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Problem 1

(a) Given,

$$\text{Area}(\Delta(0,a,b)) = \frac{1}{2}\|a\| \min_{\alpha}\|b-\alpha a\|$$

Lets assume the α which minimises $\|b-\alpha a\|$ be α^*.

(Also, we know $\alpha^* = \dfrac{a^T b}{\|a\|^2}$)

Similarly, $\text{Area}(\Delta(0,Sa,Sb)) = \frac{1}{2}\|Sa\| \, \|Sb - \hat{\alpha}(Sa)\|$

{ where $\hat{\alpha}$ is the α which minimises $\|Sb-\alpha(Sa)\|$ }

Then,

$\text{area}(\Delta(0,Sa,Sb)) - \text{area}(\Delta(0,a,b))$

$$= \frac{1}{2}\|Sa\| \, \|Sb-\hat{\alpha}Sa\| - \frac{1}{2}\|a\| \, \|b-\alpha^* a\|$$

Taking absolute value of both sides :-

$$\boxed{\begin{aligned}&\left|\text{area}(\Delta(0,Sa,Sb)) - \text{area}(\Delta(0,a,b))\right| \\ &= \left|\frac{1}{2}\|Sa\| \, \|Sb-\hat{\alpha}Sa\| - \frac{1}{2}\|a\| . \|b-\alpha^* a\|\right| \quad -(1)\end{aligned}}$$

Variation in $\|Sa\|$ as compared to $\|a\|$:-

from JL lemma :-

$\|Sa\|^2 \in (1\pm\varepsilon)\|a\|^2$ { it holds with high probability }

Taking ~~seq~~ square root of both sides:-

$\|Sa\| \in \sqrt{(1+\varepsilon)}\,\|a\|$

Let us find only the upper bound on $\|Sa\|$. so,

$\|Sa\| \leq \sqrt{(1+\varepsilon)}\,\|a\|$ — (2)

$\sqrt{1+\varepsilon} = 1 + O(\varepsilon)$

Proof: Using Taylor series expansion:-

$$(1+\varepsilon)^{1/2} = 1 + \varepsilon f'(0) + \frac{\varepsilon^2}{2!} f''(y) \quad , y \in (0, \varepsilon)$$

$$= 1 + \frac{1}{2}\varepsilon(1+0)^{-1/2} + \frac{\varepsilon^2}{2!}\left(-\frac{1}{4}\right)\frac{1}{(1+y)^{3/2}}$$

By finding bound on $(1+y)^{3/2}$ it can be found that $(1+\varepsilon)^{1/2} < 1 + \frac{3}{4}\varepsilon$

Putting this result in ~~eqn~~ relation (2):-

$\|Sa\| \leq (1 + O(\varepsilon))\,\|a\|$ — (4) {~~Now, I will directly use this result for~~ other vectors also other than a}

Finding upper bound on $\|Sb - \hat{\alpha} Sa\|$

from JL Lemma:-

For every α:-

$$(1 - O(\varepsilon))\|b - \alpha a\| < \underbrace{\|S(b - \alpha a)\| \leq (1 + O(\varepsilon))\|b - \alpha a\|}_{\text{~~we will use~~ only this part because we need upper bound only}}$$

So, it also holds for $\hat{\alpha}$:-

$\|S(b - \hat{\alpha} a)\| \leq (1 + O(\varepsilon))\|b - \hat{\alpha} a\|$ — (3)

from JL lemma:-

$$(1-\varepsilon)\|b-\hat{\alpha}a\| \le \|S(b-\hat{\alpha}a)\|$$

Since $\hat{\alpha}$ ~~is~~ minimises $\|S(b-\hat{\alpha}a)\|$, so,

$$\|S(b-\hat{\alpha}a)\| \le \|S(b-\alpha^* a)\|$$

$$\Rightarrow \|S(b-\hat{\alpha}a)\| \le (1+\varepsilon)\|b-\alpha^* a\|$$

{we use JL lemma in this step:- $\|S(b-\alpha^* a)\| \le (1+\varepsilon)\|b-\alpha^* a\|$ }

Combining these two:-

$$(1-\varepsilon)\|b-\hat{\alpha}a\| \le (1+\varepsilon)\|b-\alpha^* a\|$$

$$\Rightarrow \|b-\hat{\alpha}a\| \le \left(\frac{1+\varepsilon}{1-\varepsilon}\right)\|b-\alpha^* a\|$$

Putting this result in ~~eqn~~ inequality (3):-

$$\|S(b-\hat{\alpha}a)\| \le (1+O(\varepsilon))\left(\frac{1+\varepsilon}{1-\varepsilon}\right)\|b-\alpha^* a\| \quad -(5)$$

