CS698C 2021 August Quiz 5

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TOTAL POINTS

100 / 100

QUESTION 1

1 Problem on dimensionality reduced embedding **100 / 100**

+ **100 Point adjustment**

Excellent.

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1) Given $A \in \mathbb{R}^{n \times d}$, $B \in \mathbb{R}^{q \times p}$, $C \in \mathbb{R}^{n \times p}$

a) $X \in \mathbb{R}^{d \times p}$

Minimize $\|AX - C\|_F$

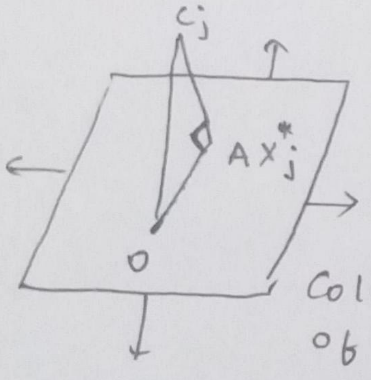
$\exists Y \in \mathbb{R}^{d \times q}$

$YB = X$

$$\|AX - C\|_F^2 = \sum_{j=1}^{p} \|Ax_j - c_j\|_2^2$$

Consider $\min \|Ax_j - c_j\|_2 \quad 0 \le j < p$

This is similar to linear regression problem p times



$$\underset{x_j}{\text{Arg Min}} \|Ax_j - c_j\|_2$$

$$= A^- c_j$$

$A^T(Ax_j^* - c_j) = 0$.

Taking $A = U_r \Sigma_r V_r^T$

$x_j^* = V_r \Sigma_r^{-1} U_r^T c_j$

$= A^- c_j$

$$\underset{x_j}{\text{Argmin}} \sum_{j=1}^{p} \|Ax_j - c_j\|_2^2.$$

$$= A^- [c_1 \cdots c_p]$$

$$= A^- C.$$

$$X^* = A^- C$$

Given $YB = X$

$$Y^* = A^- C B^-$$

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$$\min \| AX - C \|_F = \| A A^- C - C \|_F \quad X^* = A^- C$$

$$Y^* = A^- C B^-$$

$$= \| A \underbrace{A^- C B^-}_{Y^*} B - C \|_F$$

⇒ Minimum value of $\| AX - C \|_F$

P.T.O

1 b)

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→ Embedding function Matrix is used to solve the affine problem in lower dimension $\| SAx - SB \|_F$ and plug the solution in higher dimension $\| Ax - B \|_F$

→ No. of rows of the matrix depends on the distribution "D" from which it is drawn

<u>Properties of embedding function are:</u>

1) Both column space wise embedding of A and row-wise space embedding of B is done with $S \sim D_1$ and $R \sim D_2$ matrix

$\| AYB - C \|_F$ to $\| SAYBR - SCR \|_F$.

2) D_1 preserves colspace of A

$$P\left[\forall x \in \mathbb{R}^d \; \| SAx \| \in (1 \pm \epsilon) \| Ax \| \right] \geq 1 - \delta$$

3). D_2 preserves rowspace of B

$$P\left[\forall y^T \in \mathbb{R}^e \; \| y^T BR \| \in (1 \pm \epsilon/20) \| y^T B \| \right] \geq 1 - \delta$$

4) ~~Both $S \sim D_1$ $R \sim D_2$ satisfy~~

4) D_2 is an affine embedding

$$P\left[\forall Y \in \mathbb{R}^{n \times e} \; \| YBR - AY^* R \|_F \in (1 \pm \epsilon) \| YB - AY^* \| \right]$$

$C^* = A A^- C$.

