TA Session : 16th October, 2020

Principal Component Analysis

(I) **What is it?**

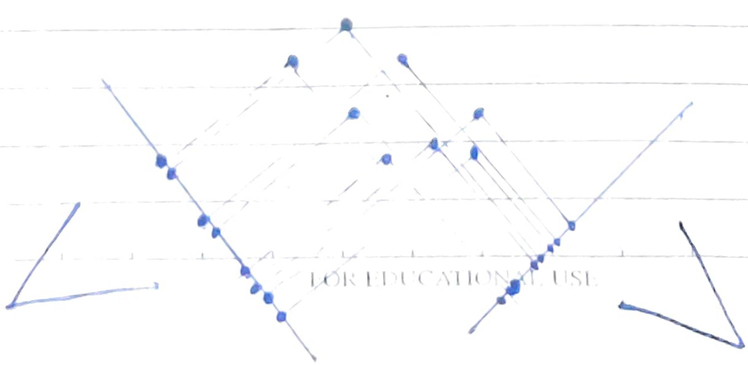
Single line : unsupervised dimensionality reduction technique.

Defn : PCA is a statistical procedure that uses an orthogonal transformation to convert a set of observations of possibly correlated variables into a set of linearly uncorrelated variables called principal components.

Why?

- It is an important technique for visualization
- Denoising and data compression.
- reduces training time?

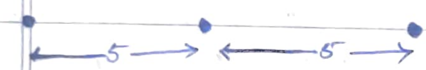
Think of it as taking a picture!



(II) Lens 1

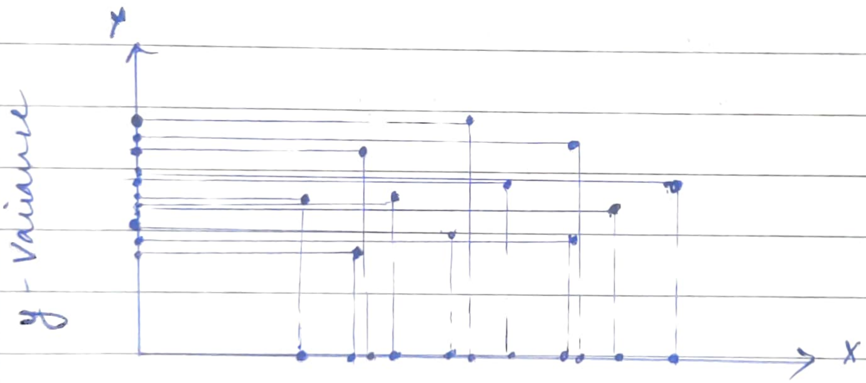
- Mean
- Variance

$$\frac{1^2+0^2+1^2}{3}$$ How spread out ?

