CS698C 2021 August Quiz 4

Anirudh Nanduri

TOTAL POINTS

92 / 150

QUESTION 1

Area of triangle 50 pts

1.1 Method and arguments 25 / 25

✓ + **25 pts** **Correct**

+ **0 pts** Incoorrect

Page3, you have dropped some terms, which is wrong, (a^{T}b can be -ve)

1.2 Analysis of m 25 / 25

✓ + **25 pts** **Correct**

+ **0 pts** Incorrect

QUESTION 2

Area of parallelopiped 50 pts

2.1 Method arguments 17 / 25

✓ + **25 pts** **Correct**

+ **0 pts** Incorrect

- 8 Point adjustment

Formula is wrong, its correct only if they are orthogonal, beta is a random variable so it doesn't work if you don't assume colspace embedding,You have explicitly not mentioned using Colspace embedding, Eq:1, on page-9 is not valid unless you use colspace embedding

2.2 Analysis of m 25 / 25

✓ + **25 pts** **Correct**

+ **0 pts** Incorrect

QUESTION 3

3 Bonus 0 / 50

+ **50 pts** Correct

✓ + **0 pts** **Incorrect or not attempted**

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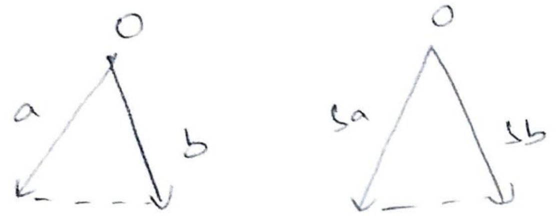
1 a)



$\leftarrow$ Mapping to low dimensions

Area of $(\Delta(0,a,b)) = \frac{1}{2}\|a\| \left\| b - \frac{a^Tb}{\|a\|^2} a \right\|$

Area of $(\Delta(0,Sa,Sb)) = \frac{1}{2}\|Sa\| \left\| Sb - \frac{(Sa)^T(Sb)}{\|Sa\|^2} Sa \right\|$

According to JL Lemma

$\rightarrow$ S is iid $N(0,\frac{1}{m})$ entries $m \times n$ matrix for any fixed vector $a \in \mathbb{R}^n$ $P\left(\left| \|Sa\|^2 - \|a\|^2 \right| \leq \epsilon \|a\|^2 \right) \geq 1-\delta$

where $m \geq O\left(\frac{1}{\epsilon^2} \log \frac{1}{\delta}\right)$

$\|Sa\|^2 = (1 \pm \epsilon)\|a\|^2$

$\|Sa\| = (1 \pm \epsilon)\|a\|$

$\|Sb\| = (1 \pm \epsilon)\|b\|$

$\boxed{\begin{aligned} &\|Sa\| = (1\pm\epsilon)^{1/2}\|a\| \\ &\text{for } \epsilon < 1/2 \\ &\|Sa\| = \left(1 \pm \tfrac{3\epsilon}{4}\right)\|a\| \\ &\|Sa\| = (1\pm\epsilon)\|a\| \end{aligned}}$

Area of $(\Delta(0,Sa,Sb))$

$= \frac{1}{2}\|Sa\| \left\| Sb - \frac{(Sa)^T(Sb)}{\|Sa\|^2} Sa \right\|$

$= \frac{1}{2}(1\pm\epsilon)\|a\| \left\| Sb - \frac{(Sa)^T(Sb)}{\|Sa\|^2} Sa \right\|$ —— ①

Using $\|Sa\| = (1\pm\epsilon)\|a\|$

Consider the term

Area of $(\Delta(0, Sa, Sb))$

$$= \frac{1}{2}(1 \pm \epsilon)\|a\| \left\| Sb - \frac{(Sa)^T(Sb)}{\|Sa\|^2} Sa \right\|$$

Consider the term

$$\left\| Sb - \frac{(Sa)^T(Sb)}{\|Sa\|^2} Sa \right\|$$

Squaring it we get

$$\left\| Sb - \frac{(Sa)^T(Sb)}{\|Sa\|^2} Sa \right\|^2$$

$$= \|Sb\|^2 + \left\| \frac{(Sa)^T(Sb)}{\|Sa\|^2} Sa \right\|^2 - 2(Sb)^T(Sa)\frac{(Sa)^T(Sb)}{\|Sa\|^2}$$

$$= \|Sb\|^2 + \frac{[(Sa)^T(Sb)]^2 \|Sa\|^2}{\|Sa\|^4} - 2\frac{[(Sa)^T(Sb)]^2}{\|Sa\|^2}$$

