MA225: Probability Theory and Random Processes

Textbook : Probability, Random Variables and Stochastic Processes

Authors : A. Papoulis and S. U. Pillai

Instructor : Yogeshmani Tripathi (315, Science Block)

3-1-0 MON → 10-11 WED → 2-3 (Tut)

TUE → 11-12

THU → 11-12

The theory of probability deals with the phenomenon which are random in nature which under some random expt. yeild outcome indicating some pattern about the quantity of interest.

Random Expt.

1. All possible outcomes are known in advanced.
2. Any particular trial will yeild an outcome that is not known in advance.
3. Expt. can be repeated under identical condition.

Sample Space (S or Ω)

Collection of all possible outcomes of a R.E.

Toss a coin once $S = \{H, T\}$.

Toss a coin ten times

$S = \{(e_1, e_2, \ldots, e_{10}) : e_i = H/T ; i = 1, 2, \ldots, 10\}$.

Lifetime of a Machine. $t \to$ life in years.

$S = \{t : t \geq 0\}$.

$S_1 = \{t : 0 \leq t \leq N\}$.

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★ Event :- Some subset of sample space.

let A and B are events in S.

★ $A \cup B$ ⇒ occurrence of at least one of events A and B.

let $A_1, A_2, \ldots A_n$ be events of S then

$\bigcup_{i=1}^{n} A_i = A_1 \cup A_2 \cup \ldots \cup A_n$ ⇒ occurrence of at least one of $A_1, A_2, \ldots, A_n$.

★ $A \cap B$ ⇒ simultaneous occurrence of A and B.

★ A^c ⇒ complement of A.

eg. A = { first toss is a head }

Once we have events of our interest we can assign probability to them.

★ Classical Defn of Probability (laplace 1812)

Consider a RE whose outcomes are strict in S which contain n points. Then prob of an event A with m favourable outcomes is given by

$$P(A) = m/n.$$

★ Cndns: ① Sample space should be finite.
② All of these n possible outcomes are equally likely to occur.

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* The theory of probability has its origin in the games of chance which can be traced back at least upto 17^{th} century.

At the request of owners of some big gambling houses mathematicians like fermat, pascal, demoivre, developed the theory of prob.

Probability has found of wide application in many fields. Tools of prob. and statistics can be used to explain and understand variations involved in areas like science, engineering, clinical trials, life testing, survival analysis, biology, computer, electronics, etc.

* Relative Frequency Def^n of Prob. (Empirical Def^n)

Consider a RE with some sample space S.

Let expt. is conducted n times and n_A be the no. of outcomes for event A then Relative frequency of A $= \frac{n_A}{n}$

The stabalization of this frequency for large sequence of trials for a R.E is known as statistical regularity.

So $P(A) = \frac{n_A}{n}$ as $n \to \infty$.

$\frac{\sqrt{n}}{n} \to 0$ as $n \to \infty$ $\quad \frac{n-\sqrt{n}}{n} \to 1$ as $n \to \infty$.

eg. 1 3 2 5 7 4 9 11 6 - - - - -

event interested is odd number.

$\frac{1}{1}$ $\frac{2}{2}$ $\frac{2}{3}$ $\frac{3}{4}$ $\frac{4}{5}$ $\frac{4}{6}$ . . . . $\rightarrow$ $\frac{2}{3}$ (can converge to)

★ Axiomatic Defn of Probability :-

(A.N. Kolmogorov (foundation of Prob. Theory) 1933)

(this is the most accepted defn of prob)

Let S be a sample space and P(A) denote the prob. of event A in S.

Then

AXIOM I : $P(A) \geq 0 \quad \forall A \in S$.

AXIOM II : $P(S) = 1$.

AXIOM III : let A and B be two disjoint events in S then $A \cap B = \phi$.

$$P(A \cup B) = P(A) + P(B).$$

AXIOM III* : let $A_1, A_2, \ldots, A_n$ be pairwise disjoint events then.

$$P(A_1 \cup A_2 \cup \_\_ \cup A_n) = P(A_1) + P(A_2) + \_\_ + P(A_n)$$

(i) $P(\phi) = 0$

$$\phi \cup \phi^c = S.$$

$$P(\phi \cup \phi^c) = P(S)$$

(Axiom III) $P(\phi) + P(\phi^c) = P(S)$

$$\Rightarrow P(\phi) + P(S) = P(S)$$

$$\Rightarrow \boxed{P(\phi) = 0}$$

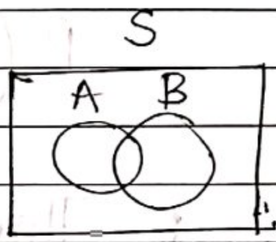
ii) $P(A^c) = 1 - P(A)$ for $A \in S$

$A \cup A^c = S$

iii) $0 \leq P(A) \leq 1$.

iv) If $A \subseteq B$ then $P(A) \leq P(B)$.

(v) $P(B^c \cap A) = P(A) - P(A \cap B)$.



$A = A \cap B \cup A \cap B^C$.

* **Add^n formula for probability :-**

let A and B two events in S. Then

$$P(A \cup B) = P(A) + P(B) - P(A \cap B).$$

$\Rightarrow A \cup B = A \cap B^C \cup A \cap B \cup A^C \cap B$.

$P(A \cup B) = P(A \cap B^c) + P(A \cap B) + P(A^c \cap B)$.

$= P(A) - \cancel{P(A \cap B)} + \cancel{P(A \cap B)} + P(A^c \cap B)$.
(using 5)

$= P(A) + P(B) - P(A \cap B)$ (using 5).

General formula for n events, $A_1, A_2, \text{---}, A_n$.

$$P\left(\bigcup_{i=1}^{n} A_i\right) = \sum_{i=1}^{n} P(A_i) - \sum_{i<j}^{n}\sum^{n} P(A_i \cap A_j)$$

$$+ \sum_{i<j<k}^{n}\sum^{n}\sum^{n} P(A_i \cap A_j \cap A_k) - - - -$$

$$+ (-1)^{n+1} P\left(\bigcap_{i=1}^{n} A_i\right).$$

Proof :- Use M.I.

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$$P\left(\bigcup_{i=1}^{n} A_i\right) = 1 - P\left(\bigcup_{i=1}^{n} A_i\right)^c$$

$$= 1 - P\left(\bigcap_{i=1}^{n} A_i^c\right).$$

MI The result holds for $n=2$.
Assume it is true for $n=m$ then
prove it is true $n=m+1$.

$$P\left(\bigcup_{i=1}^{m} A_i \cup A_{m+1}\right)$$

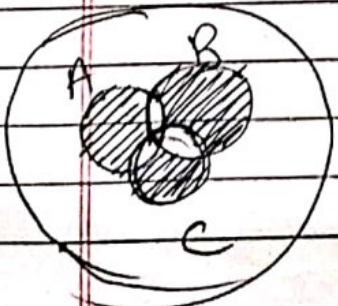
Ex. Three popular option of a car model are
A: Automatic B: new V_6 engine C: Air Condition
we have following data.

$P(A) = 0.7$ $P(A\cup B) = 0.8$ $P(A\cup B\cup C)$
$P(B) = 0.75$ $P(A\cup C) = 0.85$ $= 0.95$.
$P(C) = 0.8$ $P(B\cup C) = 0.9$

find the prob. that a buyer choose exactly one option.

$$P(A\cup B\cup C) - P(A\cap B) - P(B\cap C) - P(A\cap C) + 2P(A\cap B\cap C)$$

$$P(A\cup B\cup C) = P(A) + P(B) + P(C) - P(A\cap B) - P(B\cap C) - P(A\cap C) + \underline{P(A\cap B\cap C)}.$$



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Ex. Toss a fair die twice. Define events

A : {at least one six}.

$P(A) = 11/36$

B : { sum of faces ≥ 10 }.
when this event known then what is the $P(A) = 5/6$.

(1,1) (1,2) - - - (1,6)
(2,1) - - - (2,6)
(3,1) - - - (3,6)
(4,1) - - - (4,6)
(5,1) - - - (5,5) (5,6)
(6,1) - - (6,4) (6,5) (6,6)

★ Conditional Probability :- Consider a sample space S where let A and B are two events. Then conditional prob. of A given that event B has already occured is defined as.

① — $P(A|B) = \dfrac{P(A\cap B)}{P(B)}$; $P(B) > 0$.

If $A \subset B \Rightarrow P(A|B) = \dfrac{P(A)}{P(B)} > P(A)$.
$A \supset B \Rightarrow$ ~~PE~~ $P(A|B) = 1$.

① is a valid prob. function.
$P(\cdot|B)$ is proper.

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AXIOM I : $P(A|B) \geq 0 \quad A \in S$.

II : $P(S|B) = 1$.

III : If A and C are disjoint events in S then

$$P(A \cup C|B) = P(A|B) + P(C|B).$$

★ **Multiplication Rule :**

$$P(A \cap B) = P(B).P(A|B) = P(A).P(B|A).$$

$$P(A \cap B \cap C) = P(A).P(B|A).P(C|A \cap B).$$

General Multiplication Rule :-

Let $A_1, A_2, \ldots, A_n$ be n events in S. Then

$$P\left(\bigcap_{i=1}^{n} A_i\right) = P(A_1)\,P(A_2|A_1)\,P(A_3|A_1 \cap A_2) \ldots \ldots P\left(A_n \,\middle|\, \bigcap_{i=1}^{n-1} A_i\right).$$

★ **Independent events :-**

Two events A and B are independent

$$P(A \cap B) = P(A).P(B).$$

$$P(A|B) = P(A) \qquad P(B|A) = P(B).$$

Disjoint events with positive probabilites are dependent.

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eg. Toss a coin 3 times

$S = \{ HHH, HHT, HTH, THH, HTT, THT, TTH, TTT \}$

A: Head on first coin. $P(A) = 1/2$

B: —//— 2nd —//— $= P(B)$

C: —//— 3rd —//— $= P(C)$.

Are A, B, C independents?

In order to verify whether A, B, C are independent we need to check.

$1/4 = P(A \cap B) = P(A) \cdot P(B) = 1/4$

$1/4 = P(A \cap C) = P(A) \cdot P(C) = 1/4$

$1/4 = P(B \cap C) = P(B) \cdot P(C) = 1/4$

$1/8 = P(A \cap B \cap C) = P(A) \cdot P(B) \cdot P(C) = 1/8$.

So, A, B, C are mutually independent events.

eg. Toss a coin 2 times. Are A, B, C are independent events?

A: Head on first coin.

B: head on 2nd coin.

C: Same result.

$P(A) = 1/2 = P(B) = P(C)$.

$P(A \cap B) = P(A) \cdot P(B) = 1/4$.

$P(B \cap C) = P(B) \cdot P(C) = 1/4$

$P(A \cap C) = P(A) \cdot P(C) = 1/4$.

$P(A \cap B \cap C) = 1/4$. $P(A) \cdot P(B) \cdot P(C) = 1/8$.

A, B, C are pairwise independent only.

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Now consider n events for them to be mutually independent $A_1, A_2, \ldots, A_n$. Then we have following condition.

$^nC_2$ $\quad P(A_i \cap A_j) = P(A_i).P(A_j) \quad i<j = 1, 2, \ldots, n.$

$^nC_3$. $\quad P(A_i \cap A_j \cap A_k) = P(A_i).P(A_j).P(A_k) \quad i<j<k = 1, 2, \ldots, n$

^nC_n $\quad P\left(\bigcap_{i=1}^{n} A_i\right) = \prod_{i=1}^{n} P(A_i)$

Total $= \underline{\underline{2^n - n - 1}}$.

★ **Total Prob. Result** :-

Let $B_1, B_2, \ldots, B_n$ be n events of a sample space which form a partition of the given sample space. Now let A be any event in S then

$$P(A) = \sum_{i=1}^{n} P(B_i)\, P(A|B_i).$$

Proof: Since $B_1, B_2, \ldots, B_n$ form a partition of S hence these events are pairwise disjoint and they satisfy

$$\bigcup_{i=1}^{n} B_i = S.$$

We have,
$A = (A \cap B_1) \cup (A \cap B_2) \cup \ldots \cup (A \cap B_n).$

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Note that $A \cap B_i$ and $A \cap B_j$ are pairwise disjoint. $i, j = 1, 2, \ldots, n$

$$P(A) = P\big((A \cap B_1) \cup (A \cap B_2) \cup \ldots \cup (A \cap B_n)\big)$$

$$= P(A \cap B_1) + P(A \cap B_2) + \ldots + P(A \cap B_n).$$

$$= P(B_1).P(A|B_1) + P(B_2).P(A|B_2) + \ldots + P(B_n).P(A|B_n)$$

$$P(A) = \sum_{i=1}^{n} P(B_i).P(A|B_i)$$

$$\Rightarrow \boxed{P(A) = \sum_{i=1}^{n} P(B_i).P(A|B_i)}$$

★ **Bayes' Theorem :- (1764).**

Let $B_1, B_2, \ldots, B_n$ be a partition of S and A be any event in S. Then.

$$\boxed{P(B_i|A) = \frac{P(A \cap B_i)}{P(A)} = \frac{P(B_i).P(A|B_i)}{\sum_{i=1}^{n} P(B_i).P(A|B_i)} \qquad i = 1, 2, \ldots, n.}$$

$P(B_i) \Rightarrow$ a priori probability ; $i = 1, 2, \ldots, n$.

