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Q1)

(a) . $p_{X_1, X_2}(x_1, x_2)$ ~~$(x_3, x_4, x_5, \ldots x_{k-1})$~~

$$p_X(x_1, x_2, \ldots x_{k-1}) = \frac{n!}{x_1!\, x_2! \cdots x_{k-1}!\, x_k!} p_1^{x_1} p_2^{x_2} \cdots p_{k-1}^{x_{k-1}} p_k^{x_k}$$

$$p_{X_1 X_2}(x_1, x_2) = \sum_{x_{k-1}=0}^{n-x_3-x_4 \cdots x_{k-2}} \cdots \sum_{x_3=0}^{n} p_X(x_1, x_2 \ldots x_{k-1})$$

$$= \sum \cdots \sum \frac{n!}{x_1!\, x_2! \cdots x_{k-1}!\, x_k!} p_1^{x_1} p_2^{x_2} \cdots p_k^{x_k}$$

$$= \frac{n!\, p_1^{x_1} p_2^{x_2}}{x_1!\, x_2!\, (n-x_1-x_2)!} \underbrace{\sum \cdots \sum \frac{(n-x_1-x_2)!}{x_3!\, x_4! \cdots x_k!} p_3^{x_3} p_4^{x_4} \cdots p_k^{x_k}}_{\text{multinomial with parameters}}$$

$n' = n - x_1 - x_2$

$$\Rightarrow (p_3 + p_4 + \cdots p_k)^{n-x_1-x_2}$$

$$\Rightarrow (1 - p_1 - p_2)^{n-x_1-x_2}$$

$$p_{X_1 X_2}(x_1, x_2) = \frac{n!}{x_1!\, x_2!\, (n-x_1-x_2)!} p_1^{x_1} p_2^{x_2} (1-p_1-p_2)^{n-x_1-x_2}$$

(b) $p_{X_1, X_2 | \bar{X}}(x_1, x_2 \mid x_3, x_4 \ldots x_{k-1})$.

$$\Rightarrow \cancel{p_{X_1 X_2 X_3 \ldots}} \quad \frac{p_X(x_1\ x_2\ x_3 \ldots x_{k-1})}{p_{X_3 X_4 X_5 \ldots X_{k-1}}(x_3, x_4 \ldots x_{k-1})}.$$

$$\Rightarrow \frac{\dfrac{n!}{x_1!\, x_2! \cdots x_{k-1}!\, x_k!} p_1^{x_1} p_2^{x_2} p_3^{x_3} \cdots p_{k-1}^{x_{k-1}} p_k^{x_k}}{\dfrac{n!}{x_3!\, x_4! \cdots (n-x_3-x_4-x_5 \cdots -x_{k-1})!} p_3^{x_3} p_4^{x_4} \cdots p_{k-1}^{x_{k-1}} p_k^{x_k} (p_2+p_1)^{n-x_3-x_4 \cdots}}$$

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$$\Rightarrow \frac{(x_1+x_2)!}{x_1!\,x_2!} p_1^{x_1} p_2^{x_2} \frac{1}{(p_1+p_2)^{x_1+x_2}}$$

$$\Rightarrow \binom{x_1+x_2}{x_1} \left(\frac{p_1}{p_1+p_2}\right)^{x_1} \left(\frac{p_2}{p_1+p_2}\right)^{x_2}.$$

$\Rightarrow$ Binomial with parameter $B\left(x_1+x_2, \frac{p_1}{p_1+p_2}\right)$

Q2

$$p(x,y) = \frac{\mu^y e^{-2\mu}}{x!(y-x)!} \qquad y = 0, 1, 2 \ldots \qquad x = 0, 1, 2 \ldots y$$

$$\cancel{p_X(x) = \sum_{y=0}^{\infty} \frac{\mu^y e^{-2\mu}}{x!(y-x)!}}$$

$$\cancel{\text{let } T = X, \; U = X-Y \quad \text{then } \begin{bmatrix} T \\ U \end{bmatrix} = \begin{bmatrix} X \\ X-Y \end{bmatrix}}$$

$$\cancel{\therefore X = T, \; Y = T-U. \quad \begin{bmatrix} X \\ Y \end{bmatrix} = \begin{bmatrix} T \\ T-U \end{bmatrix}.}$$

$$\cancel{p(x,y) = p(t, t-u) = \frac{\mu^{t-u} e^{-2\mu}}{t!(-u)!}}$$

(a) $$p_X(x) = \sum_{y=0}^{\infty} \frac{\mu^y e^{-2\mu}}{(y-x)!\,x!}$$

$$= \frac{e^{-2\mu}}{x!} \cdot \mu^x \underbrace{\sum_{y=x}^{\infty} \frac{\mu^{y-x}}{(y-x)!}}_{e^{\mu}}.$$

$$\Rightarrow \frac{e^{-\mu}\mu^x}{x!}$$

(b)

$$p_{X,Y}(x,y) = \frac{\mu^y e^{-2\mu}}{x!\,(y-x)!}$$

$$E\left[e^{t[Y-X]}\right] = \sum_y \sum_x \frac{e^{ty-tx} \cdot e^{-2\mu} \mu^y}{x!\,(y-x)!}$$