Using $(Sb)^T(Sa) = (Sa)^T(Sb)$

$$= \|Sb\|^2 + \frac{[(Sa)^T(Sb)]^2}{\|Sa\|^2} - 2\frac{[(Sa)^T(Sb)]^2}{\|Sa\|^2}$$

$$= \|Sb\|^2 - \frac{[(Sa)^T(Sb)]^2}{\|Sa\|^2}$$

Using $\left| (Sa)^T(Sb) - a^T b \right| < 3\epsilon\|a\|\|b\|$ in the eqn above

$(Sa)^T(Sb) < a^T b \pm 3\epsilon\|a\|\|b\|$

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$$\leq \|Sb\|^2 - \frac{\left[(a^Tb) \pm 3\epsilon\|a\|\|b\|\right]^2}{(1\pm\epsilon)\|a\|^2}$$

Using $\|Sb\|^2 = (1\pm\epsilon)\|b\|^2$

$\|Sa\|^2 = (1\pm\epsilon)\|a\|^2$

$$= (1\pm\epsilon)\|b\|^2 - \left[\frac{(a^Tb)^2 + 9\epsilon^2\|a\|^2\|b\|^2 \pm 6\epsilon(a^Tb)\|a\|\|b\|}{(1\pm\epsilon)\|a\|^2}\right]$$

Using $\frac{1}{1\pm\epsilon} = 1\pm\epsilon$

$\epsilon(1\pm\epsilon) = \epsilon$.

$$= (1\pm\epsilon)\|b\|^2 - \left[\frac{(a^Tb)^2 + \epsilon\|a\|^2\|b\|^2 \pm \epsilon(a^Tb)\|a\|\|b\|}{(1\pm\epsilon)\|a\|^2}\right].$$

$$= (1\pm\epsilon)\|b\|^2 - (1\pm\epsilon)\frac{(a^Tb)^2}{\|a\|^2} \underbrace{- \epsilon\|b\|^2 \pm \epsilon\frac{(a^Tb)\|b\|}{\|a\|}}$$

dropping these terms as they are (net) negative.

$$\leq (1\pm\epsilon)\|b\|^2 - (1\pm\epsilon)\frac{(a^Tb)^2}{\|a\|^2}.$$

$$\leq (1\pm\epsilon)\left\|b - \frac{a^Tb}{\|a\|^2}a\right\|^2$$

$$\boxed{\text{Using } \left\|b - \frac{a^Tb}{\|a\|^2}a\right\|^2 = \|b\|^2 - \frac{(a^Tb)^2}{\|a\|^2}.}$$

Hence

$$\left\|Sb - \frac{(Sa)^T(Sb)}{\|Sa\|^2}Sa\right\| = (1\pm\epsilon)\left\|b - \frac{a^Tb}{\|a\|^2}a\right\| \quad \text{— (2)}.$$

$$\min_\alpha \|Sb - \alpha Sa\| = (1\pm\epsilon)\min_\alpha\|b - \alpha a\|$$

| height of triangle is preserved

Using (2) in area eqn.

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$$\text{Area of } (\Delta(0, Sa, Sb)) = \frac{1}{2}\|Sa\| \left\| Sb - \frac{(Sa)^T(Sb)}{\|Sa\|^2} Sa \right\|$$

$$= \frac{1}{2}(1\pm\epsilon)\|a\|(1\pm\epsilon)\left\| b - \frac{a^Tb}{\|a\|^2}a \right\|$$

$$= \frac{1}{2}(1\pm\epsilon)\|a\|\left\| b - \frac{a^Tb}{\|a\|^2}a \right\|$$

$$= (1\pm\epsilon)\,\text{area}(\Delta(0,a,b)).$$

Hence $\left|\text{area}(\Delta(0,Sa,Sb)) - \text{area}(\Delta(0,a,b))\right|$

$\leq O(\epsilon)\cdot\text{area}(\Delta(0,a,b))$

$$P\left[\left|\text{area}(\Delta(0,Sa,Sb)) - \text{area}(\Delta(0,a,b))\right| \leq O(\epsilon)\,\text{area}(\Delta(0,a,b))\right] \geq 1-3\delta$$

Here 3δ is used because we have to preserve norms of vectors $|\vec{a}|, |\vec{b}|$, $|\overrightarrow{\min_{\alpha\in\mathbb{R}} b-\alpha a}|$

$\|Sa\|^2 - \|a\|^2 = (\epsilon)\|a\|^2$ with probability $1-\delta$

$\|Sb\|^2 - \|b\|^2 = \epsilon\|b\|^2$ with probability $1-\delta$

~~$\|Sa - Sb\|^2 - \|a-b\|^2 = \epsilon\|a-b\|^2$ with probability $1-\delta$~~

$\min_{\alpha\in\mathbb{R}}\|b-\alpha a\|^2 - \|Sb-\alpha(Sa)\|^2 = \epsilon\|b-\alpha a\|^2$ with probability $1-\delta$