$P(B_i|A) \rightarrow$ posterior probability.

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Q. Let three vendors A, B, C supply some product to IIT PATNA. Suppose these vendors supply 35%, 35%, 30% of the product. Also it is known that 8%, 10% and 5% of those products tend to be defective resp. Suppose an item is selected at random and tested.

event D. ⇒ What is prob. that it is defective? <u>0.23</u>

Given that it is defective what is the probability that it was supplied by A, B and C resp..

$P(A) = 0.35 \quad P(B) = 0.35 \quad P(C) = 0.30$.

→

$P(D) = 0.08 + 0.10 + 0.05 = 0.23$.

$$P(A|D) = \frac{P(A \cap D)}{P(D)} =$$

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Ex. A Binary communication channel carries data as one of 2 types of signals 0 and 1. Due to noise, a transmitted 0 is sometimes received as 1 and viceversa. For a given channel, assume that a prob. of 0.94 that a transmitted 0 is correctly recieved as 0. & a prob. of 0.91 a transmitted 1 is correctly received as 1. Further assume that prob. of transmitting a 0 is 0.45. If signal is sent. Find the prob. that

(i) a 1 is received

(ii) a 1 was transmitted given that a 1 was received.

(iii) an error occurs.

Ans. A : 0 was transmitted.

B : 0 was received.

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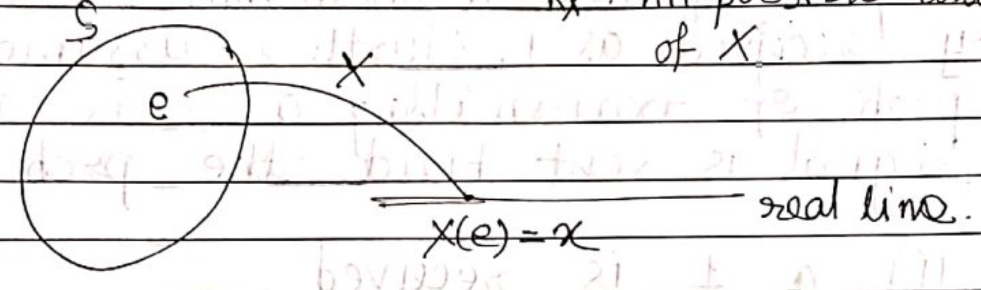
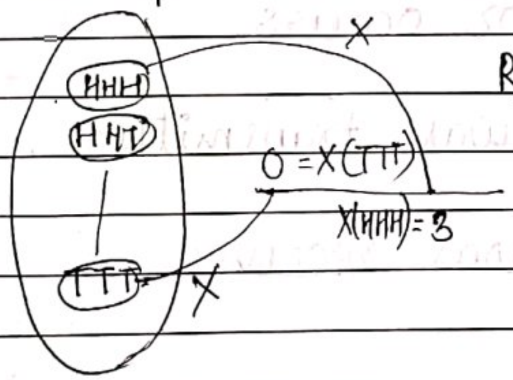
☆ Random Variables :-

Consider a R.E. with sample space S. A random variable X is a function defined over S which associates each outcome e in S to a real number.

R_X : All possible values of X.



Ex. Toss a coin 3 times.
X : no. of heads.



$R_X = \{0, 1, 2, 3\}$.

$P(X=0) = P(TTT) = 1/8$

After that we compute probability of each event in R_X.

$X = x$	0	1	2	3.
$P(X=x)$	$1/8$	$3/8$	$3/8$	$1/8$.

* Cummulative Distribution Function (CDF) :-

Let X be a Random Variable with some prob.

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distribution. Then CDF of X is defined as

$$F_X(x) = P(-\infty < X \le x) \quad \forall x \in R$$

$$= P(X \le x).$$

* **Properties of CDF :**

(i) $0 \le F_X(x) \le 1, \quad \forall x \in \mathbb{R}$

(ii) $\lim_{x\to\infty} F_X(x) = F_X(\infty) = 1$

iii) $F_X(-\infty) = 0$.

iv) $F_X(x)$ is non-decreasing in x.

v) $F_X(x)$ is right continuous at each x, $\forall x \in R$

$$\left(F_X(x+h) = F_X(x) \text{ as } h \to 0^+\right).$$

vi) $P(X > x) = 1 - F_X(x)$

vii) $P(a < X \le b) = F_X(b) - F_X(a)$.

viii) $P(a \le X \le b) = F_X(b) - F_X(a^-)$.

$X=x$	0	1	2	3
$P(X=x)$	$1/8$	$3/8$	$3/8$	$1/8$

$$F(x) = P(X \le x) = \begin{cases} 0 & x < 0 \\ 1/8 & 0 \le x < 1 \\ 1/8 + 3/8 & 1 \le x < 2 \\ 1/8 + 3/8 + 3/8 & 2 \le x < 3 \\ 1/8 + 3/8 + 3/8 + 1/8 & x \ge 3 \end{cases}$$

ix) $P(X = x) = F_X(x) - F_X(x^-)$.

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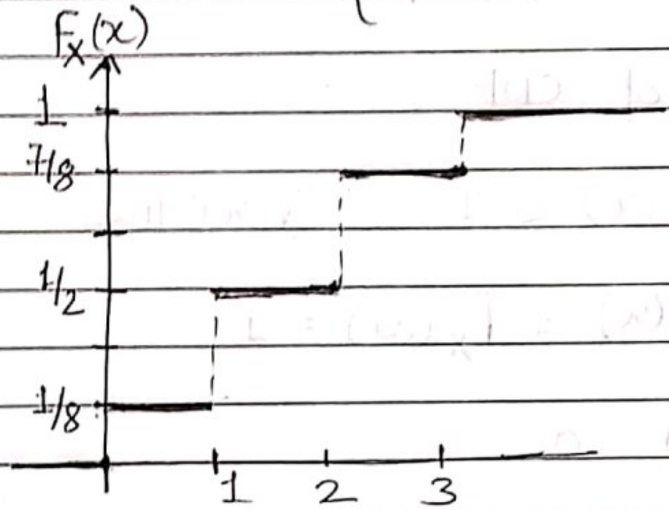
$$F_X(x) = \begin{cases} 0 & ; \quad x < 0 \\ 1/8 & ; \quad 0 \le x < 1 \\ 1/2 & ; \quad 1 \le x < 2 \\ 7/8 & ; \quad 2 \le x < 3 \\ 1 & ; \quad x \ge 3 \end{cases}$$



$P(X=0) = F_X(0) - F_X(0^-) = 1/8 - 0 = 1/8.$

$P(X=1) = F_X(1) - F_X(1^-) = 1/2 - 1/8 = 3/8.$

$P(X=2) = F_X(2) - F_X(2^-) = 7/8 - 1/2 = 3/8.$

$P(X=3) = F_X(3) - F_X(3^-) = 1 - 7/8 = 1/8.$

Ex. Toss a fair die twice.

X : Sum on upper faces.

$R_X = \{2, 3, \_\_\_, 12\}$.

Ex. Toss a coin until a head.

X : no. of tosses to get that head

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Types of RVs :-

(i) Discrete RVs
(ii) Continuous RVs
(iii) Mixed RVs.

* Discrete RV :

A RV X is said to be discrete if its range space contains at most countably infinite no. of elements like $x_1, x_2, x_3, \ldots, x_i, \ldots,$

Now let p_i be a number denoting the prob. of event $X = x_i$ i.e. $p_i = P(X = x_i)$
$= P_X(x_i)$

Then the collection $\{P_i, i \in N\}$ is referred as the Prob. Mass function (PMF) of X provided.

(i) $P_X(x_i) \geq 0 \quad \forall \; x_i \in R_X$

(ii) $\sum_{x_i \in R_X} P_X(x_i) = 1$

$$\boxed{F_X(x) = \sum_{x_i \leq x} P_X(x_i)}.$$

Ex. A shop has 10 products of a type of which 3 are defective. A buyer buys 2 products. Let X be the no. of defective products then find PMF, CDF of X.

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② **Continuous RV**

A RV X is said to be continuous if its CDF is defined as

$$F_X(x) = \int_{-\infty}^{x} f_X(t)\,dt.$$

where $f_X(x)$ is known as probability density function (PDF) of X and it satisfies the following two properties:

(i) $f_X(x) \geq 0 \quad \forall x \in \mathbb{R}$.

(ii) $\int_{-\infty}^{\infty} f_X(x)\,dx = 1$.

* $P(a < \cancel{X} \leq b) = F_X(b) - F_X(a)$.

$$= \int_a^b f_X(x)\,dx.$$

* If X is continuous then for any constant c

$$P(X=c) = 0.$$

$$\left.\begin{array}{l} a<X\leq b \\ a<X<b \\ a\leq X<b \\ a\leq X\leq b \end{array}\right\}$$ are all equivalent if X is continuous.

* CDF of a continuous RV. X is continuous.

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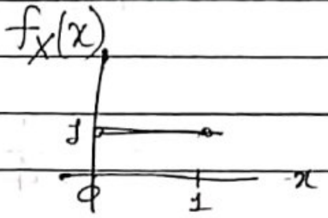
* $\boxed{\dfrac{dF_X(x)}{dx} = f_X(x)}$

eg. $f_X(x) = \begin{cases} 1 & ; \; 0 < x < 1 \\ 0 & ; \; \text{elsewhere} \end{cases}$



$$P(0 < X < 1/2) = \int_0^{1/2} f_X(x)\,dx = \int_0^{1/2} (1)\,dx = 1/2.$$

Ex

$$f_X(x) = \begin{cases} x/2 & ; \; 0 < x < 1 \\ k/2 & ; \; 1 \le x < 2 \\ (3-x)/2 & ; \; 2 \le x < 3 \\ 0 & ; \; \text{otherwise.} \end{cases}$$

Find k so that $f_X(x)$ is a PDF. Also Compute CDF of X.

HINT: To find k use the relation $\int_{-\infty}^{\infty} f_X(x) = 1$.

$$\rightarrow \int_0^1 (x/2)\,dx + \int_1^2 (k/2)\,dx + \int_2^3 (3-x)/2 \cdot dx = 1.$$

$$\left(\frac{x^2}{4}\right)_0^1 + \frac{k}{2}(x)_1^2 + \frac{(-1)}{2}\left(\frac{(3-x)^2}{2}\right)_2^3 = 1$$

$$\Rightarrow \frac{1}{4} + \frac{k}{2} - \frac{1}{4}\cancel{\times}(-1) = 1$$

$$\Rightarrow \frac{k}{2} = 1/2 \Rightarrow \boxed{k = 1}.$$

$$f_X(x) = \begin{cases} \frac{x}{2} & ; \; 0 < x < 1 \\ 1/2 & ; \; 1 \le x < 2 \\ (3-x)/2 & ; \; 2 \le x < 3. \\ 0 & ; \; \text{otherwise.} \end{cases}$$

$$F_X(x) = P(X \le x) = 0 \quad ; \; x < 0.$$

$$= \int_0^x t/2\,dt = x^2/4 \; ; \; 0 \le x < 1$$

$$= \int_0^1 (t/2)\,dt + \int_1^x (1/2)\,dt \qquad 1 \le x < 2$$

$$= \frac{1}{4} + \frac{x-1}{2} = \frac{x}{2} - \frac{1}{4}$$

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$$= \int_0^1 \frac{t}{2}\,dt + \int_1^2 \frac{1}{2}\,dt + \int_2^x \frac{(3-t)}{2}\,dt$$

$$= \frac{1}{4} + \frac{1}{2} + \frac{1}{2}\left(3t - \frac{t^2}{2}\right)_2^x$$

$$= \frac{1}{4} + \frac{1}{2} + \frac{1}{2}\left[\left(3x - \frac{x^2}{2}\right) - (6-2)\right]$$

$$= \frac{1}{4} + \frac{1}{2} + \frac{3x}{2} - \frac{x^2}{4} - 2$$

$$= \frac{3x}{2} - \frac{x^2}{4} - \frac{5}{4} \quad ; 2 < x < 3.$$

$$= 1. \quad ; x \geq 3.$$

$$F_X(x) = \begin{cases} 0 & ; x < 0 \\ x^2/4 & ; 0 \leq x < 1 \\ x/2 - 1/4 & ; 1 \leq x < 2 \\ 3x/2 - x^2/4 - 5/4 & ; 2 \leq x < 3 \\ 1 & ; x \geq 3. \end{cases}$$

