# Notes on Sketching for Low Rank Approximation

## 1 Low rank approximation of a matrix

Let $A$ be an $m \times n$ matrix over reals. In practice, it is often the case that both $m$ and $n$ are very large. For example, consider a customer-product buying matrix $A$, where, the row indices $i = 1, 2, \ldots, m$ correspond to ids given to customers and the column indices $j = 1, 2, \ldots, n$ correspond to product ids. Typically $A_{ij}$ denotes the number of times customer $i$ purchased product $j$ (or, $A_{ij}$ denotes the number of units of product $j$ purchased by customer $i$, etc.). Such matrices typically can be quite sparse and/or have "lot of noise". For example, consider the matrix

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \epsilon_1 & 0 \\ 0 & 0 & \epsilon_2 \end{bmatrix}$$

where, we assume that $|\epsilon_1|, |\epsilon_2| \ll 1$. So $A$ has full rank, although it seems to be reasonably well approximated by the rank 1 matrix

$$\hat{A} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

This is often quantified by answering the following questions:

1. What is $\|A - \hat{A}\|_F$? or,
2. What is $\|A - \hat{A}\|_2$?

In this example, it is easy to see that $\|A - \hat{A}\|_F = (\epsilon_1^2 + \epsilon_2^2)^{1/2}$ and $\|A - \hat{A}\|_2 = \max(|\epsilon_1|, |\epsilon_2|)$. The low rank approximation question is posed as follows.

Let $A$ be an $m \times n$ matrix. The best rank-$k$ approximation matrix with respect to Frobenius (resp. 2-norm) for $A$ is a matrix $\hat{A} \in \mathbb{R}^{m \times n}$ satisfying the following condition:

$$\|A - \hat{A}\|_F = \min_{\text{rank}(X) \le k} \|A - X\|_F \ .$$

The notation $\text{argmin}_{X \in S} \phi(X)$ is often used to return a value of $X^* \in S$ in the feasible space $S$ that attains the minimum value $\min\{\phi(X) \mid X \in S\}$. That is, $\phi(X^*) = min\{\phi(X) \mid X \in S\}$. In this notation,

$$\hat{A} = \text{argmin}_{\text{rank}(X) \le k} \|A - X\|_F \ .$$

Analogously, a similar definition for the best rank-$k$ approximation may be formulated with respect to the 2-norm of a matrix. This defines $\hat{A}$ to be a matrix that minimizes the $\|A - X\|_2$, among all matrices $X \in \mathbb{R}^{m \times n}$ such that $\text{rank}(X) \le k$. That is,

$$\|A - \hat{A}\|_2 = \min_{\text{rank}(X) \le k} \|A - X\|_2, \text{ or, } \hat{A} = \text{argmin}_{\text{rank}(X) \le k} \|A - X\|_2 \ .$$

*The Eckart-Young Theorem.* We now state the famous and widely applicable Eckart-Young Theorem. We first state a few notations used by this theorem. Let $A \in \mathbb{R}^{m \times n}$ be a given matrix and let $A = U \Sigma V^T$ be the classical SVD of $A$. Let $\text{rank}(A) = r$. In this notation, we assume that the corresponding left singular vectors in $U$ and right singular vectors in $V$ are arranged so that the sequential diagonal entries of $\Sigma$ are ordered in a non-increasing order, that is,

$$\sigma_1 = \Sigma_{11} \ge \sigma_2 = \Sigma_{22} \ge \cdots \ge \sigma_r = \Sigma_{rr} \ .$$

All other entries in the matrix $\Sigma$ are zeros. For $1 \le k \le r$, the Eckart-Young theorem's statement defines the notation $A_k$ to denote the matrix

$$A_k = U \Sigma_k V^T = U \begin{bmatrix} \sigma_1 & & & & \\ & \sigma_2 & & & \\ & & \ddots & & \\ & & & \sigma_k & \\ & & & & \end{bmatrix} V^T$$

where, $\Sigma_k$ has only the first $k$-diagonal entries $\sigma_1, \sigma_2, \ldots, \sigma_k$ in sequence, and all other entries including all remaining diagonal entries set to zero. Another way of viewing $A$ and the $A_k$'s are the sum of single rank matrices.

