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MODULE 1

NUMERICAL METHODS - 1

SOLUTION OF POLYNOMIAL AND TRANSCENDENTAL EQUATIONS – BISECTION METHOD, NEWTON-RAPHSON METHOD AND REGULA-FALSI METHOD

Polynomial Equation – An expression of the form $f_n(x) = a_0x^n + a_1x^{n-1} + \ldots\ldots + a_n$, where a's are constants and n is a positive integer is said to be a polynomial in x, provided $a_0 \neq 0$. The values of x which make $f_n(x)$ to zero are called as zeroes or the root of the polynomial $f_n(x)$ and polynomial of nth degree has n-zeroes.

If f (x) is any polynomial, then the equation f (x) = 0 is known as ***algebraic equation*** in x.

For example – $4x^5 + 8x^4 - 7x^3 + x^2 - 12x + 1 = 0$

and $7x^6 + 4x^4 - 10x + 4 = 0$ are algebraic equations.

Definition – When in a polynomial, a + sign follows a + sign or a – sign follows a – sign, a continuation or a permanence of signs is said to occur. But if a + sign follows a – sign or a – sign follows a + sign, then a change of sign is said to occur.

For example, the polynomial

$$f_5(x) = 4x^5 + 6x^4 - 10x^3 + 11x^2 - 14x + 3 = 0$$

has four changes and one continuation of signs.

Descarte's Rule of Signs – The number of positive real roots of $f_n(x) = 0$ cannot exceed the number of sign changes in $f_n(x)$ and the number of negative real roots of $f_n(x) = 0$ cannot exceed the number of sign changes in $f_n(-x)$.

Transcendental Equation – If f(x) also contains trigonometric, logarithmic, exponential functions, etc. then the equation f(x) = 0 is known as ***transcendental equation***.

For example – $(x^3 - 1)\, e^{-5x} + \log(1 - x^3) = 0$

and $x^3 - \sin x + e^{-x} = 0$

Solution of Algebraic and Transcendental Equations – To obtain the roots of an equation f(x) = 0, we start with a known approximate solution and

apply any of the following methods –

The Bisection Method (Bolzano Method) – If f(x) is a continuous function between a and b and f(a) and f(b) are of opposite signs, then there exists at least one root between a and b. For definiteness, let f(a) be negative and f(b) be positive. Then the root lies between a and b and let its approximate value will be given by –

$$x_0 = \frac{a+b}{2}$$

If $f(x_0) = 0$, we conclude that x_0 is a root of the equation f(x) = 0. Otherwise the root lies either between x_0 and b or between x_0 and a, depending on whether $f(x_0)$ is negative or positive. Then as before we bisect the interval and repeat the process until the root is known to the desired accuracy. The method is shown graphically in fig. 1.1.

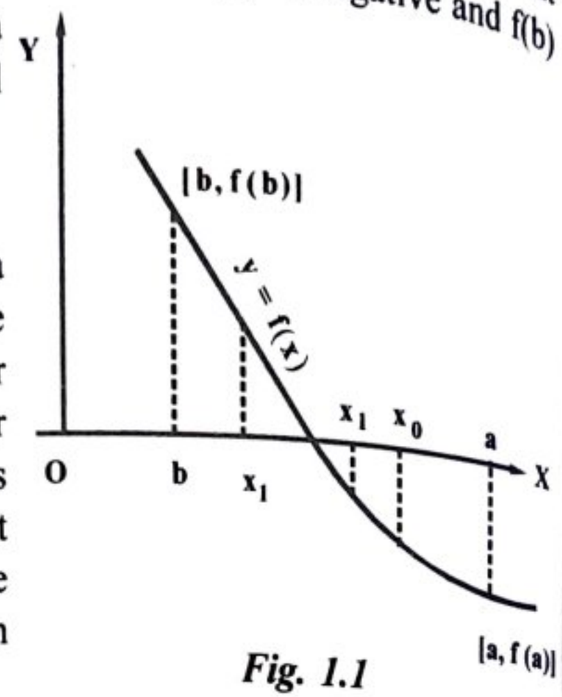

Fig. 1.1

Convergence of Bisection Method –

(Convergence of Sequence) – The sequence $\langle x_n \rangle$ of successive approximations of a root $x = \alpha$ of the equation f(x) = 0 is said to ***converge to*** $x = \alpha$ ***with order*** $q \geq 1$, if $|x_{n+1} - \alpha| \leq c\,|x_n - \alpha|\,q$, $n \geq 0$, where c is some constant greater than zero.

When q = 1 and 0 < c < 1, then the convergence is said to be of first order or linear and constant c is called the rate of linear convergence, convergence is rapid or slow according as 'c' is near 0 or 1.

Further, if $|x_{n+1} - \alpha| = c\,|x_n - \alpha|$, $n \geq 0$

$0 < c < 1$, then the convergence is called geometric.

Proceeding in this way and using principle of induction, the condition for linear convergence can be simplified of the form.

$$|x_n - \alpha| \leq c^n\,|x_0 - \alpha|,\ n \geq 0,\ 0 < c < 1.$$

Newton-Raphson Method –

Let m be a real root of an equation f(x) = 0.

Then we have

$$f(m) = 0 \qquad \text{....(i)}$$

If m lies near a point a, by Taylor's expansion we have

$$f(m) = f(a) + (m-a)f'(a) + \frac{(m-a)^2}{2!}f''(a) + \ldots..$$

Neglecting second and higher degree terms in (m – a), this expression becomes

$$f(m) = f(a) + (m-a)\,f'(a) \qquad \text{....(ii)}$$

This expression serves as the first order approximate values of f(m) near the point a, the exact value being zero as in equation (i).

From equations (i) and (ii), we obtain

$$0 = f(a) + (m-a)\,f'(a)$$

or

$$m = a - \frac{f(a)}{f'(a)} \qquad \text{....(iii)}$$

$\because$ Equation (ii) gives the first order approximate value of f (m) and equation (iii) gives the first order approximate value of m (in term of a).

Taking $a = x_0$ and $m = x_1$ in equation (iii), we obtain

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} \qquad \text{....(iv)}$$

This gives the approximate values of x_1 of m in terms of $a = x_0$.

Here x_0 is the initial approximation of m and x_1 is the first-order approximation. Replacing x_0 by x_1 and x_1 by x_2 in equation (iv), we get the following expression for x_2 which is the second order approximation of m.

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} \qquad \text{....(v)}$$

Proceeding further, we get successive approximations x_3, x_4, ... of m. The following is the formula for the $(k+1)^{th}$ order approximation of m in terms of the kth-order approximation –

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)} \qquad \text{....(vi)}$$

This formula is known as the ***Newton-Raphson formula*** or the ***Newton's iterative formula***. The process computing an approximate root by using this formula is known as Newton-Raphson method.

Method of False Position or Regula-falsi Method – This is the most oldest method of obtaining the real root of a numerical equation f(x) = 0 and closely resembles the bisection method. In this method we choose two points x_0 and x_1 such that $f(x_0)$ and $f(x_1)$ are of opposite signs. Since the graph of y = f(x) crosses the x-axis between these two points, a root must lies in between these points. This indicates that root lies between x_0 and x_1, consequently $f(x_0)\,f(x_1) < 1$.

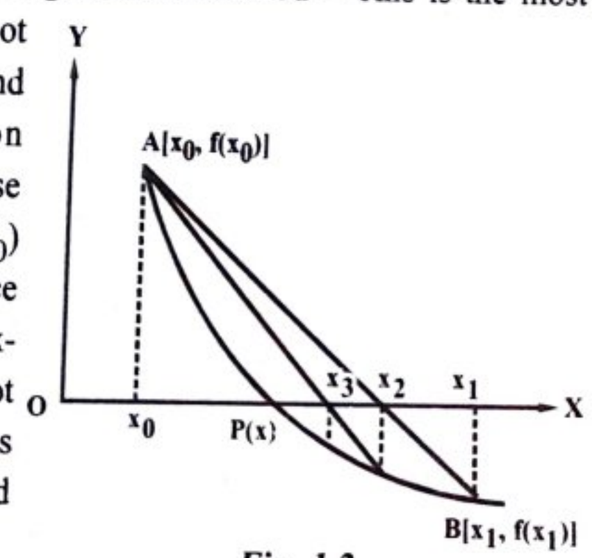

Fig. 1.2

Equation of the chord joining the A $[x_0, f(x_0)]$ and B $[x_1, f(x_1)]$ is

$$y - f(x_0) = \frac{f(x_1) - f(x_0)}{x_1 - x_0}(x - x_0) \quad \text{....(i)}$$

The method consists in replacing the curve AB, by means of the chord AB and taking the point of intersection of the chord with the x-axis as an approximation to the root. Therefore the abscissa of the point where the chord cut the x-axis (y = 0) is given by

$$x_2 = x_0 - \frac{x_1 - x_0}{f(x_1) - f(x_0)} f(x_0) \quad \text{....(ii)}$$

which is an approximation to the root.

If now $f(x_0)$ and $f(x_2)$ are of opposite signs, then the root lies between x_0 and x_2. Therefore replacing x_1 by x_2 in equation (ii), we find the next approximation x_3. (The root could as well lies between x_1 and x_2 and we would find x_3 accordingly). This method is repeated till the root is obtained to desired accuracy.

Comparison of Regula-falsi and Newton-Raphson Methods –

Table 1.1 Comparison of Methods

S. No.	Name of the Method	Iterative Formula of the Method	Rate of Convergence	Reliability of Convergence	Minimum Statements of 'c' Required for Logic Implementation	Evaluation of Functions per Iteration
(i)	Regula-falsi or False position method	$x_2 = x_0 - \frac{x_1 - x_0}{f(x_1) - f(x_0)} \times f(x_0)$	1	Guaranteed Convergence.	1	1
(ii)	Newton-Raphson method	$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}$	2	Sensitive to starting value. Convergence is fast if x_0 is close to root.	2	2

Q.1. Define algebraic and transcendental equations. ***(R.G.P.V., Dec. 2014)***

Ans. Refer to the matter given on page 3.

Q.2. Give convergence values of Regula-falsi and Newton-Raphson method. ***(R.G.P.V., June 2015)***

Ans. Refer to the matter given on page 6, under table 1.1.

NUMERICAL PROBLEMS

Prob.1. Find a root of the following equation using bisection method correct to three places of decimal –

$$f(x) = x^3 - 3x - 5$$

[R.G.P.V., May 2019 (IV-Sem)]

Sol. Let $f(x) = x^3 - 3x - 5$

$f(1) = -7$, i.e., –ve

$f(2) = -3$, i.e., –ve

and $f(3) = 13$, i.e., +ve

Thus the root lies between 2 and 3. Then first approximation to the root is

$$x_1 = \frac{2+3}{2} = 2.5$$

Now, $f(x_1) = f(2.5) = 3.125$, i.e., +ve

Hence, the root lies between 2 and 2.5. Then the second approximation to the root is

$$x_2 = \frac{2+2.5}{2} = 2.25$$

Also $f(x_2) = f(2.25) = -0.3594$, i.e., –ve

Hence, the root lies between 2.25 and 2.5.

$$\therefore \quad x_3 = \frac{2.25+2.5}{2} = 2.375$$

Now, $f(x_3) = f(2.375) = 1.2715$, i.e., +ve

Hence, the root lies between 2.25 and 2.375.

$$\therefore \quad x_4 = \frac{2.25+2.375}{2} = 2.3125$$

Now, $f(x_4) = f(2.3125) = 0.4289$, i.e., +ve

Hence, the root lies between 2.25 and 2.3125.

$$\therefore \quad x_5 = \frac{2.25+2.3125}{2} = 2.2812$$

Now, $f(x_5) = f(2.2812) = 0.0275$, i.e., +ve

Hence, the root lies between 2.25 and 2.2812.

$$\therefore \quad x_6 = \frac{2.25+2.2812}{2} = 2.2656$$

Now, $f(x_6) = f(2.2656) = -0.1676$, i.e., –ve

Hence, the root lies between 2.2656 and 2.2812.

$$\therefore \quad x_7 = \frac{2.2656 + 2.2812}{2} = 2.2734$$

Now, $f(x_7) = f(2.2734) = -0.07$, i.e., –ve

Hence, the root lies between 2.2734 and 2.2812.

$$\therefore \quad x_8 = \frac{2.2734 + 2.2812}{2} = 2.2773$$

Now, $f(x_8) = f(2.2773) = -0.0216$, i.e., –ve

Hence, root lies between 2.2773 and 2.2812.

$$\therefore \quad x_9 = \frac{2.2773 + 2.2812}{2} = 2.27925$$

Hence the root is **2.279** **Ans.**

Prob.2. Find a root of the equation $x^3 - 4x - 9 = 0$ using bisection method correct to three decimal places. ***(R.G.P.V., Nov. 2018)***

Sol. Let $f(x) = x^3 - 4x - 9$

$f(1) = -12$, i.e., –ve

$f(2) = -9$, i.e., –ve

and $f(3) = 6$, i.e., +ve

Thus the root lies between 2 and 3. Then first approximation to the root is

$$x_1 = \frac{2+3}{2} = 2.5$$

Now, $f(x_1) = f(2.5) = -3.375$, i.e., –ve

Hence, the root lies between 2.5 and 3. Then the second approximation to the root is

$$x_2 = \frac{2.5+3}{2} = 2.75$$

Also, $f(x_2) = f(2.75) = 0.7969$, i.e., +ve

Hence, the root lies between 2.5 and 2.75.

$$\therefore \quad x_3 = \frac{2.5+2.75}{2} = 2.625$$

Now, $f(2.625) = -1.4121$, i.e., –ve

Hence, the root lies between 2.625 and 2.75.

$$\therefore \quad x_4 = \frac{2.625+2.75}{2} = 2.6875$$

Now, $f(2.6875) = -0.3391$, i.e., –ve

Hence, the root lies between 2.6875 and 2.75.

$$\therefore \quad x_5 = \frac{2.6875+2.75}{2} = 2.71875$$

Now, $f(2.71875) = 0.2209$, i.e., +ve

Hence, the root lies between 2.6875 and 2.71875.

$$\therefore \quad x_6 = \frac{2.6875+2.71875}{2} = 2.7031$$

Now, $f(2.7031) = -0.0615$, i.e., –ve

Hence, the root lies between 2.7031 and 2.71875.

$$\therefore \quad x_7 = \frac{2.7031+2.71875}{2} = 2.7109$$

Now, $f(2.7109) = 0.0787$, i.e., +ve

Hence, the root lies between 2.7031 and 2.7109.

$$\therefore \quad x_8 = \frac{2.7031+2.7109}{2} = 2.707$$

Hence, the root is **2.707**. **Ans.**

Prob.3. Find a real root of the equation $f(x) = x^3 - 4x - 9 = 0$, using bisection method in four stages. ***(R.G.P.V., Dec. 2015)***

Sol. Refer to Prob.2.

Prob.4. Find the real root of the equation $x \log_{10} x = 1.2$ by bisection method correct to four decimal places. ***[R.G.P.V., May 2019 (III-Sem)]***

Sol. Let $f(x) = x \log_{10} x - 1.2$

$f(1) = -1.2$, i.e., –ve

$f(2) = 2 \log_{10} 2 - 1.2 = -0.59794$, i.e., –ve

and $f(3) = 3 \log_{10} 3 - 1.2 = 0.23136$, i.e., +ve

Therefore, the root lies between 2 and 3. Then first approximation to the root is

$$x_1 = \frac{2+3}{2} = 2.5$$

Now, $f(x_1) = f(2.5) = -0.2051$, i.e., –ve

Hence, the root lies between 2.5 and 3.

$$\therefore \quad x_2 = \frac{2.5+3}{2} = 2.75$$

Also $f(x_2) = f(2.75) = 0.00816$, i.e., +ve

Hence, the root lies between 2.5 and 2.75.

$$\therefore \quad x_3 = \frac{2.5+2.75}{2} = 2.625$$

Now, $f(x_3) = f(2.625) = -0.09978$, i.e., –ve

Hence, the root lies between 2.625 and 2.75.

$$\therefore \quad x_4 = \frac{2.625+2.75}{2} = 2.6875$$

Now, $f(x_4) = f(2.6875) = -0.04612$, i.e., –ve

Hence, the root lies between 2.6875 and 2.75.

$$\therefore \quad x_5 = \frac{2.6875 + 2.75}{2} = 2.71875$$

Now, $f(x_5) = f(2.71875) = -0.01906$, i.e., –ve

Hence, the root lies between 2.71875 and 2.75.

$$\therefore \quad x_6 = \frac{2.71875 + 2.75}{2} = 2.73437$$

Now, $f(x_6) = f(2.73437) = -0.00547$, i.e., –ve

Hence, the root lies between 2.73437 and 2.75.

$$\therefore \quad x_7 = \frac{2.73437 + 2.75}{2} = 2.74218$$

Hence the root is **2.7421.** **Ans.**

Prob.5. Find the smallest positive root of the equation $x^3 - 2x + 0.5 = 0$ by Newton-Raphson method. ***(R.G.P.V., Dec. 2014)***

Sol. Let

$$f(x) = x^3 - 2x + 0.5 \quad \text{...(i)}$$

$\therefore$ $f(0) = 0.5$, (+ve) and $f(1) = -0.5$, (–ve)

$\therefore$ Root lies between 0 and 1.

Taking $x_0 = 0.5$

Now, by Newton-Raphson formula, we have

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

From equation (i), we obtain

$$f'(x) = 3x^2 - 2$$

Therefore,

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 0.5 - \frac{f(0.5)}{f'(0.5)} = 0.5 - \frac{(-0.375)}{(-1.25)} = 0.2$$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 0.2 - \frac{f(0.2)}{f'(0.2)} = 0.2 - \frac{0.108}{(-1.88)} = 0.257$$

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} = 0.257 - \frac{f(0.257)}{f'(0.257)} = 0.257 - \frac{0.002974}{(-1.8018)} = 0.2586$$

Hence the required root is **0.2** **Ans.**

Prob.6. Using Newton-Raphson method find the real root of the equation $3x = \cos x + 1$ correct to five decimal places. ***(R.G.P.V., Dec. 2006, June 2011)***

Or

Using Newton-Raphson method, find the real root of the equation $3x = \cos x + 1$. ***(R.G.P.V., June 2016, Nov. 2018)***

Or

Solve the equation $3x = \cos x + 1$ by Newton-Raphson method. ***[R.G.P.V., Nov. 2019 (III-Sem)]***

Sol. Let $f(x) = 3x - \cos x - 1$ ...(i)

$\therefore$ $f(0) = -2$, i.e., –ve

and $f(1) = 1.4597$, i.e., +ve

$\therefore$ Root lies between 0 and 1.

From equation (i), we obtain $f'(x) = 3 + \sin x$.

By Newton-Raphson formula, we have

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

So, here taking $x_0 = 0.5$

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 0.5 - \frac{f(0.5)}{f'(0.5)} = 0.5 - \frac{(-0.37758)}{(3.479425)} = 0.60852$$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 0.60852 - \frac{f(0.60852)}{f'(0.60852)} = 0.60852 - \frac{(0.005065036)}{(3.571653754)} = 0.60710$$

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} = 0.60710 - \frac{f(0.60710)}{f'(0.60710)} = 0.60710 - \frac{(-0.0000058845)}{(3.5704881)} = 0.60710$$

Hence the required root is **0.60710.** **Ans.**

Prob.7. Using Newton-Raphson method, find a root of the equation $f(x) = x \sin x + \cos x = 0$ correct to three decimal places, assuming that the root is near to $x = \pi$. ***(R.G.P.V., June 2013)***

Sol. Given $f(x) = x \sin x + \cos x$

Take the initial approximation $x_0 = \frac{5}{6}\pi$

$$f'(x) = x \cos x + \sin x - \sin x = x \cos x$$

Starting with x_0

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = x_0 - \frac{x_0 \sin x_0 + \cos x_0}{x_0 \cos x_0}$$

$$= 2.617994 - \frac{2.617994 \times 0.5 - 0.866025}{2.617994 \times (-0.866025)}$$

$$= 2.617994 + 0.1953787 = 2.8133727$$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = x_1 - \frac{x_1 \sin x_1 + \cos x_1}{x_1 \cos x_1}$$

$$= 2.8133727 - \frac{2.8133727 \times 0.3223585 - 0.9466176}{2.8133727 \times (-0.9466176)}$$

$$= 2.8133727 - 0.0149081 = 2.7984646$$

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} = x_2 - \frac{x_2 \sin x_2 + \cos x_2}{x_2 \cos x_2}$$

$$= 2.7984646 - \frac{2.7984646 \times 0.3364344 - 0.9417069}{2.7984646 \times (-0.9417069)}$$

$$= 2.7984646 - 0.0000786 = 2.798386$$

The exact value correct to three decimal places is **2.798.** **Ans.**

Prob.8. Find a real root of the equation $x = e^{-x}$ using the Newton-Raphson method. ***[R.G.P.V., May 2019 (III-Sem)]***

Sol. Let $f(x) = x - e^{-x}$ ...(i)

$\therefore$ $f(0) = -1$, i.e., –ve

and $f(1) = 0.6321$, i.e., +ve

$\therefore$ Root lies between 0 and 1.

Let us take $x_0 = 0.5$

From equation (i), we obtain

$$f'(x) = 1 + e^{-x}$$

By Newton-Raphson formula, we have

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 0.5 - \frac{f(0.5)}{f'(0.5)}$$

$$= 0.5 - \frac{(-0.10653)}{1.60653} = 0.56631$$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 0.56631 - \frac{f(0.56631)}{f'(0.56631)}$$

$$= 0.56631 - \frac{(-0.001306)}{1.567616} = 0.56714$$

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} = 0.56714 - \frac{f(0.56714)}{f'(0.56714)}$$

$$= 0.56714 - \frac{(-0.00000516)}{1.56714} = 0.56714$$

Hence the required root is **0.56714.** **Ans.**

Prob.9. By using Newton-Raphson method. Find the root of $x^4 - x - 10 = 0$, which is near to $x = 2$ correct to three places of decimal.
(R.G.P.V., June 2008, Dec. 2010, 2011)

Sol. Let $f(x) = x^4 - x - 10$ ...(i)

$\therefore$ $f(1) = -10$, i.e., –ve

and $f(2) = 4$, i.e., +ve

Clearly root lies between 1 and 2.

Let us take $x_0 = 1.5$

Differentiating equation (i), we get

$$f'(x) = 4x^3 - 1$$

By Newton-Raphson formula, we have $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$ ...(ii)

Putting n = 0, the first approximation is

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 1.5 - \frac{f(1.5)}{f'(1.5)} = 1.5 - \frac{(-6.4375)}{12.5} = 2.015$$

Similarly putting n = 1, 2, 3 in equation (ii), we get

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 2.015 - \frac{f(2.015)}{f'(2.015)} = 2.015 - \frac{(4.4704)}{31.7254} = 1.874$$

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} = 1.874 - \frac{f(1.874)}{f'(1.874)} = 1.874 - \frac{0.45927}{25.325} = 1.856$$

$$x_4 = x_3 - \frac{f(x_3)}{f'(x_3)} = 1.856 - \frac{f(1.856)}{f'(1.856)} = 1.856 - \frac{0.010206}{24.5737} = 1.856$$

Since, $x_3 = x_4$, therefore **1.856** is a real root of the given equation correct upto three decimal places. **Ans.**

Prob.10. Evaluate $\sqrt{12}$ correct to four decimal places by Newton-Raphson method. ***(R.G.P.V., Dec. 2012)***

Sol. Let $x = \sqrt{12}$ or $x^2 = 12$

Taking $f(x) = x^2 - 12$ ...(i)

$\therefore$ $f(3) = -3$, (–ve) and $f(4) = 4$, (+ve)

$\therefore$ Roots lies between 3 and 4. Then taking $x_0 = 3$

Now by Newton-Raphson formula, we have

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \quad ...(ii)$$

From equation (i), we obtain $f'(x) = 2x$

$\therefore$ From equation (ii)

$$x_{n+1} = x_n - \frac{x_n^2 - 12}{2x_n} = \frac{x_n^2 + 12}{2x_n} \quad ...(iii)$$

Put n = 0, in equation (iii)

$$x_1 = \frac{x_0^2 + 12}{2x_0} = \frac{9 + 12}{6} = \frac{21}{6} = 3.5$$

Put n = 1,

$$x_2 = \frac{x_1^2 + 12}{2x_1} = \frac{12.25 + 12}{7} = \frac{24.25}{7} = 3.4643$$

Put n = 2,
$$x_3 = \frac{x_2^2 + 12}{2x_2} = \frac{12.0013 + 12}{6.9286} = 3.4641$$

Put n = 3,
$$x_4 = \frac{x_3^2 + 12}{2x_3} = \frac{11.9999 + 12}{6.9282} = 3.4641$$

Since $x_3 = x_4$

Hence the required root is **3.4641** correct to four decimal places. **Ans.**

Prob.11. Find a real root of the equation $x^4 - x - 13 = 0$ by Newton-Raphson method correct to three decimal places. ***(R.G.P.V., Dec. 2015)***

Sol. Let $f(x) = x^4 - x - 13$ ...(i)

$\therefore$ $f(1) = (1)^4 - 1 - 13 = -13$, i.e., –ve

and $f(2) = (2)^4 - 2 - 13 = 1$, i.e., +ve

Clearly root lies between 1 and 2.

Let us take $x_0 = 1.5$

Differentiating equation (i), we get
$$f'(x) = 4x^3 - 1$$

By Newton-Raphson formula, we have
$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \quad ...(ii)$$

Putting n = 0, the first approximation is
$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 1.5 - \frac{f(1.5)}{f'(1.5)} = 1.5 - \frac{(-9.4375)}{12.5} = 2.255$$

Similarly putting n = 1, 2, 3, 4 in equation (ii), we get
$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 2.255 - \frac{f(2.255)}{f'(2.255)} = 2.255 - \frac{10.6025}{44.8669} = 2.0187$$
$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} = 2.0187 - \frac{f(2.0187)}{f'(2.0187)} = 2.0187 - \frac{1.5881}{31.9060} = 1.9689$$
$$x_4 = x_3 - \frac{f(x_3)}{f'(x_3)} = 1.9689 - \frac{f(1.9689)}{f'(1.9689)} = 1.9689 - \frac{0.05887}{29.5303} = 1.9669$$
$$x_5 = x_4 - \frac{f(x_4)}{f'(x_4)} = 1.9669 - \frac{f(1.9669)}{f'(1.9669)} = 1.9669 - \frac{(-0.000094)}{29.4373} = 1.9669$$

Since $x_4 = x_5$, therefore **1.966** is a real root of the given equation correct upto three decimal places. **Ans.**

Prob.12. By using Newton-Raphson method, find the real roots of the equation $x^3 - x - 1 = 0$. ***(R.G.P.V., June 2012)***

Sol. Let $f(x) = x^3 - x - 1$ ...(i)

$f(1) = (1)^3 - 1 - 1 = 1 - 2 = -1$, i.e., –ve

$f(2) = (2)^3 - 2 - 1 = 8 - 3 = 5$, i.e., +ve

So a root of f(x) = 0, lies between 1 and 2

Let us take $x_0 = 1$

Differentiating equation (i), we get
$$f'(x) = 3x^2 - 1$$

Newton's method formula gives
$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{x_n^3 - x_n - 1}{3x_n^2 - 1}$$

or
$$x_{n+1} = \frac{3x_n^3 - x_n - x_n^3 + x_n + 1}{3x_n^2 - 1}$$

or
$$x_{n+1} = \frac{2x_n^3 + 1}{3x_n^2 - 1} \quad ...(ii)$$

Putting n = 0, first approximation is
$$x_1 = \frac{2x_0^3 + 1}{3x_0^2 - 1} = \frac{2 \times (1)^3 + 1}{3 \times (1)^2 - 1} = \frac{3}{2} = 1.5$$

Similarly putting n = 1, 2, 3, 4.... in equation (ii), we have
$$x_2 = \frac{2 \times (1.5)^3 + 1}{3 \times (1.5)^2 - 1} = \frac{6.75 + 1}{6.75 - 1} = \frac{7.75}{5.75} = 1.3478$$
$$x_3 = \frac{2 \times (1.3478)^3 + 1}{3 \times (1.3478)^2 - 1} = \frac{5.89673}{4.44969} = 1.3252$$
$$x_4 = \frac{2 \times (1.3252)^3 + 1}{3 \times (1.3252)^2 - 1} = \frac{5.65451}{4.26846} = 1.3247$$
$$x_5 = \frac{2 \times (1.3247)^3 + 1}{3 \times (1.3247)^2 - 1} = \frac{5.64925}{4.26449} = 1.3247$$

Clearly $x_4 = x_5$, therefore the root is **1.3247** **Ans.**

Prob.13. Solve the algebraic equation $x^3 + 2x^2 + 10x - 20 = 0$ by Newton-Raphson method. ***[R.G.P.V., May 2019 (IV-Sem)]***

Sol. Let $f(x) = x^3 + 2x^2 + 10x - 20$ ...(i)

$f(1) = 1 + 2 + 10 - 20 = -7$

and $f(1.5) = (1.5)^3 + 2(1.5)^2 + 10(1.5) - 20 = +2.875$

Since f(1) and f(1.5) are of opposite sign, the root of equation (i) lies between 1 and 1.5; f(1.5) is near to zero than f(1), so 1.5 is better appropriate root than 1.

Differentiating equation (i), we get

$$f'(x) = 3x^2 + 4x + 10$$

$$\therefore \quad f'(1.5) = 3(1.5)^2 + 4(1.5) + 10 = 22.75$$

Let 1.5 be an approximate root of equation (i),

By Newton-Raphson method

$$a_1 = a - \frac{f(a)}{f'(a)} = 1.5 - \frac{f(1.5)}{f'(1.5)} = 1.5 - \frac{2.875}{22.75}$$

$$= 1.37363$$

Now $\quad f(1.37363) = (1.37363)^3 + 2(1.37363)^2 + 10(1.37363) - 20$

$$= 0.10186$$

$$f'(1.37363) = 3(1.37363)^2 + 4(1.37363) + 10 = 27.1551$$

$$a_2 = 1.37363 - \frac{f(1.37363)}{f'(1.37363)} = 1.37363 - \frac{0.10186}{27.1551}$$

$$= 1.37363 - 0.00375 = 1.36988$$

$$\therefore \quad f(1.36988) = (1.36988)^3 + 2(1.36988)^2 + 10(1.36988) - 20$$

$$= 0.02262$$

$$f'(1.36988) = 3(1.36988)^2 + 4(1.36988) + 10 = 21.1092$$

$$\therefore \quad a_3 = 1.36988 - \frac{f(1.36988)}{f'(1.36988)} = 1.36988 - \frac{0.02262}{21.1092}$$

$$= 1.36988 - 0.00107 = 1.36881$$

$$f(1.36881) = (1.36881)^3 + 2(1.36881)^2 + 10(1.36881) - 20$$

$$= 0.0000399$$

$$f'(1.36881) = 3(1.36881)^2 + 4(1.36881) + 10 = 21.0962$$

$$\therefore \quad a_4 = 1.36881 - \frac{f(1.36881)}{f'(1.36881)} = 1.36881 - \frac{0.0000399}{21.0962}$$

$$= 1.36881 - 0.00000189 = 1.36881$$

Clearly $a_3 = a_4$

Hence the required root is **1.36881** correct to five decimal places. **Ans.**

Prob.14. Find the negative root of the equation $x^3 - 21x + 3500 = 0$ correct to 2 decimal places by Newton's-Raphson method. ***(R.G.P.V., Dec. 2013)***

Sol. Let $\quad f(x) = x^3 - 21x + 3500 \quad$ ...(i)

$$f(-15) = (-15)^3 - 21 \times (-15) + 3500 = 440, \text{ i.e., +ve}$$

$$f(-16) = (-16)^3 - 21 \times (-16) + 3500 = -260, \text{ i.e., -ve}$$

$\therefore$ Root lies between –15 and –16.

Differentiating equation (i), we get

$$f'(x) = 3x^2 - 21$$

By Newton's-Raphson formula, we have

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$= x_n - \frac{x_n^3 - 21x_n + 3500}{3x_n^2 - 21}$$

$$= \frac{3x_n^3 - 21x_n - x_n^3 + 21x_n - 3500}{3x_n^2 - 21}$$

$$= \frac{2x_n^3 - 3500}{3x_n^2 - 21} \quad \text{...(ii)}$$

Putting n = 0 and first approximation is $x_0 = -15.6$

$$x_1 = \frac{2(-15.6)^3 - 3500}{3(-15.6)^2 - 21} = \frac{-7592.832 - 3500}{730.08 - 21}$$

$$= \frac{-11092.832}{709.08} = -15.643978$$

Putting n = 1 in equation (ii) and second approximation is

$$x_2 = \frac{2(-15.643978)^3 - 3500}{3(-15.643978)^2 - 21}$$

$$= \frac{-7657.228115 - 3500}{734.202143 - 21}$$

$$= \frac{-11157.228115}{713.202143} = -15.64385$$

Since x_1 and x_2 are almost equal, the root is **– 15.64** correct to 2 decimal places. **Ans.**

Prob.15. Solve $x^3 - 3x + 1 = 0$ by Newton's method. ***(R.G.P.V., June 2017)***

Sol. Let $\quad f(x) = x^3 - 3x + 1 \quad$ ...(i)

$\therefore \quad f(1) = -1$, i.e., –ve

and $\quad f(2) = 3$, i.e., +ve

Clearly root lies between 1 and 2.

Let us take $x_0 = 1.5$

Differentiating equation (i), we get

$$f'(x) = 3x^2 - 3$$

By Newton's method, we have

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \quad \text{...(ii)}$$

Putting n = 0, the first approximation is

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 1.5 - \frac{f(1.5)}{f'(1.5)} = 1.5 - \frac{(-0.125)}{3.75} = 1.5333$$

Similarly putting n = 1, 2 in equation (ii), we get

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

$$= 1.5333 - \frac{f(1.5333)}{f'(1.5333)} = 1.5333 - \frac{0.00490193}{4.053026} = 1.53209$$

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)}$$

$$= 1.53209 - \frac{f(1.53209)}{f'(1.53209)} = 1.53209 - \frac{0.000005}{4.041899} = 1.53209$$

Since $x_2 = x_3$, therefore **1.53209** is a real root of the given equation. **Ans.**

Prob.16. Find the real root of the equation $x \log_{10} x = 1.2$ by Regula-falsi method correct to two decimal places. ***(R.G.P.V., June 2016)***

Or

Find the real root of the equation $x \log_{10} x = 1.2$ by the method of false position. ***(R.G.P.V., June 2017)***

Or

Find a real root of the equation $x \log_{10} x = 1.2$ by Regula-falsi method correct to one decimal place. ***(R.G.P.V., June 2014)***

Sol. Let $f(x) = x \log_{10} x - 1.2$, $f(1) = -1.2$, i.e., –ve

$f(2) = -0.59794$, i.e., –ve, $f(3) = 0.23136$, i.e., +ve

i.e., the root lies between 2 and 3.

Taking $x_0 = 2$, $x_1 = 3$, $f(x_0) = -0.59794$ and $f(x_1) = 0.23136$ in the method of false position, we get

$$x_2 = x_0 - \frac{f(x_0)}{f(x_1) - f(x_0)}(x_1 - x_0) \qquad ...(i)$$

$$x_2 = 2 + \frac{0.59794}{0.23136 + 0.59794} \times (3 - 2) = 2.72102$$

Now, $f(x_2) = -0.0171$

i.e., the root lies between 2.72102 and 3.

Now x_0 replace by x_2 in equation (i), we get

$$x_3 = x_2 - \frac{f(x_2)}{f(x_1) - f(x_2)}(x_1 - x_2) \qquad ...(ii)$$

$$x_3 = 2.72102 + \frac{0.0171}{0.23136 + 0.0171} \times (3 - 2.72102) = 2.74022$$

Now, $f(x_3) = -0.0003716$

i.e., the root lies between 2.74022 and 3.

Now x_0 replace by x_3 in equation (i), we get

$$x_4 = x_3 - \frac{f(x_3)}{f(x_1) - f(x_3)}(x_1 - x_3) \qquad ...(iii)$$

$$x_4 = 2.74022 + \frac{0.0003716}{0.23136 + 0.0003716} \times (3 - 2.74022)$$

$$x_4 = 2.74064$$

Hence the root is **2.74** correct to two decimal places. **Ans.**

Prob.17. Find a real root of the equation $x^3 - 9x + 1 = 0$ by method of false position. ***[R.G.P.V., Nov. 2019 (IV-Sem)]***

Sol. Let $f(x) = x^3 - 9x + 1$

then $f(1) = 1^3 - 9 \times 1 + 1 = -7$ (–ve)

$f(2) = 2^3 - 9 \times 2 + 1 = -9$ (–ve)

$f(3) = 3^3 - 9 \times 3 + 1 = 1$(+ve)

i.e., the root lies between 2 and 3.

Taking $x_0 = 2$, $x_1 = 3$, $f(x_0) = -9$ and $f(x_1) = 1$ in the method of false position, we get

$$x_2 = x_0 - \frac{x_1 - x_0}{f(x_1) - f(x_0)} f(x_0)$$

$$= 2 - \frac{3 - 2}{1 - (-9)}(-9)$$

$$= 2 - \frac{1}{10}(-9) = 2.9$$

Now $f(x_2) = (2.9)^3 - 9 \times 2.9 + 1 = -0.711$

i.e., the root lies between 2.9 and 3.

$$\therefore \quad x_3 = x_2 - \frac{x_1 - x_2}{f(x_1) - f(x_2)} f(x_2)$$

$$= 2.9 - \frac{3 - 2.9}{1 - (-0.711)}(-0.711)$$

$$= 2.9 + \frac{0.0711}{1.711} = 2.9415$$

Now $f(x_3) = (2.9415)^3 - 9 \times 2.9415 + 1 = -0.0224$

i.e., the root lies between 2.9415 and 3.

$$\therefore \quad x_4 = x_3 - \frac{x_1 - x_3}{f(x_1) - f(x_3)} f(x_3)$$

$$= 2.9415 - \frac{3 - 2.9415}{1 - (-0.0224)}(-0.0224)$$

$$= 2.9415 + \frac{0.00131}{1.0224} = 2.9428$$

Now $\quad f(x_4) = (2.9428)^3 - 9 \times 2.9428 + 1$

$$= -0.00034$$

i.e., the root lies between 2.9428 and 3.

$$\therefore \quad x_5 = x_4 - \frac{x_1 - x_4}{f(x_1) - f(x_4)} f(x_4)$$

$$= 2.9428 - \frac{3 - 2.9428}{1 - (-0.00034)}(-0.00034)$$

$$= 2.9428 + \frac{0.000019}{1.00034}$$

$$= 2.9428 + 0.00001899 = 2.9428$$

Since $x_4 = x_5$, therefore **2.9428** is a real root of the given equation. **Ans.**

Prob.18. Find a real root of the equation $x^3 - 2x - 5 = 0$, by the method of false position, correct to three decimal places.

(R.G.P.V., June 2010, Dec. 2011, 2016)

Or

Find a real root of the equation $x^3 - 2x - 5 = 0$ by the method of Regula-falsi correct to three decimal places. ***(R.G.P.V., Dec. 2013)***

Sol. Let $\quad f(x) = x^3 - 2x - 5$

then $\quad f(1) = 1^3 - 2 \times 1 - 5 = -6$ (–ve)

$f(2) = 2^3 - 2 \times 2 - 5 = -1$ (–ve)

$f(2.5) = (2.5)^3 - 2 \times 2.5 - 5 = 5.625$ (+ve)

i.e., the root lies between 2 and 2.5

Taking $x_0 = 2$, $x_1 = 2.5$, $f(x_0) = -1$ and $f(x_1) = 5.625$

by using method of false position, we get

$$x_2 = x_0 - \frac{x_1 - x_0}{f(x_1) - f(x_0)} f(x_0)$$

$$= 2 - \frac{2.5 - 2}{5.625 - (-1)}(-1) = 2 - \frac{0.5}{6.625}(-1) = 2 + \frac{0.5}{6.625} = 2.075$$

Now $f(x_2) = (2.075)^3 - 2 \times 2.075 - 5 = -0.216$

i.e., the root lies between 2.075 and 2.5

$$\therefore \quad x_3 = x_2 - \frac{x_1 - x_2}{f(x_1) - f(x_2)} f(x_2)$$

$$= 2.075 - \frac{(2.5 - 2.075)}{5.625 - (-0.216)}(-0.216) = 2.075 + \frac{0.0918}{5.841} = 2.091$$

Now $f(x_3) = (2.091)^3 - 2 \times 2.091 - 5 = -0.040$ (–ve)

i.e., the root lies between 2.091 and 2.5

$$\therefore \quad x_4 = x_3 - \frac{x_1 - x_3}{f(x_1) - f(x_3)} f(x_3)$$

$$= 2.091 - \frac{(2.5 - 2.091)}{5.625 - (-0.040)}(-0.040) = 2.091 + \frac{0.016}{5.665} = 2.094$$

Now $f(x_4) = (2.094)^3 - 2 \times 2.094 - 5 = -0.006$ (–ve)

i.e., the root lies between 2.094 and 2.5

$$\therefore \quad x_5 = x_4 - \frac{x_1 - x_4}{f(x_1) - f(x_4)} f(x_4)$$

$$= 2.094 - \frac{(2.5 - 2.094)}{5.625 - (-0.006)}(-0.006) = 2.094 + \frac{0.002}{5.631} = 2.094$$

The exact value correct to three decimal places is **2.094**. **Ans.**

Prob.19. The equation $x^6 - x^4 - x^3 - 1 = 0$, has one real root between 1.4 and 1.5. Find the root to four decimal places by the method of false position. ***(R.G.P.V., Dec. 2014)***

Sol. Let $f(x) = x^6 - x^4 - x^3 - 1$

So that $f(1.4) = (1.4)^6 - (1.4)^4 - (1.4)^3 - 1 = -0.056$, i.e., –ve

and $f(1.5) = (1.5)^6 - (1.5)^4 - (1.5)^3 - 1 = 1.953$, i.e., +ve

$\because$ A root lies between 1.4 and 1.5.

Taking, $x_0 = 1.4$, $x_1 = 1.5$, $f(x_0) = -0.056$, $f(x_1) = 1.953$ in the method of false position, we get

$$x_2 = x_0 - \frac{x_1 - x_0}{f(x_1) - f(x_0)} \times f(x_0) \qquad \text{...(i)}$$

$$= 1.4 - \frac{1.5 - 1.4}{1.953 + 0.056} \times (-0.056) = 1.4 + \frac{0.0056}{2.009} = 1.4028$$

Now $f(x_2) = f(1.4028) = -0.01258$ i.e., the root lies between 1.4028 and 1.5.

Taking $x_0 = 1.4028$, $x_1 = 1.5$, $f(x_0) = -0.01258$ in equation (i), we get

$$x_3 = 1.4028 - \frac{1.5 - 1.4028}{1.953 + 0.01258} \times (-0.01258)$$

$$= 1.4028 + 0.000622 = 1.4034$$

Now f(1.4034) = – 0.003175 i.e., the root lies between 1.4034 and 1.5.

Taking $x_0 = 1.4034$, $x_1 = 1.5$, $f(x_0) = -0.003175$ in equation (i), we get

$$x_4 = 1.4034 - \frac{1.5 - 1.4034}{1.953 + 0.003175} \times (-0.003175)$$

$$= 1.4034 - \frac{0.0966}{1.956175} \times (-0.003175) = 1.4036$$

Now f(1.4036) = – 0.0000334 i.e., the root lies between 1.4036 and 1.5.

Taking $x_0 = 1.4036$, $x_1 = 1.5$, $f(x_0) = -0.0000334$ in equation (i), we get

$$x_5 = 1.4036 - \frac{1.5 - 1.4036}{1.953 + 0.0000334} \times (-0.0000334)$$

$$= 1.4036 + \frac{0.00000322}{1.9530} = 1.4036$$

Hence the root is **1.4036** correct to four decimal places. **Ans.**

Prob.20. Using Regula-falsi method, compute the real root of equation $xe^x = 2$, correct to four decimal places. ***(R.G.P.V., June 2012)***

Sol. Let $f(x) = xe^x - 2$

$f(0) = 0.e^0 - 2 = -2$, i.e., –ve

$f(1) = 1.e^1 - 2 = 2.71828 - 2 = 0.71828$, i.e., +ve

i.e., the root lies between 0 and 1.

Taking $x_0 = 0$, $x_1 = 1$, so that

$f(x_0) = -2$ and $f(x_1) = 0.71828$ in the Regula-falsi method, we get

$$x_2 = x_0 - \frac{x_1 - x_0}{f(x_1) - f(x_0)} \times f(x_0) \qquad \text{...(i)}$$

$$= 0 - \frac{1 - 0}{0.71828 + 2} \times (-2) = \frac{2}{2.71828} = 0.73576$$

Now $f(0.73576) = 0.73576 \times e^{0.73576} - 2$

$= 0.73576 \times 2.08707 - 2 = -0.46442$

i.e., the root lies between 0.73576 and 1.

Taking $x_0 = 0.73576$ and $x_1 = 1$

$f(x_0) = -0.46442$, $f(x_1) = 0.71828$ in equation (i),

we get

$$x_3 = 0.73576 - \frac{1 - 0.73576}{0.71828 + 0.46442} \times (-0.46442)$$

$$= 0.73576 + \frac{0.12272}{1.1827} = 0.83952$$

Now $f(0.83952) = 0.83952 \times e^{0.83952} - 2 = 1.9437 - 2 = -0.0563$

i.e., the root lies between 0.83952 and 1.

Taking $x_0 = 0.83952$, $x_1 = 1$, $f(x_0) = -0.0563$, $f(x_1) = 0.71828$ in equation (i), we get

$$x_4 = 0.83952 - \frac{1 - 0.83952}{0.71828 + 0.0563} \times (-0.0563)$$

$$= 0.83952 + \frac{0.009035}{0.77458} = 0.85118$$

Now $f(0.85118) = 0.85118 \times e^{0.85118} - 2$

$= 1.9938 - 2 = -0.0062$

i.e., the root lies between 0.85118 and 1.

Taking $x_0 = 0.85118$, $x_1 = 1$, $f(x_0) = -0.0062$, $f(x_1) = 0.71828$ in equation (i), we get

$$x_5 = 0.85118 - \frac{1 - 0.85118}{0.71828 + 0.0062} \times (-0.0062)$$

$$= 0.85118 + \frac{0.0009227}{0.72448} = 0.85245$$

Now $f(0.85245) = 0.85245 \times e^{0.85245} - 2$

$= 1.999324 - 2 = -0.000676$

i.e., the root lies between 0.85245 and 1.

Taking $x_0 = 0.85245$, $x_1 = 1$, $f(x_0) = -0.000676$, $f(x_1) = 0.71828$ in equation (i), we get

$$x_6 = 0.85245 - \frac{1 - 0.85245}{0.71828 + 0.000676} \times (-0.000676)$$

$$= 0.85245 + \frac{0.00009974}{0.718956} = 0.852589$$

Now $f(0.852589) = 0.852589 \times e^{0.852589} - 2$

$= 1.999924 - 2 = -0.000072$

i.e., the root lies between 0.852589 and 1.

Taking $x_0 = 0.852589$, $x_1 = 1$, $f(x_0) = -0.000072$, $f(x_1) = 0.71828$ in equation (i), we get

$$x_7 = 0.852589 - \frac{1 - 0.852589}{0.71828 + 0.000072} \times (-0.000072)$$

$$= 0.852589 + \frac{0.0000106}{0.718352} = 0.8526$$

Now $f(0.8526) = 0.8526 \times e^{0.8526} - 2$

$= 1.999976 - 2 = -0.0000239$

i.e., the root lies between 0.8526 and 1.

Taking $x_0 = 0.8526$, $x_1 = 1$, $f(x_0) = -0.0000239$, $f(x_1) = 0.71828$ in equation (i), we get

$$x_8 = 0.8526 - \frac{1 - 0.8526}{0.71828 + 0.0000239} \times (-0.0000239)$$

$$= 0.8526 + \frac{0.00000352}{0.7183039} = 0.8526$$

Hence the root is **0.8526** correct to four decimal places. **Ans.**

FINITE DIFFERENCES, RELATION BETWEEN OPERATORS

Finite Differences – Consider a function $y = f(x)$ of an independent variable x. Let $y_0, y_1, y_2, \ldots\ldots y_r \ldots\ldots$ be the values of y corresponding to the values $x_0, x_1, x_2, \ldots\ldots x_r$ of x respectively as shown below –

x	x_0	x_1	x_2	.....	x_r	.....
y	y_0	y_1	y_2	.....	y_r	.....

Suppose that the function $y = f(x)$ is tabulated for the equally spaced values $x = x_0, x_0 + h, x_0 + 2h, \ldots\ldots x_0 + rh, \ldots\ldots$ giving by $y = y_0, y_1, y_2, \ldots, y_r, \ldots$ To obtain the values of $f(x)$ or $f'(x)$ for some intermediate values of x, the following three types of differences are found useful.

(i) Forward Differences – Let us set $\Delta y_0 = y_1 - y_0$, $\Delta y_1 = y_2 - y_1$, $\Delta y_2 = y_3 - y_2$.

Then $\Delta y_0, \Delta y_1, \Delta y_2 \ldots\ldots$ are called the first forward difference of y.

In general, first forward difference of y are defined by.

$$\Delta y_r = y_{r+1} - y_r,\ r = 0, 1, 2\ldots\ldots \qquad \ldots(i)$$

Here, the symbol Δ is called the ***forward difference operator***. The first forward differences of the first forward differences are said to be the ***second forward differences*** and are denoted by $\Delta^2 y_0, \Delta^2 y_1, \ldots\ldots$ and so on.

$$\Delta^2 y_0 = \Delta(\Delta y_0) = \Delta(y_1 - y_0) = \Delta y_1 - \Delta y_0$$

$$\Delta^2 y_1 = \Delta(\Delta y_1) = \Delta y_2 - \Delta y_1$$

In general, the second forward differences of y are defined by

$$\Delta^2 y_r = \Delta y_{r+1} - \Delta y_r,\ r = 0, 1, 2$$

Similarly the nth forward differences of y are defined by

$$\Delta^n y_r = \Delta^{n-1} y_{r+1} - \Delta^{n-1} y_r,\ r = 0, 1, 2$$

Using this formula for n = 1, we have the notation

$$\Delta^0 y_r = y_r$$

If f(x) is constant function i.e., if

$$f(x) = c, \text{ a constant then}$$

$$y_0 = y_1 = y_2 = \ldots\ldots = c, \text{ we have}$$

$$\Delta^n y_r = 0, \text{ for } n = 1, 2, 3 \ldots\ldots \text{ and } r = 0, 1, 2\ldots\ldots$$

The symbol Δ^n is referred to as the nth forward difference operator.

The forward differences are usually arranged in the tabular columns as shown in the following table, called ***forward difference tables***.

Forward Difference Table

Values of x	*Values of y*	*First Differences*	*Second Differences*	*Third Differences*	*Fourth Differences*
x_0	y_0				
		Δy_0			
x_1	y_1		$\Delta^2 y_0$		
		Δy_1		$\Delta^3 y_0$	
x_2	y_2		$\Delta^2 y_1$		$\Delta^4 y_0$
		Δy_2		$\Delta^3 y_1$	
x_3	y_3		$\Delta^2 y_2$		
		Δy_3			
x_4	y_4				

(ii) Backward Differences – Suppose $y_0, y_1, y_2, \ldots\ldots y_r, \ldots\ldots$ are the values of a function $y = f(x)$ corresponding to the values $x_0, x_1, x_2 \ldots\ldots x_r \ldots\ldots$ of x respectively. Let us write

$$\nabla y_1 = y_1 - y_0,\ \nabla y_2 = y_2 - y_1,\ \nabla y_3 = y_3 - y_2 \ldots\ldots$$

Then $\nabla y_1, \nabla y_2, \nabla y_3 \ldots\ldots$ are called the BI_I of y in general, the first backward differences of y are defined by

$$\nabla y_r = y_r - y_{r-1},\ r = 1, 2, 3\ldots\ldots \qquad \ldots(i)$$

The symbol ∇ is called the ***backward difference operator***.

Comparing equation (i) above with the expression

$$\Delta y_r = y_{r+1} - y_r,\ r = 0, 1, 2 \ldots\ldots$$

We immediately note that

$$\nabla y_1 = \Delta y_0$$

$$\nabla y_2 = \Delta y_1 \ldots\ldots$$

$$\nabla y_r = \Delta y_{r-1} \ldots\ldots,\ (r = 1, 2 \ldots\ldots\ldots) \qquad \ldots(ii)$$

The backward differences of the first backward differences are called second backward differences and are denoted by $\nabla^2 y_2, \nabla^2 y_3 \ldots, \nabla^2 y_r, \ldots$; that is

$$\nabla^2 y_2 = \nabla(\nabla y_2) = \nabla y_2 - \nabla y_1$$

$$\nabla^2 y_3 = \nabla(\nabla y_3) = \nabla y_3 - \nabla y_2, \ldots\ldots$$

In general, the second backward differences of y defined by

$$\nabla^2 y_r = \nabla y_r - \nabla y_{r-1}, \quad r = 2, 3, \ldots\ldots \qquad \ldots(iii)$$

Similarly, the nth backward differences of y are defined by

$$\nabla^n y_r = \nabla^{n-1} y_r - \nabla^{n-1} y_{r-1}, \ldots\ldots \qquad \ldots(iv)$$

$$r = n, (n+1), \ldots\ldots$$

while using this formula for n = 1, we employ the notation $\nabla^0 y_r = y_r$. If y = f(x) is a constant function, then y = c, a constant for all x, and we get.

$$\nabla^n y_r = 0, \text{ for}$$

all n and for r = n, (n + 1).....

The symbol ∇^n is referred to as the nth backward difference operator.

The backward difference can be exhibited as shown in the following table, called the ***backward difference table***.

Backward Difference Table

x	y	∇y	$\nabla^2 y$	$\nabla^3 y$
x_0	y_0			
		∇y_1		
x_1	y_1		$\nabla^2 y_2$	
		∇y_2		$\nabla^3 y_3$
x_2	y_2		$\nabla^2 y_3$	
		∇y_3		
x_3	y_3			

(iii) Central Differences – Suppose $y_0, y_1, y_2 \ldots\ldots y_r \ldots\ldots$ are the values of a function y = f(x) corresponding to the values $x_0, x_1, x_2 \ldots x_r$ of x, the first central differences

$\delta y_{1/2}, \delta y_{3/2}, \delta y_{5/2}, \ldots\ldots$ are defined,

as

$$\delta y_{1/2} = y_1 - y_0, \delta y_{3/2} = y_2 - y_1, \delta y_{5/2} = y_3 - y_2, \ldots\ldots$$

$$\delta y_{r-1/2} = y_r - y_{r-1} \ldots\ldots \qquad \ldots(i)$$

(r = 1, 2, 3, ...)

The symbol δ is said to be the ***central difference operator***.

Comparing equation (i) with equation (i) in each of the forward differences and backward differences, we get

$$\delta y_{1/2} = \Delta y_0 = \nabla y_1$$

$$\delta y_{3/2} = \Delta y_1 = \nabla y_2$$

$$\delta y_{5/2} = \Delta y_2 = \nabla y_3 \text{ and so on.}$$

Generally

$$\delta y_{r-1/2} = \Delta y_{r-1} = \nabla y_r, \ldots\ldots\ldots\ldots$$

$$r = 1, 2, 3, \ldots\ldots\ldots\ldots \qquad \ldots(ii)$$

The first central differences of the first central differences are said to be the ***second central differences*** and are denoted by

$$\delta^2 y_1, \delta^2 y_2, \delta^3 y_3, \ldots\ldots$$

Thus

$$\delta^2 y_1 = \delta y_{3/2} - \delta y_{1/2}$$

$$\delta^2 y_2 = \delta y_{5/2} - \delta y_{3/2}$$

In general,

$$\delta^2 y_r = \delta y_{r+1/2} - \delta y_{r-1/2} \qquad \ldots(iii)$$

where, r = 1, 2, 3, ...

Higher order central differences are similarly define. The nth central differences are given by

(i) **For odd, n** – $\delta^n y_{r-1/2} = \delta^{n-1} y_r - \delta^{n-1} y_{r-1}$ ....(iv)

r = 1, 2, 3 .....

(ii) **For even, n** – $\delta^n y_r = \delta^{n-1} y_{r+1/2} - \delta^{n-1} y_{r-1/2}$ ....(v)

r = 1, 2, 3 .....

While using formula (iv) for n = 1, we use the notation $\delta^0 y_r = y_r$. If y = f(x) is a constant function i.e., if y = c, a constant, then $\delta^n y_r = 0$ for all $n \geq 1$.

The symbol δ^n is referred to as the nth central difference operator.

The central differences can be displayed in the table as shown below called ***central difference table***.

Central Difference Table

x	y	δy	$\delta^2 y$	$\delta^3 y$	$\delta^4 y$
x_0	y_0				
		$\delta y_{1/2}$			
x_1	y_1		$\delta^2 y_1$		
		$\delta y_{3/2}$		$\delta^3 y_{3/2}$	
x_2	y_2		$\delta^2 y_2$		$\delta^4 y_2$
		$\delta y_{5/2}$		$\delta^3 y_{5/2}$	
x_3	y_3		$\delta^2 y_3$		
		$\delta y_{7/2}$			
x_4	y_4				

Differences of Polynomial – If f(x) be a polynomial of nth degree in x, then the nth difference of f(x) is constant and $\Delta^{n+1} f(x) = 0$

One or Missing terms (Equal Intervals) –

Sometimes we may be given a set of equidistance terms with some terms missing. The problem of estimating such terms can be easily tackled by the use of the operators E and Δ.

