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Problem 1 (a and b)

consider Area $(\Delta(0,a,b)) = \frac{1}{2}\|a\| \min_{\alpha} \|b-\alpha a\|$

0 = origin
$a, b \in \mathbb{R}^n$
$\alpha \in \mathbb{R}$

subspace preserving embeddings to preserve a column space: let A be some arbitrary but fixed n by d orthonormal matrix of reals and S be a k by n matrix of iid $N(0,\frac{1}{k})$ distributed random variables. Then:

$$P\left[\forall x \in \mathbb{R}^d, \|SAx\|_2^2 \in (1 \pm \epsilon)\|Ax\|_2^2\right] \geq 1-\delta$$

for $k = O\left(\frac{d}{\epsilon^2}\log\frac{1}{\delta}\right)$.

and if

$$\min_x \|Ax\| = \|Ax^*\|$$

$$\min_x \|SAx\| = \|SA\hat{x}\|$$

then

$$(1-\epsilon)\|Ax^*\| \leq (1-\epsilon)\|A\hat{x}\| \leq \|SA\hat{x}\| \leq \|SAx^*\| \leq (1+\epsilon)\|Ax^*\|$$

in other words

$$\|SA\hat{x}\| \in (1 \pm \epsilon)\|Ax^*\|$$

or

$$\min_x \|SAx\| \in (1 \pm \epsilon)\min_x \|Ax\|$$

for our problem, consider $A = [b\ a]_{n\times 2}$
and x of the form $x = \begin{bmatrix} 1 \\ -\alpha \end{bmatrix}_{2\times 1}$

Then by argument of subspace embedding using the random matrix $S_{m\times n}$ of iid $N(0, \frac{1}{m})$, we have:

$$P\left[\forall \alpha \in \mathbb{R},\ \|SAx\|_2^2 \in (1\pm\epsilon)\|Ax\|_2^2\right] \geq 1-\delta$$

$$\text{for } m = O\left(\frac{d}{\epsilon^2}\log\frac{1}{\delta}\right) = O\left(\frac{2}{\epsilon^2}\log\frac{1}{\delta}\right)$$

or.

$$P\left[\min_{\alpha\in\mathbb{R}} \|SAx\|_2^2 \in (1\pm\epsilon)\min_{\alpha\in\mathbb{R}}\|Ax\|_2^2\right] \geq 1-\delta$$

$$P\left[\min_{\alpha\in\mathbb{R}} \|S(b-\alpha a)\|_2^2 \in (1\pm\epsilon)\min_{\alpha\in\mathbb{R}}\|b-\alpha a\|_2^2\right] \geq 1-\delta \quad \text{--- } ①$$

We also have by J-L Lemma using the same matrix S and any arbitrary fixed vector $a \in \mathbb{R}^n$.

$$P\left[\|Sa\|_2^2 \in (1\pm\epsilon)\|a\|_2^2\right] \geq 1-\delta \quad \text{--- } ②$$

$$\text{for } m = O\left(\frac{1}{\epsilon^2}\log\frac{1}{\delta}\right)$$

combining ① & ②

$$P\left[\|Sa\|_2^2 \in (1\pm\epsilon)\|a\|_2^2 \cap \min_{\alpha\in\mathbb{R}}\|S(b-\alpha a)\|_2^2 \in (1\pm\epsilon)\min_{\alpha\in\mathbb{R}}\|b-\alpha a\|_2^2\right]$$

$$\geq (1-\delta)^2$$

$$\geq 1-2\delta$$

So with $P \geq 1-2\delta$ and $m = O\left(\frac{1}{\epsilon^2}\log\frac{1}{\delta}\right)$ we have

$$\|Sa\|_2^2 \cdot \min_{\alpha \in \mathbb{R}} \|S(b-\alpha a)\|_2^2 \in (1 \pm \epsilon)^2 \|a\|_2^2 \cdot \min_{\alpha \in \mathbb{R}} \|b-\alpha a\|_2^2 \qquad (3)$$

$(1-\epsilon)^2 = 1+\epsilon^2-2\epsilon$ $\qquad$ $(1+\epsilon)^2 = 1+\epsilon^2+2\epsilon$

