$$A = \begin{bmatrix} 1 & 4 \\ 2 & 8 \end{bmatrix}$$

$$A^T A = \begin{bmatrix} 1 & 2 \\ 4 & 8 \end{bmatrix} \begin{bmatrix} 1 & 4 \\ 2 & 8 \end{bmatrix}$$

$$= \begin{bmatrix} 5 & 20 \\ 20 & 80 \end{bmatrix}$$

$$(5-\lambda)(80-\lambda) - 20^2 = 0$$

$$\cancel{5 \times 80} - 5\lambda - 80\lambda + \lambda^2 - \cancel{20^2} = 0$$

$$\lambda(\lambda - 85) = 0$$

$$\lambda = 0, 85$$

$\underline{\underline{\lambda = 85}}$

$$\begin{bmatrix} 5-85 & 20 \\ 20 & 80-85 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

$$\Rightarrow \begin{bmatrix} -80 & 20 \\ 20 & -5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

$$20x_1 - 5x_2 = 0$$

$\Rightarrow 4x_1 = x_2$ so $\begin{bmatrix} 1 \\ 4 \end{bmatrix}$ eig vect

$\underline{\underline{\lambda = 0}}$

$$\begin{bmatrix} 5 & 20 \\ 20 & 80 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

$\Rightarrow \begin{bmatrix} -4 \\ 1 \end{bmatrix}$ eigvect

$$\text{So, } V = \begin{array}{c} \begin{bmatrix} 1/\sqrt{17} & -4/\sqrt{17} \\ 4/\sqrt{17} & 1/\sqrt{17} \end{bmatrix} \\ \hookrightarrow \lambda = 85 \qquad \hookrightarrow \lambda = 0 \end{array}$$

(Nullspace) of A

$$\text{Also, } \Sigma = \begin{array}{c} \lambda = 85 \\ \begin{bmatrix} \sqrt{85} & 0 \\ 0 & \sqrt{0} \end{bmatrix} \\ \hookrightarrow \lambda = 0 \end{array}$$

Be careful with ordering of λ & its corresponding eigvector in V

$\sigma_1 u_1 = A v_1$

$\rightarrow u_1 = \dfrac{A v_1}{\sigma_1}$

$$= \begin{bmatrix} 1 & 4 \\ 2 & 8 \end{bmatrix} \begin{bmatrix} 1/\sqrt{17} \\ 4/\sqrt{17} \end{bmatrix} \frac{1}{\sqrt{85}}$$

$$= \begin{bmatrix} 17/\sqrt{17} \\ 34/\sqrt{17} \end{bmatrix} \frac{1}{\sqrt{85}} = \begin{bmatrix} 17/17\sqrt{5} \\ 34/17\sqrt{5} \end{bmatrix} = \begin{bmatrix} 1/\sqrt{5} \\ 2/\sqrt{5} \end{bmatrix}$$

$\sigma_2 u_2 = A v_2$ → this is valid for all vectors, bcz $Av_2 = 0$ & $\sigma_2 = 0$ so, equality always holds

$\Rightarrow u_2 = \dfrac{A v_2}{\sigma_2} = \begin{bmatrix} 1 & 4 \\ 2 & 8 \end{bmatrix} \begin{bmatrix} -4/\sqrt{17} \\ 1/\sqrt{17} \end{bmatrix} \dfrac{1}{0}$ → cant do like this

We have flexibity in choosing u_2

~~We can put any~~ → So, i will choose any u_2 such that ~~not~~ u_1 & u_2 are orthogonal. Moreover matrix $U = \begin{bmatrix} | & | \\ u_1 & u_2 \\ | & | \end{bmatrix}$ should be orthonormal i.e $U^T U = U U^T = I$

Lets choose $u_2 = \begin{bmatrix} -2/\sqrt{5} \\ 1/\sqrt{5} \end{bmatrix}$.

So, $U = \begin{bmatrix} | & | \\ u_1 & u_2 \\ | & | \end{bmatrix} = \begin{bmatrix} 1/\sqrt{5} & -2/\sqrt{5} \\ 2/\sqrt{5} & 1/\sqrt{5} \end{bmatrix}$

So, $A = U \Sigma V^T$

$$= \begin{bmatrix} 1/\sqrt{5} & -2/\sqrt{5} \\ 2/\sqrt{5} & 1/\sqrt{5} \end{bmatrix} \begin{bmatrix} \sqrt{85} & 0 \\ 0 & \sqrt{0} \end{bmatrix} \begin{bmatrix} 1/\sqrt{17} & 4/\sqrt{17} \\ -4/\sqrt{17} & 1/\sqrt{17} \end{bmatrix}$$