Let us suppose that we are given (n + 1) equidistance arguments (r = 0, 1, ..., n say), but the entry y_r corresponding to any one of them, say

$(r+1)^{th}$ argument, is not given and we want to estimate it. Since we are given n entries, the data can be represented by a polynomial of $(n-1)^{th}$ degree. Hence, we may take y_x as a polynomial of $(n-1)^{th}$ degree.

$$\therefore \qquad \Delta^{n-1} y_x = \text{Constant}$$

and
$$\Delta^n y_x = 0,\ x = 0,1,2, \ldots\ldots n$$

In particular

$$\Delta^n y_0 = 0$$

$$\Rightarrow \qquad (E-1)^n y_0 = 0$$

$$[E^n - {}^nC_1 E^{n-1} + {}^nC_2 E^{n-2} - \ldots\ldots + (-1)^n] y_0 = 0 \qquad \ldots(i)$$

From this equation, the missing entry can be easily calculated.

If in a set of $(n+1)$ equidistance arguments two entries y_r and y_s are missing then the data can be represented by a polynomial of degree $n-2$. So proceeding as above we have

$$\Delta^{n-1} y_0 = 0 \text{ and } \Delta^{n-1} y_1 = 0$$

$$\Rightarrow \qquad (E-1)^{n-1} y_0 = 0 \text{ and } (E-1)^{n-1} y_1 = 0 \qquad \ldots(ii)$$

Expanding and simplifying as above the two missing terms can be estimated by solving the equations (ii). Similarly in a set of $(n+1)$ equidistance arguments the three missing terms can be estimated by solving the equations,

$$\Delta^{n-2} y_0 = 0,\ \Delta^{n-2} y_1 = 0,\ \Delta^{n-2} y_2 = 0$$

Factorial Notation – A product of the form $n(n-1)(n-2) \ldots\ldots (n-r+1)$ is denoted by $[n]^r$ is called a ***factorial***.

In particular,

$$[n] = n$$
$$[n]^2 = n(n-1)$$
$$[n]^3 = n(n-1)(n-2)$$
$$\vdots \qquad \vdots \qquad \vdots$$

The factorial notation is of special utility in the theory of finite differences. It helps in finding the successive differences of a polynomial directly by simple rule of differentiation. Similarly given any difference of a function in factorial notation, we can find the corresponding function by simple integration.

Other Difference Operators – We have already introduce the operators Δ, ∇ and δ. Besides these, there are the operators E and μ, which we define

below –

(i) Shift Operator –

The shift operator E is given by the rule

$$E\, f(x) = f(x+h)$$

also,
$$E^2 f(x) = f(x+2h)$$
$$E^3 f(x) = f(x+3h)$$
$$\ldots\ldots\ldots\ldots\ldots$$
$$E^n f(x) = f(x+nh)$$

Similarly, the inverse operator E^{-1} is defined by,

$$E^{-1} f(x) = f(x-h),\ E^{-2} f(x) = f(x-2h)$$
$$\vdots \qquad \vdots \qquad \vdots \qquad \vdots$$
$$E^{-n} f(x) = f(x-nh)$$

where n may be any real number.

(ii) Averaging Operator –

It is denoted by μ and is defined by the equation,

$$\mu\, f(x) = \frac{1}{2}\left\{ f\left(x+\frac{h}{2}\right) + f\left(x-\frac{h}{2}\right) \right\}$$

Relation between Operators –

(i) We have

$$\Delta\, f(x) = f(x+h) - f(x)$$
$$= E\, f(x) - f(x) = (E-1)\, f(x)$$
$$\therefore \qquad \mathbf{\Delta = E - 1}$$

(ii) We have

$$\nabla\, f(x) = f(x) - f(x-h)$$
$$= f(x) - E^{-1} f(x)$$
$$= (1 - E^{-1})\, f(x)$$
$$\mathbf{\nabla = 1 - E^{-1} \text{ or } E^{-1} = 1 - \nabla}$$

(iii) We have

$$\delta\, f(x) = f\left(x+\frac{h}{2}\right) - f\left(x-\frac{h}{2}\right)$$
$$= E^{1/2} f(x) - E^{-1/2} f(x)$$
$$\delta\, f(x) = (E^{1/2} - E^{-1/2})\, f(x)$$
$$\delta = \mathbf{E^{1/2} - E^{-1/2}}$$

(iv) We have

$$Ef(x) = f(x+h)$$

Expand by Taylor's series

$$Ef(x) = f(x) + hf'(x) + \frac{h^2}{2!}f''(x) + \ldots\ldots\ldots\ldots$$

$$= f(x) + h\,Df(x) + \frac{h^2}{2!}D^2 f(x) + \ldots\ldots\ldots$$

where, $Df(x) = f'(x)$ etc.

$$= \left(1 + hD + \frac{h^2D^2}{2!} + \ldots\ldots\right)f(x)$$

$$Ef(x) = e^{hD}f(x), \qquad \left(\because e^x = 1 + x + \frac{x^2}{2!} + \ldots\ldots\right)$$

or

$$E = e^{hD}$$

Hence,

We can also write

$$E = 1 + \Delta = e^{hD} \qquad \text{[from relation (i)]}$$

$$E = e^{hD} \text{ or, } E^{-1} = e^{-hD} \text{ or, } E^{-1} = 1 - \nabla = e^{-hD}$$

Again,

So,

$$\nabla = 1 - e^{-hD}$$

(v) We have

$$\mu\, f(x) = \frac{1}{2}\left\{f\left(x + \frac{h}{2}\right) + f\left(x - \frac{h}{2}\right)\right\}$$

Hence,

$$\mu\, f(x) = \frac{1}{2}\left\{E^{1/2} f(x) + E^{-1/2} f(x)\right\}$$

$$= \frac{1}{2}(E^{1/2} + E^{-1/2})\, f(x)$$

or

$$\mu = \frac{1}{2}(E^{1/2} + E^{-1/2})$$

NUMERICAL PROBLEMS

Prob.21. Evaluate –

$$\Delta^2\left(\frac{5x+12}{x^2+5x+6}\right).$$

(R.G.P.V., May/June 2006)

Sol. Here, $\Delta^2\left(\dfrac{5x+12}{x^2+5x+6}\right)$

$$= \Delta^2\left\{\frac{5x+12}{(x+2)(x+3)}\right\} = \Delta^2\left\{\frac{2}{x+2} + \frac{3}{x+3}\right\} \qquad \text{(By partial fraction}$$

$$= \Delta\left\{\Delta\left(\frac{2}{x+2}\right) + \Delta\left(\frac{3}{x+3}\right)\right\} = \Delta\left\{2\left(\frac{1}{x+3} - \frac{1}{x+2}\right) + 3\left(\frac{1}{x+4} - \frac{1}{x+3}\right)\right\}$$

$$= -2\Delta\left\{\frac{1}{(x+2)(x+3)}\right\} - 3\Delta\left\{\frac{1}{(x+3)(x+4)}\right\}$$

$$= -2\left\{\frac{1}{(x+3)(x+4)} - \frac{1}{(x+2)(x+3)}\right\} - 3\left\{\frac{1}{(x+4)(x+5)} - \frac{1}{(x+3)(x+4)}\right\}$$

$$= \frac{4}{(x+2)(x+3)(x+4)} + \frac{6}{(x+3)(x+4)(x+5)}$$

$$= \frac{\mathbf{2(5x+16)}}{\mathbf{(x+2)(x+3)(x+4)(x+5)}}$$

Ans.

Prob.22. Prove that –

$$e^x = \left(\frac{\Delta^2}{E}\right)e^x \cdot \frac{Ee^x}{\Delta^2 e^x}.$$

(R.G.P.V., Feb. 2010, June 2011, 2014)

Or

Evaluate $\left(\dfrac{\Delta^2}{E}\right)e^x \cdot \dfrac{Ee^x}{\Delta^2 e^x}$.

(R.G.P.V., June 2017)

Or

Prove that $e^x = \left\{\dfrac{\Delta^2}{E}\right\}e^x \cdot \dfrac{Ee^x}{\Delta^2 e^x}$***; the interval of differencing being h.***

[R.G.P.V., Nov. 2019 (IV-Sem)]

Sol. Taking R.H.S., we have

$$\left(\frac{\Delta^2}{E}\right)e^x \cdot \frac{Ee^x}{\Delta^2 e^x} = \Delta^2 . E^{-1} e^x \cdot \frac{Ee^x}{\Delta^2 e^x}$$

$$= \Delta^2 e^{x-h} \cdot \frac{e^{x+h}}{\Delta^2 e^x} = e^{-h}\,\Delta^2 e^x \cdot \frac{e^{x+h}}{\Delta^2 e^x}$$

$$= e^{-h} \cdot e^{x+h}$$

$$= \mathbf{e^x = L.H.S.}$$

Proved

Prob.23. Prove with usual notations that –

$$hD = \log(1 + \Delta) = -\log(1 - \nabla).$$

(R.G.P.V., Dec. 2005, 2011)

Or

Prove that $e^{hD} = 1 + \Delta$. ***(R.G.P.V., June 2011)***

Sol. We have $\quad Ef(x) = f(x + h)$

Expand by Taylor's series

$$= f(x) + h\, f'(x) + \frac{h^2}{2!} f''(x) + \ldots\ldots$$

$$= f(x) + hDf(x) + \frac{h^2D^2}{2!} f(x) + \ldots\ldots$$

$$= \left(1 + hD + \frac{h^2D^2}{2} + \ldots\ldots\ldots\right) f(x) = e^{hD} f(x)$$

$$E = e^{hD} \text{ or } \log E = hD$$

or $\quad E = 1 + \Delta \quad \ldots(i)$

also, $\quad \log(1 + \Delta) = hD$

Again

$$hD = \log E = -\log E^{-1} = -\log(1 - \nabla)$$

or $\quad hD = -\log(1 - \nabla) \quad \ldots(ii)$

From relations (i) and (ii), we have

$$\mathbf{hD = \log(1 + \Delta) = -\log(1 - \nabla)}$$ **Proved**

Prob.24. Prove that $\Delta \log f(x) = \log\left\{1 + \frac{\Delta f(x)}{f(x)}\right\}$.

[R.G.P.V., Nov. 2019 (III-Sem)]

Sol. Here, taking L.H.S.

$$\Delta \log f(x) = \log f(x + h) - \log f(x)$$

$$= \log\left\{\frac{f(x+h)}{f(x)}\right\}$$

$$= \log\left\{\frac{Ef(x)}{f(x)}\right\} \qquad [\because Ef(x) = f(x+h)]$$

$$= \log\left\{\frac{(1+\Delta)f(x)}{f(x)}\right\} \qquad [\because E = 1 + \Delta]$$

$$= \log\left\{\frac{f(x) + \Delta f(x)}{f(x)}\right\}$$

$$= \log\left\{1 + \frac{\Delta f(x)}{f(x)}\right\}$$

$$= \textbf{R.H.S.}$$ **Proved**

Prob.25. Find the value of $\left(\frac{\Delta^2}{E}\right)e^x$. ***(R.G.P.V., Dec. 2015)***

Sol.

$$\frac{\Delta^2}{E}e^x = \Delta^2 E^{-1} e^x$$

$$= \Delta^2 e^{x-h} = e^{-h}\,\Delta^2 e^x = e^{-h}\Delta(\Delta e^x)$$

$$= e^{-h}\Delta(e^{x+h} - e^x) = e^{-h}\Delta[e^x(e^h - 1)]$$

$$= e^{-h}(e^h - 1)\,\Delta e^x = e^{-h}(e^h - 1)\,(e^{x+h} - e^x)$$

$$= e^{-h}(e^h - 1)\,(e^h - 1)\,e^x = \mathbf{e^{-h}(e^h - 1)^2\, e^x}$$ **Ans.**

Prob.26. Prove with the usual notations that
$(E^{1/2} + E^{-1/2})\,(1 + \Delta)^{1/2} = 2 + \Delta$ ***(R.G.P.V., Dec. 2005, 2010)***

Sol. Here, taking L.H.S.

$$= (E^{1/2} + E^{-1/2})\,(1 + \Delta)^{1/2}$$

$$= (E^{1/2} + E^{-1/2})\,(E^{1/2}) \qquad (\because E = 1 + \Delta)$$

$$= (E + E^0) = E + 1 = 1 + \Delta + 1$$

$$= 2 + \Delta = \textbf{R.H.S.}$$ **Proved**

Prob.27. Express $y = 3x^3 + x^2 + x + 1$ ***in factorial notation and hence show that*** $\Delta^3 y = 18$. ***(R.G.P.V., Dec. 2010)***

Sol. Let $y \equiv 3x^3 + x^2 + x + 1$

$$\equiv A(x)\,(x - 1)\,(x - 2) + B(x)\,(x - 1) + C(x) + D$$

where A, B, C and D are constants to be determined.

Putting $x = 0$, we get $\quad 1 = D$

Putting $x = 1$, we get $\quad 6 = 0 + 0 + C + D \Rightarrow 6 = C + 1$

$\therefore \quad C = 5$

Putting $x = 2$, we get

$$3 \times 2^3 + 2^2 + 2 + 1 = 0 + B(2)\,(1) + C(2) + D$$

$$24 + 4 + 2 + 1 = 2B + 2 \times 5 + 1$$

$$2B = 31 - 10 - 1 = 20$$

$$\therefore \quad B = 10$$

Now, on comparing the coefficients of x^3 on both sides, we get $A = 3$

$$\therefore \quad y = \mathbf{3x^{(3)} + 10x^{(2)} + 5x^{(1)} + 1}$$ **Ans.**

Hence $\quad \Delta y = 3 \times 3x^{(2)} + 10 \times 2x^{(1)} + 5 \times 1$

$$\Delta y = 9x^{(2)} + 20x^{(1)} + 5$$

Now $\quad \Delta^2 y = 18x^{(1)} + 20$

and $\quad \Delta^3 y = \mathbf{18}$ **Proved**

Prob.28. Given that –

x	1	2	3	4	5
y	2	5	10	17	26

Find the value of $\nabla^2 y_5$. ***(R.G.P.V., June 200***

Sol. Here, $y_1 = 2, y_2 = 5, y_3 = 10, y_4 = 17$, and $y_5 = 26$

$$\nabla y_5 = y_5 - y_4$$

$\because$ $$\nabla(\nabla y_5) = \nabla(y_5 - y_4)$$

$\therefore$ $$\nabla^2 y_5 = \nabla y_5 - \nabla y_4 = (y_5 - y_4) - (y_4 - y_3)$$

or $$\nabla^2 y_5 = y_5 - 2y_4 + y_3$$

or Now putting the given values, we get

$$\nabla^2 y_5 = 26 - 2 \times 17 + 10 = \mathbf{2}$$ **An**

Prob.29. Find the function whose first difference is $9x^2 + 11x + 5$.
(R.G.P.V., Dec. 2004, June 201

Sol. Let f(x) be the required function

$$\Delta f(x) = 9x^2 + 11x + 5$$
$$= 9x(x-1) + Ax + B$$

Substituting x = 0, we get, B = 5.

Substituting x = 1, we get, A = 20.

$\therefore$ $$\Delta f(x) = 9x(x-1) + 20x + 5$$
$$= 9x^{(2)} + 20x^{(1)} + 5$$

and summing it, we get

$$f(x) = \frac{9x^{(3)}}{3} + \frac{20x^{(2)}}{2} + 5x^{(1)} + c$$

where c is any arbitrary constant

$$f(x) = 3x^{(3)} + 10x^{(2)} + 5x^{(1)} + c$$

or $$f(x) = \mathbf{3x^3 + x^2 + x + c}$$ **Ans**

Prob.30. Express $y = 2x^3 - 3x^2 + 3x - 10$ in factorial notation.
(R.G.P.V., June 2007, Dec. 2011, June 2016)

Sol. Let $y \equiv 2x^3 - 3x^2 + 3x - 10$

$$\equiv A(x)(x-1)(x-2) + B(x)(x-1) + C(x) + D \quad ...(i)$$

where A, B, C and D are constants to be determined.

Putting x = 0, we get

$$-10 = 0 + 0 + 0 + D$$

or $$D = -10$$

Putting x = 1, we get

$$2 - 3 + 3 - 10 = 0 + 0 + C + D$$

or $$C = -8 - D$$
$$= -8 + 10 = 2$$

Similarly by putting x = 2, we get

$$2 \times 8 - 3 \times 4 + 3 \times 2 - 10 = 0 + B(2)(1) + C(2) + D$$
$$0 = 2B + 4 - 10$$

$\Rightarrow$ $$6 = 2B \Rightarrow B = 3$$

Again on comparing the coefficients of x^3 on both sides, we get A = 2.

$\therefore$ $$y = \mathbf{2x^{(3)} + 3x^{(2)} + 2x^{(1)} - 10}$$ **Ans.**

Prob.31. Express $f(x) = x^3 - 2x^2 + x - 1$ into factorial notation and show that $\Delta^4 f(x) = 0$.
(R.G.P.V., June 2013)

Sol. Let

$$f(x) = x^3 - 2x^2 + x - 1$$
$$= Ax^{(3)} + Bx^{(2)} + Cx^{(1)} + D$$
$$= Ax(x-1)(x-2) + Bx(x-1) + Cx + D \quad ...(i)$$

where, A, B, C and D are constants to be found.

Putting, x = 0 and both sides of equation (i), we get

$$D = -1$$

Now putting x = 1 on both sides of equation (i), we get

$$-1 = 0 + 0 + C + D$$

or $$C = -1 - D = -1 + 1 = 0$$

Similarly by putting x = 2, we get

$$8 - 8 + 2 - 1 = 2B + 2C + D$$
$$B = 1$$

Again, comparing the coefficients of x^3 on both sides, we get A = 1.

$\therefore$ $$f(x) = \mathbf{x^{(3)} + x^{(2)} + 0x^{(1)} - 1}$$ **Ans.**

Hence $$\Delta f(x) = 3x^{(2)} + 2x^{(1)}$$
$$\Delta^2 f(x) = 6x^{(1)} + 2$$
$$\Delta^3 f(x) = 6$$

and $$\Delta^4 f(x) = \mathbf{0}$$ **Proved**

Prob.32. Find the first term of the series whose second and subsequent terms are 8, 3, 0, – 1, 0. (R.G.P.V., June 2003, Nov./Dec. 2007, June 2016)

Sol. We know that $f(1) = E^{-1} f(2)$, the interval of differencing being unity

$$= (1 + \Delta)^{-1}.f(2)$$
$$= (1 - \Delta + \Delta^2 - \Delta^3 ....) f(2)$$

Again five observations are given so that 4th difference will be constant and fifth difference will be zero.

Construct the following table –

x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$
2	8		
		–5	
3	3		2
		–3	
4	0		2
		–1	
5	–1		2
		1	
6	0		

$\therefore$ $f(1) = f(2) - \Delta f(2) + \Delta^2 f(2)$

($\because$ higher order differences are zero)

$= 8 - (-5) + 2 = \mathbf{15}$ **Ans.**

Prob.33. Find the missing values of the following data –

x :	0	5	10	15	20	25
y :	6	10	–	17	–	31

(R.G.P.V., June 2015)

Sol. The difference table is as follows –

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
0	$6 = y_0$				
		4			
5	$10 = y_1$		$y_2 - 14$		
		$y_2 - 10$		$41 - 3y_2$	
10	y_2		$27 - 2y_2$		$y_4 + 6y_2 - 102$
		$17 - y_2$		$y_4 + 3y_2 - 61$	
15	$17 = y_3$		$y_4 + y_2 - 34$		$143 - 4y_4 - 4y_2$
		$y_4 - 17$		$82 - 3y_4 - y_2$	
20	y_4		$48 - 2y_4$		
		$31 - y_4$			
25	$31 = y_5$				

As only four entries y_0, y_1, y_3, y_5 are given the function y can be represented by a third degree polynomial.

$\therefore$ $\Delta^4 y_0 = 0$ and $\Delta^4 y_1 = 0$

i.e., $y_4 + 6y_2 = 102$, and $4y_4 + 4y_2 = 143$

On solving we get $y_2 = \mathbf{13.25}$, $y_4 = \mathbf{22.5}$ **Ans.**

Prob.34. Estimate the missing term in the following table –

x :	0	1	2	3	4
$f(x)$:	1	3	9	?	81

(R.G.P.V., Dec. 2002, 2008)

Sol. Since, there are given 4 values,

$$\Delta^4 f(x) = 0, \forall x,$$
$$(E-1)^4 f(x) = 0, \forall x,$$
$$(E^4 - 4E^3 + 6E^2 - 4E + 1)\, f(x) = 0, \forall x,$$
$$E^4 f(x) - 4E^3 f(x) + 6E^2 f(x) - 4Ef(x) + f(x) = 0, \forall x,$$
$$f(x+4) - 4f(x+3) + 6f(x+2) - 4f(x+1) + f(x) = 0, \forall x,$$

here interval of differencing is 1.

Putting $x = 0$, we get

$f(4) - 4f(3) + 6f(2) - 4f(1) + f(0) = 0$

$81 - 4f(3) + 54 - 12 + 1 = 0$

or $4f(3) = 124$

$\therefore$ $f(3) = \mathbf{31}$ **Ans.**

Prob.35. Find the missing values in the table

x	45	50	55	60	65
y	3	–	2	–	–2.4

(R.G.P.V., June 2007)

Sol. The difference table is as follows –

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$
45	$3 = y_0$			
		$y_1 - 3$		
50	y_1		$-2y_1 + 5$	
		$2 - y_1$		$y_3 + 3y_1 - 9$
55	$2 = y_2$		$y_3 + y_1 - 4$	
		$y_3 - 2$		$-3y_3 - y_1 + 3.6$
60	y_3		$-2y_3 - 0.4$	
		$-2.4 - y_3$		
65	$-2.4 = y_4$			

As only three entries y_0, y_2, y_4 are given the function y can be represented by a second degree polynomial

$\therefore$ $\Delta^3 y_0 = 0$ and $\Delta^3 y_1 = 0$

i.e., $y_3 + 3y_1 = 9$ and $-3y_3 - y_1 = -3.6$

or $3y_3 + y_1 = 3.6$

On solving, we get

$y_3 = \mathbf{0.225}$, $y_1 = \mathbf{2.925}$ **Ans.**

INTERPOLATION USING NEWTON'S FORWARD AND BACKWARD DIFFERENCE FORMULAE, INTERPOLATION WITH UNEQUAL INTERVALS – NEWTON'S DIVIDED DIFFERENCE AND LAGRANGE'S FORMULAE

Interpolation and Extrapolation – Let the values of f(x) be known for certain values of x may or (may not be equispaced). The process of finding the value of f(x) for any other value of x in the given interval is called ***interpolation*** while the process of finding the value of f(x) for a value of x outside the given interval is called ***extrapolation***.

Use of Calculus of Finite Differences –

(i) Newton-Gregory Formula for Forward Interpolation – Let, $y = f(x)$ represent a function which assumes the values $f(x_0)$, $f(x_1)$, ... $f(x_n)$, we have

$$f(x) = f(x_0 + x - x_0) = f\left(x_0 + \frac{x - x_0}{h}.h\right)$$

Put $\quad u = \dfrac{x - x_0}{h}$

Thus, $\quad f(x) = f(x_0 + u\,h) = E^u f(x_0) \qquad [\because E^n f(x) = f(x + nh)]$

$$= (1 + \Delta)^u f(x_0) \qquad (\because E = 1 + \Delta)$$

$$= \left[1 + u\,\Delta + \frac{u(u-1)}{2!}\Delta^2 + \ldots\ldots\ldots\right] f(x_0)$$

$$= f(x_0) + u\Delta f(x_0) + \frac{u(u-1)}{2!}\Delta^2 f(x_0) + \ldots\ldots$$

This is known as ***Newton-Gregory formula of interpolation*** and can also be written as

$$y(x) = y_0 + u\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + \ldots\ldots$$

where $u = \dfrac{x - x_0}{h}$, $y_0 = f(x_0)$

This formula is generally applied for interpolation near the beginning of the tabulated values.

(ii) Newton-Gregory Formula for Backward Interpolation – In this case we have,

$$f(x) = f(x_n + x - x_n)$$

$$= f\left(x_n + \frac{x - x_n}{h}.h\right)$$

Put $\quad u = \dfrac{x - x_n}{h}$

Thus, $\quad f(x) = f(x_n + uh) = E^u f(x_n) = (E^{-1})^{-u} f(x_n)$

$$= (1 - \nabla)^{-u} f(x_n) \qquad (\because E^{-1} = 1 - \nabla)$$

$$= \left[1 + u\nabla + \frac{u(u+1)}{2!}\nabla^2 + \ldots\ldots\right] f(x_n)$$

$$= f(x_n) + u\nabla f(x_n) + \frac{u(u+1)}{2!}\nabla^2 f(x_n) + \ldots\ldots$$

This is known as ***Newton's backward interpolation formula.***

Interpolation with Unequal Interval –

Divided Differences – The difference defined by taking into consideration the change in the values of the arguments are called ***divided difference***.

Suppose $f(x_0)$, $f(x_1)$ ..... $f(x_n)$ are the values of the function $y = f(x)$ corresponding to the values $x_0, x_1, \ldots, x_n, x_{n-1}$ are not necessarily equal i.e., when there is a case of unequal intervals.

The first divided differences of f(x) for the arguments x_0, x_1 is defined as $\dfrac{f(x_1) - f(x_0)}{x_1 - x_0}$ and is denoted by $f(x_0, x_1)$ or by $x_1^{\Delta} f(x_0)$

Similarly the other divided differences of f(x) for the arguments x_1, x_2; x_2, x_3; ..... x_{n-1}, x_n are $f(x_1, x_2) = \dfrac{f(x_2) - f(x_1)}{x_2 - x_1} = x_2^{\Delta} f(x_1)$

$$f(x_2, x_3) = \frac{f(x_3) - f(x_2)}{x_3 - x_2} = x_3^{\Delta} f(x_2)$$

.....

$$f(x_{n-1}, x_n) = \frac{f(x_n) - f(x_{n-1})}{x_n - x_{n-1}} = x_n^{\Delta} f(x_{n-1})$$

Again second and higher divided differences are defined in terms of lower divided differences, for example the second divided differences of f(x) for the three arguments is given by

$$\frac{f(x_1, x_2) - f(x_0, x_1)}{x_2 - x_0}$$

and is denoted by $f(x_0, x_1, x_2)$ or $x_1^{\Delta^2}, x_2 f(x_0)$

Similarly nth divided difference is given by

$$\frac{f(x_1, x_2, \ldots\ldots x_n) - f(x_0, x_1 \ldots\ldots x_{n-1})}{x_n - x_0}$$

This is denoted by

$$f(x_0, x_1, \ldots\ldots x_n) \text{ or by } x_1, x_2^{\Delta^2} \ldots x_n f(x_0).$$

Lagrange's Interpolation Formula (Unequal Interval) – Let the values of f(x) be $f(x_0)$, $f(x_1)$, $f(x_2)$, $f(x_3)$, ...$f(x_n)$ at the points $x_0, x_1, x_2, \ldots\ldots x_n$, which are not necessarily equally spaced. Then f(x) can be considered as a polynomial of degree n in x. Then

$$f(x) = \frac{(x-x_1)(x-x_2)\ldots\ldots(x-x_n)}{(x_0-x_1)(x_0-x_2)\ldots\ldots(x_0-x_n)} f(x_0)$$

$$+ \frac{(x-x_0)(x-x_2)\ldots(x-x_n)}{(x_1-x_0)(x_1-x_2)\ldots\ldots(x_1-x_n)} f(x_1)$$

$$+ \frac{(x-x_0)(x-x_1)(x-x_3)\ldots\ldots(x-x_n)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)\ldots(x_2-x_n)} f(x_2)$$

$$+ \ldots\ldots + \frac{(x-x_0)(x-x_1)\ldots\ldots(x-x_{n-1})}{(x_n-x_0)(x_n-x_1)\ldots\ldots(x_n-x_{n-1})} f(x_n)$$

This is called Lagrange's formula for interpolation.

Newton's Formula for (Unequal Interval) – Suppose $f(x_0)$, $f(x_1)$ .... $f(x_n)$ be the values of f(x) corresponding to the arguments $x_0, x_1, \ldots\ldots x_n$ not necessarily equally spaced. Then

$$f(x) = f(x_0) + (x - x_0)\, f(x_0, x_1) + (x - x_0)(x - x_1)\, f(x_0, x_1, x_2) + \ldots\ldots + (x - x_0)(x - x_1) \ldots\ldots\ldots (x - x_{n-1})\, f(x_0, x_1 \ldots\ldots x_n)$$

This formula is called ***Newton's divided difference interpolation.***

Inverse Interpolation – The technique of determining the value of the argument corresponding to the given value of the function when the function lies between two tabulated values is known as ***inverse interpolation.*** To determine inverse interpolation, we usually follow the following important methods –

(i) Lagrange's method

(ii) Iteration or successive approximation method

(iii) Reversion of series method.

Lagrange's Method – This method resembles to the method of direct interpolation as discussed earlier the main difference is that here we assume x to be a polynomial in y [= f(x)], viz

$$\begin{aligned} x = {} & A_0\,[f(x) - f(x_1)]\,[f(x) - f(x_2)]\ldots\ldots[f(x) - f(x_n)] \\ & + A_1\,[f(x) - f(x_0)]\,[f(x) - f(x_2)]\ldots[f(x) - f(x_n)] \\ & + \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ & + A_n\,[f(x) - f(x_0)]\,[f(x) - f(x_1)]\ldots\ldots[f(x) - f(x_{n-1})] \end{aligned} \qquad \ldots(i)$$

where we are given n values of x and y [= f(x)], namely

$$[x_0, f(x_0)]\ldots\ldots\ldots[x_{n-1}, f(x_{n-1})]$$

In equation (i), there are (n + 1) terms and n factors in each term. The (n + 1) arbitrary constant $A_0, A_1, \ldots A_n$ are evaluated as follows –

Substituting $x = x_0$ in equation (i), we get

$$x_0 = A_0[f(x_0) - f(x_1)]\,[f(x_0) - f(x_2)]\ldots\ldots[f(x_0) - f(x_n)]$$

$$\Rightarrow \quad A_0 = \frac{x_0}{[f(x_0) - f(x_1)][f(x_0) - f(x_2)]\ldots\ldots[f(x_0) - f(x_n)]}$$

$$= \frac{x_0}{(y_0 - y_1)(y_0 - y_2)\ldots(y_0 - y_n)} \qquad [\because y = f(x)]$$

Similarly other constants can be determined. This implies

$$\begin{aligned} x = {} & \frac{(y-y_1)(y-y_2)\ldots\ldots(y-y_n)}{(y_0-y_1)(y_0-y_2)\ldots\ldots(y_0-y_n)} x_0 \\ & + \frac{(y-y_0)(y-y_2)\ldots\ldots(y-y_n)}{(y_1-y_0)(y_1-y_2)\ldots\ldots(y_1-y_n)} x_1 \\ & + \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ & + \frac{(y-y_0)(y-y_1)\ldots\ldots(y-y_{n-1})}{(y_n-y_0)(y_n-y_1)\ldots\ldots(y_n-y_{n-1})} x_n \end{aligned} \qquad \ldots(ii)$$

This formula gives the values of x when y [= f(x)] is known.

Q.3. Derive Newton's forward interpolation formula. ***(R.G.P.V., June 2014)***

Ans. Refer to the matter given on page 38.

Q.4. Define interpolation and write the Newton's forward and backward interpolation formula. ***(R.G.P.V., Dec. 2014)***

Ans. Refer to the matter given on pages 38.

NUMERICAL PROBLEMS

Prob.36. Prepare the difference table for the following data –

x :	10	20	30	40	50
y :	12	15	20	27	39

(R.G.P.V., Dec. 2015)

Sol. The difference table is –

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
10	12				
		3			
20	15		2		
		5		0	
30	20		2		3
		7		3	
40	27		5		
		12			
50	39				

Prob.37. The area 'A' of a circle of diameter 'd' is given for the following values –

d	A
80	5026
85	5674
90	6362
95	7088
100	7854

By using appropriate interpolating formulae, find approximate values for the areas of circles of diameter 82 and 91 respectively. (R.G.P.V., June 2012)

Sol. The difference table is as follows –

d	A	Δ	Δ^2	Δ^3
80	**5026**	**648**	**40**	**– 2**
85	5674	688	**38**	2
90	6362	**726**	40	
95	**7088**	766		
100	7854			

If we take $d_0 = 80$ then $A_0 = 5026$

$$\Delta A_0 = 648,\ \Delta^2 A_0 = 40 \text{ and } \Delta^3 A_0 = -2$$

Since d = 82 and h = 5,

$$u = \frac{d - d_0}{h} = \frac{82 - 80}{5} = \frac{2}{5} = 0.4$$

Using Newton's forward interpolation formula, we get

$$d_{82} = A_0 + u\Delta A_0 + \frac{u(u-1)}{2!}\Delta^2 A_0 + \frac{u(u-1)(u-2)}{3!}\Delta^3 A_0$$

$$d_{82} = 5026 + 0.4 \times 648 + \frac{0.4(0.4-1)}{2} \times 40 + \frac{0.4(0.4-1)(0.4-2)}{6} \times (-2)$$

$$= 5026 + 259.2 - 4.8 - 0.128 = 5285.2 - 4.928$$

$$= \mathbf{5280.27}$$ **Ans.**

Since d = 91 is near the end of the interval, we use Newton's backward interpolation formula.

Taking $d_n = 95$

$$u = \frac{d - d_n}{h} = \frac{91 - 95}{5} = \frac{-4}{5} = -0.8$$

$$\therefore \quad d_{91} = A_n + u\nabla A_n + \frac{u(u+1)}{2!}\nabla^2 A_n + \frac{u(u+1)(u+2)}{3!}\nabla^3 A_n$$

$$= 7088 + (-0.8) \times 726 + \frac{(-0.8)(-0.8+1)}{2} \times 38 + \frac{(-0.8)(-0.8+1)(-0.8+2)}{6} \times (-2)$$

$$= 7088 - 580.8 - 3.04 + 0.064$$

$$= 7088.064 - 583.84$$

$$= \mathbf{6504.22}$$ **Ans.**

Prob.38. Using Newton's forward interpolation formula, find the value of f(1.6) if –

x :	1	1.4	1.8	2.2
y :	3.49	4.82	5.96	6.5

[R.G.P.V., Nov. 2019 (IV-Sem)]

Sol. The difference table is as follows –

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$
1	**3.49**			
		1.33		
1.4	4.82		**– 0.19**	
		1.14		**– 0.41**
1.8	5.96		– 0.6	
		0.54		
2.2	6.5			

Here we are to compute f(1.6), so x = 1.6.

Also $\quad x_0 = 1,\ h = 0.4$

$$\therefore \quad u = \frac{x - x_0}{h} = \frac{1.6 - 1}{0.4} = 1.5$$

Using Newton's forward interpolation formula, we get

$$y = f(x) = y_0 + u\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + \frac{u(u-1)(u-2)}{3!}\Delta^3 y_0$$

$$f(1.6) = 3.49 + 1.5 \times 1.33 + \frac{1.5 \times (1.5-1)}{2!} \times (-0.19) + \frac{1.5 \times (1.5-1)(1.5-2)}{3!} \times (-0.41)$$

$$= 3.49 + 1.995 - 0.07125 + 0.025625$$

$$= \mathbf{5.439}$$ **Ans.**

Prob.39. Given sin 45° = 0.7071, sin 50° = 0.7660, sin 55° = 0.8192, sin 60° = 0.8660, find sin 52°, by using Newton's forward difference interpolation formula. ***[R.G.P.V., Nov. 2019 (III-Sem)]***

Sol. Given

x :	45°	50°	55°	60°
$f(x)$:	0.7071	0.7660	0.8192	0.8660

where f(x) = sin x

The difference table is as follows –

x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$
45	**0.7071**			
		0.0589		
50	0.7660		**– 0.0057**	
		0.0532		**– 0.0007**
55	0.8192		– 0.0064	
		0.0468		
60	0.8660			

Here we are to compute sin 52°, i.e., f(52).

So x = 52°

Also $x_0 = 45°$, h = 5

$$\therefore \quad u = \frac{x - x_0}{h} = \frac{52-45}{5} = \frac{7}{5} = 1.4$$

Using Newton's forward difference interpolation formula, we get

$$f(x) = f(x_0) + u\Delta f(x_0) + \frac{u(u-1)}{2!}\Delta^2 f(x_0) + \frac{u(u-1)(u-2)}{3!}\Delta^3 f(x_0)$$

$$f(52) = 0.7071 + 1.4 \times 0.0589 + \frac{1.4 \times (1.4-1)}{2!} \times (-0.0057) + \frac{1.4 \times (1.4-1)(1.4-2)}{3!} \times (-0.0007)$$

$$= 0.7071 + 0.08246 - 0.001596 + 0.0000392$$

$$= 0.7880032$$

Hence sin 52° = **0.7880032** **Ans.**

Prob.40. Find the number of men getting wages between ₹10 and ₹15 from the following data –

Wages in (₹) :	*0-10*	*10-20*	*20-30*	*30-40*
Frequency :	*9*	*30*	*35*	*42*

(R.G.P.V., June 2010, Dec. 2013)

Sol. First we prepare the cumulative frequency table as follows –

Wages below x :	10	20	30	40
Frequency y :	9	39	74	116

Now the difference table is as follows –

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$
10	**9**			
		30		
20	39		**5**	
		35		**2**
30	74		7	
		42		
40	116			

Here we have to find the number of men getting wages between 10-15.

$\therefore$ Taking $x_0 = 10$, x = 15

$$u = \frac{x - x_0}{h} = \frac{15-10}{10} = 0.5$$

Using Newton's forward interpolation formula, we get

$$y = y_0 + u\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + \frac{u(u-1)(u-2)}{3!}\Delta^3 y_0$$

$$\therefore \quad y_{15} = 9 + (0.5)(30) + \frac{(0.5)(0.5-1)}{2!}(5) + \frac{0.5(0.5-1)(0.5-2)}{3!}(2)$$

$$= 9 + 15 - 0.625 + 0.125 = 23.5$$

Thus the number of men getting wages below 15 = 23.5 ≈ 24

But the number of men getting wages below 10 is 9. Hence the number of men getting wages in between 10-15

= 24 – 9 = **15** **Ans.**

Prob.41. From the following table, estimate the number of students who obtained marks between 40 and 45.

Marks	*30-40*	*40-50*	*50-60*	*60-70*	*70-80*
No. of Students	*31*	*42*	*51*	*35*	*31*

(R.G.P.V., Dec. 2012, June 2015)

Sol. We prepare the difference table as follows –

Marks Below x	No. of Students y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
40	**31**				
		42			
50	73		**9**		
		51		**–25**	
60	124		–16		**37**
		35		12	
70	159		–4		
		31			
80	190				

Here we have to find the number of students who obtained marks betwe 40 and 45.

Taking $x_0 = 40$, $x = 45$

$$u = \frac{x - x_0}{h} = \frac{45 - 40}{10} = 0.5$$

Using Newton's forward interpolation formula, we get

$$y = y_0 + u\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + \frac{u(u-1)(u-2)}{3!}\Delta^3 y_0 + \frac{u(u-1)(u-2)(u-3)}{4!}\Delta^4 y_0$$

$$y_{45} = 31 + (0.5)(42) + \frac{(0.5)(0.5-1)}{2!}(9) + \frac{(0.5)(0.5-1)(0.5-2)}{3!}(-25) + \frac{(0.5)(0.5-1)(0.5-2)(0.5-3)}{4!}(37)$$

$$y_{45} = 31 + 21 - 1.125 - 1.5625 - 1.4453 = 47.867$$

Thus the number of students who obtained marks below $45 = 47.867 \approx 48$

But the number of students who obtained marks below 40 is 31. Hence the number of students who obtained marks between 40-45 = 48-31 = **17.**

Ans.

Prob.42. Find the cubic polynomial which takes the following values-

x :	0	1	2	3
$f(x)$:	1	2	1	10

(R.G.P.V., Feb. 2010, Dec. 2014, June 2017, Nov. 2018)

Sol. The difference table is as follows –

x	$f(x)$	$\nabla f(x_n)$	$\nabla^2 f(x_n)$	$\nabla^3 f(x_n)$
0	1			
		1		
1	2		–2	
		–1		**12**
2	1		**10**	
		9		
3	**10**			

Here, $u = \frac{x - x_n}{h} = \frac{x-3}{1} = (x-3)$

Then using Newton's backward formula, we have

$$f(x) = f(x_n) + u\nabla f(x_n) + \frac{u(u+1)}{2!}\nabla^2 f(x_n) + \frac{u(u+1)(u+2)}{3!}\nabla^3 f(x_n) + \ldots\ldots$$

$$= 10 + (x-3)(9) + \frac{(x-3)(x-2)}{2!} \times 10 + \frac{(x-3)(x-2)(x-1)}{3!} \times 12$$

$$= 10 + 9x - 27 + 5(x^2 - 5x + 6) + 2(x-3)(x^2 - 3x + 2)$$

$$= 9x - 17 + 5x^2 - 25x + 30 + 2x^3 - 6x^2 + 4x - 6x^2 + 18x - 12$$

$$= \mathbf{2x^3 - 7x^2 + 6x + 1}$$ **Ans.**

Prob.43. Find a polynomial which takes the following values –

x :	1	3	5	7	9	11
$f(x)$:	3	14	19	21	23	28

and hence compute $f(x)$ at $x = 2$. ***[R.G.P.V., May 2019 (IV-Sem)]***

Sol. The difference table is as follows –

x	$f(x)$	$\nabla f(x_n)$	$\nabla^2 f(x_n)$	$\nabla^3 f(x_n)$	$\nabla^4 f(x_n)$	$\nabla^5 f(x_n)$
1	3					
		11				
3	14		–6			
		5		3		
5	19		–3		0	
		2		3		**0**
7	21		0		**0**	
		2		**3**		
9	23		**3**			
		5				
11	**28**					

Here, $$u = \frac{x - x_n}{h} = \frac{x - 11}{2}$$

Then using Newton's backward formula, we have

$$f(x) = f(x_n) + u\nabla f(x_n) + \frac{n(u+1)}{2!}\nabla^2 f(x_n) + \frac{u(u+1)(u+2)}{3!}\nabla^3 f(x_n) + \ldots$$

$$= 28 + \frac{(x-11)}{2}(5) + \frac{\frac{(x-11)}{2}\left\{\frac{(x-11)}{2}+1\right\}}{2!} \times 3$$

$$+ \frac{\frac{(x-11)}{2}\left\{\frac{(x-11)}{2}+1\right\}\left\{\frac{(x-11)}{2}+2\right\}}{3!} \times$$

$$= 28 + \frac{5}{2}(x-11) + \frac{3}{8}(x-11)(x-9) + \frac{1}{16}(x-11)(x-9)(x-7)$$

$$= \frac{1}{16}\left[448 + 40(x-11) + 6(x^2 - 20x + 99) + (x^2 - 20x + 99)(x-7)\right]$$

$$= \frac{1}{16}(448 + 40x - 440 + 6x^2 - 120x + 594 + x^3 - 20x^2 + 99x$$

$$- 7x^2 + 140x - 693)$$

$$= \mathbf{\frac{1}{16}(x^3 - 21x^2 + 159x - 91)}$$ **Ans**

which is the required polynomial.

Hence, $$f(2) = \frac{1}{16}[(2)^3 - 21(2)^2 + 159(2) - 91]$$

$$= \frac{1}{16}(8 - 84 + 318 - 91) = \frac{151}{16} = \mathbf{9.4375}$$ **Ans**

Prob.44. Find a cubic polynomial in x for the following data –

x:	*0*	*1*	*2*	*3*	*4*	*5*
y:	*–3*	*3*	*11*	*27*	*57*	*107*

(R.G.P.V., June 2013)

Sol. The difference table is as follows –

x	y	∇y	$\nabla^2 y$	$\nabla^3 y$	$\nabla^4 y$	$\nabla^5 y$
0	–3					
		6				
1	3		2			
		8		6		
2	11		8		0	
		16		6		**0**
3	27		14		**0**	
		30		**6**		
4	57		**20**			
		50				
5	**107**					

Here, $$u = \frac{x - x_n}{h} = \frac{x-5}{1} = x - 5$$

Then using Newton's backward formula, we have

$$y = y_n + u\nabla y_n + \frac{u(u+1)}{2!}\nabla^2 y_n + \frac{u(u+1)(u+2)}{3!}\nabla^3 y_n$$

$$+ \frac{u(u+1)(u+2)(u+3)}{4!}\nabla^4 y_n$$

$$+ \frac{u(u+1)(u+2)(u+3)(u+4)}{5!}\nabla^5 y_n + \ldots..$$

$$y = 107 + (x-5)(50) + \frac{(x-5)(x-4)}{2!}(20) + \frac{(x-5)(x-4)(x-3)}{3!}(6)$$

$$+ \frac{(x-5)(x-4)(x-3)}{4!}(0) + \frac{(x-5)(x-4)(x-3)(x-2)}{5!}(0)$$

$$y = 107 + 50x - 250 + 10(x^2 - 9x + 20) + (x-5)(x^2 - 7x + 12)$$

$$y = 107 + 50x - 250 + 10x^2 - 90x + 200 + x^3 - 7x^2 + 12x$$

$$- 5x^2 + 35x - 60$$

$$y = \mathbf{x^3 - 2x^2 + 7x - 3}$$ **Ans.**

Prob.45. Find a polynomial satisfied by (– 4, 1245), (–1, 33), (0, 5), (2, 9) and (5, 1335) by Newton's divided difference formula. (R.G.P.V., Dec. 2015)

Sol. The divided difference table is as follows –

x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$	$\Delta^4 f(x)$
– 4	1245				
		$\frac{33-1245}{-1+4} = -404$			
–1	33		$\frac{-28+404}{0+4} = 94$		
		$\frac{5-33}{0+1} = -28$		$\frac{10-94}{2+4} = -14$	
0	5		$\frac{2+28}{2+1} = 10$		$\frac{13+14}{5+4} = 3$
		$\frac{9-5}{2-0} = 2$		$\frac{88-10}{5+1} = 13$	
2	9		$\frac{442-2}{5-0} = 88$		
		$\frac{1335-9}{5-2} = 442$			
5	1335				

Newton's divided difference formula is

$$f(x) = f(x_0) + (x - x_0)\,\Delta f(x_0) + (x - x_0)(x - x_1)\,\Delta^2 f(x_0) + (x - x_0)(x - x_1)(x - x_2)\,\Delta^3 f(x_0) + (x - x_0)(x - x_1)(x - x_2)(x - x_3)\,\Delta^4 f(x_0) \quad ...(i)$$

Here $x_0 = -4,\ x_1 = -1,\ x_2 = 0,\ x_3 = 2,\ x_4 = 5$

$f(x_0) = 1245,\ \Delta f(x_0) = -404,\ \Delta^2 f(x_0) = 94$

$\Delta^3 f(x_0) = -14,\ \Delta^4 f(x_0) = 3$

Substituting these values in equation (i), we get

$$f(x) = 1245 + (x + 4)(-404) + (x + 4)(x + 1)(94) + (x + 4)(x + 1)\,x\,(-14) + (x + 4)(x + 1)\,x\,(x - 2)(3)$$

$$= 1245 - 404x - 1616 + 94(x^2 + 5x + 4) - 14x(x^2 + 5x + 4) + 3x(x - 2)(x^2 + 5x + 4)$$

$$= 1245 - 404x - 1616 + 94x^2 + 470x + 376 - 14x^3 - 70x^2 - 56x + 3x^4 + 15x^3 + 12x^2 - 6x^3 - 30x^2 - 24x$$

$$= 3x^4 + (-14 + 15 - 6)x^3 + (94 - 70 + 12 - 30)x^2 + (-404 + 470 - 56 - 24)x + (1245 - 1616 + 376)$$

$$= \mathbf{3x^4 - 5x^3 + 6x^2 - 14x + 5}$$ **Ans.**

Prob.46. Use Newton's divided difference formula to find the form of f(x), given –

x	0	2	3	6
$f(x)$	648	704	729	792

(R.G.P.V., May/June 2006, June 2015)

Sol. The divided difference table is

x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$
0	648			
		$\frac{704-648}{2-0} = 28$		
2	704		$\frac{25-28}{3-0} = -1$	
		$\frac{729-704}{3-2} = 25$		$\frac{-1+1}{6-0} = 0$
3	729		$\frac{21-25}{6-2} = -1$	
		$\frac{792-729}{6-3} = 21$		
6	792			

Newton's divided difference formula is

$$f(x) = f(x_0) + (x - x_0)\,\Delta f(x_0) + (x - x_0)(x - x_1)\,\Delta^2 f(x_0) + (x - x_0)(x - x_1)(x - x_2)\,\Delta^3 f(x_0)$$

$$f(x) = 648 + (x - 0)(28) + (x - 0)(x - 2)(-1) + (x - 0)(x - 2)(x - 3)(0)$$

$$= 648 + 28x - x^2 + 2x + 0$$

$$\mathbf{f(x) = -x^2 + 30x + 648}$$ **Ans.**

Prob.47. The following table is given

x :	0	1	2	5
y :	2	3	12	147

What is the form of the function ?

[R.G.P.V., May 2019 (III-Sem)]

Sol. The divided difference table is

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$
0	2			
		$\frac{3-2}{1-0} = 1$		
1	3		$\frac{9-1}{2-0} = 4$	
		$\frac{12-3}{2-1} = 9$		$\frac{9-4}{5-0} = 1$
2	12		$\frac{45-9}{5-1} = 9$	
		$\frac{147-12}{5-2} = 45$		
5	147			

Newton's divided difference formula is

$$y = y_0 + (x - x_0)\,\Delta y_0 + (x - x_0)(x - x_1)\,\Delta^2 y_0 + (x - x_0)(x - x_1)(x - x_2)\,\Delta^3 y_0$$

$$y = 2 + (x - 0)(1) + (x - 0)(x - 1)(4) + (x - 0)(x - 1)(x - 2)(1)$$

$$y = 2 + x + 4x^2 - 4x + x(x^2 - 3x + 2)$$

$$y = 2 + x + 4x^2 - 4x + x^3 - 3x^2 + 2x$$

$$y = \mathbf{x^3 + x^2 - x + 2}$$ **Ans.**

Prob.48. By means of Newton's divided difference formula, find f(8) and f(9) from the following data –

x	4	5	7	10	11	13
f(x)	48	100	294	900	1210	2028

(R.G.P.V., Nov. 2018)

Sol. The divided difference table is given as follows –

x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$	$\Delta^4 f(x)$
4	48	$\frac{100-48}{5-4}=52$			
			$\frac{97-52}{7-4}=15$		
5	100			$\frac{21-15}{10-4}=1$	
		$\frac{294-100}{7-5}=97$			
			$\frac{202-97}{10-5}=21$		$\frac{1-1}{11-4}=0$
7	294			$\frac{27-21}{11-5}=1$	
		$\frac{900-294}{10-7}=202$			
			$\frac{310-202}{11-7}=27$		$\frac{1-1}{13-5}=0$
10	900			$\frac{33-27}{13-7}=1$	
		$\frac{1210-900}{11-10}=310$			
			$\frac{409-310}{13-10}=33$		
11	1210				
		$\frac{2028-1210}{13-11}=409$			
13	2028				

Now applying Newton's divided difference formula

$$f(x) = f(x_0) + (x - x_0)\,\Delta f(x_0) + (x - x_0)(x - x_1)\,\Delta^2 f(x_0)$$
$$+ (x - x_0)(x - x_1)(x - x_2)\,\Delta^3 f(x_0)$$
$$+ (x - x_0)(x - x_1)(x - x_2)(x - x_3)\,\Delta^4 f(x_0) \quad \text{...(i)}$$

Here $x_0 = 4, x_1 = 5, x_2 = 7, x_3 = 10, x_4 = 11, x_5 = 13$ and $f(x_0) = 48$,

$\Delta f(x_0) = 52,\ \Delta^2 f(x_0) = 15,\ \Delta^3 f(x_0) = 1,\ \Delta^4 f(x_0) = 0.$

Substituting these values in equation (i), we get

$$f(x) = 48 + (x - 4)(52) + (x - 4)(x - 5)(15)$$
$$+ (x - 4)(x - 5)(x - 7)(1)$$
$$+ (x - 4)(x - 5)(x - 7)(x - 10)(0) \quad \text{...(ii)}$$

Putting $x = 8$ in equation (ii), we have

$$f(8) = 48 + (8 - 4)(52) + (8 - 4)(8 - 5)(15)$$
$$+ (8 - 4)(8 - 5)(8 - 7)(1)$$
$$+ (8 - 4)(8 - 5)(8 - 7)(8 - 10)(0)$$

or $f(8) = 48 + (4)(52) + (4)(3)(15) + (4)(3)(1)(1) + 0$

or $f(8) = \mathbf{448}$ **Ans.**

Putting $x = 9$ in equation (ii), we have

$$f(9) = 48 + (9 - 4)(52) + (9 - 4)(9 - 5)(15) + (9 - 4)(9 - 5)$$
$$(9 - 7)(1) + (9 - 4)(9 - 5)(9 - 7)(9 - 10)(0)$$

or $f(9) = 48 + (5)(52) + (5)(4)(15) + (5)(4)(2)(1) + 0$

$= 48 + 260 + 300 + 40$

$= \mathbf{648}$ **Ans.**

Prob.49. Find f(8) by using Newton's divided difference formula from the following data –

x :	4	5	7	10	11	13
f(x) :	48	100	294	900	1210	2028

[R.G.P.V., Nov. 2019 (III-Sem)]

Sol. Refer to Prob.48.

Prob.50. Using Newton's divided difference formula, evaluate f(8) and f(15) from the following data –

x	4	5	7	10	11	13
f(x)	48	100	294	900	1210	2028

(R.G.P.V., June 2012, Dec. 2012)

Sol. **f(8)** – Refer to Prob.48.

Putting $x = 15$ in equation (ii), we have

$$f(15) = 48 + (15 - 4)(52) + (15 - 4)(15 - 5)(15) + (15 - 4)(15 - 5)$$
$$(15 - 7)(1) + (15 - 4)(15 - 5)(15 - 7)(15 - 10)(0)$$

$= 48 + (11)(52) + (11)(10)(15) + (11)(10)(8)(1) + 0$

$= \mathbf{3150}$ **Ans.**

Prob.51. Using Newton's divided difference formula, evaluate $f(9)$ from the following values of x and y –

x	5	7	11	13	17
y	150	392	1452	2366	5202

(R.G.P.V., June 2010, 2011, Dec. 2013)

Sol. The divided difference table is given as follows –

x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$	$\Delta^4 f(x)$
5	150				
		$\frac{392-150}{7-5} = 121$			
7	392		$\frac{265-121}{11-5} = 24$		
		$\frac{1452-392}{11-7} = 265$		$\frac{32-24}{13-5} = 1$	
11	1452		$\frac{457-265}{13-7} = 32$		$\frac{1-1}{17-5} = 0$
		$\frac{2366-1452}{13-11} = 457$		$\frac{42-32}{17-7} = 1$	
13	2366		$\frac{709-457}{17-11} = 42$		
		$\frac{5202-2366}{17-13} = 709$			
17	5202				

Newton's divided difference formula is

$$f(x) = f(x_0) + (x - x_0)\,\Delta f(x_0) + (x - x_0)(x - x_1)\,\Delta^2 f(x_0) + (x - x_0)(x - x_1)(x - x_2)\,\Delta^3 f(x_0) + (x - x_0)(x - x_1)(x - x_2)(x - x_3)\,\Delta^4 f(x_0)$$

$$\therefore \quad f(9) = 150 + (9-5)(121) + (9-5)(9-7)(24) + (9-5)(9-7)(9-11)(1) + (9-5)(9-7)(9-11)(9-13)(0)$$

$$f(9) = 150 + 484 + 192 - 16 + 0$$

$$f(9) = \mathbf{810}$$

Ans

Prob.52. Apply Newton's divided difference formula to find the value of $f(9)$ from the following table –

x	5	7	11	13	17
$f(x)$	150	392	1452	2368	5202

(R.G.P.V., June 2014)

Sol. This problem can be solved in a similar way as Prob.51.