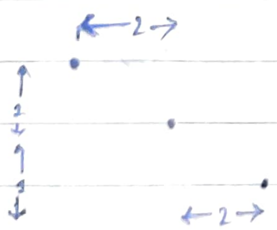


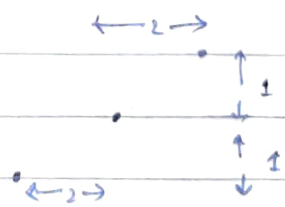
$$\frac{5^2+0^2+5^2}{3}$$

2D



x - variance

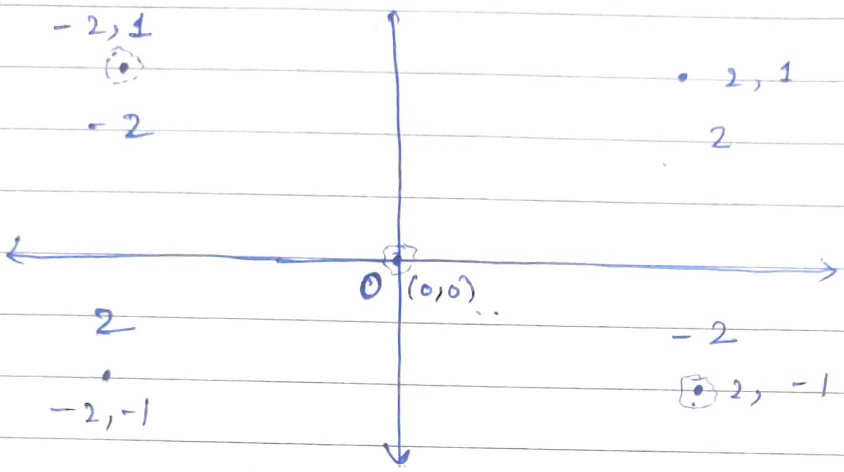




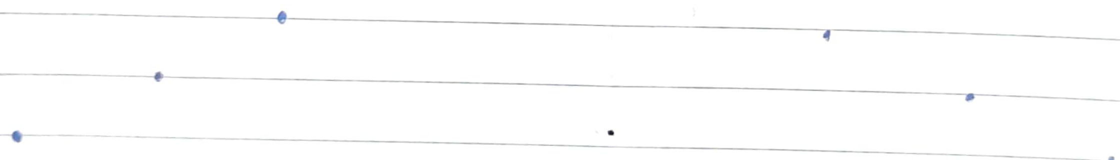
x - variance $= \frac{2^2+0^2+2^2}{3}$

y - variance $= \frac{1^2+0^2+1^2}{3}$

Covariance



Sum of product of co-ordinates

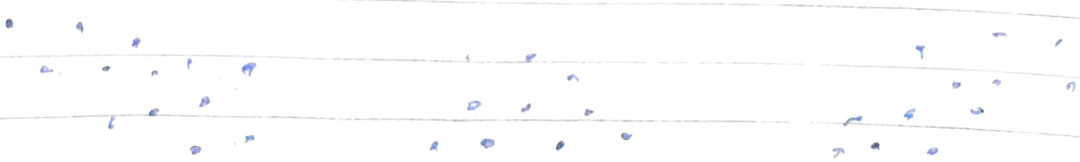


covariance : $\frac{2 + 0 + 2}{3}$

$= 4/3$

covariance : $\frac{(-2) + 0 + (-2)}{3}$

$= -4/3$

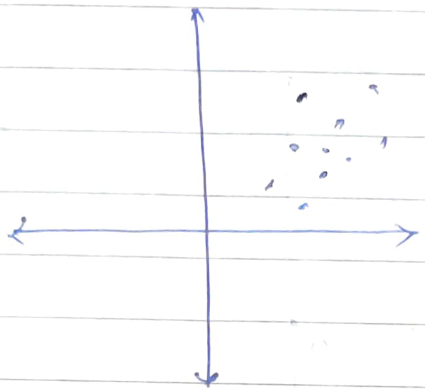


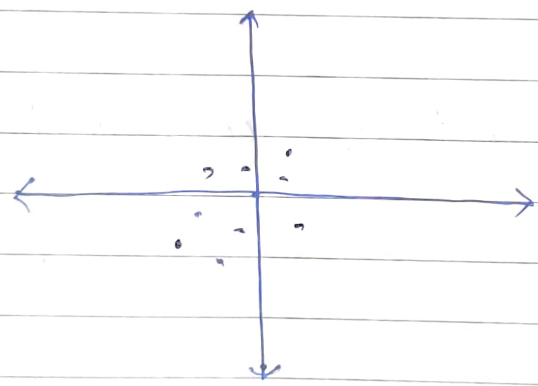
negative covariance

zero covariance

positive covariance

PCA (2D)