Inequalities (4) & (5) give upper bounds on the value of $\|Sa\|$ & $\|S(b-\hat{\alpha}a)\|$ respectively.

from eqn (1):-

$$|\text{area}(\Delta(0,Sa,Sb)) - \text{area}(\Delta(0,a,b))|$$

$$= \left|\frac{1}{2}\|Sa\|\,\|S(b-\hat{\alpha}a)\| - \frac{1}{2}\|a\|\,\|b-\alpha^* a\|\right|$$

$$\le \left|\frac{1}{2}(1+O(\varepsilon))\|a\|\,(1+O(\varepsilon))\left(\frac{1+\varepsilon}{1-\varepsilon}\right)\|b-\alpha^* a\| - \frac{1}{2}\|a\|\,\|b-\alpha^* a\|\right|$$

$$= \left|\frac{1}{2}\frac{(1+\varepsilon)^3}{1-\varepsilon}\|a\|\,\|b-\alpha^* a\| - \frac{1}{2}\|a\|\,\|b-\alpha^* a\|\right|$$

{$\because$ I have put $1+O(\varepsilon) = 1+\varepsilon$ for ease in calculation in next step & also because ε is small}

$$= \left| \left(\frac{(1+\varepsilon)^3}{(1-\varepsilon)} - 1 \right) \frac{1}{2} \|a\| \, \|b - \alpha^* a\| \right|$$

$$= \left| \frac{(1+\varepsilon)^3}{(1-\varepsilon)} - 1 \right| \frac{1}{2} \|a\| \, \|b - \alpha^* a\|$$

$$\frac{(1+\varepsilon)^3}{(1-\varepsilon)} = (1+\varepsilon)^3 (1 + \varepsilon + \varepsilon^2 + \dots) = 1 + O(\varepsilon)$$

So, $\left| \text{area}(\Delta(0, Sa, Sb)) - \text{area}(\Delta(0,a,b)) \right|$

$$\leq |1 + O(\varepsilon) - 1| \; \text{area}(\Delta(0,a,b))$$

$$\leq O(\varepsilon) \; \text{area}(\Delta(0,a,b))$$ {assuming ε is positive}

The conditions under which this property holds is already mentioned ~~it~~ during the arguments of proof.

(b) In the above property, ~~the~~ I have shown that norm of a vector is approximately preserved ~~of~~ before & after multiplying with random normal matrix S. of dimension $m \times n$. Here I will show why that is true with $m = O(\varepsilon^{-2} \log 1/\delta)$.

$$\underset{m \times n}{\underbrace{\begin{bmatrix} s_{11} & s_{12} & \cdots & s_{1n} \\ \vdots & & \cdots & \vdots \\ s_{m1} & s_{m2} & \cdots & s_{mn} \end{bmatrix}}}_{S} \; \underset{n \times 1}{\underbrace{\begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}}}_{A} = \underset{m \times 1}{\underbrace{\begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{bmatrix}}}_{Y}$$

$m \times 1 \downarrow$ vector in lower dimension

$s_{ij} \sim N(0, 1)$ $\forall i = 1, 2, \dots m$ $\forall j = 1, 2, \dots n$

{Without loss of generality, I have assumed A to be a unit vector}

I will show that square norm of Y is approximately preserved (and hence norm is also preserved) i.e. $\|Y\|^2 \in (1 \pm \varepsilon)\|A\|^2$

Notice that $\|Y\|^2$ follows χ^2 distribution. It is because:-

$$\|Y\|^2 = y_1^2 + y_2^2 + \dots + y_m^2$$
$$= (S_{11}a_1 + S_{12}a_2 + \dots + S_{1n}a_n)^2 + (S_{21}a_1 + \dots + S_{2n}a_n)^2 + \dots + (S_{m1}a_1 + \dots + S_{mn}a_n)^2$$

Each term is standard normal R.V. $\sim N(0,1)$

Sum of squares of standard normal R.V is χ^2 distribution. $E[\|Y\|^2] = m$ (degrees of freedom)

$Var(\|Y\|^2) = 2m$.