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5) $P_{S|R}\left[\|SC^*R\|_F \in (1\pm\epsilon)\|C^*R\|_F\right] \geq 1-\delta$

6) $P\left[\|C^*R\|_F \in (1\pm\epsilon)\|C^*\|_F\right] \geq 1-\delta$

7) $P_{S|R}\left[\|SB^*R\|_F \in (1\pm\epsilon)\|B^*R\|_F\right] \geq 1-\delta$

$B^* = A A^- C(B^-B - I)$

8) $P\left[\|B^*R\|_F \in (1\pm\epsilon)\|B^*\|_F\right] \geq 1-\delta$

9) ~~Trace~~ Inner products

$j \in \{0, P\}$

$P\left| (S(AX - AX^*)R)_j \, S(AX^*R - CR)_j \right.$

$- (AYB - AX^*)R_j \, (AX^* - c)R_{j'} \left.\right|$

$\leq \epsilon \|AYB - AX^*)R_j\| \, \|AX^* - c)R_{j'}\| \geq 1-\delta$

$j = 1 \ldots d \quad j' = 1 \ldots d$

10) $P\left[(SA_j)^T (SC^*R_{j'}) \leq \epsilon \|A_j\| \|C^*R_{j'}\|\right] \geq 1-\delta$

11) $P\left[\|C^*R\|_F \in (1\pm\epsilon)\|C^*\|_F\right] > 1-\delta$

12) $P\left[\|SAB^*\|_F \in (1\pm\epsilon)\|AB^*\|\right] 1-\delta$

Given,

$A \in \mathbb{R}^{n\times d}$

$X \in \mathbb{R}^{d\times p}$

$C \in \mathbb{R}^{n\times p}$

$B \in \mathbb{R}^{q\times p}$

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(5)

$$\min \|AX - C\|_F$$

$\exists Y \in \mathbb{R}^{d\times q}$ s.t $YB = X$

$$\min \|\underbrace{\underbrace{A}_{n\times d}\underbrace{Y}_{d\times q}\underbrace{B}_{q\times p}}_{n\times p} - C\|_F$$

Dimensionality reduction on both n and p is done by using random matrices S and R respectively $S \sim D_1$, $R \sim D_2$

The problem $\|SAYBR - SCR\|_F$ needs to be solved

Required to prove: $\|SAYBR - SCR\|_F \in (1\pm\epsilon)\|AYB - C\|_F$ (embeddings, belongs to)

S $k\times n$

R $p\times l$

$$\|SAYBR - SCR\|_F^2$$

$X^* = A^+C$

$$= \|SAYBR - SCR - SA\underbrace{A^+C}_{X^*}R + SA A^+ CR\|_F^2$$

$$= \|SAYBR - SAX^*R - SCR + SAX^*R\|_F^2$$

$$= \|SA(YB - X^*)R + SAX^*R - SCR\|_F^2$$ opening $\|\cdot\|_F$ norm

$$= \|SA(YB - X^*)R\|_F^2 + \|SAX^*R - SCR\|_F^2 + 2\,tr\left[(SA(YB - X^*)R)^T(SAX^*R - SCR)\right]$$

$\|A+B\|_F^2 = \|A\|_F^2 + \|B\|_F^2 + \sum_j A_j B_j \ldots tr(\ldots)$

$= \| SAYBR - SAX^*R + SAY^*BR - SAY^*BR \|_F^2$ | $X^* = A^{-}C$, $Y^* = A^{-}CB^{-}$

$+ \| SAX^*R - SCR \|_F^2 + 2tr((S(AYB - AX^*)R)^T (SAX^*R - SCR))$

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$= \| SA(Y - Y^*)BR + SAY^*BR - SAX^*R \|_F^2$

$+ \| SAX^*R - SCR \|_F^2 + 2tr((S(AYB - AX^*)R)^T (SAX^*R - SCR))$

$= \| SA(Y - Y^*)BR \|_F^2 + \| SA(Y^*B - X^*)R \|_F^2$ } opening $\| \|_F$

$+ 2tr[(SA(Y - Y^*)BR)^T (SA(Y^*B - X^*)R)]$

$+ \| SAX^*R - SCR \|_F^2 + 2tr[((S(AYB - AX^*)R)^T (SAX^*R - SCR)]$

Rearranging terms.