<u>H.W.</u> $P(X < 5/2 \mid X > 1)$

$$= \frac{P((X < 5/2) \cap (X > 1))}{P(X > 1)}.$$

$$= \frac{P(1 < X < 5/2)}{P(X > 1)} = \frac{P(1 < X < 5/2)}{1 - P(X \leq 1)}$$

$$= \int_1^{5/2} f_X(x)\,dx \Big/ 1 - \int_{-\infty}^1 f_X(x)\,dx.$$

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$$\int_1^{5/2} f_X(x)\,dx = \int_1^2 f_X(x)\,dx + \int_2^{5/2} f_X(x)\,dx$$

$$= \int_1^2 \left(\tfrac{1}{2}\right) dx + \int_2^{5/2} \left(\frac{3-x}{2}\right) dx$$

$$= \frac{1}{2} + \frac{1}{2}\left(3x - \frac{x^2}{2}\right)_2^{5/2}$$

$$= \frac{1}{2} + \frac{1}{2}\left(3\left(\frac{1}{2}\right) - \frac{1}{2}\left(\frac{9}{4}\right)\right)$$

$$= \frac{1}{2} + \frac{1}{2}\left(\frac{3}{2} - \frac{9}{8}\right) = \frac{1}{2} + \frac{1}{2}\left(\frac{3}{8}\right)$$

$$= \frac{1}{2} + \frac{3}{16} = \frac{11}{16}$$

$$\int_{-\infty}^{1} f_X(x)\,dx = \int_{-\infty}^{0} f_X(x)\,dx + \int_0^1 f_X(x)\,dx$$

$$= 0 + \int_0^1 \left(\frac{x}{2}\right) dx$$

$$= \frac{1}{2} \cdot \frac{(x^2)_0^1}{2} = \underline{\frac{1}{4}}$$

$$P(X < 5/2 \mid X > 1) = \frac{11/16}{1 - 1/4}$$

$$= \frac{11/\cancel{16}\,4}{3/\cancel{4}} = \frac{11}{12}.$$

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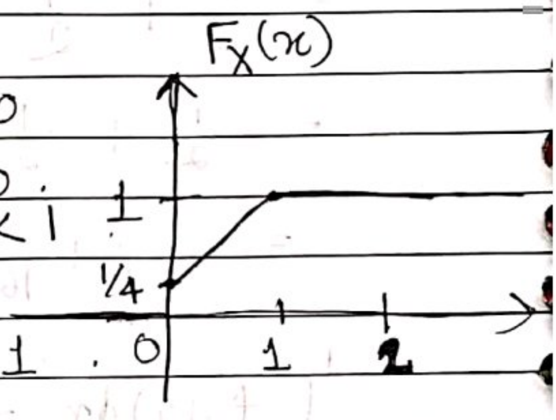
⑧ Mixed Type RV.

X : Waiting time for service at counter.

$P(X=0) = 1/4$.

$$f_X(x) = \begin{cases} 3/4 ; & 0 \le x < 1 \\ 0 ; & \text{otherwise} \end{cases}$$

$$\begin{aligned} F_X(x) &= 0 ; & x < 0 \\ &= 1/4 ; & x = 0 . \\ & \tfrac{1}{4} + \left(\tfrac{3}{4}\right)x ; & 0 < x < 1 \\ &= 1 ; & x \ge 1 . \end{aligned}$$



$$F_X(x) = \begin{cases} 0 ; & x < 0 \\ \tfrac{1}{4} + \left(\tfrac{3}{4}\right) \cdot x ; & 0 \le x < 1 . \\ 1 ; & x \ge 1 . \end{cases}$$

$$P_X(0) + \int_0^1 \frac{3}{4} . dx = 1 .$$

$$P(X=0) = F_X(0) - F_X(0^-) = \frac{1}{4} - 0 = \frac{1}{4}$$

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Tut. 1

(28) A [5 black, 6 white] B [8 black, 4 white]

Two balls are transferred

drawn a ball.

Case I : Two white E_1
II : Two black E_2
III : One white and one black E_3
W : Drawing a white ball.

(i) P(a white is drawn).

$= P(E_1) \times P(W|E_1)$
$+ P(E_2) \times P(W|E_2)$
$+ P(E_3) \times P(W|E_3)$.

$= \frac{^4C_2}{^{12}C_2} \times \frac{8}{13} + \frac{^8C_2}{^{12}C_2} \times \frac{6}{13} + \frac{^4C_1 \cdot {^8C_1}}{^{12}C_2} \times \frac{7}{13}$

$P(E_1) = \frac{^4C_2}{^{12}C_2}$

$P(E_2) = \frac{^8C_2}{^{12}C_2}$

$P(E_3) = \frac{^4C_1 \times {^8C_1}}{^{12}C_2}$

(ii) W_1 : At least one white ball was transferred to A (case I and III).

$P(W_1|W) = \frac{P(W_1 \cap W)}{P(W)}$.

$= \frac{P(W_1) \cdot P(W|W_1)}{P(W)} = \frac{[P(E_1)+P(E_3)] \cdot \left[\frac{P(E_1) \cdot P(W|E_1) + P(E_3) \cdot P(W|E_3)}{P(E_1)+P(E_3)}\right]}{P(W)}$.

$= \left(\frac{^4C_2}{^{12}C_2} + \frac{^8C_1 \cdot {^4C_1}}{^{12}C_2}\right) \cdot \left(\frac{8}{13} \times \frac{^4C_2}{^{12}C_2} + \frac{7}{13} \times \frac{^8C_1 \cdot {^4C_1}}{^{12}C_2}\right) / P(W)$

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(21) ⇒

$\boxed{\begin{array}{l} m \text{ white} \\ n \text{ black} \end{array}} \longrightarrow$ k balls are drawn.

W : Drawing at least one white balls.
W^c : Drawing zero white balls

Ans. $= 1 - P(W^c)$.

$$= 1 - \frac{{}^{n}C_{k}}{{}^{n+m}C_{k}}$$

(20) ⇒

$\boxed{\begin{array}{l} n \text{ balls (identical)} \\ \text{no. 1 to } n. \end{array}} \longrightarrow$ k balls are drawn in succession

(i) ⇒ Probability that m is the largest number drawn.

So, one will be m and for other (k-1) balls we have 1, 2, ..., m-1 as options.

$$\text{possible ways} = {}^{(m-1)}C_{(k-1)} \times 1.$$

$$\text{Total Prob.} = {}^{(m-1)}C_{(k-1)} \Big/ {}^{n}C_{k}.$$

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(ii) ⇒ Probability that the largest number drawn is less than or equal to m.

$\{1, 2, \ldots, m\}$

$$\frac{^{m-1}C_{k-1}}{^{n}C_k} + \frac{^{m-2}C_{k-1}}{^{n}C_k} + \cdots + \frac{^{m-k+1}C_{k-1}}{^{n}C_k}.$$

or

$$\frac{^{k-1}C_{k-1}}{^{n}C_k} + \frac{^{k}C_{k-1}}{^{n}C_k} + \cdots + \frac{^{m-1}C_{k-1}}{^{n}C_k}$$

(19)

$$\boxed{\begin{array}{l} m \text{ white} \\ n \text{ black.} \end{array}}$$

Player A
Player B.

Let Player A starts the game.

P(Player A wins the game).

$$\cancel{= \frac{^{m}C_1}{^{(m+n)}C_1} + \frac{^{n}C_2 \cdot {^{m}C_1}}{^{m+n}C_3} + \frac{^{n}C_4 \cdot {^{m}C_1}}{^{m+n}C_5} + \cdots + \frac{^{n}C_{m+n-1}\,{^{m}C_1}}{^{m+n}C_{m+n}}}$$

$$P(W) = \frac{m}{m+n}$$

~~(18)~~

$$\cancel{\begin{array}{l} m \text{ white} \\ n \text{ black} \end{array}}$$

$$P(BBW) = \left(\frac{n}{m+n}\right)\left(\frac{n-1}{m+n-1}\right)\left(\frac{m}{m+n-2}\right)$$

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$P(BBBBW)$

$$= \left(\frac{n}{m+n}\right)\left(\frac{n-1}{m+n-1}\right)\left(\frac{n-2}{m+n-2}\right)\left(\frac{n-3}{m+n-3}\right)\left(\frac{m}{m+n-4}\right)$$

$P(\text{Player A wins})$.

$$= \frac{m}{m+n}\left(1 + \frac{n(n-1)}{(m+n)(m+n-1)} + \cdots\right)$$

(18) $\boxed{\begin{matrix} m \text{ white} \\ n \text{ black} \end{matrix}}$

$P(\text{Encountering a white ball by the } k^{th} \text{ draw})$.

$= P(1^{st} \text{ draw}) + P(2^{nd} \text{ draw}) + P(3^{rd} \text{ draw}) + \cdots + P(k^{th} \text{ draw})$.

$$= \frac{{}^{m}C_1}{{}^{m+n}C_1} +$$

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Today we look at some summary of information which a RV can provide such as, Mean Variance, Median, quantile, etc.

★ Mathematical Expectation (Average Value) of R.V X :-

let X be a RV with some prob distribution, then expected value of X is defined as

$$E(X) = \int_{-\infty}^{\infty} x \cdot f_X(x) \cdot dx \quad \left(\begin{array}{c} X \text{ is continuous} \\ \text{with PDF } f_X(x) \end{array} \right)$$

provided $\int_{-\infty}^{\infty} |x| \cdot f_X(x) \cdot dx < \infty$.

If X is discrete with prob. $P_X(x)$ then

$$E(X) = \sum_{x_i \in R_X} x_i P_X(x_i)$$

provided $\sum_{x_i} |x_i| P_X(x_i) < \infty$.

⊛ Mean of RV X may not exist.

★ In general,

$$E(g(x)) \xrightarrow{\text{continuous}} \int_{-\infty}^{\infty} g(x) \cdot f_X(x) \cdot dx$$

$\Downarrow$ discrete

$$\sum_{x_i} g(x_i) \cdot P_X(x_i).$$

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Ex. Mean may not exist.

Cauchy distⁿ $f_X(x) = \frac{\beta}{\pi} \frac{1}{\beta^2+(x-\alpha)^2}$

$-\infty < x < \infty$
$-\infty < \alpha < \infty$
$0 < \beta < \infty$

Take $\alpha = 0$; $\beta = 1$.

$$\boxed{f_X(x) = \frac{1}{\pi} \cdot \frac{1}{1+x^2} \; ; \; -\infty < x < \infty}$$

Std. form of Cauchy Distribution.

$$E(X) = \int_{-\infty}^{\infty} x \cdot f_X(x)\,dx = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{x}{1+x^2}\,dx$$

$$\frac{1}{\pi} \int_0^{\infty} \frac{x}{1+x^2}\,dx \longrightarrow \infty$$

$E(|X|)$ does not exist.

eg. (1) $f_X(x) = \begin{cases} 1/x^2 \; ; & x \geq 1 \\ 0 \; ; & \text{otherwise} \end{cases}$

is a PDF because $f_X(x) \geq 0 \;\; \forall x$

$$\int_{-\infty}^{\infty} f_X(x)\,dx = 1.$$

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$$E(X) = \int_{-\infty}^{\infty} x . f_X(x)\, dx = \int_{1}^{\infty} x . \frac{1}{x^2}\, dx = \int_{1}^{\infty} \frac{1}{x}\, dx \to \infty$$

Mean does not exist.

eg. ② $f_X(x) = 1 \; ; \; 0 < x < 1.$
$\quad\quad\quad = 0 \; ; \;$ elsewhere.

$$E(X) = \int_0^1 x . f_X(x)\, dx.$$

$$= \int_0^1 x.(1).dx = \frac{1}{2}.$$

eg. ③. $f(x) = \frac{1}{2} . e^{-x/2} \; ; \; x > 0.$

$$E(X) = \frac{1}{2} \int_0^{\infty} e^{-x/2} . x\, dx = 2.$$

eg. ①

x	0	1	2	3.
X	$1/8$	$3/8$	$3/8$	$1/8$.

