# CS698C 2021 August Quiz 3

Anirudh Nanduri

TOTAL POINTS

**74 / 100**

QUESTION 1

## Multinomial 20 pts

### 1.1 joint pdf of X1 and X2 8 / 8

+ **0 pts** Correct
+ **8 Point adjustment**

### 1.2 Conditional pdf of X1,X2 given x3... x_{k-1} 5 / 12

+ **0 pts** Correct
+ **5 Point adjustment**

QUESTION 2

## Joint Poisson 20 pts

### 2.1 marginal pdf of X 10 / 10

✓ + **10 pts Correct**

+ **0 pts** Incorrect

### 2.2 marginal pdf of Y-X 10 / 10

+ **0 pts** InCorrect

✓ + **10 pts Correct**

QUESTION 3

## Poisson parameter conditional on Gamma 20 pts

### 3.1 Joint distribution 5 / 10

+ **0 pts** Correct
+ **0 pts** Click here to replace this description.
+ **0 pts** Click here to replace this description.
+ **5 Point adjustment**

### 3.2 Marginal pdf of X 6 / 10

+ **0 pts** Correct
+ **6 Point adjustment**

QUESTION 4

## n variate Normal distribution partitioned into X1 and X2 20 pts

### 4.1 Choose C appropriately so that X1 and W are indepednent. 10 / 10

+ **0 pts** Incorrect

✓ + **10 pts Corrrect solution**

### 4.2 Distribution of W? 5 / 5

✓ + **5 pts Correct**

+ **0 pts** Incorrect

### 4.3 Distribution of Y| X=x 0 / 5

✓ + **0 pts Incorrect**

+ **5 pts** Correct

QUESTION 5

## Normal distr 20 pts

### 5.1 Distribution of AZ 10 / 10

+ **0 pts** Incorrect or not attempted

✓ + **10 pts Correct Solution**

+ **5 pts** Partially Correct

### 5.2 Distribution of norm{AZ}^2 5 / 10

+ **0 pts** Incorrect or not attempted
+ **10 pts** Correct answer

✓ + **5 pts Partially correct**

**1.1 joint pdf of X1 and X2 8 / 8**

+ **0 pts** Correct

+ **8 Point adjustment**

### 1.2 Conditional pdf of X1,X2 given x3... x_{k-1} **5 / 12**

+ **0 pts** Correct

+ **5 Point adjustment**

## 2.1 marginal pdf of X **10 / 10**

✓ **+ 10 pts** **Correct**

**+ 0 pts** Incorrect

## 2.2 marginal pdf of Y-X **10 / 10**

+ **0 pts** InCorrect

✓ + **10 pts** Correct

### 3.1 Joint distribution **5 / 10**

- \+ **0 pts** Correct
- \+ **0 pts** Click here to replace this description.
- \+ **0 pts** Click here to replace this description.

\+ **5 Point adjustment**

### 3.2 Marginal pdf of X **6 / 10**

+ **0 pts** Correct

+ **6 Point adjustment**

5) $A\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ cx_1+x_2 \end{bmatrix}$ (9)

a) $\operatorname{cov}(x_1, cx_1+x_2) \overset{=0}{\cancel{\text{are}}}$ ~~indep~~

$c\operatorname{cov}(x_1, x_1) + \operatorname{cov}(x_1, x_2) = 0$

$c\Sigma_{11} + \Sigma_{12} = 0$

$c\Sigma_{11} = -\Sigma_{12}$

$c = -\Sigma_{12}\Sigma_{11}^{-1}$

b). $W = -\Sigma_{12}\Sigma_{11}^{-1}x_1 + x_2$

W is Normal distrib.

$E[x] = \text{Mean} = -\Sigma_{12}\Sigma_{11}^{-1}\mu_1 + \mu_2$

$\sigma^2 = -\Sigma_{12}\Sigma_{11}^{-1}\Sigma_{11} + \Sigma_{22}$

$= -\Sigma_{12} + \Sigma_{22}$

**4.1** Choose C appropriately so that X1 and W are indepednent. **10 / 10**

+ **0 pts** Incorrect

✓ + **10 pts** Corrrect solution

5) $A\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ cx_1 + x_2 \end{bmatrix}$

a) $\operatorname{cov}(x_1, cx_1 + x_2) \overset{=0}{\cancel{\text{are}}}$ ~~indep~~

$c\operatorname{cov}(x_1, x_{\cancel{A}}) + \operatorname{cov}(x_1, x_2) = 0$

$c\,\Sigma_{11} + \Sigma_{12} = 0$

$c\Sigma_{11} = -\Sigma_{12}$

$c = -\Sigma_{12}\Sigma_{11}^{-1}$

b). $W = -\Sigma_{12}\Sigma_{11}^{-1}x_1 + x_2$

W is Normal distrib.