Prob.53. Given $\log_{10}654 = 2.8156$, $\log_{10}658 = 2.8182$, $\log_{10}659 = 2.8189$, $\log_{10}661 = 2.8202$. Find by using Newton's divided difference formula, the value of $\log_{10}656$. **[R.G.P.V., May 2019 (IV-Sem)]**

Sol. The divided difference table is given as follows –

x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$
654	2.8156			
		$\frac{2.8182-2.8156}{658-654} = 0.00065$		
658	2.8182		$\frac{0.0007-0.00065}{659-654} = 0.00001$	
		$\frac{2.8189-2.8182}{659-658} = 0.0007$		$\frac{-0.0000167-0.00001}{661-654} = -0.0000038$
659	2.8189		$\frac{0.00065-0.0007}{661-658} = -0.0000167$	
		$\frac{2.8202-2.8189}{661-659} = 0.00065$		
661	2.8202			

Newton's divided difference formula is

$$f(x) = f(x_0) + (x - x_0)\,\Delta f(x_0) + (x - x_0)(x - x_1)\,\Delta^2 f(x_0) + (x - x_0)(x - x_1)(x - x_2)\,\Delta^3 f(x_0) \quad \text{...(i)}$$

Here $x_0 = 654,\ x_1 = 658,\ x_2 = 659,\ x_3 = 661$

and $f(x_0) = 2.8156,\ \Delta f(x_0) = 0.00065,\ \Delta^2 f(x_0) = 0.00001,$

$\Delta^3 f(x_0) = -0.0000038$

Substituting these values in equation (i), we get

$$f(x) = 2.8156 + (x - 654)(0.00065) + (x - 654)(x - 658)(0.00001) + (x - 654)(x - 658)(x - 659)(-0.0000038)$$

$$\therefore f(656) = 2.8156 + (656 - 654)(0.00065) + (656 - 654)(656 - 658)(0.00001) + (656 - 654)(656 - 658)(656 - 659)(-0.0000038)$$

$$= 2.8156 + 2 \times 0.00065 + (2)(-2)(0.00001) + (2)(-2)(-3)(-0.0000038)$$

$$= 2.8156 + 0.0013 - 0.00004 - 0.0000456$$

$$= 2.8168$$

Hence $\log_{10}656$ = **2.8168 Approx.** **Ans.**

Prob.54. For the following values of x and y, find the first derivative at x = 4.

x :	1	2	4	8
y :	0	1	5	21

(R.G.P.V., May/June 2006)

Sol. In this case the value of the argument are not equally spaced so we shall use Newton's divided difference formula. The divided difference table is

x	$y = f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$
1	0			
		1		
2	1		1/3	
		2		0
4	5		1/3	
		4		
8	21			

Newton's divided difference formula is

$$y = f(x) = f(a) + (x - a)\,\Delta f(a) + (x - a)(x - b)\,\Delta^2 f(a) + (x - a)(x - b)(x - c)\,\Delta^3 f(a)$$

Differentiating this w.r.t. x and putting x = c, we have

$$\frac{dy}{dx} = f'(x) = \Delta f(a) + (2c - a - b)\,\Delta^2 f(a) + (c - a)(c - b)\,\Delta^3 f(a)$$

Putting a = 1, b = 2, c = 4, d = 8 and the values of difference from the table, we get

$$f'(4) = 1 + (8 - 1 - 2)\left(\frac{1}{3}\right) + (4 - 1)(4 - 2)(0)$$

$$= 1 + \frac{5}{3} = \frac{8}{3} = \mathbf{2.67}$$ **Ans.**

Prob.55. The values of x and y are given as below –

x :	5	6	9	11
y :	12	13	14	16

Find the value of y, when x = 10 by using Lagranges interpolation formula. ***[R.G.P.V., Nov. 2019 (IV-Sem)]***

Sol. Here $x_0 = 5$, $x_1 = 6$, $x_2 = 9$ and $x_3 = 11$

$y_0 = 12$, $y_1 = 13$, $y_2 = 14$ and $y_3 = 16$

By Lagranges interpolation formula, we have

$$y = \frac{(x - x_1)(x - x_2)(x - x_3)}{(x_0 - x_1)(x_0 - x_2)(x_0 - x_3)} \times y_0 + \frac{(x - x_0)(x - x_2)(x - x_3)}{(x_1 - x_0)(x_1 - x_2)(x_1 - x_3)} \times y_1 + \frac{(x - x_0)(x - x_1)(x - x_3)}{(x_2 - x_0)(x_2 - x_1)(x_2 - x_3)} \times y_2 + \frac{(x - x_0)(x - x_1)(x - x_2)}{(x_3 - x_0)(x_3 - x_1)(x_3 - x_2)} \times y_3$$

$\because$ x = 10

$$\therefore y = \frac{(10-6)(10-9)(10-11)}{(5-6)(5-9)(5-11)} \times 12 + \frac{(10-5)(10-9)(10-11)}{(6-5)(6-9)(6-11)} \times 13 + \frac{(10-5)(10-6)(10-11)}{(9-5)(9-6)(9-11)} \times 14 + \frac{(10-5)(10-6)(10-9)}{(11-5)(11-6)(11-9)} \times 16$$

$$= \frac{-4}{-24} \times 12 + \frac{-5}{15} \times 13 + \frac{-20}{-24} \times 14 + \frac{20}{60} \times 16 = 2 - \frac{13}{3} + \frac{35}{3} + \frac{16}{3}$$

$$= \mathbf{14.67}$$ **Ans.**

Prob.56. Apply Lagrange's formula to find f(15), if –

x :	10	12	14	16	18	20
f(x) :	2420	1942	1497	1109	790	540

(R.G.P.V., Dec. 2010)

Sol. Here $x_0 = 10$, $x_1 = 12$, $x_2 = 14$, $x_3 = 16$, $x_4 = 18$ and $x_5 = 20$.

$f(x_0) = 2420$, $f(x_1) = 1942$, $f(x_2) = 1497$, $f(x_3) = 1109$, $f(x_4) = 790$ and $f(x_5) = 540$.

By Lagrange's formula, we have

$$f(x) = \frac{(x-x_1)(x-x_2)(x-x_3)(x-x_4)(x-x_5)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)(x_0-x_4)(x_0-x_5)} \times f(x_0)$$

$$+\frac{(x-x_0)(x-x_2)(x-x_3)(x-x_4)(x-x_5)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)(x_1-x_4)(x_1-x_5)} \times f(x_1)$$

$$+\frac{(x-x_0)(x-x_1)(x-x_3)(x-x_4)(x-x_5)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)(x_2-x_4)(x_2-x_5)} \times f(x_2)$$

$$+\frac{(x-x_0)(x-x_1)(x-x_2)(x-x_4)(x-x_5)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)(x_3-x_4)(x_3-x_5)} \times f(x_3)$$

$$+\frac{(x-x_0)(x-x_1)(x-x_2)(x-x_3)(x-x_5)}{(x_4-x_0)(x_4-x_1)(x_4-x_2)(x_4-x_3)(x_4-x_5)} \times f(x_4)$$

$$+\frac{(x-x_0)(x-x_1)(x-x_2)(x-x_3)(x-x_4)}{(x_5-x_0)(x_5-x_1)(x_5-x_2)(x_5-x_3)(x_5-x_4)} \times f(x_5)$$

Putting above values, we get

$$f(15) = \frac{(15-12)(15-14)(15-16)(15-18)(15-20)}{(10-12)(10-14)(10-16)(10-18)(10-20)} \times 2420$$

$$+\frac{(15-10)(15-14)(15-16)(15-18)(15-20)}{(12-10)(12-14)(12-16)(12-18)(12-20)} \times 1942$$

$$+\frac{(15-10)(15-12)(15-16)(15-18)(15-20)}{(14-10)(14-12)(14-16)(14-18)(14-20)} \times 1497$$

$$+\frac{(15-10)(15-12)(15-14)(15-18)(15-20)}{(16-10)(16-12)(16-14)(16-18)(16-20)} \times 1109$$

$$+\frac{(15-10)(15-12)(15-14)(15-16)(15-20)}{(18-10)(18-12)(18-14)(18-16)(18-20)} \times 790$$

$$+\frac{(15-10)(15-12)(15-14)(15-16)(15-18)}{(20-10)(20-12)(20-14)(20-16)(20-18)} \times 540$$

$$f(15) = \frac{-45}{-3840} \times 2420 + \frac{(-75)}{768} \times 1942 + \frac{(-225)}{(-384)} \times 1497 + \frac{225}{384} \times 1109$$

$$+\frac{75}{(-768)} \times 790 + \frac{45}{3840} \times 540$$

$$= 28.36 - 189.65 + 877.15 + 649.81 - 77.15 + 6.33$$

$$= \mathbf{1294.85}$$

Ans.

Prob.57. The following table gives the normal weights of babies during the first 12 months of life.

Age in Months:	0	2	5	8	10	12
Weights in lbs:	7½	10¼	15	16	18	21

Estimate the weight of the baby at the age of 7 months.

(R.G.P.V., Dec. 2014)

Sol. Here $x_0 = 0$, $x_1 = 2$, $x_2 = 5$, $x_3 = 8$, $x_4 = 10$ and $x_5 = 12$.

$f(x_0) = 7\frac{1}{2} = \frac{15}{2}$, $f(x_1) = 10\frac{1}{4} = \frac{41}{4}$, $f(x_2) = 15$, $f(x_3) = 16$, $f(x_4) = 18$ and $f(x_5) = 21$.

By Lagranges formula, we have

$$f(x) = \frac{(x-x_1)(x-x_2)(x-x_3)(x-x_4)(x-x_5)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)(x_0-x_4)(x_0-x_5)} \times f(x_0)$$

$$+\frac{(x-x_0)(x-x_2)(x-x_3)(x-x_4)(x-x_5)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)(x_1-x_4)(x_1-x_5)} \times f(x_1)$$

$$+\frac{(x-x_0)(x-x_1)(x-x_3)(x-x_4)(x-x_5)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)(x_2-x_4)(x_2-x_5)} \times f(x_2)$$

$$+\frac{(x-x_0)(x-x_1)(x-x_2)(x-x_4)(x-x_5)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)(x_3-x_4)(x_3-x_5)} \times f(x_3)$$

$$+\frac{(x-x_0)(x-x_1)(x-x_2)(x-x_3)(x-x_5)}{(x_4-x_0)(x_4-x_1)(x_4-x_2)(x_4-x_3)(x_4-x_5)}\times f(x_4)$$

$$+\frac{(x-x_0)(x-x_1)(x-x_2)(x-x_3)(x-x_4)}{(x_5-x_0)(x_5-x_1)(x_5-x_2)(x_5-x_3)(x_5-x_4)}\times f(x_5)$$

Putting above values, we get

$$f(7)=\frac{(7-2)(7-5)(7-8)(7-10)(7-12)}{(0-2)(0-5)(0-8)(0-10)(0-12)}\times\frac{15}{2}$$

$$+\frac{(7-0)(7-5)(7-8)(7-10)(7-12)}{(2-0)(2-5)(2-8)(2-10)(2-12)}\times\frac{41}{4}$$

$$+\frac{(7-0)(7-2)(7-8)(7-10)(7-12)}{(5-0)(5-2)(5-8)(5-10)(5-12)}\times 15$$

$$+\frac{(7-0)(7-2)(7-5)(7-10)(7-12)}{(8-0)(8-2)(8-5)(8-10)(8-12)}\times 16$$

$$+\frac{(7-0)(7-2)(7-5)(7-8)(7-12)}{(10-0)(10-2)(10-5)(10-8)(10-12)}\times 18$$

$$+\frac{(7-0)(7-2)(7-5)(7-8)(7-10)}{(12-0)(12-2)(12-5)(12-8)(12-10)}\times 21$$

$$f(7)=\frac{-150}{-9600}\times\frac{15}{2}+\frac{(-210)}{2880}\times\frac{41}{4}+\frac{(-525)}{(-1575)}\times 15+\frac{1050}{1152}\times 16$$

$$+\frac{350}{(-1600)}\times 18+\frac{210}{6720}\times 21$$

$$=\frac{15}{128}-\frac{287}{384}+5+\frac{175}{12}-\frac{63}{16}+\frac{21}{32}$$

$$=0.117-0.747+5+14.583-3.937+0.656$$

$$=\mathbf{15.67\ lbs}$$

Ans.

Prob.58. Using Lagrange's interpolation formula, find the cubic polynomial that takes the following values.

x	*0*	*1*	*2*	*4*
y	*1*	*1*	*2*	*5*

(R.G.P.V., Nov. 2018)

Sol. Here $x_0=0$, $x_1=1$, $x_2=2$, $x_3=4$, $f(x_0)=1$, $f(x_1)=1$, $f(x_2)=2$, $f(x_3)=5$.

By Lagrange's interpolation formula, we have

$$f(x)=\frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)}f(x_0)+\frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)}f(x_1)$$

$$+\frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)}f(x_2)+\frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)}f(x_3)$$

Putting above values, we get

$$f(x)=\frac{(x-1)(x-2)(x-4)}{(0-1)(0-2)(0-4)}\times 1+\frac{(x-0)(x-2)(x-4)}{(1-0)(1-2)(1-4)}\times 1$$

$$+\frac{(x-0)(x-1)(x-4)}{(2-0)(2-1)(2-4)}\times 2+\frac{(x-0)(x-1)(x-2)}{(4-0)(4-1)(4-2)}\times 5$$

$$f(x)=-\frac{1}{8}(x-1)(x-2)(x-4)+\frac{1}{3}x(x-2)(x-4)$$

$$-\frac{1}{2}x(x-1)(x-4)+\frac{5}{24}x(x-1)(x-2)$$

$$=\frac{1}{24}\Big[-3(x^2-3x+2)(x-4)+8x(x^2-6x+8)$$

$$-12x(x^2-5x+4)+5x(x^2-3x+2)\Big]$$

$$=\frac{1}{24}\Big[-3(x^3-7x^2+14x-8)+8x^3-48x^2+64x-12x^3$$

$$+60x^2-48x+5x^3-15x^2+10x\Big]$$

$$=\frac{1}{24}\Big[-2x^3+18x^2-16x+24\Big]$$

$$=\mathbf{-\frac{1}{12}x^3+\frac{3}{4}x^2-\frac{2}{3}x+1}$$

Ans.

Prob.59. Apply Lagrange's interpolation method to find the values of x when f(x) = 15 from the given data –

x :	*5*	*6*	*9*	*11*
f(x) :	*12*	*13*	*14*	*16*

(R.G.P.V., June 2005, Dec. 2011)

Or

Apply Lagrange's method to find the value of x, when f(x) = 15, from the given data –

x	5	6	9	11
y	12	13	14	16

(R.G.P.V., June 2016)

Sol. Here

$x_0 = 5,\ x_1 = 6,\ x_2 = 9,\ x_3 = 11,\ f(x_0) = 12,\ f(x_1) = 13,\ f(x_2) = 14,$

$f(x_3) = 16$ and $f(x) = 15$

Putting these values in Lagrange's formula for inverse interpolation, we get

$$x = \frac{(15-13)(15-14)(15-16)}{(12-13)(12-14)(12-16)} \times 5 + \frac{(15-12)(15-14)(15-16)}{(13-12)(13-14)(13-16)} \times 6$$

$$+ \frac{(15-12)(15-13)(15-16)}{(14-12)(14-13)(14-16)} \times 9 + \frac{(15-12)(15-13)(15-14)}{(16-12)(16-13)(16-14)} \times 11$$

$$\Rightarrow \quad x = \frac{2 \times 1 \times (-1)}{(-1) \times (-2) \times (-4)} \times 5 + \frac{3 \times 1 \times (-1)}{1 \times (-1) \times (-3)} \times 6$$

$$+ \frac{3 \times 2 \times (-1)}{2 \times 1 \times (-2)} \times 9 + \frac{3 \times 2 \times 1}{4 \times 3 \times 2} \times 11$$

$$\Rightarrow \quad x = \frac{10}{8} + \frac{(-18)}{3} + \frac{54}{4} + \frac{66}{24}$$

$$\Rightarrow \quad x = 1.25 - 6 + 13.5 + 2.75$$

$$= \mathbf{11.5}$$ **Ans.**

MODULE 2

NUMERICAL METHODS - 2

NUMERICAL DIFFERENTIATION

Numerical Differentiation – The process of finding the derivative or derivatives of a function at some values of the independent variable, when we know the values of the function corresponding to the given values of the independent variable, is called ***numerical differentiation.***

Formulae for Derivatives –

Consider the function $y = f(x)$ which is tabulated for the values,

$$x_n = (x_0 + nh),\ n = 0, 1, 2, \ldots.$$

(i) Derivatives Using Forward Difference Formula – Newton's forward interpolation formula is,

$$y = y_0 + u\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + \frac{u(u-1)(u-2)}{3!}\Delta^3 y_0 + \frac{u(u-1)(u-2)(u-3)}{4!}\Delta^4 y_0 + \ldots\ldots \quad \ldots(i)$$

Here, $$u = \frac{x - x_0}{h} \quad \ldots(ii)$$

Then

$$\left(\frac{dy}{dx}\right)_{x=x_0} = \frac{1}{h}\left[\Delta y_0 - \frac{\Delta^2 y_0}{2} + \frac{\Delta^3 y_0}{3} - \frac{\Delta^4 y_0}{4} + \ldots\right]$$

$$\left(\frac{d^2y}{dx^2}\right)_{x=x_0} = \frac{1}{h^2}\left[\Delta^2 y_0 - \Delta^3 y_0 + \frac{11}{12}\Delta^4 y_0 - \frac{5}{6}\Delta^5 y_0 + \ldots\right]$$

and $$\left(\frac{d^3y}{dx^3}\right)_{x=x_0} = \frac{1}{h^3}\left[\Delta^3 y_0 - \frac{3}{2}\Delta^4 y_0 + \ldots\right]$$

(ii) Derivatives Using Backward Difference Formula – Newton's backward formula is,

$$y = y_n + u\nabla y_n + \frac{u(u+1)}{2!}\nabla^2 y_n + \frac{u(u+1)(u+2)}{3!}\nabla^3 y_n + \ldots\ldots \quad \ldots(iii)$$

Here, $u = \dfrac{x - x_n}{h}$ ....(iv)

$$\therefore \quad \frac{dy}{dx} = \frac{1}{h}\left[\nabla y_n + (2u+1)\frac{\nabla^2 y_n}{2!} + (3u^2+6u+2)\frac{\nabla^3 y_n}{3!} + (4u^3+18u^2+22u+6)\frac{\nabla^4 y_n}{4!} + \ldots\ldots\right]$$

and $$\left(\frac{d^2y}{dx^2}\right)_{x=x_n} = \frac{1}{h^2}\left[\nabla^2 y_n + \nabla^3 y_n + \frac{11}{12}\nabla^4 y_n + \frac{5}{6}\nabla^5 y_n + \ldots\right]$$

Similarly, $$\left(\frac{d^3y}{dx^3}\right)_{x=x_n} = \frac{1}{h^3}\left[\nabla^3 y_n + \frac{3}{2}\nabla^4 y_n + \ldots\right]$$

where, $\dfrac{du}{dx} = \dfrac{1}{h}$

(iii) Derivatives Using Central Difference Formula – Stirling's formula is,

$$y = y_0 + \frac{u}{1!}\left(\frac{\Delta y_0 + \Delta y_{-1}}{2}\right) + \frac{u^2}{2!}\Delta^2 y_{-1} + \frac{u(u^2-1^2)}{3!}\left(\frac{\Delta^3 y_{-1} + \Delta^3 y_{-2}}{2}\right) + \frac{u^2(u^2-1^2)}{4!}\Delta^4 y_{-2} + \ldots \quad \ldots(v)$$

Here, $u = \dfrac{x - x_0}{h}$

$\therefore \quad \dfrac{du}{dx} = \dfrac{1}{h}$

Now, differentiating equation (v) both sides w.r.t. 'x', we get

$$\frac{dy}{dx} = \left(\frac{\Delta y_0 + \Delta y_{-1}}{2}\right)\frac{du}{dx} + \frac{2u}{2!}\Delta^2 y_{-1}\frac{du}{dx} + \left(\frac{3u^2-1}{3!}\right)\frac{du}{dx}\left(\frac{\Delta^3 y_{-1} + \Delta^3 y_{-2}}{2}\right) + \left(\frac{4u^3-2u}{4!}\right)\frac{du}{dx}\Delta^4 y_{-2} + \ldots$$

$$= \frac{1}{h}\left[\left(\frac{\Delta y_0 + \Delta y_{-1}}{2}\right) + u\Delta^2 y_{-1} + \left(\frac{3u^2-1}{6}\right)\left(\frac{\Delta^3 y_{-1} + \Delta^3 y_{-2}}{2}\right) + \left(\frac{2u^3-u}{12}\right)\Delta^4 y_{-2} + \ldots\ldots\right]$$

At $x = x_0$, $u = 0$

Hence, $$\left(\frac{dy}{dx}\right)_{x=x_0} = \frac{1}{h}\left[\frac{\Delta y_0 + \Delta y_{-1}}{2} - \frac{1}{12}(\Delta^3 y_{-1} + \Delta^3 y_{-2}) + \frac{1}{60}(\Delta^5 y_{-2} + \Delta^5 y_{-3}) + \ldots\ldots\right]$$

Similarly, $$\left(\frac{d^2y}{dx^2}\right)_{x=x_0} = \frac{1}{h^2}\left[\Delta^2 y_{-1} - \frac{1}{12}\Delta^4 y_{-2} + \frac{1}{90}\Delta^6 y_{-3} + \ldots\right]$$

NUMERICAL PROBLEMS

Prob.1. A rod is rotating in a plane. The following table gives the angle θ(radians) through which the rod has turned, for various values of the time t second –

t :	*0.0*	*0.2*	*0.4*	*0.6*	*0.8*	*1.0*	*1.2*
θ :	*0.00*	*0.12*	*0.49*	*1.12*	*2.02*	*3.20*	*4.67*

Calculate the angular velocity and acceleration of the rod when t = 0.6 second. ***(R.G.P.V., June 2010, Dec. 2012)***

Sol. As t = 0.6 is the middle value, we shall use the central difference formulae, noting that a = 0.6 and h = 0.2.

$$\mu\delta f(0.6) = \frac{1}{2}(0.63 + 0.9) = 0.765$$

$$\mu\delta^3 f(0.6) = \frac{1}{2}(0.01 + 0.01) = 0.01$$

$$\therefore \quad f'(0.6) = \frac{1}{0.2}\left[0.765 - \frac{1}{6}(0.01)\right] = 3.82 \text{ radian/sec}^2$$

Also $\delta^2 f(0.6) = 0.27$, $\delta^4 f(0.6) = 0$

$$\therefore \quad f''(0.6) = \frac{1}{(0.2)^2}[0.27] = 6.75 \text{ radian/sec}^2$$

Hence the required angular velocity is **3.82 radian/sec** and acceleration is **6.75 radian/sec²**. **Ans.**

Prob.2. *A slider in a machine moves along a fixed straight rod. Its distance x cm, along the rod is given below for various values of the time t seconds. Find the velocity of the slider and its acceleration when t = 0.3 second.*

t	x
0	30.13
0.1	31.62
0.2	32.87
0.3	33.64
0.4	33.95
0.5	33.81
0.6	33.24

(R.G.P.V., Dec. 2004, June 2012, Dec. 2013)

Sol. As t = 0.3 is the middle value, we will use the central difference formulae, noting that, a = 0.3 and h = 0.1.

The central difference table is

t	x	δ	δ^2	δ^3	δ^4	δ^5	δ^6
0	30.13						
		1.49					
0.1	31.62		–0.24				
		1.25		–0.24			
0.2	32.87		–0.48		0.26		
		0.77		**0.02**		–0.27	
0.3	**33.64**		**–0.46**		**–0.01**		**0.29**
		0.31		**0.01**		**0.02**	
0.4	33.95		–0.45		0.01		
		–0.14		0.02			
0.5	33.81		–0.43				
		–0.57					
0.6	33.24						

$$\mu\delta f(0.3) = \frac{1}{2}(0.77 + 0.31) = 0.54$$

$$\mu\delta^3 f(0.3) = \frac{1}{2}(0.02 + 0.01) = 0.015$$

and
$$\mu\delta^5 f(0.3) = \frac{1}{2}(-0.27 + 0.02) = -0.125$$

$\therefore$
$$f'(0.3) = \frac{1}{0.1}\left[0.54 - \frac{1}{6}(0.015) + \frac{1}{30}(-0.125)\right] = \mathbf{5.33}$$

Also
$$\delta^2 f(0.3) = -0.46,\ \delta^4 f(0.3) = -0.01,\ \delta^6 f(0.3) = 0.29$$

$\therefore$
$$f''(0.3) = \frac{1}{(0.1)^2}\left[-0.46 - \frac{1}{12}(-0.01) + \frac{1}{90}(0.29)\right] = \mathbf{-45.6}$$

Hence the required velocity is **5.33** cm/sec and acceleration is **–45.6** cm/sec^2. **Ans.**

Prob.3. *Given*

x :	0.1	0.2	0.3	0.4
$y = f(x)$:	1.10517	1.22140	1.34986	1.49182

Find $\frac{dy}{dx}$ *and* $\frac{d^2y}{dx^2}$ *at x = 0.4.*

[R.G.P.V., Nov. 2019 (IV-Sem)]

Sol. Since the derivatives is required at x = 0.4, which is near the end of the table, therefore we shall use Newton's backward formula.

The difference table is given below –

x	y	∇y	$\nabla^2 y$	$\nabla^3 y$
0.1	1.10517			
		0.11623		
0.2	1.22140		0.01223	
		0.12846		**0.00127**
0.3	1.34986		**0.0135**	
		0.14196		
0.4	**1.49182**			

We have
$$\frac{dy}{dx} = \frac{1}{h}\left[\nabla y_n + \frac{\nabla^2 y_n}{2} + \frac{\nabla^3 y_n}{3} + \ldots\right]$$

Here , h = 0.1, $\nabla y_n = 0.14196$, $\nabla^2 y_n = 0.0135$, $\nabla^3 y_n = 0.00127$

Thus
$$\left(\frac{dy}{dx}\right)_{x=0.4} = \frac{1}{0.1}\left[0.14196 + \frac{0.0135}{2} + \frac{0.00127}{3}\right] = \mathbf{1.4913} \quad \textbf{Ans.}$$

and
$$\frac{d^2y}{dx^2} = \frac{1}{h^2}\left[\nabla^2 y_n + \nabla^3 y_n + \frac{11}{12}\nabla^4 y_n + \ldots\right]$$

$$\left(\frac{d^2y}{dx^2}\right)_{x=0.4} = \frac{1}{(0.1)^2}[0.0135 + 0.00127] = \mathbf{1.477} \quad \textbf{Ans.}$$

Prob.4. Given that

x	y
1.0	7.989
1.1	8.403
1.2	8.781
1.3	9.129
1.4	9.451
1.5	9.750
1.6	10.031

Find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ at (i) $x = 1.1$ (ii) $x = 1.6$.

(R.G.P.V., Nov./Dec. 2007, Dec. 2011)

Sol. (i) The difference table is

x	y	Δ	Δ^2	Δ^3	Δ^4	Δ^5	Δ^6
1.0	7.989						
		0.414					
1.1	**8.403**		−0.036				
		0.378		0.006			
1.2	8.781		**−0.030**		−0.002		
		0.348		**0.004**		0.001	
1.3	9.129		−0.026		**−0.001**		**0.002**
		0.322		0.003		**0.003**	
1.4	9.451		−0.023		**0.002**		
		0.299		**0.005**			
1.5	9.750		**−0.018**				
		0.281					
1.6	**10.031**						

We have $$\left(\frac{dy}{dx}\right)_{x_0} = \frac{1}{h}\left[\Delta y_0 - \frac{1}{2}\Delta^2 y_0 + \frac{1}{3}\Delta^3 y_0 - \frac{1}{4}\Delta^4 y_0 + \ldots\right] \quad \text{....(i)}$$

and $$\left(\frac{d^2y}{dx^2}\right)_{x_0} = \frac{1}{h^2}\left[\Delta^2 y_0 - \Delta^3 y_0 + \frac{11}{12}\Delta^4 y_0 - \frac{5}{6}\Delta^5 y_0 + \ldots\right] \quad \text{....(ii)}$$

Here

$h = 0.1,\ x_0 = 1.1,\ \Delta y_0 = 0.378,\ \Delta^2 y_0 = -0.03$ etc.

Substituting these values in equations (i) and (ii), we get

$$\left(\frac{dy}{dx}\right)_{1.1} = \frac{1}{0.1}\left[0.378 - \frac{1}{2}(-0.03) + \frac{1}{3}(0.004) - \frac{1}{4}(-0.001) + \frac{1}{5}(0.003)\right]$$

$$= \mathbf{3.952}$$

Ans.

$$\left(\frac{d^2y}{dx^2}\right)_{1.1} = \frac{1}{(0.1)^2}\left[-0.03 - (0.004) + \frac{11}{12}(-0.001) - \frac{5}{6}(0.003)\right]$$

$$= \mathbf{-3.7417}$$

Ans.

(ii) We use the above difference table and the backward difference operator ∇ instead of Δ

$$\left(\frac{dy}{dx}\right)_{x_n} = \frac{1}{h}\left[\nabla y_n + \frac{1}{2}\nabla^2 y_n + \frac{1}{3}\nabla^3 y_n + \frac{1}{4}\nabla^4 y_n + \ldots\right] \quad \text{....(iii)}$$

and $$\left(\frac{d^2y}{dx^2}\right)_{x_n} = \frac{1}{h^2}\left[\nabla^2 y_n + \nabla^3 y_n + \frac{11}{12}\nabla^4 y_n + \frac{5}{6}\nabla^5 y_n + \ldots\right] \quad \text{....(iv)}$$

Here

$h = 0.1,\ x_n = 1.6,\ \nabla y_n = 0.281,\ \nabla^2 y_n = -0.018$ etc.

Substituting these values in equations (iii) and (iv), we get

$$\left(\frac{dy}{dx}\right)_{1.6} = \frac{1}{0.1}\left[0.281 + \frac{1}{2}(-0.018) + \frac{1}{3}(0.005) + \frac{1}{4}(0.002) + \frac{1}{5}(0.003)\right]$$

$$= \mathbf{2.7477}$$

Ans.

$$\left(\frac{d^2y}{dx^2}\right)_{1.6} = \frac{1}{(0.1)^2}\left[-0.018 + 0.005 + \frac{11}{12}(0.002) + \frac{5}{6}(0.003)\right]$$

$$= \mathbf{-0.867}$$

Ans.

Prob.5. Given that –

x :	1.0	1.1	1.2	1.3	1.4	1.5
y :	7.989	8.403	8.781	9.129	9.451	9.750

Find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ at $x = 1.6$. ***(R.G.P.V., June 2013)***

Sol. Similar as Prob.4.

Prob.6. Find $\frac{dy}{dx}$ at $x = 0.1$ from the following table –

x :	0.1	0.2	0.3	0.4
y :	0.9975	0.9900	0.9776	0.9604

[R.G.P.V., Dec. 2016, May 2019 (III-Sem)]

Sol. Here the step length h = 0.1. At first form the following difference table is given by

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$
0.1	0.9975			
		−0.0075		
0.2	0.9900		**−0.0049**	
		−0.0124		**0.0001**
0.3	0.9776		−0.0048	
		−0.0172		
0.4	0.9604			

We have

$$\left(\frac{dy}{dx}\right)_{x_0} = \frac{1}{h}\left[\Delta y_0 - \frac{1}{2}\Delta^2 y_0 + \frac{1}{3}\Delta^3 y_0 - \cdots\cdots\right] \quad \text{...(i)}$$

Here

$h = 0.1$, $x_0 = 0.1$, $\Delta y_0 = -0.0075$, $\Delta^2 y_0 = -0.0049$ and $\Delta^3 y_0 = 0.0001$

Substituting these values in equation (i), we get

$$\left(\frac{dy}{dx}\right)_{0.1} = \frac{1}{0.1}\left[-0.0075 - \frac{1}{2}(-0.0049) + \frac{1}{3}(0.0001)\right]$$

$$= \frac{1}{0.1}(-0.0075 + 0.00245 + 0.000033)$$

$$= \frac{1}{0.1}(-0.005017)$$

$$= \mathbf{-0.05017}$$ **Ans.**

Prob.7. Find $\frac{dy}{dx}$ at x = 1.1 from the following table –

x	1.0	1.2	1.4	1.6	1.8	2.0
y	0	0.128	0.544	1.296	2.432	4.000

(R.G.P.V., Feb. 2010, June 2014)

Sol. Since the derivative is required at x = 1.1, which is near the beginning of the table, therefore we shall use Newton's forward formula.

The difference table is as follows –

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
1.0	0				
		0.128			
1.2	0.128		**0.288**		
		0.416		**0.048**	
1.4	0.544		0.336		**0**
		0.752		0.048	
1.6	1.296		0.384		0
		1.136		0.048	
1.8	2.432		0.432		
		1.568			
2.0	4.000				

Newton's forward difference formula is

$$y = y_0 + u\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + \frac{u(u-1)(u-2)}{3!}\Delta^3 y_0$$

(upto 3rd differences, since 4th and higher differences are zero)

Differentiating with respect to u, we get

$$\frac{dy}{dx} = \frac{1}{h}\left[\Delta y_0 + \frac{(2u-1)}{2}\Delta^2 y_0 + \frac{(3u^2 - 6u + 2)}{6}\Delta^3 y_0\right] \quad \text{...(i)}$$

Here $h = 0.2$, $x_0 = 1.0$, taking $u = 1/2$ so that $x_0 + uh = 1.1$

Putting these values in equation (i), we get

$$\left(\frac{dy}{dx}\right)_{1.1} = \frac{1}{0.2}\left[0.128 + \frac{\left(2\times\frac{1}{2} - 1\right)}{2}\times 0.288 + \frac{\left(3\times\frac{1}{4} - 6\times\frac{1}{2} + 2\right)}{6}\times 0.048\right]$$

$$= \frac{1}{0.2}\left[0.128 + 0 - \frac{1}{4\times 6}\times 0.048\right]$$

$$= \frac{1}{0.2}[0.128 - 0.002] = \frac{0.126}{0.2} = \mathbf{0.63}$$ **Ans.**

Prob.8. Calculate the first and second derivatives of the function tabulated below, at the point $x = 1.1$.

x :	1.0	1.2	1.4	1.6	1.8	2.0
$f(x)$:	0	0.128	0.544	1.296	2.432	4.0

(R.G.P.V., June 2009, Dec. 2015)

Sol. Since the derivative is required at $x = 1.1$, which is near the beginning of the table, therefore we shall use Newton's forward formula.

Here the step length $h = 0.2$. At first form the following difference table.

x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$	$\Delta^4 f(x)$	$\Delta^5 f(x)$
1.0	0					
		0.128				
1.2	0.128		**0.288**			
		0.416		**0.048**		
1.4	0.544		0.336		**0.05**	
		0.752		0.098		**– 0.25**
1.6	1.296		0.434		– 0.2	
		1.186		– 0.102		
1.8	2.482		0.332			
		1.518				
2.0	4.0					

We have

$$\left(\frac{df(x)}{dx}\right)_{x_0} = \frac{1}{h}\left[\Delta f_0(x_0) - \frac{1}{2}\Delta^2 f_0(x_0) + \frac{1}{3}\Delta^3 f_0(x_0) - \frac{1}{4}\Delta^4 f_0(x_0) + \ldots\right] \quad \ldots(i)$$

and
$$\left(\frac{d^2 f(x)}{dx^2}\right)_{x_0} = \frac{1}{h^2}\left[\Delta^2 f_0(x_0) - \Delta^3 f_0(x_0) + \frac{11}{12}\Delta^4 f_0(x_0) - \frac{5}{6}\Delta^5 f_0(x_0) + \ldots\right] \quad \ldots(ii)$$

Here $x_0 = 1.1$, $\Delta f_0(x_0) = 0.128$, $\Delta^2 f_0(x_0) = 0.288$, $\Delta^3 f_0(x_0) = 0.048$, $\Delta^4 f_0(x_0) = 0.05$, and $\Delta^5 f_0(x_0) = -0.25$

Substituting these values in equations (i) and (ii), we get

$$\left(\frac{df(x)}{dx}\right)_{x=1.1} = \frac{1}{0.2}\left[0.128 - \frac{1}{2}\times 0.288 + \frac{1}{3}\times 0.048 - \frac{1}{4}\times 0.05 + \frac{1}{5}(-0.25)\right]$$

$$= \frac{1}{0.2}[0.128 - 0.144 + 0.016 - 0.0125 - 0.05]$$

$$= 5 \times [-0.0625] = \mathbf{-0.3125}$$ **Ans.**

and
$$\left(\frac{d^2 f(x)}{dx^2}\right)_{x=1.1} = \frac{1}{(0.2)^2}\left[0.288 - 0.048 + \frac{11}{12}\times 0.05 - \frac{5}{6}\times(0.25)\right]$$

$$= \frac{1}{0.04}[0.24 + 0.046 + 0.21] = \frac{1}{0.04}[0.496]$$

$$= \mathbf{12.4}$$ **Ans.**

Prob.9. Find $f'(x)$ and $f''(x)$ at $x = 6$, given that –

x :	4.5	5.0	5.5	6.0	6.5	7.0	7.5
$y = f(x)$:	9.69	12.90	16.71	21.18	26.37	32.34	39.15

(R.G.P.V., Dec. 2014)

Sol. Since we want to find the derivative at a point near the middle of the table, we may use any central difference formula. In this case we shall use Bessel's formula. Here $x_0 = 6$ and $h = 0.5$ consider a new variate $u = \frac{x-6}{h}$ so that when $x = 6$, we have $u = 0$.

Thus we are to find first derivative at $u = 0$

Also $$\frac{d}{dx}f(u) = \frac{d}{du}\frac{du}{dx}f(u) = \frac{d}{du}f(u).\frac{1}{h} = \frac{f'(u)}{h} \quad \ldots(i)$$

and $$\frac{d^2}{dx^2}f(u) = \frac{d}{du}\frac{du}{dx}\frac{d}{dx}f(u) = \frac{d}{du}\frac{1}{h}\frac{f'(u)}{h} = \frac{f''(u)}{h^2} \quad \ldots(ii)$$

The difference table is as under –

u	x	$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$	$\Delta^4 f(x)$
–3	4.5	9.69				
			3.21			
–2	5.0	12.90		0.6		
			3.81		0.06	
–1	5.5	16.71		0.66		0
			4.47		0.06	
0	6.0	**21.18**		**0.72**		0
			5.19		**0.06**	
1	6.5	26.37		**0.78**		0
			5.97		0.06	
2	7.0	32.34		0.84		
			6.81			
3	7.5	39.15				

Bessel's formula is

$$f(u) = \frac{1}{2}[f(0) + f(1)] + \left(u - \frac{1}{2}\right)\Delta f(0) + \frac{u(u-1)}{2!}\frac{1}{2}[\Delta^2 f(-1) + \Delta^2 f(0)]$$

$$+ \frac{\left(u - \frac{1}{2}\right)u(u-1)}{3!}\Delta^3 f(-1) \quad \ldots(iii)$$

(upto 3rd difference)

Differentiating equation (iii) w.r.t. u and substituting u = 0, we have

$$f'(0) = \Delta f(0) - \frac{1}{4}[\Delta^2 f(-1) + \Delta^2 f(0)] + \frac{1}{12}\Delta^3 f(-1)$$

$$= 5.19 - \frac{1}{4}[0.72 + 0.78] + \frac{1}{12}[0.06]$$

$$= 5.19 - \frac{1.5}{4} + \frac{0.06}{12} = \frac{57.84}{12}$$

From equation (i), we have

$$\left[\frac{d}{dx} f(u)\right]_{u=0} = \frac{f'(0)}{h} = \frac{57.84}{12 \times 0.5} = \frac{57.84}{6} = 9.64$$

Hence f'(x) at x = 6 is **9.64** **Ans.**

Differentiating equation (iii) w.r.t. u twice and substituting u = 0, we have

$$f''(0) = \frac{1}{2}[\Delta^2 f(-1) + \Delta^2 f(0)] - \frac{1}{2}\Delta^3 f(-1)$$

$$= \frac{1}{2}[0.72 + 0.78] - \frac{1}{2}[0.06]$$

$$= \frac{1.5}{2} - \frac{0.06}{2} = \frac{1.44}{2} = 0.72$$

From equation (ii), we have

$$\left[\frac{d^2}{dx^2} f(u)\right]_{u=0} = \frac{f''(0)}{h^2} = \frac{0.72}{(0.5)^2} = 2.88$$

Hence f''(x) at x = 6 is **2.88** **Ans.**

Prob.10. Find the first, second and third derivatives of the function tabulated below, at the point x = 1.5 –

x	*f(x)*
1.5	*3.375*
2.0	*7.0*
2.5	*13.625*
3.0	*24.0*
3.5	*38.875*
4.0	*59.0*

(R.G.P.V., June 2008, 2011)

Sol. Since required derivatives are at a point near the beginning of the table, therefore we shall use Newton-Gregory forward formula. The difference table is as below –

x	*f(x)*	$\Delta f(x)$	$\Delta^2 f(x)$	$\Delta^3 f(x)$	$\Delta^4 f(x)$
1.5	3.375				
		3.625			
2.0	7.000		3.000		
		6.625		0.750	
2.5	13.625		3.750		0
		10.375		0.750	
3.0	24.000		4.500		0
		14.875		0.750	
3.5	38.875		5.250		
		20.125			
4.0	59.000				

Newton-Gregory formula is

$$f(a + xh) = f(a) + {}^xC_1\Delta f(a) + {}^xC_2\Delta^2 f(a) + {}^xC_3\Delta^3 f(a)$$

(upto 3rd difference)

$$= f(a) + x\Delta f(a) + \frac{x^2 - x}{2}\Delta^2 f(a) + \frac{x^3 - 3x^2 + 2x}{6}\Delta^3 f(a)$$

Differentiating w. r. to x thrice and then putting x = 0 in the equations obtained, we get

$$hf'(a) = \Delta f(a) - \frac{1}{2}\Delta^2 f(a) + \frac{1}{3}\Delta^3 f(a)$$

$$h^2 f''(a) = \Delta^2 f(a) - \Delta^3 f(a)$$

$$h^3 f'''(a) = \Delta^3 f(a)$$

Here a = 1.5, h = 0.5

$$\therefore \quad f'(1.5) = \frac{1}{0.5}\left[3.625 - \frac{1}{2}(3.000) + \frac{1}{3}(0.750)\right] = \mathbf{4.750}$$

$$f''(1.5) = \frac{1}{0.25}[3.000 - 0.750] = \mathbf{9.000}$$

and $$f'''(1.5) = \frac{1}{(0.5)^3}[0.750] = \mathbf{6.000}$$ **Ans.**

Prob.11. Find the first and second order derivative of f(x) at x = 1.5 if

x :	*1.5*	*2.0*	*2.5*	*3.0*	*3.5*	*4.0*
f(x) :	*3.375*	*7.000*	*13.625*	*24.000*	*38.875*	*59.000*

[R.G.P.V., May 2019 (IV-Sem)]

Or

Find the first and second order derivative at the point x = 1.5 for the following data –

x :	1.5	2.0	2.5	3.0	3.5	4.0
f(x) :	3.375	7.000	13.625	24.000	38.875	59.000

[R.G.P.V., Nov. 2019 (III-Sem)]

Sol. Refer to Prob.10.

NUMERICAL INTEGRATION – TRAPEZOIDAL RULE AND SIMPSON'S $\frac{1}{3}$rd AND $\frac{3}{8}$th RULES

Numerical Integration – The process of evaluating or obtaining a definite integral, $I = \int_a^b f(x)\,dx$ from a set of numerical value of integrand f(x) is called numerical integration. This process when applied to a function of a single variable is known as ***quadrature***. Thus to evaluate the definite integral $\int_a^b f(x)\,dx$, first express the function f(x) by an interpolation formula say p(x) and then integrate f(x) between the limits a and b.

i.e., $$\int_a^b f(x)\,dx = \int_a^b p(x)\,dx$$

The error in E in such type of approximation is given by

$$\int_a^b f(x)\,dx - \int_a^b p(x)\,dx = \int_a^b [f(x) - p(x)]\,dx$$

A General Quadrature Formula for Equidistant Ordinates –

Suppose $I = \int_a^b y\,dx$, where $y = f(x)$. Suppose f(x) is given for certain equally distant values of arguments, say x_0, $x_0 + h$, $x_0 + 2h$..... Let the range (b – a) be divided into n equal parts, each of which is of width h i.e., b – a = nh. Suppose $x_0 = a$, $x_1 = x_0 + h = a + h$, $x_2 = a + 2h$, ... $x_n = a + nh = b$, we have assumed that the (n + 1) ordinate $y_0, y_1, \ldots, y_n$ are at equal intervals.

$$\therefore \qquad I = \int_a^b y\,dx = \int_{x_0}^{x_0+nh} y_x\,dx = \int_0^n y_{x_0+nh}\,h\,du$$

where new variate,

$$u = \frac{x - x_0}{h}; \quad dx = h\,du$$

$$I = h\int_0^n \left[y_0 + u\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + \frac{u(u-1)(u-2)}{3!}\Delta^3 y_0 + \ldots + \frac{u(u-1)(u-n+1)}{n!}\Delta^n y_0 \right] du$$

$$= h\left[ny_0 + \frac{n^2}{2}\Delta y_0 + \left(\frac{n^3}{3} - \frac{n^2}{2}\right)\frac{\Delta^2 y_0}{2!} + \left(\frac{n^4}{4} - n^3 + n^2\right)\frac{\Delta^3 y}{3!} + \ldots \right.$$

$$\left. + \text{upto } (n+1) \text{ terms} \right] \qquad \ldots(i)$$

This is the general quadrature formula, we can obtain a number of quadrature formula from this by putting n = 1, 2.

(i) Trapezoidal Rule – Substituting n = 1 in equation (i) and taking the curve through (x_0, y_0) and (x_1, y_1) as a straight line i.e., a polynomial of first order so that differences of order higher than first become zero, we get

$$\int_{x_0}^{x_0+h} f(x)\,dx = h\left(y_0 + \frac{1}{2}\Delta y_0\right) = \frac{h}{2}(y_0 + y_1)$$

Similarly $$\int_{x_0+h}^{x_0+2h} f(x)\,dx = h\left(y_1 + \frac{1}{2}\Delta y_1\right) = \frac{h}{2}(y_1 + y_2)$$

$$\int_{x_0+(n-1)h}^{x_0+nh} f(x)\,dx = \frac{h}{2}(y_{n-1} - y_n)$$

Adding these n integrals, we obtain

$$\int_{x_0}^{x_0+nh} f(x)\,dx = \frac{h}{2}[(y_0 + y_n) + 2(y_1 + y_2 + \ldots y_{n-1}] \qquad \ldots(ii)$$

This is known as Trapezoidal rule.

(ii) Simpson's One-third Rule – Substituting n = 2 in equation (i) above and taking the curve through (x_0, y_0), (x_1, y_1) and (x_2, y_2) as a parabola, i.e., a polynomial of second order so that difference of higher than second vanish, we get

$$\int_{x_0}^{x_0+2h} f(x)\,dx = 2h\left(y_0 + \Delta y_0 + \frac{1}{6}\Delta^2 y_0\right) = \frac{h}{3}(y_0 + 4y_1 + y_2)$$

Similarly,

$$\int_{x_0+2h}^{x_0+4h} f(x)\,dx = \frac{h}{3}(y_2 + 4y_3 + y_4)$$

.....

$$\int_{x_0+(n-1)h}^{x_0+nh} f(x)\,dx = \frac{h}{3}(y_{n-2} + 4y_{n-1} + y_n), \text{ n being even.}$$

Adding all these integrals, we have when n is even.

$$\int_{x_0}^{x_0+nh} f(x)\,dx = \frac{h}{3}[(y_0+y_n)+4(y_1+y_3+\ldots+y_{n-1}) +2(y_2+y_4+\ldots+y_{n-2})] \quad \ldots(iii)$$

This is known as the Simpson's one-third rule or simply Simpson's rule and is most commonly used.

(iii) Simpson's Three-eight Rule – Substituting n = 3 in equation (i) above taking the curve through $(x_i, y_i) : i = 0, 1, 2, 3$ as a, polynomial of third order so that difference above the third order vanish, we get

$$\int_{x_0}^{x_0+3h} f(x)\,dx = 3h\left(y_0+\frac{3}{2}\Delta y_0+\frac{3}{4}\Delta^2 y_0+\frac{1}{8}\Delta^3 y_0\right)$$

$$= \frac{3}{8}h(y_0+3y_1+3y_2+y_3)$$

Similarly $\int_{x_0+3h}^{x_0+6h} f(x)\,dx = \frac{3h}{8}(y_3+3y_4+3y_5+y_6)$

and so on.

Adding all such expressions from x_0 to $x_0 + nh$, where n is a multiple of 3, we obtain

$$\int_{x_0}^{x_0+nh} f(x)\,dx = \frac{3h}{8}[(y_0+y_n)+3(y_1+y_2+y_4 +y_5+\ldots+y_{n-1})+2(y_3+y_6+y_{n-3})] \quad \ldots(iv)$$

which is knowns on Simpson's three-eighth rule.

NUMERICAL PROBLEMS

Prob.12. Evaluate $\int_4^{5.2} \log_e x\,dx$ by Trapezoidal rule.

Sol. We have $f(x) = \log_e x$, a = 4, b = 5.2

Here the number of equal interval is not given so we take n = 6

$$\therefore \quad h = \frac{b-a}{n} = \frac{5.2-4}{6} = 0.2$$

We obtain the following table –

x :	4	4.2	4.4	4.6	4.8	5.0	5.2
y = f(x) :	1.3863	1.4351	1.4816	1.5261	1.5686	1.6094	1.6487

By Trapezoidal's rule, we have

$$\int_4^{5.2} \log_e x\,dx = h/2\,[(y_0+y_6)+2(y_1+y_2+y_3+y_4+y_5)]$$

$$= \frac{0.2}{2}[(1.3863+1.6487)+2(1.4351+1.4816+1.5261+1.5686+1.6094)]$$

$$= \mathbf{1.82766}$$ **Ans.**

Prob.13. Use Trapezoidal rule to evaluate $\int_0^1 x^3 dx$ considering give sub-intervals. ***(R.G.P.V., Dec. 2016)***

Sol. We have $f(x) = x^3$, a = 0, b = 1

Here the range of integration (0, 1) is subdivided into five equal parts.

$$\therefore \quad h = \frac{b-a}{n} = \frac{1-0}{5} = 0.2$$

We obtain the following table –

x :	0	0.2	0.4	0.6	0.8	1
y = f(x) :	0	0.008	0.064	0.216	0.512	1

By Trapezoidal's rule, we have

$$\int_0^1 x^3 dx = \frac{h}{2}[(y_0+y_5)+2(y_1+y_2+y_3+y_4)]$$

$$= \frac{0.2}{2}[(0+1)+2(0.008+0.064+0.216+0.512)]$$

$$= 0.1\,(1+2\times 0.8) = 0.1\,(1+1.6) = 0.1\times 2.6 = \mathbf{0.26}$$ **Ans.**

Prob.14. Calculate $\int_1^2 \frac{dx}{x}$ by,

(i) Simpson's $\frac{1}{3}$ rule with h = 0.5 (ii) Simpson's $\frac{1}{3}$ rule with h = 0.25.

(R.G.P.V., June 2013)

Sol. We have $f(x) = \frac{1}{x}$, a = 1, b = 2

(i) By Simpson's $\frac{1}{3}$ rule with h = 0.5

x	1	1.5	2.0
f(x)	1	0.667	0.5

By Simpson's $\frac{1}{3}$ rule, we have

$$\int_1^2 \frac{dx}{x} = \frac{h}{3}[(f_0+f_2)+4f_1] = \frac{0.5}{3}[(1+0.5)+4(0.667)] = \mathbf{0.6947} \quad \textbf{Ans.}$$

(ii) By Simpson's $\frac{1}{3}$ rule with h = 0.25

x	1	1.25	1.5	1.75	2
$f(x)$	1	0.8	0.667	0.571	0.5

By Simpson's $\frac{1}{3}$ rule, we have

$$\int_1^2 \frac{dx}{x} = \frac{h}{3}[(f_0+f_4)+4(f_1+f_3)+2f_2]$$

$$= \frac{0.25}{3}[(1+0.5)+4(0.8+0.571)+2\times 0.667]$$

$$= \frac{0.25}{3}[1.5+5.484+1.334] = \mathbf{0.6932} \quad \textbf{Ans.}$$

Prob.15. Evaluate $\int_0^1 \frac{dx}{1+x^2}$ using Simpson's $\frac{1}{3}$ rule, taking $h = \frac{1}{4}$.

[R.G.P.V., May 2019 (III-Sem)]

Sol. We have $f(x) = \frac{1}{1+x^2}$, a = 0, b = 1

Here $h = \frac{1}{4} = 0.25$

Now we obtain the following table –

x :	0	0.25	0.5	0.75	1
$f(x)$:	1	0.9412	0.8	0.64	0.5

By Simpson's $\frac{1}{3}$ rule, we have

$$\int_0^1 \frac{dx}{1+x^2} = \frac{h}{3}[(f_0+f_4)+4(f_1+f_3)+2f_2]$$

$$= \frac{0.25}{3}[(1+0.5)+4(0.9412+0.64)+2\times 0.8]$$

$$= \frac{0.25}{3}[1.5+6.3248+1.6]$$

$$= \frac{0.25\times 9.4248}{3} = \frac{2.3562}{3} = \mathbf{0.7854} \quad \textbf{Ans.}$$

Prob.16. Find $\int_0^{0.6} e^{-x^2}dx$ by taking seven ordinates using Simpson's 1/3 rule.

(R.G.P.V., June 2009)

Or

Evaluate the integral $\int_0^{0.6} e^{-x^2}dx$ by Simpson 1/3 rule.

(R.G.P.V., June 2014)

Sol. Let $I = \int_0^{0.6} e^{-x^2}dx$

Here $f(x) = e^{-x^2}$, a = 0, b = 0.6

Since h is not given, so we take n = 6

$$h = \frac{b-a}{n} = \frac{0.6-0}{6} = 0.1$$

∴ The values of f(x) are given below –

x:	0	0.1	0.2	0.3	0.4	0.5	0.6
$f(x)$:	1	0.99	0.961	0.914	0.85	0.78	0.7

By Simpson's $\frac{1}{3}$ rule, we have

$$I = \frac{h}{3}[(f_0+f_6)+4(f_1+f_3+f_5)+2(f_2+f_4)]$$

$$= \frac{0.1}{3}[(1+0.7)+4(0.99+0.914+0.78)+2(0.961+0.85)]$$

$$= \frac{0.1}{3}[1.7+4(2.684)+2(1.811)] = \frac{1}{30}[16.058] = \mathbf{0.5353} \quad \textbf{Ans.}$$

Prob.17. Evaluate $\int_0^{\pi/2} \sqrt{\cos\theta}\, d\theta$ by using Simpson's $\frac{1}{3}$ rules.

(R.G.P.V., June 2011)

Sol. Here, take n = 6, $h = \frac{1}{6}(\pi/2-0) = \pi/12$ and compute the vlaue of $\phi = f(\theta) = \sqrt{\cos\theta}$, for $\theta_0 = 0, \theta_1 = \frac{\pi}{12}, \theta_2 = \frac{\pi}{6}, \theta_3 = \frac{\pi}{4}, \theta_4 = \frac{\pi}{3}, \theta_5 = \frac{5\pi}{12}$ and $\theta_6 = \frac{\pi}{2}$.

These values are shown in the following table

θ :	0	$\pi/12$	$\pi/6$	$\pi/4$	$\pi/3$	$5\pi/12$	$\pi/2$
$\phi = f(\theta)$:	1	0.9828	0.9306	0.8409	0.7071	0.5087	0

By Simpson's $\frac{1}{3}$ rule, we get

$$\int_0^{\pi/2}\sqrt{\cos\theta} = \int_{\theta_0}^{\theta_6} f(\theta)\,d\theta = \frac{1}{3}h[(\phi_0+\phi_6)+4(\phi_1+\phi_3+\phi_5)+2(\phi_2+\phi_4)]$$

$$= \frac{1}{3}\times\frac{\pi}{12}[(1+0)+4(0.9828+0.8409+0.5087)$$

$$+2(0.9306+0.7071)]$$

$$= \frac{1}{3}\times\frac{\pi}{12}[1+(4\times 2.3324)+(2\times 1.6377)]$$

$$= \frac{\pi}{36}[1+9.3296+3.2754]$$

$$= \frac{\pi}{36}[13.605] = \pi\,(0.37792) = \mathbf{1.1873}$$ **Ans.**

Prob.18. Using Simpson's $\frac{1}{3}$ rule, obtain $\int_{-3}^{3} x^4 dx$ by taking seven ordinates. Compare it with exact values. ***(R.G.P.V., Nov. 2018)***

Sol. We have

$$y = f(x) = x^4 \text{ and } a = -3,\ b = 3$$

Using 7 ordinates, so we take n = 6, then

$$h = \frac{b-a}{n} = \frac{3-(-3)}{6} = 1$$

We obtain the following table –

x :	−3	−2	−1	0	1	2	3
y = f(x) :	81	16	1	0	1	16	81

By Simpson's $\frac{1}{3}$ rule, we have

$$\int_{-3}^{3} x^4 dx = \frac{h}{3}[(y_0+y_6)+4(y_1+y_3+y_5)+2(y_2+y_4)]$$

$$= \frac{1}{3}[(81+81)+4(16+0+16)+2(1+1)]$$

$$= \frac{1}{3}(162+128+4) = \frac{294}{3} = \mathbf{98}$$ **Ans.**

Again the exact value

$$\int_{-3}^{3} x^4 dx = \left[\frac{x^5}{5}\right]_{-3}^{3}$$

$$= \frac{1}{5}[(3)^5-(-3)^5] = \frac{243+243}{5} = \frac{486}{5} = \mathbf{97.2}$$ **Ans.**

Prob.19. Calculate by Simpson's $\frac{1}{3}$rd rule an approximate value of $\int_0^1 \frac{dx}{1+x}$ by taking seven ordinates. Compare it with exact value and deduce the value of $\log_e 2$. ***(R.G.P.V., June 2012)***

Sol. Let $I = \int_0^1 \frac{dx}{1+x}$

Here $f(x) = \frac{1}{1+x}$, a = 0, b = 1

Since h is not given, so we take n = 6

$$h = \frac{b-a}{n} = \frac{1-0}{6} = \frac{1}{6}$$

∴ The value of f(x) are given below –

x:	0	$\frac{1}{6}$	$\frac{2}{6}$	$\frac{3}{6}$	$\frac{4}{6}$	$\frac{5}{6}$	1
f(x):	1	$\frac{6}{7}$	$\frac{6}{8}$	$\frac{6}{9}$	$\frac{6}{10}$	$\frac{6}{11}$	$\frac{1}{2}$

By Simpson's $\frac{1}{3}$ rule, we have

$$I = \frac{h}{3}[(f_0+f_6)+4(f_1+f_3+f_5)+2(f_2+f_4)]$$

$$= \frac{1/6}{3}\left[\left(1+\frac{1}{2}\right)+4\left(\frac{6}{7}+\frac{6}{9}+\frac{6}{11}\right)+2\left(\frac{6}{8}+\frac{6}{10}\right)\right]$$

$$= \frac{1}{18}\left[\frac{3}{2}+4\left(\frac{1434}{693}\right)+2\left(\frac{54}{40}\right)\right]$$

$$= \frac{1}{18}[1.5+8.277+2.7] = \mathbf{0.693167}$$

The value of integral $= \int_0^1 \frac{dx}{1+x} = [\log(1+x)]_0^1$

$$= \log 2 - \log 1$$

$$= \log 2 = 0.693147$$

Comparing we get $\log_e 2 = 0.693167$

Hence the error = 0.693147 − 0.693167 = **− 0.00002** **Ans.**

Prob.20. Evaluate $\int_0^1 \frac{dx}{1+x}$ ***by using Simpson's*** $\frac{1}{3}$***rd rule, take n = 6.***

(R.G.P.V., Dec. 2015)

Or

Apply Simpson's $\frac{1}{3}$ ***rule, to evaluate the integration*** $\int_0^1 \frac{dx}{1+x}$.