X: no. of heads in three tosses.

$$E(X) = \sum_{x=0}^{3} x . P_X(x)$$

$$= 0.\left(\frac{1}{8}\right) + 1.\left(\frac{3}{8}\right) + 2.\left(\frac{3}{8}\right) + 3.\left(\frac{1}{8}\right).$$

$$= 3/2$$

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* Variance :-

Let X be a RV with some prob. distⁿ.

$$Var(X) \text{ or } V(X) = E(X - E(X))^2.$$

* $E(X)$ is linear.
- $E(ax+b) = aE(X) + b$.
- $E(c) = c$ where c is a constant.

$$= E\left(X^2 - 2XE(X) + (E(X))^2\right).$$

$$= E(X^2) - 2E(X).E(X) + (E(X))^2.$$

$$\boxed{Var(X) \text{ or } V(X) = E(X^2) - (E(X))^2}$$

* $V(x)$ is always non-negative.

$$E(X^2) \xRightarrow{\text{discrete}} \sum_{x_i} x_i^2 P_X(x_i).$$

⇓ continuous

$$\int_{-\infty}^{\infty} x^2 . f_X(x) . dx.$$

$E(X) = 3/2$

eg.

x	0	1	2	3
X	1/8	3/8	3/8	1/8

$$E(X^2) = (0)^2 . 1/8 + (1)^2 . 3/8 + (2)^2 . 3/8 + (3)^2 . 1/8$$

$$E(X^2) = 3$$

$$Var(X) = 3 - (3/2)^2 = 3 - \frac{9}{4}$$

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Mean $\Rightarrow$ Central Tendency.

Variance $\Rightarrow$ Dispersion.

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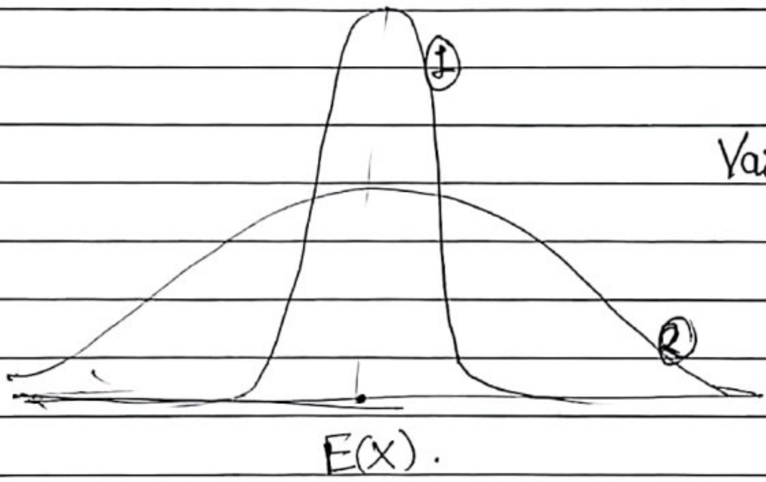
* $Var(X) = 0$ if RV. X is constant with prob. 1.

eg. $f(x) = 1 \; ; \; 0 < x < 1$
$\quad\quad\quad 0 \; ;$ elsewhere.

$$E(X) = \frac{1}{2} \quad E(X^2) = \int_0^1 x^2 \cdot dx = \frac{1}{3}$$

$$V(X) = \frac{1}{3} - \left(\frac{1}{2}\right)^2 = \frac{1}{12} > 0.$$

*.



$Var(1) < Var(2)$

$$E(g(x)) \Rightarrow \int g(x) \cdot f_X(x) \cdot dx.$$

$\Downarrow$

$$\sum_{x_i} g(x_i) \cdot f_X(x_i) \cdot dx.$$

eg. $P_X(0) = 3/4 \; ; \quad f_X(x) = 1/4 \; ; \; 0 < x < 1.$

$$\Rightarrow E(X) = 0 \cdot P_X(0) + \int_0^1 x \cdot \left(\frac{1}{4}\right) \cdot dx. = \frac{(x^2)_0^1}{8} = \frac{1}{8}$$

(Mixed RV).

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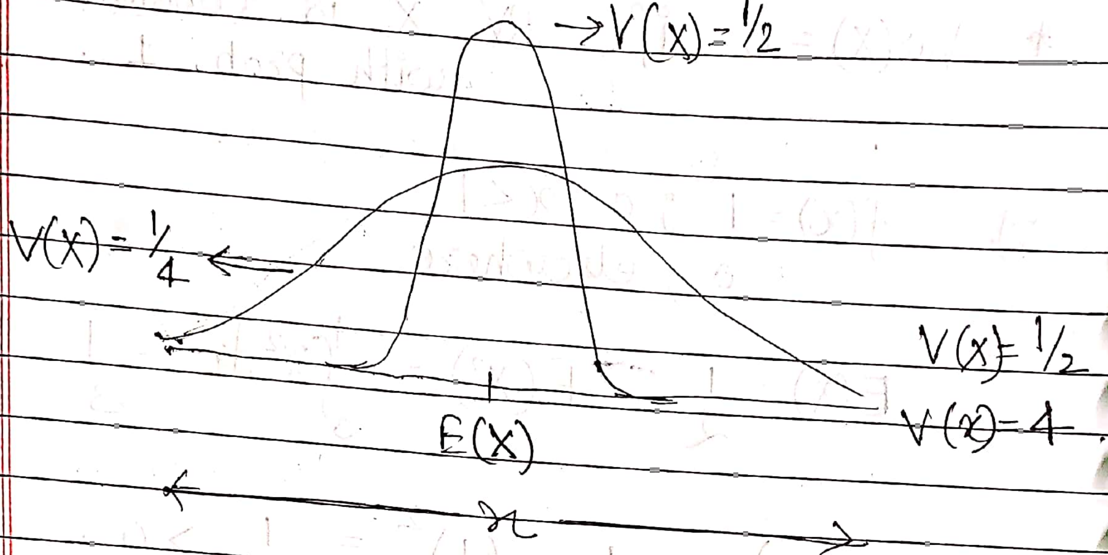


We know that

$$E(aX+b) = aE(X) + b \; ; \; a, b \text{ given consts}$$

Now,

$$V(aX+b) = a^2V(X)$$

In particular,

$$V(X+b) = V(X)$$
$$V(aX) = a^2V(X).$$

$\to E[aX+b - E(aX+b)]^2$ (Hint: for Proof)

* Mean is largely affected by xtreme values.
* Mean & Variance may not exist always.

Defⁿ :- A RV X is said to have a symmetric probability distribution about a point α if

$$P(X \geq \alpha + x) = P(X \leq \alpha - x)$$
$$\forall x \in R_X.$$

$$F_X(x) = P(X \le x)$$

$$F_X(\alpha - x) = 1 - F_X(\alpha + x) + P(X = \alpha + x)$$

$\Rightarrow$ If X is continuous $\boxed{\Rightarrow P(X=c) = 0}$.

$$F_X(\alpha - x) = 1 - F_X(\alpha + x).$$

Differentiating;

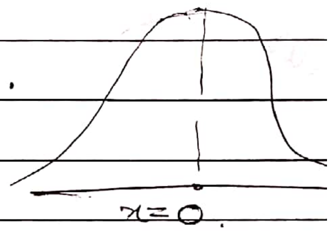
$$f_X(\alpha - x) = f_X(\alpha + x) \quad \forall\, x \in R_X$$

If $\alpha = 0$;

$$f_X(-x) = f_X(x). \quad \forall\, x \in R_X$$

Ex. 1 $f_X(x) = \dfrac{1}{\pi} \cdot \dfrac{1}{1+x^2}$; $-\infty < x < \infty$

it is symmetrix about $x = 0$.



2 $f_X(x) = \dfrac{\beta}{\pi} \cdot \dfrac{1}{\beta^2 + (x-\alpha)^2}$; $-\infty < x < \infty$

is symmetric about $x = \alpha$.

3 $P_X(-1) = \dfrac{1}{4} = P_X(1)$; $P_X(0) = \dfrac{1}{2}$.

this distn is symmetric about $x=0$.

$$P(X \ge x) = P(X \le -x). \quad \forall\, x \in R.$$

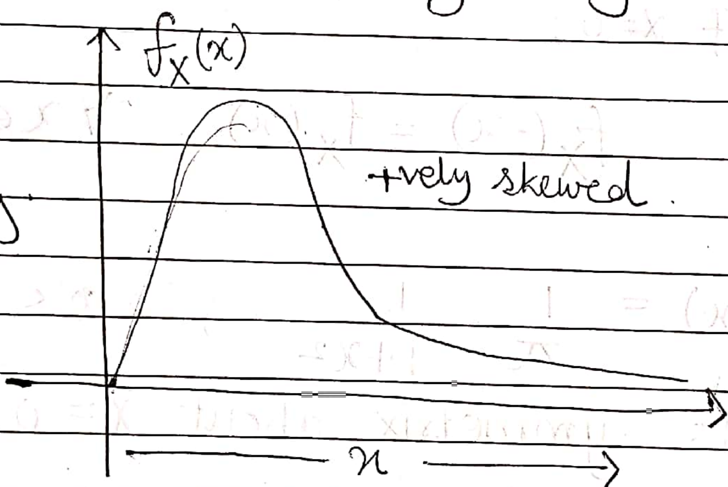
* Coefficient of Skewness :-

$$\beta_1 = \frac{\mu_3}{\sigma^3} = \frac{E(X - E(X))^3}{\sigma^3}$$

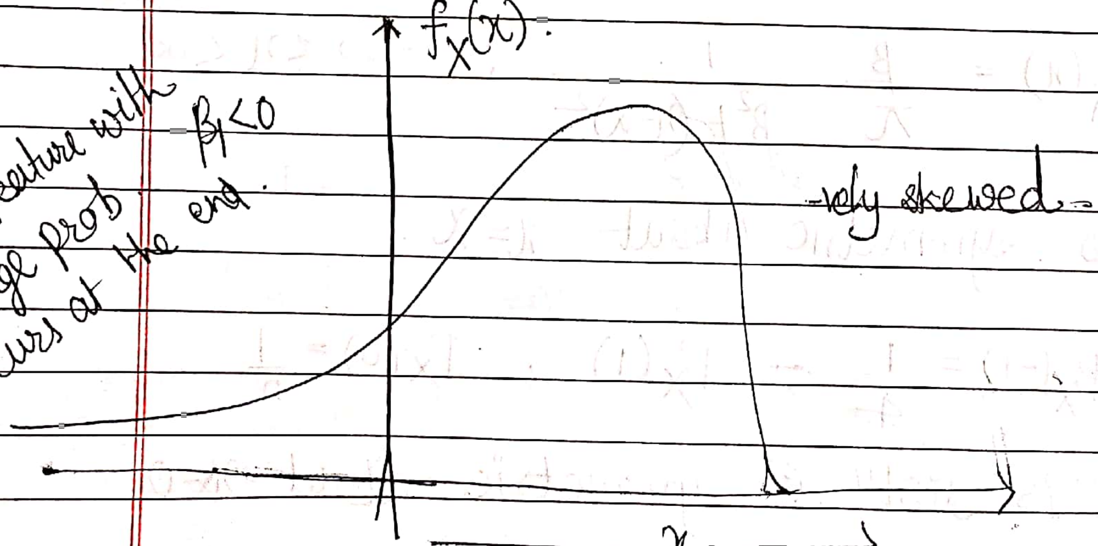
(σ) Standard Deviation $\Rightarrow$ $S.d(X) = \sqrt{V(x)}$

$$\beta_1 = \begin{cases} 0 \; ; & \text{symmetric} \\ >0 \; ; & \text{positively skewed} \\ <0 \; ; & \text{negatively skewed} \end{cases}$$

feature with large prob. at the beginning occurs. $\beta_1 > 0$



feature with large prob occurs at the end. $\beta_1 < 0$



$\beta_1 = 0$; feature with large prob. occurs at the middle (somewhere).

* Quantiles :-

Let X be a RV with some prob. distribution.

A number x_p is said to be p^{th} quantile of this distribution if.

$$P(X \leq x_p) \geq p$$

$$; \quad 0 < p < 1 .$$

$$P(X \geq x_p) \geq 1-p$$

$$\Rightarrow \quad F_X(x_p) \geq p . \quad \text{—— (1)}$$

$$1 - P(X < x_p) \geq 1-p .$$

$$P(X < x_p) \leq p .$$

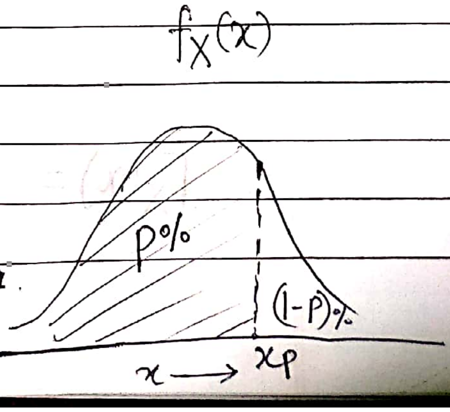
$$F_X(x_p) \leq p + P(X = x_p) . \quad \text{—— (2)}$$

from (1) & (2) ;

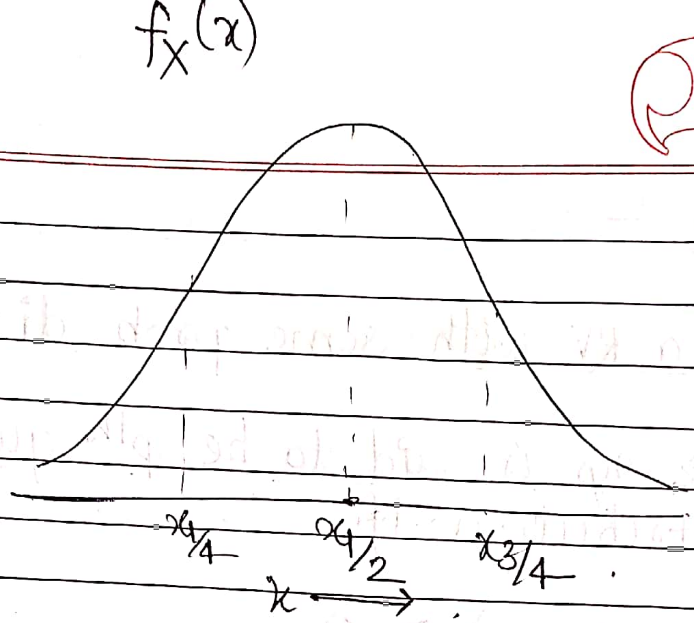
$$\boxed{p \leq F_X(x_p) \leq p + P(X = x_p) .}$$

if X is continuous ; then.

$$\boxed{F_X(x_p) = p}$$



x_p is a no. of real line below which p% of total observations occur. & (1-p)% occur beyond it.