By concentration property of χ^2 distribution:-

$$P(|\|Y\|^2 - m| > t) \leq 2e^{-t^2/8m}, \quad t < 2m$$

$\Rightarrow$ Let $t = \varepsilon m$, $0 < \varepsilon < 1$ is accuracy parameter

$$\Rightarrow P(|\|Y\|^2 - m| > \varepsilon m) \leq 2e^{-\varepsilon^2 m/8}$$

divide by m:-

$$\Rightarrow P\left(\left|\frac{\|Y\|^2}{m} - 1\right| > \varepsilon\right) \leq 2e^{-\varepsilon^2 m/8} \leq \delta$$ {δ = error probability}

For this to hold:-

$$2e^{-\varepsilon^2 m/8} \leq \delta$$ {also $\delta < 1$}

$$\Rightarrow -\frac{\varepsilon^2 m}{8} \leq \log\frac{\delta}{2}$$

$$\Rightarrow \boxed{m \geq \frac{8}{\varepsilon^2}\log\frac{2}{\delta}} \quad \text{or} \quad \boxed{m \doteq O(\varepsilon^{-2}\log 1/\delta)}$$

Even though I have preserved two norms $\|a\|$ & $\|b - \alpha a\|$ but within $(1 \pm \varepsilon)$ I have approx. preserved $\|a\|$ only then according approx preserved ~~preserved~~ $\|b - \alpha a\|$ in terms of that ε. So $m \doteq O(\varepsilon^{-2}\log 1/\delta)$ is sufficient for Problem-1 (a) property to hold.

$m = O\left(\frac{\log 1/\delta}{\varepsilon^2}\right)$ can also be shown by union bound method.

Let E_1 be fail probability of ~~pres~~ approx-preservance of $\|Sa\|$. Then

$E_1 = \|Sa\| \notin (1 \pm \varepsilon)\|a\|$

$P(E_1) \leq \delta$

Let E_2 be $\|S(b - \alpha a)\| \notin (1 \pm \varepsilon)\|b - \alpha a\|$

$P(E_2) \leq \delta$

$P(E_1 \cup E_2) \leq P(E_1) + P(E_2) \leq \delta + \delta = 2\delta$

$\Rightarrow P(\overline{\overline{E_1} \cap \overline{E_2}}) \leq 2\delta$

$\Rightarrow 1 - P(\overline{E_1} \cap \overline{E_2}) \leq 2\delta$

$\Rightarrow \underbrace{P(\overline{E_1} \cap \overline{E_2})}_{} \geq 1 - 2\delta$ → Probability that both satisfy approx norm preservance

putting $\delta = \frac{\delta}{2}$

$\Rightarrow P(\overline{E_1} \cap \overline{E_2}) \geq 1 - \delta$

so, replace $\delta = \delta/2$ in $m \geq \frac{8}{\varepsilon^2} \log \frac{2}{\delta/2} = \frac{8}{\varepsilon^2} \log \frac{4}{\delta}$

$\Rightarrow m = O\left(\varepsilon^{-2} \log \frac{1}{\delta}\right)$

Problem-2

(a) $\left| \mathrm{vol}(0, Sa, Sb, Sc) - \mathrm{vol}(0, a, b, c) \right|$

$= \left| \cancel{\|a\| \, \|b - \alpha^* a\| \, \|c - \beta^*} \right.$

$= \left| \, \|Sa\| \, \|Sb - \hat{\alpha}(Sa)\| \, \|Sc - \hat{\beta}\, SA\| - \|a\| \, \|b - \alpha^* a\| \, \|c - \beta^* A\| \, \right|$

Just like problem-1, here $\hat{\alpha}$ & $\hat{\beta}$ minimise $\|Sb - \alpha Sa\|$ & $\|Sc - \beta SA\|$ respectively and α^* & β^* minimise $\|b - \alpha a\|$ & $\|c - \beta A\|$ respectively

I have already calculated the upper bounds for $\|Sa\|$ & $\|S(b - \hat{\alpha} a)\|$. So I will directly replace them here.

So, $\left| \mathrm{vol}(0, Sa, Sb, Sc) - \mathrm{vol}(0, a, b, c) \right|$

$$\leq \left| \left((1 + O(\epsilon)) \|a\| \right) \left((1 + O(\epsilon)) \frac{(1+\epsilon)}{(1-\epsilon)} \|b - \alpha^* a\| \right) \left((1 + O(\epsilon)) \frac{(1+\epsilon)}{(1-\epsilon)} \|c - \beta^* A\| \right) \right.$$

$$\left. - \|a\| \, \|b - \alpha^* a\| \, \|c - \beta^* A\| \right|$$

$$= \left| \frac{(1 + O(\epsilon))^3 (1+\epsilon)^2}{(1-\epsilon)^2} \|a\| \, \|b - \alpha^* a\| \, \|c - \beta^* A\| \right.$$

$$\left. - \|a\| \, \|b - \alpha^* a\| \, \|c - \beta^* A\| \right|$$

$$= \left| \left(\frac{(1+\epsilon)^5}{(1-\epsilon)^2} - 1 \right) \|a\| \, \|b - \alpha^* a\| \, \|c - \beta^* A\| \right|$$

