

① $Z = 3X - 2Y$
 $E[Z] = E[3X - 2Y] = 3E[X] - 2E[Y]$
 $= 3(1) - 2(4) = 3 - 8 = -5$
 , by linearity of expectation

$Var[Z] = Var[3X - 2Y]$
 $= Var[3X] - Var[2Y]$, because X & Y are independent.

$= 9 Var[X] - 4 Var[Y]$
 $= 9(4) - 4(6)$
 $= 36 - 24 = 12$

② (a) $E[S] = E[a_1 r_1 + \dots + a_n r_n]$
 $= E[a_1 r_1] + \dots + E[a_n r_n]$, by linearity of expectation

$= a_1 E[r_1] + \dots + a_n E[r_n]$
 $= a_1 \left(1\left(\frac{1}{2}\right) + (-1)\left(\frac{1}{2}\right) \right) + \dots + a_n \left(1\left(\frac{1}{2}\right) + (-1)\left(\frac{1}{2}\right) \right)$
 $= a_1(0) + \dots + a_n(0)$
 $= 0$

(b) $Var[S] = Var[a_1 r_1 + \dots + a_n r_n]$
 $= a_1^2 Var[r_1] + \dots + a_n^2 Var[r_n]$ - (i)
 , bcz $r_i \forall i \in 1$ to n are independent.

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$$\begin{aligned}\text{Var}[x_i] &= E[x_i^2] - (E[x_i])^2 \\ &= \cancel{x_i^2} \left((1)^2 \frac{1}{2} + (-1)^2 \frac{1}{2} \right) - (0)^2 \\ &= \frac{1}{2} + \frac{1}{2} = 1\end{aligned}$$

from eqn. (1) :-

$$\begin{aligned}\text{Var}[S] &= a_1^2(1) + \dots + a_n^2(1) \\ &= a_1^2 + a_2^2 + \dots + a_n^2\end{aligned}$$

$$\begin{aligned}\textcircled{4} \textcircled{a) } E[(X_1 - \mu_1) | X_2 = x_2] \\ &= \sum_{x_1} (x_1 - \mu_1) P(X_1 = x_1 | X_2 = x_2) \\ &= \sum_{x_1} x_1 P(X_1 = x_1 | X_2 = x_2) - \mu_1\end{aligned}$$

But given that $E[(X_1 - \mu_1) | X_2 = x_2] = a + bx_2$
Taking expectation of both sides :-

$$E[E[(X_1 - \mu_1) | X_2 = x_2]] = E[a + bx_2] = a + b\mu_2$$

$$\Rightarrow E\left[\sum_{x_1} x_1 P(X_1 = x_1 | X_2 = x_2) - \mu_1\right] = a + b\mu_2$$

$$\Rightarrow E\left[\sum_{x_1} x_1 P(X_1 = x_1 | X_2 = x_2)\right] - E[\mu_1] = a + b\mu_2$$

$$\Rightarrow \mu_1 - \mu_1 = a + b\mu_2$$

$$\Rightarrow a + b\mu_2 = 0$$

(3) ~~$\text{Var}[Y]$~~
 ~~$\text{Var}[Y] = \text{Var}[a_1 X_1 + \dots + a_n X_n]$~~
 ~~$\text{Var}[Y]$~~

$$\text{Var}[Y] = E[Y^2] - (E[Y])^2 \quad \text{--- (1)}$$

$$\begin{aligned} E[Y] &= E[a_1 X_1 + \dots + a_n X_n] \\ &= a_1 E[X_1] + \dots + a_n E[X_n] \end{aligned}$$

$$= \begin{bmatrix} a_1 & a_2 & \dots & a_n \end{bmatrix} \begin{bmatrix} E[X_1] \\ E[X_2] \\ \vdots \\ E[X_n] \end{bmatrix}$$

$$\begin{aligned} E[Y^2] &= E[(a_1 X_1 + \dots + a_n X_n)^2] \\ &= E[a_1^2 X_1^2 + \dots + a_n^2 X_n^2] + \sum_j \sum_i \text{Cov}(X_i, X_j) \\ &= a_1^2 E[X_1^2] + \dots + a_n^2 E[X_n^2] + \sum_j \sum_i \text{Cov}(X_i, X_j) \end{aligned}$$

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Ques 3 continued

From eqn (1)

$$\text{Var}[Y] = a_1^2 E[X_1^2] + \dots + a_n^2 E[X_n^2] + \sum_j \sum_i \text{Cov}(X_i, X_j) - (a_1 E[X_1] + \dots + a_n E[X_n])^2$$

$$= a_1^2 E[X_1^2] + \dots + a_n^2 E[X_n^2] + \sum_j \sum_i \text{Cov}(X_i, X_j)$$

$$- a_1^2 E[X_1]^2 - \dots - a_n^2 E[X_n]^2 - \sum_j \sum_i E[X_i] E[X_j]$$

$$- \sum_j \sum_i \sum_k E[X_i] E[X_j] E[X_k]$$

$$\dots - \sum \sum \sum \dots \sum (n \text{ times}) E[X_i] \dots E[X_n]$$

$$= a_1^2 [E[X_1^2] - E[X_1]^2] + \dots + a_n^2 [E[X_n^2] - E[X_n]^2]$$

$$+ \sum_j \sum_i \text{Cov}(X_i, X_j) - \sum_j \sum_i E[X_i] E[X_j]$$

$$\dots - \sum \sum \dots n \text{ times } E[X_i] \dots E[X_n]$$

$$= a_1^2 \text{Var}[X_1] + \dots + a_n^2 \text{Var}[X_n]$$

$$+ \sum_j \sum_i \text{Cov}(X_i, X_j) - \sum_j \sum_i E[X_i] E[X_j]$$

$$\dots - \sum \sum \dots n \text{ times } E[X_i] \dots E[X_n]$$

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Ques 3 continued
Similarly,

$$\text{Var}[Y] = b_1^2 \text{Var}[X_1] + \dots + b_n^2 \text{Var}[X_n]$$

$$+ \sum_j \sum_i \text{Cov}(X_i, X_j) - \sum_j \sum_i E[X_j] E[X_i]$$

$$\dots - \sum \dots n \text{ times } E[X_i] \dots E[X_n]$$

(2)

from (1) & (2)

$$\text{Var}[Y] = b_1^2 \text{Var}[X_1] + \dots + b_n^2 \text{Var}[X_n]$$

$$+ \sum_j \sum_i \text{Cov}(X_i, X_j)$$

$$= a_1^2 \text{Var}[X_1] + \dots + a_n^2 \text{Var}[X_n]$$

$$+ \sum_j \sum_i \text{Cov}(X_i, X_j)$$

$$= \text{Var}[Y]$$