Ques 1

(a) The pmf of multinomial distribution is:-

$$P(x_1, x_2, \dots, x_k) = \frac{n!}{x_1!\, x_2! \dots x_k!} p_1^{x_1} p_2^{x_2} \dots p_k^{x_k}$$

The joint pmf of X_1 and X_2 will be :-

$$P(x_1, x_2) = \sum_{x_3}^{n-x_1-x_2} \sum_{x_4}^{n-x_1-x_2-x_3} \dots \sum_{x_{k-1}}^{n-(x_1+x_2+\dots+x_{k-2})} \frac{n!}{x_1!\, x_2! \dots x_k!} p_1^{x_1} p_2^{x_2} \dots p_k^{x_k}$$

$$= {}^{n}C_{x_1}\, {}^{n-x_1}C_{x_2} \sum_{(x_3, x_4 \dots x_k)} \frac{(n-x_1-x_2)!}{x_3!\, x_4! \dots x_k!} p_1^{x_1} p_2^{x_2} \dots p_k^{x_k}$$

$$= {}^{n}C_{x_1}\, {}^{n-x_1}C_{x_2}\, p_1^{x_1} p_2^{x_2} \sum_{\substack{(x_3, x_4, \dots, x_k \geq 0 \\ \text{and} \\ x_3 + x_4 + \dots x_k \leq n - x_1 - x_2)}} \frac{(n-x_1-x_2)!}{x_3!\, x_4! \dots x_k!} p_3^{x_3} p_4^{x_4} \dots p_k^{x_k}$$

$$= {}^{n}C_{x_1}\, {}^{n-x_1}C_{x_2}\, p_1^{x_1} p_2^{x_2} (p_3 + p_4 + \dots p_k)^{n-x_1-x_2}$$

$$= {}^{n}C_{x_1}\, {}^{n-x_1}C_{x_2}\, p_1^{x_1} p_2^{x_2} (1 - p_1 - p_2)^{n-x_1-x_2}$$

Q1 (b) $P(X_1, X_2 \mid X_3 = x_3 \dots, X_{k-1} = x_{k-1})$

$$= \frac{P(X_1, X_2 \dots X_{k-1})}{P(X_3 = x_3 \dots X_{k-1} = x_{k-1})}$$

$$= \frac{\frac{n!}{x_1! \dots x_k!} p_1^{x_1} p_2^{x_2} \dots p_k^{x_k}}{\sum\limits_{\substack{x_1, x_2 \geq 0 \\ x_1 + x_2 \leq n - x_3 \dots x_k}} \frac{(n - x_1 - x_2)}{x_3! \, x_4! \dots x_k!} p_3^{x_3} p_4^{x_4} \dots p_k^{x_k}}$$

$$= \frac{{}^{n}C_{x_3, x_4 \dots x_k} p_3^{x_3} p_4^{x_4} \dots p_k^{x_k} \; {}^{n - x_3 - \dots x_k}C_{x_1, x_2} p_1^{x_1} p_2^{x_2} (1 - p_1 - p_2)}{\underbrace{{}^{n}C_{x_3, x_4 \dots x_k} p_3^{x_3} p_4^{x_4} \dots p_k^{x_k} (1 - p_3 - p_4 - \dots p_k)^{x_1 + x_2}}_{\text{Probability of occurrence of } X_3, \dots X_k}}$$

This is the probability of occurrence of X_1, X_2 given that $X_3, \dots X_k$ has occured

So, $P(X_1, X_2 \mid X_3 = x_3 \dots X_{k-1} = x_{k-1})$

$$= {}^{n - x_3 - \dots x_k}C_{x_1, x_2} p_1^{x_1} p_2^{x_2} (1 - p_1 - p_2)^{n - x_1 - \dots x_k}$$

$$= {}^{n - x_3 - \dots x_k}C_{x_1, x_2} p_1^{x_1} p_2^{x_2} (1 - p_1 - p_2)^{0}$$

Q2 (a) $P_X(x) = \sum_{y=0}^{\infty} \frac{\mu^y e^{-2\mu}}{x!\,(y-x)!}$

$$= \frac{e^{-2\mu}}{x!} \sum_{y=0}^{\infty} \frac{\mu^y}{(y-x)!}$$

$$= \frac{e^{-2\mu}}{x!} \left[\sum_{y=x}^{\infty} \frac{\mu^y}{(y-x)!} \right]$$

↳ because factorial of negative numbers is assumed as 0.

$$= \frac{e^{-2\mu}}{x!} \left[\frac{\mu^x}{0!} + \frac{\mu^{x+1}}{1!} + \frac{\mu^{x+2}}{2!} + \dots \right]$$

$$= \frac{e^{-2\mu}}{x!} \mu^x \left[\frac{1}{0!} + \frac{\mu}{1!} + \frac{\mu^2}{2!} + \dots \right]$$

$$= \frac{e^{-2\mu}}{x!} \mu^x e^{\mu}$$

$$= \cancel{\frac{e^{-\mu}}{x!}\mu} \quad \frac{e^{-\mu}}{x!} \mu^x$$

Q2(b)