$(1-\epsilon)^2 \geq 1+\epsilon^2-2\epsilon-(\epsilon+\epsilon^2)$ $\qquad$ $(1+\epsilon)^2 \leq 1+\epsilon+2\epsilon$

$(1-\epsilon)^2 \geq 1-3\epsilon$ $\qquad$ $(1+\epsilon)^2 \leq 1+3\epsilon$

for $0<\epsilon<1$

$$1-3\epsilon \leq (1-\epsilon)^2 \leq (1+\epsilon)^2 \leq 1+3\epsilon$$

$$\therefore \quad (1 \pm \epsilon)^2 \sim 1 \pm O(\epsilon)$$

Also by taking square root of (3)

$$\|Sa\|_2 \cdot \min_{\alpha \in \mathbb{R}} \|S(b-\alpha a)\|_2 \in (1 \pm \epsilon) \|a\| \cdot \min_{\alpha \in \mathbb{R}} \|b-\alpha a\|_2$$

i.e.

$$\text{Area}(\Delta(0, Sa, Sb)) \in (1 \pm O(\epsilon))\, \text{Area}(\Delta(0,a,b))$$

with probability $1-\delta'$ $(\delta' = 2\delta)$ and $m = O\left(\frac{1}{\epsilon^2}\log\frac{2}{\delta'}\right)$

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Problem 2: (a and b)

$$vol\ (o,a,b,c) = \|a\| \cdot \min_{\alpha} \|b-\alpha a\| \cdot \min_{\beta} \|c-A\beta\|$$

$a, b, c \in \mathbb{R}^n$.
$\alpha \in \mathbb{R}$
$\beta \in \mathbb{R}^2$
$A = n\times 2 = [a \quad b]$

This can be written as :

$$vol(o,a,b,c) = \|a\| \cdot \min_{\alpha} \left\| \underset{n\times 2}{[b \quad a]} \underset{2\times 1}{\begin{bmatrix} 1 \\ -\alpha \end{bmatrix}} \right\| \min_{\beta} \left\| \underset{n\times 3}{[c \quad a \quad b]} \underset{3\times 1}{\begin{bmatrix} 1 \\ -\beta \end{bmatrix}} \right\|$$

To preserve the volume of parallelepiped, it is sufficient to preserve the ~~matrix [c a b]~~ column space of matrix $[c \quad a \quad b]$ or $[a \quad b \quad c]$

Then automatically the column space of $[b \quad a]$ and $[a]$ is preserved to within the same factor.

[Since column space of $[b \quad a]$ and $[a]$ are the subspaces of column space of $[a \quad b \quad c]$]

Let S be a $m\times n$ random matrix of iid $N(0,\frac{1}{m})$, then by the argument of subspace embedding, we have

$$P\left[\forall x \in \mathbb{R}^3, \|SYx\|_2^2 \in (1\pm\epsilon)\|Yx\|_2^2\right] \geq 1-\delta \quad (1)$$

$\left(Y = [a \quad b \quad c]_{n\times 3}\right)$. for $m = O\left(\frac{d}{\epsilon^2}\log\frac{1}{\delta}\right) = O\left(\frac{3}{\epsilon^2}\log\frac{1}{\delta}\right)$

① implies that [given ① holds]

$$P\left[\forall x' \in \mathbb{R}^2, \|SY'x'\|_2^2 \in (1\pm\epsilon)\|Y'x'\|_2^2\right] = 1$$

where $Y' = [a\ b]$

$$P\left[\|Sa\|_2^2 \in (1\pm\epsilon)\|a\|_2^2\right] = 1$$

Therefore :

(i) $$\|Sa\|_2^2 \in (1\pm\varepsilon)\|a\|_2^2$$

(ii) $$\min_{\alpha\in\mathbb{R}} \left\|S[b\ a]\begin{bmatrix}1\\-\alpha\end{bmatrix}\right\| \in (1\pm\varepsilon)\min_{\alpha\in\mathbb{R}}\left\|[b\ a]\begin{bmatrix}1\\-\alpha\end{bmatrix}\right\|$$

(iii) $$\min_{\beta\in\mathbb{R}^2} \left\|S[c\ a\ b]\begin{bmatrix}1\\-\beta\end{bmatrix}\right\| \in (1\pm\varepsilon)\min_{\beta\in\mathbb{R}^2}\left\|[c\ a\ b]\begin{bmatrix}1\\-\beta\end{bmatrix}\right\|$$