$= \underbrace{\| SA(Y - Y^*)BR \|_F^2}_{\text{term1}} + \underbrace{\| SA(Y^*B - X^*)R \|_F^2}_{\text{term2}} + \underbrace{\| SAX^*R - SCR \|_F^2}_{\text{term3}}$

$+ 2tr[(SA(Y - Y^*)BR)^T (SA(Y^*B - X^*)R)] \leftarrow$ term4

$+ 2tr[\,[S(AYB - AX^*)R]^T (SAX^*R - SCR)] \leftarrow$ term5 — (1)

term 1

$\| SA(Y - Y^*)BR \|_F^2 \Rightarrow \sum_{j=1}^{d} \| SA(Y - Y^*)BR_j \|_2^2$

$(1 \pm \epsilon) \sum_{j=1}^{d} \| A(Y - Y^*)BR_j \|_2^2$

$\Downarrow$

$= (1 \pm \epsilon) \| A(Y - Y^*)BR \|_F^2$

$= (1 \pm \epsilon)^2 \| A(Y - Y^*)B \|_F^2$ — (2)

$= (1 \pm \epsilon) \| A(Y - Y^*)B \|_F^2$

Using

$\| SA \|_F^2 = (1 \pm \epsilon) \| A \|_F^2$

$\| AR \|_F^2 = (1 \pm \epsilon) \| A \|_F^2$

$(1 \pm \epsilon)^2 \approx (1 \pm \epsilon)$

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Similarly

term 2

$$\| SA(Y^*B - X^*)R\|_F^2$$

$$= (1\pm\epsilon)\|A(Y^*B - X^*)\|_F^2 \quad —③$$

$X^* = A^- C$

$Y^* = A^- C B^-$

$X^* B^- = Y^*$

~~Assuming~~

$$\| SA(Y^*B - X^*)R\|_F^2 = (1\pm\epsilon)\|A(Y^*B - X^*)R\|_F^2$$

$$\| A(Y^*B - X^*)R\|_F^2 = (1\pm\epsilon)\|A(Y^*B - X^*)\|_F^2$$

term 3

$$\| SAX^*R - SCR\|_F^2$$

$$= (1\pm\epsilon)\|AX^* - C\|_F^2 \quad —④$$

term 4

$$2\,tr\left[(SA(Y - Y^*)BR)^T(SA(Y^*B - X^*)R)\right]$$

Consider the term

$$(A(Y - Y^*)B)^T(A(Y^*B - X^*))]$$

Using $X^* B^- = Y^*$

$$= (A(Y - Y^*)B)^T(A(X^* B^- B - X^*))$$

$\sum_j A_j B_j = tr(A^T B)$

$$= \underbrace{(A(Y - Y^*)B)^T}_{\text{rowspace } B}\underbrace{(-AX^*(I - B^- B))}_{\text{orthogonal to rowspace } B} = 0$$

~~Preserving inner product~~

$$\left[~~(SA(Y-Y^*)BR)^T(SA(Y^*B-X^*)R)~~\right] - (A(Y-Y^*)B)^T(A(Y^*B-X^*))\right] \swarrow 0$$

$$\leq ~~O(\epsilon)\|A(Y-Y^*)B\|~~$$

for some $0 \leq j < k$

$$((A(Y-Y^*)B)R_j)^T(A(Y^*B - X^*)R_j)$$

$$= R_j^T(A(Y-Y^*)B)^T(A(Y^*B - X^*)R_j) = 0 \quad —⑤$$

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S $k \times n$

R $p \times d$.