[R.G.P.V., May 2019 (IV-Sem)]

Sol. Refer to Prob.19.

Prob.21. Find the value of log 2 from $\int_0^1 \frac{x^2}{1+x^3}dx$***, using Simpson's*** $\frac{1}{3}$ ***rule by dividing it, into four equal parts. [R.G.P.V., Nov. 2019 (III-Sem)]***

Sol. Here the range of integration (0, 1) is subdivided into four equal parts. So we take n = 4, then

$$h = \frac{b-a}{n} = \frac{1-0}{4} = \frac{1}{4} = 0.25$$

∴ The value of f(x) are given below –

x :	0	0.25	0.5	0.75	1
f(x) :	0	0.06154	0.22222	0.3956	0.5

Using the Simpson's $\frac{1}{3}$ rule, the approximate value of the integral is

$$I = \frac{h}{3}\left[(f_0 + f_4) + 4(f_1 + f_3) + 2f_2\right]$$

$$\int_0^1 \frac{x^2}{1+x^3}dx = \frac{0.25}{3}\left[(0+0.5) + 4(0.06154 + 0.3956) + 2 \times 0.22222\right]$$

$$= \frac{0.25}{3}(0.5 + 1.82856 + 0.44444) = \frac{0.25}{3} \times 2.773$$

$$= 0.23108 \qquad \text{...(i)}$$

The value of integral,

$$I_r = \int_0^1 \frac{x^2}{1+x^3}dx = \frac{1}{3}\left[\log(1+x^3)\right]_0^1$$

$$= \frac{1}{3}\left[\log 2 - \log 1\right] = \frac{1}{3}\log 2 \qquad \text{...(ii)}$$

$$= \frac{1}{3} \times 0.693147 = 0.23105$$

By equations (i) and (ii), we get

$$\frac{1}{3} \times \log 2 = 0.23108$$

$$\log 2 = \mathbf{0.69324\ approx.}$$

Ans.

Prob.22. Calculate the value of $\int_0^{\pi/2} \sin x\, dx$ ***by Simpson's 1/3 rule using 11 ordinates.***

(R.G.P.V., Dec. 2013)

Sol. We have $y = f(x) = \sin x$ and $a = 0,\ b = \frac{\pi}{2}$

Using 11 ordinates, so we take n = 10, then

$$h = \frac{b-a}{n} = \frac{\pi}{20}$$

and from trigonometric tables we get the following values –

x :	0	π/20	π/10	3π/20	π/5	π/4	3π/10	7π/20	2π/5	9π/20	π/2
y :	0	0.1564	0.3090	0.4540	0.5878	0.7071	0.8090	0.8910	0.9511	0.9877	1.000

Using the Simpson's 1/3 rule, the approximate value of the integral is

$$I = \frac{h}{3}\left[(y_0 + y_{10}) + 4(y_1 + y_3 + y_5 + y_9) + 2(y_2 + y_4 + y_6 + y_8)\right]$$

$$= \frac{\pi}{60}\left[(0+1) + 4(0.1564 + 0.4540 + 0.7071 + 0.8910 + 0.9877)\right.$$

$$\left. + 2(0.3090 + 0.5878 + 0.8090 + 0.9511)\right]$$

$$= \mathbf{1.00000036}$$

Ans.

Prob.23. Evaluate $\int_{0.5}^{0.7} x^{1/2}e^{-x}dx$ ***approximately by using a suitable formula.***

(R.G.P.V., June 2003, Dec. 2006)

Sol. Let us divide the range of integration (0.5, 0.7) into 4 equal parts each of width $\frac{0.7-0.5}{4} = 0.05$; h = 0.05. The values of y for each point of sub-division are as given below –

x	$y = x^{1/2}e^{-x}$
$x_0 = 0.50$	0.4288818
$x_0 + h = 0.55$	0.4278774
$x_0 + 2h = 0.60$	0.4251076
$x_0 + 3h = 0.65$	0.4208867
$x_0 + 4h = 0.70$	0.4154730

Hence using Simpson's 1/3 rule, we obtain

$$\int_{0.5}^{0.7} x^{1/2}e^{-x}dx = \frac{h}{3}[(y_0 + y_4) + 4(y_1 + y_3) + 2y_2]$$

$$= \frac{0.05}{3}[(0.4288818 + 0.4154730) + 4(0.4278774 + 0.4208867) + 2(0.4251076)]$$

$$= \frac{0.05}{3}[0.8443548 + 3.3950564 + 0.8502152]$$

$$= \frac{0.05}{3}(5.0896264) = \mathbf{0.0848271}$$ **Ans.**

Prob.24. The following table gives the velocity v of a particle at time t –

t (sec)	*v (m/sec)*
0	*4*
2	*6*
4	*16*
6	*34*
8	*60*
10	*94*
12	*136*

Find the distance moved by the particle in 12 seconds and also the acceleration at t = 2 second. ***(R.G.P.V., Feb. 2010)***

Sol. Let s be the distance moved by the particle in 12 seconds.

We know that $$v = \frac{ds}{dt}$$

$\therefore$ $$s = \int_0^{12} v\,dt$$

We have given

t	0	2	4	6	8	10	12
v	4	6	16	34	60	94	136

By Simpson's 1/3 rule, we have

$$s = \frac{h}{3}[(v_0 + v_6) + 4(v_1 + v_3 + v_5) + 2(v_2 + v_4)]$$

$$= \frac{2}{3}[(4 + 136) + 4(6 + 34 + 94) + 2(16 + 60)]$$

$$= \frac{2}{3}[140 + 4 \times 134 + 2 \times 76]$$

$$= \frac{2}{3}[140 + 536 + 152] = \frac{2}{3} \times 828 = 552 \text{ m}$$

Hence the distance moved by the particle in 12 seconds is **552 m** **Ans.**

We also know that

$$a = \frac{v}{t}$$

Acceleration 'a' at t = 2 second is

$$a = \frac{6}{2} = \mathbf{3\ m/sec^2}$$ **Ans.**

Prob.25. Find the value of $\int_1^2 \frac{dx}{x}$ by Simpson's $\frac{3}{8}$th rule. Hence obtain approximation value of $\log_e 2$. ***(R.G.P.V., May/June 2006)***

Sol. Let $$I = \int_1^2 \frac{dx}{x}$$

Here $f(x) = \frac{1}{2}$, a = 1, b = 2

Divide the interval (1, 2) into six parts each of width $h = \frac{1}{6}$. The values of $f(x) = \frac{1}{x}$ are given below –

x:	1	7/6	8/6	9/6	10/6	11/2	2
f(x):	1	6/7	6/8	6/9	6/10	6/11	1/2

By Simpson's 3/8 rule

$$\int_1^2 \frac{dx}{x} = \frac{3}{8} \times h[(y_0 + y_6) + 3(y_1 + y_2 + y_4 + y_5) + 2y_3]$$

$$= \frac{3}{8} \times \frac{1}{6}\left[\left(1 + \frac{1}{2}\right) + 3\left(\frac{6}{7} + \frac{6}{8} + \frac{6}{10} + \frac{6}{11}\right) + 2 \times \frac{6}{9}\right]$$

$$= \frac{3}{48}\left[\frac{3}{2} + 3\left(\frac{8478}{3080}\right) + \frac{12}{9}\right] = \frac{3}{48}[1.5 + 8.26 + 1.33]$$

$$= \mathbf{0.693125}$$ **Ans.**

By direct integration, we get

$$\int_1^2 \frac{dx}{x} = [\log_e x]_1^2 = \log_e 2 = \mathbf{0.69314718}$$ **Ans.**

Prob.26. A river is 80 ft. wide. The depth d in feet at a distance x ft. fromone bank is given by the following table –

x	*0*	*10*	*20*	*30*	*40*	*50*	*60*	*70*	*80*
d	*0*	*4*	*7*	*9*	*12*	*15*	*14*	*8*	*3*

Find approximately the area of cross-section.

(R.G.P.V., June 2010, Dec. 2012)

Sol. Given table is

x :	0	10	20	30	40	50	60	70	80
d :	0	4	7	9	12	15	14	8	3

By Simpson's 3/8 rule, we know

$$\int_0^{80} f(x)\,dx = \frac{3h}{8}[\{y_0 + y_{80}\} + 3\{y_{10} + y_{20} + y_{40} + y_{50} + y_{70}\} + 2\{y_{30} + y_{60}\}]$$

$$= \frac{3 \times 10}{8}[(0 + 3) + 3(4 + 7 + 12 + 15 + 8) + 2(9 + 14)]$$

[Here y = d and h = 10]

$$= \frac{30}{8}[3 + 138 + 46] = 701.25 \text{ square feets}$$

Hence the area of cross-section of the river is **701** square feets (approx). **Ans.**

Prob.27. Evaluate the integration $\int_0^1 \frac{dx}{1+x^2}$ ***by using Simpson's*** $\frac{1}{3}$ ***and*** $\frac{3}{8}$ ***rule.*** ***[R.G.P.V., Nov. 2019 (IV-Sem)]***

Sol. We have $f(x) = \frac{1}{1+x^2}$, a = 0, b = 1

Here the number of equal interval is not given, so we take n = 6.

$$\therefore \quad h = \frac{b-a}{n} = \frac{1-0}{6} = \frac{1}{6} = 0.167$$

Now we obtain the following table –

x :	0	0.167	0.333	0.5	0.667	0.833	1
f(x) :	1	0.9729	0.9002	0.8	0.6921	0.5903	0.5

Simpson's 1/3 Rule – By Simpson's 1/3 rule, we have

$$\int_0^1 \frac{dx}{1+x^2} = \frac{h}{3}[(f_0 + f_6) + 4(f_1 + f_3 + f_5) + 2(f_2 + f_4)]$$

$$= \frac{(1/6)}{3}[(1 + 0.5) + 4(0.9729 + 0.8 + 0.5903) + 2(0.9002 + 0.6921)]$$

$$= \frac{1}{18}[1.5 + 9.4528 + 3.1846]$$

$$= \frac{14.1374}{18} = \mathbf{0.785411}$$ **Ans.**

Simpson's 3/8 Rule – By Simpson's 3/8 rule, we have

$$\int_0^1 \frac{dx}{1+x^2} = \frac{3h}{8}[(f_0 + f_6) + 3(f_1 + f_2 + f_4 + f_5) + 2f_3]$$

$$= \frac{3 \times (1/6)}{8}[(1 + 0.5) + 3(0.9729 + 0.9002 + 0.6921 + 0.5903) + 2 \times 0.8]$$

$$= \frac{1}{16}[1.5 + 9.4665 + 1.6] = \frac{12.5665}{16} = \mathbf{0.785406}$$ **Ans.**

Prob.28. Evaluate $\int_0^6 \frac{dx}{1+x^2}$ ***by using (i) Trapezoidal rule (ii) Simpson's 1/3 rule (iii) Simpson's 3/8 rule.*** ***(R.G.P.V., Dec. 2002)***

Sol. We have $f(x) = \frac{1}{1+x^2}$, a = 0, b = 6

Here the number of equal interval is not given, so we take n = 6

$$\therefore \quad h = \frac{b-a}{n} = \frac{6-0}{6} = 1$$

Now we obtain the following table –

x :	0	1	2	3	4	5	6
f(x) :	1	0.5	0.2	0.1	0.0588	0.0385	0.027

(i) Trapezoidal Rule – By Trapezoidal's rule, we have

$$\int_0^6 \frac{dx}{1+x^2} = \frac{h}{2}[(f_0 + f_6) + 2(f_1 + f_2 + f_3 + f_4 + f_5)]$$

$$= \frac{1}{2}[(1 + 0.027) + 2(0.5 + 0.2 + 0.1 + 0.0588 + 0.0385] = \mathbf{1.4108}$$ **Ans.**

(ii) Simpson's 1/3 Rule – By Simpson's 1/3 rule, we have

$$\int_0^6 \frac{dx}{1+x^2} = \frac{h}{3}[(f_0 + f_6) + 4(f_1 + f_3 + f_5) + 2(f_2 + f_4)]$$

$$= \frac{1}{3}[(1 + 0.027) + 4(0.5 + 0.1 + 0.0385) + 2(0.2 + 0.0588)]$$

$$= \frac{1}{3}(1.027 + 2.554 + 0.5176) = \mathbf{1.3662}$$ **Ans.**

(iii) Simpson's 3/8 Rule – By Simpson's 3/8 rule, we have

$$\int_0^6 \frac{dx}{1+x^2} = \frac{3h}{8}[(f_0 + f_6) + 3(f_1 + f_2 + f_4 + f_5) + 2f_3]$$

$$= \frac{3}{8}[(1 + 0.027) + 3(0.5 + 0.2 + 0.0588 + 0.0385) + 2 \times 0.1]$$

$$= \frac{3}{8}(1.027 + 2.3919 + 0.2) = \mathbf{1.3571}$$ **Ans.**

Prob.29. Evaluate $\int_0^6 \frac{1}{1+x^2}dx$ ***by using Simpson's 1/3 rule.***

(R.G.P.V., June 2015, 2017)

Sol. Refer to Prob.28 (ii).

Prob.30. Evaluate $\int_0^6 \frac{dx}{1+x^2}$ ***by using Simpson's*** $\frac{3}{8}$ ***rule.***

[R.G.P.V., May 2019 (III-Sem)]

Sol. Refer to Prob.28 (iii).

SOLUTION OF SIMULTANEOUS LINEAR ALGEBRAIC EQUATIONS BY GAUSS'S ELIMINATION, GAUSS'S JORDAN, CROUT'S METHODS, JACOBI'S, GAUSS-SEIDAL AND RELAXATION METHOD

Solution of Simultaneous Linear Algebraic Equation – Consider the following m first degree equations of n known $x_1, x_2, x_3 \ldots x_n$.

$$\left.\begin{array}{l} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1j}x_j + \ldots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2j}x_j + \ldots + a_{2n}x_n = b_2 \\ \ldots \\ a_{i1}x_1 + a_{i2}x_2 + \ldots + a_{ij}x_j + \ldots + a_{in}x_n = b_i \\ \ldots \\ a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mj}x_j + \ldots + a_{mn}x_n = b_m \end{array}\right\} \quad \ldots(i)$$

The above system can be written in a single matrix equation as

$$AX = B.$$

where

$$A = \begin{bmatrix} a_{11} & a_{12} \ldots a_{1j} \ldots a_{1n} \\ a_{21} & a_{22} \ldots a_{2j} \ldots a_{2n} \\ \ldots & \ldots \\ a_{i1} & a_{i2} \ldots a_{ij} \ldots a_{in} \\ \ldots & \ldots \\ a_{m1} & a_{m2} \ldots a_{mj} \ldots a_{mn} \end{bmatrix}, X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \text{ and } B = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

By finding a solution of a system of equations we mean to find a set of values of $x_1, x_2 \ldots x_n$ which satisfies all these equations simultaneously. The above system of equations is said to be ***homogeneous*** if all the b_i $(i = 1, \ldots m)$ vanish otherwise the system is called ***nonhomogeneous*** system.

Different Methods of Obtaining the Solution –

We have $AX = B$ ...(i)

$$\text{where } A = (a_{ij})_{m \times n},\ X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}_{n\times 1},\ B = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}_{m\times 1}$$

where A is the coefficient matrix, X is the solution vector and B is called right-hand-side vector.

The necessary and sufficient condition for the existance of a solution of the system $AX = B$, is that $r(A) = r(A\ B)$ i.e., in other words rank of the coefficient matrix is the same as the rank of the augmented matrix.

We now give some rules to give information about the existance of a solution of the system –

(i) If $B = 0$, $\det A = 0$, then there exist infinite number of non-trivial solutions besides trival solution $X = 0$.

(ii) If $B = 0$ and $\det A \neq 0$, then the system has the only unique trivial solution $X = 0$. In this case $r(A) = m$.

(iii) If $B \neq 0$ and $\det A \neq 0$; then system has only unique solution and in this case we have $r(A) = m$.

(iv) If $B \neq 0$, $\det A = 0$; then there exist infinite number of solutions provided the equations are consistent. In this case we have $r(A) < m$.

The Elimination Method by Gauss – We have seen that any system of linear algebraic equations can be solved by the use of determinants. If the order of the determinant is large, then the evaluation becomes tedious. To avoid these unnecessary computations, from the practical computation point of view various mathematician have tried to develop simpler and less time consuming procedures, and various methods for solving system of linear equations,

$$\left.\begin{array}{l} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n = b_2 \\ \ldots \\ \ldots \\ a_{n1}x_1 + a_{n2}x_2 + \ldots + a_{nn}x_n = b_n \end{array}\right\} \quad \ldots(i)$$

have been suggested. One of the most important method is the so called ***Gauss-algorithm***, which is a systematic elimination method. In this method for solving

system (i) we proceed stepwise as follows –

First Step – ***Elimination of x_1 from the second, third ..... nth equation.***

We assume here that the order of the equation and the order of the unknowns in each equations are such that $a_{11} \neq 0$ the variable x_1 can then be eliminated from the second equation by subtracting $\left(\frac{a_{21}}{a_{11}}\right)$ times the first equation from the second equation, $\left(\frac{a_{31}}{a_{11}}\right)$ times the first equation from the third equation, etc. This gives new system as follows –

$$\left.\begin{aligned} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n &= b_1 \\ a'_{22}x_2 + \ldots + a'_{2n}x_n &= b'_2 \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ a'_{n2}x_2 + \ldots + a'_{nn}x_n &= b'_n \end{aligned}\right\} \quad \ldots\text{(ii)}$$

Second Step – ***Elimination of x_2 from the third.....nth equation in (ii).***

If the coefficient a'_{22} .....a'_{nn} in (ii) are not all zero, we may assume that the order of equation and the unknowns is such that $a'_{22} \neq 0$. Then we may eliminate x_2 from the third, fourth ..... nth equations of (ii) by subtracting.

(a'_{33}/a'_{22}) times the second equation from the third equation (a'_{42}/a'_{22}) times the second equation from the fourth equation etc. The further steps are now obvious. In the third step we eliminate x_3 and in the fourth step, we eliminate x_4 etc.

By successive elimination, we arrive at a single equation in the unknown x_n which can be solved and putting this in the preceding equation we find the value of $x_{(n-1)}$. In this manner we obtain x_1 when the elimination is completed. Also when the elimination is complete, the system takes the form,

$$\left.\begin{aligned} c_{11}x_1 + c_{12}x_2 + \ldots\ldots + c_{1n}x_n &= d_1 \\ c_{22}x_2 + \ldots\ldots + c_{2n}x_n &= d_2 \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ c_{nn}x_n &= d_n \end{aligned}\right\} \quad \text{Triangular form}$$

In this case these exists a unique solution. The new coefficient matrix is an upper triangular matrix; the diagonal element c_{ii} are usually equal to 1.

Jordan's Method – It is a modification of the method due to Gauss. In this method, the elimination is performed not only in the equations below but also in the equations above. In this way we finally find a diagonal (even unit) coefficient matrix, and we have the solution with without further computation.

Crout's Method – This method is advantageous over Gauss method in the sense that it requires less computation. We shall explain this method by taking a set of three equations.

Consider the system

$$AX = B \quad \ldots\text{(i)}$$

i.e.,
$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} \Rightarrow \left\{\begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 &= b_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 &= b_2 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 &= b_3 \end{aligned}\right\}$$

The augmented matrix of equation (i) is

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & b_1 \\ a_{21} & a_{22} & a_{23} & b_2 \\ a_{31} & a_{32} & a_{33} & b_3 \end{bmatrix} \quad \ldots\text{(ii)}$$

Then we consider derived matrix

$$\begin{bmatrix} a'_{11} & a'_{12} & a'_{13} & b'_1 \\ a'_{21} & a'_{22} & a'_{23} & b'_2 \\ a'_{31} & a'_{32} & a'_{33} & b'_3 \end{bmatrix} \quad \ldots\text{(iii)}$$

which is obtained in order as given below –

(i) To determine the first column, $a'_{i1} = a_{i1}$, for i i.e., the first column of the augmented matrix is as the first column of the coefficient matrix A.

(ii) To determine the first row to the right of the first column

$$a'_{1j} = \frac{a_{1j}}{a_{11}};\ j = 2, 3,\ b'_1 = \frac{b_1}{a_{11}}$$

i.e., the first row except the first element is found by dividing the corresponding elements of the first row of the matrix (ii) by the first element of that row.

(iii) To determine the elements of second column except a'_{21}

$$a'_{i2} = a_{i2} - a'_{12}\, a'_{i1};\ i = 2, 3$$

(iv) To determine the elements of second row except a'_{21}, a'_{22}

$$a'_{2j} = \frac{a_{2j} - a'_{1j}\, a'_{21}}{a'_{22}},\ j = 3 \text{ and } b'_2 = \frac{b_2 - b'_1 . a'_{21}}{a'_{22}}$$

(v) To determine the elements of third column except a'_{13}, a'_{23}

$$a'_{33} = a_{33} - a'_{23}\, a'_{32} - a'_{13}\, a'_{31}$$

(vi) To determine the remaining elements of third row

$$b'_3 = \frac{b_3 - b'_2\, a'_{32} - b'_1\, a'_{31}}{a'_{33}}$$

The solution is given by

$$x_3 = b'_3,\ x_2 = b'_2 - a'_{23}\, x_3,\ x_1 = b'_1 - a'_{13}\, x_3 - a'_{12}\, x_2.$$

Hence the elements of the derived matrix are found as follows –

First obtain first column, then elements of the first row to the right of the first column. After this compute elements of second column below the first row and then elements of the second row to the right of second column etc.

Method of Factorization – This method is also called triangularization method. This method is based on the fact that a square matrix A can be factorized into the form LU where L is unit lower triangular matrix and U is upper triangular, if all the principal minors of A are non-singular.

i.e., if $a_{11} \neq 0$, $\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \neq 0$, $\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \neq 0$, etc.

We shall explain this method by taking a set of three equations,

$$\left.\begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 &= b_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 &= b_2 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 &= b_3 \end{aligned}\right\} \quad \text{...(i)}$$

Putting system (i) in the form

$$AX = B \quad \text{...(ii)}$$

Let A = LU, where $L = \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix}$ and $U = \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}$

Writing A = LU in equation (ii) we have

$$LUX = B \quad \text{...(iii)}$$

Substituting UX = Y, the equation (iii) becomes

$$LY = B \quad \text{...(iv)}$$

Equation (iv) is equivalent to the system

$$\left.\begin{aligned} y_1 &= b_1 \\ l_{21}y_1 + y_2 &= b_2 \\ l_{31}y_1 + l_{32}y_2 + y_3 &= b_3 \end{aligned}\right\} \quad \text{...(v)}$$

The values of y_1, y_2, y_3 are found by the forward substitution.

When we know Y, the system UX = Y gives –

$$\left.\begin{aligned} u_{11}x_1 + u_{12}x_2 + u_{13}x_3 &= y_1 \\ u_{22}x_2 + u_{23}x_3 &= y_2 \\ u_{33}x_3 &= y_3 \end{aligned}\right\} \quad \text{....(vi)}$$

By the backward substitution, we get the values of x_1, x_2 and x_3.

The procedure of computing the matrices L and U is discussed below.

From the relation A = LU, we have

$$\begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

On multiplying the matrices on the left hand side and then equating the corresponding elements on both sides, we get

$$u_{11} = a_{11};\ u_{12} = a_{12};\ u_{13} = a_{13}$$

$$l_{21}u_{11} = a_{21} \Rightarrow l_{21} = a_{21}/a_{11}$$

$$l_{31}u_{11} = a_{31} \Rightarrow l_{31} = a_{31}/a_{11}$$

$$l_{21}u_{12} + u_{22} = a_{22} \Rightarrow u_{22} = a_{22} - (a_{21}/a_{11})\,a_{12}$$

$$l_{21}u_{13} + u_{23} = a_{23} \Rightarrow u_{23} = a_{23} - (a_{21}/a_{11})\,a_{13}$$

$$l_{31}u_{12} + l_{32}u_{22} = a_{32} \Rightarrow l_{32} = \frac{a_{32} - (a_{31}/a_{11})\,a_{12}}{u_{22}}$$

and $l_{31}u_{13} + l_{32}u_{23} + u_{33} = a_{33}$, which gives u_{33}.

Hence in a systematic way the elements of L and U can be found.

Suppose the lower and upper triangular matrices are denoted by L and U respectively, then we have

$$LA = U \Rightarrow L^{-1}LA = L^{-1}U \Rightarrow A = L^{-1}U$$

But L^{-1} is also a lower triangular matrix, we can obtain a factorization of A as the product of one L-matrix and one R-matrix. Replacing L^{-1} by L, we get LU = A. Conveniently, we choose diagonal elements of one as 1. Thus the system Ax = y is resolved into two simplar system.

$$Lz = y,\ Ux = z$$

Jacobi's Iteration Method –

Consider the equations,

$$\left.\begin{aligned} a_1x + a_2y + a_3z &= d_1 \\ b_1x + b_2y + b_3z &= d_2 \\ c_1x + c_2y + c_3z &= d_3 \end{aligned}\right\} \quad \text{....(i)}$$

If a_1, b_2, c_3 are large as compared to other coefficients, then solving these for x, y, z respectively, the system can be written in the form

$$\left.\begin{aligned} x &= k_1 - l_1y - m_1z \\ y &= k_2 - l_2x - m_2z \\ z &= k_3 - l_3x - m_3y \end{aligned}\right\} \quad \text{....(ii)}$$

Let us start with the initial approximations x_0, y_0, z_0 (each = 0) for the values of x, y, z, putting these values on the right sides, we get the first approximations.

$$x_1 = k_1,\ y_1 = k_2,\ z_1 = k_3$$

Hence the elements of the derived matrix are found as follows –

First obtain first column, then elements of the first row to the right of the first column. After this compute elements of second column below the first row and then elements of the second row to the right of second column etc.

Method of Factorization – This method is also called triangularization method. This method is based on the fact that a square matrix A can be factorized into the form LU where L is unit lower triangular matrix and U is upper triangular, if all the principal minors of A are non-singular.

i.e., if $a_{11} \neq 0$, $\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \neq 0$, $\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \neq 0$, etc.

We shall explain this method by taking a set of three equations,

$$\left.\begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 &= b_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 &= b_2 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 &= b_3 \end{aligned}\right\} \quad ...(i)$$

Putting system (i) in the form

$$AX = B \quad ...(ii)$$

Let A = LU, where $L = \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix}$ and $U = \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}$

Writing A = LU in equation (ii) we have

$$LUX = B \quad ...(iii)$$

Substituting UX = Y, the equation (iii) becomes

$$LY = B \quad ...(iv)$$

Equation (iv) is equivalent to the system

$$\left.\begin{aligned} y_1 &= b_1 \\ l_{21}y_1 + y_2 &= b_2 \\ l_{31}y_1 + l_{32}y_2 + y_3 &= b_3 \end{aligned}\right\} \quad ...(v)$$

The values of y_1, y_2, y_3 are found by the forward substitution.

When we know Y, the system UX = Y gives –

$$\left.\begin{aligned} u_{11}x_1 + u_{12}x_2 + u_{13}x_3 &= y_1 \\ u_{22}x_2 + u_{23}x_3 &= y_2 \\ u_{33}x_3 &= y_3 \end{aligned}\right\} \quad ....(vi)$$

By the backward substitution, we get the values of x_1, x_2 and x_3.

The procedure of computing the matrices L and U is discussed below. From the relation A = LU, we have

$$\begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

On multiplying the matrices on the left hand side and then equating the corresponding elements on both sides, we get

$$u_{11} = a_{11};\ u_{12} = a_{12};\ u_{13} = a_{13}$$
$$l_{21}u_{11} = a_{21} \Rightarrow l_{21} = a_{21}/a_{11}$$
$$l_{31}u_{11} = a_{31} \Rightarrow l_{31} = a_{31}/a_{11}$$
$$l_{21}u_{12} + u_{22} = a_{22} \Rightarrow u_{22} = a_{22} - (a_{21}/a_{11})\,a_{12}$$
$$l_{21}u_{13} + u_{23} = a_{23} \Rightarrow u_{23} = a_{23} - (a_{21}/a_{11})\,a_{13}$$
$$l_{31}u_{12} + l_{32}u_{22} = a_{32} \Rightarrow l_{32} = \frac{a_{32} - (a_{31}/a_{11})\,a_{12}}{u_{22}}$$

and $l_{31}u_{13} + l_{32}u_{23} + u_{33} = a_{33}$, which gives u_{33}.

Hence in a systematic way the elements of L and U can be found.

Suppose the lower and upper triangular matrices are denoted by L and U respectively, then we have

$$LA = U \Rightarrow L^{-1}LA = L^{-1}U \Rightarrow A = L^{-1}U$$

But L^{-1} is also a lower triangular matrix, we can obtain a factorization of A as the product of one L-matrix and one R-matrix. Replacing L^{-1} by L, we get LU = A. Conveniently, we choose diagonal elements of one as 1. Thus the system Ax = y is resolved into two simplar system.

$$Lz = y,\ Ux = z$$

Jacobi's Iteration Method –

Consider the equations,

$$\left.\begin{aligned} a_1x + a_2y + a_3z &= d_1 \\ b_1x + b_2y + b_3z &= d_2 \\ c_1x + c_2y + c_3z &= d_3 \end{aligned}\right\} \quad ....(i)$$

If a_1, b_2, c_3 are large as compared to other coefficients, then solving these for x, y, z respectively, the system can be written in the form

$$\left.\begin{aligned} x &= k_1 - l_1y - m_1z \\ y &= k_2 - l_2x - m_2z \\ z &= k_3 - l_3x - m_3y \end{aligned}\right\} \quad ....(ii)$$

Let us start with the initial approximations x_0, y_0, z_0 (each = 0) for the values of x, y, z, putting these values on the right sides, we get the first approximations.

$$x_1 = k_1,\ y_1 = k_2,\ z_1 = k_3$$

Putting these on the right sides of equation (ii), we get the second approximations.

$$x_2 = k_1 - l_1y_1 - m_1z_1$$
$$y_2 = k_2 - l_2x_1 - m_2z_1$$
$$z_2 = k_3 - l_3x_1 - m_3y_1$$

This process is repeated till the difference between two consecutive approximations is negligible.

Gauss-Seidel Method – It is a simple modification of Jacobi's method. As before, we start with initial approximations x_0, y_0, z_0 (each = 0) for x, y, z respectively. Substituting $y = y_0$, $z = z_0$ in the first of the equation (ii), we get

$$x_1 = k_1$$

Then putting $x = x_1$, $z = z_0$ in the second of the equation (ii), we get

$$y_1 = k_2 - l_2x_1 - m_2z_0$$

Next substituting $x = x_1$, $y = y_1$, in the third of the equation (ii), we get

$$z_1 = k_3 - l_3x_1 - m_3z_1$$

This process of iteration is continued till convergency to the desired degree of accuracy is obtained.

Relaxation Method –

Consider the equations,

$$a_1x + a_2y + a_3z = d_1$$
$$b_1x + b_2y + b_3z = d_2$$
$$c_1x + c_2y + c_3z = d_3$$

We define the residuals R_x, R_y, R_z by the relations,

$$\left.\begin{aligned} R_x &= d_1 - a_1x - a_2y - a_3z \\ R_y &= d_2 - b_1x - b_2y - b_3z \\ R_z &= d_3 - c_1x - c_2y - c_3z \end{aligned}\right\} \quad ...(i)$$

To start with we assume $x = y = z = 0$ and calculate the initial residuals. Then the residuals are reduced step by step, by giving increments to the variables, now we construct the following operation table.

	δR_x	δR_y	δR_z
$\delta x = 1$	$-a_1$	$-b_1$	$-c_1$
$\delta y = 1$	$-a_2$	$-b_2$	$-c_2$
$\delta z = 1$	$-a_3$	$-b_3$	$-c_3$

We note from the equations (i) that if x is increased by equation (i) (keeping y and z constants), R_x, R_y and R_z decrease by a_1, b_1, c_1 respectively. This is shown in the above table alongwith the effects on the residuals when y and z are given unit increments.

At each step, the numerically largest residuals is reduced to almost zero. To reduce a particular residual, the value of the corresponding variable is changed; e.g. to reduce R_x by p, x should be increased by p/a_1.

When all the residuals have been reduced to almost zero, the increments in x, y, z are added separately to give the desired solution.

Q.1. Write the procedure of Gauss-elimination method for solving the system of simultaneous linear equations. ***(R.G.P.V., Dec. 2015)***

Ans. Refer to the matter given on page 91.

NUMERICAL PROBLEMS

Prob.31. Apply Gauss elimination method to solve the equations –

$$2x + y + z = 10$$
$$3x + 2y + 3z = 18$$
$$x + 4y + 9z = 16$$

(R.G.P.V., June 2012)

Sol. We write the equation, first, has unity as coefficient of x otherwise divide the equation by the coefficient of x to make it unity. Thus

$$x + 4y + 9z = 16 \quad ...(i)$$
$$2x + y + z = 10 \quad ...(ii)$$
$$3x + 2y + 3z = 18 \quad ...(iii)$$

To eliminate x, subtract suitable multiples of first equation from the remaining equation, and we get

$$x + 4y + 9z = 16 \quad ...(iv)$$
$$-7y - 17z = -22 \quad ...(v) \quad [(ii) - 2(i)]$$
$$-10y - 24z = -30 \quad ...(vi) \quad [(iii) - 3(i)]$$

The coefficient of y is made unity in none of the resulting equation, we have

$$x + 4y + 9z = 16 \quad ...(vii)$$
$$y + \frac{17}{7}z = \frac{22}{7} \quad ...(viii)$$
$$5y + 12z = 15 \quad ...(ix)$$

To eliminate y, from (ix), operate (ix) – 5 (viii), we get

$$x + 4y + 9z = 16$$
$$y + \frac{17}{7}z = \frac{22}{7}$$
$$-\frac{1}{7}z = -\frac{5}{7}$$

Start from bottom and substitute

$$-\frac{1}{7}z = -\frac{5}{7} \text{ or } z = 5$$

$$y + \frac{85}{7} = \frac{22}{7}$$

$$y = \frac{22}{7} - \frac{85}{7} = \frac{-63}{7} = -9$$

$$x + 4y + 9z = 16$$

$$x + 4(-9) + 9 \times 5 = 16$$

$$x - 36 + 45 = 16$$

$$x + 9 = 16$$

$$x = 7$$

Hence $x = 7$, $y = -9$, $z = 5$ **Ans.**

Prob.32. Solve the following system of equations by Gauss elimination method the equations are

$$2x + y + z = 12$$
$$3x + 2y + 3z = 18$$
$$x + 4y + 9z = 16$$

(R.G.P.V., Dec. 2016)

Sol. We write the equation, first, has unity as coefficient of x, otherwise divide the equation by the coefficient of x to make it unity. Thus

$$x + 4y + 9z = 16 \quad ...(i)$$
$$2x + y + z = 12 \quad ...(ii)$$
$$3x + 2y + 3z = 18 \quad ...(iii)$$

To eliminate x, subtract suitable multiples of first equation from the remaining equation, and we get

$$x + 4y + 9z = 16 \quad ...(iv)$$
$$-7y - 17z = -20 \quad ...(v)\ [(ii) - 2(i)]$$
$$-10y - 24z = -30 \quad ...(vi)\ [(iii) - 3(i)]$$

The coefficient of y is made unity in none of the resulting equation, we have

$$x + 4y + 9z = 16 \quad ...(vii)$$
$$y + \frac{17}{7}z = \frac{20}{7} \quad ...(viii)$$
$$5y + 12z = 15 \quad ...(ix)$$

To eliminate y, from (ix), operate (ix) – 5 (viii), we get

$$x + 4y + 9z = 16$$
$$y + \frac{17}{7}z = \frac{20}{7}$$
$$-\frac{1}{7}z = \frac{5}{7}$$

Start from bottom and substitute

$$-\frac{1}{7}z = \frac{5}{7}$$

$$z = -5$$

or

$$y - \frac{85}{7} = \frac{20}{7}$$

$$y = \frac{20}{7} + \frac{85}{7}$$

$$y = \frac{105}{7}$$

$$y = 15$$

$$x + 4y + 9z = 16$$

$$x + 4 \times 15 + 9 \times (-5) = 16$$

$$x + 60 - 45 = 16$$

$$x + 15 = 16$$

$$x = 1$$

Hence $x = 1$, $y = 15$, $z = -5$ **Ans.**

Prob.33. Using Gauss elimination method solve

$$2x - y + 3z = 9$$
$$x + y + z = 6$$
$$x - y + z = 2$$

(R.G.P.V., June 2017)

Sol. We write the equation, first, which has unity as coefficient of x, otherwise divide the equation by the coefficient of x to make it unity. Thus

$$x + y + z = 6 \quad ...(i)$$
$$2x - y + 3z = 9 \quad ...(ii)$$
$$x - y + z = 2 \quad ...(iii)$$

To eliminate x, subtract suitable multiples of first equation from the remaining equation, and we get

$$x + y + z = 6 \quad ...(iv)$$
$$-3y + z = -3 \quad ...(v)\ [(ii) - 2(i)]$$
$$-2y = -4 \quad ...(vi)\ [(iii) - (i)]$$

Start from bottom and substitute

$$-2y = -4$$

or

$$y = 2$$

$$-3y + z = -3$$

$$-6 + z = -3$$

$$z = 3$$

$$x + y + z = 6$$

$$x + 2 + 3 = 6$$

$$x + 5 = 6$$

$$x = 1$$

Hence $x = 1$, $y = 2$, $z = 3$ **Ans.**

Prob.34. Solve the equation by Gauss-elimination method –

$$2x + y + 4z = 12$$
$$8x - 3y + 2z = 23$$
$$4x + 11y - z = 33$$

[R.G.P.V., Nov. 2019 (III-Sem)]

Sol. Given equations are

$$2x + y + 4z = 12 \quad ...(i)$$
$$8x - 3y + 2z = 23 \quad ...(ii)$$
$$4x + 11y - z = 33 \quad ...(iii)$$

To eliminate x, we are operating (ii) – 4(i), (iii) – 2 (i), we obtain

$$-7y - 14z = -25$$
$$9y - 9z = 9$$

i.e., $$y + 2z = \frac{25}{7} \quad ...(iv)$$
$$y - z = 1 \quad ...(v)$$

To eliminate y, we are operating (v) – (iv), we obtain

$$-3z = -\frac{18}{7}$$

or $$z = \frac{6}{7} = 0.8571$$

Putting z = 0.8571 in equation (v), we get

$$y - 0.8571 = 1$$
$$y = 1.8571$$

Putting y = 1.8571, z = 0.8571 in equation (i), we get

$$2x + 1.8571 + 4 \times 0.8571 = 12$$
$$2x + 5.2855 = 12$$
$$2x = 12 - 5.2855$$
$$2x = 6.7145$$
$$x = 3.3572$$

Hence x = **3.3572**, y = **1.8571**, z = **0.8571** **Ans.**

Prob.35. Solve the following system by Gauss elimination method –

$$6x + 3y + 2z = 6$$
$$6x + 4y + 3z = 0$$
$$20x + 15y + 12z = 0$$

(R.G.P.V., Nov. 2018)

Sol. Given equations are

$$6x + 3y + 2z = 6 \quad ...(i)$$
$$6x + 4y + 3z = 0 \quad ...(ii)$$
$$20x + 15y + 12z = 0 \quad ...(iii)$$

To eliminate x, we are operating (ii) – (i), (iii) $-\frac{20}{6}$ (i), we obtain

$$y + z = -6 \quad ...(iv)$$
$$5y + \frac{16}{3}z = -20 \quad ...(v)$$

To eliminate y, we are operating (v) – 5 (iv), we get

$$\frac{1}{3}z = 10$$
$$z = 30$$

Putting z = 30 in equation (iv), we get

$$y + 30 = -6$$

or $$y = -6 - 30$$

or $$= -36$$

Putting y = – 36, z = 30 in equation (i), we obtain

$$6x + 3(-36) + 2(30) = 6$$
$$6x - 108 + 60 = 6$$
$$6x = 6 + 108 - 60$$
$$6x = 54$$
$$x = 9$$

Hence x = **9**, y = **– 36**, z = **30** **Ans.**

Prob.36. Solve by Gauss elimination method –

$$10x + y + 2z = 13$$
$$3x + 10y + z = 14$$
$$2x + 3y + 10z = 15$$

(R.G.P.V., June 2002, 2015)

Sol. Given equation are

$$10x + y + 2z = 13 \quad ...(i)$$
$$3x + 10y + z = 14 \quad ...(ii)$$
$$2x + 3y + 10z = 15 \quad ...(iii)$$

To eliminate x, we are operating (ii) $-\frac{3}{10}$(i), (iii) $-\frac{1}{5}$ (i), we obtain

$$\frac{97}{10}y + \frac{2}{5}z = \frac{101}{10} \quad ...(iv)$$
$$\frac{14}{5}y + \frac{48}{5}z = \frac{62}{5} \quad ...(v)$$

To eliminate y, we are operating $\frac{97}{2}(v) - 14(iv)$, we get

$$460z = 460 \qquad \ldots(vi)$$

$$z = 1$$

Putting z = 1 in equation (iv), we get

$$\frac{97}{10}y + \frac{2}{5} = \frac{101}{10} \quad \text{or} \quad \frac{97}{10}y = \frac{97}{10} \quad \text{or} \quad y = 1$$

Putting y = 1, z = 1 in equation (i), we obtain

$$10x + 1 + 2 = 13$$

$$\Rightarrow \qquad x = 1$$

Hence $\qquad x = \mathbf{1}, y = \mathbf{1}, z = \mathbf{1}$ **Ans.**

Prob.37. Apply Gauss-Jordan method to solve the equations –
$10x + y + z = 9, x + 10y + z = 12, x + y + 10z = 12$
(R.G.P.V., June 2012)

Sol. We have

$$x + y + 10z = 12 \qquad \ldots(i)$$

$$x + 10y + z = 12 \qquad \ldots(ii)$$

$$10x + y + z = 9 \qquad \ldots(iii)$$

To eliminate x, we are operating (ii) – (i) and (iii) – 10(i), we have

$$9y - 9z = 0 \text{ or } y - z = 0 \qquad \ldots(iv)$$

$$-9y - 99z = -111 \qquad \ldots(v)$$

To eliminate y, we are operating (iv) + (v), we have

$$-108z = -111 \qquad \ldots(vi)$$

$$z = \frac{111}{108} = 1.0278$$

Putting z = 1.0278 in equation (iv), we get

$$y - 1.0278 = 0$$

or $\qquad y = 1.0278$

Putting y = 1.0278, z = 1.0278 in equation (i), we obtain

$$x + 1.0278 + 10.278 = 12$$

$$x = 12 - 11.3058 = 0.6942$$

Hence the solution is x = **0.6942**, y = **1.0278**, z = **1.0278.** **Ans.**

Prob.38. Solve the equations $10x + y + z = 12$, $2x + 10y + z = 13$ and $x + y + 5z = 7$ by Gauss-Jordan method. [R.G.P.V., May 2019 (IV-Sem)]

Or

Apply Gauss-Jordan method to find the solution of the following system of equations– $10x + y + z = 12$, $2x + 10y + z = 13$, $x + y + 5z = 7$.
(R.G.P.V., Dec. 2014)

Sol. The given system is equivalent to

$$AX = B$$

or
$$\begin{bmatrix} 10 & 1 & 1 \\ 2 & 10 & 1 \\ 1 & 1 & 5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 12 \\ 13 \\ 7 \end{bmatrix}$$

The augmented matrix is

$$[A|B] = \left[\begin{array}{ccc|c} 10 & 1 & 1 & 12 \\ 2 & 10 & 1 & 13 \\ 1 & 1 & 5 & 7 \end{array}\right]$$

Operating $R_2 \to 5R_2 - R_1$, $R_3 \to 10R_3 - R_1$

$$[A|B] = \left[\begin{array}{ccc|c} 10 & 1 & 1 & 12 \\ 0 & 49 & 4 & 53 \\ 0 & 9 & 49 & 58 \end{array}\right]$$

Operating $R_1 \to 9R_1 - R_3$, $R_2 \to 49R_2 - 4R_3$

$$[A|B] = \left[\begin{array}{ccc|c} 90 & 0 & -40 & 50 \\ 0 & 2365 & 0 & 2365 \\ 0 & 9 & 49 & 58 \end{array}\right]$$

Operating $R_3 \to R_3 - \frac{9}{2365}R_2$

$$[A|B] = \left[\begin{array}{ccc|c} 90 & 0 & -40 & 50 \\ 0 & 2365 & 0 & 2365 \\ 0 & 0 & 49 & 49 \end{array}\right]$$

Operating $R_1 \to 49R_1 + 40R_3$

$$[A|B] = \left[\begin{array}{ccc|c} 4410 & 0 & 0 & 4410 \\ 0 & 2365 & 0 & 2365 \\ 0 & 0 & 49 & 49 \end{array}\right]$$

Operating $R_1 \to \frac{1}{4410}R_1$, $R_2 \to \frac{1}{2365}R_2$ and $R_3 \to \frac{1}{49}R_3$

$$[A|B] = \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{array}\right]$$

From this, we get $\qquad x = \mathbf{1}, y = \mathbf{1}, z = \mathbf{1}$ **Ans.**

Prob.39. Apply Crout's method to solve the equations –
$3x + 2y + 7z = 4$, $2x + 3y + z = 5$, $3x + 4y + z = 7$.
(R.G.P.V., Dec. 2004, 2012, June 2015)

Sol. Coefficient matrix is given by

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} 3 & 2 & 7 \\ 2 & 3 & 1 \\ 3 & 4 & 1 \end{bmatrix}$$

The augmented matrix of the coefficient matrix is given by

$$\begin{bmatrix} a'_{11} & a'_{12} & a'_{13} & b'_1 \\ a'_{21} & a'_{22} & a'_{23} & b'_2 \\ a'_{31} & a'_{32} & a'_{33} & b'_3 \end{bmatrix}$$

where, $a'_{i1} = a_{i1}$, i.e., $a'_{11} = 3$, $a'_{21} = 2$, $a'_{31} = 3$ ...(i)

$$a'_{12} = \frac{a_{12}}{a_{11}} = \frac{2}{3}, a'_{13} = \frac{a_{13}}{a_{11}} = \frac{7}{3}$$

$$b'_1 = \frac{b_1}{a_{11}} = \frac{4}{3} \quad \text{...(ii)}$$

$$a'_{22} = a_{22} - a'_{12}\, a'_{21} = 3 - \frac{2}{3} \times 2 = \frac{5}{3}$$

$$a'_{32} = a_{32} - a'_{12}\, a'_{31} = 4 - \frac{2}{3} \times 3 = 2 \quad \text{...(iii)}$$

$$a'_{23} = \frac{a_{23} - a'_{13}\, a'_{21}}{a'_{22}} = \frac{1 - \frac{7}{3} \times 2}{\frac{5}{3}} = -\frac{11}{3} \times \frac{3}{5} = -\frac{11}{5}$$

$$b'_2 = \frac{b_2 - b'_1\, a'_{21}}{a'_{22}} = \frac{5 - \frac{4}{3} \times 2}{\frac{5}{3}} = \frac{15-8}{3} \times \frac{3}{5} = \frac{7}{5} \quad \text{...(iv)}$$

$$a'_{33} = a_{33} - a'_{23}\, a'_{32} - a'_{13}\, a'_{31}$$

$$= 1 - \left(-\frac{11}{5}\right) \times 2 - \frac{7}{3} \times 3 = 1 + \frac{22}{5} - 7 = -\frac{8}{5} \quad \text{...(v)}$$

$$b'_3 = \frac{b_3 - b'_2\, a'_{32} - b'_1\, a'_{31}}{a'_{33}} = \frac{7 - \left(\frac{7}{5}\right) \times 2 - \left(\frac{4}{3}\right) \times 3}{-\frac{8}{5}} = \frac{1}{5} \times -\frac{5}{8} = -\frac{1}{8}$$

Thus the derived matrix is $\begin{bmatrix} 3 & \frac{2}{3} & \frac{7}{3} & \frac{4}{3} \\ 2 & \frac{5}{3} & -\frac{11}{5} & \frac{7}{5} \\ 3 & 2 & -\frac{8}{5} & -\frac{1}{8} \end{bmatrix}$

Hence, $z = b'_3 = -\frac{1}{8}$, $y = b'_2 - a'_{23}\, z = \frac{7}{5} - \left(-\frac{11}{5}\right) \times \left(-\frac{1}{8}\right) = \frac{9}{8}$,

$$x = b'_1 - a'_{13}\, z - a'_{12}\, y = \frac{4}{3} - \frac{7}{3} \times \left(-\frac{1}{8}\right) - \frac{2}{3} \times \frac{9}{8} = \frac{4}{3} + \frac{7}{24} - \frac{18}{24} = \frac{7}{8}$$

Hence, $\mathbf{x = \frac{7}{8}, y = \frac{9}{8}, z = -\frac{1}{8}}$ **Ans.**

Prob.40. Solve the following system of equations by Crout's method –
$x + y + z = 3$, $2x - y + 3z = 16$, $3x + y - z = -3$.
[R.G.P.V., May 2019 (III-Sem)]

Sol. Coefficient matrix is given by

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 2 & -1 & 3 \\ 3 & 1 & -1 \end{bmatrix}$$

The augmented matrix of the coefficient matrix is given by

$$\begin{bmatrix} a'_{11} & a'_{12} & a'_{13} & b'_1 \\ a'_{21} & a'_{22} & a'_{23} & b'_2 \\ a'_{31} & a'_{32} & a'_{33} & b'_3 \end{bmatrix}$$

where, $a'_{i1} = a_{i1}$, i.e., $a'_{11} = 1$, $a'_{21} = 2$, $a'_{31} = 3$

$$a'_{12} = \frac{a_{12}}{a_{11}} = \frac{1}{1} = 1$$

$$a'_{13} = \frac{a_{13}}{a_{11}} = \frac{1}{1} = 1$$

$$b'_1 = \frac{b_1}{a_{11}} = \frac{3}{1} = 3$$

$$a'_{22} = a_{22} - a'_{12} a'_{21} = -1 - 1 \times 2 = -3$$

$$a'_{32} = a_{32} - a'_{12} a'_{31} = 1 - 1 \times 3 = -2$$

$$a'_{23} = \frac{a_{23} - a'_{13}\, a'_{21}}{a'_{22}} = \frac{3 - 1 \times 2}{-3} = -\frac{1}{3}$$

$$b'_2 = \frac{b_2 - b'_1 a'_{21}}{a'_{22}} = \frac{16 - 3\times 2}{-3} = -\frac{10}{3}$$

$$a'_{33} = a_{33} - a'_{23}a'_{32} - a'_{13}a'_{31}$$

$$= -1 - \left(-\frac{1}{3}\right)(-2) - 1\times 3 = -1 - \frac{2}{3} - 3 = -\frac{14}{3}$$

$$b'_3 = \frac{b_3 - b'_2 a'_{32} - b'_1 a'_{31}}{a'_{33}}$$

$$= \frac{-3 - \left(-\frac{10}{3}\right)(-2) - 3\times 3}{\left(-\frac{14}{3}\right)} = \frac{-3 - \frac{20}{3} - 9}{-\frac{14}{3}}$$

$$= \left(-\frac{56}{3}\right)\times\left(-\frac{3}{14}\right) = 4$$

Thus the derived matrix is $\begin{bmatrix} 1 & 1 & 1 & 3 \\ 2 & -3 & -\frac{1}{3} & -\frac{10}{3} \\ 3 & -2 & -\frac{14}{3} & 4 \end{bmatrix}$

Hence, $z = b'_3 = 4$

$$y = b'_2 - a'_{23}z$$

$$= -\frac{10}{3} - \left(-\frac{1}{3}\right)(4) = -\frac{10}{3} + \frac{4}{3} = -2$$

$$x = b'_1 - a'_{13}z - a'_{12}y$$

$$= 3 - 1\times 4 - 1\times(-2)$$

$$= 3 - 4 + 2 = 1$$

Hence, $x = 1$, $y = -2$, $z = 4$ **Ans.**

Prob.41. Solve the following equations by Crout's method –

$$10x + y + z = 12,\ 2x + 10y + z = 13,\ 2x + 2y + 10z = 14$$

(R.G.P.V., Dec. 2013, June 2016)

Sol. Coefficient matrix is given by

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} 10 & 1 & 1 \\ 2 & 10 & 1 \\ 2 & 2 & 10 \end{bmatrix}$$

The augmented matrix of the coefficient matrix is given by

$$\begin{bmatrix} a'_{11} & a'_{12} & a'_{13} & b'_1 \\ a'_{21} & a'_{22} & a'_{23} & b'_2 \\ a'_{31} & a'_{32} & a'_{33} & b'_3 \end{bmatrix}$$

where, $a'_{i1} = a_{i1}$, i.e., $a'_{11} = 10$, $a'_{21} = 2$, $a'_{31} = 2$

$$a'_{12} = \frac{a_{12}}{a_{11}} = \frac{1}{10},\ a'_{13} = \frac{a_{13}}{a_{11}} = \frac{1}{10},\ b'_1 = \frac{b_1}{a_{11}} = \frac{12}{10} = \frac{6}{5}$$

$$a'_{22} = a_{22} - a'_{12}\,a'_{21} = 10 - \frac{1}{10}\times 2 = \frac{49}{5}$$

$$a'_{32} = a_{32} - a'_{12}\,a'_{31} = 2 - \frac{1}{10}\times 2 = \frac{9}{5}$$

$$a'_{23} = \frac{a_{23} - a'_{13}a'_{21}}{a'_{22}} = \frac{1 - \frac{1}{10}\times 2}{\frac{49}{5}} = \frac{4}{5}\times\frac{5}{49} = \frac{4}{49}$$

$$b'_2 = \frac{b_2 - b'_1 a'_{21}}{a'_{22}} = \frac{13 - \frac{6}{5}\times 2}{\frac{49}{5}} = \frac{\frac{53}{5}}{\frac{49}{5}} = \frac{53}{49}$$

$$a'_{33} = a_{33} - a'_{23}\,a'_{32} - a'_{13}\,a'_{31}$$

$$= 10 - \frac{4}{49}\times\frac{9}{5} - \frac{1}{10}\times 2 = 10 - \frac{36}{245} - \frac{1}{5} = \frac{2365}{245} = \frac{473}{49}$$

$$b'_3 = \frac{b_3 - b'_2 a'_{32} - b'_1 a'_{31}}{a'_{33}} = \frac{14 - \frac{53}{49}\times\frac{9}{5} - \frac{6}{5}\times 2}{\frac{473}{49}}$$

$$= \frac{14 - \frac{477}{245} - \frac{12}{5}}{\frac{473}{49}} = \frac{\frac{2365}{245}}{\frac{473}{49}} = 1$$

Thus the derived matrix is $\begin{bmatrix} 10 & \frac{1}{10} & \frac{1}{10} & \frac{6}{5} \\ 2 & \frac{49}{5} & \frac{4}{49} & \frac{53}{49} \\ 2 & \frac{9}{5} & \frac{473}{49} & 1 \end{bmatrix}$

Hence, $z = b'_3 = 1,\ y = b'_2 - a'_{23}z = \frac{53}{49} - \frac{4}{49} \times 1 = 1$

$$x = b'_1 - a'_{13}z - a'_{12}y$$

$$= \frac{6}{5} - \frac{1}{10} \times 1 - \frac{1}{10} \times 1 = \frac{6}{5} - \frac{1}{10} - \frac{1}{10} = 1$$

Hence, $x = 1,\ y = 1,\ z = 1$ **Ans.**

Prob.42. Apply Crout's factorization method to solve the system of equations –

$$x - y = 0$$
$$-2x + 4y - 2z = -1$$
$$-y + 2z = 1.5$$

(R.G.P.V., June 2014)

Sol. Let $\begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 \\ -2 & 4 & -2 \\ 0 & -1 & 2 \end{bmatrix}$ (i.e., A)

So that, (i) $u_{11} = 1,\ u_{12} = -1,\ u_{13} = 0$

(ii) $l_{21}u_{11} = -2 \Rightarrow l_{21} = -2$

$l_{31}u_{11} = 0 \Rightarrow l_{31} = 0$

(iii) $l_{21}u_{12} + u_{22} = 4 \Rightarrow u_{22} = 2$

$l_{21}u_{13} + u_{23} = -2 \Rightarrow u_{23} = -2$

(iv) $l_{31}u_{12} + l_{32}u_{22} = -1 \Rightarrow l_{32} = -\frac{1}{2}$

(v) $l_{31}u_{13} + l_{32}u_{23} + u_{33} = 2 \Rightarrow u_{33} = 1$

Thus, $A = LU$

$$A = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & -\frac{1}{2} & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 & 0 \\ 0 & 2 & -2 \\ 0 & 0 & 1 \end{bmatrix}$$

Writing $UX = V$, the given system becomes $LV = B$

i.e., $$\begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & -\frac{1}{2} & 1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 1.5 \end{bmatrix}$$

Solving this system, we have

$$v_1 = 0$$
$$-2v_1 + v_2 = -1 \text{ or } v_2 = -1$$
$$-\frac{1}{2}v_2 + v_3 = 1.5 \text{ or } v_3 = 1$$

Hence, the original system becomes

$$\begin{bmatrix} 1 & -1 & 0 \\ 0 & 2 & -2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$$

i.e., $$x - y = 0$$
$$2y - 2z = -1$$
$$z = 1$$

By back-substitution, we have

$$z = 1,\ y = \frac{1}{2},\ x = \frac{1}{2}$$ **Ans.**

Prob.43. Solve by Jacobi's iteration method, the equations

$$20x + y - 2z = 17,\ 3x + 20y - z = -18,\ 2x - 3y + 20z = 25.$$

(R.G.P.V., Dec. 2012)

Sol. $\because$ There is no need of pivoting.

