Chapter - 3

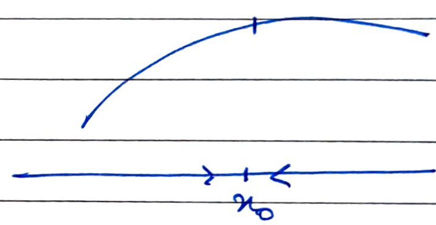
FUNCTIONS

- $f : D \to \mathbb{R}$

  Domain $(f) \subset D$

→ LIMIT AND CONTINUITY

- $\lim\limits_{x \to x_0} f(x) = f(x_0)$



- Cauchy Criterion

- If $\{x_n\} \to x_0$ and $\Rightarrow f(x_n) \to f(x_0)$

  then $f(x)$ is continuous

- eg- $f(x) = 2x^2 + 1$

  $\lim\limits_{n \to \infty} x_n = x_0$

  $f(x_n) = 2x_n^2 + 1$

  $\lim\limits_{n \to \infty} f(x_n) = f(x_0)$

- ## Definition of limit

$$\lim_{x \to x_0} f(x) = L$$

means when

$$0 < |x - x_0| < \delta$$

↳ can't be equal for limit but $\leq$ for definition of continuity

then $|f(x) - f(x_0)| < \epsilon$

$\Rightarrow \lim_{x \to x_0} f(x) = L$

where δ and ϵ are arbitrarily small quantities.

For continuity the conditions become

$$|x - x_0| < \delta$$

then $|f(x) - f(x_0)| < \epsilon$

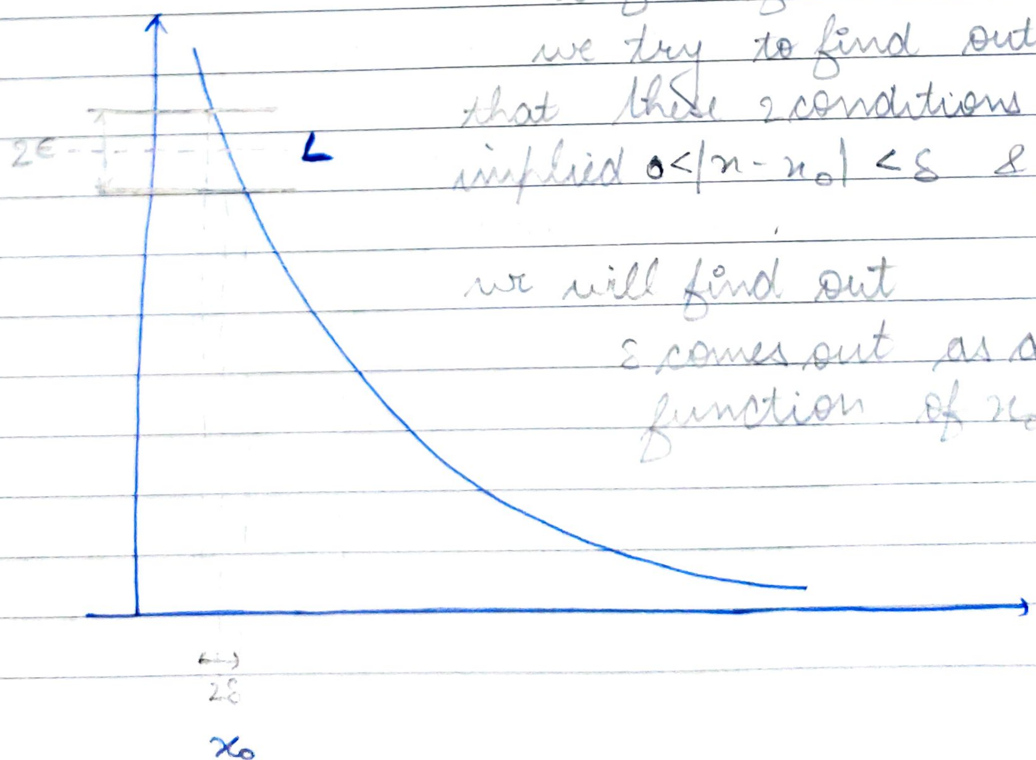
$\Rightarrow f(x)$ is continuous at $x = x_0$