$$A = U \Sigma V^T = \sigma_1 u_1 v_1^T + \sigma_2 u_2 v_2^T + \cdots + \sigma_r u_r v_r^T$$

where, $\sigma_1 \ge \sigma_2 \ge \cdots \ge \sigma_r > 0$. $A_k$ is the sum of the first $k$ terms in the above summation, that is,

$$A_k = U \Sigma_k V^T = \sigma_1 u_1 v_1^T + \cdots + \sigma_k u_k v_k^T \ .$$

It follows that

$$\begin{aligned} A - A_k &= U(\Sigma - \Sigma_k) V^T \\ &= \sigma_{k+1} u_{k+1} v_{k+1}^T + \cdots \sigma_r u_r v_r^T \\ &= U \begin{bmatrix} 0 & & & & \\ & \ddots & & & \\ & & \sigma_{k+1} & & \\ & & & \ddots & \\ & & & & \sigma_r \\ & & & & \end{bmatrix} V^T \ . \end{aligned}$$

$\Sigma - \Sigma_k$ has top $k$ diagonal entries as zeros, followed by the diagonal entries $\sigma_{k+1}, \cdots, \sigma_r$. All other entries of $\Sigma - \Sigma_k$ are zeros.

It follows that

1. $$\|A - A_k\|_2 = \sigma_{k+1}$$
2. $$\|A - A_k\|_F = \left(\sigma_{k+1}^2 + \cdots + \sigma_r^2\right)^{1/2}$$

We now state the Eckart-Young Theorem.

**Theorem 1.** *Eckart-Young Theorem*

1. $$\|A - A_k\|_F = \min_{\text{rank}(X)\le k} \|A - X\|_F, \qquad A_k = \text{argmin}_{\text{rank}(X)\le k}\|A - X\|_F \ .$$
2. $$\|A - A_k\|_2 = \min_{\text{rank}(X)\le k} \|A - X\|_2, \qquad A_k = \text{argmin}_{\text{rank}(X)\le k}\|A - X\|_2 \ .$$

The Eckart-Young theorem states that the best rank-$k$ or less approximation matrix for $A$ is $A_k$, for both Frobenius norm and 2-norm.

There are varied advantages of approximating $A$ by a low rank $A_k$, for example,

1. Space complexity of storing the approximation is less $O((m+n)k)$ real numbers.
2. Data is more interpretable and has less noise, etc.. High rank can be because of noise.

# 2 Low rank approximation

The best rank-$k$ approximation of a matrix $A$ by $A_k$ is universally obtained by using the Eckart-Young theorem. This requires the computation of the SVD of $A$. Typically this requires time $O(\min(mn^2, m^2n))$. In this section, we consider the problem of using sketching to obtain low rank approximation of a matrix $A$ more efficiently than that given above, while obtaining bounds on how much sub-optimal it can get as compared to $A_k$ (for rank-$k$ approximation).

## 2.1 Motivation

Consider as a first step the linear regression problem

$$\min_X \|A_k X - A\|_F \ .$$

First, we note that by the Eckart-Young's theorem, $X = I_n$ is clearly an optimal solution, since, $A_k X$ has rank at most $\text{rank}(A_k) = k$. Therefore,

$$\|A_k \cdot I - A\|_F \ge \min_X \|A_k X - A\|_F \ge \min_{\text{rank}(Y)\le k} \|Y - A\|_F = \|A_k - A\|_F$$

since, $Y = A_k$ is an optimal solution to the latter problem above, by Eckart-Young theorem.

Suppose we write the matrix of right singular vectors $V$ as

$$V = \begin{bmatrix} v_1 & v_2 & \dots, v_n \end{bmatrix} = \begin{bmatrix} V_k & V_k^\perp \end{bmatrix}$$

where, $V_k = \begin{bmatrix} v_1 & \cdots & v_k \end{bmatrix}$ and $V_k^\perp = \begin{bmatrix} v_{k+1} & \cdots & v_n \end{bmatrix}$. Note that the optimal solution to the linear regression problem $\min_X \|A_k X - A\|_F$ is

$$X^* = A_k^- A = V\Sigma_k^- U^T U \Sigma V^T = V\Sigma_k^- \Sigma V^T = \begin{bmatrix} V_k & V_k^\perp \end{bmatrix} \begin{bmatrix} I_k & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} V_k^T \\ (V_k^\perp)^T \end{bmatrix} = \begin{bmatrix} V_k & V_k^\perp \end{bmatrix} \begin{bmatrix} V_k^T \\ 0 \end{bmatrix} = V_k V_k^T$$