Check

$$\begin{bmatrix} \sqrt{85}/\sqrt{5} & 0 \\ 2\sqrt{85}/\sqrt{5} & 0 \end{bmatrix} \begin{bmatrix} 1/\sqrt{17} & 4/\sqrt{17} \\ -4/\sqrt{17} & 1/\sqrt{17} \end{bmatrix} = \begin{bmatrix} \sqrt{85}/\sqrt{85} & 4\sqrt{85}/\sqrt{85} \\ 2\sqrt{85}/\sqrt{85} & 8\sqrt{85}/\sqrt{85} \end{bmatrix}$$

→ equals A

360

Ques 2 $A = \begin{bmatrix} 4 & 3 \\ 8 & 6 \end{bmatrix}$

$$A^T A = \begin{bmatrix} 4 & 8 \\ 3 & 6 \end{bmatrix}\begin{bmatrix} 4 & 3 \\ 8 & 6 \end{bmatrix} = \begin{bmatrix} 80 & 60 \\ 60 & 45 \end{bmatrix}$$

$$\cancel{(20-\lambda)(45-\lambda) - 24^2 = 0}$$

$$\cancel{20\times 45 - 20\lambda - 45\lambda + \lambda^2 - 24^2 = 0}$$

$$(80-\lambda)(45-\lambda) - 60^2 = 0$$

$$80\times 45 - 80\lambda - 45\lambda + \lambda^2 - 60^2 = 0$$

$$\lambda^2 - 125\lambda = 0$$

$$\lambda = 0, 125$$

$\underline{\lambda = 125}$

$$\begin{bmatrix} -45 & 60 \\ 60 & -80 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

$\Rightarrow -45x_1 + 60x_2 = 0$

$\Rightarrow -3x_1 + 4x_2 = 0$

$\Rightarrow 3x_1 = 4x_2$

So, eig vect is $\begin{bmatrix} 4 \\ 3 \end{bmatrix}$

$\underline{\lambda = 0}$

$$\begin{bmatrix} 80 & 60 \\ 60 & 45 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

eigvect is $\begin{bmatrix} -3 \\ 4 \end{bmatrix}$

$$\text{So, } V = \begin{array}{c} \text{for } \lambda=125 \quad \lambda = 0 \\ \begin{bmatrix} 4/5 & -3/5 \\ 3/5 & 4/5 \end{bmatrix} \end{array}$$

Be careful about this ordering. Now Σ will be $\begin{bmatrix} \sqrt{125} & \\ & \sqrt{0} \end{bmatrix}$ & not $\begin{bmatrix} \sqrt{0} & 0 \\ 0 & \sqrt{125} \end{bmatrix}$

Similarly u_1 will be vector corresponding to $\lambda = 125$ & u_2 corresp. to $\lambda = 0$

$u_1 = \frac{Av_1}{\sigma_1} = \begin{bmatrix} 4 & 3 \\ 8 & 6 \end{bmatrix} \begin{bmatrix} 4/5 \\ 3/5 \end{bmatrix} \frac{1}{\sqrt{125}}$

$= \begin{bmatrix} 5 \\ 10 \end{bmatrix} \frac{1}{\sqrt{125}} = \begin{bmatrix} 1/\sqrt{5} \\ 2/\sqrt{5} \end{bmatrix}$

$\sigma_2 u_2 = Av_2 \rightarrow$ always holds b/z $\sigma_2 = 0$ & Av_2 is also zero.

This means we have flexibility in choosing u_2. Lets choose $u_2 = \begin{bmatrix} -2/\sqrt{5} \\ 1/\sqrt{5} \end{bmatrix}$

(You may have a doubt here: we could have also choosen $u_2 = \begin{bmatrix} 2/\sqrt{5} \\ -1/\sqrt{5} \end{bmatrix}$. But note that

then $U = \begin{bmatrix} 1/\sqrt{5} & 2/\sqrt{5} \\ 2/\sqrt{5} & -1/\sqrt{5} \end{bmatrix} \rightarrow$ its columns are perpendicular but its rows are not. And ~~he be~~ for U to be orthornal (wikipedia) both rows & columns should be perpendicular i.e. $UU^T = U^TU = I$.)

So, $U = \begin{bmatrix} 1/\sqrt{5} & -2/\sqrt{5} \\ 2/\sqrt{5} & 1/\sqrt{5} \end{bmatrix}$

corresponding to $\lambda = 125$

$\lambda = 0$ } Being careful with the ordering

$\therefore A = \begin{bmatrix} 1/\sqrt{5} & -2/\sqrt{5} \\ 2/\sqrt{5} & 1/\sqrt{5} \end{bmatrix} \begin{bmatrix} \sqrt{125} & 0 \\ 0 & \sqrt{0} \end{bmatrix} \begin{bmatrix} 4/5 & 3/5 \\ -3/5 & 4/5 \end{bmatrix}$

<u>Check</u>

$\begin{bmatrix} 1/\sqrt{5} & 0 \\ 2/\sqrt{5} & 0 \end{bmatrix} \begin{bmatrix} 4/5 & 3/5 \\ -3/5 & 4/5 \end{bmatrix} = \cancel{4/5}$

Sorry no space left. But since you are hero you can check it's correct