{putting $1 + O(\epsilon) = 1 + \epsilon$ ~~for ease in~~ calculation}

$(1+\varepsilon)^5(1+\varepsilon+\varepsilon^2+\dots)$

$= (1+\varepsilon)(1+\varepsilon)(1+\varepsilon)(1+\varepsilon)(1+\varepsilon)(1+\varepsilon+\varepsilon^2+\dots)$

neglecting ε^2 & higher power terms

$= 1 + O(\varepsilon)$

So,

$$\boxed{\begin{aligned} &|\mathrm{vol}(0, Sa, Sb, Sc) - \mathrm{vol}(0,a,b,c)| \\ &= |(1+O(\varepsilon) - 1)|\, \|a\|\, \|b-\alpha^* a\|\, \|c-\beta^* A\| \\ &= O(\varepsilon)\, \|a\|\, \|b-\alpha^* a\|\, \|c-\beta^* A\| \quad \{\text{since } \varepsilon \text{ is positive}\} \\ &= O(\varepsilon)\, \mathrm{vol}(0,a,b,c) \end{aligned}}$$

(b) Similar to problem-1, here also the use of $m \times n$ random normal matrix S is that has preserved these norms:-

$$\|Sa\| \in (1 \pm O(\varepsilon))\|a\|$$
$$\|Sb - \alpha Sa\| \in (1 + O(\varepsilon))\|b - \alpha a\|$$
$$\|Sc - \beta SA\| \in (1 + O(\varepsilon))\|c - \beta A\|$$

<u>Union-bound</u>

Let E_1, E_2 & E_3 be failing probabilities of the three above conditions.

$P(E_1) \leq \delta \qquad P(E_2) \leq \delta \qquad P(E_3) \leq \delta$

$\Rightarrow P(E_1 \cup E_2 \cup E_3) \leq P(E_1) + P(E_2) + P(E_3) = 3\delta$

None of three fails (i.e. all three satisfy) :-

$P(\bar{E}_1 \cap \bar{E}_2 \cap \bar{E}_3) = 1 - 3\delta$

replace $\delta \to \frac{\delta}{3}$

$P(\bar{E}_1 \cap \bar{E}_2 \cap \bar{E}_3) = 1 - 3\left(\frac{\delta}{3}\right) = 1 - \delta$

Accordingly. replace $\delta \to \frac{\delta}{3}$ in m.

$$m = \frac{8}{\varepsilon^2} \log \frac{2}{\delta} = \frac{8}{\varepsilon^2} \log \frac{2}{\delta/3} = \frac{8}{\varepsilon^2} \log \frac{6}{\delta}$$

$$\Rightarrow m = \cancel{\Theta(\log} \; O\left(\varepsilon^{-2} \log \frac{1}{\delta}\right)$$

Problem-3

I have shown the formal arguments for two & three vectors. ~~The~~ But in this, I will first ~~show~~ present an intuitive argument.

~~We~~ If S is a random normal $m \times n$ matrix and a is a $n \times 1$ vector, then norm of a is approx. preserved with high probability before and after multiplication. i.e.

$$P\left(\left| \|Sa\|^2 - \|a\|^2 \right| > \varepsilon \|a\|^2 \right) < \delta$$

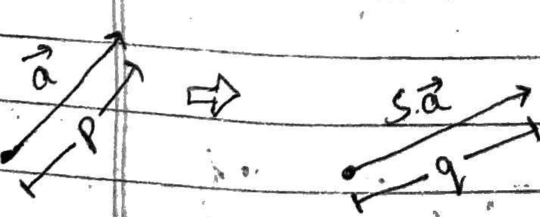
error probability (δ is small)

Also, if A is a ~~an~~ matrix having several $n \times 1$ vectors, then the angles are also approx. preserved before & after embedding.

$$\underset{S}{\begin{bmatrix} & & & \\ & & & \end{bmatrix}} \underset{A}{\begin{bmatrix} | & | & | \\ a_1 & a_2 & a_3 \\ | & | & | \end{bmatrix}} = \underset{Y}{\begin{bmatrix} | & | & | \\ y_1 & y_2 & y_3 \\ | & | & | \end{bmatrix}}$$

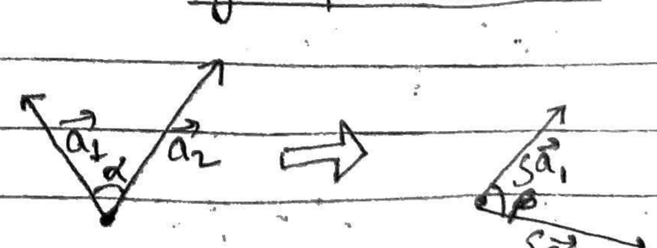
Angle between any two vectors in Y is approximately preserved as corresponding vectors in A with high probability.