Further,

$$\|X^*\|_F = \|V_k V_k^T\|_F = \|V_k\|_F = \sqrt{k} \ .$$

Above, it was suggested that $X = I$ is an optimal solution to $\min_X \|A_kX - A\|_F$?? Below, we try to connect the dots (if any left unspecified). Suppose we write the left singular vector matrix $U$ as

$$U = \begin{bmatrix} u_1 & u_2 & \cdots & u_m \end{bmatrix} = \begin{bmatrix} U_k & U_k^{\perp} \end{bmatrix}$$

where, $U_k = \begin{bmatrix} u_1 & \cdots & u_k \end{bmatrix}$ and $U_k^{\perp} = \begin{bmatrix} u_{k+1} & \cdots & u_m \end{bmatrix}$. Then,

$$A_kX^* = (U_k\Sigma_kV_k^T)(V_kV_k^T) = U_k\Sigma_kV_k^T = A_k \ .$$

In other words, the optimal solution value to $\min_X \|A_kX - A\|_F$ is $\|A_k - A\|$. The solution to the problem

$$\text{argmin}_X \|X\|_F \text{ subject to } \|A_kX - A\|_F = \|A_k - A\|_F$$

is $X^*$, based on linear regression. The general solution to $A_kX = A_k$ is $X = V_kV_k^T + (V_k^{\perp})Y$, where, $Y \in \mathbb{R}^{(n-k)\times n}$ is any matrix. In particular, a solution is $Y = (V_k^{\perp})^T$, which gives $X = V_kV_k^T + V_k^{\perp}(V_k^{\perp})^T = I_n$. Further, the general solution $X = V_kV_k^T + (V_k^{\perp})^TY$ has Frobenius norm

$$\begin{aligned}
\|X\|_F^2 &= \|V_kV_k^T + V_k^{\perp}Y\|_F^2 \\
&= \|V_kV_k^T\|_F^2 + \|V_k^{\perp}Y\|_F^2 + 2\text{tr } V_kV_k^TV_k^{\perp}Y \\
&= \|V_k^T\|_F^2 + \|Y\|_F^2 \\
&= k + \|Y\|_F^2
\end{aligned}$$

since, step 3 follows from $V_k^TV_k^{\perp} = 0_{k\times(n-k)}$.

With this detour, we now return to our problem $\min_X \|A_kX - A\|_F$.

## Affine subspace embeddings

Suppose we consider solving the problem $\min_X \|A_kX - A\|_F$ approximately using affine subspace embeddings. Let $S$ be a random matrix drawn from a distribution $\mathcal{D}$ such that $S$ approximately preserves the norms of vectors in the affine subspace $A_kX - A$, that is,

$$\|SA_kX - SA\|_F \in (1 \pm \epsilon)\|A_kX - A\|_F, \qquad \text{for all } X \in \mathbb{R}^{n\times n} \ .$$

It would then follow that

$$\min_X \|SA_kX - SA\|_F \leq (1+\epsilon) \min_X \|A_kX - A\|_F = (1+\epsilon)\|A_k - A\|_F \ .$$

The solution

$$\text{argmin}_X \|SA_kX - SA\|_F \text{ is } \hat{X} = (SA_k)^{-}SA$$

and therefore, by affine subspace embedding norm preservation properties,

$$(1-\epsilon)\|A_k\hat{X} - A\|_F \leq \|SA_k\hat{X} - SA\|_F \leq (1+\epsilon)\|A_k - A\|_F \ .$$

Since, $\hat{X} = (SA_k)^{-}SA$, substituting in the above,

$$\|A_k(SA_k)^{-}SA - A\|_F \leq \left(\frac{1+\epsilon}{1-\epsilon}\right)\|A_k - A\|_F = (1+3\epsilon)\|A_k - A\|_F$$

for $\epsilon \leq \frac{1}{9}$.

Note that $A_k(SA_k)^-SA$ is a matrix with rank at most $k$ (since it has $A_k$ as a factor) and whose rowspace is a subspace of the row space of $SA$.

*Conclusion.* Consider a solution to the problem

$$\min_{\mathrm{rank}(X)\le k} \|XSA - A\|_F \ .$$

Then,

$$\min_{\mathrm{rank}(X)\le k} \|XSA - A\|_F \le \|A_k(SA_k)^-SA - A\|_F \le (1+3\epsilon)\|A_k - A\|_F$$

where, the first inequality follows by setting $X = A_k(SA_k)^-SA$ and the last inequality follows from the previous paragraph. Further, since $\mathrm{rank}(XSA) \le \mathrm{rank}(X) \le k$,

$$\min_{\mathrm{rank}(X)\le k} \|XSA - A\|_F \ge \min_{\mathrm{rank}(Y)\le k} \|Y - A\|_F = \|A_k - A\|_F \ .$$

Putting the two equations together, we have,

$$\|A_k - A\|_F \le \min_{\mathrm{rank}(X)\le k} \|XSA - A\|_F \le (1+3\epsilon)\|A_k - A\|_F \ .$$

So, we now turn towards trying to solve the optimization problem $\min_{\mathrm{rank}(X)\le k}\|XSA - A\|_F$.