1.1 Method and arguments **25 / 25**

✓ **+ 25 pts** **Correct**

+ 0 pts Incoorrect

Page3, you have dropped some terms, which is wrong, (a^{T}b can be -ve)

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(6)

1 b) Using JL lemma and from result proved in part (a)

$$P\left[\,\left|\text{area}(\Delta(0, Sa, Sb)) - \text{area}(\Delta(0,a,b))\right| \le O(\epsilon)\,\text{area}(\Delta(0,a,b))\right] \ge 1-3\delta$$

Size of S matrix $m \times n$

iid $(0, \frac{1}{m})$

From $\chi^2(m)$ tail distribution.

$$P\left[\,|w-m| > \epsilon m\right] \le 2e^{-\epsilon^2 m/8}$$

Considering $2e^{-\epsilon^2 m/8} \le \delta$.

$$e^{-\epsilon^2 m/8} \le \delta/2$$

$$-\epsilon^2 m/8 \le \log(\delta/2)$$

Taking log on both sides. log is a monotonic function Hence, comparison sign does not change

$$\epsilon^2 m/8 \ge \log\left(\frac{2}{\delta}\right)$$

$$m \ge \frac{8}{\epsilon^2}\log\left(\frac{2}{\delta}\right)$$

$$m = O\left(\frac{1}{\epsilon^2}\log\frac{1}{\delta}\right)$$

$$Sa = \begin{bmatrix} - & s_1^T & - \\ & \vdots & \\ - & s_m^T & - \end{bmatrix} a$$

$$= \begin{bmatrix} - & s_1^T a & - \\ - & s_2^T a & - \\ & \vdots & \\ - & s_m^T a & - \end{bmatrix}$$

$$\|Sa\|^2 = (s_1^T a)^2 + \cdots (s_m^T a)^2$$

$$= \chi^2(m)$$

1.2 Analysis of m 25 / 25

- ✓ + **25 pts** **Correct**
- \+ **0 pts** Incorrect

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2a) Volume of parallelopiped with sides $\bar{a}, \bar{b}, \bar{c}$ is

$$\text{vol}(0, a, b, c) = \|a\| \left\| b - \frac{a^T b}{\|a\|^2} a \right\| \left\| c - \frac{a^T c}{\|a\|^2} a - \frac{b^T c}{\|b\|^2} b \right\|$$

$$= \|a\| \min_{\alpha} \|b - \alpha a\| \min_{\beta} \| c - \beta A \|$$

$$A = [a, b]$$

Volume in reduced dimension having sides $\bar{Sa}, \bar{Sb}, \bar{Sc}$ is

$$\text{vol}(0, Sa, Sb, Sc) = \|Sa\| \left\| Sb - \frac{(Sa)^T (Sb)}{\|Sa\|^2} Sa \right\| \left\| Sc - \frac{(Sa)^T (Sc)}{\|Sa\|^2} Sa - \frac{(Sb)^T (Sc)}{\|Sb\|^2} Sb \right\|$$

$$= \|Sa\| \min_{\alpha} \|Sb - \alpha Sa\| \min_{\beta} \| Sc - \beta A]$$

$$A = [Sa, Sb].$$

~~Using~~ JL ~~Lemma~~

Consider the term

$$\min_{\alpha} \|b - \alpha a\|$$

Here $\alpha = \frac{a^T b}{\|a\|^2}$.

$$\min_{\beta} \| Sb - \beta Sa\|$$

Here $\beta = \frac{(Sa)^T Sb}{\|Sa\|^2}$

Using JL lemma and Taylor Series

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$\|Sa\| = (1 \pm \epsilon)\|a\|$

$\|Sb\| = (1 \pm \epsilon)\|b\|$

$\|b - \alpha a\| = (1 \pm \epsilon)\|Sb - \alpha Sa\|$ Here $\boxed{\alpha = \frac{a^T b}{\|a\|^2}}$

$\|b - \beta a\| = (1 \pm \epsilon)\|Sb - \beta Sa\|$ Here $\boxed{\beta = \frac{(Sa)^T Sb}{\|Sa\|^2}}$

$$\boxed{(1-\epsilon)\|b - \beta a\| \leq \|Sb - \beta Sa\| \leq \|Sb - \alpha Sa\| \leq (1 \pm \epsilon)\|b - \alpha a\|}$$

$(1-\epsilon)\|b - \beta a\| = (1+\epsilon)\|b - \alpha a\|$

$\|b - \beta a\| = (1 + O(\epsilon))\|b - \alpha a\|$

$\|Sb - \beta Sa\| = (1+\epsilon)\|b - \alpha a\|$ —— (1)

Similarly

$\|Sc - \beta A'\| = (1+\epsilon)\|c - \alpha A\|$ —— (2)

$A = [a, b]$

$A' = [Sa, Sb]$

Using (1) and (2) in the volume eqn.

