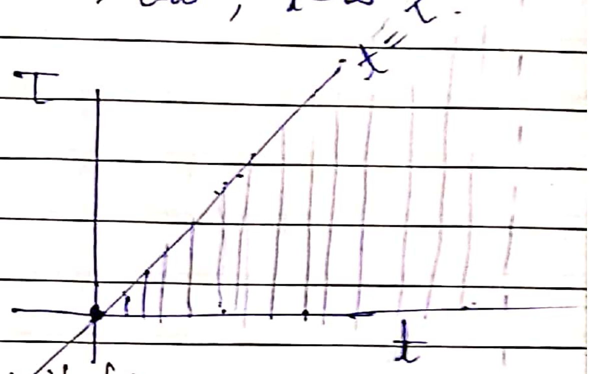
take $\tau + \sigma = t$;

$\Rightarrow \quad d\sigma = dt \; ; \quad \tau \leq t \leq \infty$

$$= \int_0^\infty \left[\int_\tau^\infty f(t-\tau).e^{-st}.dt \right] g(\tau).d\tau \qquad \begin{array}{c} dt.d\tau \\ \downarrow \\ d\tau.dt \end{array}$$

Changing order of Integration. Limits for $dt.d\tau \Rightarrow \tau=0 \; ; \; t=\tau$; $\tau=\infty \; ; \; t=\infty$

$$= \int_0^\infty \left[\int_0^t f(t-\tau).g(\tau).e^{-st}.d\tau \right].dt$$



$$= \int_0^\infty \left[\int_0^t f(t-\tau).g(\tau).d\tau \right].e^{-st}.dt$$

Limits for $d\tau.dt \Rightarrow \tau=0 \quad t=0 \quad t=\infty$

$\tau = t \quad t=\infty.$

Here; $\left[f(t) = \int_0^t f(t-\tau).g(\tau).d\tau \right] = \int_0^\infty \left[\int_0^t f(\tau).g(t-\tau).d\tau \right].dt$

$$= L\left[f*g(t) \right]$$

b. $\Rightarrow \quad L(f*g) = F(S).G(S).$

$L(g*f) = G(S).F(S).$

$\Rightarrow \quad f*g = g*f.$