$\therefore$ Given equations can be written as

$$\left.\begin{aligned} x &= \frac{1}{20}(17 - y + 2z) \\ y &= \frac{1}{20}(-18 - 3x + z) \\ z &= \frac{1}{20}(25 - 2x + 3y) \end{aligned}\right\} \quad \text{...(i)}$$

Starting with $x_0 = y_0 = z_0 = 0$, substituting these values in the right side of the equation (i), we get

$$x_1 = 0.85,\quad y_1 = -0.9,\quad z_1 = 1.25$$

Substituting these values in the right side of the equation (i), we get

$$x_2 = \frac{1}{20}(17 - y_1 + 2z_1) = 1.0200$$

$$y_2 = \frac{1}{20}(-18 - 3x_1 + z_1) = -0.9650$$

$$z_2 = \frac{1}{20}(25 - 2x_1 + 3y_1) = 1.0300$$

Substituting these values in the right side of the equation (i), we get

$$x_3 = \frac{1}{20}(17 - y_2 + 2z_2) = 1.0013$$

$$y_3 = \frac{1}{20}(-18 - 3x_2 + z_2) = -1.0015$$

$$z_3 = \frac{1}{20}(25 - 2x_2 + 3y_2) = 1.0033$$

Substituting these values in the right side of the equation (i), we get

$$x_4 = \frac{1}{20}(17 - y_3 + 2z_3) = 1.0004$$

$$y_4 = \frac{1}{20}(-18 - 3x_3 + z_3) = -1.0000$$

$$z_4 = \frac{1}{20}(25 - 2x_3 + 3y_3) = 0.9996$$

Substituting these values in the right side of the equation (i), we get

$$x_5 = \frac{1}{20}(17 - y_4 + 2z_4) = 1.0000$$

$$y_5 = \frac{1}{20}(-18 - 3x_4 + z_4) = -1.0001$$

$$z_5 = \frac{1}{20}(25 - 2x_4 + 3y_4) = 1.0000$$

Hence, $x = 1$, $y = -1$, and $z = 1$ **Ans.**

Prob.44. Solve the following equations by Jacobi's method –

$$10x + 2y + 3z = 15$$
$$x + 7y + 2z = 10$$
$$4x + 2y + 10z = 16.$$

[R.G.P.V., Nov. 2019 (III-Sem)]

Sol. $\because$ There is no need of pivoting.

$\therefore$ Given equations can be written as

$$\left.\begin{aligned} x &= \frac{1}{10}(15 - 2y - 3z) \\ y &= \frac{1}{7}(10 - x - 2z) \\ z &= \frac{1}{10}(16 - 4x - 2y) \end{aligned}\right\} \quad ...(i)$$

Starting with $x_0 = y_0 = z_0 = 0$, substituting these values in the right side of the equation (i), we get

$$x_1 = 1.5,\ y_1 = 1.429,\ z_1 = 1.6$$

Substituting these values in the right side of the equation (i), we get

$$x_2 = \frac{1}{10}(15 - 2y_1 - 3z_1) = \frac{1}{10}(15 - 2\times1.429 - 3\times1.6) = 0.7342$$

$$y_2 = \frac{1}{7}(10 - x_1 - 2z_1) = \frac{1}{7}(10 - 1.5 - 2\times1.6 = 0.7571$$

$$z_2 = \frac{1}{10}(16 - 4x_1 - 2y_1) = \frac{1}{10}(16 - 4\times1.5 - 2\times1.429) = 0.7142$$

Substituting these values in the right side of the equation (i), we get

$$x_3 = \frac{1}{10}(15 - 2y_2 - 3z_2) = \frac{1}{10}(15 - 2\times0.7571 - 3\times0.7142) = 1.1343$$

$$y_3 = \frac{1}{7}(10 - x_2 - 2z_2) = \frac{1}{7}(10 - 0.7342 - 2\times0.7142) = 1.1196$$

$$z_3 = \frac{1}{10}(16 - 4x_2 - 2y_2) = \frac{1}{10}(16 - 4\times0.7342 - 2\times0.7571) = 1.1549$$

Substituting these values in the right side of the equation (i), we get

$$x_4 = \frac{1}{10}(15 - 2y_3 - 3z_3) = \frac{1}{10}(15 - 2\times1.1196 - 3\times1.1549) = 0.9296$$

$$y_4 = \frac{1}{7}(10 - x_3 - 2z_3) = \frac{1}{7}(10 - 1.1343 - 2\times1.1549) = 0.9366$$

$$z_4 = \frac{1}{10}(16 - 4x_3 - 2y_3) = \frac{1}{10}(16 - 4\times1.1343 - 2\times1.1196) = 0.9224$$

Substituting these values in the right side of the equation (i), we get

$$x_5 = \frac{1}{10}(15 - 2y_4 - 3z_4) = \frac{1}{10}(15 - 2\times0.9366 - 3\times0.9224) = 1.036$$

$$y_5 = \frac{1}{7}(10 - x_4 - 2z_4) = \frac{1}{7}(10 - 0.9296 - 2\times0.9224) = 1.0322$$

$$z_5 = \frac{1}{10}(16 - 4x_4 - 2y_4) = \frac{1}{10}(16 - 4\times0.9296 - 2\times0.9366) = 1.0408$$

Substituting these values in the right side of the equation (i), we get

$$x_6 = \frac{1}{10}(15 - 2y_5 - 3z_5) = \frac{1}{10}(15 - 2\times1.0322 - 3\times1.0408) = 0.9813$$

$$y_6 = \frac{1}{7}(10 - x_5 - 2z_5) = \frac{1}{7}(10 - 1.036 - 2\times1.0408) = 0.9832$$

$$z_6 = \frac{1}{10}(16 - 4x_5 - 2y_5) = \frac{1}{10}(16 - 4\times1.036 - 2\times1.0322) = 0.9792$$

Substituting these values in the right side of the equation (i), we get

$$x_7 = \frac{1}{10}(15 - 2y_6 - 3z_6) = \frac{1}{10}(15 - 2\times0.9832 - 3\times0.9792) = 1.0096$$

$$y_7 = \frac{1}{7}(10 - x_6 - 2z_6) = \frac{1}{7}(10 - 0.9813 - 2\times0.9792) = 1.0086$$

$$z_7 = \frac{1}{10}(16 - 4x_6 - 2y_6) = \frac{1}{10}(16 - 4\times0.9813 - 2\times0.9832) = 1.0108$$

Hence, $x = 1$, $y = 1$, $z = 1$ **Ans.**

Prob.45. Solve by Gauss-Seidel method the equations –

$$54x + y + z = 110$$
$$2x + 15y + 6z = 72$$
$$-x + 6y + 27z = 85$$

(R.G.P.V., June 2013)

Sol. We solve each equation of the system for the unknown with the largest coefficient in terms of the remaining unknowns, viz.

$$x = \frac{1}{54}(110 - y - z) \quad ...(i)$$

$$y = \frac{1}{15}(72 - 6z - 2x) \quad ...(ii)$$

and

$$z = \frac{1}{27}(85 + x - 6y) \quad ...(iii)$$

Putting $y = z = 0$ in the right side of the equation (i), we get

$$x_1 = \frac{1}{54}(110 - 0 - 0) = \frac{110}{54} = 2.037$$

Now putting $x = 2.037$, $z = 0$ in the equation (ii), we get

$$y_1 = \frac{1}{15}(72 - 6 \times 0 - 2 \times 2.037) = \frac{1}{15}(72 - 4.074) = \frac{67.926}{15} = 4.5284$$

Again putting $x = 2.037$, $y = 4.5284$ in the equation (iii), we get

$$z_1 = \frac{1}{27}(85 + 2.037 - 6 \times 4.5284) = \frac{59.87}{27} = 2.22$$

Now we proceed to obtain the second approximation.

$$x_2 = \frac{1}{54}(110 - y_1 - z_1) \quad ...(iv)$$

$$y_2 = \frac{1}{15}(72 - 6z_1 - 2x_2) \quad ...(v)$$

and

$$z_2 = \frac{1}{27}(85 + x_2 - 6y_2) \quad ...(vi)$$

Putting values of y_1 and z_1 in equation (iv), we have

$$x_2 = \frac{1}{54}(110 - 4.5284 - 2.22) = \frac{103.252}{54} = 1.912$$

Again putting the values of x_2 and z_1 in equation (v), we have

$$y_2 = \frac{1}{15}(72 - 6 \times 2.22 - 2 \times 1.912) = \frac{54.86}{15} = 3.66$$

Now putting $x_2 = 1.912$ and $y_2 = 3.66$ in equation (vi), we get

$$z_2 = \frac{1}{27}(85 + 1.912 - 6 \times 3.66) = \frac{64.952}{27} = 2.41$$

Now we proceed to obtain the third approximation

$$x_3 = \frac{1}{54}(110 - y_2 - z_2) \quad ....(vii)$$

$$y_3 = \frac{1}{15}(72 - 6z_2 - 2x_3) \quad ...(viii)$$

and

$$z_3 = \frac{1}{27}(85 + x_3 - 6y_3) \quad ...(ix)$$

Putting values of y_2 and z_2 in equation (vii), we get

$$x_3 = \frac{1}{54}(110 - 3.66 - 2.41) = 1.925$$

Now putting values of z_2 and x_3 in equation (viii), we get

$$y_3 = \frac{1}{15}(72 - 6 \times 2.41 - 2 \times 1.925) = 3.58$$

Again putting values of x_3 and y_3 in equation (ix), we get

$$z_3 = \frac{1}{27}(85 + 1.925 - 6 \times 3.58) = 2.424$$

Similarly

$$x_4 = \frac{1}{54}(110 - y_3 - z_3) = 1.926$$

$$y_4 = \frac{1}{15}(72 - 6z_3 - 2x_4) = 3.574$$

and

$$z_4 = \frac{1}{27}(85 + x_4 - 6y_4) = 2.425$$

These values are sufficiently close to x_3, y_3 and z_3, respectively. Hence the values 1.926, 3.574, 2.425 can be considered as the solution of the given system

$$x = \mathbf{1.926}, \; y = \mathbf{3.574}, \; z = \mathbf{2.425} \qquad \textbf{Ans.}$$

Prob.46. Solve the following system by Gauss-Seidel method –

$$10x + 2y + z = 9$$
$$2x + 20y - 2z = -44$$
$$2x + 3y + 10z = 22.$$

(R.G.P.V., Dec. 2015)

Sol. We write the given equations in the form

$$x = \frac{1}{10}(9 - 2y - z) \quad ....(i)$$

$$y = \frac{1}{10}(-22 - x + z) \quad ....(ii)$$

$$z = \frac{1}{10}(22 - 2x - 3y) \quad ....(iii)$$

Putting $y = z = 0$ in the right side of the equation (i), we get

$$x_1 = \frac{9}{10} = 0.9$$

Putting $x = x_1$, $z = 0$ in equation (ii), we get

$$y_1 = \frac{1}{10}(-22 - x_1) = -2.29$$

Putting $x = x_1$, $y = y_1$ in equation (iii), we get

$$z_1 = \frac{1}{10}(22 - 2x_1 - 3y_1) = 2.707$$

For the second iteration, we have

$$x_2 = \frac{1}{10}(9 - 2y_1 - z_1) = 1.0873,\ y_2 = \frac{1}{10}(-22 - x_2 + z_1) = -2.03803$$

$$z_2 = \frac{1}{10}(22 - 2x_2 - 3y_2) = 2.59395$$

For the third iteration, we have

$$x_3 = \frac{1}{10}(9 - 2y_2 - z_2) = 1.048211,\ y_3 = \frac{1}{10}(-22 - x_3 + z_2) = -2.045426$$

$$z_3 = \frac{1}{10}(22 - 2x_3 - 3y_3) = 2.603986$$

For the fourth iteration, we have

$$x_4 = \frac{1}{10}(9 - 2y_3 - z_3) = 1.0486866,\ y_4 = \frac{1}{10}(-22 - x_4 + z_3) = -2.04447$$

$$z_4 = \frac{1}{10}(22 - 2x_4 - 3y_4) = 2.60360$$

For the fifth iteration, we have

$$x_5 = \frac{1}{10}(9 - 2y_4 - z_4) = 1.0485,\ y_5 = \frac{1}{10}(-22 - x_5 + z_4) = -2.04449$$

$$z_5 = \frac{1}{10}(22 - 2x_5 - 3y_5) = 2.60365$$

These values are sufficiently close to x_4, y_4 and z_4 respectively. Hence the values 1.0485, – 2.04449, 2.60365 can be considered as the solution of the given system

$$x = \mathbf{1.0485},\ y = -\mathbf{2.04449},\ z = \mathbf{2.60365}$$ **Ans.**

Prob.47. Solve the equations –

$$10x + 2y + z = 9$$
$$-2x + 3y + 10z = 22$$
$$x + 10y - z = -22$$

By Gauss-Seidel method.

(R.G.P.V., June 2016)

Sol. We have

$$10x + 2y + z = 9$$
$$x + 10y - z = -22$$
$$-2x + 3y + 10z = 22$$

We write the given equations in the form

$$x = \frac{1}{10}(9 - 2y - z) \quad \text{...(i)}$$

$$y = \frac{1}{10}(-22 - x + z) \quad \text{...(ii)}$$

$$z = \frac{1}{10}(22 + 2x - 3y) \quad \text{...(iii)}$$

Putting $y = z = 0$ in the right side of the equation (i), we get

$$x_1 = \frac{9}{10} = 0.9$$

Putting $x = x_1$, $z = 0$ in equation(ii), we get

$$y_1 = \frac{1}{10}(-22 - x_1) = -2.29$$

Putting $x = x_1$, $y = y_1$ in equation (iii), we get

$$z_1 = \frac{1}{10}(22 + 2x_1 - 3y_1) = 3.067$$

For the second iteration, we have

$$x_2 = \frac{1}{10}(9 - 2y_1 - z_1) = 1.0513$$

$$y_2 = \frac{1}{10}(-22 - x_2 + z_1) = -1.9984$$

$$z_2 = \frac{1}{10}(22 + 2x_2 - 3y_2) = 3.0098$$

For the third iteration, we have

$$x_3 = \frac{1}{10}(9 - 2y_2 - z_2) = 0.9987$$

$$y_3 = \frac{1}{10}(-22 - x_3 + z_2) = -1.9989$$

$$z_3 = \frac{1}{10}(22 + 2x_3 - 3y_3) = 2.9994$$

For the fourth iteration, we have

$$x_4 = \frac{1}{10}(9 - 2y_3 - z_3) = 0.9998$$

$$y_4 = \frac{1}{10}(-22 - x_4 + z_3) = -2.0000$$

$$z_4 = \frac{1}{10}(22 + 2x_4 - 3y_4) = 2.99996$$

For the fifth iteration, we have

$$x_5 = \frac{1}{10}(9 - 2y_4 - z_4) = 1.0000$$

$$y_5 = \frac{1}{10}(-22 - x_5 + z_4) = -2.0000$$

$$z_5 = \frac{1}{10}(22 + 2x_5 - 3y_5) = 3.0000$$

For the sixth iteration, we have

$$x_6 = \frac{1}{10}(9 - 2y_5 - z_5) = 1.0000$$

$$y_6 = \frac{1}{10}(-22 - x_6 + z_5) = -2.0000$$

$$z_6 = \frac{1}{10}(22 + 2x_6 - 3y_6) = 3.0000$$

The values are same in fifth and sixth iteration.

Hence the solution is $x = 1, y = -2, z = 3$ **Ans.**

Prob.48. Apply Gauss-Seidel iteration method to solve the system of equations –

$$20x + y - 2z = 17$$
$$3x + 20y - z = -18$$
$$2x - 3y + 20z = 1.5$$

(R.G.P.V., June 2014)

Sol. We solve each equation of the system for the unknown with the largest coefficient in terms of the remaining unknowns, viz.

$$x = \frac{1}{20}(17 - y + 2z) \quad ...(i)$$

$$y = \frac{1}{20}(-18 + z - 3x) \quad ...(ii)$$

and

$$z = \frac{1}{20}(1.5 - 2x + 3y) \quad ...(iii)$$

Putting $y = z = 0$ in the right side of the equation (i), we get

$$x_1 = \frac{17}{20} = 0.85$$

Now putting $x = 0.85, z = 0$ in the equation (ii), we get

$$y_1 = \frac{1}{20}(-18 + 0 - 3 \times 0.85) = \frac{-20.55}{20} = -1.0275$$

Again putting $x = 0.85, y = -1.0275$ in the equation (iii), we get

$$z_1 = \frac{1}{20}(1.5 - 2 \times 0.85 + 3 \times (-1.0275)] = \frac{-3.2825}{20}$$
$$= -0.1641$$

Now we proceed to obtain the second approximation,

$$x_2 = \frac{1}{20}(17 - y_1 + 2z_1) \quad ...(iv)$$

$$y_2 = \frac{1}{20}(-18 + z_1 - 3x_2) \quad ...(v)$$

$$z_2 = \frac{1}{20}(1.5 - 2x_2 + 3y_2) \quad ...(vi)$$

and

Putting values of y_1 and z_1 in equation (iv), we have

$$x_2 = \frac{1}{20}[17 - (-1.0275) + 2(-0.1641)] = \frac{17.6993}{20}$$
$$= 0.8849$$

Again putting the values of x_2 and z_1 in equation (v), we have

$$y_2 = \frac{1}{20}(-18 - 0.1641 - 3 \times 0.8849) = \frac{-20.8188}{20} = -1.0409$$

Now putting the values of x_2 and y_2 in equation (vi), we get

$$z_2 = \frac{1}{20}[1.5 - 2 \times 0.8849 + 3 \times (-1.0409)] = \frac{-3.3925}{20} = -0.1696$$

Now we proceed to obtain the third approximation

$$x_3 = \frac{1}{20}(17 - y_2 + 2z_2) \quad ...(vii)$$

$$y_3 = \frac{1}{20}(-18 + z_2 - 3x_3) \quad ...(viii)$$

and

$$z_3 = \frac{1}{20}(1.5 - 2x_3 + 3y_3) \quad ...(ix)$$

Putting values of y_2 and z_2 in equation (vii), we get

$$x_3 = \frac{1}{20}[17 - (-1.0409) + 2 \times (-0.1696)] = \frac{17.7017}{20} = 0.8851$$

Now putting values of z_2 and x_3 in equation (viii), we get

$$y_3 = \frac{1}{20}[-18 - 0.1696 - 3 \times 0.8851] = \frac{-20.8249}{20} = -1.0412$$

Again putting values of x_3 and y_3 in equation (ix), we get

$$z_3 = \frac{1}{20}[1.5 - 2 \times 0.8851 + 3 \times (-1.0412)] = \frac{-3.3938}{20} = -0.1697$$

Similarly

$$x_4 = \frac{1}{20}(17 + 1.0412 - 2 \times 0.1697) = 0.8851$$

$$y_4 = \frac{1}{20}(-18 - 0.1697 - 3 \times 0.8851) = -1.0412$$

and $z_4 = \frac{1}{20}(1.5 - 2 \times 0.8851 - 3 \times 1.0412) = -0.1697$

These values are same to x_3, y_3 and z_3 respectively.

Hence, $x = \mathbf{0.8851}$, $y = \mathbf{-1.0412}$, $z = \mathbf{-0.1697}$ **Ans.**

Prob.49. Solve the following system of equations $8x - y + z = 18$; $2x + 5y - 2z = 3$; $x + 2y - 3z = -6$ using Gauss-Seidel iterative method.

(R.G.P.V., Dec. 2014)

Sol. We solve each equation of the system for the unknown with the largest coefficient in terms of the remaining unknowns, viz.

$$x = \frac{1}{8}(18 + y - z) \quad \text{...(i)}$$

$$y = \frac{1}{5}(3 + 2z - 2x) \quad \text{...(ii)}$$

and

$$z = \frac{1}{3}(6 + x + 2y) \quad \text{...(iii)}$$

Putting $y = z = 0$ in the right side of the equation (i), we get

$$x_1 = \frac{1}{8}(18 + 0 - 0) = \frac{18}{8} = 2.25$$

Now putting $x = 2.25$, $z = 0$ in the equation (ii), we get

$$y_1 = \frac{1}{5}(3 + 2 \times 0 - 2 \times 2.25)$$

$$= \frac{1}{5}(3 - 4.5) = -\frac{1.5}{5} = -0.3$$

Again putting $x = 2.25$, $y = -0.3$ in the equation (iii), we get

$$z_1 = \frac{1}{3}[6 + 2.25 + 2 \times (-0.3)] = \frac{7.65}{3} = 2.55$$

Now we proceed to obtain the second approximation,

$$x_2 = \frac{1}{8}(18 + y_1 - z_1) \quad \text{...(iv)}$$

$$y_2 = \frac{1}{5}(3 + 2z_1 - 2x_2) \quad \text{...(v)}$$

$$z_2 = \frac{1}{3}(6 + x_2 + 2y_2) \quad \text{...(vi)}$$

Putting values of y_1 and z_1 in equation (iv), we have

$$x_2 = \frac{1}{8}(18 - 0.3 - 2.55) = \frac{15.15}{8} = 1.8937$$

Again putting the values of x_2 and z_1 in equation (v), we have

$$y_2 = \frac{1}{5}(3 + 2 \times 2.55 - 2 \times 1.8937) = \frac{4.3126}{5} = 0.8625$$

Now putting $x_2 = 1.8937$ and $y_2 = 0.8625$ in equation (vi), we get

$$z_2 = \frac{1}{3}(6 + 1.8937 + 2 \times 0.8625) = \frac{9.6187}{3} = 3.2062$$

For the third iteration, we have

$$x_3 = \frac{1}{8}(18 + 0.8625 - 3.2062) = 1.9570$$

$$y_3 = \frac{1}{5}(3 + 2 \times 3.2062 - 2 \times 1.9570) = 1.0997$$

$$z_3 = \frac{1}{3}(6 + 1.9570 + 2 \times 1.0997) = 3.3855$$

For the fourth iteration, we have

$$x_4 = \frac{1}{8}(18 + 1.0997 - 3.3855) = 1.9643$$

$$y_4 = \frac{1}{5}(3 + 2 \times 3.3855 - 2 \times 1.9643) = 1.1685$$

$$z_4 = \frac{1}{3}(6 + 1.9643 + 2 \times 1.1685) = 3.4338$$

For the fifth iteration, we have

$$x_5 = \frac{1}{8}(18 + 1.1685 - 3.4338) = 1.9668$$

$$y_5 = \frac{1}{5}(3 + 2 \times 3.4338 - 2 \times 1.9668) = 1.1868$$

$$z_5 = \frac{1}{3}(6 + 1.9668 + 2 \times 1.1868) = 3.4468$$

For the sixth iteration, we have

$$x_6 = \frac{1}{8}(18 + 1.1868 - 3.4468) = 1.9675$$

$$y_6 = \frac{1}{5}(3 + 2 \times 3.4468 - 2 \times 1.9675) = 1.1917$$

$$z_6 = \frac{1}{3}(6 + 1.9675 + 2 \times 1.1917) = 3.4503$$

For the seventh iteration, we have

$$x_7 = \frac{1}{8}(18 + 1.1917 - 3.4503) = 1.9677$$

$$y_7 = \frac{1}{5}(3 + 2 \times 3.4503 - 2 \times 1.9677) = 1.1930$$

$$z_7 = \frac{1}{3}(6 + 1.9677 + 2 \times 1.1930) = 3.4512$$

For the eighth iteration, we have

$$x_8 = \frac{1}{8}(18 + 1.1930 - 3.4512) = 1.9677$$

$$y_8 = \frac{1}{5}(3 + 2 \times 3.4512 - 2 \times 1.9677) = 1.1934$$

$$z_8 = \frac{1}{3}(6 + 1.9677 + 2 \times 1.1934) = 3.4515$$

The values in 7th and 8th iterations being practically same.

Hence the solution is

$$x = \mathbf{1.9677},\ y = \mathbf{1.1934},\ z = \mathbf{3.4515}$$ **Ans.**

Prob.50. Solve the following system of equations using Gauss-Seidel method

$$27x + 6y - z = 85$$
$$6x + 15y + 2z = 72$$
$$x + y + 54z = 110$$

(R.G.P.V., Dec. 2002, June 2008, 2011, Dec. 2011, June 2017, Nov. 2018)

Sol. We solve each equation of the system for the unknown with the largest coefficient in terms of the remaining unknowns, viz.

$$x = \frac{1}{27}(85 - 6y + z) \quad ....(i)$$

$$y = \frac{1}{15}(72 - 6x - 2z) \quad ....(ii)$$

$$z = \frac{1}{54}(110 - x - y) \quad ....(iii)$$

Putting $y = z = 0$ in the right side of the equation (i), we get

$$x_1 = \frac{85}{27} = 3.15$$

Now putting $x = 3.15$, $z = 0$ in the equation (ii), we get

$$y_1 = \frac{1}{15}(72 - 18.90) = 3.54$$

Again putting $x = 3.15$, $y = 3.54$ in the equation (iii), we get

$$z_1 = \frac{1}{54}(110 - 3.15 - 3.54) = 1.91$$

Now we proceed to obtain the second approximation,

$$x_2 = \frac{1}{27}(85 - 6y_1 + z_1) = 2.43,\quad y_2 = \frac{1}{15}(72 - 6x_2 - 2z_1) = 3.573,$$

$$z_2 = \frac{1}{54}(110 - x_2 - y_2) = 1.926$$

Similarly

$$x_3 = \frac{1}{27}(85 - 6y_2 + z_2) = 2.426,\quad y_3 = \frac{1}{15}(72 - 6x_3 - 2z_2) = 3.573$$

and

$$z_3 = \frac{1}{54}(110 - x_3 - y_3) = 1.926$$

These values are sufficiently close to x_2, y_2, z_2 respectively.

Hence, the values 2.426, 3.572, 1.926 can be considered as the solution of the given system.

$$x = \mathbf{2.426},\ y = \mathbf{3.573},\ z = \mathbf{1.926}$$ **Ans.**

Prob.51. Solve the following equations by Gauss-Seidel iteration method –
$10x + y + z = 12,\ x + 10y + z = 12,\ x + y + 10z = 12.$

[R.G.P.V., May 2019 (IV-Sem)]

Or

Solve by Gauss-Seidel method the equations

$$10x + y + z = 12$$
$$x + 10y + z = 12$$
$$x + y + 10z = 12$$

(R.G.P.V., Dec. 2016)

Or

Solve the following equations by Gauss-Seidel method –
$10x + y + z = 12,\ x + 10y + z = 12,\ x + y + 10z = 12.$

[R.G.P.V., Nov. 2019 (IV-Sem)]

Sol. We write the given equations in the form

$$x = \frac{1}{10}(12 - y - z) \quad ...(i)$$

$$y = \frac{1}{10}(12 - x - z) \quad ...(ii)$$

$$z = \frac{1}{10}(12 - x - y) \quad ...(iii)$$

Putting $y = z = 0$ in the right side of the equation (i), we get

$$x_1 = \frac{12}{10} = 1.2$$

Putting $x = x_1$, $z = 0$ in equation (ii), we get

$$y_1 = \frac{1}{10}(12 - x_1) = 1.08$$

Puttig $x = x_1$, $y = y_1$ in equation (iii), we get

$$z_1 = \frac{1}{10}(12 - x_1 - y_1) = 0.972$$

For the second iteration, we have

$$x_2 = \frac{1}{10}(12 - y_1 - z_1) = 0.9948$$

$$y_2 = \frac{1}{10}(12 - x_2 - z_1) = 1.0033$$

$$z_2 = \frac{1}{10}(12 - x_2 - y_2) = 1.0002$$

For the third iteration, we have

$$x_3 = \frac{1}{10}(12 - y_2 - z_2) = 0.9996$$

$$y_3 = \frac{1}{10}(12 - x_3 - z_2) = 1.0000$$

$$z_3 = \frac{1}{10}(12 - x_3 - y_3) = 1.0000$$

For the fourth iteration, we have

$$x_4 = \frac{1}{10}(12 - y_3 - z_3) = 1$$

$$y_4 = \frac{1}{10}(12 - x_4 - z_3) = 1$$

$$z_4 = \frac{1}{10}(12 - x_4 - y_4) = 1$$

Hence $x = 1, y = 1, z = 1$ **Ans.**

Prob.52. Solve the following equation by Gauss-Seidel method –
$20x + y - 2z = 17,\ 3x + 20y - z = -18,\ 2x - 3y + 20z = 25.$
(R.G.P.V., Dec. 2013)

Sol. We write the given equation in the form

$$x = \frac{1}{20}(17 - y + 2z) \quad \text{....(i)}$$

$$y = \frac{1}{20}(-18 - 3x + z) \quad \text{....(ii)}$$

$$z = \frac{1}{20}(25 - 2x + 3y) \quad \text{....(iii)}$$

Putting $y = z = 0$ in the right side of the equation (i), we get

$$x_1 = \frac{17}{20} = 0.85$$

Putting $x = x_1$, $z = 0$, in equation (ii), we get

$$y_1 = \frac{1}{20}(-18 - 3x_1) = -1.0275$$

Putting $x = x_1$, $y = y_1$ in equation (iii), we obtain

$$z_1 = \frac{1}{20}(25 - 2x_1 + 3y_1) = 1.01$$

For the second iteration, we have

$$x_2 = \frac{1}{20}(17 - y_1 + 2z_1) = 1$$

$$y_2 = \frac{1}{20}(-18 - 3x_2 + z_1) = -0.9995$$

$$z_2 = \frac{1}{20}(25 - 2x_2 + 3y_2) = 1$$

For the third iteration, we have

$$x_3 = \frac{1}{20}(17 - y_2 + 2z_2) = 0.999975$$

$$y_3 = \frac{1}{20}(-18 - 3x_3 + z_2) = -0.99999$$

$$z_3 = \frac{1}{20}(25 - 2x_3 + 3y_3) = 1$$

For the fourth iteration, we have

$$x_4 = \frac{1}{20}(17 - y_3 + 2z_3) = 1$$

$$y_4 = \frac{1}{20}(-18 - 3x_4 + z_3) = -1$$

$$z_4 = \frac{1}{20}(25 - 2x_4 + 3y_4) = 1$$

Hence, $\mathbf{x = 1, y = -1, z = 1}$ **Ans.**

Prob.53. Solve the following equations by relaxation method.
$10x - 2y - 2z = 6, -x + 10y - 2z = 7, -x - y + 10z = 8$
[R.G.P.V., Nov. 2019 (IV-Sem)]

Sol. The residuals are given by,

$$R_x = 6 - 10x + 2y + 2z$$
$$R_y = 7 + x - 10y + 2z$$
$$R_z = 8 + x + y - 10z$$

The operation table is

	δR_x	δR_y	δR_z
$\delta x = 1$	–10	1	1
$\delta y = 1$	2	–10	1
$\delta z = 1$	2	2	–10

The relaxation table is –

	R_x	R_y	R_z	Steps
$x = y = z = 0$	6	7	8	I
$\delta z = 8/10 = 0.8$	7.6	8.6	0	II
$\delta y = 8.6/10 = 0.86$	9.32	0	0.86	III
$\delta x = 9.32/10 = 0.932$	0	0.932	1.792	IV
$\delta z = 1.792/10 = 0.1792$	0.3584	1.2904	0	V
$\delta y = 1.2904/10 = 0.129$	0.6164	0.0004	0.129	VI
$\delta x = 0.6164/10 = 0.062$	– 0.0036	0.0624	0.191	VII
$\delta z = 0.0191$	0.0346	0.1006	0	VIII
$\delta y = 0.01$	0.0546	0.0006	0.01	IX
$\delta x = 0.005$	0.0046	0.0056	0.015	X
$\delta z = 0.0015$	0.0076	0.0086	0	XI
$\delta y = 0.0009$	0.0094	– 0.0004	0.0009	XII
$\delta x = 0.0009$	0.0004	0.0005	0.0018	XIII

Hence

$$x = \sum \delta x = 0.932 + 0.062 + 0.005 + 0.0009 = 0.9999 = 1$$
$$y = \sum \delta y = 0.86 + 0.129 + 0.01 + 0.0009 = 0.9999 = 1$$
$$z = \sum \delta z = 0.8 + 0.1792 + 0.0191 + 0.0015 = 0.9998 = 1$$
$$x = y = z = 1$$

Ans.

Prob.54. Solve the equations $10x - 2y - 3z = 205$, $-2x + 10y - 2z = 154$, $-2x - y + 10z = 120$ by relaxation method.

Sol. The residuals are given by,

$$R_x = 205 - 10x + 2y + 3z,$$
$$R_y = 154 + 2x - 10y + 2z,$$
$$R_z = 120 + 2x + y - 10z$$

The operation table is –

	δR_x	δR_y	δR_z
$\delta x = 1$	–10	2	2
$\delta y = 1$	2	–10	1
$\delta z = 1$	3	2	–10

The relaxation table is –

	R_x	R_y	R_z	*Steps*
$x = y = z = 0$	205	154	120	I
$\delta x = 20$	5	194	160	II
$\delta y = 19$	43	4	179	III
$\delta z = 18$	97	40	–1	IV
$\delta x = 10$	–3	60	19	V
$\delta y = 6$	9	0	25	VI
$\delta z = 2$	15	4	5	VII
$\delta x = 2$	–5	8	9	VIII
$\delta z = 1$	2	10	–1	IX
$\delta y = 1$	0	0	0	X

Hence

$$x = \sum \delta x = \mathbf{32}$$
$$y = \sum \delta y = \mathbf{26}$$
$$z = \sum \delta z = \mathbf{21}$$

Ans.

Explanation – In step I, the largest residual is 205. To reduce it, we give an increment $\delta x = 20$ and the resulting residuals are shown in step II of these $R_y = 194$ is the largest and we give an increment $\delta y = 19$ to get the results in step III similarly the other steps have been carried out.

OO

MODULE 3

NUMERICAL METHODS - 3

ORDINARY DIFFERENTIAL EQUATIONS – TAYLOR'S SERIES, EULER AND MODIFIED EULER'S METHODS, RUNGE-KUTTA METHOD OF FOURTH ORDER FOR SOLVING FIRST AND SECOND ORDER EQUATIONS, MILNE'S AND ADAM'S PREDICTOR-CORRECTOR METHODS

Solution of Ordinary Differential Equation – The important methods of solving ordinary differential equations of first order, numerically are as follows –

(i) Picard's method
(ii) Taylor's series method
(iii) Euler's method
(iv) Modified Euler's method
(v) Runge-Kutta method.

In case of ordinary differential equations of first order, one arbitrary constant quantity comes in the solution and this will be determined by the conditions given to us.

(i) Picard's Method – Consider the following differential equation

$$\frac{dy}{dx} = f(x, y) \qquad \text{....(i)}$$

Initial condition is that $y = y_0$ at $x = x_0$

Integrating the above equation between the limits x_0 and x, we get

$$\int_{y_0}^{y} dy = \int_{x_0}^{x} f(x, y)\, dx \quad \text{or} \quad y = y_0 + \int_{x_0}^{x} f(x, y)\, dx \qquad \text{....(ii)}$$

As first approximation y_1 to the solution , we put $y = y_0$ in f(x, y) and integrate, equation (ii) giving,

$$y_1 = y_0 + \int_{x_0}^{x} f(x, y_0)\, dx$$

For second approximation y_2, we put $y = y_1$ in f(x, y) and integrate equation (ii), we have

$$y_2 = y_0 + \int_{x_0}^{x} f(x, y_1)\, dx$$

Similarly, the third approximation is,

$$y_3 = y_0 + \int_{x_0}^{x} f(x, y_2)\, dx$$

$$\vdots \qquad \vdots$$

n^{th} approximation $\quad y_n = y_0 + \int_{x_0}^{x} f(x, y_{n-1})\, dx$

Hence, this method gives a sequence of approximations $y_1, y_2, \ldots, y_n$ each giving a better result than the preceding one.

As a matter of fact the process is stopped when the two values of y viz. y_{n-1} and y_n are sensible the same i.e., they are same to the desired degree of accuracy.

Picard's method is of considerable theoretical value in general, it is unsatisfactory as a practical mean of approximation because of difficulties which aries in performing the necessary integration.

(ii) Taylor's Series Method – We consider the differential equation

$$\frac{dy}{dx} = f(x, y),\ y(x_0) = y_0 \qquad \ldots(i)$$

Let, $\quad y = F(x)$ be a solution of equation (i) such that $F(x_0) \neq 0$.

Then expanding it by Taylor's series about the point x_0, we get

$$y = y_0 + (x - x_0)y_0' + \frac{(x - x_0)^2}{2!} y_0'' + \ldots.. \qquad \ldots(ii)$$

Putting, $\quad x = x_1 = x_0 + h$ in equation (ii), we get

$$y_1 = y_0 + hy_0' + \frac{h^2}{2!} y_0'' + \ldots \qquad \ldots(iii)$$

which is a convergent series in the range $x_0 \le x \le x_n$.

If the values of y_0', y_0'', ... are known, then equation (ii) gives a power series for y_1. The coefficients y_0', y_0'', ... can be found from equation (i). We can rewritten equation (iii) as

$$y' = f(x, y) = g_1(x, y), \text{ say} \qquad \ldots(iv)$$

Differentiating equation (iv) successively, we get

$$y'' = \frac{\partial g_1}{\partial x} + \frac{\partial g_1}{\partial y} y' = g_2(x, y, y'), \text{ say}$$

$$y''' = \frac{\partial g_2}{\partial x} + \frac{\partial g_2}{\partial y} y' + \frac{\partial g_2}{dy'} y'' = g_3(x, y, y', y''), \text{ say}$$

Proceeding in this way, we have

$$y^{(n)} = \frac{\partial g_{n-1}}{\partial x} + \frac{\partial g_{n-1}}{\partial y} y' + \frac{\partial g_{n-1}}{dy'} y'' + \ldots + \frac{\partial g_{n-1}}{\partial y^{(n-2)}} y^{(n-1)}$$

$$= g_n(x, y, y', \ldots, y^{(n-1)}), \text{ (say)}$$

Putting, $\quad x = x_0, y = y_0$ in the above result, we get

$$= y_0', y_0'', y_0''', \ldots, y_0^{(n)}$$

Practically this method is not of much importance because of its need of partial derivatives. It also suffers from the serious disadvantages that h should be small enough so that successive terms of the series diminish quite rapidly.

(iii) Euler's Method – It is one of the oldest and simplest methods but also the crudest.

Let the differential equation be

$$\frac{dy}{dx} = f(x, y),\ y(x_0) = y_0 \qquad \ldots(i)$$

Integrating equation (i), we get a relation between y and x which can be written in the form,

$$y = f(x) \qquad \ldots(ii)$$

In the xy-plane the equation (ii), represents a curve. Practically a smooth curve is straight for a short distance from any point on it . Hence, we have the approximate relation,

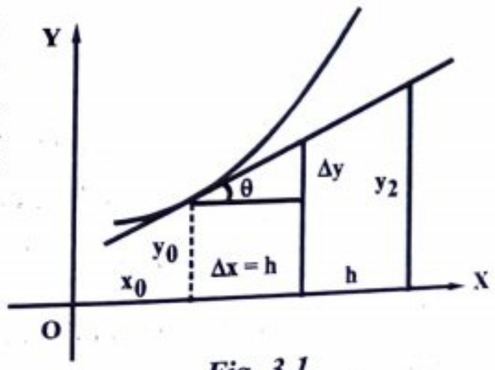

Fig. 3.1

$$\Delta y \approx \Delta x \tan\theta$$

$$= \Delta x \left(\frac{dy}{dx}\right)_0 = \Delta x . f(x_0, y_0)$$

$$\therefore \quad y_1 = y_0 + \Delta x . f(x_0, y_0)$$

or

$$y_1 = y_0 + h\, f(x_0, y_0)$$

This y_1 is the approximate value of y for $x = x_1 = x_0 + h$

Similarly, the values of y corresponding to

$x_2 = x_1 + h,\ x_3 = x_2 + h$, etc. are given by

$$y_2 = y_1 + h\, f(x_1, y_1)$$
$$y_3 = y_2 + h\, f(x_2, y_2) \text{ etc.}$$

In general, $y_{n+1} = y_n + h\, f(x_n, y_n), \quad n = 0, 1, 2, \ldots.$

Taking h small enough and corresponding in this way we could get the integral of equation (i) as a set of corresponding values of x and y.

(iv) Modified Euler's Method – In each step of this method, we first compute the auxiliary value

$$y^*_{n+1} = y_n + h\, f(x_n, y_n) \qquad \ldots(i)$$

and then the new value

$$y_{n+1} = y_n + \frac{h}{2}\left[f(x_n, y_n) + f(x_{n+1}, y^*_{n+1})\right] \qquad \ldots(ii)$$

In fact, we approximate the solution y by the straight line through (x_n, y_n) with slope $f(x_n, y_n)$ in the interval $\left(x_n, x_0 + \frac{h}{2}\right)$ and then we continue along the straight line with slope $f\left(x_{n+1}, y^*_{n+1}\right)$ untill x reaches x_{n+1}. (In the fig. 3.2, n = 0).

The modified method is a predictor method, because in each step we first predict a value by equation (i) and then correct it by equation (ii).

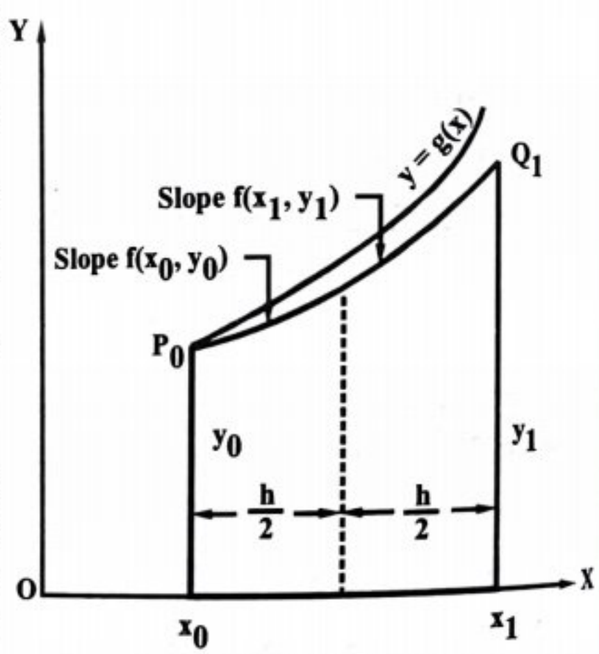

Fig. 3.2

(v) Runge-Kutta Method (Fourth Order) – It is one of the most widely used method. This method does not involve the derivatives of f(x, y). There is a set of methods called Runge-Kutta methods. But the most widely used method is Runge-Kutta of fourth order and so this method is generally given the name Runge-Kutta method.

We consider the differential equation,

$$\frac{dy}{dx} = f(x, y)$$

with the initial condition $\quad y(x_0) = y_0$

Let 'h' be the interval between equidistant values of x. Then the first increment in y is computed from the formula.

$$k_1 = h\, f(x_0, y_0),\ k_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right)$$

$$k_3 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right)$$

$$k_4 = hf(x_0 + h, y_0 + k_3)$$

$$\therefore \quad k = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) \qquad \ldots(i)$$

Then, $\quad x_1 = x_0 + h$ and $y_1 = y_0 + k$

In a similar manner, the increment in y for the second interval is computed by means of the formulae.

$$k_1 = h\, f(x_1, y_1),\ k_2 = hf\left(x_1 + \frac{h}{2}, y_1 + \frac{k_1}{2}\right)$$

$$k_3 = hf\left(x_1 + \frac{h}{2}, y_1 + \frac{k_2}{2}\right),\ k_4 = hf(x_1 + h, y_1 + k_3)$$

$$\therefore \quad k = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

$$\therefore \quad y_2 = y_1 + k$$

and similar for the next intervals.

Predictor Corrector Methods – In this method, to solve a differential equation over a single interval (x_i, x_i+1), we require, information only at the beginning of the interval at $x = x_i$. Predictor-corrector methods require function values at $x_i, x_{i-1}, x_{i-2}\ldots.$ for finding the function value at x_{i+1}. A predictor formula is applied to predict the value of y at x_{i+1} and the corrector formula is applied to improve the value of y_{i+1}. We now explain two such methods –

(i) Milne's Predictor-Corrector Method – Let us consider the first order differential equation

$$\frac{dy}{dx} = f(x, y) \qquad \ldots(i)$$

and $y = y_0$ when $x = x_0$ for finding an approximate value of y for $x = x_0 + nh$ by Milne's method, we proceed as follows –

As $y_0 = y(x_0)$ so $y_1 = y(x_0 + h)$, $y_2 = y(x_0 + 2h)$ and $y_3 = y(x_0 + 3h)$ by Picard's or Taylor's series method.

Now we calculate,

$$f_0 = f(x_0, y_0),\ f_1 = f(x_0 + h, y_1),$$
$$f_2 = f(x_0 + 2h, y_2) \text{ and } f_3 = f(x_0 + 3h, y_3)$$

For finding the value of $y_4 = y(x_1 + 4h)$, we now substitute Newton's forward interpolation formula.

$$f(x, y) = f_0 + n\Delta f_0 + \frac{n(n-1)}{2}\Delta^2 f_0 + \frac{n(n-1)(n-2)}{6}\Delta^3 f_0 + \ldots$$

in the relation $\quad y_4 = y_0 + \int_{x_0}^{x_0+4h} f(x,y)\,dy$

or

$$y_4 = y_0 + \int_{x_0}^{x_0+4h}\left(f_0 + n\Delta f_0 + \frac{n(n-1)}{2}\Delta^2 f_0 + \ldots\right)dx$$

[put $x = x_0 + nh$, then $dx = h\,dn$]

$$= y_0 + h\int_0^4\left(f_0 + n\Delta f_0 + \frac{n(n-1)}{2}\Delta^2 f_0 + \ldots\right)dn$$

$$= y_0 + h\left[4f_0 + 8\Delta f_0 + \frac{20}{3}\Delta^2 f_0 + \frac{8}{3}\Delta^3 f_0 + \ldots\right]$$

Now expressing the first, second and third differences in term of the function value while neglecting fourth and higher order derivatives, we get

$$y_4 = y_0 + \frac{4h}{3}(2f_1 - f_2 + 2f_3)$$ which is called a ***predictor***.

Having found y_4 we find a first approximation $f_4 = f(x_0 + 4h, y_4)$

By Simpson's rule, we find the better value of y_4 as

$$y_4 = y_2 + \frac{h}{3}(f_2 + 4f_3 + f_4)$$, which is called a ***corrector***.

Now an improved value of y_4 is computed and again-applying the corrector to find a still better value of y_4. This procedure is repeated until y_4 remains unchanged.

Once y_4 and f_4 is found to desired degree of accuracy, we find $y_5 = y(x_0 + 5h)$ by use of predictor as

$$y_5 = y_1 + \left\{\frac{4h}{3}\right\}(2f_2 - f_3 + 2f_4)$$

and $f_5 = f(x_0 + 5h, y_5)$ is calculated. Then a better approximation to value of y_5 is found by use of corrector as

$$y_5 = y_3 + \frac{h}{3}(f_3 + 4f_4 + f_5)$$

We continue this procedure till y_5 becomes stationary and we, then proceed to calculate y_6 as before.

In this method for greater accuracy of result, we must first improve the accuracy of the starting value and after sub-divide the interval.

(ii) Adam's Predictor-corrector Method –

Given, $\quad \frac{dy}{dx} = f(x, y)$ and $y_0 = y(x_0)$

Now we find the value of

$y_{-1} = y(x_0 - h, y_{-1})$, $y_{-2} = y(x_0 - 2h, y_{-2})$ and

$y_{-3} = y(x_0 - 3h, y_{-3})$ by Taylor's series or Euler's method or Runge-Kutta method.

Next we find

$$f_{-1} = f(x_0 - h_1, y_{-1}),\ f_{-2} = f(x_1 - 2h, y_{-2})$$

and $\quad f_{-3} = f(x_0 - 3h, y_{-3})$

We know that Newton's backward difference interpolation formula is

$$f(x,y) = f_0 + n\nabla f_0 + \frac{n(n+1)}{2}\nabla^2 f_0 + \frac{n(n+1)(n+2)}{6}\nabla^3 f_0 + \ldots \quad \ldots(i)$$

where $\quad n = \frac{x - x_0}{h}$ and $f_0 = f(x_1, y_0)$

Now, if this formula is substitute in the following equation

$$y_1 = y_0 + \int_{x_0}^{x_1} f(x,y)\,dx \quad \ldots(ii)$$

We obtain

$$y_1 = y_0 + \int_{x_0}^{x_1}\left[f_0 + n\nabla f_0 + \frac{n(n+1)}{2}\nabla^2 f_0 + \ldots\right]dx$$

[put $x = x_0 + nh$, then $dx = h\,dn$]

$$= y_0 + h\int_0^1\left[f_0 + n\nabla f_0 + \frac{n(n+1)}{2}\nabla^2 f_0 + \ldots\right]dn$$

$$= y_0 + h\left[1 + \frac{1}{2}\nabla + \frac{5}{12}\nabla^2 + \frac{3}{8}\nabla^3 + \frac{251}{720}\nabla^4 + \ldots\right]f_0$$

Expressing first, second and fourth differences in terms of the function values, we get

$$y_1 = y_0 + \frac{h}{24}(55f_0 - 59f_{-1} + 37f_{-2} - 9f_{-3})$$

This is called ***Adams-Bashforth predictor formula***.

Remark –

$$y_1 = y_0 + \frac{h}{24}(9f_1 + 19f_0 - 5f_{-1} + f_{-2})$$

is called ***Adams-Moulton corrector formula***.

Here $\quad [f_1 = f(x_0 + h, y_1)]$

Q.1. Write the working rule of Runge-Kutta method of fourth order for the numerical solution of differential equation. (R.G.P.V., June 2014)

Or

Write the steps of Runge-Kutta method. (R.G.P.V., Dec. 2014)

Ans. Refer to the matter given on page 128.

NUMERICAL PROBLEMS

Prob.1. If $\frac{dy}{dx} = f(x, y)$ with initial condition $y = y_0$ at $x = x_0$, then write first two approximations using Picard's method. (R.G.P.V., Dec. 2015)

Sol. Refer Picard's method, given on page 125.

Prob.2. Use Picard's method to approximate the value of y, when x = 0.1, 0.2 and 0.3, given that y = 1 at x = 0 and $\frac{dy}{dx} = 1 + xy$, correct to three decimal places. [R.G.P.V., May 2019 (IV-Sem)]

Sol. Let $f(x, y) = 1 + xy$, $x_0 = 0$, $y_0 = 1$.

Now, first approximation,

$$y_1 = y_0 + \int_{x_0}^{x} f(x, y_0)dx$$

$$= 1 + \int_0^x (1+x)dx = 1 + x + \frac{x^2}{2}$$

Second approximation,

$$y_2 = y_0 + \int_{x_0}^{x} f(x, y_1)dx$$

$$= 1 + \int_0^x \left[1 + x\left(1 + x + \frac{x^2}{2}\right)\right]dx$$

$$= 1 + \int_0^x \left(1 + x + x^2 + \frac{x^3}{2}\right)dx$$

$$= 1 + x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{8}$$

When x = 0.1, we get

$$y_1 = 1 + 0.1 + \frac{(0.1)^2}{2} = 1.105$$

$$y_2 = 1 + 0.1 + \frac{(0.1)^2}{2} + \frac{(0.1)^3}{3} + \frac{(0.1)^4}{8} = 1.1053$$

Hence y = **1.1053** at x = 0.1 **Ans.**

When x = 0.2, we get

$$y_1 = 1 + 0.2 + \frac{(0.2)^2}{2} = 1.22$$

$$y_2 = 1 + 0.2 + \frac{(0.2)^2}{2} + \frac{(0.2)^3}{3} + \frac{(0.2)^4}{8} = 1.2229$$

Hence y = **1.2229** at x = 0.2 **Ans.**

When x = 0.3, we get

$$y_1 = 1 + 0.3 + \frac{(0.3)^2}{2} = 1.345$$

$$y_2 = 1 + 0.3 + \frac{(0.3)^2}{2} + \frac{(0.3)^3}{3} + \frac{(0.3)^4}{8} = 1.355$$

Hence y = **1.355** at x = 0.3 **Ans.**

Prob.3. Using Picard's method to obtain y for x = 0.1, if $\frac{dy}{dx} = 1 + xy$, with y(0) = 1. (R.G.P.V., June 2015)

Sol. Refer to Prob.2.

Prob.4. Using Picard's method, find an approximate value of y when x = 0.1 if $\frac{dy}{dx} = x - y^2$ and y = 1 at x = 0. (R.G.P.V., Dec. 2012)

Sol. Let $f(x, y) = x - y^2$, $x_0 = 0$, $y_0 = 1$.

Now, first approximation,

$$y_1 = y_0 + \int_{x_0}^{x} f(x, y_0)dx = 1 + \int_0^x (x-1)dx = 1 + \frac{x^2}{2} - x$$

Second approximation,

$$y_2 = y_0 + \int_{x_0}^{x} f(x, y_1)dx = 1 + \int_0^x x - \left(1 - x + \frac{x^2}{2}\right)^2 dx$$

$$= 1 + \int_0^x \left(-1 + 3x - 2x^2 + x^3 - \frac{x^4}{4}\right)dx$$

$$= 1 - x + \frac{3x^2}{2} - \frac{2x^3}{3} + \frac{x^4}{4} - \frac{x^5}{20}$$

When x = 0.1, we get

$y_1 = 0.9050$, $y_2 = 0.9144$ **Ans.**

Hence, y = **0.9144**

Prob.5. Use Picard's method to approximate y when x = 0.1 given that y = 1, when x = 0 and $\frac{dy}{dx} = \frac{y-x}{y+x}$. ***(R.G.P.V., Dec. 2014)***

Or

Using Picard's method find y for x = 0.1. Given that

$$\frac{dy}{dx} = \frac{y-x}{y+x},\ y(0) = 1.$$

(R.G.P.V., Nov. 2018)

Sol. Let $f(x, y) = \frac{y-x}{y+x}$, $x_0 = 0$, $y_0 = 1$

Now first approximation,

$$y_1 = y_0 + \int_{x_0}^{x} f(x, y_0)dx$$

$$= 1 + \int_0^x \left(\frac{1-x}{1+x}\right)dx$$

$$= 1 + [2\log(1+x) - x]_0^x = 1 - x + 2\log(1+x)$$

Second approximation,

$$y_2 = y_0 + \int_{x_0}^{x} f(x, y_1)dx$$

$$= 1 + \int_0^x \frac{[1-x+2\log(1+x)]-x}{[1-x+2\log(1+x)]+x}dx$$

$$= 1 + \int_0^x \frac{1+2\log(1+x)-2x}{1+2\log(1+x)}dx$$

$$= 1 + \int_0^x \left[1 - \frac{2x}{1+2\log(1+x)}\right]dx$$

$$= 1 + x - 2\int_0^x \frac{x\,dx}{1+2\log(1+x)}$$

$$= 1 + x - 2\int_0^t \frac{e^t(e^t-1)}{1+2t}dt \quad [\text{where } t = \log(1+x)]$$

$$= 1 + x - 2\int_0^t \frac{e^{2t}}{1+2t}dt + 2\int_0^t \frac{e^t}{1+2t}dt$$

which is not easily integrable.

Therefore, we can get the value of y only by first approximation.

Hence at x = 0.1,

$$y = 1 - 0.1 + 2\log(1 + 0.1)$$

$$= 0.9 + 2\log(1.1) = \mathbf{1.09062}$$

Ans.

Prob.6. Perform two iterations of Picard's method to find an approximate solution of the initial value problem

$$\frac{dy}{dx} = x + y^2;\ y(0) = 1.$$

(R.G.P.V., June 2008, Dec. 2011)

Sol. Here $\frac{dy}{dx} = x + y^2$ ...(i)

$\because$ $y(0) = 1$

Integrating equation (i), with respect to x, between 0 and x, we have

$$\int_0^y dy = \int_0^x (x + y^2)dx$$

or

$$y = y_0 + \int_0^x (x + y^2)dx \quad \text{...(ii)}$$

On putting y(0) = 1, for y on R.H.S. of equation (ii), we have

$$y_1 = 1 + \int_0^x (x+1)\,dx = 1 + \frac{x^2}{2} + x$$

On substituting the value of y_1 for y on R.H.S. of equation (ii), we get

$$y_2 = 1 + \int_0^x \left[x + \left(1 + x + \frac{x^2}{2}\right)^2\right]dx$$

$$y_2 = 1 + \int_0^x \left(x + 1 + x^2 + \frac{x^4}{4} + 2x + x^2 + x^3\right)dx$$

or

$$y_2 = 1 + \int_0^x \left(1 + 3x + 2x^2 + x^3 + \frac{x^4}{4}\right)dx$$

or

$$y_2 = \mathbf{1 + x + \frac{3x^2}{2} + \frac{2x^3}{3} + \frac{x^4}{4} + \frac{x^5}{20}}$$

Ans.