$E[x] = \text{Mean} = -\Sigma_{12}\Sigma_{11}^{-1}\mu_1 + \mu_2$

$\sigma^2 = -\Sigma_{12}\Sigma_{11}^{-1}\Sigma_{11} + \Sigma_{22}$

$= -\Sigma_{12} + \Sigma_{22}$

### 4.2 Distribution of W? 5 / 5

✓ **+ 5 pts** Correct

**+ 0 pts** Incorrect

### 4.3 Distribution of Y| X=x **0 / 5**

✓ **+ 0 pts** Incorrect

**+ 5 pts** Correct

(A)

(4).

b)

a) Normal distributions are rotational invariant

$(A^T A = I)$ orthogonal

Let $y = AZ$

$\text{cov}(Y) = \text{cov}(AZ) = A^T \text{cov}(Z) A = A^T I A = I$

Let $\text{cov}(Z) = \Sigma = I$ (covariance of standard normal distribution).

Hence $\text{cov}(AZ) = \text{cov}(Z)$. — ①

For standard normal distribution $Z$ mean $\mu = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

Hence mean $AZ = E[AZ] = A[Z] = 0$.

Hence $E[AZ] = E[Z]$ — ②.

$y = AZ$
$Z = A^{-1} y$.

from ① and ② distribution of

$AZ =$ distribution of $Z := N_m(0, I_m)$.

pdf $\exp\left\{ -\frac{1}{2} A^{-1} Z (A^{-1} Z)^T \right\} = \exp\{(-1/2) | A$

(B).

~~$\|A^{-1} y\| = y^T (A^{-1})^T A^{-1} y =$~~

$\|A^{-1} Z\| = Z^T (A^{-1})^T A^{-1} Z = \|Z\|_2^2$.

## 5.1 Distribution of AZ **10 / 10**

- **+ 0 pts** Incorrect or not attempted
- ✓ **+ 10 pts** Correct Solution
- **+ 5 pts** Partially Correct

2111011

8

$z = n \times 1$
$A = n \times n$

6 b)

$$\|Az\|_2^2 = \sum_{i=1}^{n} (a_i^T z)^2$$

$$= \sum_{i=1}^{n} a_i z z^T a_i^T$$

$$= \|z\|^2 \quad \text{as } a \text{ is orthogow.}$$

## 5.2 Distribution of norm{AZ}^2 **5 / 10**

- \+ **0 pts** Incorrect or not attempted
- \+ **10 pts** Correct answer
- ✓ + **5 pts** Partially correct

21111011

Anirudh.N

1

2) $P(x,y) = \dfrac{\mu^y e^{-2\mu}}{x!\,(y-x)!}$

a) Marginal pmf of $X$

$$P(x) = \sum_{y=x}^{\infty} \frac{\mu^y e^{-2\mu}}{x!\,(y-x)!}$$

$e^{e^x}$

$$= e^{-2\mu} \sum_{y=x}^{\infty} \frac{\mu^y}{x!\,(y-x)!}$$

$$= \frac{e^{-2\mu}}{x!} \sum_{y=x}^{\infty} \frac{\mu^y}{(y-x)!}$$

$$= \frac{e^{-2\mu}}{x!} \left[ \frac{\mu^x}{0!} + \frac{\mu^{x+1}}{1!} + \frac{\mu^{x+2}}{2!} + \cdots \right].$$

$$= \frac{e^{-2\mu}}{x!} \mu^x \left[ \frac{1}{0!} + \frac{\mu}{1!} + \frac{\mu^2}{2!} + \cdots \right]$$

$$= \frac{e^{-2\mu}}{x!} \mu^x e^{\mu}$$

$$= \frac{\mu^x e^{-\mu}}{x!}$$

b) Marginal pmf of $Y-X$

$P(Y-X)$

2111011

(2)

$$P(x,y) = \frac{\mu^y e^{-2\mu}}{x!\,(y-x)!}$$

y ranges from x to $\infty$

y-x ranges from 0 to $\infty$

$$= e^{-2\mu}\left[\frac{\mu^y}{\mu^x}\cdot\frac{\mu^x}{x!\,(y-x)!}\right]$$

$$= e^{-2\mu}\left[\frac{\mu^{y-x}}{(y-x)!}\ \frac{\mu^x}{x!}\right]$$

Let $z = y - x$

$$= \frac{e^{-\mu}\mu^{y-x}}{(y-x)!}\ \frac{e^{-\mu}\mu^x}{x!}$$

$\text{Poisson}(y-x, x) = \text{Poisson}(y-x, \mu) \times \text{Poisson}(x, \mu)$

Hence y-x and x are independent.

$$P(y-x) = \frac{e^{-\mu}\mu^{y-x}}{(y-x)!}\sum_{x=0}^{\infty}\frac{e^{-\mu}\mu^x}{x!}$$

$$P(y-x) = \frac{e^{-\mu}\mu^{y-x}}{(y-x)!}$$

2111011

(4) (3)