$$2\,tr\left[(SA(Y-Y^*)BR)^T (SA(Y^*B-X^*)R) \right].$$

$$= 2\sum_{j=1}^{d} (SA(Y-Y^*)BR)_j^T (SA(Y^*B-X^*)R)_j$$

$$= 2\sum_{j=1}^{d} \Big| (SA(Y-Y^*)BR)_j^T \cdot (SA(Y^*B-X^*)R)_j - \underbrace{(A(Y-Y^*)BR)_j^T (A(Y^*B-X^*)R)_j}_{\Downarrow\ 0 \text{ proved above}} \Big|$$

preserving inner product

$$\leq 2\sum_{j=1}^{d} \epsilon \,\|(A(Y-Y^*)BR)_j\|_2 \,\|(A(Y^*B-X^*)R)_j\|_2 .$$

$$\leq \epsilon \sum_{j=1}^{d} \|(A(Y-Y^*)BR)_j\|_2 \sum_{j=1}^{d} \|(A(Y^*B-X^*)R)_j\|_2$$

Using Cauchy Schwarz inequality

$$\Big|\sum_{i=1}^{n} u_i \bar{v}_i\Big|^2 \leq \sum_{i=1}^{n} |u_i|^2 \sum_{i=1}^{n} |v_i|^2$$

$$\leq \epsilon \,\|A(Y-Y^*)BR\|_F \,\|A(Y^*B-X^*)R\|_F$$

$2 * \epsilon = \epsilon$

$$\leq \epsilon \,\|A(Y-Y^*)B\|_F \,\|A(Y^*B-X^*)\|_F . \quad (6)$$

Using $\|AR\|_F \in (1 \pm \epsilon)\|A\|_F$

Similarly

<u>term 5</u>

$$tr\left[(S(AYB-AX^*)R)^T (SAX^*R-SCR) \right]$$

$$\leq \epsilon \,\|AYB-AX^*\|_F \,\|AX^*-C\|_F \quad (7)$$

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(9)

Hence by combining the results of term(1) term(2) term(3) term(4) term(5) i.e.,

$x^* = A^- C$

$Y^* = A^- C B^-$

$$\| SA(Y-Y^*)BR \|_F^2 = (1 \pm \epsilon) \| A(Y-Y^*)B \|_F^2 \quad \text{— (a)}$$

$$\| SA(Y^*B - X^*)R \|_F^2 = (1 \pm \epsilon) \| A(Y^*B - X^*) \|_F^2 \quad \text{— (b)}$$

$$\| SAX^*R - SCR \|_F^2 = (1 \pm \epsilon) \| AX^* - C \|_F^2 \quad \text{— (c)}$$

$$2tr\left[(SA(Y-Y^*)BR)^T (SA(Y^*B - X^*)R) \right] \leq \epsilon \| A(Y-Y^*)B \|_F \| A(Y^*B - X^*) \|_F \quad \text{— (d)}$$

$$2tr\left[S(AYB - AX^*)R)^T (SAX^*R - SCR) \right] \leq \epsilon \| AYB - AX^* \|_F \| AX^* - C \|_F \quad \text{— (e)}$$

$$\| SAYBR - SCR \|_F^2 = \| SA(Y-Y^*)BR \|_F^2 + \| SA(Y^*B - X^*)R \|_F^2 + \| SAX^*R - SCR \|_F^2 + 2tr\left[(SA(Y-Y^*)BR)^T (SA(Y^*B - X^*)R) \right] + 2tr\left[S(AYB - AX^*)R)^T (SAX^*R - SCR) \right]$$

$$\leq (1 \pm \epsilon) \| A(Y-Y^*)B \|_F^2 + (1 \pm \epsilon) \| A(Y^*B - X^*) \|_F^2 + (1 \pm \epsilon) \| AX^* - C \|_F^2 + \epsilon \underbrace{\| A(Y-Y^*)B \|_F \| A(Y^*B - X^*) \|_F}_{\text{Using AM} \geq \text{GM}} + \epsilon \underbrace{\| AYB - AX^* \|_F \| AX^* - C \|_F}_{\text{Using AM} \geq \text{GM}}$$