Prob.7. Using Picard's process of successive approximation, obtain a solution upto the fifth approximation of the equation $\frac{dy}{dx} = y + x$, ***such that y = 1 when x = 0.*** ***(R.G.P.V., June 2012, Dec. 2016)***

Sol. Here, we are given that

$$\frac{dy}{dx} = y + x \Rightarrow f(x, y) = x + y,\ x_0 = 0,\ y_0 = 1$$

Now, first approximation,

$$y_1 = y_0 + \int_{x_0}^{x} f(x, y_0)\,dx = 1 + \int_0^x (x + y_0)\,dx$$

$$= 1 + \int_0^x (x+1)\,dx = \frac{x^2}{2} + x + 1$$

Second approximation,

$$y_2 = y_0 + \int_{x_0}^{x} f(x, y_1)\,dx$$

$$= 1 + \int_0^x \left[x + \left(1 + x + \frac{x^2}{2}\right)\right] dx = 1 + \int_0^x \left(\frac{x^2}{2} + 2x + 1\right) dx$$

$$= 1 + \left[\frac{x^3}{6} + x^2 + x\right]_0^x = \frac{x^3}{6} + x^2 + x + 1$$

Third approximation,

$$y_3 = y_0 + \int_{x_0}^{x} f(x, y_2)\,dx$$

$$= 1 + \int_0^x \left[x + \left(\frac{x^3}{6} + x^2 + x + 1\right)\right] dx$$

$$= 1 + \int_0^x \left(\frac{x^3}{6} + x^2 + 2x + 1\right) dx = \frac{x^4}{24} + \frac{x^3}{3} + x^2 + x + 1$$

Fourth approximation,

$$y_4 = y_0 + \int_{x_0}^{x} f(x, y_3)\,dx$$

$$= 1 + \int_0^x \left(x + \frac{x^4}{24} + \frac{x^3}{3} + x^2 + x + 1\right) dx$$

$$= \frac{x^5}{120} + \frac{x^4}{12} + \frac{x^3}{3} + x^2 + x + 1$$

Fifth approximation,

$$y_5 = y_0 + \int_{x_0}^{x} f(x, y_4)\,dx$$

$$= 1 + \int_0^x \left(x + \frac{x^5}{120} + \frac{x^4}{12} + \frac{x^3}{3} + x^2 + x + 1\right) dx$$

$$= 1 + \int_0^x \left(\frac{x^5}{120} + \frac{x^4}{12} + \frac{x^3}{3} + x^2 + 2x + 1\right) dx$$

$$= 1 + \frac{x^6}{720} + \frac{x^5}{60} + \frac{x^4}{12} + \frac{x^3}{3} + x^2 + x$$

or $$y_5 = \frac{x^6}{720} + \frac{x^5}{60} + \frac{x^4}{12} + \frac{x^3}{3} + x^2 + x + 1$$ **Ans.**

Prob.8. Find by Taylor's series method the value y(0.1) correct to three decimal places from the differential equation –

$$\frac{dy}{dx} = x^2 y - y, \quad y(0) = 1$$

(R.G.P.V., June 2014)

Sol. Given that $\frac{dy}{dx} = x^2y - y$

or $$y' = x^2y - y \qquad ...(i)$$

Also $$x_0 = 0,\ y_0 = 1$$

$\therefore$ $$y_0' = 0 - 1 = -1$$

Differentiating equation (i) successively, we get

$$y'' = 2xy + x^2y' - y', \quad y_0'' = 0 + 0 - (-1) = 1$$

$$y''' = 2y + 2xy' + 2xy' + x^2y'' - y''$$

$$= 2y + 4xy' + x^2y'' - y''$$

$$y_0''' = 2 \times 1 + 0 + 0 - 1 = 1$$

$$y^{iv} = 2y' + 4y' + 4xy'' + 2xy'' + x^2y''' - y'''$$

$$= 6y' + 6xy'' + x^2y''' - y'''$$

$$y_0^{iv} = 6 \times (-1) - 1 = -6 - 1 = -7 \text{ and so on}$$

Now by the Taylor's series, we have

$$y(x) = y_0 + (x - x_0)y_0' + \frac{(x - x_0)^2}{2!}y_0'' + \frac{(x - x_0)^3}{3!}y_0''' + ...$$

or $$y(x) = 1 + x(-1) + \frac{x^2}{2!}.1 + \frac{x^3}{3!}.1 + \frac{x^4}{4!}(-7) + ...$$

$$y(x) = 1 - x + \frac{x^2}{2!} + \frac{x^3}{3!} - \frac{7}{4!}x^4 + ...$$

Hence $$y(0.1) = 1 - 0.1 + \frac{(0.1)^2}{2!} + \frac{(0.1)^3}{3!} - \frac{7}{4!}(0.1)^4 + ...$$

$$= 1 - 0.1 + 0.005 + 0.0001667 - 0.00002917 + ...$$

$$= \mathbf{0.905}$$ **Ans.**

Prob.9. Solve the given equation for y(1.1) using Taylor series method –

$$\frac{dy}{dx} = 2y + 3e^x, \quad y(0) = 0.$$

(R.G.P.V., Nov. 2018)

Sol. Given that $\frac{dy}{dx} = 2y + 3e^x$

or $$y' = 2y + 3e^x \qquad ...(i)$$

Also $x_0 = 0,\ y_0 = 0$

$\therefore \quad y'_0 = 2y_0 + 3e^0 = 0 + 3.1 = 3$

Differentiating equation (i) successively, we get

$$y'' = 2y' + 3e^x,\ y''_0 = 2y'_0 + 3e^0 = 2\times 3 + 3\times 1 = 9$$

$$y''' = 2y'' + 3e^x,\ y'''_0 = 2y''_0 + 3e^0 = 2\times 9 + 3\times 1 = 21$$

$$y^{iv} = 2y''' + 3e^x,\ y^{iv}_0 = 2y'''_0 + 3e^0 = 2\times 21 + 3\times 1 = 45$$

and so on.

Now by the Taylor's series, we have

$$y(x) = y_0 + (x - x_0)y'_0 + \frac{(x-x_0)^2}{2!}y''_0 + \frac{(x-x_0)^3}{3!}y'''_0 + \ldots..$$

or $$y(x) = 0 + x(3) + \frac{x^2}{2!}(9) + \frac{x^3}{3!}(21) + \frac{x^4}{4!}(45) + \ldots..$$

or $$y(x) = 3x + \frac{9}{2!}x^2 + \frac{21}{3!}x^3 + \frac{45}{4!}x^4 + \ldots..$$

Hence $$y(1.1) = 3(1.1) + \frac{9}{2!}(1.1)^2 + \frac{21}{3!}(1.1)^3 + \frac{45}{4!}(1.1)^4 + \ldots..$$

$$= 3.3 + 5.445 + 4.6585 + 2.7451875 + \ldots..$$

$$= \mathbf{16.1487\ (approx.)}$$ **Ans.**

Prob.10. Solve $\frac{dy}{dx} = 1 - 2xy$ given that $y(0) = 0$, by Taylor's method.

(R.G.P.V., June 2016)

Or

Using Taylor's series method, solve $\frac{dy}{dx} = 1 - 2xy$ given that $y(0) = 0$.

(R.G.P.V., June 2017)

Or

Solve $\frac{dy}{dx} = 1 - 2xy$, given that $y(0) = 0$, by Taylor's series method.

[R.G.P.V., Nov. 2019 (IV-Sem)]

Sol. Given differential equation is

$$\frac{dy}{dx} = 1 - 2xy$$

or $$y' = 1 - 2xy \text{ and } x_0 = 0,\ y_0 = 0.$$

We have by Taylor series

$$y(x) = y_0 + (x - x_0)\,y'_0 + \frac{(x-x_0)^2}{2!}y''_0 + \ldots \qquad \ldots(i)$$

Now $$y'(x) = 1 - 2xy,\ y'_0 = 1 - 2x_0y_0 = 1$$

$\therefore$ $$y''(x) = -2y - 2xy',\ y''_0 = -2y_0 - 2x_0y'_0 = 0$$

$$y'''(x) = -2y' - 2xy'' - 2y' = -4y' - 2xy'',$$

$$y'''_0 = -4y'_0 - 2x_0y''_0 = -4$$

$$y^{(iv)}(x) = -4y'' - 2xy''' - 2y'' = -6y'' - 2xy''',\ y^{iv}_0 = 0$$

$$y^{(v)}(x) = -6y''' - 2xy^{iv} - 2y''' = -8y''' - 2xy^{iv},$$

$$y^{(v)}_0 = 32 \quad \text{and so on}$$

Hence $$y(x) = 0 + x.1 + 0 + \frac{x^3}{3!}(-4) + 0 + \frac{x^5}{5!}(32) + \ldots$$

$$= \mathbf{x - \frac{2x^3}{3} + \frac{4}{15}x^5}$$ **Ans.**

Prob.11. Find by Taylor's series method, the values of y at $x = 0.1$ and $x = 0.2$ to five places of decimal from $\frac{dy}{dx} = x^2y - 1$, $y(0) = 1$. (R.G.P.V., Dec. 2013)

Sol. Given that $$\frac{dy}{dx} = x^2y - 1$$

or $$y' = x^2y - 1 \qquad \ldots(i)$$

Also, $x_0 = 0,\ y_0 = 1$

$\therefore \quad y'_0 = 0 - 1 = -1$

Differentiating equation (i) successively, we get

$$y'' = 2xy + x^2y',\ y''_0 = 0$$

$$y''' = 2y + 2xy' + 2xy' + x^2y'' = 2y + 4xy' + x^2y''$$

$$y'''_0 = 2\times 1 + 0 = 2$$

$$y^{iv} = 2y' + 4y' + 4xy'' + 2xy'' + x^2y'''$$

$$= 6y' + 6xy'' + x^2y'''$$

$$y^{iv}_0 = 6\times -1 + 0 = -6 \text{ and so on}$$

Now by the Taylor's series, we have

$$y(x) = y_0 + (x - x_0)y'_0 + \frac{(x-x_0)^2}{2!}y''_0 + \frac{(x-x_0)^3}{3!}y'''_0 + \ldots.$$

$$y(x) = 1 + x.(-1) + \frac{x^2}{2!}.(0) + \frac{x^3}{3!}.2 + \frac{x^4}{4!}(-6) + \ldots.$$

$$y(x) = 1 - x + \frac{x^3}{3} - \frac{x^4}{4} + \ldots.$$

Hence $$y(0.1) = 1 - (0.1) + \frac{(0.1)^3}{3} - \frac{(0.1)^4}{4} + \ldots.$$

$$= 1 - 0.1 + 0.000333 - 0.000025 + \ldots..$$

$$= \mathbf{0.90031}$$ **Ans.**

and $$y(0.2) = 1-(0.2)+\frac{(0.2)^3}{3}-\frac{(0.2)^4}{4}+\ldots$$
$$= 1 - 0.2 + 0.002667 - 0.0004 + \ldots$$
$$= \mathbf{0.80227}$$

Ans.

Prob.12. Using Euler's method, compute y(0.04) for the differential equation $\frac{dy}{dx} = -y$, y(0) = 1. Take h = 0.01. [R.G.P.V., May 2019 (III-Sem)]

Sol. Here, $f(x, y) = -y$, $x_0 = 0$, $y_0 = 1$ and $h = 0.01$.

Also $x_1 = x_0 + h = 0 + 0.01 = 0.01$

$x_2 = x_1 + h = 0.01 + 0.01 = 0.02$

$x_3 = x_2 + h = 0.02 + 0.01 = 0.03$

and $x_4 = x_3 + h = 0.03 + 0.01 = 0.04$

Using Euler's formula, we get

$$y_1 = y_0 + hf(x_0, y_0)$$
$$= y_0 + h(-y_0)$$
$$= 1 + 0.01(-1) = 1 - 0.01 = 0.99$$
$$y_2 = y_1 + hf(x_1, y_1)$$
$$= y_1 + h(-y_1)$$
$$= 0.99 + 0.01(-0.99)$$
$$= 0.99 - 0.0099 = 0.9801$$
$$y_3 = y_2 + hf(x_2, y_2)$$
$$= y_2 + h(-y_2)$$
$$= 0.9801 + 0.01(-0.9801) = 0.9703$$
$$y_4 = y_3 + hf(x_3, y_3)$$
$$= y_3 + h(-y_3)$$
$$= 0.9703 + 0.01(-0.9703) = 0.9606$$

Hence $y(0.04) = \mathbf{0.9606}$

Ans.

Prob.13. Use Euler's method to find y(0.4) from the differential equation $\frac{dy}{dx} = xy$, y(0) = 1, where step size is h = 0.1.

[R.G.P.V., Nov. 2019 (III-Sem)]

Sol. Here $f(x, y) = xy$, $x_0 = 0$, $y_0 = 1$ and $h = 0.1$.

Also $x_1 = x_0 + h = 0 + 0.1 = 0.1$

$x_2 = x_1 + h = 0.1 + 0.1 = 0.2$

$x_3 = x_2 + h = 0.2 + 0.1 = 0.3$

and $x_4 = x_3 + h = 0.3 + 0.1 = 0.4$

Using Euler's formula, we get

$$y_1 = y_0 + hf(x_0, y_0)$$
$$= y_0 + h\,(x_0 y_0)$$
$$= 1 + 0.1\,(0 \times 1) = 1$$
$$y_2 = y_1 + h\,f(x_1, y_1)$$
$$= y_1 + h\,(x_1\, y_1)$$
$$= 1 + 0.1\,(0.1 \times 1)$$
$$= 1 + 0.01 = 1.01$$
$$y_3 = y_2 + hf(x_2, y_2)$$
$$= y_2 + h\,(x_2\, y_2)$$
$$= 1.01 + 0.1\,(0.2 \times 1.01)$$
$$= 1.01 + 0.0202 = 1.0302$$
$$y_4 = y_3 + hf(x_3, y_3) = y_3 + h\,(x_3\, y_3)$$
$$= 1.0302 + 0.1\,(0.3 \times 1.0302)$$
$$= 1.0302 + 0.030906 = 1.061106$$

Hence $y(0.4) = \mathbf{1.061106}$

Ans.

Prob.14. Use Euler's method to find y(0.2) from the differential equation $\frac{dy}{dx} = xy$, y(0) = 1 take h = 0.1. ***(R.G.P.V., Dec. 2015)***

Sol. Here $f(x, y) = xy$, $x_0 = 0$, $y_0 = 1$ and $h = 0.1$.

Also $x_1 = x_0 + h = 0 + 0.1 = 0.1$

and $x_2 = x_1 + h = 0.1 + 0.1 = 0.2$

Using Euler's formula, we get

$$y_1 = y_0 + hf(x_0, y_0)$$
$$= y_0 + h(x_0\, y_0)$$
$$= 1 + 0.1\,(0 \times 1)$$
$$= 1 + 0 = 1$$
$$y_2 = y_1 + hf(x_1, y_1)$$
$$= y_1 + h(x_1 y_1)$$
$$= 1 + 0.1\,(0.1 \times 1)$$
$$= 1.01$$

Hence $y(0.2) = \mathbf{1.01}$

Ans.

Prob.15. Find y(2.2) using Euler's method for the equation $\frac{dy}{dx} = -xy^2$ with y(2) = 1. **(R.G.P.V., June 2011, 2016)**

Sol. Here $f(x, y) = -xy^2$, $x_0 = 2$, $y_0 = 1$.

Taking h = 0.1

Also $x_1 = x_0 + h = 2 + 0.1 = 2.1$

and $x_2 = x_1 + h = 2.1 + 0.1 = 2.2$

Using Euler's formula, we get

$$y_1 = y_0 + hf(x_0, y_0)$$

$$= y_0 + h(-x_0 y_0^2)$$

$$= 1 + 0.1(-2 \times 1^2)$$

$$= 1 + 0.1(-2) = 1 - 0.2 = 0.8$$

$$y_2 = y_1 + hf(x_1, y_1)$$

$$= y_1 + h(-x_1 y_1^2)$$

$$= 0.8 + 0.1(-2.1 \times 0.8^2)$$

$$= 0.8 + 0.1(-1.344)$$

$$= 0.8 - 0.1344 = 0.6656$$

Hence y (2.2) = **0.6656** **Ans.**

Prob.16. Using modified Euler's method, find an approximate value of y, when x = 0.3, given that

$$\frac{dy}{dx} = x + y$$

and y = 1, when x = 0.

[R.G.P.V., Nov./Dec. 2007, June 2010, May 2019 (IV-Sem)]

Or

Using modified Euler's method, find the value of y(0.3) from the equation –

$$\frac{dy}{dx} = x + y, \quad y(0) = 1$$

(R.G.P.V., June 2014)

Sol. Taking h = 0.1, the various calculations are arranged as follows –

x	$x + y = y'$	***Mean Slop***	***Old y + 0.1 (Mean Slop) = New y***
0.0	0 + 1	–	1.00 + 0.1(1.00) = 1.10
0.1	0.1 + 1.1	$\frac{1}{2}(1 + 1.2)$	1.00 + 0.1(1.1) = 1.11
0.1	0.1 + 1.11	$\frac{1}{2}(1 + 1.21)$	1.00 + 0.1(1.105) = 1.1105
0.1	0.1 + 1.1105	$\frac{1}{2}(1 + 1.2105)$	1.00 + 0.1(1.1052) = 1.1105
0.1	1.2105	–	1.1105 + 0.1(1.2105) = 1.2316
0.2	0.2 + 1.2316	$\frac{1}{2}(1.2105 + 1.4316)$	1.1105 + 0.1(1.3210) = 1.2426
0.2	0.2 + 1.2426	$\frac{1}{2}(1.2105 + 1.4426)$	1.1105 + 0.1(1.3266) = 1.2432
0.2	0.2 + 1.2432	$\frac{1}{2}(1.2105 + 1.4432)$	1.1105 + 0.1(1.3268) = 1.2432
0.2	1.4432	–	1.2432 + 0.1(1.4432) = 1.3875
0.3	0.3 + 1.3875	$\frac{1}{2}(1.4432 + 1.6875)$	1.2432 + 0.1(1.5654) = 1.3997
0.3	0.3 + 1.3997	$\frac{1}{2}(1.4432 + 1.6997)$	1.2432 + 0.1(1.5715) = 1.4003
0.3	0.3 + 1.4003	$\frac{1}{2}(1.4432 + 1.7003)$	1.2432 + 0.1(1.5718) = 1.4004
0.3	0.3 + 1.4004	$\frac{1}{2}(1.4432 + 1.7004)$	1.2432 + 0.1(1.5718) = 1.4004

Hence, y(0.3) = **1.4004** approximately. **Ans.**

Prob.17. Given $\frac{dy}{dx} = 1 + \frac{y}{x}$, y = 2 at x = 1. Find approximate value of y at x = 1.4 by taking step size h = 0.2, apply modified Euler's method. **(R.G.P.V., Dec. 2014)**

Sol. Here $f(x, y) = 1 + \frac{y}{x}$, $x_0 = 1$, $y_0 = 2$ and h = 0.2.

Since $x_{n+1} = x_n + h \Rightarrow x_1 = 1.2$, $x_2 = 1.4$.

Euler's modified formula is

$$y_{n+1} = y_n + \frac{h}{2}[f(x_n, y_n) + f(x_{n+1}, y^*_{n+1})] \quad \text{...(i)}$$

where $y^*_{n+1} = y_n + hf(x_n, y_n)$

From equation (i), for n = 0, we have

$$y_1^* = y_0 + hf(x_0, y_0)$$

$$= 2 + (0.2)\left(1 + \frac{2}{1}\right) = 2 + 0.2 \times 3 = 2.6$$

and

$$y_1 = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^*)\right]$$

$$= 2 + \frac{0.2}{2}\left[\left(1 + \frac{2}{1}\right) + \left(1 + \frac{2.6}{1.2}\right)\right]$$

$$= 2 + 0.1(3 + 3.167) = 2.6167$$

From equation (i), for n = 1, we have

$$y_2^* = y_1 + hf(x_1, y_1)$$

$$= 2.6167 + 0.2\left(1 + \frac{2.6167}{1.2}\right) = 2.6167 + 0.6361 = 3.2528$$

and

$$y_2 = y_1 + \frac{h}{2}\left[f(x_1, y_1) + f(x_2, y_2^*)\right]$$

$$= 2.6167 + \frac{0.2}{2}\left[\left(1 + \frac{2.6167}{1.2}\right) + \left(1 + \frac{3.2528}{1.4}\right)\right]$$

$$= 2.6167 + 0.6504 = 3.2671$$

Hence y(1.4) = **3.2671 approximately** **Ans.**

Prob.18. Given that $\frac{dy}{dx} = \log_{10}(x + y)$, y(0) = 1, find y(0.2) using modified Euler's method. ***[R.G.P.V., May 2019 (III-Sem)]***

Sol. Here $f(x, y) = \log_{10}(x + y)$, $x_0 = 0$, $y_0 = 1$.

Taking h = 0.1

Since $x_{n+1} = x_n + h \Rightarrow x_1 = 0.1$, $x_2 = 0.2$.

Euler's modified formula is

$$y_{n+1} = y_n + \frac{h}{2}[f(x_n, y_n) + f(x_{n+1}, y_{n+1}^*)] \quad ...(i)$$

where $y_{n+1}^* = y_n + hf(x_n, y_n)$

From equation (i), for n = 0, we have

$$y_1^* = y_0 + hf(x_0, y_0)$$

$$= 1 + (0.1)\log_{10}(0 + 1) = 1$$

and

$$y_1 = y_0 + \frac{h}{2}[f(x_0, y_0) + f(x_1, y_1^*)]$$

$$= 1 + \frac{0.1}{2}[\log_{10}(0+1) + \log_{10}(0.1+1)] = 1.0021$$

From equation (i), for n = 1, we have

$$y_2^* = y_1 + hf(x_1, y_1)$$

$$= 1.0021 + (0.1)[\log_{10}(0.1 + 1.0021)] = 1.0063$$

and

$$y_2 = y_1 + \frac{h}{2}[f(x_1, y_1) + f(x_2, y_2^*)]$$

$$= 1.0021 + \frac{0.1}{2}[\log_{10}(0.1+1.0021) + \log_{10}(0.2+1.0063)]$$

$$= 1.0083$$

Hence y(0.2) = **1.0083 approximately** **Ans.**

Prob.19. Use Euler's modified form to obtain y(0.2), y(0.4) and y(0.6) correct to three decimal places, given that $\frac{dy}{dx} = y - x^2$ with initial condition y(0) = 1. ***(R.G.P.V., June 2013)***

Sol. Here $f(x, y) = y - x^2$, $x_0 = 0$, $y_0 = 1$ and h = 0.2 and $x_0 = 0$, $x_1 = 0.2$, $x_2 = 0.4$, $x_3 = 0.6$.

By Euler's modified formula

$$y_{n+1} = y_n + \frac{h}{2}\left[f(x_n, y_n) + f(x_{n+1}, y_{n+1}^*)\right] \quad ...(i)$$

where $y_{n+1}^* = y_n + hf(x_n, y_n)$

From equation (i), for n = 0, we have

$$y_1^* = y_0 + 0.2(y_0 - x_0^2) = 1 + 0.2(1 - 0) = 1.2$$

and

$$y_1 = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^*)\right]$$

$$= 1 + \frac{0.2}{2}[(1 - 0) + (1.2 - 0.04)] = 1.216$$

i.e., y(0.2) = **1.216** **Ans.**

From equation (i), for n = 1, we have

$$y_2^* = y_1 + hf(x_1, y_1)$$

$$= 1.216 + 0.2(1.216 - 0.04) = 1.4512$$

and

$$y_2 = y_1 + \frac{h}{2}\left[f(x_1, y_1) + f(x_2, y_2^*)\right]$$

$$= 1.216 + \frac{0.2}{2}[(1.216 - 0.04) + (1.4512 - 0.16)] = 1.4627$$

i.e., y(0.4) = **1.4627** **Ans.**

From equation (i), for n = 2, we have

$$y_3^* = y_2 + hf(x_2, y_2)$$

$$= 1.4627 + 0.2(1.4627 - 0.16) = 1.7232$$

and $\quad y_3 = y_2 + \frac{h}{2}\left[f(x_2, y_2) + f(x_3, y_3^*)\right]$

$$= 1.4627 + \frac{0.2}{2}\left[(1.4627 - 0.16) + (1.7232 - 0.36)\right] = 1.7293$$

i.e., $y(0.6) = \mathbf{1.7293}$ **Ans.**

Prob.20. Apply Runge-Kutta method to find an approximate value of y when x = 0.2, given that $\frac{dy}{dx} = x + y$ and y = 1 when x = 0.

(R.G.P.V., June 2010, Dec. 2013)

Sol. We have $f(x, y) = x + y$,

$x_0 = 0$, $y_0 = 1$ and $h = 0.2$

Then $\quad k_1 = h\, f(x_0, y_0)$

$= 0.2(x_0 + y_0) = 0.2(0 + 1) = 0.2$

$$k_2 = h\, f\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right) = 0.2\left(0 + \frac{0.2}{2} + 1 + \frac{0.2}{2}\right)$$

$= 0.2(1.2) = 0.24$

$$k_3 = h\, f\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right) = 0.2\left(0 + \frac{0.2}{2} + 1 + \frac{0.24}{2}\right)$$

$= 0.2(0.1 + 1 + 0.12) = 0.2(1.22) = 0.244$

$k_4 = h\, f(x_0 + h, y_0 + k_3)$

$= 0.2(0 + 0.2 + 1 + 0.244) = 0.2(1.444) = 0.289$

Hence $\quad k = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$

$= \frac{1}{6}\{0.2 + (2 \times 0.24) + (2 \times 0.244) + 0.289\}$

$= \frac{1}{6}\{0.2 + 0.48 + 0.488 + 0.289\} = \frac{1}{6}\{1.457\} = 0.243$

Finally $\quad y(0.2) = y_1 = y_0 + k$

$= 1 + 0.243 = \mathbf{1.243}$ **Ans.**

Prob.21. Use Runge-Kutta method to approximate y, when x = 0.1, given that x = 0, when y = 1 and $\frac{dy}{dx} = x + y$. ***[R.G.P.V., Nov. 2019 (III-Sem)]***

Sol. We have $f(x, y) = x + y$,

$x_0 = 0$, $y_0 = 1$ and $h = 0.1$

Then $\quad k_1 = h\, f(x_0, y_0)$

$= 0.1\,(x_0 + y_0) = 0.1\,(0 + 1) = 0.1$

$$k_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right)$$

$$= 0.1f\left(0 + \frac{0.1}{2}, 1 + \frac{0.1}{2}\right) = 0.1\ f(0.05, 1.05)$$

$= 0.1\,(0.05 + 1.05) = 0.1 \times 1.1 = 0.11$

$$k_3 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right)$$

$$= 0.1f\left(0 + \frac{0.1}{2}, 1 + \frac{0.11}{2}\right)$$

$= 0.1\ f(0.05, 1.055)$

$= 0.1\,(0.05 + 1.055)$

$= 0.1 \times 1.105 = 0.1105$

$k_4 = hf(x_0 + h, y_0 + k_3)$

$= 0.1\ f(0 + 0.1, 1 + 0.1105)$

$= 0.1\ f(0.1, 1.1105)$

$= 0.1\,(0.1 + 1.1105)$

$= 0.1 \times 1.2105 = 0.12105$

Hence $\quad k = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$

$$= \frac{1}{6}(0.1 + 2 \times 0.11 + 2 \times 0.1105 + 0.12105)$$

$$= \frac{1}{6}(0.1 + 0.22 + 0.221 + 0.12105)$$

$$= \frac{1}{6} \times 0.66205 = 0.11034$$

Hence the value of y at x = 0.1

$y = y_0 + K$

$= 1 + 0.11034 = \mathbf{1.11034}$ **Ans.**

Prob.22. Use Runge-Kutta method of fourth order to solve –

$$\frac{dy}{dx} = xy \text{ for } x = 1.2$$

Initially x = 1, y = 2 (take h = 0.1) ***(R.G.P.V., June 2017)***

Or

Using Runge-Kutta method of fourth order solve the differential equation

$$\frac{dy}{dx} = xy \text{ for } x = 1.2$$

Given that y(1) = 2 (take h = 0.1) ***(R.G.P.V., Nov. 2018)***

Sol. Given $f(x, y) = xy$

$$x_0 = 1,\ y_0 = 2,\ h = 0.1$$

Now $\quad k_1 = hf(x_0, y_0)$

$$= (0.1)\ f(1, 2)$$

$$= (0.1)\ (1 \times 2) = 0.2$$

$$k_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right)$$

$$= (0.1) f\left(1 + \frac{0.1}{2}, 2 + \frac{0.2}{2}\right) = (0.1)\ f\ (1.05, 2.1)$$

$$= 0.1\ (1.05 \times 2.1) = 0.1\ (2.205) = 0.2205$$

$$k_3 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right)$$

$$= (0.1) f\left(1 + \frac{0.1}{2}, 2 + \frac{0.2205}{2}\right) = (0.1)\ f\ (1.05, 2.11025)$$

$$= 0.1\ (1.05 \times 2.11025) = 0.221576$$

$$k_4 = h\ f\ (x_0 + h, y_0 + k_3)$$

$$= (0.1)\ f\ (1 + 0.1, 2 + 0.221576) = (0.1)\ f\ (1.1, 2.221576)$$

$$= 0.1\ (1.1 \times 2.221576) = 0.24437$$

and finally $\quad k = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$

$$= \frac{1}{6}(0.2 + 2 \times 0.2205 + 2 \times 0.221576 + 0.24437) = 0.22142$$

$\therefore \quad y\ (1.1) = y_1 = y_0 + k$

$$= 2 + 0.22142 = 2.22142$$

Next Step –

$$x_1 = x_0 + h = 1 + 0.1 = 1.1$$

$$y_1 = 2.22142,\ h = 0.1$$

Now

$$k_1 = hf\ (x_1, y_1)$$

$$= (0.1)\ f\ (1.1, 2.22142)$$

$$= (0.1)\ (1.1 \times 2.22142) = 0.24436$$

$$k_2 = hf\left(x_1 + \frac{h}{2}, y_1 + \frac{k_1}{2}\right)$$

$$= (0.1)\ f\left(1.1 + \frac{0.1}{2}, 2.22142 + \frac{0.24436}{2}\right)$$

$$= (0.1)\ f\ (1.15, 2.3436) = (0.1)\ (1.15 \times 2.3436) = 0.26951$$

$$k_3 = hf\left(x_1 + \frac{h}{2}, y_1 + \frac{k_2}{2}\right)$$

$$= (0.1) f\left(1.1 + \frac{0.1}{2}, 2.22142 + \frac{0.26951}{2}\right)$$

$$= (0.1)\ f(1.15, 2.3562) = 0.1\ (1.15 \times 2.3562) = 0.27096$$

$$k_4 = h\ f(x_1 + h, y_1 + k_3)$$

$$= (0.1)\ f(1.1 + 0.1, 2.22142 + 0.27096)$$

$$= (0.1)\ f(1.2, 2.49238)$$

$$= 0.1\ (1.2 \times 2.49238) = 0.2991$$

Thus $\quad k = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$

$$= \frac{1}{6}(0.24436 + 2 \times 0.26951 + 2 \times 0.27096 + 0.2991) = 0.27073$$

So that,

$$y\ (1.2) = y_2 = y_1 + k$$

$$= 2.22142 + 0.27073 = \mathbf{2.49215} \qquad \textbf{Ans.}$$

Prob.23. Use Runge-Kutta method to approximate the value of y when $x = 0.1$ given that $y(0) = 1$ and $\frac{dy}{dx} = 3x + y^2$. ***(R.G.P.V., Dec. 2008)***

Or

Use Runge-Kutta method to fourth order to approximate y, when $x = 0.1$, given that $y = 1$ at $x = 0$ and $\frac{dy}{dx} = 3x + y^2$. ***(R.G.P.V., Dec. 2016)***

Sol. Here $x_0 = 0$, $y_0 = 1$ and $f(x_0, y_0) = 1$.

Let $h = 0.1$. Then

$$k_1 = hf(x_0, y_0) = 0.1 \times 1 = 0.1$$

$$k_2 = hf\left(x_0 + \frac{1}{2}h, y_0 + \frac{1}{2}k_1\right)$$

$$= (0.1)\ f(0 + 0.05, 1 + 0.05) = (0.1)\ f(0.05, 1.05)$$

$$= (0.1)\ [3(0.05) + (1.05)^2] = 0.125$$

$$k_3 = hf\left(x_0 + \frac{1}{2}h, y_0 + \frac{1}{2}k_2\right)$$

$$= (0.1)\ f\left[0 + 0.05, 1 + \frac{1}{2}(0.125)\right]$$

$$= (0.1)\ f(0.05, 1.0625)$$

$$= (0.1)\ [3(0.05) + (1.0625)^2] = 0.1279$$

$$k_4 = hf(x_0 + h, y_0 + k_3)$$
$$= (0.1)\ f(0 + 0.1, 1 + 0.1279)$$
$$= (0.1)\ f(0.1, 1.1279)$$
$$= (0.1)\ [3(0.1) + (1.1279)^2] = 0.1572$$

$\therefore$
$$k = \frac{1}{2}(k_1 + 2k_2 + 2k_3 + k_4)$$
$$= \frac{1}{6}[0.1 + 2(0.125) + 2(0.1279) + 0.1572]$$
$$= \frac{1}{6} \times 0.763 = 0.127$$

Hence the value of y at x = 0.1

$$y = y_0 + k$$
$$= 1 + 0.127 = \mathbf{1.127}$$

Ans.

Prob.24. Use Runge-Kutta method to solve the equation $\frac{dy}{dx} = 1 + y^2$ for x = 0.2 and x = 0.4 with h = 0.2, given that y(0) = 0.5. ***(R.G.P.V., Dec. 2015)***

Sol. Given,

$$f(x, y) = 1 + y^2,\ x_0 = 0,\ y_0 = 0.5 \text{ and } h = 0.2.$$

Now
$$k_1 = hf(x_0, y_0)$$
$$= (0.2)\ f(0, 0.5) = (0.2)\ [1 + (0.5)^2] = 0.25$$
$$k_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right)$$
$$= (0.2)\ f(0.1, 0.625) = (0.2)\ [1 + (0.625)^2] = 0.278125$$
$$k_3 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right)$$
$$= (0.2)\ f(0.1, 0.63906)$$
$$= (0.2)\ [1 + (0.63906)^2] = 0.28168$$
$$k_4 = hf(x_0 + h, y_0 + k_3)$$
$$= (0.2)\ f(0.2, 0.78168)$$
$$= (0.2)\ [1 + (0.78168)^2] = 0.3222$$

Thus
$$k = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$
$$= \frac{1}{6}(0.25 + 2 \times 0.278125 + 2 \times 0.28168 + 0.3222)$$
$$= 0.28197$$

$\therefore$
$$y_1 = y_0 + k$$

or
$$y(0.2) = 0.5 + 0.28197 = 0.78197$$

To find y(0.4)

Here,
$$x_1 = 0.2,\ y_1 = 0.78197,\ h = 0.2$$

Also
$$k_1 = hf(x_1, y_1)$$
$$= (0.2)\ f(0.2, 0.78197)$$
$$= (0.2)\ [1 + (0.78197)^2] = 0.32229$$
$$k_2 = hf\left(x_1 + \frac{h}{2}, y_1 + \frac{k_1}{2}\right)$$
$$= (0.2)\ f(0.3, 0.943115)$$
$$= (0.2)\ [1 + (0.943115)^2] = 0.37789$$
$$k_3 = hf\left(x_1 + \frac{h}{2}, y_1 + \frac{k_2}{2}\right)$$
$$= (0.2)\ f(0.3, 0.970915)$$
$$= (0.2)\ [1 + (0.970915)^2] = 0.388535$$
$$k_4 = hf(x_1 + h, y_1 + k_3)$$
$$= (0.2)\ f(0.4, 1.170505)$$
$$= (0.2)\ [1 + (1.170505)^2] = 0.474016$$

Thus
$$k = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$
$$= \frac{1}{6}(0.32229 + 2 \times 0.37789 + 2 \times 0.388535 + 0.474016)$$
$$= 0.38819$$

So
$$y_2 = y_1 + k$$

or
$$y(0.4) = 0.78197 + 0.38819 = \mathbf{1.17016}$$

Ans.

Prob.25. Apply Runge-Kutta method to find an approximate value of y when x = 0.2 in steps of 0.1 if $\frac{dy}{dx} = x + y^2$, given that y = 1 when x = 0.

(R.G.P.V., Feb. 2010, June 2011, Dec. 2012)

Or

Apply Runge-Kutta method (Fourth order) to find an approximate value of y when x = 0.2, given that $\frac{dy}{dx} = x + y^2$ and y = 1 when x = 0.

(R.G.P.V., Dec. 2010, June 2012)

Sol. We have $f(x, y) = x + y^2,\ x_0 = 0,\ y_0 = 1$ and $h = 0.1$.

Then,
$$k_1 = h\ f(x_0, y_0) = (0.1)\ f(0, 1) = (0.1)\ [0 + (1)^2] = 0.1$$

$$k_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right) = (0.1)\, f(0.05, 1.05)$$

$$= (0.1)[(0.05) + (1.05)^2] = 0.11525$$

$$k_3 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right) = (0.1)\, f(0.05, 1.057625)$$

$$= (0.1)[(0.05) + (1.057625)^2] = 0.11686$$

$$k_4 = hf(x_0 + h, y_0 + k_3) = (0.1)\, f(0.1, 1.11686)$$

$$= (0.1)\,[(0.1) + (1.11686)^2] = 0.1347$$

and finally $k = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$

$$= \frac{1}{6}(0.1 + 2 \times 0.11525 + 2 \times 0.11686 + 0.1347) = 0.1165$$

$\therefore$ $y(0.1) = y_1 = y_0 + k = 1 + 0.1165 =$ **1.1165**

Next step –

$x_1 = x_0 + h = 0.1,\ y_1 = 1.1165,\ h = 0.1$

Now $k_1 = h\, f(x_1, y_1) = (0.1) f(0.1, 1.1165)$

$$= (0.1)\,[(0.1) + (1.1165)^2] = 0.1347$$

$$k_2 = hf\left(x_1 + \frac{h}{2}, y_1 + \frac{k_1}{2}\right)$$

$$= (0.1) f(0.15, 1.1838) = (0.1)[(0.15) + (1.1838)^2] = 0.1551$$

$$k_3 = hf\left(x_1 + \frac{h}{2}, y_1 + \frac{k_2}{2}\right) = (0.1) f(0.15, 1.194)$$

$$= (0.1)\,[(0.15) + (1.194)^2] = 0.1576$$

$$k_4 = hf(x_1 + h, y_1 + k_3) = (0.1) f(0.2, 1.2741)$$

$$= (0.1)[(0.2) + (1.2741)^2] = 0.1823$$

Thus, $k = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) = 0.1571$

So that, $y(0.2) = y_2 = y_1 + k =$ **1.2736** **Ans.**

Prob.26. Apply Runge-Kutta method to find a approximate value of y when x = 0.1. If $\frac{dy}{dx} = x + y^2$ ***given that y(0) = 1.*** ***(R.G.P.V., June 2015)***

Sol. Refer to Prob.25.

Prob.27. Using Runge-Kutta method of fourth order solve $\frac{dy}{dx} = \frac{y^2 - x^2}{y^2 + x^2}$ ***with y(0) = 1, at x = 0.2 and 0.4.***

(R.G.P.V., June 2004, 2008, 2016)

Sol. Given

$$f(x, y) = \frac{y^2 - x^2}{y^2 + x^2} \quad \text{and} \quad x_0 = 0,\ y_0 = 1,\ h = 0.2$$

Now, $k_1 = h\, f(x_0, y_0) = (0.2)\, f(0, 1) = (0.2)\left[\frac{(1)^2 - 0}{(1)^2 + 0}\right] = 0.2$

$$k_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right) = (0.2)\, f(0.1, 1.1) = 0.19672$$

$$k_3 = h\, f\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right) = (0.2)\, f(0.1, 1.09836) = 0.1967$$

$$k_4 = h\, f(x_0 + h, y_0 + k_3) = (0.2)\, f(0.2, 1.1967) = 0.1891$$

Thus, $k = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$

$$= \frac{1}{6}[0.2 + 2 \times (0.19672) + 2 \times (0.1967) + 0.1891] = 0.19599$$

$\therefore$ $y_1 = y_0 + k$

or $y(0.2) = 1 + 0.19599 = 1.19599 \approx 1.196.$

To find y(0.4)

Here, $x_1 = 0.2,\ y_1 = 1.196,\ h = 0.2$

Also, $k_1 = h\, f(x_1, y_1) = 0.2\, f(0.2, 1.196)$

$$= (0.2)\left[\frac{(1.196)^2 - (0.2)^2}{(1.196)^2 + (0.2)^2}\right] = 0.1891$$

$$k_2 = hf\left(x_1 + \frac{h}{2}, y_1 + \frac{k_1}{2}\right) = (0.2)\, f(0.3, 1.2906) = 0.1795$$

$$k_3 = h\, f\left(x_1 + \frac{h}{2}, y_1 + \frac{k_2}{2}\right) = (0.2)\, f(0.3, 1.2858) = 0.1793$$

and $k_4 = h\, f(x_1 + h, y_1 + k_3) = (0.2)\, f(0.4, 1.3753) = 0.1688$

Thus, $k = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$

$$= \frac{1}{6}[0.1891 + 2 \times (0.1795) + 2 \times (0.1793) + 0.1688] = 0.1793$$

So, $y_2 = y_1 + k$

or $y(0.4) = 1.196 + 0.1793 =$ **1.3753** **Ans.**

Prob.28. Solve the differential equation $\frac{dy}{dx} = x + y$ by the Milne's method for x = 0.2, given that –

x :	0	0.05	0.10	0.15
y :	1	1.0525	1.1103	1.1737

[R.G.P.V., Nov. 2019 (IV-Sem)]

Sol. Here we have

x	y	$y' = \frac{dy}{dx} = x + y$
$x_0 = 0.00$	$y_0 = 1.0000$	$y'_0 = 1.0000$
$x_1 = 0.05$	$y_1 = 1.0525$	$y'_1 = 1.1025$
$x_2 = 0.10$	$y_2 = 1.1103$	$y'_2 = 1.2103$
$x_3 = 0.15$	$y_3 = 1.1737$	$y'_3 = 1.3237$

and h = 0.05

By Milne's predictor formula

or $$y_4 = y_0 + \frac{4h}{3}(2y'_1 - y'_2 + 2y'_3)$$

$$= 1 + \frac{4 \times 0.05}{3}(2 \times 1.1025 - 1.2103 + 2 \times 1.3237) = 1.2428$$

$\therefore$ $$y'_4 = x_4 + y_4 = 0.2 + 1.2428 = 1.4428$$

y_4 is the predicted value of y corresponding to x = 0.2 which is corrected by

$$y_4^{(1)} = y_2 + \frac{h}{3}[y'_2 + 4y'_3 + y'_4]$$

$$= 1.1103 + \frac{0.05}{3}[1.2103 + 4 \times 1.3237 + 1.4428] = 1.2428$$

which is same as the predicted value.

$\therefore$ $$y_4 = y_{0.2} = 1.2428$$

and $$y'_4 = \mathbf{1.4428}$$ **Ans.**

Prob.29. Apply Milne's method to find a solution of the differential equation $dy/dx = x - y^2$ in the range $0 \le x \le 1$ for the boundary condition $y = 0$ at $x = 0$. ***(R.G.P.V., Dec. 2005, 2013)***

Sol. Using Picard's method, we have

$$y = y(0) + \int_0^x f(x,y)\,dx, \text{ where } f(x, y) = x - y^2.$$

To get the first approximation, we put y = 0 in f(x, y), giving

$$y_1 = 0 + \int_0^x x\,dx = \frac{x^2}{2}$$

To obtain the second approximation, we put $y = \frac{x^2}{2}$ in f(x, y), giving

$$y_2 = \int_0^x \left(x - \frac{x^4}{4}\right)dx = \frac{x^2}{2} - \frac{x^5}{20}$$

Similarly, the third approximation is

$$y_3 = \int_0^x \left[x - \left(\frac{x^2}{2} - \frac{x^5}{20}\right)^2\right]dx = \frac{x^2}{2} - \frac{x^5}{20} + \frac{x^8}{160} - \frac{x^{11}}{4400} \quad ...(i)$$

Now let us determine the starting values of the Milne's method from equation (i), by choosing h = 0.2

x = 0.0	$y_0 = 0.0$	$f_0 = 0.0$
x = 0.2	$y_1 = 0.02$	$f_1 = 0.1996$
x = 0.4	$y_2 = 0.0795$	$f_2 = 0.3937$
x = 0.6	$y_3 = 0.1762$	$f_3 = 0.5689$

Using the predictor, $y_4 = y_0 + \frac{4h}{3}(2f_1 - f_2 + 2f_3)$, $x = 0.8, y_4 = 0.3049$, $f_4 = 0.7070$ and the corrector

$$y_4 = y_2 + \frac{h}{3}(f_2 + 4f_3 + f_4), \text{ yields}$$

$$y_4 = 0.3046, \quad f_4 = 0.7072 \quad .....(ii)$$

Again using the corrector $y_4 = 0.3046$,

which is same as in equation (ii).

Now, using the predictor

$$y_5 = y_1 + \frac{4h}{3}(2f_2 - f_3 + 2f_4)$$

$$x = 1.0, y_5 = 0.4553, f_5 = 0.7927$$

and the corrector, $$y_5 = y_3 + \frac{h}{3}(f_3 + 4f_4 + f_5) \text{ gives}$$

$$y_5 = 0.4555, \quad f_5 = 0.7925$$

Again using the corrector, $y_5 = 0.4555$ a value which is the same as before.

Hance $$y(1) = \mathbf{0.4555}$$ **Ans.**

Prob.30. Use Milne's predictor-corrector method to find y(0.3) from $\frac{dy}{dx} = x^2 + y^2$, y(0) = 1. Find the initial values y(– 0.1), y(0.1) and y(0.2) from the Taylor's series method. ***(R.G.P.V., May/June 2006, June 2007)***

Sol. Here $(y)_0 = 1$ and $y' = x^2 + y^2$, $(y')_0 = 1$

Differentiating successively and substituting, we get

$$y'' = 2x + 2yy', \ (y'')_0 = 2$$

$$y''' = 2 + 2yy'' + 2y'^2, \ (y''')_0 = 8$$

$$y^{iv} = 2yy''' + 2y'y'' + 4y'y'', \ (y^{iv})_0 = 28$$

Putting these values in the Taylor's series –

$$y(x) = y_0 + xy'(0) + \frac{x^2}{2!}y''(0) + \frac{x^3}{3!}y'''(0) + \frac{x^4}{4!}y^{iv}(0) + \ldots$$

or $$y(x) = 1 + x + x^2 + \frac{4}{3}x^3 + \frac{7}{6}x^4 + \ldots$$

Now, to obtain initial values, put $x = -0.1$, 0.1 and 0.2 in above equation, we get

$$y(-0.1) = 0.909, \ y(0.1) = 1.1115$$

and $$y(0.2) = 1.2525$$

Here $f(x) = x^2 + y^2$, choosing $h = 0.1$

$\therefore$ $x = 0$	$y_0 = 1$	$f_0 = 1$
$x = -0.1$	$y_1 = 0.909$	$f_1 = 0.8363$
$x = 0.1$	$y_2 = 1.1115$	$f_2 = 1.2454$
$x = 0.2$	$y_3 = 1.2525$	$f_3 = 1.6088$

Using the predictor,

$$y_4 = y_0 + \frac{4h}{3}(2f_1 - f_2 + 2f_3)$$

Here $x = 0.3$

$$y_4 = 1 + \frac{4(0.1)}{3}[2 \times 0.8363 - 1.2454 + 2 \times 1.6088]$$

or $$y_4 = 1.4860$$

$\therefore$ $x = 0.3$, $y_4 = 1.4860$, $f_4 = 2.2982$

and using corrector

$$y_4 = y_2 + \frac{h}{3}(f_2 + 4f_3 + f_4)$$

$$= 1.1115 + \frac{0.1}{3}[1.2454 + 4 \times 1.6088 + 2.2982] = 1.4441$$

$\therefore$ $x = 0.3$, $y_4 = 1.4441$, $f_4 = 2.1754$

Using the corrector $y_4 = 1.444$, a value which is same.

Hence $y(0.3) = \mathbf{1.444}$ **Ans.**

Prob.31. Apply Milne's method to find the solution of $\frac{dy}{dx} = x + y$ given $y(0) = 0$ for $0.4 < x \le 1.0$ with $h = 0.1$.

(R.G.P.V., Dec. 2012)

Sol. Given equation is

$$\frac{dy}{dx} = x + y$$

$$\Rightarrow \quad \frac{dy}{dx} - y = x \qquad \therefore \ PI = e^{\int -dx} = e^{-x}$$

Thus, the solution is

$$ye^{-x} = \int xe^{-x}dx + C = -xe^{-x} - e^{-x} + C$$

$$\therefore \quad y = -x - 1 + Ce^{x} \qquad \ldots(i)$$

Initially, $x = 0$, $y = 0$

From equation (i), $C = 1$

$x_0 = 0$,	$y_0 = 0$,	$y'_0 = 0$
$x_{0.1} = 0.1$,	$y_1 = 0.0052$,	$y'_1 = 0.1052$
$x_{0.2} = 0.2$,	$y_2 = 0.0214$,	$y'_2 = 0.2214$
$x_{0.3} = 0.3$,	$y_3 = 0.0498$,	$y'_3 = 0.3498$

By Milne predictor formula

$$y_4 = y_0 + \frac{4h}{3}(2y'_1 - y'_2 + 2y'_3)$$

$$= 0 + \frac{0.4}{3}[2(0.1052) - (0.2214) + 2(0.3498)] = 0.0918$$

$$y'_4 = x_4 + y_4 = 0.4918$$

y_4 is the predicted value of y corresponding to $x = 0.4$ which is corrected by

$$y_4^{(1)} = y_2 + \frac{h}{3}[y'_2 + 4y'_3 + y'_4]$$

$$= 0.0214 + \frac{0.1}{3}[0.2214 + 4(0.3498) + (0.4918)] = 0.09181$$

Now, we are to find the values of y corresponding to $x = 0.5, 0.6 \ldots 1$

i.e.; $y_5, y_6, y_7, y_8, y_9, y_{10}$

Milne's predictor formula for y_5 is

$$y_5 = y_1 + \frac{4h}{3}[2y'_2 - y'_3 + 2y'_4]$$

$$= 0.0052 + \frac{0.4}{3}[2(0.2214) - 0.3498 + 2(0.4918)] = 0.1487,$$

$$y'_5 = 0.6487$$

which is corrected by

$$y_5^{(1)} = y_3 + \frac{h}{3}[y'_3 + 4y'_4 + y'_5]$$

$$= 0.0498 + \frac{0.1}{3}[0.3498 + 4(0.4918) + 0.6487] = 0.1487$$

Milne's predictor formula for y_6 is

$$y_6 = y_2 + \frac{4h}{3}[2y'_3 - y'_4 + 2y'_5]$$

$$= 0.0214 + \frac{0.4}{3}[2(0.3498) - 0.4918 + 2(0.6487)] = 0.22209$$

$$y'_6 = 0.82209$$

$$y_6^{(1)} = y_4 + \frac{h}{3}[y'_4 + 4y'_5 + y'_6]$$

$$= 0.09181 + \frac{0.1}{3}[0.4918 + 4(0.6487) + (0.82209)] = 0.2221$$

Milne's predictor formula for y_7 is

$$y_7 = y_3 + \frac{4h}{3}[2y'_4 - y'_5 + 2y'_6]$$

$$= 0.0498 + \frac{0.4}{3}[2(0.4918) - 0.6487 + 2(0.82209)] = 0.31368$$

$$y'_7 = 1.01368$$

Corrected value of y_7

$$y_7^{(1)} = y_5 + \frac{h}{3}[y'_5 + 4y'_6 + y'_7]$$

$$= 0.1487 + \frac{0.1}{3}[0.6487 + 4(0.82209) + 1.01368] = 0.313725$$

Milne's predictor formula for y_8 is

$$y_8 = y_4 + \frac{4h}{3}[2y'_5 - y'_6 + 2y'_7]$$

$$= 0.09181 + \frac{0.4}{3}[2(0.6487) - 0.82209 + 2(1.01368)] = 0.4255$$

$$y'_8 = 1.2255$$

Corrected value of y_8

$$y_8^{(1)} = y_6 + \frac{h}{3}[y'_6 + 4y'_7 + y'_8]$$

$$= 0.2221 + \frac{0.1}{3}[0.82209 + 4(1.01368) + 1.2255] = 0.4255$$

Milne's predictor formula for y_9 is

$$y_9 = y_5 + \frac{4h}{3}[2y'_6 - y'_7 + 2y'_8]$$

$$= 0.1487 + \frac{0.4}{3}[2(0.82209) - 1.01368 + 2(1.2255)] = 0.55957$$

$$y'_9 = 1.45957$$

Corrected value of y_9

$$y_9^{(1)} = y_7 + \frac{h}{3}[y'_7 + 4y'_8 + y'_9]$$

$$= 0.313725 + \frac{0.1}{3}[1.01368 + 4(1.2255) + 1.45957] = 0.5596$$

Milne's predictor formula for y_{10} is

$$y_{10} = y_6 + \frac{4h}{3}[2y'_7 - y'_8 + 2y'_9]$$

$$= 0.2221 + \frac{0.4}{3}[2(1.01368) - 1.2255 + 2(1.45957)] = 0.7182$$

$$y'_{10} = 1.7182$$

Corrected value of y_{10}

$$y_{10}^{(1)} = y_8 + \frac{h}{3}[y'_8 + 4y'_9 + y'_{10}]$$

$$= 0.4255 + \frac{0.1}{3}[1.2255 + 4(1.45957) + 1.7182] = 0.7182$$

Hence, the results is as follows –

x	y	dy/dx
0.4	0.09181	0.4918
0.5	0.1487	0.6487
0.6	0.2221	0.82209
0.7	0.313725	1.01368
0.8	0.4255	1.2255
0.9	0.5596	1.45957
1.0	0.7182	1.7182

Prob.32. Given $\frac{dy}{dx} = x^2(1+y)$ ***and*** $y(1) = 1$, $y(1.1) = 1.233$, $y(1.2) = 1.548$, $y(1.3) = 1.979$. ***Evaluate y(1.4) by Adams-Bashforth method.***

Sol. Given that

$$\frac{dy}{dx} = x^2(1+y) \text{ and } y(1) = 1,\ y(1.1) = 1.233$$

$$y(1.2) = 1.548,\ y(1.3) = 1.979$$

Here, $f(x, y) = x^2(1+y)$ ...(i)

Now, let us determine the starting values of the Adams-Bashforth method with h = 0.1, are

$$x = 1.0,\ y_{-3} = 1.000,\ f_{-3} = (1.0)^2(1+1.000) = 2.000$$

$$x = 1.1,\ y_{-2} = 1.233,\ f_{-2} = (1.1)^2(1+1.233) = 2.702$$

$$x = 1.2,\ y_{-1} = 1.548,\ f_{-1} = (1.2)^2(1+1.548) = 3.669$$

and $$x = 1.3,\ y_0 = 1.979,\ f_0 = (1.3)^2(1+1.979) = 5.035$$

Now using the predictor

$$y_1 = y_0 + \frac{h}{24}(55f_0 - 59f_{-1} + 37f_{-2} - 9f_{-3}) \quad ...(ii)$$

Putting the above values in equation (ii), we have

$$y_1 = 1.979 + \frac{(0.1)}{24}\{55(5.035) - 59(3.669) + 37(2.702) - 9(2)\}$$

$$= 1.979 + \frac{(0.1)}{24}(276.925 - 216.471 + 99.974 - 18)$$

or $$y_1 = 2.5725$$

Putting x = 1.4 and $y_1 = 2.5725$ in equation (i), we have

$$f_1 = x^2(1+y_1)$$

$$= (1.4)^2(1+2.5725) = 7.0021$$

Now using the corrector

$$y_1 = y_0 + \frac{h}{24}(9f_1 + 19f_0 - 5f_{-1} + f_{-2}) \quad ...(iii)$$

Putting the above values in equation (iii), we have

$$y_1 = 1.979 + \frac{(0.1)}{24}\{9(7.0021) + 19(5.035) - 5(3.699) + 2.702\}$$

$$y_1 = 2.57438$$

Hence $y(1.4) = \mathbf{2.574}$

Ans.

Prob.33. Using Adams-Bashforth formula determine y(0.4) given the differential equation $\frac{dy}{dx} = \frac{1}{2}xy$ ***and data –***

x :	***0***	***0.1***	***0.2***	***0.3***
y :	***1***	***1.0025***	***1.0101***	***1.0228***

Sol. Here $f(x, y) = \frac{1}{2}xy$ ...(i)

Now, let us determine the starting values of the Adams-Bashforth method with h = 0.1, are

$$x = 0,\ y_{-3} = 1.000,\ f_{-3} = \frac{1}{2} \times 0 \times 1 = 0$$

$$x = 0.1,\ y_{-2} = 1.0025,\ f_{-2} = \frac{1}{2} \times 0.1 \times 1.0025 = 0.050125$$

$$x = 0.2,\ y_{-1} = 1.0101,\ f_{-1} = \frac{1}{2} \times 0.2 \times 1.0101 = 0.10101$$

$$x = 0.3,\ y_0 = 1.0228,\ f_0 = \frac{1}{2} \times 0.3 \times 1.0228 = 0.15342$$

Now using the predictor

$$y_1 = y_0 + \frac{h}{24}(55f_0 - 59f_{-1} + 37f_{-2} - 9f_{-3}) \quad ...(ii)$$

Putting the above values in equation (ii), we have

$$y_1 = 1.0228 + \frac{(0.1)}{24}[55(0.15342) - 59(0.10101) + 37(0.050125 - 9(0)]$$

$$y_1 = 1.0228 + \frac{(0.1)}{24}(8.4381 - 5.95959 + 1.854625 - 0)$$

$$y_1 = 1.0228 + 0.018055$$

$$= 1.040855$$

Putting x = 0.4 and $y_1 = 1.040855$ in equation(i), we have

$$f_1 = \frac{1}{2}xy_1 = \frac{1}{2} \times 0.4 \times 1.040855 = 0.208171$$

Now using the corrector

$$y_1 = y_0 + \frac{h}{24}(9f_1 + 19f_0 - 5f_{-1} + f_{-2}) \quad ...(iii)$$

Putting the above values in equation(iii), we have

$$y_1 = 1.0228 + \frac{(0.1)}{24}(9 \times 0.208171 + 19 \times 0.15342 - 5 \times 0.10101 + 0.050125)$$

$$= 1.0228 + \frac{(0.1)}{24}(1.873539 + 2.91498 - 0.50505 + 0.050125)$$

$$= 1.0228 + 0.0180566$$

$$= 1.0408566$$

Hence $y(0.4) = \mathbf{1.04085}$

Ans.