- Graphical meaning

~~For limit~~

We first fix an ϵ and then we try to find out δ such that these 2 conditions are implied $0 < |x - x_0| < \delta$ & $|f(x) - f(x_0)| < \epsilon$

we will find out δ comes out as a function of x_0 & ϵ



given

$\epsilon > 0$

$\exists\ \delta > 0$ such that

$0 < |x - x_0| < \delta$

$\Rightarrow |f(x) - f(x_0)| < \epsilon$

$\Rightarrow \lim_{x \to x_0} f(x) = L$

& $|x - x_0| < \delta$

$\Rightarrow |f(x) - f(x_0)| < \epsilon$

$\Rightarrow$ $f(x)$ is continuous at $x = x_0$

- Consider previous example

$f(x) = 2x^2 + 1$

$\therefore |f(x) - f(x_0)| = 2|x - x_0|\,|x + x_0|$

we say for once

$|x - x_0| < 1$ you can assume any number. (i)

We know

$\Rightarrow \underbrace{||x| - |x_0|| \le |x - x_0| < 1}$

$\Rightarrow |x| < 1 + |x_0|$

$|x + x_0| \le |x| + |x_0| < 1 + 2|x_0|$

NOTE: For conti. our aim is

if $|x - x_0| < \delta$

$\Rightarrow |f(x) - f(x_0)| < \epsilon$

$\Rightarrow |f(x) - f(x_0)| < 2(1 + 2|x_0|)\,|x - x_0| < \epsilon$

$$\Rightarrow |x - x_0| < \frac{\epsilon}{2(1+2|x_0|)} \qquad (ii)$$

$\therefore$ we can write from (i) & (ii)

$$\delta = \min\left\{1, \frac{\epsilon}{2(1+2|x_0|)}\right\}$$

$\Rightarrow |f(x) - f(x_0)| < \epsilon$ for this δ

Q→ $f(x) = \frac{1}{x^2}$, in $(0, \infty)$ prove continuity

NOTE: In case of limit, we never reach at the point i.e

$$\lim_{x \to x_0} f(x) = L$$

This doesn't necessarily mean that

$f(x_0) = L$ $\quad$ $f(x_0)$ may be $\neq L$

But if it is continuous then

$\lim_{x \to x_0} f(x) = L$ & $f(x_0) = L$

Q→ $f(x) = \begin{cases} x^2 \sin \frac{1}{x} & ; x \neq 0 \\ 0 & ; x = 0 \end{cases}$

Check continuity at $x=0$ using ϵ-δ definition.

Ans.

$|f(x) - f(0)| = |f(x)| = \left|x^2 \sin \frac{1}{x}\right| \leq |x|^2$

& we know

$|x - x_0| < \delta \Rightarrow |x| < \delta$

$\therefore |f(x) - f(0)| \leq |x|^2 < \delta^2 = \epsilon$

$\Rightarrow \delta = \sqrt{\epsilon}$

Q: $f(x) = \begin{cases} \frac{1}{x} \sin \frac{1}{x} & ; x \neq 0 \\ 0 & ; x = 0 \end{cases}$