3)

a) Let the pdf of gamma distribution be $f_X(x, \cancel{\alpha})$ and pmf of poisson be $g_W(w)$.

$$f(x,w) = \frac{1}{\Gamma(\alpha)} \left[ f(x/w)\, g(w) \right] \quad , \quad \beta = 1$$

$$= \frac{1}{\Gamma(\alpha)} \left[ e^{-x} x^{\alpha-1} \frac{e^{-w} w^{x}}{x!} \right]$$

$$= \frac{1}{\Gamma(\alpha)} \left[ \frac{e^{-x-w} x^{\alpha-1} w^{x}}{x!} \right].$$

b) Marginal pmf of $X$.

$P_X(a)$.

$$P_X(a) = \int_0^\infty f(a,w)\, dw.$$

$$= \frac{1}{\Gamma(\alpha)} \int_0^\infty \frac{e^{-a-w} a^{\alpha-1} w^{a}}{a!} dw$$

$$= \frac{a^{\alpha-1}}{\Gamma(\alpha) a!} \int_0^\infty e^{-a-w} w^{a}\, dw$$

21111011

5

3b) contd.

$$P_X(a) = \frac{a^{\alpha-1} e^{-a}}{\Gamma(\alpha)\, a!} \int_0^\infty e^{-w} w^a \, dw.$$

$$P_X(a) \; \frac{a^{\alpha-1} e^{-a}}{\Gamma(\alpha)\, a!} \Gamma(a+1).$$

$$\int_0^\infty e^{-x} x^{\alpha-1} = \Gamma(\alpha)$$

$\alpha - 1 = a$

$\alpha = a+1$

$$P_X(a) = \frac{\Gamma(a+1)}{\Gamma(\alpha)} \frac{a^{\alpha-1} e^{-a}}{a!}$$

21111011

(5)

1)

Given

$$P_X(x_1, x_2, \ldots, x_{k-1}) = \frac{n!}{x_1! x_2! \cdots x_{k-1}! x_k!} P_1^{x_1} \cdots P_{k-1}^{x_{k-1}} P_k^{x_k}.$$

$x_1 + x_2 + \cdots x_{k-1} \le n$.

$0 \le x_1 \ldots x_{k-1}$

a) Joint pmf $x_1, x_2$

$$P(x_1, x_2) = {}^{n}C_{x_1} P_1^{x_1} \; {}^{n-x_1}C_{x_2} P_2^{x_2} (1 - P_1 - P_2)^{n - x_1 - x_2}$$

Here $P_1$ is probability of $x_1$

$P_2$ is probability of $x_2$.

$$P(x_1, x_2) = {}^{n}C_{x_1} \; {}^{n-x_1}C_{x_2} P_1^{x_1} P_2^{x_2} (1 - P_1 - P_2)^{n - x_1 - x_2}$$

$0 \le x_1 + x_2 \le n$.

b) $P(x_1, x_2 / (X_3 = x_3, X_4 = x_4 \ldots))$

$$\frac{P(x_1, x_2, X_3 = x_3, X_4 = x_4 \ldots X_k = x_k)}{P(X_3 = x_3, X_4 = x_4 \ldots).}$$

$P(A|B) = \frac{P(A, B)}{P(B)}.$

Denominator

$P(X_3 = x_3, X_4 = x_4, X_5 = x_5 \cdots)$.

$$= \frac{n!}{x_3! x_4! \cdots x_k!} P_3^{x_3} P_4^{x_4} \cdots P_{k-1}^{x_{k-1}} P_k^{x_k}.$$

2|11|011

(b)

$$x_3 + x_4 + \dots + x_k = 1 - x_1 - x_2$$

$$P(X_1, X_2 \mid X_3 = x_3 \dots X_k = x_k)$$

$$= \cancel{P(X_1 \mid X_3 = x_3 \dots X_k = x_k)\, P(X_2 \mid X_3 = x_3 \dots X_k = x_k)}$$

independent

$$= \cancel{{}^{n}C_{x_1}\, p_1^{x_1}\, {}^{n-x_1} p_2^{x_2}}$$

$$= {}^{n-(x_3+\dots x_k)}C_{x_1}\, p_1^{x_1}\; {}^{n-(x_1+x_3+\dots x_k)}C_{x_2}\, p_2^{x_2}$$

$$= {}^{n-1+x_1+x_2}C_{x_1}\, p_1^{x_1}\; {}^{n-\cancel{x_1}-1+\cancel{x_1}+x_2}C_{x_2}\, p_2^{x_2}$$

$$= {}^{n-1+x_1+x_2}C_{x_1}\; {}^{n+x_2-1}C_{x_2}\, p_1^{x_1}\, p_2^{x_2}$$