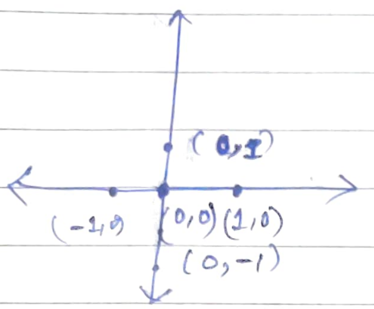


Covariance Matrix

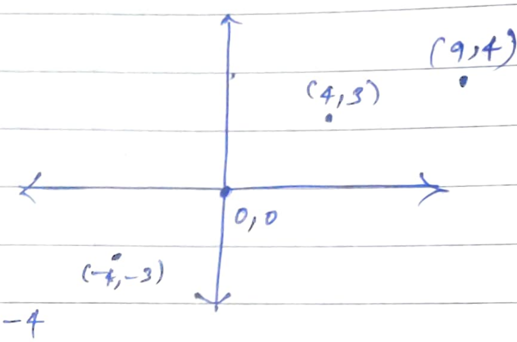
[illegible] $V = \begin{bmatrix} Var(X) & Cov(X,Y) \\ Cov(X,Y) & Var(Y) \end{bmatrix}$

$$\left\{ \begin{bmatrix} 9 & 4 \\ 4 & 3 \end{bmatrix} \right\}$$

Linear Transformation

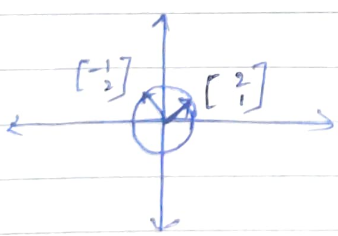


$$\begin{bmatrix} 9 & 4 \\ 4 & 3 \end{bmatrix} \longrightarrow$$

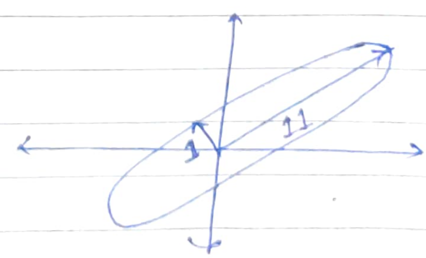


$$(x, y) \longrightarrow (9x + 4y,\ 4x + 3y)$$

$$\begin{aligned}
(0,0) &\longrightarrow (0,0) \\
(1,0) &\longrightarrow (9,4) \\
(0,1) &\longrightarrow (4,3) \\
(-1,0) &\longrightarrow (-9,-4) \\
(0,-1) &\longrightarrow (-4,-3)
\end{aligned}$$



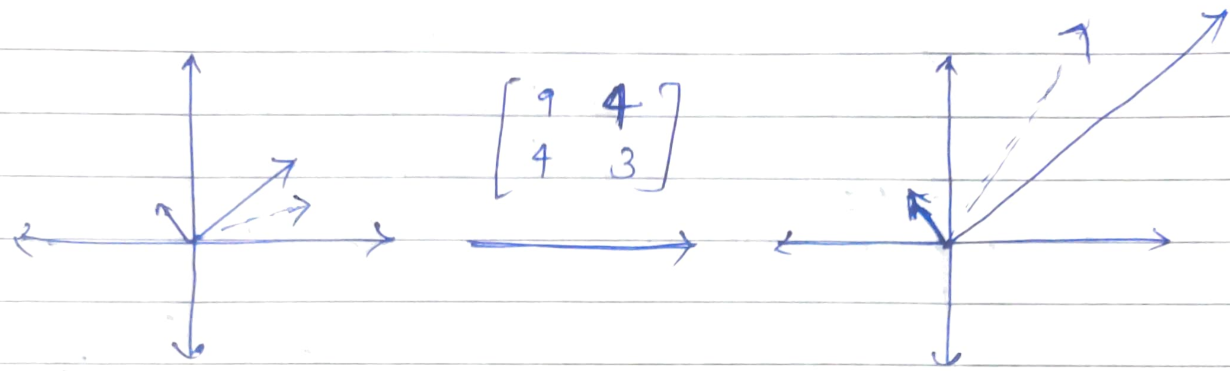
$$\begin{bmatrix} 9 & 4 \\ 4 & 3 \end{bmatrix} \longrightarrow$$



Eigen vectors $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ $\begin{bmatrix} -1 \\ 2 \end{bmatrix}$

Eigen values 11 1

Why eigenvectors?



$$\begin{bmatrix} 9 & 4 \\ 4 & 3 \end{bmatrix} \underbrace{v}_{\text{eigenvector}} = \underbrace{\lambda}_{\text{eigenvalue}} v$$