Ans: let $x = \frac{1}{n}$

$\Rightarrow$ for $x \neq 0$ $\quad f\left(\frac{1}{n}\right) = n \sin n$

now

$\lim_{n \to \infty} f\left(\frac{1}{n}\right) \neq f(0)$

$\therefore$ discontinuous

Prove by $\varepsilon - \delta$ method also ↓

- If g & f is continuous at $x_0 \in \text{Dom}(f)$, then

a) $|f|$, kf (k is some real no.) is continuous at x_0

b) $f+g$ are also continuous at x_0

c) fg is continuous at x_0 & f/g is also continuous at x_0 provided $g(x_0) \neq 0$

d) $g(f(x))$ is continuous at x_0 if g is cont. at $f(x_0)$

e) $\max(f,g)$ & $\min(f,g)$ is continuous

$$\max(f,g) = \frac{1}{2}(f+g) + \frac{1}{2}|f-g|$$

$$\min(f,g) = \frac{1}{2}(f,g) - \frac{1}{2}|f-g|$$

Q→ Discuss cont. of $\frac{1}{x^2}$ in $(0, \infty)$

Ans: $$\left|\frac{1}{x^2} - \frac{1}{x_0^2}\right| = \frac{|x - x_0|\,|x + x_0|}{x^2 x_0^2}$$

we assume

$$\frac{|x - x_0|\,|x + x_0|}{x^2 x_0^2} < M|x - x_0|$$

ie simply $\delta = \frac{\epsilon}{M}$

$\therefore$ we need to find bound of $\frac{|x + x_0|}{x^2 x_0^2} = M$

Assuming

$$|x - x_0| < \frac{x_0}{2}$$

$$\Rightarrow \frac{x_0}{2} < |x| < \frac{3x_0}{2} \quad (i)$$

$$\Rightarrow |x + x_0| \leq |x| + |x_0| < \frac{5x_0}{2}$$

$\therefore$ $|x + x_0| < \frac{5x_0}{2}$ (considering $x_0 > 0$)

& for x, from (i)

$$\frac{1}{x} < \frac{2}{x_0}$$

$$\therefore \frac{|x + x_0|}{x^2 x_0^2} < \frac{5x_0 \cdot 4}{2\, x_0^2 x_0^2} = \frac{10}{x_0^3}$$

$$\therefore |f(x) - f(x_0)| < \frac{10}{x_0^3} |x - x_0| < \epsilon$$

ie $|x - x_0| < \frac{\epsilon x_0^3}{10}$

$$\therefore M = \frac{10}{x_0^3}$$

$$\therefore \delta = \frac{\epsilon x_0^3}{10}$$

- Again consider

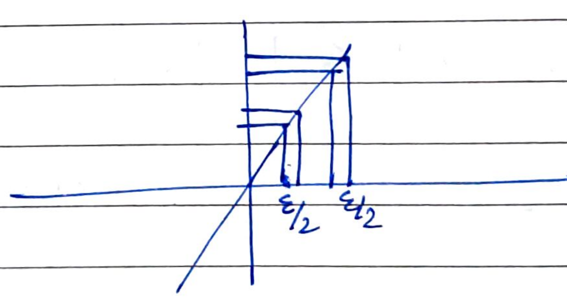
$f(x) = 2x + 1$

$\therefore$ we will get

$$2|x - x_0| < \varepsilon$$

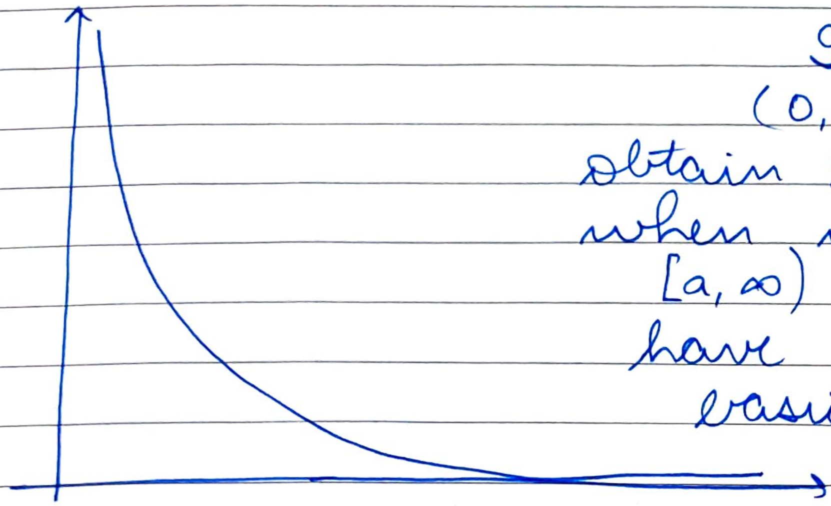
$$\Rightarrow |x - x_0| < \frac{\epsilon}{2} = \delta$$