## 2.2 How to solve $\|XSA - A\|_F$ subject to $\mathrm{rank}(X) \le k$

Let us consider how to solve the problem

$$\min_{\mathrm{rank}(X)\le k} \|XSA - A\|_F$$

and the problems encountered in doing so. Let $X^*$ denote the least Frobenius solution to the regression problem $\min_X \|XSA - A\|_F$. Then, by linear regression,

$$X^* = A(SA)^- \quad \text{and} \quad X^*SA = A(SA)^-SA \ .$$

In particular,

$$X^*SA - A = A(SA)^-SA - A = -A(I - (SA)^-SA)$$

has a row space that lies in the orthogonal complement to the row space of $SA$. (For this discussion, keep $S$ fixed. In other words, the discussion is conditional on $S$).

$$\begin{aligned}\|XSA - A\|_F^2 &= \|(X - X^*)SA + X^*SA - A\|_F^2 \\ &= \|(X - X^*)SA\|_F^2 + \|X^*SA - A\|_F^2\end{aligned}$$

since, $(X - X^*)SA$ has a rowspace which is a subspace of the row space of $SA$ and $X^*SA - A$ has a row space in the orthogonal complement space of $SA$ of $\mathbb{R}^n$. Therefore,

$$\min_{\mathrm{rank}(X)\le k} \|XSA - A\|_F^2 = \|X^*SA - A\|_F^2 + \min_{\mathrm{rank}(X)\le k} \|(X - X^*)SA\|_F^2$$

since, $\|X^*SA - A\|_F$ is not a function of $X$.

We may use the Eckart-Young theorem for the following problem.

$$\min_{\text{rank}(X)\le k} \|XSA - X^*SA\|_F = \min_{\text{rank}(X)\le k} \|XSA - A(SA)^-SA\|_F$$

Let $SA = U\Sigma V^T$ in SVD form. Then, letting $Y = XU\Sigma$,

$$\begin{aligned}\min_{\text{rank}(X)\le k} \|XSA - A(SA)^-SA\|_F &= \min_{\text{rank}(X)\le k} \|XU\Sigma V^T - A(SA)^-U\Sigma V^T\|_F \\ &= \min_{\text{rank}(X)\le k} \|XU\Sigma - A(SA)^-U\Sigma\|_F \\ &= \min_{\text{rank}(Y)\le k} \|Y - A(SA)^-U\Sigma\|_F \ .\end{aligned}$$

The minimization step has optimal solution $Y^* = [A(SA)^-U\Sigma]_k$ by the Eckart-Young theorem. The final solution $X^* = Y^*\Sigma^-U^T$.

The problem that arises is how to find the SVD of $A(SA)^-U\Sigma$ more efficiently, since it is an $m \times n$ matrix.

### 2.3 Solving Affine Embedding

Consider the problem considered in the subsection above, namely,

$$\min_{\text{rank}(X)\le k} \|XSA - A\|_F \ .$$

Conditioned on $S$, we can consider an embedding $R$ such that it approximately preserves norms for the affine embedding:

$$\|XSAR - AR\|_F \in (1 \pm \epsilon)\|XSA - F\|_F, \qquad \text{for all } X \in \mathbb{R}^{m\times k}.$$

Following the steps in the previous subsection, and letting $\hat{X} = AR(SAR)^-$ we have,

$$\min_{\text{rank}(X)\le k} \|XSAR - AR\|_F^2 = \|X^*SAR - AR\|_F^2 + \min_{\text{rank}(X)\le k} \|XSAR - X^*SAR\|_F \ .$$

Letting $SAR = U\Sigma V^T$, we have,

$$\|XSAR - X^*SAR\|_F = \|(X - X^*)U\Sigma V^T\|_F = \|(X - X^*)U\Sigma\|_F \ .$$

Therefore,

$$\begin{aligned}\min_{\text{rank}(X)\le k} \|XSAR - X^*SAR\|_F &= \min_{\text{rank}(X)\le k} \|XU\Sigma - AR(SAR)^-U\Sigma\|_F \\ &= \min_{\text{rank}(Y)\le k} \|Y - AR(SAR)^-U\Sigma\|_F \ .\end{aligned}$$

The solution is obtained by letting $Y^* = [AR(SAR)^-U\Sigma]_k$ and obtaining

$$X^* = Y^*\Sigma^-U^T = Y^*(\Sigma_k)^{-1}U_k^T \ .$$

Since $\text{rank}(Y^*) \le k$, $\text{rank}(X^*) \le k$.