$$\Rightarrow \sum_{y=0}^{\infty} \sum_{x=0}^{y} \frac{e^{ty-tx} e^{-2\mu} \mu^y}{x!\,(y-x)!}$$

$$\cancel{= \sum_{y=0}^{\infty} \frac{e^{-tx-2\mu}}{\ } \sum_{x=0} \frac{e^{ty}\mu^y}{(y-x)!}}$$

$$\Rightarrow \sum_{y=0}^{\infty} \frac{e^{ty-2\mu}\mu^y}{y!} \sum_{x=0}^{y} \frac{e^{-tx} \cdot y!}{x!\,(y-x)!}$$

$$= \sum_{y=0}^{\infty} \frac{e^{ty-2\mu}\mu^y}{y!} \left(1+e^{-t}\right)^y$$

$$= e^{-2\mu} \sum_{y=0}^{\infty} \frac{\left[\mu e^t (1+e^{-t})\right]^y}{y!}$$

$$\Rightarrow \exp\{-2\mu + \mu e^t + \mu\}$$

$$\Rightarrow \exp\{\mu(e^t - 1)\}.$$

$\Rightarrow$ mgf of poisson's.

$$p(y-x) = p(t) = \frac{\mu^t e^{-\mu}}{t!} = \frac{\mu^{y-x} e^{-\mu}}{(y-x)!}$$

Q3 $\quad X|\omega \sim \text{Poisson}(\omega)$

$\omega \sim \Gamma(\alpha=1, \beta=1)$

(a) $$P_{X,\omega}(x,\omega) = P_{X|\omega}(x|\omega) \cdot p(\omega).$$

$$= \left(\frac{(\omega)^x e^{-\omega}}{x!}\right) \cdot \frac{1}{\Gamma(1)(1)^1} (\omega)^{\cancel{1}\alpha-1} e^{-\omega/1}.$$

$$= \frac{\omega^x e^{-\omega}}{x!} \cdot \omega^0 e^{-\omega}$$

$$= \frac{\omega^x e^{-2\omega}}{x!}$$

(b) $$p_X(x) = \int_0^\infty p_{X,\omega}(x,\omega)\, d\omega.$$

$$= \int_0^\infty \frac{\omega^x e^{-2\omega}}{x!}\, d\omega.$$

$$\Rightarrow \cancel{\frac{1}{2^x} \int_0^\infty \frac{(2\omega)^x e^{-2\omega}}{x!}\, d\omega}$$

$$\Rightarrow \frac{1}{x!} \underbrace{\int_0^\infty \frac{\omega^{(x+1-1)} e^{-\omega/1/2}}{\Gamma(x+1)\left(\frac{1}{2}\right)^{x+1}}\, d\omega}_{\substack{\text{gamma distribution} \\ =1}} \cdot * \Gamma(x+1)\left(\frac{1}{2}\right)^{x+1}$$

$$p_X(x) \Rightarrow \frac{\Gamma(x+1)}{x!}\left(\frac{1}{2}\right)^{x+1}$$

Q5 a

$$[\operatorname{cov}(AX)]_{ij} = [A\Sigma A^T]_{ij} \Rightarrow [0]_{ij} \text{ for } i \neq j.$$

$$\Rightarrow \begin{bmatrix} I_m & 0 \\ C & I_{n-m} \end{bmatrix} \begin{bmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{bmatrix} \begin{bmatrix} I_m & C \\ 0 & I_{n-m} \end{bmatrix}.$$

$$\Rightarrow \begin{bmatrix} \Sigma_{11} & \Sigma_{12} \\ C\Sigma_{11}+\Sigma_{21} & C\Sigma_{12}+\Sigma_{22} \end{bmatrix} \begin{bmatrix} I_m & C \\ 0 & I_{n-m} \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} \Sigma_{11} & C\Sigma_{11}+\Sigma_{12} \\ C\Sigma_{11}+\Sigma_{21} & C^2\Sigma_{11}+C\Sigma_{21}+C\Sigma_{12}+\Sigma_{22} \end{bmatrix} \Rightarrow 0.$$

$$C\Sigma_{11}+\Sigma_{12} = 0.$$

$$C = -\Sigma_{12}\Sigma_{11}^{-1}$$

b

$p_w(w)$

$$\begin{bmatrix} X_1 \\ W \end{bmatrix} = AX = \begin{bmatrix} I_m & 0 \\ -\Sigma_{12}\Sigma_{11}^{-1} & I_{n-m} \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix}$$

$$= \begin{bmatrix} X_1 \\ -\Sigma_{12}\Sigma_{11}^{-1}X_1 + X_2 \end{bmatrix}.$$

$$W = -\Sigma_{12}\Sigma_{11}^{-1}X_1 + X_2$$

(linear combination of the Normal random variable

distribution:

$$N\left(\mu_2 - \Sigma_{21}\Sigma_{11}^{-1}\mu_1;\ \Sigma_{22} - \Sigma_{21}\Sigma_{11}^{-1}\Sigma_{12}\right)$$

$\Rightarrow p(x_2|x_1)$

$$\sim N\left(\mu_2 - \Sigma_{21}\Sigma_{11}^{-1}\mu_1 + \Sigma_{21}\Sigma_{11}^{-1}x_1,\ \Sigma_{22} - \Sigma_{21}\Sigma_{11}^{-1}\Sigma_{12}\right).$$