$\therefore$ here δ is independent of x_0



• UNIFORM CONTINUITY

consider function $\frac{1}{x^2}$



If we take interval $(0,\infty)$ then we obtain $\delta(x_0,\epsilon)$ but when we take interval $[a,\infty)$ then we have $\delta(a,\epsilon)$ ie basically $\delta(\epsilon)$.

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- In previous examples,

$f(x) = 2x^2+1$ on $\mathbb{R}$ $\delta = \min\left\{1, \frac{\epsilon}{2(1+|x_0|)}\right\}$

$f(x) = \frac{1}{x^2}$ on $(0,\infty)$ $\delta = \min\left\{\frac{x_0}{2}, \frac{\epsilon x_0^3}{10}\right\}$

- To check continuity at x_0,

a) we have $\rightarrow$ $f(x)$ & x_0

b) we fix $\longrightarrow$ ϵ

c) we find $\rightarrow \delta = \delta(x_0, \epsilon)$

- Uniform continuity - It is a property of a function over some interval or a set.

To check uniform continuity on a set 'S'

a) we have $\longrightarrow$ $f(x)$ on some set S or on an interval

b) we fix ϵ

c) we find $\delta = \delta(\epsilon)$

∴ as as mathematical definition, we have

UNIFORM CONTINUITY on 'S'	CONTINUITY at x_0
given $\epsilon > 0$ $\exists$ $\delta = \delta(\epsilon)$ such that $\forall x, y \in S$, $\lvert x - y \rvert < \delta$ $\Rightarrow \lvert f(x) - f(y) \rvert < \epsilon$	given $\epsilon > 0$ $\exists$ $\delta > 0$: such that ($\delta = \delta(x_0, \epsilon)$) $x_0 \in Dom(f)$ $\lvert x - x_0 \rvert < \delta$ $\Rightarrow \lvert f(x) - f(x_0) \rvert < \epsilon$

- eg- for function $\frac{1}{x^2}$ on $[a, \infty)$ we check uniform cont.

$$\left| \frac{1}{x^2} - \frac{1}{y^2} \right| = \frac{|x-y||x+y|}{x^2 y^2}$$

$y, x \in [a, \infty)$
now

$x \geq a, \quad y \geq a$

$\Rightarrow \frac{1}{x} \leq \frac{1}{a}, \quad \frac{1}{y} \leq \frac{1}{a}$

$$\therefore \left| \frac{1}{x^2} - \frac{1}{y^2} \right| = \left| \frac{1}{xy^2} + \frac{1}{x^2 y} \right| |x-y| \leq \frac{2}{a^3} |x-y|$$

since

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We can write

$\frac{2}{a^3}|x-y| < \frac{2}{a^3}\delta = \epsilon$

we write this since we are dead sure that

$\left|\frac{1}{x^2}-\frac{1}{y^2}\right| < \frac{2}{a^3}\delta$

$\therefore \frac{2}{a^3}\delta = \epsilon$

$\frac{2}{a^3}\delta = \epsilon$

$\Rightarrow \boxed{\delta = \frac{a^3\epsilon}{2}}$

$\therefore$ δ independent of x_0 ie for fixed ϵ δ is fixed.