* $x_{1/4}, x_{1/2}, x_{3/4}$ quartiles of the distribution
* $x_{1/2}$ is the median of the distribution.
~~(measure of central tendency which always exists)~~

Ex.① $f_X(x) = \frac{\beta}{\pi} \cdot \frac{1}{\beta^2 + (x-\alpha)^2}$; $-\infty < x < \infty$
$\alpha \in R$
$\beta > 0$.

find quartiles of the distribution.

⇒ CDF

$$F_X(x) = \int_{-\infty}^{x} f_X(t)\, dt.$$

$$= \int_{-\infty}^{x} \frac{\beta}{\pi} \cdot \frac{1}{\beta^2 + (t-\alpha)^2}\, dt$$

$$F_X(x) = \frac{1}{\pi}\left[\tan^{-1}\left(\frac{x-\alpha}{\beta}\right) + \frac{\pi}{2}\right]$$

$x_{1/4} \Rightarrow$ first quartile

$x_{1/2} \Rightarrow$ Median or Second Quartile

$x_{3/4} \Rightarrow$ Third quartile.

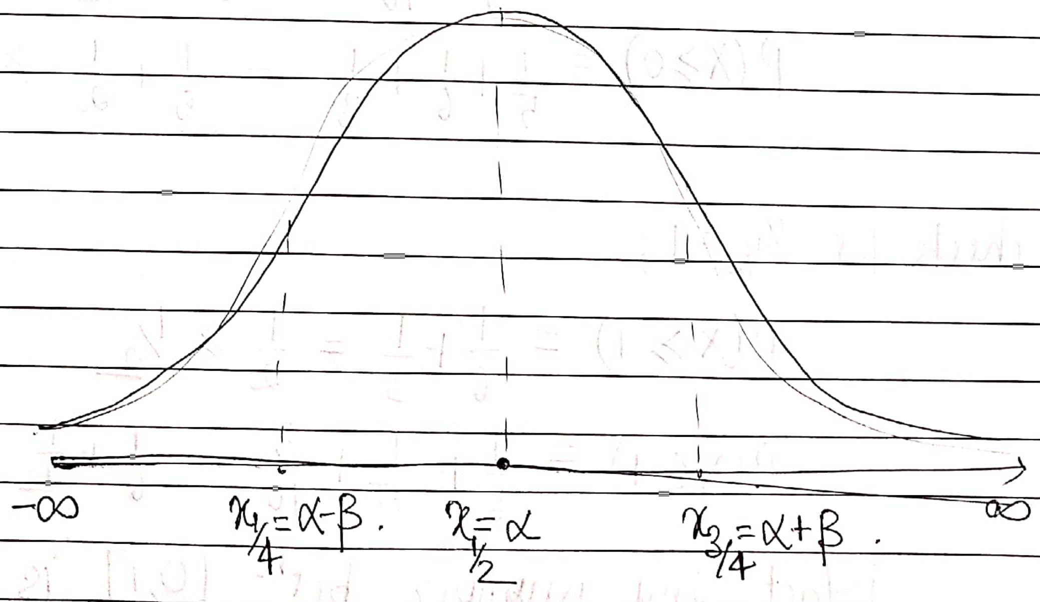
Range of distribution $(x_{3/4} - x_{1/4})$.

To find the first quartile $x_{1/4}$ we need to solve.

$$\frac{1}{\pi}\left[\tan^{-1}\left(\frac{x_{1/4}-\alpha}{\beta}\right)+\frac{\pi}{2}\right] = \frac{1}{4}$$

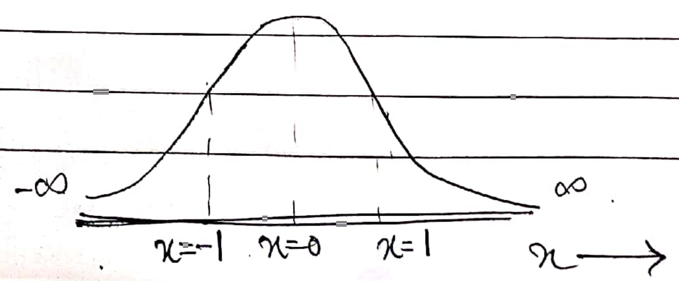
$$\boxed{x_{1/4} = \alpha - \beta}$$

Similarly; $\boxed{x_{1/2} = \alpha}$; $\boxed{x_{3/4} = \alpha + \beta}$



if $\alpha = 0$
$\beta = 1$

Standard prob. distn.



Ex. $P_X(-2) = \frac{3}{10}$; $P_X(0) = 1/5$.

$P_X(1) = \frac{1}{6}$; $P_X(2) = \frac{1}{3}$.

Find median of this distribution.

$\Rightarrow$ We need $x_{1/2}$ such that

$$P(X \leq x_{1/2}) \geq 1/2$$

$$P(X \geq x_{1/2}) \geq 1/2.$$

let check for $x_{1/2} = 0$;

$$P(X \leq 0) = \frac{1}{5} + \frac{3}{10} = \frac{1}{2} \geq \frac{1}{2}$$

$$P(X \geq 0) = \frac{1}{5} + \frac{1}{6} + \frac{1}{3} = \frac{1}{5} + \frac{1}{2} \geq \frac{1}{2}$$

check for $x_{1/2} = 1$;

$$P(X \geq 1) = \frac{1}{6} + \frac{1}{3} = \frac{1}{2} \geq 1/2$$

$$P(X \leq 1) = \frac{1}{6} + \frac{1}{5} + \frac{3}{10} = \frac{1}{6} + \frac{1}{2} \geq \frac{1}{2}.$$

Infact any number betn $[0,1]$ is a median.

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* **Moment Generating Function :- (MGF).**

Suppose X is a RV with a given probability distⁿ. Then MGF of X is given by

$M_X(t) = E[e^{tX}]$ provided this expectation exists in some neighbourhood of $t \neq 0$.

discrete $\swarrow$ continuous $\downarrow$

$\sum_{x_i \in R_X} e^{tx_i} p_X(x_i)$ $\int_{-\infty}^{\infty} e^{tx} f_X(x)\,dx$

* $M_X(t)$ may not exist

eg $f_X(x) = \frac{1}{\pi} \cdot \frac{1}{1+x^2}$; $-\infty < x < \infty$.

* $M_{aX+b}(t) = E[e^{t(aX+b)}]$

a, b are constants.

$M_{aX+b}(t) = e^{bt} M_X(at)$.

eg. $M_{2X+3}(t) = e^{3t} \cdot M_X(2t)$.

*
$$\left.\frac{d^n M_X(t)}{dt^n}\right|_{t=0} = E(X^n) \; ; \; n = 1, 2, \ldots$$

eg. Discrete Case.

X	0	1	2	3
$P_X(x)$	1/8	3/8	3/8	1/8

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$$M_X(t) = E(e^{tX})$$

$$= \sum_{x=0}^{3} e^{tx} \cdot P_X(x)$$

$$= 1 \cdot P_X(0) + e^{t} P_X(1) + e^{2t} P_X(2) + e^{3t} \cdot P_X(3)$$

$$= 1 \left(\frac{1}{8}\right) + e^{t} \cdot \left(\frac{3}{8}\right) + e^{2t} \cdot \left(\frac{3}{8}\right) + e^{3t} \left(\frac{1}{8}\right)$$

$$M_X(t) = \frac{1 + 3e^{t} + 3 \cdot e^{2t} + e^{3t}}{8}$$

$$E[X] = \frac{d}{dt} M_X(t) \Big|_{t=0} = \frac{3}{8} e^{t} + \frac{6}{8} e^{2t} + \frac{3}{8} e^{3t}$$

$$E[X] = \frac{12}{8} = \frac{3}{2} = 1.5$$

eg.
$$f_X(x) = \begin{cases} 3 \cdot e^{-3x} & ; \quad x > 0 \\ 0 & ; \quad \text{otherwise} \end{cases}$$

$$M_X(t) = 3 \int_0^{\infty} e^{tx} e^{-3x} \, dx$$

$$= 3 \int_0^{\infty} e^{-(3-t)x} \, dx$$

$$= \frac{3}{-(3-t)} e^{-(3-t) \cdot x} \Big|_0^{\infty}$$

$$M_X(t) = 3/(3-t) \; ; \quad t < 3$$

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* $E(X) = 3 \cdot \int_0^{\infty} x \cdot e^{-3x} \, dx = \frac{1}{3}$.

* $M_X'(t)\Big|_{t=0} = \frac{1}{3}$.

$$V(x) = E(X^2) - (E(x))^2$$

$$V(x) = M_X''(0) - (M_X'(0))^2$$

* For any RVX CDF, Mean, Variance, β_1, β_2, quant-iles (Median), mode, MGF.

* MGF uniquely determines a prob. $dist^n$ if it exists.

* Prob. Inequalities *

① Markov Inequality : Let X be a nonnegative RV with finite mean. If $a>0$, a given const. then.

$$\boxed{P(X \geq a) \leq \frac{E(X)}{a}}$$

$\Rightarrow$ $E(X) = \int_{-\infty}^{\infty} x \cdot f_X(x) \, dx$.

$= \int_0^{\infty} x \cdot f_X(x) \, dx$ (Non-negative RV. X)

$$= \int_0^a x \cdot f_X(x) \, dx + \int_a^{\infty} x \, f_X(x) \, dx$$

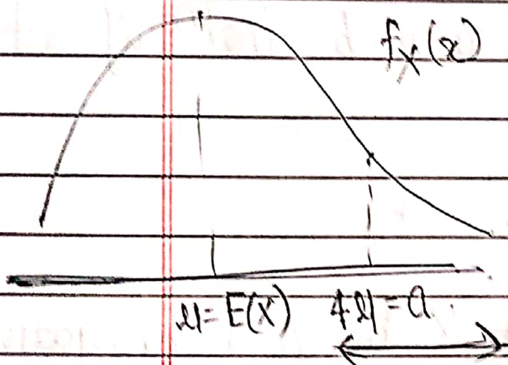
$$\geq \int_a^{\infty} x \cdot f_X(x) \cdot dx .$$

$$\geq a \cdot \int_a^{\infty} f_X(x) \cdot dx .$$

$$= a \cdot P(X>a) .$$

$$E(X) \geq a \cdot P(X>a) .$$

$$\boxed{P(X>a) \leq \frac{E(X)}{a}}$$



let; $a = \frac{\mu}{2}$; $E(X) = \mu$

$$P\left(X>\frac{\mu}{2}\right) \leq \frac{2\mu}{\mu}$$

$$P\left(X>\frac{\mu}{2}\right) \leq 2$$

This result is useful for finding limitations on tail probabilites.

e.g. $P(X>4\mu) \leq \frac{\cancel{4}\mu}{4\mu} \leq 0.25$.

eg. $f_X(x) = 3 \cdot e^{-3x}$; $x>0$

$\Rightarrow$ $P(X>2) = \int_2^{\infty} 3 \cdot e^{-3x} \cdot dx$ | $P(X>2) \leq \frac{E(X)}{2}$

$= e^{-6}$ | $P(X>2) \leq \frac{1}{6}$

Accurate | Markov Inequality

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Tchebyshev Inequality :-

Let X be a RV whose mean and variance exist. If $k>0$ is a given const. then

$$P(|X-\mu| \geq k) \leq \frac{\sigma^2}{k^2}$$

$\sigma^2 \Rightarrow$ Variance of X.

$\mu \Rightarrow$ Mean of X.