How to calculate?

$$\begin{bmatrix} 9 & 4 \\ 4 & 3 \end{bmatrix}$$

Find eigen values

- Characteristic polynomial

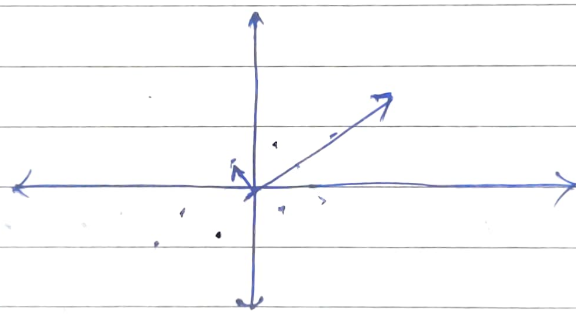
$$\begin{vmatrix} (x-9) & -4 \\ -4 & (x-3) \end{vmatrix} = (x-9)(x-3) - (-4)(-4)$$
$$= x^2 - 12x + 11$$
$$= (x-11)\cdot(x-1)$$

Eigenvalues : 11 and 1

Eigenvectors

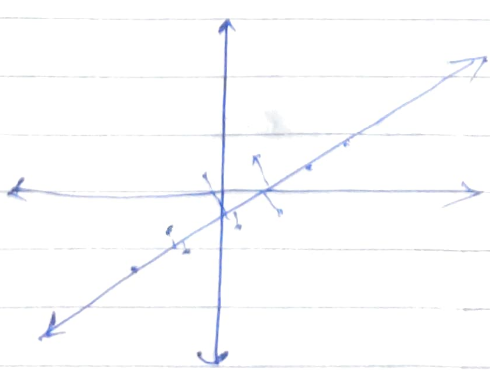
$$\begin{bmatrix} 9 & 4 \\ 4 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 11 \begin{bmatrix} x \\ y \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$



- Eigenvalues are real.
- The eigenvectors are perpendicular.

Why ?



PCA (Multi - dimension)

	att 1	att 2	att 3	att 4	att 5
N_1					
N_2					
N_3					
⋮					

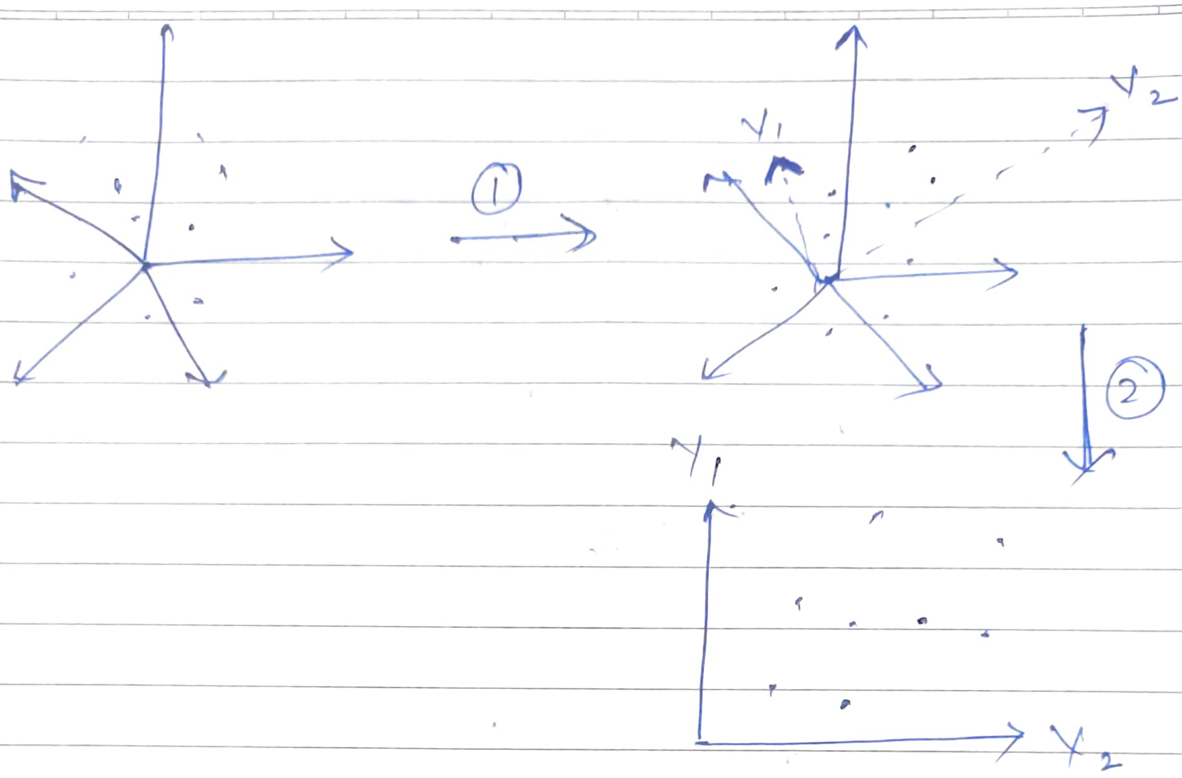
Covariance Matrix (5 x 5)