$\therefore$ given $\epsilon > 0$ $\exists$

$\delta = \frac{a^3\epsilon}{2}$

st. $\forall x, y \in [a, \infty)$

and $|x-y| < \delta$

$\Rightarrow \left|\frac{1}{x^2} - \frac{1}{y^2}\right| < \epsilon$

$\therefore$ uniformly continuous

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• Again considering $f(x)=2x^2+1$ in interval $[-M, M]$

$$|2x^2+1-2y^2-1| = 2|x+y||x-y| \le 4M|x-y|$$

since $x,y \in [-M,M]$

$\therefore$ we can write

$$4M|x-y| < 4M\delta$$

$$\therefore \quad |f(x)-f(y)| < 4M\delta \text{ for sure}$$

$$\therefore \quad 4M\delta = \epsilon$$

$$\Rightarrow \quad \delta = \frac{\epsilon}{4M}$$

Given $\epsilon > 0$ $\exists$

$$\delta = \frac{\cancel{\epsilon}\,\epsilon}{4M}$$

s.t. $\forall\, x,y \in [-M, M]$

and $|x-y| < \delta$

$$\rightarrow \left|\frac{1}{x^2} - \frac{1}{y^2}\right| < \epsilon$$

$\therefore$ uniformly continuous

- Another eg -

$$f(x) = x+1 \quad \text{on } \mathbb{R}$$

$$|f(x) - f(y)| = |x-y| < \delta = \epsilon$$

$\therefore \delta = \epsilon$

Here no interval is defined yet it is uniformly continuous.

→ **THEOREM** →

- A continuous function in a closed & bounded interval is uniformly continuous eg - $f(x) = 2x^2+1$ on $[-M, M]$

imp

- The inverse is not true. eg $f(x) = x+1$ on $\mathbb{R}$ is uniformly cont. but neither closed nor bounded
- complicated proof.

→ **THEOREM** →

- If a function is uniformly continuous on a set S and $\{x_n\}$ is Cauchy sequence in S then $f\{x_n\}$ is Cauchy.

- $f: S \to R$

  given $\epsilon > 0$ $\exists$ δ $\forall$ $x, y \in S$

  s.t. $|x-y| < \delta \Rightarrow |f(x) - f(y)| < \epsilon$

given $\epsilon > 0$ $\exists N$, $m, n > N \Rightarrow |x_m - x_n| < \delta$

$$\Rightarrow |f(x_m) - f(x_n)| < \epsilon$$

ie $f(x_n)$ is cauchy sequence.

imp. If a function does not satisfy

$|f(x_m) - f(x_n)| < \epsilon$ where x_n is cauchy,

we can directly conclude that $f(x)$ is not uniformly continuous on S.

eg- $\frac{1}{x^2}$ on $(0, \infty)$

Let $x_n = \frac{1}{n}$

$\therefore f(x_n) = n^2$

which is not cauchy at all

$\therefore f(x)$ is not uniformly continuous.

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→ **THEOREM →**

- Let $f(x)$ be continuous on $[a,b]$ (bounded or unbounded interval) and f is differentiable on (a,b) and f' is bounded on (a,b) then $f(x)$ is uniformly continuous on $[a,b]$

- $|f(x)-f(y)| = |f'(\xi_1)|\,|x-y| < M|x-y| < \epsilon$

$x < \xi_1 < y$ $\quad \therefore \boxed{\delta = \frac{\epsilon}{M}}$

~~where $f'(\xi_1)$ is slope calculated according to mean value theorem.~~

where $f'(\xi_1)$ is maximum slope in interval $[x,y]$ (Think!!)

- eg $\sin x$ is uniform continuous on $\mathbb{R}$. (Prove it!)