All these 3 conditions hold true with probability $1-\delta$ for $m = O\left(\frac{1}{\varepsilon^2}\log\frac{1}{\delta}\right)$

Therefore volume estimation of parallelepiped $[vol(0,a,b,c)]$ by using $0, Sa, Sb, Sc$:

$$[vol(0,Sa,Sb,Sc)]^2 = \|Sa\|_2^2 \cdot \min_\alpha \|S(b-\alpha a)\|_2^2 \min_\beta \|S(c-A\beta)\|_2^2$$

$$\in (1\pm\varepsilon)^3 \|a\|_2^2 \min_\alpha \|b-\alpha a\|_2^2 \min_\beta \|c-A\beta\|_2^2$$

$$\in (1\pm\varepsilon)^3 [vol(0,a,b,c)]^2$$

$$vol(0,Sa,Sb,Sc] \in (1\pm\varepsilon)^{3/2}\, vol(0,a,b,c).$$

for $\epsilon \in [0,\frac{1}{2}]$, we have $1-\frac{3}{4}\varepsilon \le (1\pm\varepsilon)^{1/2} \le 1+\frac{3}{4}\varepsilon$

$$(1-\varepsilon)\left(1-\frac{3}{4}\varepsilon\right) \leq (1 \pm \varepsilon)^{3/2} \leq (1+\varepsilon)\left(1+\frac{3\varepsilon}{4}\right).$$

$$\frac{3\varepsilon^2}{4} - \frac{7}{4}\varepsilon + 1 \leq (1 \pm \varepsilon)^{3/2} \leq \frac{3}{4}\varepsilon^2 + \frac{7}{4}\varepsilon + 1$$

$$\frac{3\varepsilon^2}{4} - \frac{7}{4}\varepsilon + 1 \geq \frac{3\varepsilon^2}{4} - \frac{7\varepsilon}{4} + 1 - \frac{3}{4}(\varepsilon + \varepsilon^2).$$

$$= 1 - \frac{10\varepsilon}{4} = 1 - \frac{5}{2}\varepsilon.$$

$$\frac{3\varepsilon^2}{4} + \frac{7}{4}\varepsilon + 1 \leq \frac{3\varepsilon}{4} + \frac{7\varepsilon}{4} + 1$$

$$= 1 + \frac{10\varepsilon}{4} = 1 + \frac{5}{2}\varepsilon.$$

$$1 - \frac{5}{2}\varepsilon \leq (1 \pm \varepsilon)^{3/2} \leq 1 + \frac{5}{2}\varepsilon \quad \text{(which holds for } \varepsilon \leq 1/20\text{)}$$

$\therefore$

$\text{vol}(0, Sa, Sb, Sc) \in (1 \pm O(\varepsilon))\, \text{vol}(0, a, b, c)$

with probability $1 - \delta$ and $m = O\left(\frac{1}{\varepsilon^2} \log \frac{1}{\delta}\right)$.

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Problem 3.

Volume of d-dimensional parallelopiped $a^1, a^2 \ldots a^d \in \mathbb{R}^n$

$$vol(0, a^1, a^2 \ldots a^d) = \|a^1\| \cdot \prod_{j=1}^{d-1} \min_{\alpha^j \in \mathbb{R}^j} \|a^{j+1} - A^j \alpha^j\|$$

$$A^j = [a^1 \; a^2 \ldots \; a^j]_{n \times j}$$

$$vol(0, a^1, a^2 \ldots a^d) = \|a^1\| \prod_{j=1}^{d-1} \min_{\alpha^j \in \mathbb{R}^j} \left\| [a^{j+1} \; a^1 \; a^2 \ldots a^j]_{n \times (j+1)} \begin{bmatrix} 1 \\ -\alpha^j \end{bmatrix}_{(j+1) \times 1} \right\|$$

let $M^j = [a^{j+1} \; a^1 \; a^2 \ldots a^j]$ $\quad x^j = [1 \; — (-\alpha^j) —]^T$

$$vol(0, a^1, a^2 \ldots a^d] = \|a^1\| \prod_{j=1}^{d-1} \min_{\alpha^j \in \mathbb{R}^j} \|M^j x^j\|$$

To preserve the volume, it is sufficient to preserve the largest [column space of] matrix M^j i.e. M^{d-1}

Since all other M^j has subset of columns of M^{d-1} and hence, the col. space of all M^j are subspaces of col. space of M^{d-1}.