Norm Preservation



p is nearly equal to q with high probability

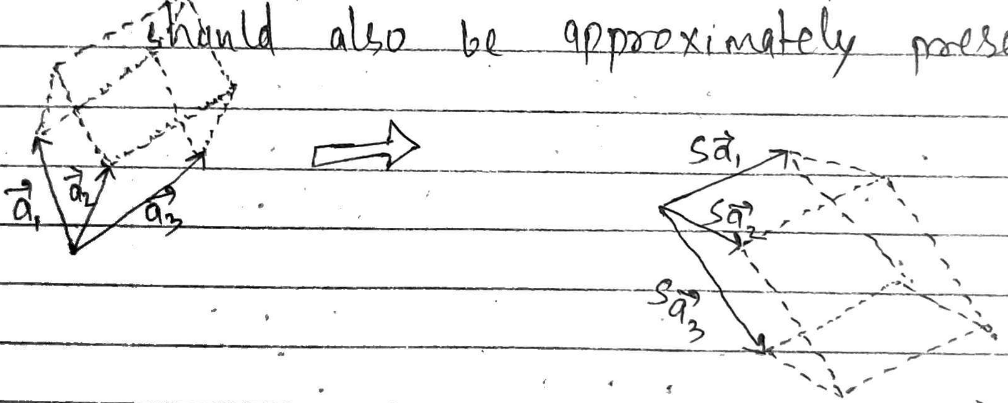
Angle preservation



α is nearly equal to β with high probability.

This means if there is a d-dimensional parallelopiped ~~formed~~ defined by vectors $a_1, a_2, \dots a_d$ and we put these vectors $a_1, a_2, \dots a_d$ in a matrix A and then we multiply with random matrix S to ~~its~~ left of A then the norms of each of $a_1, a_2, \dots, a_d$ will be approximately preserved and also the angles between them will be approximately preserved.

This shows that geometry is preserved (approx.) Hence, the volume enclosed within these vectors should also be approximately preserved.



After embedding the whole space has been embedded into another space but geometry is approx. preserved. ~~(Hence volume is~~ also "somewhat" preserved)

Formal argument

$$\left| vol(0, a^1, \dots a^d) - vol(0, b^1, \dots, b^d) \right|$$

$$= \left| \|a^1\| \prod_{j=1}^{d-1} \|a^{j+1} - A^j \alpha^{j*}\| - \|b^1\| \prod_{j=1}^{d-1} \|b^{j+1} - A^j \hat{\alpha}^j\| \right|$$

$$\leq \left| \cancel{(1 \pm O(\epsilon))} \|a^1\| \prod_{j=1}^{d-1} \|a^{j+1} - A^j \alpha^{j*}\| - (1+\epsilon)\|a^1\| \prod_{j=1}^{d-1} (1+\epsilon) \|a^{j+1} - A^j \alpha^{j*}\| \right|$$

$$= \left| \|a^1\| \prod_{j=1}^{d-1} \|a^{j+1} - A^j \alpha^{j*}\| - (1+\epsilon)^d \|a^1\| \|a^{j+1} - A^j \alpha^{j*}\| \right|$$

$$= \left| 1-(1+\varepsilon)^d \right| \, \| a^1 \| \prod_{j=1}^{d-1} \| a^{j+1} - A^j \alpha^{j*} \|$$

$$= \left| 1-1-O(\varepsilon) \right| \, vol(0, a^1, \ldots, a^d)$$

$$= O(\varepsilon) vol(0, a^1, \ldots, a^d)$$

Let $E_1, E_2, \ldots E_d$ be fail events of approx norm preservation.

$P(E_1 \cup E_2 \ldots E_d) \leq d\delta$

$\Rightarrow P(\bar{E}_1 \cap \bar{E}_2 \ldots \bar{E}_d) \geq 1-d\delta$

replace $\delta \to \frac{\delta}{d}$

$P(\bar{E}_1 \cap \bar{E}_2 \ldots \bar{E}_d) \geq 1-\delta$

so, $m = \frac{8}{\varepsilon^2} \log \frac{2}{\delta/d} = \frac{8}{\varepsilon^2} \log \frac{2d}{\delta}$

$\Rightarrow m = O\left(\varepsilon^{-2} \log \frac{d}{\delta}\right)$