Q→ Show that

$$f(x) = \begin{cases} 1 & , x \in \mathbb{Q} \\ 0 & , x \in \mathbb{R} \setminus \mathbb{Q} \end{cases}$$

is not continuous

Ans:

$$\left.\begin{array}{l} x_n \to x_0 \\ \Rightarrow f(x_n) \to f(x_0) \end{array}\right\} \longleftrightarrow \text{continuous}$$

$x_{2n} \in \mathbb{Q} \Rightarrow f(x_{2n}) \to 1$

$x_{2n+1} \in \mathbb{R}\setminus\mathbb{Q} \Rightarrow f(x_{2n+1}) \to 0$

$\therefore f$ is not continuous

we can also use dense's theorem

$|x - x_0| < \delta$

$x_0 \in \mathbb{Q}$

$\therefore x$ is an irrational (in neighbourhood of $\mathbb{Q}$)

$\therefore |f(x) - f(x_0)| = 1$

$\therefore$ not continuous.

Q. A function $f(x)$ $f: \mathbb{R} \to \mathbb{R}$ is continuous at $x=0$, and satisfies the following condition $f(0)=0$, $f(x_1+x_2) \le f(x_1)+f(x_2)$ $\forall x_1, x_2 \in \mathbb{R}$
Prove f is uniformly continuous on $\mathbb{R}$

Ans. given $f(x)$ is cont. at $x=0$

$\Rightarrow$ given $\epsilon > 0$ $\exists$ $\delta > 0$ such that

$$|x - 0| < \delta \Rightarrow |f(x) - \underset{\to 0}{f(0)}| < \epsilon$$

$$\left.\begin{aligned} &\Rightarrow |x| < \delta \\ &\Rightarrow |f(x)| < \epsilon \end{aligned}\right] \text{---} (i)$$

Now consider

$$f(t+x) - f(x) \le f(t)$$

From (i) we can write

if $|t| < \delta$

$$\Rightarrow f(t+x) - f(x) \le f(t) < \epsilon$$

Here replacing $x \to x-t$

& $t \to -t$

$$\Rightarrow f(x) - f(x+t) \le f(-t) < \epsilon \quad (\text{since } |-t| < \delta)$$

$\therefore |t| < \delta \Rightarrow -\epsilon < f(t+x) - f(x) < \epsilon$

$\Rightarrow |f(t+x) - f(x)| < \epsilon$

~~t~~ is any number. $\therefore t+x$ can denote any number

$\therefore$ now replace $t+x = y$
& $x = x$

$\therefore$ ~~f(y)~~ $|x-y| < \delta$
$\Rightarrow |f(y) - f(x)| < \epsilon$

→ THEOREM : (IMVT - Intermediate value theorem)

- If a function is continuous on a closed interval, then it is bounded and it has maximum & minimum defined in that interval.

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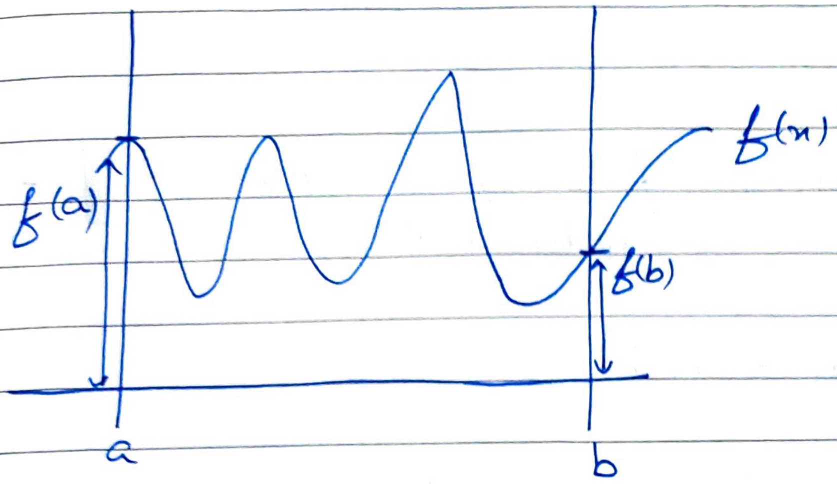

$f(x)$ is continuous on closed interval $[a, b]$

If we take $f(a) < y_0 < f(b)$

or $f(b) < y_0 < f(a)$

Then there exists an x_0 such that

$$x_0 \in [a, b] \;\&\; f(x_0) = y_0$$

This is intermediate value theorem.