Proof :- Let X be a continuous RV with PDF $f_X(x)$.

$$\sigma^2 = E(X-\mu)^2$$

$$= \int_{-\infty}^{\infty} (x-\mu)^2 \cdot f_X(x) \cdot dx$$

$$= \int_{|x-\mu|<k} (x-\mu)^2 \cdot f_X(x) \cdot dx + \int_{|x-\mu|\geq k} (x-\mu)^2 f_X(x) \cdot dx$$

$$\sigma^2 \geq \int_{|x-\mu|\geq k} (x-\mu)^2 \cdot f_X(x) \cdot dx$$

$$\geq k^2 \cdot P(|X-\mu| \geq k)$$

$$\Rightarrow \boxed{P(|X-\mu| \geq k) \leq \frac{\sigma^2}{k^2}}$$

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Ex. $P_X(0) = 1 - \frac{1}{k^2}$; $k > 1$.

$P_X(\pm 1) = \frac{1}{2k^2}$.

Find an upper bound for the parts $P(|X| \geq 1)$ using Tchebyshev Ineq.

$\Rightarrow$

$$E(X) = 0\left(1 - \frac{1}{k^2}\right) + 1 \cdot \left(\frac{1}{2k^2}\right) - 1 \cdot \left(\frac{1}{2k^2}\right).$$

$$\boxed{E(X) = 0}$$

$$E(X^2) = 0^2 \cdot \left(1 - \frac{1}{k^2}\right) + 1^2 \cdot \left(\frac{1}{2k^2}\right) + (-1)^2 \left(\frac{1}{2k^2}\right).$$

$$E(X^2) = \frac{1}{k^2}.$$

$$\boxed{V(X) = \frac{1}{k^2}}$$

$$P(|X - E(X)| \geq 1) \leq \sigma^2$$

$$\boxed{P(|X| \geq 1) \leq \frac{1}{k^2}}.$$

Accurate

$P(|X| \geq 1) = \frac{1}{k^2}$

Other form of above ineq. $P(|X - \mu| \leq k\sigma) \leq \frac{1}{k^2}$

Ex $f_X(x) = \frac{1}{2\sqrt{3}}$; $-\sqrt{3} \leq x < \sqrt{3}$.

Find an Upperbound for the prob. $P(|X| \geq \frac{3}{2})$. Compare with exact prob.

Some Known Discrete Prob. Distribution.

Discrete Uniform
Binomial
Poisson.
Geometric
Negative Binomial.
Hyper Geometric.

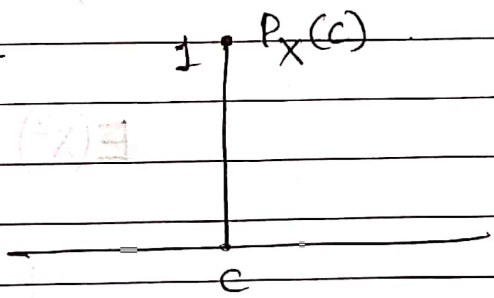
* Degeneration Probability Distribution :-

$$P(X=c) = 1$$

$$E(X) = c.$$

$$V(X) = EX^2 - (E(X))^2$$

$$= c^2 - c^2 = 0.$$



$$M_X(t) = e^{ct}$$

$$F_X(x) = \begin{cases} 0 \ ; & x < c \\ 1 \ ; & x \geq c. \end{cases}$$

* Discrete Uniform Distribution :-

A RV X is said to have a discrete uniform distribution if its PMF is given by

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$$P_X(x) = \frac{1}{n} \; ; \; x = 1, 2, \ldots, n.$$

$$X \sim DU[1, n]$$

(read as X follows Discrete uniform prob dist^n with parameter n,)

$$\boxed{\sum_{x=1}^{n} P_X(x) = 1}$$

$$E(X) = \sum_{x=1}^{n} x . P_X(x) = \sum_{x=1}^{n} \frac{x}{n}$$

$$= \frac{1}{n} . \sum_{x=1}^{n} x$$

$$= \frac{\cancel{n}(n+1)}{2\cancel{n}} = \frac{n+1}{2}$$

$$\Rightarrow \boxed{E(X) = \frac{n+1}{2}}$$

$$E(X^2) = \sum_{x=1}^{n} x^2 . P_X(x) = \frac{1}{\cancel{n}} . \frac{\cancel{n}(n+1)(2n+1)}{6}$$

$$\boxed{E(X^2) = \frac{(n+1)(2n+1)}{6}.}$$

$$V(X) = E(X^2) - [E(X)]^2$$

$$= \frac{(n+1)(2n+1)}{6} - \frac{n+1}{2} . \frac{n+1}{2}$$

$$= \frac{n+1}{2}\left(\frac{2n+1}{3} - \frac{n+1}{2}\right)$$

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$$= \frac{n+1}{2} \cdot \frac{4n+2-3n-3}{6}$$

$$= \frac{(n+1)(n-1)}{12} = \frac{n^2-1}{12}$$

$$\boxed{V(X) = \frac{n^2-1}{12}}$$

$$M_X(t) = \sum_{x=1}^{n} e^{tx} \cdot P_X(x)$$

$$= \frac{1}{n} \cdot \sum_{x=1}^{n} e^{tx}$$

$$M_X(t) = \begin{cases} \frac{e^t}{n}\left(\frac{e^{nt}-1}{e^t-1}\right) & ; t \neq 0 \\ 1 & ; t = 0 \end{cases}$$

$$F_X(x) = \begin{cases} 0 & ; x < 1 \\ 1/n & ; 1 \leq x < 2 \\ 2/n & ; 2 \leq x < 3 \\ \vdots & \vdots \quad \vdots \\ 1 & ; x \geq n \end{cases}$$

☆ **Bernoullian Experiment :-**

1) Each trial results in two mutually disjoint and exhaustive outcomes known as success and failure.

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(2) Trails are conducted independent of each other.

(3) Prob. of getting a success at each trial remains fixed call it p.

(4) * For binomial distribution no. of trials is fixed.

Application of DV :

Ex There are 100 slips of paper in a hat each of which has one of numbers 1, 2, —, 100, written on it. Five such slips are drawn at random one at a time (with replacement). Find prob. distribution of the value of j^{th} draw $(1 \leq j \leq 5)$. Find the prob that no. 100 is drawn at least once.

$X_j \Rightarrow$ value of j^{th} drawn ; $j = 1, 2, 3, 4, 5$

$X_j \sim DU[1, 100]$.

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Bernoulli Dist^n :-

Conduct a Bernoullian Expt.

X : no. of success in a single Bernoulli trial.

R_X : 0, 1

$P(X=1) = p$; $P(X=0) = 1-p$.

$$P_X(x) = p^x (1-p)^{1-x} ; \quad x = 0, 1 \quad 0 < p < 1$$

$$P_X(x) = p^x . q^{1-x} ; \quad x=0,1 \quad 0<p<1 \quad p+q=1 .$$

$$E(X) = \sum_{x=0}^{1} x . P_X(x) .$$

$$= 0 . P_X(0) + 1 . P_X(1)$$

$$\boxed{E(X) = p}$$

$$E(X^2) = \sum_{x=0}^{1} x^2 . P_X(x)$$

$$= (0)^2 . P_X(0) + (1)^2 . P_X(1)$$

$$\boxed{E(X^2) = p}$$

$$V(X) = E(X^2) - [E(X)]^2$$

$$= p - p^2$$

$$= p(1-p) = pq$$

$$\boxed{V(X) = pq}$$

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$$M_X(t) = E[e^{tx}]$$

$$= \sum_{x=0}^{1} e^{tx} \cdot P_X(x)$$

$$= (1-p) + e^t \cdot p$$

$$\boxed{M_X(t) = q + p \cdot e^t}$$

$$\boxed{X \sim Ber(p)}$$

X follows Bernoulli distⁿ with parameter p.

$$F_X(x) = \begin{cases} 0 \; ; & x < 0 \\ 1-p \; ; & 0 \leq x < 1 \\ 1 \; ; & x \geq 1 \end{cases}$$

★ Binomial Distribution :-

Conduct Bernoullian expt. n number of times.

X : no. of successes in these n trials.

R_X : 0, 1, 2, - - - , n.

$$P_X(x) = \binom{n}{x} \cdot p^x \cdot (1-p)^{n-x} \; ; \; x = 0, 1, \ldots, n \; ; \; 0 < p < 1.$$

$$\boxed{P_X(x) = \binom{n}{x} \cdot p^x q^{n-x} \; ; \; x = 0, 1, \ldots, n \; ; \; 0 < p < 1 \; ; \; p+q = 1}$$

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PMF of Binomial Dist^n.

$$P_X(x) = \binom{n}{x} p^x q^{n-x} ; \quad \begin{array}{l} x = 0, 1, \_\_\_, n \\ 0 < p < 1 \\ p + q = 1 \end{array}$$

$X \sim Bin(n, p)$.

X follows binomial distribution with param-eters n and p.

$$\sum_{x=0}^{n} \binom{n}{x} . p^x . q^{n-x} = (p+q)^n = 1$$

$$E(X) = \sum_{x=0}^{n} x . \binom{n}{x} . p^x . q^{n-x} .$$

$$= \sum_{x=1}^{n} x \binom{n}{x} p^x q^{n-x}$$

$$= \sum_{x=1}^{n} x . \frac{n!}{x! (n-x)!} p^x . q^{n-x}$$

$$= \sum_{x=1}^{n} \frac{n!}{(x-1)! (n-x)!} . p^x . q^{n-x} .$$

$$= \sum_{y=0}^{n-1} \frac{n!}{y! (n-1-y)!} . p^{y+1} q^{n-1-y} .$$

$$= np . \sum_{y=0}^{n-1} \binom{n-1}{y} p^y . q^{n-1-y} .$$

$$= np . (p+q)^{n-1} = np . \quad \boxed{E(X) = np}$$

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Nxt we compute Variance ;

$$V(X) = E(X^2) - (E(X))^2$$

In discrete case,

$$E(X^2) = E[X(X-1)+X]$$
$$= E[X(X-1)] + E[X].$$

k^{th} Factorial Moment

$$E[X(X-1)(X-2) \text{---} (X-k+1)].$$

$$= E[X(X-1)] + E(X) - (E(X))^2$$

$$= E[X(X-1)] + np - n^2p^2$$

$$E[X(X-1)] = \sum_{x=0}^{n} (x)(x-1) \frac{n!}{x!\,(n-x)!} p^x q^{n-x}$$

$$= \sum_{x=2}^{n} \frac{n!}{(x-2)!\,(n-x)!} p^x . q^{n-x}$$

$x-2=y$.

$$= \sum_{y=0}^{n} \frac{n!}{y!\,(n-2-y)!} p^{y+2} q^{n-2-y}.$$

$$= n(n-1).p^2 \sum_{y=0}^{n-2} \binom{n-2}{y} p^y . q^{n-2-y}.$$

$$= n(n-1)p^2 . (p+q)^{n-2}$$

$$= n(n-1)p^2.$$

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$$V(X) = E[X(X-1)] + np - n^2p^2$$
$$= n(n-1)p^2 + np - n^2p^2$$
$$= \cancel{n^2p^2} - np^2 + np - \cancel{n^2p^2}$$
$$= np(1-p) = npq$$

$$\boxed{V(X) = npq}$$

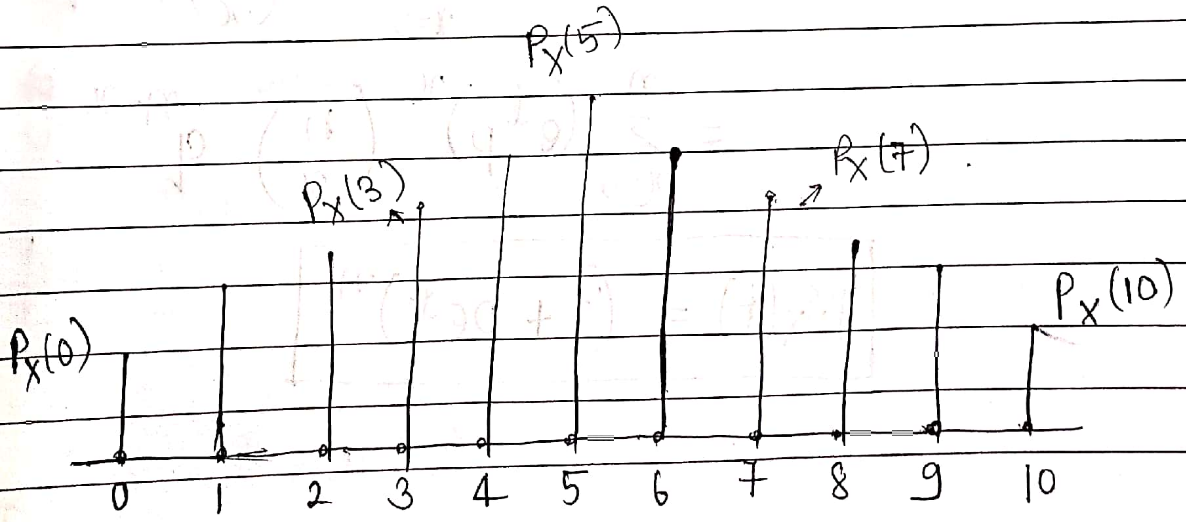
$$\boxed{\beta_1 = \frac{E[X - E(X)]^3}{\sigma^3}} = \frac{npq(1-2p)}{(npq)^{3/2}}$$

$$\beta_1 = \frac{1-2p}{\sqrt{npq}} = \begin{cases} 0 & ; \quad p = \tfrac{1}{2} \\ >0 & ; \quad p < \tfrac{1}{2} \\ <0 & ; \quad p > \tfrac{1}{2} \end{cases}$$

β_1 is coefficient of skewness.

