Ayush Sahni

Page - 1

(1) $Z = 3X - 2Y$

$E[Z] = E[3X - 2Y] = 3E[X] - 2E[Y]$

$= 3(1) - 2(4) = 3 - 8 = -5$

, by linearity of expectation

$Var[Z] = Var[3X - 2Y]$

$= Var[3X] - Var[2Y]$, because X & Y are independent.

$= 9 Var[X] - 4 Var[Y]$

$= 9(4) - 4(6)$

$= 36 - 24 = 12$

(2)

(a) $E[S] = E[a_1 r_1 + \dots + a_n r_n]$

$= E[a_1 r_1] + \dots + E[a_n r_n]$, by linearity of expectation

$= a_1 E[r_1] + \dots + a_n E[r_n]$

$= a_1 \left(1\left(\frac{1}{2}\right) + (-1)\left(\frac{1}{2}\right) \right) + \dots + a_n \left(1\left(\frac{1}{2}\right) + (-1)\frac{1}{2} \right)$

$= a_1(0) + \dots + a_n(0)$

$= 0$

(b) $Var[S] = Var[a_1 r_1 + \dots + a_n r_n]$ — (i)

$= a_1^2 Var[r_1] + \dots + a_n^2 Var[r_n]$

, bcz $r_i \ \forall i \in 1$ to n are independent.

$$Var[r_i] = E[r_i^2] - (E[r_i])^2$$
$$= \left((1)^2\frac{1}{2} + (-1)^2\frac{1}{2}\right) - (0)^2$$
$$= \frac{1}{2} + \frac{1}{2} = 1$$

from eqn. (1) :-

$$Var[S] = a_1^2(1) + \dots + a_n^2(1)$$
$$= a_1^2 + a_2^2 + \dots + a_n^2$$

(4) (a) $E[(X_1 - \mu_1) \mid X_2 = x_2]$
$$= \sum_{x_1} (x_1 - \mu_1) P(X_1 = x_1 \mid X_2 = x_2)$$
$$= \sum_{x_1} x_1 P(X_1 = x_1 \mid X_2 = x_2) - \mu_1$$

But given that $E[(X_1 - \mu_1) \mid X_2 = x_2] = a + bx_2$

Taking expectation of both sides :-

$$E[E[(X_1 - \mu_1) \mid X_2 = x_2]] = E[a + bx_2] = a + b\mu_2$$
$$\Rightarrow E\left[\sum_{x_1} x_1 P(X_1 = x_1 \mid X_2 = x_2) - \mu_1\right] = a + b\mu_2$$
$$\Rightarrow E\left[\sum_{x_1} x_1 P(X_1 = x_1 \mid X_2 = x_2)\right] - E[\mu_1] = a + b\mu_2$$
$$\Rightarrow \mu_1 - \mu_1 = a + b\mu_2$$
$$\Rightarrow a + b\mu_2 = 0$$

Page-3

(3) ~~$Var[Y]$~~

~~$Var[Y] = Var[a_1X_1 + \dots + a_nX_n]$~~

~~$Var[Y]$~~

$$Var[Y] = E[Y^2] - (E[Y])^2 \quad -(1)$$

$$E[Y] = E[a_1X_1 + \dots + a_nX_n]$$
$$= a_1E[X_1] + \dots + a_nE[X_n]$$

$$= \begin{bmatrix} a_1 & a_2 & \dots & a_n \end{bmatrix} \begin{bmatrix} E[X_1] \\ E[X_2] \\ \vdots \\ E[X_n] \end{bmatrix}$$

$$E[Y^2] = E[(a_1X_1 + \dots + a_nX_n)^2]$$
$$= E[a_1^2X_1^2 + \dots + a_n^2X_n^2] + \sum_j\sum_i Cov(X_i, X_j)$$
$$= a_1^2E[X_1^2] + \dots + a_n^2E[X_n^2] + \sum_j\sum_i Cov(X_i, X_j)$$

Page-4

Ques 3 continued

From eqn ①

$$Var[Y] = a_1^2 E[X_1^2] + \dots + a_n^2 E[X_n^2] + \sum_j \sum_i Cov(X_i, X_j)$$

$$- (a_1 E[X_1] + \dots + a_n E[X_n])^2$$

$$= a_1^2 E[X_1^2] + \dots + a_n^2 E[X_n^2] + \sum_j \sum_i Cov(X_i, X_j)$$

$$- a_1^2 E[X_1]^2 - \dots - a_n^2 E[X_n]^2 - \sum_j \sum_i E[X_i] E[X_j]$$

$$- \sum_j \sum_i \sum_k E[X_i] E[X_j] E[X_k]$$

$$\dots - \sum \sum \sum \dots \sum (n \text{ times}) E[X_i] \dots E[X_n]$$

$$= a_1^2 [E[X_1^2] - E[X_1]^2] + \dots + a_n^2 [E[X_n^2] - E[X_n]^2]$$

$$+ \sum_j \sum_i Cov(X_i, X_j) - \sum_j \sum_i E[X_i] E[X_j]$$

$$\dots - \sum \sum \dots n \text{ times } E[X_i] \dots E[X_n]$$

$$= a_1^2 Var[X_1] + \dots + a_n^2 Var[X_n]$$

$$+ \sum_j \sum_i Cov(X_i, X_j) - \sum_j \sum_i E[X_i] E[X_j]$$

$$\dots - \sum \sum \dots n \text{ times } E[X_i] \dots E[X_n]$$

Ques 3 continued

Similarly,

$$Var[Y] = b_1^2 Var[X_1] + \dots + b_n^2 Var[X_n]$$

$$+ \sum_j \sum_i Cov(X_i, X_j) - \sum_j \sum_i E[X_j]E[X_i]$$

$$\dots - \sum\sum \dots n \text{ times } E[X_i] \dots E[X_n]$$

(2)

from (1) & (2)

$$Var[Y] - b_1^2 Var[X_1] + \dots + b_n^2 Var[X_n]$$

$$- \sum_j \sum_i Cov(X_i, X_j)$$

$$= a_1^2 Var[X_1] + \dots + a_n^2 Var[X_n]$$

$$+ \sum_j \sum_i Cov(X_i, X_j)$$

$$- Var[Y]$$