→ THEOREM: CAUCHY'S MEAN VALUE THEOREM

- If $f, g \in C[a, b]$ ($\in C[a,b]$ means f & g are cont. here)

& f', g' exist in (a, b)

and $g'(x) \neq 0$ in (a, b)

then $\exists\, c$

$$\boxed{\frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f'(c)}{g'(c)}}$$

- **PROOF:**

Consider

$$h(x) = \frac{f(b)-f(a)}{g(b)-g(a)}[g(x)-g(a)] - [f(x)-f(a)]$$

$\therefore h(a) = h(b) = 0$

now using Rolle's theorem,

$h'(c) = 0$

$$\Rightarrow \frac{f(b)-f(a)}{g(b)-g(a)} = \frac{f'(c)}{g'(c)}$$

NOTE: If $g(x) = x$, then we get LMVT.

→ **THEOREM:** Inverse function theorem

If function $f:(a,b) \longrightarrow (c,d)$ is continuous and one one and onto (ie $f[(a,b)] = (c,d)$) and if f is differentiable at x_0 and $f'(x_0) \neq 0$

Then f^{-1} is differentiable at $y_0 = f(x_0)$

and $(f^{-1})'(y_0) = \dfrac{1}{f'(x_0)}$

$$\boxed{(f^{-1})'(y_0) = \frac{1}{f'(x_0)}}$$

- eg- consider $\sin x$

$$\left(-\pi/2, \pi/2\right) \to x_0$$

$$(-1, 1) \to y_0$$

$$y_0 \in (-1, 1)$$

$$(f^{-1})'(y_0) = \frac{1}{\cos x_0}$$

$$\Rightarrow (f^{-1})'(y_0) = \frac{1}{\sqrt{1-\sin^2 x_0}} = \frac{1}{\sqrt{1-y_0^2}}$$

→ TAYLOR SERIES →

- Taylor series is a power series defined around a point, say x_0 as

$$\boxed{f(x) = \sum_{0}^{\infty} a_n (x-x_0)^n} \quad ; \quad a_n = \frac{f^{(n)}(x_0)}{n!}$$

- $$f(x) = f(x_0) + \frac{f'(x_0)}{1!}(x-x_0) + \frac{f''(x_0)}{2!}(x-x_0)^2 + \dots + \frac{f^{(n)}(x_0)(x-x_0)^n}{n!} + \dots$$

Analytic: A function ~~not~~ differentiable at a point as well as its neighbourhood.

To write Taylor series the func. must be infinitely diff at neighbourhood

- Even if a function is infinitely differentiable, its Taylor series may not be valid.

- For Taylor series to be valid, function must be analytical.

- $$f(x) = \underbrace{f(x_0) + \frac{f'(x_0)(x-x_0)}{1!} + \dots + \frac{f^{(n)}(x_0)(x-x_0)^n}{n!}}_{P_n}$$

$$f(x) = \underbrace{P_n(x)}_{\text{Taylor's polynomial}} + \underbrace{R_n(x)}_{\text{Remainder Term}}$$

We can approximate $R_n(x)$ to

$$\boxed{R_n(x) = \frac{f^{(n+1)}(\xi)\,(x-x_0)^{n+1}}{(n+1)!}}$$

where ξ lies between x and x_0

- eg - take n upto 0.

$$\therefore f(x) \approx f(x_0) + (x-x_0) f'(\xi)$$

which is LMVT

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• $\lim_{n\to\infty} R_n(x) = 0$

$\Rightarrow \lim_{n\to\infty} P_n(x) = f(x)$

imp Q→ calculate value of 'e' accurate to 7 decimal places. Find n for P_n.