PARTIAL DIFFERENTIAL EQUATIONS – FINITE DIFFERENCE SOLUTION, TWO DIMENSIONAL LAPLACE EQUATION AND POISSON EQUATION, IMPLICIT AND EXPLICIT METHODS FOR ONE DIMENSIONAL HEAT EQUATION (BENDER-SCHMIDT AND CRANK-NICOLSON METHODS), FINITE DIFFERENCE EXPLICIT METHOD FOR WAVE EQUATION

Partial Differential Equations –

A general second-order linear partial differential equation is of the type –

$$A\frac{\partial^2 u}{\partial x^2} + B\frac{\partial^2 u}{\partial x \partial y} + C\frac{\partial^2 u}{\partial y^2} + D\frac{\partial u}{\partial x} + E\frac{\partial u}{\partial y} + Fu = G \qquad \ldots(i)$$

which can be written as

$$Au_{xx} + Bu_{xy} + Cu_{yy} + Du_x + Eu_y + Fu = G$$

where A, B, C, D, E, F, G are all functions of x and y. This equation can be classified with respect to the sign of the discriminant

$$\Delta_s = B^2 - 4AC$$

(i) If $\Delta_s < 0$, at a point in the (x, y) plane, the equation is said to be of ***elliptic type***.

(ii) If $\Delta_s > 0$, at a point in the (x, y) plane, the equation is said to be of ***hyperbolic type***.

(iii) If $\Delta_s = 0$, at a point in the (x, y) plane, the equation is said to be of ***parabolic type***.

We now consider the following partial differential equations.

$$u_{xx} + u_{yy} = 0 \text{ (Laplace's equation)} \qquad \ldots(ii)$$

$$u_{xx} - \frac{1}{c^2}u_{tt} = 0 \text{ (Wave equation)} \qquad \ldots(iii)$$

$$u_{xx} - u_t = 0 \text{ (Heat conduction equation)} \qquad \ldots(iv)$$

and where (x, y) are the space co-ordinates and t is the time co-ordinate. In equation (i) we have

$$A = 1, B = 0, C = 1$$

$\therefore$ $\Delta_s = -$ve. Hence Laplace's equation is of elliptic type.

In equation (iii), we have $\Delta_s = +$ve. Hence the equation is hyperbolic type.

In equation (iv), we have $\Delta_s = 0$. Hence the heat equation is parabolic type.

To obtain Finite-difference Approximations of Partial Derivatives

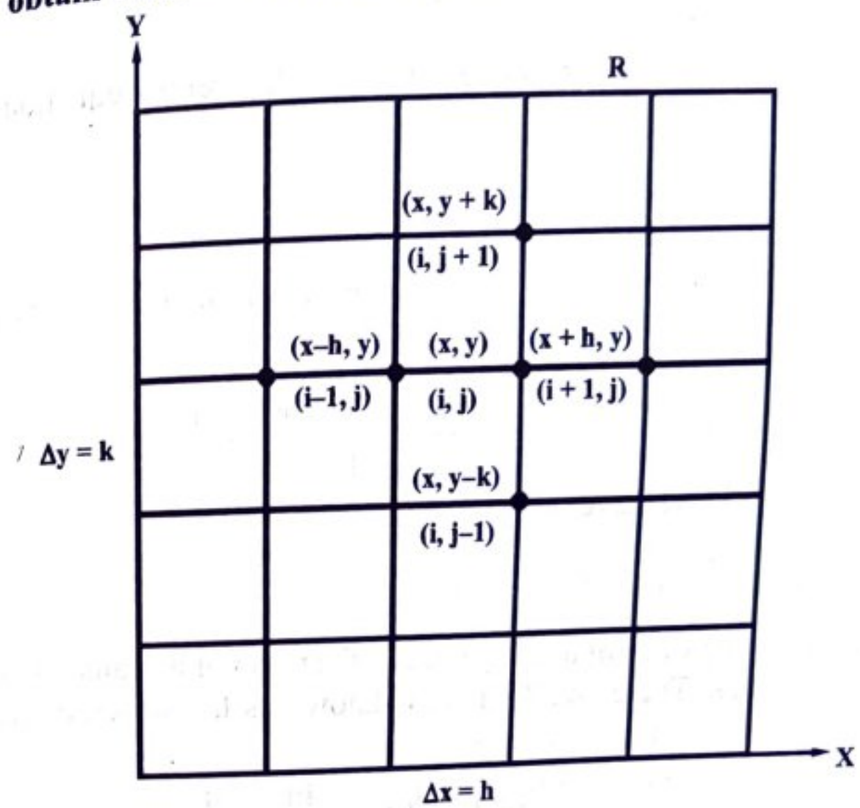

Fig. 3.3

Let the (x, y) plane be divided into a network of rectangles of sides $\Delta x = h$, $\Delta y = k$ by drawing the sets of lines $x = ih$ $(i = 0, 1, 2, \ldots)$ and $y = jk$ $(j = 0, 1, \ldots)$. The points of intersection of these families of lines are called ***mesh points***, ***lattice points*** or ***grid points***, Then we have

$$u_x = \frac{\partial u}{\partial x} = \frac{u_{i+1,j} - u_{i,j}}{h} + O(h) \qquad \ldots(v)$$

$$= \frac{u_{i,j} - u_{i-1,j}}{h} + O(h) \qquad \ldots(vi)$$

$$= \frac{u_{i+1,j} - u_{i-1,j}}{2h} + O(h^2) \qquad \ldots(vii)$$

and $$u_{xx} = \frac{u_{i-1,j} - 2u_{i,j} + u_{i+1,j}}{h^2} + O(h^2) \qquad \ldots(viii)$$

Similarly, we have the approximations

$$u_y = \frac{\partial u}{\partial y} = \frac{u_{i,j+1} - u_{i,j}}{k} + O(k) \quad \ldots(ix)$$

$$= \frac{u_{i,j} - u_{i,j-1}}{k} + O(k) \quad \ldots(x)$$

$$= \frac{u_{i,j+1} - u_{i,j-1}}{2k} + O(k^2) \quad \ldots(xi)$$

and $$u_{yy} = \frac{u_{i,j-1} - 2u_{i,j} + u_{i,j+1}}{k^2} + O(k^2) \quad \ldots(xii)$$

To obtain finite-difference analogues of partial differential equation.

Laplace's equation $$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \quad \ldots(xiii)$$

Putting the values of $\frac{\partial^2 u}{\partial x^2}$ and $\frac{\partial^2 u}{\partial y^2}$ from equations (viii) and (xii) in equation (xiii), we get

$$\frac{u_{i-1,j} - 2u_{i,j} + u_{i+1,j}}{h^2} + \frac{u_{i,j-1} - 2u_{i,j} + u_{i,j+1}}{k^2} = 0$$

If $h = k$, we have

$$u_{i,j} = \frac{1}{4}\left[u_{i+1,j} + u_{i-1,j} + u_{i,j+1} + u_{i,j-1}\right] \quad \ldots(xiv)$$

which shows that the value of u at any point is the mean of its values at the four neighbouring points. The above formula is known as the standard five-point formula and is written some time as –

$$u_{i+1,j} + u_{i-1,j} + u_{i,j+1} + u_{i,j-1} - 4u_{i,j} = 0 \quad \ldots(xv)$$

Again, expanding the terms on the right hand side of equation (xiv) by Taylor's series method, we obtain

$$u_{i+1,j} - u_{i-1,j} + u_{i,j+1} + u_{i,j-1} - 4u_{i,j}$$

$$= h^2\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) - \frac{1}{6}h^4 \frac{\partial^4 u}{\partial x^2 \partial y^2} + O(h^6) \quad \ldots(xvi)$$

$$= -\frac{1}{6}h^4 \frac{\partial^4 u}{\partial x^2 \partial y^2} + O(h^6)$$

Instead of equation (xiv), one may also use the formula

$$u_{i,j} = \frac{1}{4}\left[u_{i-1,j-1} + u_{i+1,j-1} + u_{i+1,j+1} + u_{i-1,j+1}\right] \quad \ldots(xvii)$$

This formula uses the functional values at the diagonal-points and hence is known as the ***diagonal-five-points formula.***

To Solve Laplace's Equation $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ in the Bounded Region R with Boundary C –

Laplace's equation $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$, can be either replaced by standard five-point formula or by the diagonal five-point formula. Using diagonal five-point formula, the approximate function vlaues at the interior mesh points can be easily computed. We thus compute u_5, u_7, u_9, u_1 and u_3 in this order as follows –

$$u_5 = \frac{1}{4}(C_1 + C_5 + C_9 + C_{13});$$

$$u_7 = \frac{1}{4}(C_{15} + u_5 + C_{11} + C_{13});$$

$$u_9 = \frac{1}{4}(u_5 + C_7 + C_9 + C_{11});$$

$$u_1 = \frac{1}{4}(C_1 + C_3 + u_5 + C_{15})$$

$$u_3 = \frac{1}{4}(C_3 + C_5 + C_7 + u_5)$$

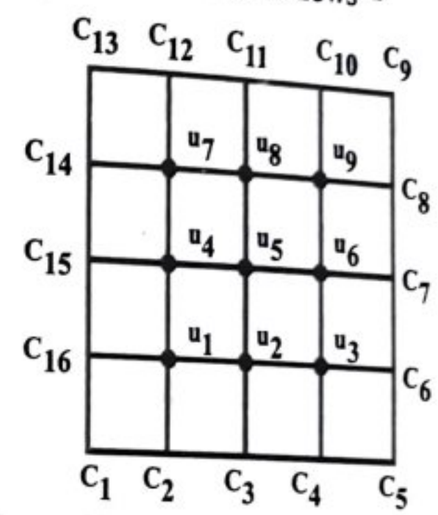

Fig. 3.4

Then we proceed to compute in order, the remaining quantities; viz u_8, u_4, u_6, u_2 by the standard five-point formula (xiv) and thus we have

$$u_8 = \frac{1}{4}(u_5 + u_9 + C_{11} + u_7); \quad u_4 = \frac{1}{4}(u_1 + u_5 + u_7 + C_{15});$$

$$u_6 = \frac{1}{4}(u_3 + C_7 + u_9 + u_5); \quad u_2 = \frac{1}{4}(C_3 + u_3 + u_5 + u_1);$$

When once all u_i's $(i = 1, 2, \ldots, 9)$ are calculated their accuracy is improved by any of the following iterative methods.

(i) Point-Jacobi Iterative Method –

If $u_{i,j}^{(n)}$ denotes the nth iterative value of $u_{i,j}$ then an iterative procedure to solve $u_{i+1,j} + u_{i-1,j} + u_{i,j+1} + u_{i,j-1} - 4u_{i,j} = 0$ is given by

$$u_{i,j}^{(n+1)} = \frac{1}{4}\left[u_{i-1,j}^{(n)} + u_{i+1,j}^{(n)} + u_{i,j-1}^{(n)} + u_{i,j+1}^{(n)}\right] \quad \ldots(xviii)$$

for the interior mesh points and is known as ***point-Jacobi's method.***

(ii) Gauss-Seidel Method –

The method uses the latest iterative values available and scans the mesh point systematically from left to right along successive rows. The iterative formula is

$$u_{i,j}^{(n+1)} = \frac{1}{4}\left[u_{i-1,j}^{(n+1)} + u_{i+1,j}^{(n)} + u_{i,j-1}^{(n+1)} + u_{i,j+1}^{(n)}\right] \quad \ldots(xix)$$

This method converges twice as fast as the Jacobi's scheme.

(iii) Successive-over-relaxation Method (S.O.R.) – By Gauss-Seidel method, we have

$$u_{i,j}^{(n+1)} = \frac{1}{4}\left[u_{i-1,j}^{(n+1)} + u_{i+1,j}^{(n)} + u_{i,j-1}^{(n+1)} + u_{i,j+1}^{(n)}\right]$$

$$= \frac{1}{4}\left[\left\{4u_{i,j}^{(n)} + u_{i-1,j}^{(n+1)} + u_{i+1,j}^{(n)} + u_{i,j-1}^{(n+1)} + u_{i,j+1}^{(n)}\right\} - 4u_{i,j}^{(n)}\right]$$

$$= u_{i,j}^{(n)} + \frac{1}{4}\left[u_{i-1,j}^{(n+1)} + u_{i+1,j}^{(n)} + u_{i,j-1}^{(n+1)} + u_{i.j+1}^{(n)} - 4u_{i,j}^{(n)}\right]$$

or

$$u_{i,j}^{(n+1)} = u_{i,j}^{(n)} + \frac{1}{4}R_{i,j} \qquad \ldots(xx)$$

where

$$R_{i,j} = \left[u_{i-1,j}^{(n+1)} + u_{i+1,j}^{(n)} + u_{i,j-1}^{(n+1)} + u_{i,j+1}^{(n)} - 4u_{i,j}^{(n)}\right]$$

This clearly shows that $R_{i,j}$ is nothing but the change in the vlaue of $u_{i,j}$ for Gauss-Seidel iteration. In S.O.R. method a larger change than the above is given to $u_{i,j}^{(n)}$ and the corresponding iteration formula becomes

$$u_{i,j}^{(n+1)} = u_{i,j}^{(n)} + \frac{1}{4}\omega R_{i,j} = u_{i,j}^{(n)}.(1-\omega) + \frac{\omega}{4}\left[u_{i-1,j}^{(n+1)} + u_{i+1,j}^{(n)} + u_{i,j-1}^{(n+1)} + u_{i,j+1}^{(n)}\right] \qquad \ldots(xxi)$$

The rate of convergence of the above formula depends upon the choice of ω, which is called the accelerating factor and lies between 1 and 2.

Solution of Poisson's Equation –

$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f(x,y) \qquad \ldots(xxii)$$

This is an elliptic equation which can be solved numerically at the interior mesh points of a square network when the boundary values are known. The standard five-point formula for equation (xxii) takes the form

$$u_{i-1,j} + u_{i+1,j} + u_{i,j+1} + u_{i,j-1} - 4\,u_{i,j} = h^2 f\,(ih, jh) \qquad \ldots(xxiii)$$

By applying equation (xxiii) at each mesh-point, we arrive at linear equations in the pivotal values i, j. These equations can be solved by Gauss-Seidel iteration method.

Bendre-Schmidt Method to Solve Parabolic Equations – Parabolic equations are connected with some kind of diffusion, for example heat conduction equation in one dimension, which is

$$\frac{\partial u}{\partial t} = c^2\frac{\partial^2 u}{\partial x^2}, \text{ where } c^2 = \frac{k}{s\rho} \qquad \ldots(i)$$

Generally the solution u(x, t) of equation (i) is a function of x and temperature t, satisfying boundary and initial conditions.

We know that

$$\frac{\partial u}{\partial t} = \frac{u(x,t+k) - u(x,t)}{k}$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{u(x+h,t) - 2u(x,t) + u(x-h,t)}{h^2}$$

Putting the above values in equation (i), we get

$$\frac{u(x,t+k) - u(x,t)}{k} = c^2\,\frac{u(x+h,t) - 2u(x,t) + u(x-h,t)}{h^2}$$

or

$$u\,(x, t+k) = \frac{c^2k}{h^2}u(x+h,t) - \frac{2c^2k}{h^2}u(x,t) + \frac{c^2k}{h^2}u(x-h,t) + u(x,t)$$

Putting $\frac{c^2k}{h^2} = \alpha$, above equation becomes

$$u\,(x, t+k) = \alpha u(x+h, t) + u(x, t) - 2\,\alpha u(x, t) + \alpha u(x-h, t)$$

$$u\,(x, t+k) = \alpha u(x+h, t) + (1-2\alpha)\,u\,(x, t) + \alpha u\,(x-h, t) \qquad \ldots(ii)$$

In case $\alpha = \frac{1}{2}$, equation (ii) becomes

$$u\,(x, t+k) = \frac{1}{2}u(x+h,t) + \frac{1}{2}u(x-h,t)$$

or

$$u_{i,j+1} = \frac{1}{2}(u_{i+1,j} + u_{i-1,j}) \qquad \ldots(iii)$$

Equation (iii) shows that the value of u at the (i, j + 1)th mesh is the mean of the values u at (i + 1, j)th mesh and (i – 1, j)th mesh.

This relation is known as ***Bendre-Schmidt recurrence relation***.

Crank-Nicolson Method to Solve Parabolic Equations –

Consider the heat conduction equation, viz.

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, \text{ where } u = u(x, t) \qquad \ldots(i)$$

Let us divide the (x, t) plane into a number of smaller rectangles by means of the lines x = 0, x = h, x = 2h (i.e. x = ih; i = 0, 1, 2, …) and t = 0, k, 2k, … (i.e. t = jk; j = 0, 1, 2, …). Then by relation

$$y(x_0) = y'\,(x_0) = A$$

and

$$y(x_n) = y''(x_n) = B$$

and

$$y'(x) = \frac{y(x+h) - y(x)}{h} + O(h)$$

We readily obtain $\dfrac{\partial^2 u}{\partial x^2} = \dfrac{1}{h^2}\left[u_{i-1,j} - 2u_{i,j} + u_{i+1,j}\right]$

and $\dfrac{\partial u}{\partial t} = \dfrac{1}{k}\left[u_{i,j+1} - u_{i,j}\right]$

Putting these values in equation (i), we get

$$\frac{1}{k}\left[u_{i,j+1} - u_{i,j}\right] = \frac{1}{h^2}\left[u_{i-1,j} - 2u_{i,j} + u_{i+1,j}\right]$$

or, $u_{i,j+1} - u_{i,j} = r\,(u_{i-1,j} - 2u_{i,j} + u_{i+1,j})$ (where $r = k/h^2$)

or $u_{i,j+1} = u_{i,j} + r\,[u_{i-1,j} - 2u_{i,j} + u_{i+1,j}]$ ...(ii)

This formula given the value of the unknown function at the $(i, j+1)^{th}$ interior point in terms of the known functional values and hence is known as ***explicit formula*** and remains valid only for $0 < r \le \frac{1}{2}$. It is important to note here that for the approximation of $\left(\dfrac{\partial^2 u}{\partial x^2}\right)$, we have used the functional values along the jth row only. In 1947 Crank and Nicolson suggested the method according to which $\left(\dfrac{\partial^2 u}{\partial x^2}\right)$ is replaced by the average value of its finite-difference approximation on the jth and $(j+1)^{th}$ rows implying.

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{2}\left[\frac{u_{i-1,j} - 2u_{i,j} + u_{i+1,j}}{h^2} + \frac{u_{i-1,j+1} - 2u_{i,j+1} + u_{i+1,j+1}}{h^2}\right]$$

Now putting this value of $\dfrac{\partial^2 u}{\partial x^2}$ and $\dfrac{\partial u}{\partial t}$ in equation (i), we get

$$\frac{1}{k}\left\{u_{i,j+1} - u_{i,j}\right\} = \frac{1}{2h^2}\left\{u_{i-1,j} - 2u_{i,j} + u_{i+1,j} + u_{i-1,j+1} - 2u_{i,j+1} + u_{i+1,j+1}\right\}$$

$$\Rightarrow -ru_{i-1,j+1} + (2+2r)u_{i,j+1} - ru_{i+1,j+1} = ru_{i-1,j} + 2(1-r)u_{i,j} + ru_{i+1,j}$$

where $r = \dfrac{k}{h^2}$ ...(iii)

On the L.H.S. of this equation we have three unknowns while on R.H.S. all the three are known. The implicit formula (iii) is known as Crank-Nicolson formula and remains convergent for all finite values of r. If in each row, we have "m" internal mesh points then the above formula gives "m" simultaneous equations for "m" unknown in terms of known boundary values. On the same lines, one can calculate the internal mesh points on all rows.

Solution of Wave Equation –

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} \quad \text{...(i)}$$

Subject to the initial conditions $u = f(x)$, $\dfrac{\partial u}{\partial t} = g(x), 0 \le x \le 1$ at $t = 0$ ...(ii)

and the boundary conditions

$$\left.\begin{aligned} u(0,t) &= \phi(t) \\ u(1,t) &= \psi(t) \end{aligned}\right\} \quad \text{...(iii)}$$

Consider a rectangular mesh in the x-t plane spacing h along x direction and k along time t direction. Denoting a mesh point (x, t) = (ih, jk) as simply i, j, we have

$$\frac{\partial^2 u}{\partial x^2} = \frac{u_{i-1,j} - 2u_{i,j} + u_{i+1,j}}{h^2}$$

and $$\frac{\partial^2 u}{\partial t^2} = \frac{u_{i,j-1} - 2u_{i,j} + u_{i,j+1}}{k^2}$$

Replacing the derivatives in equation (i) by their above approximations, we obtains

$$u_{i,j-1} - 2u_{i,j} + u_{i,j+1} = \frac{c^2k^2}{h^2}\left(u_{i-1,j} - 2u_{i,j} + u_{i+1,j}\right)$$

or $$u_{i,j+1} = 2(1-\alpha^2c^2)u_{i,j} + \alpha^2c^2\left(u_{i-1,j} + u_{i+1,j}\right) - u_{i,j-1} \quad \text{...(iv)}$$

where, $\alpha = \dfrac{k}{h}$

Now replacing the derivative in equation (ii) by its central difference approximation, we get

$$\frac{u_{i,j+1} - u_{i,j-1}}{2k} = \frac{\partial u}{\partial t} = g(x)$$

or $$\left.\begin{aligned} u_{i,j+1} &= u_{i,j-1} + 2k\,g(x) \text{ at } t = 0 \\ \text{i.e.} \quad u_{i,1} &= u_{i,-1} + 2k\,g(x), \text{ for } j = 0 \end{aligned}\right\} \quad \text{....(v)}$$

Also initial condition $u = f(x)$ at $t = 0$, becomes ...(vi)

$$u_{i,0} = f(x)$$

Combining equations (v) and (vi), we have ...(vii)

$$u_{i,1} = f(x) + k\,g(x)$$

Also equation (iii), gives $u_{0,j} = \phi(t)$ and $u_{1,j} = \psi(t)$.

Hence the explicit form equation (iv) gives the values of $u_{i,\,j+1}$ at the (j + 1)th level when the nodal values at (j – 1) th and jth levels are known from equations (vi) and (vii) as shown in following fig. 3.5.

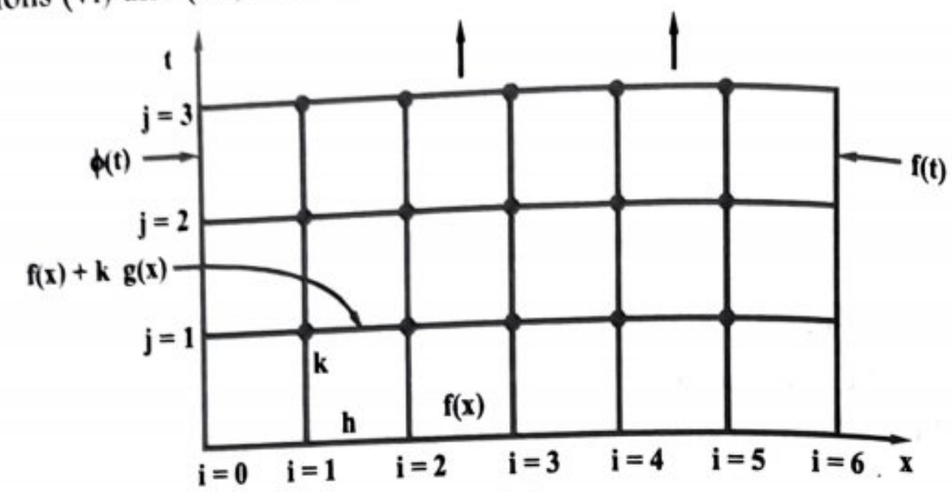

Fig. 3.5

A Special Case – The coefficient of $u_{i,\,j}$ in equation (iv) will vanish, if

$$\alpha = \frac{1}{c} \text{ or } k = \frac{h}{c}$$

Then equation (iv) reduces to the simple form

$$u_{i,\,j+1} = u_{i-1,\,j} + u_{i+1,\,j} - u_{i,\,j-1} \qquad \text{...(viii)}$$

This result provides an explicit scheme for the solution of the wave equation.

Obs. (i) For $\alpha = 1/c$, the solution of equation (iv) is stable and coincide, with the solution of equation (i).

For $\alpha < 1/c$, the solution is stable but inaccurate

For $\alpha > 1/c$, the solution is unstable

(ii) The equation (iv) coverages for, $\alpha \le 1$, i.e., for $k \le h$.

NUMERICAL PROBLEMS

Prob.34. What is the classification of the equation

$$\frac{\partial^2 u}{\partial x^2} + 4\frac{\partial^2 u}{\partial x \partial y} + 4\frac{\partial^2 u}{\partial y^2} - \frac{\partial u}{\partial x} + 2\frac{\partial u}{\partial y} = 0.$$

(R.G.P.V., June 2007)

Sol. The general partial differential is of the form

$$A(x,y)\frac{\partial^2 u}{\partial x^2} + B(x,y)\frac{\partial^2 u}{\partial x \partial y} + C(x,y)\frac{\partial^2 u}{\partial y^2} + D(x,y)\frac{\partial u}{\partial x}$$

$$+ E(x,y)\frac{\partial u}{\partial y} + F(x,y)u + G(x,y) = 0$$

Here, $A = 1, B = 4, C = 4, D = -1$ and $E = 2, F = 0, G = 0$

Now $B^2 - 4AC = (4)^2 - 4 \times 1 \times 4 = 0$

Hence given partial differential equation is **parabolic**. **Ans.**

Prob.35. Classify the following partial differential equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial x \partial y} - 2\frac{\partial^2 u}{\partial y^2} = 0$$

Sol. The general partial differential is of the form

$$A(x,y)\frac{\partial^2 u}{\partial x^2} + B(x,y)\frac{\partial^2 u}{\partial x \partial y} + C(x,y)\frac{\partial^2 u}{\partial y^2} + D(x,y)\frac{\partial u}{\partial x}$$

$$+E(x,y)\frac{\partial u}{\partial y} + F(x,y)u + G(x,y) = 0$$

Here $A = 1, B = 1, C = -2, D = 0, E = 0, F = 0, G = 0$

Now $B^2 - 4AC = (1)^2 - 4 \times 1 \times (-2) = 9$

i.e., $B^2 - 4AC > 0$

Hence given partial differential equation is **hyperbolic**. **Ans.**

Prob.36. Determine the type of given equation.

$$x^2\frac{\partial^2 u}{\partial y^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial x^2} = 0$$

Sol. The general partial differential is of the form

$$A(x,y)\frac{\partial^2 u}{\partial x^2} + B(x,y)\frac{\partial^2 u}{\partial x \partial y} + C(x,y)\frac{\partial^2 u}{\partial y^2} + D(x,y)\frac{\partial u}{\partial x}$$

$$+E(x,y)\frac{\partial u}{\partial y} + F(x,y)u + G(x,y) = 0$$

Here $A = y^2, B = 2xy, C = x^2, D = 0, E = 0, F = 0, G = 0$

Now $B^2 - 4AC = (2xy)^2 - 4y^2x^2 = 0$

Hence given partial differential equation is **parabolic**. **Ans.**

Prob.37. Determine the type of given equation

$$x^2\frac{\partial^2 u}{\partial x^2} + (1 - y^2)\frac{\partial^2 u}{\partial y^2} = 0, -\infty < x < \infty, -1 < y < 1.$$

Sol. The general partial differential is of the form

$$A(x,y)\frac{\partial^2 u}{\partial x^2} + B(x,y)\frac{\partial^2 u}{\partial x \partial y} + C(x,y)\frac{\partial^2 u}{\partial y^2} + D(x,y)\frac{\partial u}{\partial x}$$

$$+E(x,y)\frac{\partial u}{\partial y} + F(x,y)u + G(x,y) = 0$$

Here $A = x^2, B = 0, C = 1 - y^2, D = 0, E = 0, F = 0, G = 0$

Now $B^2 - 4AC = 0 - 4x^2(1 - y^2) = 4x^2(y^2 - 1)$

For all x between $-\infty$ and ∞, x^2 is positive

For all y between -1 and 1, $y^2 < 1$

Hence $B^2 - 4AC < 0$

Hence given partial differential equation is **elliptic**.

Ans.

Prob.38. Solve the partial differential equation $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$, ***for fig. 3.6 given below by (i) Jacobi's method (ii) Gauss-Seidel's method.***

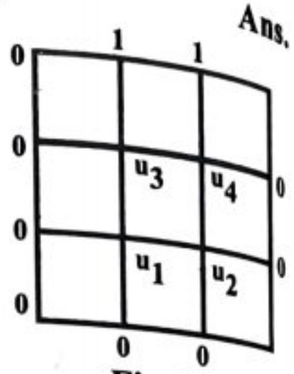

Fig. 3.6

Sol. The approximate values of u_1, u_2, u_3, u_4 are calculated as follows –

$$u_1 = \frac{1}{4}(0+0+0+1) = 0.25, \quad u_2 = \frac{1}{4}(0+0+0+1) = 0.25,$$

$$u_3 = \frac{1}{4}(1+1+0+0) = 0.5, \quad u_4 = \frac{1}{4}(1+1+0+0) = 0.5$$

Let these values be denoted by $u_1^{(1)}, u_2^{(1)}, u_3^{(1)}$ and $u_4^{(1)}$ respectively; so we get

$$u_1^{(1)} = 0.25,\ u_2^{(1)} = 0.25,\ u_3^{(1)} = 0.5,\ u_4^{(1)} = 0.5$$

(i) Using formula of Jacobi's method

$$u_{i,j}^{(n+1)} = \frac{1}{4}\left[u_{i-1,j}^{(n)} + u_{i+1,j}^{(n)} + u_{i,j-1}^{(n)} + u_{i,j+1}^{(n)}\right]$$

$$u_1^{(2)} = \frac{1}{4}\left[0 + u_2^{(1)} + 0 + u_3^{(1)}\right] = \frac{1}{4}(0.25+0.5) = 0.1875$$

$$u_2^{(2)} = \frac{1}{4}\left[u_1^{(1)} + 0 + 0 + u_4^{(1)}\right] = \frac{1}{4}(0.25+0.5) = 0.1875$$

$$u_3^{(2)} = \frac{1}{4}\left[0 + u_4^{(1)} + u_1^{(1)} + 1\right] = \frac{1}{4}(0.5+0.25+1) = 0.4375$$

$$u_4^{(2)} = \frac{1}{4}\left[u_3^{(1)} + 0 + u_2^{(1)} + 1\right] = \frac{1}{4}(0.5+0.25+1) = 0.4375$$

$$u_1^{(3)} = \frac{1}{4}\left[0 + u_2^{(2)} + 0 + u_3^{(2)}\right] = 0.15625$$

$$u_2^{(3)} = \frac{1}{4}\left[u_1^{(2)} + 0 + 0 + u_4^{(2)}\right] = 0.15625$$

$$u_3^{(3)} = \frac{1}{4}\left[0 + u_4^{(2)} + u_1^{(2)} + 1\right] = 0.40625$$

$$u_4^{(3)} = \frac{1}{4}\left[u_3^{(2)} + 0 + u_2^{(2)} + 1\right] = 0.40625$$

$$u_1^{(4)} = \frac{1}{4}\left[0 + u_2^{(3)} + 0 + u_3^{(3)}\right] = 0.14062$$

$$u_2^{(4)} = \frac{1}{4}\left[u_1^{(3)} + 0 + 0 + u_4^{(3)}\right] = 0.14062$$

$$u_3^{(4)} = \frac{1}{4}\left[0 + u_4^{(3)} + u_1^{(3)} + 1\right] = 0.39062$$

$$u_4^{(4)} = \frac{1}{4}\left[u_3^{(3)} + 0 + u_2^{(3)} + 1\right] = 0.39062$$

$$u_1^{(5)} = \frac{1}{4}\left[0 + u_2^{(4)} + 0 + u_3^{(4)}\right] = 0.13281$$

$$u_2^{(5)} = \frac{1}{4}\left[u_1^{(4)} + 0 + 0 + u_4^{(4)}\right] = 0.13281$$

$$u_3^{(5)} = \frac{1}{4}\left[0 + u_4^{(4)} + u_1^{(4)} + 1\right] = 0.38281$$

$$u_4^{(5)} = \frac{1}{4}\left[u_3^{(4)} + 0 + u_2^{(4)} + 1\right] = 0.38281$$

$$u_1^{(6)} = \frac{1}{4}\left[0 + u_2^{(5)} + 0 + u_3^{(5)}\right] = \frac{1}{4}[0.13281 + 0.38281] = 0.12891$$

$$u_2^{(6)} = \frac{1}{4}\left[u_1^{(5)} + 0 + 0 + u_4^{(5)}\right] = \frac{1}{4}[0.13281 + 0.38281] = 0.12891$$

$$u_3^{(6)} = \frac{1}{4}\left[0 + u_4^{(5)} + u_1^{(5)} + 1\right] = \frac{1}{4}[0.38281 + 0.13281 + 1] = 0.37891$$

$$u_4^{(6)} = \frac{1}{4}\left[u_3^{(5)} + 0 + u_2^{(5)} + 1\right] = \frac{1}{4}[0.38281 + 0 + 0.13281 + 1]$$
$$= 0.37891$$

These are five successive iterates by Jacobi's scheme

(ii) Using Gauss-Seidel method, the five successive iterates are given as follows –

$$u_{i,j}^{(n+1)} = \frac{1}{4}\left[u_{i-1,j}^{(n+1)} + u_{i+1,j}^{(n)} + u_{i,j-1}^{(n+1)} + u_{i,j+1}^{(n)}\right]$$

$$u_1^{(1)} = 0.25,\ u_2^{(1)} = 0.3125,\ u_3^{(1)} = 0.5625,\ u_4^{(1)} = 0.46875$$

$$u_1^{(2)} = \frac{1}{4}\left[0 + u_2^{(1)} + 0 + u_3^{(1)}\right] = 0.21875$$

$$u_2^{(2)} = \frac{1}{4}\left[u_1^{(2)} + 0 + 0 + u_4^{(1)}\right] = 0.17187$$

$$u_3^{(2)} = \frac{1}{4}\left[0 + u_4^{(1)} + u_1^{(2)} + 1\right] = 0.42187$$

$$u_4^{(2)} = \frac{1}{4}\left[u_3^{(2)} + 0 + u_2^{(2)} + 1\right] = 0.398435$$

$$u_1^{(3)} = \frac{1}{4}\left[u_2^{(2)} + u_3^{(2)}\right] = 0.148435$$

$$u_2^{(3)} = \frac{1}{4}\left[u_1^{(3)} + u_4^{(2)}\right] = 0.13672$$

$$u_3^{(3)} = \frac{1}{4}\left[u_4^{(2)} + u_1^{(3)} + 1\right] = 0.38672,$$

$$u_4^{(3)} = \frac{1}{4}\left[u_3^{(3)} + u_2^{(3)} + 1\right] = 0.38086$$

$$u_1^{(4)} = \frac{1}{4}\left[u_2^{(3)} + u_3^{(3)}\right] = 0.13086,$$

$$u_2^{(4)} = \frac{1}{4}\left[u_1^{(4)} + u_4^{(3)}\right] = \frac{1}{4}(0.13086 + 0.38086) = 0.12793$$

$$u_3^{(4)} = \frac{1}{4}\left[u_4^{(3)} + u_1^{(4)} + 1\right] = 0.37793$$

$$u_4^{(4)} = \frac{1}{4}\left[u_3^{(4)} + u_2^{(4)} + 1\right] = 0.37646$$

$$u_1^{(5)} = \frac{1}{4}\left[u_2^{(4)} + u_3^{(4)}\right] = 0.12646$$

$$u_2^{(5)} = \frac{1}{4}\left[u_1^{(5)} + u_4^{(4)}\right] = 0.12573$$

$$u_3^{(5)} = \frac{1}{4}\left[u_4^{(4)} + u_1^{(5)} + 1\right] = 0.37573$$

$$u_4^{(5)} = \frac{1}{4}\left[u_3^{(5)} + u_2^{(5)} + 1\right] = 0.37536$$

$$u_1^{(6)} = \frac{1}{4}\left[u_2^{(5)} + u_3^{(5)}\right] = 0.12536$$

$$u_2^{(6)} = \frac{1}{4}\left[u_1^{(6)} + u_4^{(5)}\right] = 0.12518$$

$$u_3^{(6)} = \frac{1}{4}\left[u_4^{(5)} + u_1^{(6)} + 1\right] = 0.37518$$

$$u_4^{(6)} = \frac{1}{4}\left[u_3^{(6)} + u_2^{(6)} + 1\right] = 0.37509$$

Prob.39. Find the values of u(x, y) satisfying the Laplaces equation $\nabla^2 u = 0$ at the pivotal points of a square region with boundary values as shown in the following figure –

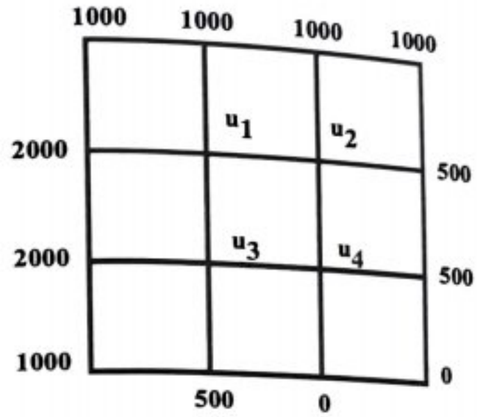

Fig. 3.7

(R.G.P.V., Dec. 2004)

Sol. To get the initial values of u_1, u_2, u_3, u_4 we assume $u_4 = 0$. Then

$$u_1 = \frac{1}{4}(1000 + 0 + 1000 + 2000) = 1000$$

(by diagonal formula)

$$u_2 = \frac{1}{4}(1000 + 500 + 1000 + 0) = 625$$

(standard five-point formula)

$$u_3 = \frac{1}{4}(2000 + 0 + 1000 + 500) = 875$$

and $$u_4 = \frac{1}{4}(875 + 0 + 625 + 0) = 375$$

We carry out the successive iterating, using the formula.

$$u_1^{(n+1)} = \frac{1}{4}\left[2000 + u_2^n + 1000 + u_3^{(n)}\right]$$

$$u_2^{(n+1)} = \frac{1}{4}\left[u_1^{(n+1)} + 500 + 1000 + u_4^{(n)}\right]$$

$$u_3^{(n+1)} = \frac{1}{4}\left[2000 + u_4^{(n)} + u_1^{(n+1)} + 500\right]$$

and $$u_4^{(n+1)} = \frac{1}{4}\left[u_3^{(n+1)} + 0 + u_2^{(n+1)} + 0\right]$$

For first iteration – (Put n = 0)

$$u_1^{(1)} = \frac{1}{4}\left[2000 + u_2^{(0)} + 1000 + u_3^{(0)}\right]$$

or $$u_1^{(1)} = \frac{1}{4}(2000+625+1000+875) = 1125$$

$$u_2^{(1)} = \frac{1}{4}(1125+500+1000+375) = 750$$

$$u_3^{(1)} = \frac{1}{4}(2000+375+1125+500) = 1000$$

and $$u_4^{(1)} = \frac{1}{4}(1000+0+750+0) = 438$$

Second iteration – (Put n = 1)

$$u_1^{(2)} = \frac{1}{4}(2000+750+1000+1000) \approx 1188$$

$$u_2^{(2)} = \frac{1}{4}(1188+500+1000+438) \approx 782$$

$$u_3^{(2)} = \frac{1}{4}(2000+438+1188+500) \approx 1032$$

and $$u_4^{(2)} = \frac{1}{4}(1032+0+782+0) \approx 454$$

Third iteration – (Put n = 2)

$$u_1^{(3)} = \frac{1}{4}(2000+782+1000+1032) \approx 1204$$

$$u_2^{(3)} = \frac{1}{4}(1204+500+1000+454) \approx 790$$

$$u_3^{(3)} = \frac{1}{4}(2000+454+1204+500) \approx 1040$$

and $$u_4^{(3)} = \frac{1}{4}(1040+0+790+0) \approx 458$$

Similarly $u_1^{(4)} \approx 1207, u_2^{(4)} \approx 791, u_3^{(4)} \approx 1041$ and $u_4^{(4)} \approx 458$

and $u_1^{(5)} \approx 1208, u_2^{(5)} \approx 791.5, u_3^{(5)} \approx 1041.5$ and $u_4^{(5)} \approx 458.25$

Thus there is no significant difference between the fourth and fifth iteration values.

Hence $u_1 = \mathbf{1208}$, $u_2 = \mathbf{792}$, $u_3 = \mathbf{1042}$, $u_4 = \mathbf{458}$. **Ans.**

Prob.40. Solve the Poisson's equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = -10(x^2 + y^2 + 10)$$ ***over the square with sides $x = 0 = y$, $x = 3 = y$ with $u(x, y) = 0$ on the boundary and mesh length = 1.***

Or

Solve the partial differential equation

$\nabla^2 u = -10(x^2 + y^2 + 10)$ ***over the square with sides $x = 0 = y$, $x = 3 = y$ with $u = 0$ on the boundary and mesh length = 1.*** ***(R.G.P.V., June 2004)***

Sol. Here h = 1

The standard five-point formula for the given equation is

$$u_{i-1,j} + u_{i+1,j} + u_{i,j+1} + u_{i,j-1} - 4u_{i,j} = -10(x^2 + y^2 + 10) \quad ...(i)$$

For u_1 (i = 1, j = 2), equation (i) becomes

$$0 + u_2 + 0 + u_3 - 4u_1 = -10\,(1 + 4 + 10)$$

$$u_2 + u_3 - 4u_1 = -150$$

or $$u_1 = \frac{1}{4}(u_2 + u_3 + 150) \quad ...(ii)$$

or

For u_2 (i = 2, j = 2), equation (i) gives

$$u_1 + 0 + 0 + u_4 - 4u_2 = -10\,(4 + 4 + 10)$$

$$\Rightarrow \quad u_1 + u_4 - 4u_2 = -180$$

or $$u_2 = \frac{1}{4}(u_1 + u_4 + 180) \quad ...(iii)$$

For u_3 (i = 1, j = 1), equation (i) gives

$$0 + u_4 + u_1 + 0 - 4u_3 = -10\,(1 + 1 + 10)$$

$$\Rightarrow \quad u_4 + u_1 - 4u_3 = -120$$

or $$u_3 = \frac{1}{4}(u_1 + u_4 + 120) \quad ...(iv)$$

For u_4 (i = 2, j = 1), equation (i) gives

$$u_3 + 0 + u_2 + 0 - 4u_4 = -10\,(4 + 1 + 10)$$

$$\Rightarrow \quad u_3 + u_2 - 4u_4 = -150$$

or $$u_4 = \frac{1}{4}(u_2 + u_3 + 150) \quad ...(v)$$

Equations (ii) and (v) we have

$$u_4 = u_1$$

Thus the above equations reduce to

$$u_1 = \frac{1}{4}(u_2 + u_3 + 150) \quad \ldots(vi)$$

$$u_2 = \frac{1}{4}(u_1 + u_1 + 180) = \frac{1}{2}(u_1 + 90) \quad \ldots(vii)$$

$$u_3 = \frac{1}{4}(u_1 + u_1 + 120) = \frac{1}{2}(u_1 + 60) \quad \ldots(viii)$$

Now let us solve these equations by Gauss-Seidel method.

For first iteration, putting $u_2 = u_3 = 0$ in equation (vi), we obtain

$$u_1^{(1)} = \frac{150}{4} = 37.5$$

Putting $u_1 = u_1^{(1)}$ in equation (vii), we obtain

$$u_2^{(1)} = \frac{1}{2}(37.5 + 90) = 64$$

Putting $u_1 = u_1^{(1)}$ in equation (viii), we obtain

$$u_3^{(1)} = \frac{1}{2}(37.5 + 60) = 49$$

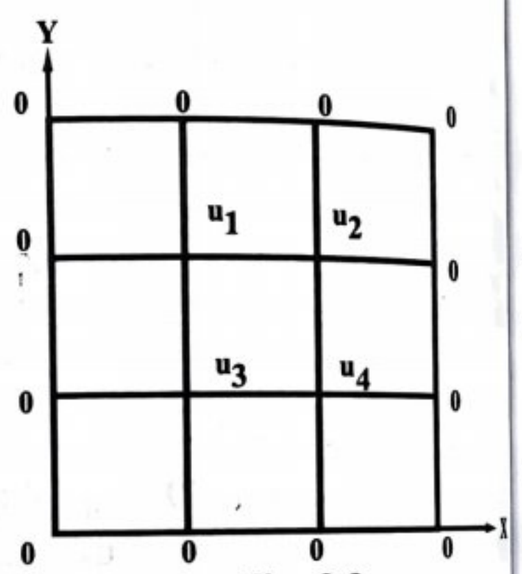

Fig. 3.8

For the second iteration, we have

$$u_1^{(2)} = \frac{1}{4}[u_2^{(1)} + u_3^{(1)} + 150]$$

$$= \frac{1}{4}(64 + 49 + 150) = 66$$

$$u_2^{(2)} = \frac{1}{2}[u_1^{(2)} + 90] = \frac{1}{2}(66 + 90) = 78$$

$$u_3^{(2)} = \frac{1}{2}[u_1^{(2)} + 60] = \frac{1}{2}(66 + 60) = 63$$

For the third iteration, we have

$$u_1^{(3)} = \frac{1}{4}[u_2^{(2)} + u_3^{(2)} + 150] = \frac{1}{4}(78 + 63 + 150) = 73$$

$$u_2^{(3)} = \frac{1}{2}[u_1^{(3)} + 90] = \frac{1}{2}(73 + 90) = 82$$

$$u_3^{(3)} = \frac{1}{2}[u_1^{(3)} + 60] = \frac{1}{2}(73 + 60) = 67$$

For the fourth iteration, we have

$$u_1^{(4)} = \frac{1}{4}[u_2^{(3)} + u_3^{(3)} + 150] = \frac{1}{4}(82 + 67 + 150) = 75$$

$$u_2^{(4)} = \frac{1}{2}[u_1^{(4)} + 90] = \frac{1}{2}(75 + 90) = 82.5$$

$$u_3^{(4)} = \frac{1}{2}[u_1^{(4)} + 60] = \frac{1}{2}(75 + 60) = 67.5$$

For the fifth iteration, we have

$$u_1^{(5)} = \frac{1}{4}[u_2^{(4)} + u_3^{(4)} + 150] = \frac{1}{4}(82.5 + 67.5 + 150) = 75$$

$$u_2^{(5)} = \frac{1}{2}[u_1^{(5)} + 90] = \frac{1}{2}(75 + 90) = 82.5$$

$$u_3^{(5)} = \frac{1}{2}[u_1^{(5)} + 60] = \frac{1}{2}(75 + 60) = 67.5$$

Since these values are the same as those of fourth iteration, we have

$u_1 = \mathbf{75}$, $u_2 = \mathbf{82.5}$, $u_3 = \mathbf{67.5}$ and $u_4 = \mathbf{75}$. **Ans.**

Prob.41. Solve the heat conduction equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial t^2}, \quad 0 \le x \le 1, t \ge 0$$

subject to boundary conditions $u = 0$ at $x = 0$ and $\frac{\partial u}{\partial x} = 0$ at $x = 1$, for all t and initial condition $u(x, 0) = \sin\frac{3\pi x}{2}$

Using the Crank-Nicolson method, choosing $h = 0.1$ and $k = 0.0025$ so that $r = \frac{1}{4}$, obtain the solution for one time level and compare it with the explicit solution.

Sol. Crank-Nicolson finite-difference representation of the given equation is

$$\frac{u_{i,j+1} - u_{i,j}}{2.\frac{k}{2}} = \frac{1}{2h^2}\left[(u_{i-1,j} - 2u_{i,j} + u_{i+1,j}) + (u_{i-1,j+1} - 2u_{i,j+1} + u_{i+1,j+1})\right]$$

or $2(u_{i,j+1} - u_{i,j}) = r(u_{i-1,j} - 2u_{i,j} + u_{i+1,j} + u_{i-1,j+1} - 2u_{i,j+1} + u_{i+1,j+1})$

$$\left(\because r = \frac{k}{h^2}\right)$$

or $-r\,u_{i-1,j+1} + (2 + 2r)\,u_{i,j+1} - r\,u_{i+1,j+1} = r\,u_{i-1,j} + (2 - 2r)\,u_{i,j} + r\,u_{i+1,j}$ ...(i)

Initial condition is $u(x, 0) = \sin\frac{3\pi x}{2}$

Boundary conditions are $u_{0,j} = 0$ and $\left(\frac{\partial u}{\partial x}\right)_{N,j} = 0; N = 10$

Replacing L.H.S. by backward difference

$$\frac{u_{N-1,j} - u_{N,j}}{h} = 0$$

or $$u_{N-1,j} - u_{N,j} = 0$$

or $$u_{N,j} = u_{N-1,j}$$

or $$u_{10,j} = u_{9,j} \quad \text{...(ii)}$$

Putting j = 0 in equation (i), we get

$$-ru_{i-1,1} + (2+2r)u_{i,1} - ru_{i+1,1} = ru_{i-1,0} + (2-2r)u_{i,0} + ru_{i+1,0} \quad \text{...(iii)}$$

Now putting i = 1, 2, 3, ..., 9 in equation (iii), we get 9 equations in 10 unknowns. So with the help of equation (ii), we have 10 equations in 10 unknowns.

The corresponding equations are

$$2.5\ u_{1,1} - 0.25u_{2,1} = 1.5\ u_{1,0} = 0.25\ u_{2,0}$$

$$-0.25\ u_{1,1} + 2.5\ u_{2,1} - 0.25\ u_{3,1} = 0.25\ u_{1,0} + 1.5\ u_{2,0} + 0.25\ u_{3,0}$$

$$-0.25\ u_{2,1} + 2.5\ u_{3,1} - 0.25\ u_{4,1} = 0.25\ u_{2,0} + 1.5\ u_{3,0} + 0.25\ u_{4,0}$$

$$-0.25\ u_{3,1} + 2.5\ u_{4,1} - 0.25\ u_{5,1} = 0.25\ u_{3,0} + 1.5\ u_{4,0} + 0.25\ u_{5,0}$$

$$-0.25\ u_{4,1} + 2.5\ u_{5,1} - 0.25\ u_{6,1} = 0.25\ u_{4,0} + 1.5\ u_{5,0} + 0.25\ u_{6,0}$$

$$-0.25\ u_{5,1} + 2.5\ u_{6,1} - 0.25\ u_{7,1} = 0.25\ u_{5,0} + 1.5\ u_{6,0} + 0.25\ u_{7,0}$$

$$-0.25\ u_{6,1} + 2.5\ u_{7,1} - 0.25\ u_{8,1} = 0.25\ u_{6,0} + 1.5\ u_{7,0} + 0.25\ u_{8,0}$$

$$-0.25\ u_{7,1} + 2.5\ u_{8,1} - 0.25\ u_{9,1} = 0.25\ u_{7,0} + 1.5\ u_{8,0} + 0.25\ u_{9,0}$$

$$-0.25\ u_{8,1} + 2.5\ u_{9,1} - 0.25\ u_{10,1} = 0.25\ u_{8,0} + 1.5\ u_{9,0} + 0.25\ u_{10,0} \quad \text{...(iv)}$$

With the help of equation (ii), the last equation (iv) becomes

$$-0.25\ u_{8,1} + 2.25\ u_{9,1} = 0.25\ u_{8,0} + 1.75\ u_{9,0}$$

On solving this system of equation we get values at the first time level as follows –

	i=0	i=1	i=2	i=3	i=4	i=5	i=6	i=7	i=8	i=9	i=10
	x=0	x=0.1	x=0.1	x=0.1	x=0.1	x=0.1	x=0.1	x=0.1	x=0.1	x=0.1	x=0.1
j=0	0	0.0082	0.0164	0.0247	0.0329	0.0411	0.0493	0.0575	0.0657	0.0740	0.0740
j=1	0	0.0082	0.0164	0.0247	0.0329	0.0411	0.0493	0.0575	0.0656	0.0722	0.0722

Comparison between the results obtained by Crank-Nicolson and Explicit methods –

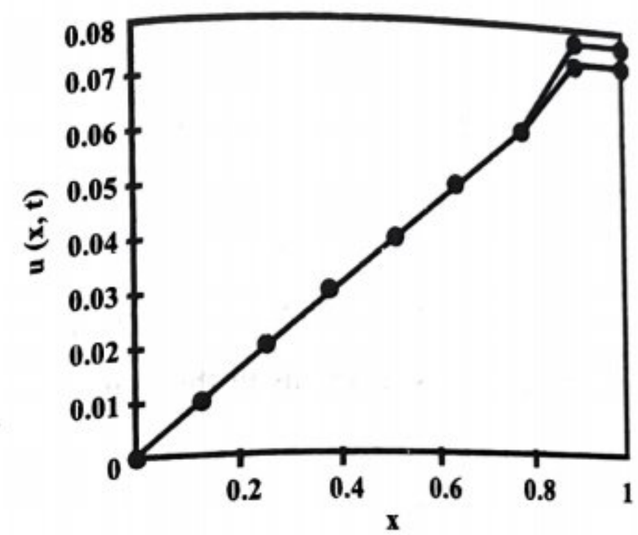

Fig. 3.9

Prob.42. Evaluate the pivotal values of the equation $u_{tt} = 16\ u_{xx}$, taking $h = 1$ upto $t = 1.25$. The boundary conditions are $u(0, t) = u(5, t) = 0$, $u_t(x, 0) = 0$ and $u(x, 0) = x^2(5 - x)$.

Sol. The given equation is

$$u_{tt} = 16\ u_{xx}$$

or $$\frac{\partial^2 u}{\partial t^2} = 16\frac{\partial^2 u}{\partial x^2}$$

Here $$C^2 = 16$$

∴ The difference equation for the given equation is

$$u_{i,j+1} = 2(1-16\alpha^2)u_{i,j} + 16\alpha^2(u_{i-1,j} + u_{i+1,j}) - u_{i,j-1} \quad \text{...(i)}$$

where, $\alpha = \dfrac{k}{h}$

Taking h = 1, we choose k so that the coefficient of $u_{i,j}$ vanishes, we get

$$1 - 16\ \alpha^2 = 0$$

$$1 - 16\left(\frac{k}{h}\right)^2 = 0 \qquad (\because h = 1)$$

$$1 - 16\ k^2 = 0$$

$$\therefore\ k = \frac{1}{4} = 0.25$$

We are to find the solution from $t = 0$ to $t = 1.25$. Since $t = jk$, therefore we are to find solution from $jk = 0$ to $jk = 1.25$.

$(\because k = 0.25)$

i.e. $j = 0$ to $j = 5$

Now putting $16\alpha^2 = 1$ in equation (i), we get

$$u_{i,j+1} = u_{i-1,j} + u_{i+1,j} - u_{i,j-1}$$

$$u(0, t) = 0,\ u(5, t) = 0$$

$\because$ $$u_{0,j} = 0,\ u_{5,j} = 0 \text{ for all values of } j \qquad \text{...(iii)}$$

$\therefore$ i.e., the entries of the first and last columns in the table are zero.

$$u(x, 0) = x^2(5 - x)$$

$\because$ $$u_{i,0} = i^2(5 - i)$$

$\therefore$ $$u_{1,0} = (1)^2(5 - 1) = 4$$

$$u_{2,0} = (2)^2(5 - 2) = 12$$

$$u_{3,0} = (3)^2(5 - 3) = 18$$

$$u_{4,0} = (4)^2(5 - 4) = 16$$

These are the entries of the first row in the table.

Finally the initial condition $u_t(x,0) = 0$ gives

$$\frac{u_{i,j+1} - u_{i,j-1}}{2k} = 0$$

$$\Rightarrow \quad u_{i,j+1} - u_{i,j-1} = 0$$

$$\Rightarrow \quad u_{i,j+1} = u_{i,j-1}$$

$$\Rightarrow \quad u_{i,1} = u_{i,-1} \text{ for } j = 0 \qquad \text{...(iv)}$$

Putting $j = 0$ in equation (ii), we get

$$u_{i,1} = u_{i-1,0} + u_{i+1,0} - u_{i,-1}$$

or $$u_{i,1} = u_{i-1,0} + u_{i+1,0} - u_{i,1} \qquad \text{[By equation (iv)]}$$

or $$2u_{i,1} = u_{i-1,0} + u_{i+1,0}$$

or $$u_{i,1} = \frac{1}{2}\left(u_{i-1,0} + u_{i+1,0}\right) \qquad \text{...(v)}$$

Putting $i = 1, 2, 3, 4$ in equation (v), we get

$$u_{1,1} = \frac{1}{2}\left(u_{0,0} + u_{2,0}\right) = \frac{1}{2}(0 + 12) = 6$$

$$u_{2,1} = \frac{1}{2}\left(u_{1,0} + u_{3,0}\right) = \frac{1}{2}(4 + 18) = 11$$

$$u_{3,1} = \frac{1}{2}\left(u_{2,0} + u_{4,0}\right) = \frac{1}{2}(12 + 16) = 14$$

$$u_{4,1} = \frac{1}{2}\left(u_{3,0} + u_{5,0}\right) = \frac{1}{2}(18 + 0) = 9$$

These are the entries of the second row in the table.