Q2 (b) Let $Z = Y - X$

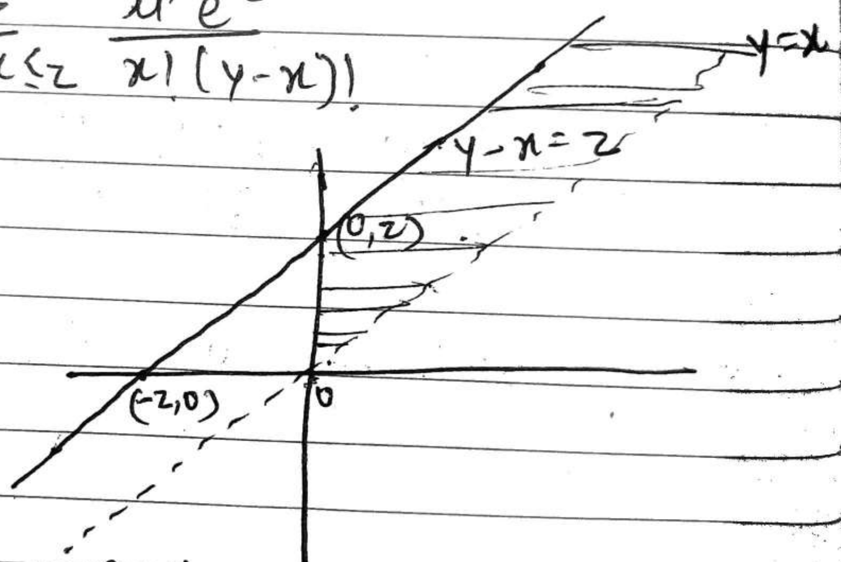
$$\cancel{P_Z(z) = P_Z(Y - X = z)}$$

Let CDF of Z be $F_Z(z)$

$$F_Z(z) = \cancel{F_Z(y)}\ P(Z \leq z)$$

$$= P(Y - X \leq z)$$

$$= \sum_{y - x \leq z} \frac{\mu^y e^{-2\mu}}{x!\,(y-x)!}$$



$$F_Z(z) = \sum_{y=0}^{\infty} \sum_{x=0}^{y} \frac{\mu^y e^{-2\mu}}{x!\,(y-x)!}$$

$$= \sum_{x=0}^{\infty} \sum_{y=x}^{x+z} \frac{\mu^y e^{-2\mu}}{x!\,(y-x)!}$$

$$= \sum_{x=0}^{\infty} \frac{e^{-2\mu}}{x!} \sum_{y=x}^{x+z} \frac{\mu^y}{(y-x)!}$$

Q2(b) continued

$$f_Z(z) = \sum_{x=0}^{\infty} \frac{e^{-2\mu}}{x!}\left[\frac{\mu^{x}}{0!} + \frac{\mu^{x+1}}{1!} + \dots + \frac{\mu^{x+z}}{z!}\right]$$

$$P_Z(z) = \frac{d}{dz} f_Z(z)$$

Q3 (a) Poisson distribution with parameter λt is

$$P_{Pos} = \frac{(\lambda t)^n e^{-\lambda t}}{n!}$$

Let $G(t)$ be gamma distr. which parameter $\alpha, \beta > 0$. Density of $G(t)$ is

$$\frac{\beta^{\alpha t}}{\Gamma(\alpha t)} x^{\alpha t - 1} e^{-\beta x} \qquad \forall x \geq 0$$

Compound distr. is

$$P_{comp}(t) = P(X(t) = k)$$

$$= \frac{e^{-\lambda t}}{k!} \frac{\lambda_1^k}{(\lambda_1 + \beta)^k} \sum_{n=1}^{\infty} \frac{(\lambda t \beta^{\alpha})^n \Gamma(\alpha n + k)}{(\lambda_1 + \beta)^{\alpha n} n! \Gamma(\alpha n)}$$

$k \geq 1$

$$p(t) = e^{-\lambda t\left(1 - \frac{\beta^{\alpha}}{(\lambda_1 + \beta)^{\alpha}}\right)}$$

Q6 (a) $AZ = \begin{bmatrix} a_1^T \\ a_2^T \\ \vdots \\ a_m^T \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ z_n \end{bmatrix}$

$= \begin{bmatrix} \cdots a_1^T \cdots \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ z_n \end{bmatrix} + \begin{bmatrix} \cdots a_2^T \cdots \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ z_n \end{bmatrix}$

$+ \cdots \begin{bmatrix} \cdot a_m^T \cdots \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ z_n \end{bmatrix}$

(b) $\|AZ\|^2 = \begin{bmatrix} \|a_1^T Z\| \\ \|a_2^T Z\| \\ \vdots \\ \|a_m^T Z\| \end{bmatrix}$

$E\left[\|AZ\|^2\right] = E\begin{bmatrix} Z^T a_1 a_1^T Z \\ Z^T a_2 a_2^T Z \\ \vdots \\ Z^T a_m a_m^T Z \end{bmatrix}$

$= E\begin{bmatrix} z_1^2 + z_2^2 + \cdots z_n^2 \\ \vdots \\ z_1^2 + z_2^2 + \cdots z_n^2 \end{bmatrix} = \begin{bmatrix} 1+1+\cdots \\ \\ 1+1+\cdots \end{bmatrix} = n \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}$

Q6(b) continued

$$Var(\|AZ\|^2) = n^2\begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}$$

Q5 (a) $A = \begin{bmatrix} I_m & 0 \\ C & I_{n-m} \end{bmatrix}$

$$\begin{bmatrix} X_1 \\ W \end{bmatrix} = AX = A \begin{bmatrix} X_1 \\ X_2 \end{bmatrix}$$

$$= \begin{bmatrix} I_m & 0 \\ C & I_{n-m} \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = \begin{bmatrix} X_1 \\ CX_1 + I X_2 \end{bmatrix}$$

$$W = CX_1 + X_2$$

(b) $W \cong N(\mu_1 C + \mu_2, C^T \Sigma C + \Sigma)$

(a) C can be $\begin{bmatrix} 0 & \cdots & 0 \\ 0 & \cdots & 0 \\ \vdots & & \\ 0 & \cdots & 0 \end{bmatrix}$