Ans. $|\text{exact} - \text{approx}| < 10^{-7}$

$\Rightarrow |f(x) - P_n(x)| < 10^{-7}$

$\Rightarrow |R_n(x)| < 10^{-7}$

$\Rightarrow \left| \frac{(x)^{n+1} f'(\varepsilon_1)}{(n+1)!} \right| < 10^{-7}$

$\Rightarrow \left| \left(\frac{f'(\varepsilon_1)}{(n+1)!} \right) \right| < 10^{-7}$

$f'(\varepsilon_1) = e^{\varepsilon_1}$

maxima in $[0,1]$ is at 1

ie $f'(\varepsilon_1) = e$

we know $e < 3$ for sure $\therefore$ 3 is upper bound for $f'(\varepsilon_1)$

$\therefore \left| \frac{3}{(n+1)!} \right| < 10^{-7}$

DARBOUX INTEGRAL →

- Let f be bounded on $[a,b]$ on any set S
- ~~Also,~~

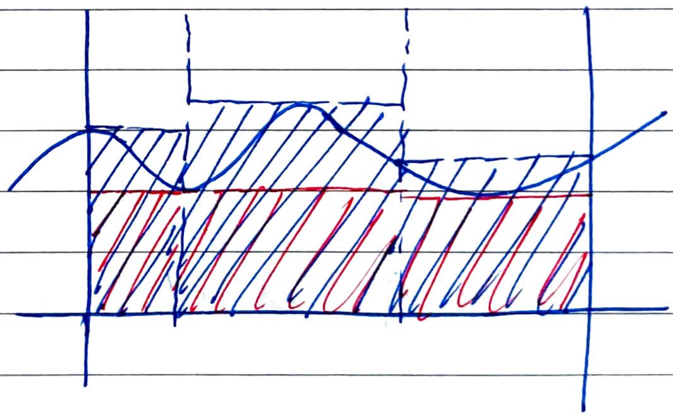
$$M(f, S) = \sup\{f(x) : x \in S\}$$

$$m(f, S) = \inf\{f(x) : x \in S\}$$

- We now partition ~~domain~~ $[a,b]$ in parts like

$$\text{Partition} = \{a = t_0 < t_1 < t_2 < \cdots < t_n = b\}$$

of $[a,b]$

- 

UPPER DARBOUX SUM

$$U(f, P) = \sum_{k=1}^{n} M(f, [t_{k-1}, t_k])\,(t_k - t_{k-1})$$

* This represents sum of areas under supremas in the intervals

LOWER DARBOUX SUM $\quad L(f,P) = \sum_{k=1}^{M} m(f,[t_{k-1},t_k])\,(t_k - t_{k-1})$

* This represents sum of areas under infimas in the intervals

we can say

$$L(f,P) < \int_a^b f\,dx < U(f,P)$$

$$\therefore \quad \boxed{U(f) = \inf U(f,P) = \overline{\int_a^b} f}$$

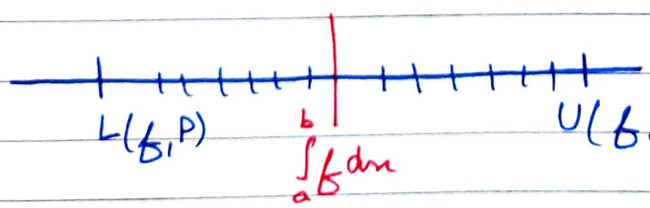
$$\boxed{L(f) = \sup L(f,P) = \underline{\int_a^b} f}$$

Here we make U indep of P

- when no of partitions increase both $U(f,P)$ & $L(f,P)$ will tend towards the integral. $\therefore$ for many such cases,

$U(f)$ would be minimum of such upper darboux sums (ie when we vary P)

Same case for $L(f)$



· Consider

$$Q > P$$

$$\therefore \quad U(f,P) \geq U(f,Q)$$

$$L(f,P) < L(f,Q)$$

$$\therefore \quad \boxed{L(f,P) \leq L(f,Q) \leq U(f,Q) \leq U(f,P)}$$