$$\begin{array}{c} \\ x \\ y \\ z \\ u \end{array}\begin{bmatrix} a & b & c & d & e \\ & & & & \\ & & & & \\ & & & & \\ & & & & \end{bmatrix}$$

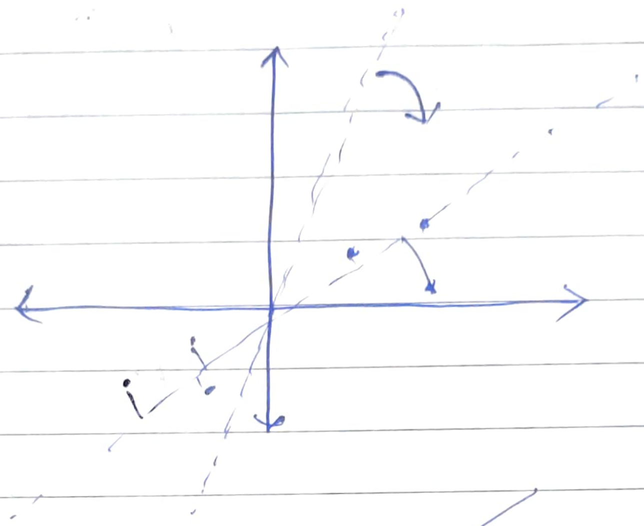
5 eigenvectors & eigenvalues.

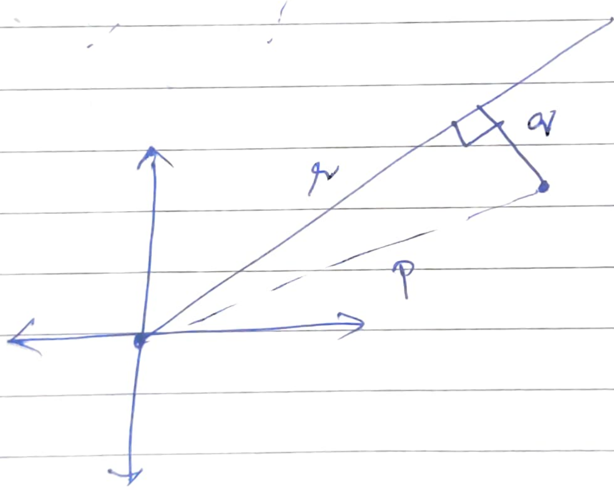
V_1	λ_1
V_2	λ_2
V_3	λ_3
V_4	λ_4
V_5	λ_5

(descending order)