Using AM $\geq$ GM

$\frac{a^2+b^2}{2} \geq \sqrt{a^2 b^2}$

$$\le (1+\epsilon)\|A(Y-Y^*)B\|_F^2 + (1+\epsilon)\|A(Y^*B-X^*)\|_F^2 + (1+\epsilon)\|AX^*-C\|_F^2$$

$$+ \ \epsilon\left(\tfrac{1}{2}\right)\|A(Y-Y^*)B\|_F^2 + \epsilon\left(\tfrac{1}{2}\right)\|A(Y^*B-X^*)\|_F^2 + \epsilon\|AX^*-C\|_F^2$$

$$\le (1\pm\epsilon)\left[\|A(Y-Y^*)B\|_F^2 + \|A(Y^*B-X^*)\|_F^2 + \|AX^*-C\|_F^2\right]$$

$Y^* = A^- C B^-$

$X^* = A^- C'$

Consider

$$\|AYB - C\|_F^2 = \|AYB - AY^*B + AY^*B - C\|_F^2$$

$$= \|AYB - AY^*B\|_F^2 + \|AY^*B - C\|_F^2$$

trace terms are $\perp$ so zero

$$= \|AYB - AY^*B\|_F^2 + \|AY^*B - AX^* + AX^* - C\|_F^2$$

$$= \|AYB - AY^*B\|_F^2 + \|A(Y^*B - X^*)\|_F^2 + \|AX^* - C\|_F^2$$

$$\le (1\pm\epsilon)\|AYB - C\|_F^2$$

~~Hen~~ Solution

Y $\cancel{\|S A \underbrace{YBR}_{t} - SCR\|}$

$t = \cancel{(SA)^- SCR(SB)^-}$

1 c) $\|AYB - C\|_f$ reduced to

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$\|SAYBR - SCR\|_F$ $S \sim D_1$ $R \sim D_2$.

S $k \times d$ A $n \times d$ B $q \times p$

R $p \times d$ Y $d \times q$ C $n \times p$.

1) Consider dimensions of S matrix $k \times n$.

S is in column space of A.

$$\|S A \underbrace{YBR}_{k_1} - S \underbrace{CR}_{k_2}\|_F$$

$\|SAk_1 - Sk_2\|_F$.

We need to preserve column space of A and columns of C. Considering $S_{ij} \sim N(0, \frac{1}{k})$, $R_{ij} \sim N(0, \frac{1}{d})$

Using γ net argument for A and p times J-L lemma for C.

$$k \geq O\left(\frac{d}{\epsilon^2} \log \frac{1}{\delta} + \frac{1}{\epsilon^2} \log \frac{p}{\delta}\right).$$

2) Consider dimensions of R matrix $p \times d$.

R is in row space of $SAYB$. q rows

Preserving row space of $S\widetilde{AYB}$ gives $O\left(\frac{q}{\epsilon^2} \log \frac{1}{\delta}\right)$.

Preserving $\overset{n}{\cancel{q}}$ rows of $\overset{C}{\cancel{B}}$ | 2111011 Anirudh (12)

$\frac{1}{\epsilon^2}\log\frac{n}{\delta}$

$$l = O\left(\frac{\cancel{q}}{\epsilon^2}\log\frac{1}{\delta} + \frac{1}{\epsilon^2}\log\frac{n}{\delta}\right)$$

Optimal solution

of $\| SAYBR - SCR \|_F$

$= \| SAXR - SCR \|_F$

~~X~~

Taking $\| SAX - SC \|_F \Rightarrow \| SA\underbrace{YB}_{X}\cancel{R} - S\underbrace{CR}_{C} \|_F$ ignoring R

$\hat{Y} = (SA)^{-} S\cancel{C}_{n\times p}$

$\| \underbrace{SAYB}R - \underbrace{SC}R \|$

ignoring S

~~Taking~~ $\|$

$\hat{Y} = R\cancel{C}^{-}(CR)^{-}(AYB)$

$\hat{Y} = (SA)^{-}SC(CR)^{-}(AYB)$

~~Time complexity is~~

Using

$\| XA\cancel{R} - B \|$

$B^{-}A$

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+ 100 **Point adjustment**

Excellent.