$\text{vol}(0, Sa, Sb, Sc)$

$= \|Sa\| \min_\beta \|Sb - \beta Sa\| \min_\beta \|Sc - \beta A'\|$ $A' = [Sa, Sb]$

$= \cancel{(1 \pm \epsilon)\|a\| (1+\epsilon) \min_\alpha \|b - \alpha a\|} (1+\epsilon)\|b - \alpha A\|$
$\Uparrow$ ~~using (1)~~

$= (1 \pm \epsilon)\|a\| (1+\epsilon) \min_\alpha \|b - \alpha a\| (1+\epsilon) \min_\alpha \|c - \alpha A\|$

$\Uparrow$ using (1) $\Uparrow$ using (2)

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$$= (1 \pm \epsilon) \|a\| \min_{\alpha} \|b - \alpha a\| \min_{\alpha} \|c - \beta A\|$$

$$= (1 \pm \epsilon) \, vol(0, a, b, c).$$

$$= (1 \pm O(\epsilon)) \, vol(0, a, b, c)$$

Hence

$$|vol(0, Sa, Sb, Sc) - vol(0, a, b, c)| \leq O(\epsilon) \, vol(0, a, b, c)$$

We need to preserve norm of vectors $|a|, |b|, |c|$,

$\|b - \alpha a\|$ Here $\left(\alpha = \frac{a^T b}{\|a\|^2}\right)$, $\|b - \beta a\|$ Here $\left(\beta = \frac{(Sa)^T Sb}{\|Sa\|^2}\right)$,

$\|c - \beta_1 [a, b]\|$ $\quad \beta_1 = \left[\frac{(a)^T c}{\|a\|^2} \quad \frac{b^T c}{\|b\|^2}\right]$

$\|c - \beta_2 [a, b]\|$ $\quad \beta_2 = \left[\frac{Sa^T Sc}{\|Sa\|^2} \quad \frac{Sb^T Sc}{\|Sb\|^2}\right].$

Hence

$$\Pr\left[|vol(0, Sa, Sb, Sc) - vol(0, a, b, c)| \leq O(\epsilon) \, vol(0, a, b, c)\right]$$

$$\geq 1 - 5\delta$$

2.1 Method arguments 17 / 25

✓ + 25 pts Correct

+ 0 pts Incorrect

- 8 Point adjustment

Formula is wrong, its correct only if they are orthogonal, beta is a random variable so it doesn't work if you don't assume colspace embedding,You have explicitly not mentioned using Colspace embedding, Eq:1, on page-9 is not valid unless you use colspace embedding

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2b) S is a random matrix with entries $(0, \frac{1}{m})$.

$$Sa = \begin{bmatrix} — s_1^T — \\ \vdots \\ — s_m^T — \end{bmatrix} = \begin{bmatrix} — s_1^T a — \\ \vdots \\ — s_m^T a — \end{bmatrix}$$

$\|Sa\|^2 = (s_1^T a)^2 + \dots (s_m^T a)^2 \in \chi^2(m)$ distribution.

Using JL lemma and from result provided in part 2a

$$P\left[|\text{vol}(0, Sa, Sb, Sc) - \text{vol}(0, a, b, c)| \leq O(\epsilon)\, \text{vol}(0, a, b, c) \right] \geq 1 - 5\delta \quad \text{—} \quad (1)$$

from $\chi^2(m)$ tail distribution

$$P\left[|w - m| > \epsilon m \right] \leq 2e^{-\epsilon^2 m/8}$$

$$2e^{-\epsilon^2 m/8} \leq 5\delta' \quad \text{from } (1).$$

$$e^{-\epsilon^2 m/8} \leq 5\delta/2$$

$$-\epsilon^2 m/8 \leq \log\left(\frac{5\delta}{2}\right)$$

$$\epsilon^2 m/8 \geq \log\left(\frac{2}{5\delta}\right)$$

$$m \geq \frac{8}{\epsilon^2} \log\left(\frac{2}{5\delta}\right)$$

$$m = O\left(\frac{1}{\epsilon^2} \log \frac{1}{\delta}\right)$$

2.2 Analysis of m 25 / 25

- ✓ **+ 25 pts** Correct
- **+ 0 pts** Incorrect

3 Bonus 0 / 50

+ **50 pts** Correct

✓ + **0 pts** **Incorrect or not attempted**