1
Y_1
Y_2
2
Y_1
Y_2

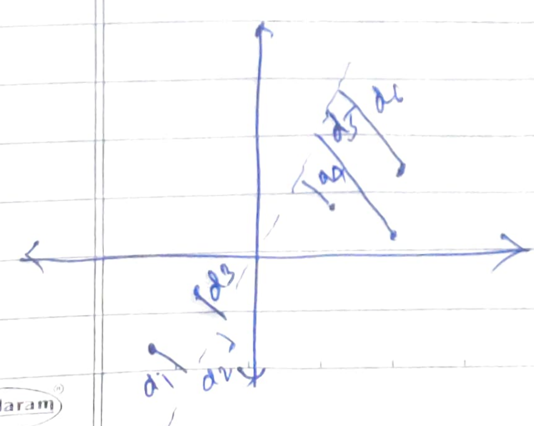
(III) Lens 2





$$r^2 = p^2 + q^2$$

Either minimize q
or maximize r

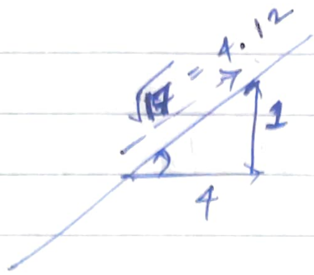


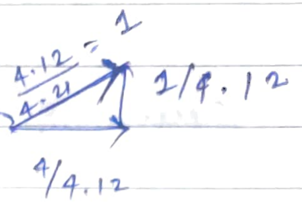
$S = d_1^2 + d_2^2 + d_3^2 + d_4^2 + d_5^2 + d_6^2$

line → max (S)

↓

Principal Component 1





unit vector

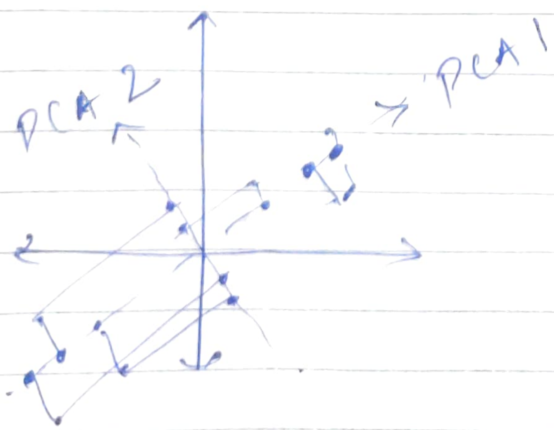
⇒ eigenvector for PC1

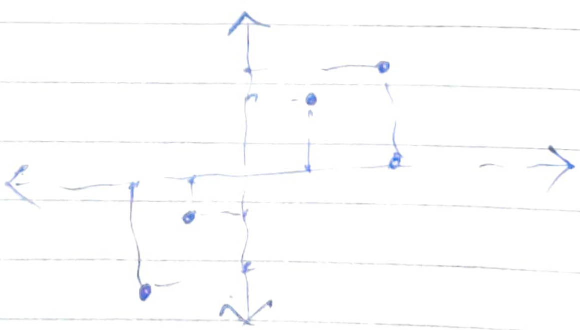
$d_1^2 + d_2^2 + d_3^2 + d_4^2 + d_5^2 + d_6^2$

⇒ eigenvalue for PC1

?

→ Variance





Variation for PC1 :

$$\frac{\text{Sum of squared distances}}{n-1}$$

Var 1 $\longrightarrow$ PCA 1
Var 2 $\longrightarrow$ PCA 2

$$\text{PCA 1 } (\%) = \frac{\text{Var 1}}{\text{Var 1} + \text{Var 2}}$$

(IV) Connecting the lenses

Eigenvalues of original covariance matrix is equal to the variances of the reduced space.

- Covariance of original space

$$C_X = \frac{1}{n} X^T X$$

- PCA : Eigendecomposition of covariance of the original space.

$$C_X = \frac{1}{n} X^T X = U \Lambda U^T$$

- Projected Data : $Y = XU$
  (U has eigenvectors as columns)

- Covariance of the reduced space

$$C_Y = \frac{1}{n} Y^T Y$$

$$= \frac{1}{n}(XU)(XU)^T$$

$$= \frac{1}{n} U^T X^T X U$$

$$= U^T \left(\frac{1}{n} X^T X \right) U$$

$$C_y = U^T C_X U$$

$$= U^T U \Lambda U^T U$$

$$= \Lambda$$ → which was the diagonal matrix consisting of eigenvalues