* eg $\boxed{X \sim Bin(10, \tfrac{1}{2})}$

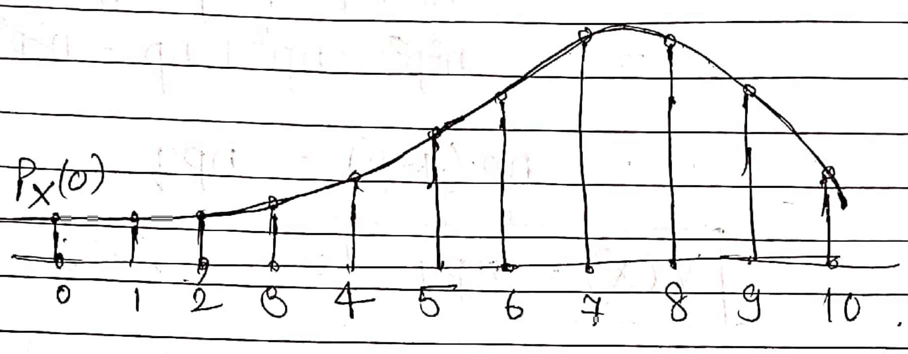
$$P_X(x) = \binom{10}{x}\left(\frac{1}{2}\right)^x \left(\frac{1}{2}\right)^{10-x} ; \quad x = 0, 1, \ldots, 10.$$



summetric distribution.

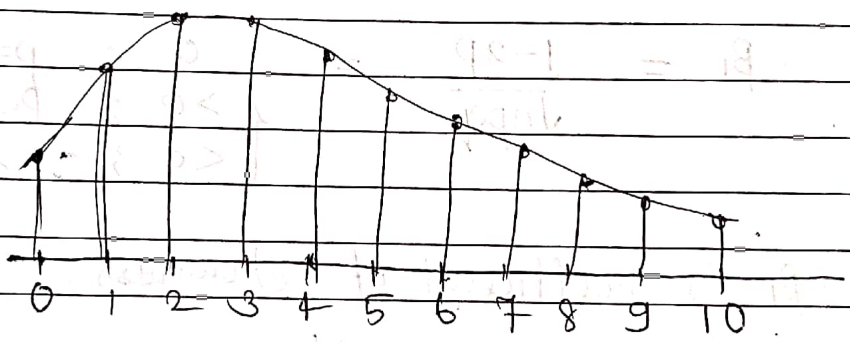
eg. ② $X \sim Bin(10, 4/5)$; $p > 1/2$.

negatively skewed.



eg. ③ $X \sim Bin(10, 1/5)$.

positively skewed.



MGF of a $Bin(n,p)$ distn.

$$M_X(t) = E(e^{tx}) = \sum_{x=0}^{n} e^{tx} \binom{n}{x} p^x q^{n-x}$$

$$= \sum_{x=0}^{n} (e^t p)^x \binom{n}{x} q^{n-x}$$

$$\boxed{M_X(t) = (q + pe^t)^n}$$

Ex. Let MGF of RV X is given by.

$$M_X(t) = \left(\frac{1}{5} + \frac{4}{5}e^t\right)^{10}$$ find $P(X \geq 3)$

* Coefficient of peakedness (β_2) :-

$$\boxed{\beta_2 = \frac{E\left(X - E(X)\right)^4}{\sigma^4} - 3}$$

* Mode of a Bin(n,p) dist^n :-

Value of x which is most likely to occur on performing the expt.

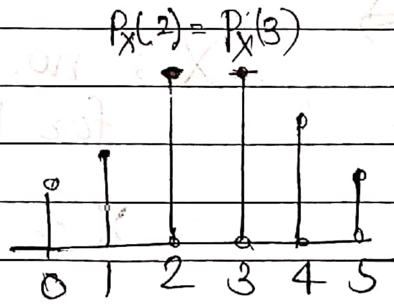
(1) If $(n+1).p$ is not an integer, then $[(n+1).p]$ is the mode. $[x]$ denotes integral part of x

(2) If $(n+1).p$ is an integer, then $(n+1).p$ and $(n+1).p - 1$ are modes.

eg. $X \sim Bin(5, \frac{1}{2})$.

$(n+1)p = (5+1)(\frac{1}{2}) = 3$.

$\boxed{\text{Modes} = 2, 3}$



eg. $X \sim Bin(10, \frac{1}{2})$.

$(n+1)p = (11)(\frac{1}{2}) = 5.5$ $\boxed{\text{Modes} = 5}$.

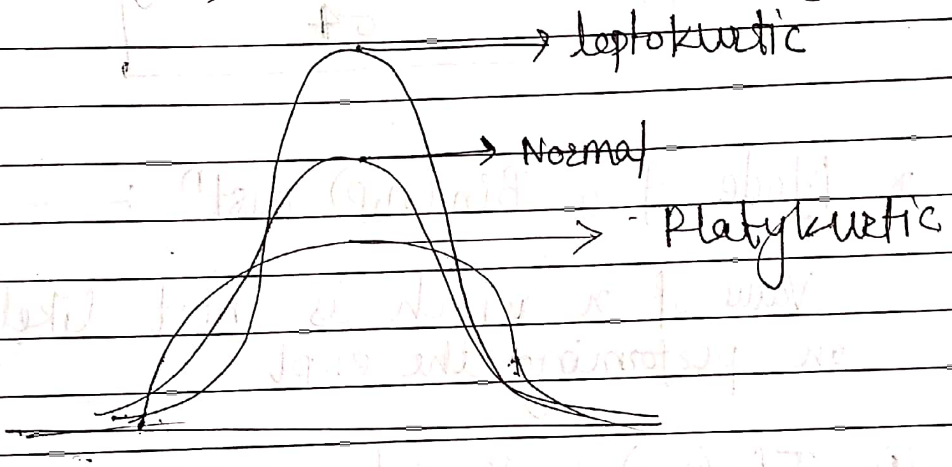
Coeff. of peakedness :-

$$\beta_2 = \frac{E(X-E(X))^4}{\sigma^4} - 3$$

$= 0$; peak is normal.

> 0 ; more than normal (leptokurtic)

< 0 ; less than normal (platykurtic)



Ex. An airline company knows that 5% of the people booking ticket do not turn up for a flight. So it sells 52 tickets for a 50 seat flight. What is the prob. that every passenger who turned up for the flight will get a seat.

→ X : no. of passengers who turned up for the flight.

$X \sim Bin(52, 0.95)$

$$P_X(x) = \binom{52}{x} (0.95)^x (0.05)^{n-x}$$

$x = 0, 1, 2, \text{———}, 52$.

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$$P(X \le 50) = 1 - P(X > 50)$$

$$= 1 - P_X(51) - P_X(52).$$

Poisson Distribution

A RV X is said to have a Poisson $dist^n$ if its PMF is of the following form:

$$P_X(x) = \frac{e^{-\lambda}.\lambda^x}{x!}; \quad x = 0,1,2, \text{——} \quad \lambda > 0.$$

$$\boxed{X \sim P(\lambda).}$$

* In practice, Poisson $dist^n$ is used to model rare events / phenomenon.

(1) No. of defective products in a lot.
(2) No. of clicks on a website.
(3) No. of suicide reported.
(4) A particular type of car passing a road interval.

$$\sum_{x=0}^{\infty} P_X(x) = e^{-\lambda} \sum_{x=0}^{\infty} \frac{\lambda^x}{x!} = e^{-\lambda}.e^{\lambda} = 1.$$

$$E(X) = e^{-\lambda} \sum_{x=0}^{\infty} x.\frac{\lambda^x}{x!} = e^{-\lambda} \sum_{x=1}^{\infty} \frac{\lambda^x}{(x-1)!}$$

$$= e^{-\lambda} \sum_{y=0}^{\infty} \frac{\lambda^{y+1}}{y!} = \lambda.$$

$$\boxed{E(X) = \lambda}$$

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$$E(X^2) = E\left(X(X-1)\right) + E(X).$$

$$= e^{-\lambda} \sum_{x=2}^{\infty} x(x-1) \cdot \frac{\lambda^x}{x!} + \lambda.$$

$$= e^{-\lambda} \sum_{y=0}^{\infty} \frac{\lambda^{y+2}}{y!} + \lambda = \lambda^2 + \lambda.$$

$$V(X) = E(X^2) - [E(X)]^2$$

$$= \lambda^2 + \lambda - (\lambda)^2$$

$$\boxed{V(X) = \lambda}$$

* Poisson distⁿ is the unique prob. distⁿ with same mean & variance.

$$\text{CDF} \Rightarrow F_X(x) = \sum_{i=0}^{x} \frac{e^{-\lambda} \cdot \lambda^i}{i!} \; ; \; x = 0, 1, 2.$$

* MGF of a $P(\lambda)$ distⁿ ;

$$M_X(t) = E(e^{tX}) = \sum_{x=0}^{\infty} e^{tx} \cdot \frac{e^{-\lambda} \lambda^x}{x!}$$

$$= e^{-\lambda} \sum_{x=0}^{\infty} \frac{(\lambda e^t)^x}{x!}.$$

$$= e^{-\lambda} \cdot e^{\lambda e^t}$$

$$\boxed{M_X(t) = e^{\lambda(e^t - 1)}}$$

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eg $M_X(t) = e^{2(e^t-1)} \Rightarrow X \sim P(2)$.

* $$\beta_1 = \frac{E\left(X-E(X)\right)^3}{\sigma^3}$$

$$= \frac{E(X-\lambda)^3}{\lambda^{3/2}} = \frac{1}{\sqrt{\lambda}} > 0$$

positively skewed.

* **Poisson Approximation to Binomial Distribution :-**

Let $X \sim Bin(n,p)$ suppose n becomes very large and prob. of a success p at each trial is very small such that $np \to \lambda$.

$$Bin(n,p) \longrightarrow P(\lambda) \quad \text{as } n \to \infty.$$

$\Rightarrow$ $$P_X(x) = \binom{n}{x} \cdot p^x \cdot (1-p)^{n-x}$$

$$= \frac{n!}{(n-x)!\,x!} \left(\frac{\lambda}{n}\right)^x \left(1-\frac{\lambda}{n}\right)^{n-x}.$$

$$= \frac{\lambda^x}{x!} \cdot \frac{n(n-1)\ldots(n-x+1)}{n^x} \cdot \frac{\left(1-\lambda/n\right)^n}{\left(1-\lambda/n\right)^x}$$

$$\boxed{P_X(x) = \frac{\lambda^x}{x!} \cdot e^{-\lambda} \quad \text{as } n \to \infty.}$$

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$$M_X(t) = (q+pe^t)^n$$

$$= (1-p+pe^t)^n$$

$$= (1+p(e^t-1))^n$$

$$= \left(1+\frac{\lambda}{n}(e^t-1)\right)^n$$

as $n \to \infty \Rightarrow M_X(t) = e^{\lambda(e^t-1)}$

* $(n \geq 100 \; ; \; p \leq 0.01)$.

Ex. Suppose 1 in 5000 light bulbs are defective. Let X denotes no. of defective light bulbs in a lot of 10,000 bulbs. What is the prob. that at least 3 of them are defective?

Ans. :

$$X \sim Bin\left(10{,}000, \frac{1}{5000}\right)$$

$$P_X(x) = \binom{10000}{x}\left(\frac{1}{5000}\right)^x \left(1-\frac{1}{5000}\right)^{10000-x} ; \; x = 0, 1, 2, \ldots$$

$$P(X \geq 3) = 1 - P(X<3)$$
$$= 1 - P_X(0) - P_X(1) - P_X(2).$$

Accurate Prob.

$$= 1 - (0.9998)^{10000} - 10000(0.0002)(0.9998)^{10000-1} - \binom{10000}{2}(0.0002)^2(0.9998)^{10000-2} \doteq 0.32.$$

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Approximated Prob.

$X \sim P(\lambda)$; $\lambda = np = (10000)\left(\frac{1}{5000}\right) = 2$.
$\sim P(2)$.

$P_X(x) = e^{-2} \cdot \frac{2^x}{x!}$; $x = 0, 1, 2, \_\_\_$

$P(X \geq 3) = 1 - P_X(0) - P_X(1) - P_X(2)$.