With $S_{m \times n}$ random matrix of iid $N(0, \frac{1}{m})$, using argument for subspace embedding, we have:

$$P\left[\forall x \in \mathbb{R}^d, \|SM^{d-1}x\|_2^2 \in (1 \pm \varepsilon) \|M^{d-1}x\|_2^2\right] \geq 1 - \delta \quad (1)$$

$(M^{d-1} = [a^1 \; a^2 \ldots a^d])$ for $m = O\left(\frac{d}{\varepsilon^2} \log \frac{1}{\delta}\right)$

given (1) holds, $P\left[\forall x \in \mathbb{R}^{j+1}, \|SM^j x\|_2^2 \in (1 \pm \varepsilon) \|M^j x\|_2^2\right] = 1$ for all $j < d$.

$$\left[vol(0, Sa^1, Sa^2 \ldots Sa^d)\right]^2$$

$$\cancel{= \|Sa^1\| \cdot \prod_{j=1}^{d-1} \|SM^j x^j\|}$$

$$= \|Sa^1\|_2^2 \cdot \prod_{j=1}^{d-1} \min_{x^j \in \mathbb{R}^j} \|SM^j x^j\|_2^2$$

$$\in (1 \pm \varepsilon)^d \; \|a^1\|_2^2 \prod_{j=1}^{d-1} \min_{x^j \in \mathbb{R}^j} \|M^j x^j\|_2^2$$

$$\in \left[vol(0, a^1, a^2 \ldots a^d)\right]^2 \cdot (1 \pm \varepsilon)^d$$

$$\left[vol(0, Sa^1, Sa^2 \ldots Sa^d)\right] \in (1 \pm \varepsilon)^{d/2} \left[vol(0, a^1, a^2 \ldots a^d)\right]$$

for $d > 2$ & $\varepsilon < \frac{1}{2d}$, we have :

$$1 - \frac{4}{3}\varepsilon d \le (1 \pm \varepsilon)^d \le 1 + \frac{4}{3}\varepsilon d .$$

$$\left(1 - \frac{4}{3}\varepsilon d\right)^{1/2} \le (1 \pm \varepsilon)^{d/2} \le \left(1 + \frac{4}{3}\varepsilon d\right)^{1/2} \quad \text{— (2)}$$

for $\frac{4}{3}\varepsilon d \in (0, \frac{1}{2}]$ $\quad \left(1 \pm \frac{4}{3}\varepsilon d\right)^{1/2} \in \left(1 \pm \frac{3}{4} \cdot \frac{4}{3}\varepsilon d\right)$

(using $\varepsilon \in (0, \frac{1}{2}]$ $\;1 - \frac{3}{4}\varepsilon \le (1 \pm \varepsilon)^{1/2} \le 1 + \frac{3}{4}\varepsilon$)

$$(1 - \varepsilon d) \le \left(1 \pm \frac{4}{3}\varepsilon d\right)^{1/2} \le (1 + \varepsilon d) \quad \text{— (3)}$$

so (2) & (3) gives $\quad (1 - O(\varepsilon d)) \le (1 \pm \varepsilon)^{d/2} \le (1 + O(\varepsilon d))$

$$\mathrm{vol}(0, Sa^1, Sa^2 \ldots Sa^d) \in (1 \pm O(d\epsilon))\, \mathrm{vol}(0, a^1, a^2 \ldots a^d)$$

with prob $1-\delta$ for $m \geq O\left(\frac{d}{\epsilon^2} \log \frac{1}{\delta}\right)$

if we replace ϵ by ϵ/d, we get

$$\mathrm{vol}(0, Sa^1, Sa^2 \ldots Sa^d) \in (1 \pm O(\epsilon))\, \mathrm{vol}(0, a^1, a^2 \ldots a^d)$$

with prob $1-\delta$ for $m \geq O\left(\frac{d^3}{\epsilon^2} \log \frac{1}{\delta}\right)$