$= 1 - e^{-2} - 2 \cdot e^{-2} - 2 \cdot e^{-2}$.
$= 1 - 5e^{-2} \doteq 0.32$.

Tut 3

$$F_X(x) = P(X \leq x) = \begin{cases} c\pi x^2 & ; \ 0 \leq x \leq 25 \\ 1 & ; \ x > 25 . \end{cases}$$

(i) $c\pi(25)^2 = 1$
$\Rightarrow \quad c = \frac{1}{\pi(25)^2}$

(ii) $$f(x) = \frac{d}{dx} F(x) = \begin{cases} 2c\pi x & ; \ 0 \leq x \leq 25 \\ 0 & ; \ \text{otherwise} . \end{cases}$$

(iii) $E(x) = \int_{-\infty}^{\infty} x . f(x) . dx$

$= \int_0^{25} x (2c\pi x) . dx$

$= 2c\pi . \frac{(x^3)_0^{25}}{3}$

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iv) $P(X \le 10 \mid X \ge 5)$

$= P(X \le 10) - P(X < 5) + \underbrace{P(X=5) - P(X=5)}_{=0}$

$= c\pi(10)^2 - c\pi(5)^2 + 0$

$= 75c\pi$

v) Amount spent by player = 1 \$
Average gain = 0.25 \$

$$\begin{cases} 10\$ & \text{when } X \le r \\ 1\$ & \text{when } r < X \le 2r \\ 0\$ & \text{when } 2r < X \le 25 \end{cases}$$

Avg. amount earned.

$= 10 \times P(X \le r) + 1 \times P(r < X \le 2r) + 0 \times P(2r < X \le 25)$

$= 10 \int_0^r 2c\pi x\,dx + \int_r^{2r} 2c\pi x\,dx = 1.25$

Q③. $\dfrac{9(3+2x)}{x^2(3+x)^2} I_{(3,\infty)}(x) + \dfrac{x(6+x)}{3(3+x)^2} I_{(0,3]}(x)$

is a probability density function (PDF)

$$f(x) = \begin{cases} \dfrac{9(3+2x)}{x^2(3+x^2)} & ; x > 3 \\ \dfrac{x(6+x)}{3(3+x^2)} & ; 0 < x \le 3 \\ 0 & ; \text{otherwise} \end{cases}$$

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$$I_A(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A . \end{cases}$$

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Show that $\int_{-\infty}^{\infty} f(x) \, dx = 1$.

(1) ⇒

$$f(x) = \begin{cases} \theta^2 x e^{-\theta x} & ; \quad x > 0 . \\ 0 & ; \quad x \le 0 . \end{cases}$$

$$\int_{-\infty}^{\infty} f(x) = 1 .$$

$$P(X \ge 1) = 1 - P(X < 1)$$
$$= 1 - F(1)$$

$$F = P(X \le x) = \begin{cases} 0 & ; \quad x \le 0 . \\ \int_0^x \theta^2 x e^{-\theta x} & ; \quad x > 0 . \end{cases}$$

$$\int_0^x \theta^2 \underset{u}{x} \, \underset{v}{e^{-\theta x}} = \theta^2 \left[x . \frac{e^{-\theta x}}{-\theta} \Big|_0^x - \int_0^x \frac{e^{-\theta x}}{-\theta} . dx \right]$$

$$= \theta^2 \left[x . \left(\frac{e^{-\theta x}}{-\theta} + \frac{1}{\theta} \right) + \frac{1}{\theta} . \frac{\left(e^{-\theta x} \right)_0^x}{-\theta} \right]$$

$$= \theta^2 \left[\frac{1}{\theta} . \left(1 - x e^{-\theta x} \right) - \frac{1}{\theta^2} \left(e^{-\theta x} - 1 \right) \right]$$

$$= \theta \left(1 - x e^{-\theta x} \right) + \left(1 - e^{-\theta x} \right) .$$

(5) (i) $F(x) = \begin{cases} 0 \; ; & x \le 0 \\ x/2 \; ; & 0 \le x < 1 \\ 1/2 & 1 \le x < 2 \\ x/4 & 2 \le x < 4 \\ 1 & x \ge 4 \end{cases}$

$f(x) = \begin{cases} 1/2 \; ; & 0 \le x < 1 \\ 0 \; ; & 1 \le x < 2 \\ 1/4 \; ; & 2 \le x < 4 \\ 0 \; ; & x \ge 4 \end{cases}$

(ii) $F(x) = \begin{cases} 0 \; ; & x < 1 \\ \frac{(x-1)^2}{8} \; ; & 1 \le x < 3 \\ 1 \; ; & x \ge 3 \end{cases}$ not continuous at $x = 3$.

$f(x) = \begin{cases} 0 \; ; & x < 1 \\ (x-1)/4 \; ; & 1 \le x < 3 \\ 1/2 \; ; & x = 3 \end{cases}$

$$P(X=3) = P(X \le 3) - P(X < 3)$$

$$= F(3) - F(3^-)$$

$$= 1 - \frac{1}{2} = \frac{1}{2}.$$

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$$P(|X-\mu| \geq k) \leq \frac{\sigma^2}{k^2}$$

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6). ~~$P(X=0)=0$~~

X	0	1	2	3	4	5	6	7
P(X)	0	k	2k	2k	3k	k^2	$2k^2$	$7k^2+k$

(i) $k+2k+2k+3k+k^2+2k^2+7k^2+k=1$

7). Two fair dice are thrown.

X	2	3	4	5	6	7	8	9	10	11	12
P(X)	$\frac{1}{36}$	$\frac{2}{36}$	$\frac{3}{36}$	$\frac{4}{36}$	$\frac{5}{36}$	$\frac{6}{36}$	$\frac{5}{36}$	$\frac{4}{36}$	$\frac{3}{36}$	$\frac{2}{36}$	$\frac{1}{36}$

$$E(X) = \sum_{x=2}^{12} x \cdot P(X=x) = 7$$

$$E(X^2) = \sum_{x=2}^{12} x^2 P(X=x) =$$

$$\frac{4+18+48+100+180+294+320+324+300+242+144}{36}$$

$$Var(X) = E(X^2) - (E(X))^2$$

$$P(|X-7| \geq 4)$$

$$= P(X-7 \leq -4;\ X-7 \geq 4)$$

$$= P(X \leq 3) + P(X \geq 7)$$

$$P(|X-E(X)| \geq k) \leq \frac{Var(x)}{k^2}$$

9) X be a RV s.t. $E|X| < \infty$. Show that $E|X-c|$ is minimized if we choose c to be a median of the distribution of X.

If c is median of X

$$\Rightarrow \int_{-\infty}^{c} f(x).dx = \frac{1}{2} = \int_{c}^{\infty} f(x).dx.$$

$$E|x-c| = \int_{-\infty}^{\infty} |x-c|.f(x).dx.$$

$$= \int_{-\infty}^{c} (c-x) f(x).dx + \int_{c}^{\infty} (x-c).f(x).dx.$$

For minimum,

$$\frac{d}{dc} E|X-C| =$$

$$f(x) = \int_{a(t)}^{b(t)} f(x,t).dx.$$

Geometric Distribution

Conduct Bernoullian Trials till first success is acheived.

Define RV X : no. of trials required to get first success.

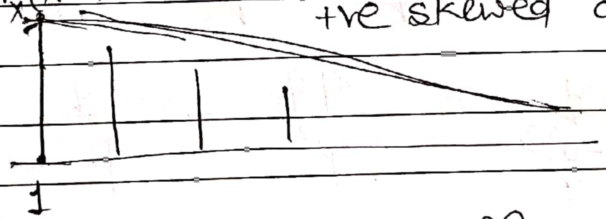
Range Space R_X : 1, 2, 3, ——

$$\boxed{P_X(x) = P(X=x) = q^{x-1}p \quad ; \quad x = 1,2,3, \text{——} \quad 0<p<1 \quad p+q=1.}$$

It is a proper prob. $dist^n$ because

$$\sum_{x=1}^{\infty} P_X(x) = p\sum_{x=1}^{\infty} q^{x-1} = \frac{p}{1-q} = 1$$

$P_X(X=1)$



+ve skewed $dist^n$.

$$E(X) = \sum_{x=1}^{\infty} x . P_X(x) = \sum_{x=1}^{\infty} x . q^{x-1} . p$$

$$= p.[1+2q+3q^2+ \text{——}]$$

$$= \frac{p}{(1-q)^2} = \frac{1}{p}. \qquad \boxed{E(X) = \frac{1}{p}}$$

$$\boxed{X \sim Geo(p)}$$

$$V(X) = E(X^2) - (E(X))^2$$

For discrete case ;

$$= E(X(X-1)) + E(X) - E(X)^2$$

$$= \frac{2q}{p^2} + \frac{p}{p^2} - \frac{1}{p^2} = \frac{q}{p^2}$$

$$E[X(X-1)] = \sum_{x=1}^{\infty} x(x-1).P_X(x)$$

$$= \sum_{x=1}^{\infty} x(x-1).q^{x-1}.p$$

$$= p[2.1q + 3.2.q^2 + 4.3q^3 + ---]$$

$$= 2pq[1 + 3q^2 + 6q^3 + ..]$$

$$= \frac{2pq}{(1-q)^3} = \frac{2pq}{p^3} = \frac{2q}{p^2}.$$

$$\boxed{V(X) = q/p^2}$$

MGF of Geometric (p) dist^n.

$$M_X(t) = E(e^{tX}) = p.\sum_{x=1}^{\infty} e^{tx}.q^{x-1}$$

$$= p[e^t + e^{2t}q + e^{3t}q^2 + \_\_\_]$$

$$= \frac{p.e^t}{1-qe^t}; \quad qe^t < 1, \text{ or } ~~the~~ \ t < -\ln(q)$$

$$\boxed{\begin{aligned} M_X(t) &= \frac{p\cdot e^t}{1-qe^t} \quad ; \quad t < -\ln q \\ &= 1 \quad ; \quad t=0 . \end{aligned}}$$

Q. Let $X \sim Geo(p)$ then find $P(X>b)$; where b is a given positive integer.

$\Rightarrow$ $P(X>b) = 1 - P(X \leq b)$.

$$= 1 - \sum_{x=1}^{b} q^{x-1}\cdot p$$

$$= 1 - p\cdot\left(1+q+q^2+ \dots + q^{b-1}\right).$$

$$= 1 - p\cdot\left((1)\,\frac{q^b-1}{q-1}\right)$$

$$= \cancel{q-1}$$

$$= \frac{q-1-pq^b+p}{q-1} \quad = \frac{\cancel{+}pq^b}{\cancel{+p}}$$

$$\boxed{P(X>b) = q^b}$$

★ <u>Memoryless Property</u> :-

Let $X \sim Geo(p)$ dis^{tn} and m & n are given +ve integer then.

$$P(X>m+n \mid X>m) = P(X>n).$$

Proof :- L.H.S. $= P(X > m+n \mid X > m)$

$$= \frac{P(X > m+n \cap X > m)}{P(X > m)}$$

$$= \frac{P(X > m+n)}{P(X > m)}$$

$$= \frac{q^{m+n}}{q^m} = q^n = P(X > n)$$

Hence proved.

☆ Geometric distⁿ is the unique discrete prob. distⁿ with memoryless property.

Alternative Representation of Geo. distⁿ.

Define X : no. of trials preceeding the first success.

R_X : 0, 1, 2, ____ .

$$P(X=x) = P_X(x) = q^x \cdot p \qquad x = 0, 1, 2, \text{---}$$

$$0 < p < 1 \; ; \; p + q = 1.$$

$E(X) = q/p$. $\qquad M_X(t) = \dfrac{p}{1 - qe^t} \; ; \; t < -\ln q$.

$V(X) = q/p^2$.

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Ex. Suppose independent trials are conducted on a monkey to develop a vaccine. If the prob. of success is $\frac{1}{3}$ in each trial then find the prob that at least 5 trials are req. to get the first success

→

X : no. of trials to get first success.

$$P_X(x) = \left(\frac{2}{3}\right)^{x-1} \frac{1}{3} \quad ; \quad x = 1, 2, 3, \_\_$$

$$P(X \geq 5) = P(X > 4) = \left(\frac{2}{3}\right)^4$$

★ Inverse/ Negative Binomial Dist^n :-

Conduct Bernoullian Expt. till a fixed r no. of successes are acheived

X : no. of trials req. to get r no. of successes.

R_X : $r, r+1, r+2, \_\_$

$$P_X(x) = \binom{x-1}{r-1} p^{r-1} p \cdot q^{x-r}$$

$$\boxed{P_X(x) = \binom{x-1}{r-1} p^r \cdot q^{x-r} \quad ; \quad x = r, r+1, r+2, \_\_ \quad 0 < p < 1 \quad p + q = 1.}$$

$$\boxed{X \sim NBin(r, p)}$$

for $r = 1$ NBin ⇒ Geo.