Putting $j = 1$ in equation (ii), we get

$$u_{i,2} = u_{i-1,1} + u_{i+1,1} - u_{i,0}$$

$\therefore$ $$u_{1,2} = u_{0,1} + u_{2,1} - u_{1,0} = 0 + 11 - 4 = 7$$

$$u_{2,2} = u_{1,1} + u_{3,1} - u_{2,0} = 6 + 14 - 12 = 8$$

$$u_{3,2} = u_{2,1} + u_{4,1} - u_{3,0} = 11 + 9 - 18 = 2$$

$$u_{4,2} = u_{3,1} + u_{5,1} - u_{4,0} = 14 + 0 - 16 = -2$$

These are the entries of the third row in the table.

Putting $j = 2$ in equation (ii), we get

$$u_{i,3} = u_{i-1,2} + u_{i+1,2} - u_{i,1}$$

$\therefore$ $$u_{1,3} = u_{0,2} + u_{2,2} - u_{1,1} = 0 + 8 - 6 = 2$$

$$u_{2,3} = u_{1,2} + u_{3,2} - u_{2,1} = 7 + 2 - 11 = -2$$

$$u_{3,3} = u_{2,2} + u_{4,2} - u_{3,1} = 8 - 2 - 14 = -8$$

$$u_{4,3} = u_{3,2} + u_{5,2} - u_{4,1} = 2 + 0 - 9 = -7$$

These are the entries of the fourth row in the table.

Putting $j = 3$ in equation (ii), we get

$$u_{i,4} = u_{i-1,3} + u_{i+1,3} - u_{i,2}$$

$\therefore$ $$u_{1,4} = u_{0,3} + u_{2,3} - u_{1,2} = 0 - 2 - 7 = -9$$

$$u_{2,4} = u_{1,3} + u_{3,3} - u_{2,2} = 2 - 8 - 8 = -14$$

$$u_{3,4} = u_{2,3} + u_{4,3} - u_{3,2} = -2 - 7 - 2 = -11$$

$$u_{4,4} = u_{3,3} + u_{5,3} - u_{4,2} = -8 + 0 + 2 = -6$$

These are the entries of the fifth row in the table.

Putting j = 4 in equation (ii), we get

$$u_{i,5} = u_{i-1,4} + u_{i+1,4} - u_{i,3}$$

$$\therefore \quad u_{1,5} = u_{0,4} + u_{2,4} - u_{1,3} = 0 - 14 - 2 = -16$$

$$u_{2,5} = u_{1,4} + u_{3,4} - u_{2,3} = -9 - 11 + 2 = -18$$

$$u_{3,5} = u_{2,4} + u_{4,4} - u_{3,3} = -14 - 6 + 8 = -12$$

$$u_{4,5} = u_{3,4} + u_{5,4} - u_{4,3} = -11 + 0 + 7 = -4$$

These are the entries of the sixth row in the following table –

$j \downarrow$ \ $i \rightarrow$	0	1	2	3	4	5
0	0	4	12	18	16	0
1	0	6	11	14	9	0
2	0	7	8	2	−2	0
3	0	2	−2	−8	−7	0
4	0	−9	−14	−11	−6	0
5	0	−16	−18	−12	−4	0

MODULE 4

TRANSFORM CALCULUS

LAPLACE TRANSFORM, PROPERTIES OF LAPLACE TRANSFORM, EVALUATION OF INTEGRALS BY LAPLACE TRANSFORM, LAPLACE TRANSFORM OF PERIODIC FUNCTIONS

Laplace Transform – The method of Laplace transforms has the advantage of directly giving the solution of differential equations with given boundary values without the necessity of first finding the general solution and then evaluating from it the arbitrary constants.

Definition – Suppose f(t) is a real valued function defined over the interval $(-\infty, \infty)$ such that $f(t) = 0, \forall\, t < 0$.

The ***Laplace transform*** of f(t), denoted by L{f(t)} is defined as

$$L\{f(t)\} = \int_0^\infty e^{-st} f(t)dt. \qquad \text{...(i)}$$

We also write $L\{f(t)\} = F(s) = \int_0^\infty e^{-st} f(t)dt$

Here L is said to be the ***Laplace transformation operator.*** The parameter s is a real or complex number. In general, the parameter s is taken to be a real positive number.

If functions are denoted by lower case letters, f(t), g(t), h(t),... then their Laplace transforms are denoted by

$\bar{f}(s), \bar{g}(s), \bar{h}(s), \ldots$ respectively or $\bar{f}(p), \bar{g}(p), \bar{h}(p), \ldots$

Laplace Transform of Elementary Functions (Some Standard Results of Laplace Transform) –

(i) $L(1) = \dfrac{1}{s}, s > 0$

(ii) $L(t^n) = \dfrac{n!}{s^{n+1}}, \left(\text{or } \dfrac{\Gamma(n+1)}{s^{n+1}}\right), \quad \text{if } s > 0, n > -1$

(iii) $L(e^{at}) = \dfrac{1}{s-a}, \quad \text{if } s > a$

(iv) $L(\sin at) = \dfrac{a}{s^2+a^2}$, if $s > 0$

(v) $L(\cos at) = \dfrac{s}{s^2+a^2}$, if $s > 0$

(vi) $L(\sinh at) = \dfrac{a}{s^2-a^2}$, if $s > |a|$

(vii) $L(\cosh at) = \dfrac{s}{s^2-a^2}$, if $s > |a|$

(viii) $L(e^{at}t^n) = \dfrac{n!}{(s-a)^{n+1}}$ $\left[\because L(t^n) = \dfrac{n!}{s^{n+1}}\right]$

(ix) $L(e^{at}\sin bt) = \dfrac{b}{(s-a)^2+b^2}$

(x) $L(e^{at}\cos bt) = \dfrac{s-a}{(s-a)^2+b^2}$

(xi) $L(e^{at}\sinh bt) = \dfrac{b}{(s-a)^2-b^2}$

(xii) $L(e^{at}\cosh bt) = \dfrac{s-a}{(s-a)^2-b^2}$, where in each case $s > a$.

Properties of Laplace Transform –

(i) Linear Property **– Suppose $F_1(s)$ and $F_2(s)$ are Laplace transforms of $f_1(t)$ and $f_2(t)$ respectively. Then**

$$L\{c_1f_1(t) + c_2f_2(t)\} = c_1L\{f_1(t)\} + c_2L\{f_2(t)\}$$

where c_1 and c_2 are any constants.

Proof. Suppose $L\{f_1(t)\} = F_1(s) = \int_0^\infty e^{-st}f_1(t)dt$

and $L\{f_2(t)\} = F_2(s) = \int_0^\infty e^{-st}f_2(t)dt$

Also suppose c_1 and c_2 are arbitrary constants. We claim

$$L\{c_1f_1(t) + c_2f_2(t)\} = c_1L\{f_1(t)\} + c_2L\{f_2(t)\}$$

Taking L.H.S.

$$L\{c_1f_1(t) + c_2f_2(t)\} = \int_0^\infty e^{-st}[c_1f_1(t) + c_2f_2(t)]dt$$

$$= c_1\int_0^\infty e^{-st}f_1(t)dt + c_2\int_0^\infty e^{-st}f_2(t)dt$$

$$= c_1L\{f_1(t)\} + c_2L\{f_2(t)\}$$ **Proved**

(ii) First Shifting Property **– If $L\{f(t)\} = F(s)$, then $L\{e^{at}f(t)\} = F(s-a)$.**

Proof. Suppose $L\{f(t)\} = F(s) = \int_0^\infty e^{-st}f(t)dt$

Then $L\{e^{at}f(t)\} = \int_0^\infty e^{-st}f(t)dt = \int_0^\infty e^{-(s-a)t}f(t)dt$

$$= \int_0^\infty e^{-ut}f(t)dt, \text{ where } u = s-a > 0$$

$$= F(u) = F(s-a)$$ **Proved**

(iii) Second Shifting Property –

If $L\{f(t)\} = F(s)$ and $g(t) = \begin{cases} f(t-a), & t > a \\ 0, & t < a \end{cases}$

then $L\{g(t)\} = e^{-as}F(s)$ or $L\{f(t-a)\,h(t-a)\} = e^{-as}F(s)$.

Proof. Suppose

$$L\{f(t)\} = F(s)$$

and $g(t) = \begin{cases} f(t-a), & t > a \\ 0, & t < a \end{cases}$

$$\therefore \quad L\{g(t)\} = \int_0^\infty e^{-st}g(t)dt = \int_0^\infty e^{-st}g(t)dt + \int_0^\infty e^{-st}g(t)dt$$

$$= \int_0^a 0.e^{-st}dt + \int_0^\infty e^{-st}f(t-a)dt = 0 + \int_0^\infty e^{-st}f(t-a)dt$$

Put $t - a = p$, so that $dt = dp$

If $t = a$, then $p = t - a = a - a = 0$

If $t = \infty$, then $p = \infty - a = \infty$.

$$\therefore \quad L\{g(t)\} = \int_0^\infty e^{-s(p+a)}f(p)dp$$

$$= e^{-sa}\int_0^\infty e^{-st}f(p)dp = e^{-sa}F(s)$$ **Proved**

(iv) Change of Scale Property –

If $L\{f(t)\} = F(s)$, then $L\{f(at)\} = \dfrac{1}{a}F\left(\dfrac{s}{a}\right)$

Proof. Suppose $L\{f(t)\} = F(s)$, then

$$L\{f(at)\} = \int_0^\infty e^{-st}f(at)dt,$$

$$= \int_0^\infty e^{-sx/a}f(x)\frac{dx}{a},$$ (Put, $at = x$)

$$\int_s^\infty F(s)\,ds = \int_0^\infty \left[\int_s^\infty e^{-st}\,ds\right] f(t)\,dt = \int_0^\infty \left[\frac{e^{-st}}{-t}\right]_s^\infty f(t)\,dt$$

$$= \int_0^\infty e^{-st}\frac{f(t)}{t}\,dt = L\left\{\frac{f(t)}{t}\right\}$$

$$\therefore \quad \mathbf{L\left\{\frac{f(t)}{t}\right\} = \int_s^\infty F(s)\,ds}$$

Proved

Evaluation of Integrals by Laplace Transform –

Let, $L[f(t)] = F(s)$, i.e., $\int_0^\infty e^{-st} f(t)\,dt = F(s)$, taking the limit of it as $s \to 0$, we have

$$\int_0^\infty f(t)\,dt = F(0)\text{, provided the integral is convergent.}$$

Initial Value Theorem –

To prove that $\lim\limits_{t\to 0} f(t) = \lim\limits_{s\to\infty} sF(s)$.

Proof. We know that, $L[f'(t)] = sF(s) - f(0)$

or $$\int_0^\infty e^{-st} f'(t)\,dt = sF(s) - f(0)$$

Taking limit $s \to \infty$, on both sides, we get

$$\lim_{s\to\infty}\int_0^\infty e^{-st} f'(t)\,dt = \lim_{s\to\infty}\{sF(s) - f(0)\}$$

or $$\lim_{s\to\infty} sF(s) = f(0) + \int_0^\infty \left(\lim_{s\to\infty} e^{-st}\right) f'(t)\,dt$$

or $$\lim_{s\to\infty} sF(s) = f(0) + \int_0^\infty 0.f'(t) = f(0) + 0 = f(0) = \lim_{t\to 0} f(t)$$

Hence $$\mathbf{\lim_{s\to\infty} sF(s) = \lim_{t\to\infty} f(t)}$$

Proved

Final Value Theorem –

To prove that, $\lim\limits_{t\to\infty} f(t) = \lim\limits_{s\to 0} sF(s)$.

Proof. We know that, $L\{f'(t)\} = sF(s) - f(0)$

or $$\int_0^\infty e^{-st} f'(t)\,dt = sF(s) - f(0)$$

Taking limit $s \to 0$, on both sides, we get

$$\lim_{s\to 0} sF(s) - f(0) = \lim_{s\to 0}\int_0^\infty e^{-st} f'(t)\,dt = \int_0^\infty \left(\lim_{s\to 0} e^{-st}\right) f'(t)\,dt$$

$$= \int_0^\infty f'(t)\,dt = \int_0^\infty \frac{df(t)}{dt}.dt = [f(t)]_{t=0}^\infty$$

$$= \lim_{t\to\infty} f(t) - f(0) \text{ or } \mathbf{\lim_{s\to 0} sF(s) = \lim_{t\to\infty} f(t)}$$

Proved

Laplace Transform of the Laplace Transform –

Since $$L[f(t)] = F(s) \text{ or } L[f(t)] = \int_0^\infty e^{-st} f(t)\,dt$$

then $$L[L\{f(t)\}] = L\int_0^\infty e^{-st} f(t)\,dt = \int_0^\infty e^{-us}\left[\int_0^\infty e^{-st} f(t)\,dt\right] ds$$

The area of integration being the whole positive quadrant. Now changing the order of integration, we get

$$L[L\{f(t)\}] = \int_0^\infty f(t)\,dt \int_0^\infty e^{-s(t+u)}\,ds = \int_0^\infty f(t)\,dt \left[\frac{e^{-s(t+u)}}{e^{-(t+u)}}\right]_0^\infty$$

Hence $$\mathbf{L[L\{f(t)\}] = \int_0^\infty \frac{f(t)}{t+u}\,dt}$$

Proved

Laplace Transform of some Special Functions –

(i) The Error Function –

If $\operatorname{erf}(\sqrt{t}) = \frac{2}{\sqrt{\pi}}\int_0^{\sqrt{t}} e^{-x^2}\,dx$, then $L\{\operatorname{erf}\sqrt{t}\} = \frac{1}{s\sqrt{s+1}}$

Proof. We have $\operatorname{erf}(\sqrt{t}) = \frac{2}{\sqrt{\pi}}\int_0^{\sqrt{t}} e^{-x^2}\,dx$

$$= \frac{2}{\sqrt{\pi}}\int_0^{\sqrt{t}}\left[1 - x^2 + \frac{x^4}{2!} - \frac{x^6}{3!} + \ldots\ldots\right] dx$$

$$= \frac{2}{\sqrt{\pi}}\left[x - \frac{x^3}{3} + \frac{x^5}{5.2!} - \frac{x^7}{7.3!} + \ldots\ldots\right]_0^{\sqrt{t}}$$

or $$\operatorname{erf}(\sqrt{t}) = \frac{2}{\sqrt{\pi}}\left[t^{1/2} - \frac{t^{3/2}}{3} + \frac{t^{5/2}}{5.2!} - \frac{t^{7/2}}{7.3!} + \ldots\ldots\right] \quad \ldots(i)$$

Taking Laplace transform of both sides of equation (i), we get

$$L\{\operatorname{erf}\sqrt{t}\} = L\left[\frac{2}{\sqrt{\pi}}\left(t^{1/2} - \frac{t^{3/2}}{3} + \frac{t^{5/2}}{5.2!} - \frac{t^{7/2}}{7.3!} + \ldots\ldots\right)\right]$$

or $$L\{\operatorname{erf}\sqrt{t}\} = \frac{2}{\sqrt{\pi}}\left[\frac{\Gamma 3/2}{s^{3/2}} - \frac{\Gamma 5/2}{3s^{5/2}} + \frac{\Gamma 7/2}{5.2!\,s^{7/2}} - \frac{\Gamma 9/2}{7.3!\,s^{9/2}} + \ldots\ldots\right]$$

$$= \frac{1}{s^{3/2}}\left[1 - \frac{1}{2}.\frac{1}{s} + \frac{1}{2}.\frac{3}{4}.\frac{1}{s^2} - \frac{1}{2}.\frac{3}{4}.\frac{5}{6}.\frac{1}{s^3} + \ldots\ldots\right] = \frac{1}{s^{3/2}}\left(1 + \frac{1}{s}\right)^{-1/2}$$

Hence $$\mathbf{L\{\operatorname{erf}\sqrt{t}\} = \frac{1}{s(s+1)^{1/2}}}$$

Proved

(ii) Dirac-delta Function –

If $F_\varepsilon(t)$ is Dirac-delta function defined by

$$F_\varepsilon(t) = \begin{cases} \frac{1}{\varepsilon}, & 0 \le t \le \varepsilon \\ 0, & t > \varepsilon \end{cases}, \text{ show that } L\{F_\varepsilon(t)\} = \frac{1}{s\varepsilon}(1 - e^{s\varepsilon})$$

Proof. We have

$$L\{F_\varepsilon(t)\} = \int_0^\infty e^{-st} F_\varepsilon(t)\, dt = \int_0^\varepsilon e^{-st} F_\varepsilon(t)\, dt + \int_\varepsilon^\infty e^{-st} F_\varepsilon(t)\, dt$$

$$= \int_0^\varepsilon e^{-st} . \frac{1}{\varepsilon} dt + \int_\varepsilon^\infty e^{-st} .0 dt = \frac{1}{\varepsilon}\left[\frac{e^{-st}}{-s}\right]_0^\varepsilon$$

Hence $\mathbf{L\{F_\varepsilon(t)\} = \frac{1}{s\varepsilon}\left(1 - e^{-s\varepsilon}\right)}$ **Proved**

(iii) Heaviside's Unit Step Function –

Suppose $H(t-a)$ is Heaviside's unit step function defined by

$$H(t-a) = \begin{cases} 1, & t > a \\ 0, & t < a \end{cases}, \text{ then prove that } L\{H(t-a)) = \frac{e^{-sa}}{s}$$

Proof. We have

$$L\{H(t-a)\} = \int_0^\infty e^{-st} H(t-a)\, dt$$

$$= \int_0^a e^{-st} H(t-a)\, dt + \int_a^\infty e^{-st} H(t-a)\, dt$$

$$= \int_0^a e^{-st} .0 dt + \int_a^\infty e^{-st} .1 dt = 0 + \int_a^\infty e^{-st}\, dt = \left[\frac{e^{-st}}{-s}\right]_a^\infty$$

Hence $\mathbf{L\{H(t-a)\} = \frac{e^{-sa}}{s}}$ **Proved**

Laplace Transform of Periodic Function –

Fundamental Theorem – Let $f(t)$ be a periodic function with period T, i.e.

$$f(t + rT) = f(t), \text{ for } r = 1, 2, 3, \ldots$$

then prove that $L\{f(t)\} = \left(1 - e^{-sT}\right)^{-1} \int_0^T f(t) e^{-st}\, dt$

Proof. By definition of Laplace transform

$$L\{f(t)\} = \int_0^\infty e^{-st} f(t)\, dt = \sum_{r=0}^\infty \int_{rT}^{(r+1)T} e^{-st} f(t)\, dt$$

$$= \sum_{r=0}^\infty \int_0^T e^{-s(x+Tr)} f(x + rT)\, dx \quad (\text{put, } t = x + Tr \Rightarrow dt = dx)$$

$$= \sum_{r=0}^\infty e^{-srT} \int_0^T e^{-sx} f(x)\, dx \qquad [\because f(x + rT) = f(x) \text{ for, } r = 1, 2, 3]$$

$$= \int_0^T e^{-st} f(t)\, dt \left(1 + e^{-sT} + e^{-2sT} + \ldots\ldots\right) = \int_0^T e^{-st} f(t)\, dt \left(\frac{1}{1 - e^{-sT}}\right)$$

$$= \mathbf{(1 - e^{-sT})^{-1} \int_0^T e^{-st} f(t)\, dt}$$

Proved

Laplace Transform of Bessel Functions $J_0(x)$ and $J_1(x)$ – The Bessel function of the first kind of orders zero and one which are denoted by $J_0(x)$ and $J_1(x)$ respectively are given by

$$J_0(x) = 1 - \frac{x^2}{2^2} + \frac{x^4}{2^2.4^2} - \frac{x^6}{2^2.4^2.6^2} \cdots \qquad \ldots(i)$$

$$J_1(x) = \frac{x}{2} - \frac{x^3}{2^2.4} + \frac{x^5}{2^2.4^2.6} \cdots \qquad \ldots(ii)$$

From relations (i) and (ii), we see that

$$J_0'(x) = -J_1(x) \text{ and } J_0(0) = 1$$

and $J_1(0) = 0.$

$$\therefore \quad L\{J_0(x)\} = \frac{1}{s} - \frac{1}{2^2} . \frac{2!}{s^3} + \frac{1}{2^2.4^2} . \frac{4!}{s^5} - \frac{1}{2^2.4^2.6^2} . \frac{6!}{s^7} + \ldots$$

$$= \frac{1}{s}\left\{1 - \frac{1}{2}\left(\frac{1}{s^2}\right) + \frac{1}{2} . \frac{3}{4}\left(\frac{1}{s^4}\right) - \frac{1}{2} . \frac{3}{4} . \frac{5}{6} . \left(\frac{1}{s^6}\right) + \ldots\right\}$$

$$= \frac{1}{s}\left[1 + \left(-\frac{1}{2}\right)\left(\frac{1}{s^2}\right) + \frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{2!}\left(\frac{1}{s^2}\right)^2 + \frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)}{3!}\left(\frac{1}{s^2}\right)^3 + \ldots\ldots\right]$$

$$= \frac{1}{s}\left(1 + \frac{1}{s^2}\right)^{-1/2} = \frac{1}{\sqrt{s^2 + 1}} \qquad \ldots(iii)$$

$$\because \quad J_0'(x) = -J_1(x)$$

$$\therefore \quad L\{J_1(x)\} = -L\{J_0'(x)\} = -\left[sL\{J_0(x)\} - J_0(0)\right]$$

$$= -\left[s . \frac{1}{\sqrt{s^2 + 1}} - 1\right] = 1 - \frac{s}{\sqrt{s^2 + 1}} \qquad \ldots(iv)$$

Impulse Function –

When a large force acts for a short time, then the product of the force and the time is said to be the ***impulse*** in applied mechanics. The unit impulse function is the limit function.

$$\delta(t-1) = \begin{cases} \frac{1}{\varepsilon}, & a < t < a+\varepsilon \\ 0, & \text{otherwise} \end{cases}$$

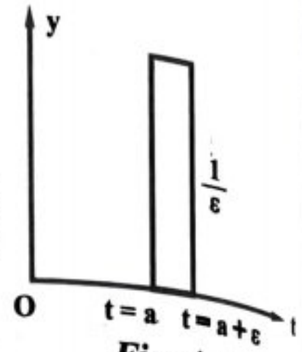

Fig. 4.1

The value of the function (height of the strip in the fig. 4.1) infinite as $\varepsilon \to 0$ and the area of the rectangle is unity.

The Unit Impulse Function – The unit impulse function is defined as follows –

$$\delta(t-a) = \begin{cases} \infty, & \text{for } t = a \\ 0, & \text{for } t \neq a \end{cases}$$

and $\int_0^\infty \delta(t-a)\,dt = 1$ $\quad$ ($\because$ area of strip = 1)

Laplace Transform of Unit Impulse Function –

$$\int_0^\infty f(t)\,\delta(t-a)\,dt = \int_a^{a+\varepsilon} f(t).\frac{1}{\varepsilon}dt = (a+\varepsilon-a)f(\eta).\frac{1}{\varepsilon} = f(\eta)$$

where $a < \eta < a+\varepsilon$ $\quad$ (by mean value theorem for integrals)

Property 1 : $\int_0^\infty f(t)\,\delta(t-a)\,dt = f(a)$, as $\varepsilon \to 0$

Note. Let $f(t) = e^{-st}$ and $L[\delta(t-a)] = e^{-as}$

Property II : $\int_{-\infty}^\infty f(t)\,\delta'(t-a)\,dt = -f'(a)$

Proof. Since

$$\int_{-\infty}^\infty f(t)\,\delta'(t-a)\,dt = [f(t).\delta(t-a)]_{-\infty}^{\infty} - \int_{-\infty}^\infty f'(t)\,\delta(t-a)dt$$

$$= 0 - 0 - f'(a) = -f'(a)$$

Q.1. Write three properties of Laplace transform.
[R.G.P.V., May 2019 (III-Sem)]

Ans. Refer to the matter given on page 186, under same heading.

NUMERICAL PROBLEMS

Prob.1. Find the Laplace transform of

$$f(t) = t^3\, e^{-3t}.$$

Sol. Since $\quad L(e^{-3t}) = \dfrac{1}{s+3}$

$$\therefore \quad L(t^3\, e^{-3t}) = (-1)^3 \frac{d^3}{ds^3}\left(\frac{1}{s+3}\right) \left[\because L\{t^n f(t)\} = (-1)^n \frac{d^n}{ds^n} L\{f(t)\}\right]$$

$$= (-1)\frac{(-1)\,3!}{(s+3)^4} = \frac{6}{(s+3)^4} \qquad \textbf{Ans.}$$

Prob.2. Find L{2 sin t cos t}.

Sol. $L(2\sin t \cos t) = L(\sin 2t)$

$$= \frac{2}{s^2+4}, s > 0 \qquad \textbf{Ans.}$$

Prob.3. Find L{sin t + 3cos 2t + t^3 + $t^{3/2}$ + $3e^{-2t}$ – e^{3t}}.
[R.G.P.V., May 2019 (IV-Sem)]

Sol. $L\{\sin t + 3\cos 2t + t^3 + t^{3/2} + 3e^{-2t} - e^{3t}\}$

$$= L\{\sin t\} + 3L(\cos 2t) + L\{t^3\} + L\{t^{3/2}\} + 3L\{e^{-2t}\} - L\{e^{3t}\}$$

$$= \frac{1}{s^2+1} + 3.\frac{s}{s^2+2^2} + \frac{3!}{s^4} + \frac{\Gamma\left(\frac{5}{2}\right)}{s^{5/2}} + 3.\frac{1}{s+2} - \frac{1}{s-3}$$

$s > 0$, $n > -1$, $s > -2$ and $s > 3$

$$= \frac{1}{s^2+1} + \frac{3s}{s^2+4} + \frac{6}{s^4} + \frac{\left(\frac{3}{2}\right)\left(\frac{1}{2}\right)\sqrt{\pi}}{s^{5/2}} + \frac{3}{s+2} - \frac{1}{s-3}$$

$$= \frac{1}{s^2+1} + \frac{3s}{s^2+4} + \frac{6}{s^4} + \frac{3\sqrt{\pi}}{4s^{5/2}} + \frac{3}{s+2} - \frac{1}{s-3} \qquad \textbf{Ans.}$$

Prob.4. Find L{$3t^4$ – $2t^3$ + $4e^{-3t}$ – 2 sin 5t + 3 cos 2t}.

Sol. $L\{3t^4 - 2t^3 + 4e^{-3t} - 2\sin 5t + 3\cos 2t\}$

$$= 3L\{t^4\} - 2L\{t^3\} + 4L\{e^{-3t}\} - 2L\{\sin 5t\} + 3L\{\cos 2t\}$$

$$= 3.\frac{4!}{s^5} - 2.\frac{3!}{s^4} + 4.\frac{1}{s+3} - 2.\frac{5}{s^2+5^2} + 3.\frac{s}{s^2+2^2}$$

$s > 0$ and $s > -3$ i.e. $s > 0$

$$= \frac{72}{s^5} - \frac{12}{s^4} + \frac{4}{s+3} - \frac{10}{s^2+25} + \frac{3s}{s^2+4}, s > 0 \qquad \textbf{Ans.}$$

Prob.5. Find Laplace transform of the following function –

(i) $2t^3 - 6t + 5$ (ii) $2e^{3t} - e^{-3t}$.

Sol. (i) $L(2t^3 - 6t + 5) = 2L(t^3) - 6L(t) + 5L(1)$ [From property (i)]

$$= 2.\frac{3!}{s^4} - 6.\frac{1!}{s^2} + 5.\frac{1}{s}$$

$$= \frac{12}{s^4} - \frac{6}{s^2} + \frac{5}{s}, s > 0$$ **Ans.**

(ii) $L(2e^{3t} - e^{-3t}) = 2L(e^{3t}) - L(e^{-3t})$

$$= 2.\frac{1}{s-3} - \frac{1}{s+3} = \frac{2s+6-s+3}{(s-3)(s+3)}$$

$$= \frac{s+9}{s^2-9}$$ **Ans.**

Prob.6. Find the Laplace transform of $\frac{1-e^t}{t}$.

Sol. Here, $L(1 - e^t) = L\{1\} - L\{e^t\} = \frac{1}{s} - \frac{1}{s-1}$

$$\therefore L\left\{\frac{1-e^t}{t}\right\} = \int_s^\infty \left(\frac{1}{s} - \frac{1}{s-1}\right) ds = [\log s - \log(s-1)]_s^\infty$$

$$= \left[\log\frac{s}{s-1}\right]_s^\infty = \left[0 - \log\frac{s}{s-1}\right] = -\log\frac{s}{s-1} = \log\frac{s-1}{s}$$ **Ans.**

Prob.7. Find the Laplace transform of $e^{-4t}\sin 3t$.

Sol. Let, $A = e^{-4t}\sin 3t$

Taking Laplace transform on both sides, we have

$$L(A) = L(e^{-4t}\sin 3t)$$

$$= \frac{3}{(s+4)^2 + (3)^2} = \frac{3}{s^2 + 8s + 16 + 9}$$

or $$L(A) = \frac{3}{s^2 + 8s + 25}$$ **Ans.**

Prob.8. Find the Laplace transform of – $t^2 \cos at$.

Sol. We know

$$L\{\cos at\} = \frac{s}{s^2 + a^2}$$

Now

$$L\{t^2 \cos at\} = (-1)^2 \frac{d^2}{ds^2}\left(\frac{s}{s^2 + a^2}\right)$$

$$= \frac{d}{ds}\left\{\frac{(s^2 + a^2)1 - s.2s}{(s^2 + a^2)^2}\right\} = \frac{d}{ds}\left\{\frac{a^2 - s^2}{(s^2 + a^2)^2}\right\}$$

$$= \left\{\frac{(s^2 + a^2)^2.(-2s) - (a^2 - s^2).2(s^2 + a^2).2s}{(s^2 + a^2)^4}\right\}$$

$$= \frac{(s^2 + a^2)(-2s) - (a^2 - s^2).4s}{(s^2 + a^2)^3}$$

$$= \frac{-2s\{(s^2 + a^2) + 2a^2 - 2s^2\}}{(s^2 + a^2)^3} = -2s\left\{\frac{3a^2 - s^2}{(s^2 + a^2)^3}\right\}$$ **Ans.**

Prob.9. Find $L\left\{\frac{1 - \cos 2t}{t}\right\}$. ***[R.G.P.V., Nov. 2019 (IV-Sem)]***

Sol. Here, $L(1 - \cos 2t) = L(1) - L(\cos 2t) = \frac{1}{s} - \frac{s}{s^2 + 4}$

$$\therefore L\frac{(1 - \cos 2t)}{t} = \int_s^\infty \left(\frac{1}{s} - \frac{s}{s^2 + 4}\right) ds = \left[\log s - \frac{1}{2}\log(s^2 + 4)\right]_s^\infty$$

$$= \frac{1}{2}\left[\log s^2 - \log(s^2 + 4)\right]_s^\infty = \frac{1}{2}\left[\log\frac{s^2}{s^2 + 4}\right]_s^\infty$$

$$= \frac{1}{2}\left[0 - \log\frac{s^2}{s^2 + 4}\right] = -\frac{1}{2}\log\frac{s^2}{s^2 + 4}$$ **Ans.**

Prob.10. Find the Laplace transform of $\frac{1 - \cos t}{t^2}$.

Sol. Here, $L(1 - \cos t) = L(1) - L(\cos t)$

$$= \frac{1}{s} - \frac{s}{s^2 + 1}$$

$$\therefore L\left(\frac{1 - \cos t}{t}\right) = \int_s^\infty \left(\frac{1}{s} - \frac{s}{s^2 + 1}\right) ds = \left[\log s - \frac{1}{2}\log(s^2 + 1)\right]_s^\infty$$

$$= \frac{1}{2}\left[\log s^2 - \log(s^2 + 1)\right]_s^\infty = \frac{1}{2}\left[\log\left(\frac{s^2}{s^2 + 1}\right)\right]_s^\infty$$

$$= \frac{1}{2}\left[\log\frac{s^2}{s^2\left(1+\frac{1}{s^2}\right)}\right]_s^\infty = \frac{1}{2}\left[\log\left(\frac{1}{1+\frac{1}{s^2}}\right)\right]_s^\infty$$

$$= \frac{1}{2}\left[\log 1 - \log\left(\frac{1}{1+\frac{1}{s^2}}\right)\right] = \frac{1}{2}\left[0 - \log\left(\frac{s^2}{s^2+1}\right)\right]$$

$$= -\frac{1}{2}\log\left(\frac{s^2}{s^2+1}\right)$$

Again $\quad L\left(\frac{1-\cos t}{t^2}\right) = -\frac{1}{2}\int_s^\infty \log\left(\frac{s^2}{s^2+1}\right)ds = -\frac{1}{2}\int_s^\infty\left[\log\left(\frac{s^2}{s^2+1}\right).1\right]ds$

Integrating by parts, taking '1' as a second function, we have

$$L\left(\frac{1-\cos t}{t^2}\right) = -\frac{1}{2}\left[\log\left(\frac{s^2}{s^2+1}\right).s - \int\frac{1}{s^2/(s^2+1)}.\left\{\frac{(s^2+1).2s - s^2.2s}{(s^2+1)^2}\right\}.s\,ds\right]_s^\infty$$

$$= -\frac{1}{2}\left[s\log\left(\frac{s^2}{s^2+1}\right) - \int\frac{(s^2+1)}{s^2}.\frac{2s.s}{(s^2+1)^2}ds\right]_s^\infty$$

$$= -\frac{1}{2}\left[s\log\left(\frac{s^2}{s^2+1}\right) - 2\int\frac{1}{s^2+1}.ds\right]_s^\infty = -\frac{1}{2}\left[s\log\frac{s^2}{s^2+1} - 2\tan^{-1}s\right]_s^\infty$$

$$= -\frac{1}{2}\left[0 - 2.\left(\frac{\pi}{2}\right) - s\log\left(\frac{s^2}{s^2+1}\right) + 2\tan^{-1}s\right]$$

$$= \frac{\pi}{2} + \frac{s}{2}\log\left(\frac{s^2}{s^2+1}\right) - \tan^{-1}s = \left(\frac{\pi}{2} - \tan^{-1}s\right) + \frac{s}{2}\log\left(\frac{s^2}{s^2+1}\right)$$

$$= \cot^{-1}s + \frac{s}{2}\log\left(\frac{s^2}{s^2+1}\right)$$ **Ans.**

Prob.11. Find the Laplace transform of $\sin\sqrt{t}$.

Sol. Since $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots\ldots$

$$\therefore\ L\{\sin\sqrt{t}\} = L\left\{\sqrt{t} - \frac{(\sqrt{t})^3}{3!} + \frac{(\sqrt{t})^5}{5!} - \frac{(\sqrt{t})^7}{7!} + \ldots\ldots\right\}$$

$$= L\left\{t^{1/2} - \frac{t^{3/2}}{3!} + \frac{t^{5/2}}{5!} - \frac{t^{7/2}}{7!} + \ldots\ldots\right\}$$

$$= L(t^{1/2}) - \frac{1}{3!}L(t^{3/2}) + \frac{1}{5!}L(t^{5/2}) - \frac{1}{7!}L(t^{7/2}) + \ldots\ldots$$

$$= \frac{\Gamma(3/2)}{s^{3/2}} - \frac{1}{3!}\frac{\Gamma(5/2)}{s^{5/2}} + \frac{1}{5!}\frac{\Gamma(7/2)}{s^{7/2}} - \frac{1}{7!}\frac{\Gamma(9/2)}{s^{9/2}} + \ldots\ldots$$

$$= \frac{\frac{1}{2}\sqrt{\pi}}{s^{3/2}} - \frac{1}{6}\frac{\frac{3}{2}\frac{1}{2}\sqrt{\pi}}{s^{5/2}} + \frac{1}{120}\frac{\frac{5}{2}\frac{3}{2}\frac{1}{2}\sqrt{\pi}}{s^{7/2}} - \frac{1}{5040}\frac{\frac{7}{2}\frac{5}{2}\frac{3}{2}\frac{1}{2}\sqrt{\pi}}{s^{9/2}}\ldots\ldots$$

$$= \frac{\sqrt{\pi}}{2s^{3/2}}\left[1 - \left(\frac{1}{4s}\right) + \frac{1}{2!}\left(\frac{1}{4s}\right)^2 - \frac{1}{3!}\left(\frac{1}{4s}\right)^3 + \ldots\ldots\right] = \frac{\sqrt{\pi}}{2s^{3/2}}e^{-\frac{1}{4s}}$$

Ans.

Prob.12. If $f(t) = \frac{e^{at} - \cos bt}{t}$, find the Laplace transform of $f(t)$.

Sol. Here, $\quad f(t) = \frac{e^{at} - \cos bt}{t} = \frac{e^{at}}{t} - \frac{\cos bt}{t}$

We know, $\quad L(e^{at} - \cos bt) = \left(\frac{1}{s-a} - \frac{s}{s^2+b^2}\right)$

$$\therefore\ L\left(\frac{e^{at} - \cos bt}{t}\right) = \int_s^\infty\left(\frac{1}{s-a} - \frac{s}{s^2+b^2}\right)ds$$

$$= \left[\log(s-a) - \frac{1}{2}\log(s^2+b^2)\right]_s^\infty = \left[\frac{2\log(s-a) - \log(s^2+b^2)}{2}\right]_s^\infty$$

$$= \frac{1}{2}\left[\log(s-a)^2 - \log(s^2+b^2)\right]_s^\infty = \frac{1}{2}\left[\log\frac{(s-a)^2}{(s^2+b^2)}\right]_s^\infty$$

$$= \frac{1}{2}\left[\log\left\{\frac{\left(1-\frac{a}{s}\right)^2}{\left(1+\frac{b^2}{s^2}\right)}\right\}\right]_s^\infty = \frac{1}{2}\left[0 - \log\frac{\left(1-\frac{a}{s}\right)^2}{\left(1+\frac{b^2}{s^2}\right)}\right] = \frac{1}{2}\left[\log\frac{(s^2+b^2)}{(s-a)^2}\right]$$

Ans.

Prob.13. Find the L.T. of

(i) $f(t) = \begin{cases} t, & 0 \le t \le 1 \\ 1, & t > 1 \end{cases}$ **(ii)** $f(t) = t^2 \sin 2t$.

Sol. (i) $L\{f(t)\} = \int_0^\infty e^{-st}.f(t)\,dt = \int_0^1 e^{-st}.t\,dt + \int_1^\infty e^{-st}.1\,dt$

$$= \left[\left\{t.\left(\frac{e^{-st}}{-s}\right)\right\}_0^1 - \int_0^1 1.\left(\frac{e^{-st}}{-s}\right)dt\right] + \left[\frac{e^{-st}}{-s}\right]_1^\infty$$

$$= \left[-\frac{e^{-s}}{s} + 0 + \frac{1}{s}\left\{\frac{e^{-st}}{-s}\right\}_0^1\right] - \frac{1}{s}\left[0 - e^{-s}\right]$$

$$= -\frac{e^{-s}}{s} - \frac{1}{s^2}(e^{-s} - 1) + \frac{e^{-s}}{s} = -\frac{1}{s^2}(e^{-s} - 1) = \frac{1 - e^{-s}}{s^2} \quad \textbf{Ans.}$$

(ii) Since $L(\sin 2t) = \dfrac{2}{s^2 + 2^2}$

Now $L(t^2 \sin 2t) = (-1)^2 \dfrac{d^2}{ds^2}\left(\dfrac{2}{s^2 + 2^2}\right)$ $\left[\because L\{t^n f(t)\} = \dfrac{(-1)^n d^n}{ds^n} L\{f(t)\}\right]$

$$= \frac{d}{ds}\left[\frac{(s^2 + 2^2).0 - 2.2s}{(s^2 + 2^2)^2}\right]$$

$$= -4\frac{d}{ds}\left[\frac{s}{(s^2 + 4)^2}\right]$$

$$= -4\left[\frac{(s^2 + 4)^2.1 - s.2(s^2 + 4).2s}{(s^2 + 4)^4}\right]$$

$$= -4\left[\frac{(s^2 + 4)\{s^2 + 4 - 4s^2\}}{(s^2 + 4)^4}\right]$$

$$= -4\left[\frac{-3s^2 + 4}{(s^2 + 4)^3}\right] = \frac{4(3s^2 - 4)}{(s^2 + 4)^3} \quad \textbf{Ans.}$$

Prob.14. Find the Laplace transform of

$$f(t) = \begin{cases} 1, & 0 \le t < 2 \\ t - 2, & 2 \le t \end{cases}$$

[R.G.P.V., May 2019 (III-Sem)]

Sol. $L\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt = \int_0^2 e^{-st}.1dt + \int_2^\infty e^{-st}(t - 2)\,dt$

$$= \left[\frac{e^{-st}}{-s}\right]_0^2 + \left[\left\{(t - 2)\left(\frac{e^{-st}}{-s}\right)\right\}_2^\infty - \int_2^\infty 1.\left(\frac{e^{-st}}{-s}\right)dt\right]$$

$$= -\frac{1}{s}[e^{-2s} - 1] + \left[0 + \frac{1}{s}\left(\frac{e^{-st}}{-s}\right)_2^\infty\right]$$

$$= -\frac{1}{s}(e^{-2s} - 1) - \frac{1}{s^2}(0 - e^{-2s})$$

$$= -\frac{1}{s}e^{-2s} + \frac{1}{s} + \frac{1}{s^2}e^{-2s} = \frac{1}{s^2}e^{-2s}(1 - s) + \frac{1}{s} \quad \textbf{Ans.}$$

Prob.15. Find the Laplace transform of f(t),

$$\text{where } f(t) = \begin{cases} 2, & 0 \le t \le 2 \\ t - 1, & 2 \le t \le 3 \\ 7, & t > 3 \end{cases}$$

Sol. $L\{f(t)\} = \int_0^2 e^{-st}.2dt + \int_2^3 e^{-st}(t - 1)dt + \int_3^\infty e^{-st}.7dt$

$$= 2\left[\frac{e^{-st}}{-s}\right]_0^2 + \left[\left\{(t - 1).\frac{e^{-st}}{-s}\right\}_2^3 - \int_2^3 1.\frac{e^{-st}}{-s}dt\right] + 7\left[\frac{e^{-st}}{-s}\right]_3^\infty$$

$$= 2\left[\frac{e^{-2s} - 1}{-s}\right] + \left[\left\{(3 - 1).\frac{e^{-3s}}{-s} - (2 - 1)\frac{e^{-2s}}{-s}\right\} - \left\{\frac{e^{-st}}{s^2}\right\}_2^3\right] + 7\left[\frac{0 - e^{-3s}}{-s}\right]$$

$$= \frac{-2(e^{-2s} - 1)}{s} - \frac{2e^{-3s}}{s} + \frac{e^{-2s}}{s} - \left(\frac{e^{-3s} - e^{-2s}}{s^2}\right) + 7\frac{e^{-3s}}{s}$$

$$= \frac{(-2e^{-2s} + 2 - 2e^{-3s} + e^{-2s} + 7e^{-3s})}{s} - \frac{(e^{-3s} - e^{-2s})}{s^2}$$

$$= \frac{1}{s}(5e^{-3s} - e^{-2s} + 2) - \frac{1}{s^2}(e^{-3s} - e^{-2s}) \quad \textbf{Ans.}$$

Prob.16. Evaluate $L\{te^{-t}\sin at\}$.

Sol. Here, $L(\sin at) = \dfrac{a}{s^2+a^2}$

therefore, $L(t\sin at) = -\dfrac{d}{ds}\left(\dfrac{a}{s^2+a^2}\right) = \dfrac{2as}{(s^2+a^2)^2}$

Now using the first shifting property, we get

$$L(e^{-t}\,t\sin at) = \frac{2a(s+1)}{\left[(s+1)^2+a^2\right]^2} = \frac{\mathbf{2a(s+1)}}{\mathbf{(s^2+2s+a^2+1)^2}} \quad \textbf{Ans.}$$

Prob.17. Find $L\left\{e^{-t}\sin^2 t + t\cos t\right\}$ ***[R.G.P.V., Nov. 2019 (IV-Sem)]***

Sol. Since $L(\cos t) = \dfrac{s}{s^2+1}$

$$\therefore \quad L\{t\cos t\} = -\frac{d}{ds}\left(\frac{s}{s^2+1}\right) = -\frac{s^2+1-s.2s}{(s^2+1)^2} = \frac{s^2-1}{(s^2+1)^2}$$

and
$$L\{e^{-t}\sin^2 t\} = \frac{1}{2}L\{e^{-t}(1-\cos 2t)\} = \frac{1}{2}L\{e^{-t}\} - \frac{1}{2}L\{e^{-t}\cos 2t\}$$
$$= \frac{1}{2}.\frac{1}{s+1} - \frac{1}{2}.\frac{s+1}{(s+1)^2+2^2}$$
$$= \frac{1}{2}\left[\frac{4}{(s+1)(s^2+2s+5)}\right] = \frac{2}{(s+1)(s^2+2s+5)}$$

Now $L\{e^{-t}\sin^2 t + t\cos t\}$
$$= L\{e^{-t}\sin^2 t\} + L\{t\cos t\}$$
$$= \frac{\mathbf{2}}{\mathbf{(s+1)(s^2+2s+5)}} + \frac{\mathbf{s^2-1}}{\mathbf{(s^2+1)^2}} \quad \textbf{Ans.}$$

Prob.18. Find the Laplace transform of $te^{-4t}\sin 3t$.

Sol. Here, $L(\sin 3t) = \dfrac{3}{s^2+9}$

therefore, $L(t\sin 3t) = -\dfrac{d}{ds}\left(\dfrac{3}{s^2+9}\right) = \dfrac{6s}{(s^2+9)^2}$

Now using the first shifting property, we get

$$L(e^{-4t}\,t\sin 3t) = \frac{6(s+4)}{[(s+4)^2+9]^2} = \frac{\mathbf{6(s+4)}}{\mathbf{(s^2+8s+25)^2}} \quad \textbf{Ans.}$$

Prob.19. Evaluate by using Laplace transform –

(i) $\int_0^\infty t\,e^{-4t}\sin t\,dt$ ***(ii)*** $\int_0^\infty e^{-t}\dfrac{\sin t}{t}dt$.

Sol. (i) Here, $L(\sin t) = \dfrac{1}{s^2+1}$

therefore, $L(t\sin t) = -\dfrac{d}{ds}\left(\dfrac{1}{s^2+1}\right) = \dfrac{2s}{(s^2+1)^2}$

$$\therefore \quad \int_0^\infty te^{-4t}\sin t\,dt = \frac{2\times 4}{(4^2+1)^2} = \frac{8}{(16+1)^2} = \frac{8}{(17)^2} = \frac{\mathbf{8}}{\mathbf{289}} \quad \textbf{Ans.}$$

(ii) Since,
$$L(\sin t) = \frac{1}{s^2+1}$$
$$\therefore \quad L\left(\frac{\sin t}{t}\right) = \int_s^\infty \frac{ds}{s^2+1}$$
$$= \left[\tan^{-1}s\right]_s^\infty = \tan^{-1}\infty - \tan^{-1}s = \frac{\pi}{2} - \tan^{-1}s$$

or
$$\int_0^\infty e^{-st}\left(\frac{\sin t}{t}\right)dt = \frac{\pi}{2} - \tan^{-1}s$$
(By definition of Laplace transform)

Now, taking limits on both sides as $s \to 1$, we get
$$\int_0^\infty e^{-t}\left(\frac{\sin t}{t}\right)dt = \frac{\pi}{2} - \tan^{-1}(1) = \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4} \quad \textbf{Ans.}$$

Prob.20. Find $L\left\{\dfrac{\sin t}{t}\right\}$ ***[R.G.P.V., May 2019 (IV-Sem)]***

Sol. Refer to Prob.19 (ii).

Prob.21. Find $L\{f(t)\}$ if $f(t) = \begin{cases} \sin\left(t-\dfrac{\pi}{3}\right), & t > \dfrac{\pi}{3} \\ 0, & t < \dfrac{\pi}{3} \end{cases}$.

Sol. Let, $f(t) = \sin t$

$$\therefore \quad L\{F(t)\} = L\{\sin t\} = \frac{1}{s^2+1}, s>0 = F(s), \text{ say}$$

$\therefore$ Using second shifting property,

$$L\{f(t)\} = e^{-\left(\frac{\pi}{3}\right)s}.F(s) = e^{-\left(\frac{\pi}{3}\right)s}.\frac{1}{s^2+1}, s>0 \quad \textbf{Ans.}$$

Prob.22. Evaluate $L\left\{\int_0^t \frac{\sin t}{t}dt\right\}$. **[R.G.P.V., Nov. 2019 (III-Sem)]**

Sol. Since $L\{\sin t\} = \frac{1}{s^2+1}$

Then
$$L\left\{\frac{\sin t}{t}\right\} = \int_s^\infty \frac{1}{s^2+1}ds$$
$$= \left[\tan^{-1}s\right]_s^\infty$$
$$= \tan^{-1}\infty - \tan^{-1}s$$
$$= \frac{\pi}{2} - \tan^{-1}s$$
$$= \cot^{-1}s$$

Hence $L\left\{\int_0^t \frac{\sin t}{t}dt\right\} = \mathbf{\frac{1}{s}\cot^{-1}s}$ (By transform of integral)

Ans.

Prob.23. Find the followings –

(i) $L\{t^2 \sin at\}$ (ii) $L\left\{\int_0^t e^{-t}\cos t\, dt\right\}$

Sol. (i) We know that **(R.G.P.V., Nov. 2018)**
$$L\{\sin at\} = \frac{a}{s^2+a^2}$$

Now
$$L\{t^2 \sin at\} = (-1)^2 \frac{d^2}{ds^2}\left(\frac{a}{s^2+a^2}\right)$$
$$\left[\because L\{t^n f(t)\} = (-1)^n \frac{d^n}{ds^n}L\{f(t)\}\right]$$
$$= \frac{d}{ds}\left[\frac{(s^2+a^2).0 - a.2s}{(s^2+a^2)^2}\right] = -2a\frac{d}{ds}\left\{\frac{s}{(s^2+a^2)^2}\right\}$$
$$= -2a\left\{\frac{(s^2+a^2)^2.1 - s.2(s^2+a^2).2s}{(s^2+a^2)^4}\right\}$$
$$= -2a\left\{\frac{(s^2+a^2)(s^2+a^2-4s^2)}{(s^2+a^2)^4}\right\}$$
$$= -2a\left\{\frac{-3s^2+a^2}{(s^2+a^2)^3}\right\} = \mathbf{\frac{2a(3s^2-a^2)}{(s^2+a^2)^3}}$$

Ans.

(ii) We know that by transform of integral
$$L\left(\int_0^t f(x)dx\right) = \frac{1}{s}F(s)$$
$$L\left(\int_0^t e^{-t}\cos t\, dt\right) = \frac{1}{s}L\left(e^{-t}\cos t\right) = \frac{1}{s}.\frac{s+1}{(s+1)^2+1}$$
$$\therefore \quad = \frac{s+1}{s[(s^2+2s+1)+1]} = \mathbf{\frac{s+1}{s(s^2+2s+2)}}$$

Ans.

Prob.24. Find the Laplace transform of $\int_0^t \frac{e^t \sin t}{t}dt$.

Sol. Since, $L\left(\frac{\sin t}{t}\right) = \int_s^\infty \frac{ds}{s^2+1} = \left[\tan^{-1}s\right]_s^\infty = \frac{\pi}{2} - \tan^{-1}s = \cot^{-1}s$

$\therefore \quad L\left\{e^t\left(\frac{\sin t}{t}\right)\right\} = \cot^{-1}(s-1)$ ($\because$ by first shifting property)

Hence,
$$L\left\{\int_0^t\left\{e^t\left(\frac{\sin t}{t}\right)\right\}dt\right\} = \mathbf{\frac{1}{s}\cot^{-1}(s-1)}$$ (by transform of integral)

Ans.

Prob.25. Find $L\left\{e^{3t}\sin t + \frac{\sin t}{t}\right\}$. **[R.G.P.V., Nov. 2019 (III-Sem)]**

Sol. $L\left\{e^{3t}\sin t + \frac{\sin t}{t}\right\} = L\left\{e^{3t}\sin t\right\} + L\left\{\frac{\sin t}{t}\right\}$
$$= \frac{1}{(s-3)^2+(1)^2} + \int_s^\infty \frac{ds}{s^2+1}$$
$$= \frac{1}{s^2-6s+9+1} + \left[\tan^{-1}s\right]_s^\infty$$
$$= \frac{1}{s^2-6s+10} + \left[\tan^{-1}\infty - \tan^{-1}s\right]$$
$$= \frac{1}{s^2-6s+10} + \left[\frac{\pi}{2} - \tan^{-1}s\right]$$
$$= \mathbf{\frac{1}{s^2-6s+10} + \cot^{-1}s}$$

Ans.

Prob.26. Prove that $\int_0^\infty \frac{e^{-t}-e^{-3t}}{t}dt = \log 3$

Sol. $\int_0^\infty \frac{e^{-t}-e^{-3t}}{t}dt = \int_0^\infty e^{-t}.\frac{1}{t}dt - \int_0^\infty e^{-3t}.\frac{1}{t}dt$

$$= \lim_{s\to 1} L\left[\frac{1}{t}\right] - \lim_{s\to 3}\left[\frac{1}{t}\right] = \lim_{s\to 1}\int_s^\infty \frac{1}{s}ds - \lim_{s\to 3}\int_s^\infty \frac{1}{s}ds$$

$$= \lim_{s\to 1}[\log s]_s^\infty - \lim_{s\to 3}[\log s]_s^\infty$$

$$= \log\infty - \log 1 - \log\infty + \log 3 = \mathbf{\log 3}$$

Proved

Prob.27. Show that $\int_0^\infty \frac{\cos 6t - \cos 4t}{t} = \log\frac{2}{3}$.

Sol. Let, $f(t) = \cos 6t - \cos 4t$

$$L\{f(t)\} = L\{\cos 6t - \cos 4t\}$$

$$= L\{\cos 6t\} - L\{\cos 4t\}$$

$$= \frac{s}{s^2+6^2} - \frac{s}{s^2+4^2} \qquad \left[\because L\{\cos at\} = \frac{s}{s^2+a^2}\right]$$

$$= F(s), \text{ say}$$

Now $\int_0^\infty \frac{f(t)}{t}dt = \int_0^\infty f(x)\,dx$, where $L\{f(t)\} = F(s)$

$$\Rightarrow \int_0^\infty\left[\frac{\cos 6t - \cos 4t}{t}\right]dt = \int_0^\infty\left[\frac{x}{x^2+6^2} - \frac{x}{x^2+4^2}\right]dx$$

$$= \left[\frac{1}{2}\log(x^2+6^2) - \frac{1}{2}\log(x^2+4^2)\right]_0^\infty$$

$$= \frac{1}{2}\left[\log\frac{x^2+36}{x^2+16}\right]_0^\infty$$

$$= \frac{1}{2}\left[\lim_{x\to\infty}\left\{\log\frac{x^2+36}{x^2+16}\right\} - \log\frac{36}{16}\right]$$

$$= \frac{1}{2}\left[\lim_{x\to\infty}\left\{\log\frac{1+(36/x^2)}{1+(16/x^2)}\right\} - \log\frac{9}{4}\right]$$

$$= \frac{1}{2}\left[\log 1 - \log\left(\frac{3}{2}\right)^2\right] = \frac{1}{2}\left[0 - 2\log\frac{3}{2}\right]$$

$$= -\log\frac{3}{2} = \mathbf{\log\frac{2}{3}}$$

Proved

Prob.28. Find the Laplace transform of $(t-1)^2\,u(t-1)$.

Sol. Here,

$$L[(t-1)^2 u(t-1)] = \int_0^\infty e^{-st}[(t-1)^2 u(t-1)]\,dt$$

$$= \int_0^1 e^{-st}(t-1)^2.0\,dt + \int_1^\infty e^{-st}(t-1)^2.1\,dt$$

$$= 0 + \left[-\frac{(t-1)^2 e^{-st}}{s}\right]_{t=1}^\infty + \frac{2}{s}\int_1^\infty (t-1)e^{-st}\,dt$$

$$= 0 + 0 + \frac{2}{s}\left[-\frac{(t-1)}{s}e^{-st}\right]_1^\infty + \frac{2}{s^2}\left(\frac{e^{-st}}{-s}\right)_1^\infty$$

$$= \frac{2}{s}(0) + \frac{2}{s^3}e^{-s} = \mathbf{\frac{2}{s^3}e^{-s}}$$

Ans.

Prob.29. Find the Laplace transform of –

(i) $f(t) = \int_0^t e^{-t}\frac{\sin t}{t}dt$ **(ii)** $f(t) = t^2 e^{-2t}\cos 3t$.

Sol. (i) Since,

$$L\left(\frac{\sin t}{t}\right) = \int_s^\infty \frac{ds}{s^2+1} = \left[\tan^{-1}s\right]_s^\infty = \frac{\pi}{2} - \tan^{-1}s = \cot^{-1}s$$

$\therefore$ $L\left(e^{-t}\frac{\sin t}{t}\right) = \cot^{-1}(s+1)$ ($\because$ By first shifting property)

Hence, $L\left\{\int_0^t\left(e^{-t}\frac{\sin t}{t}\right)dt\right\} = \mathbf{\frac{1}{s}\cot^{-1}(s+1)}$

(By transform of integral) **Ans.**

(ii) Since, $L[\cos 3t] = \frac{s}{s^2+3^2}$

$$\therefore \quad L[t^2\cos 3t] = \frac{d}{ds}\left[\frac{1.(s^2+3^2) - 2s.s}{(s^2+3^2)^2}\right]$$

$$= \frac{d}{ds}\left[\frac{9-s^2}{(s^2+9)^2}\right] = \frac{-2s.(s^2+9)^2 - 2(s^2+9).2s.(9-s^2)}{(s^2+9)^4}$$

$$= \frac{(s^2+9)\left[-2s(s^2+9) - 4s(9-s^2)\right]}{(s^2+9)^4}$$

$$= \frac{-2s^3 - 18s - 36s + 4s^3}{(s^2+9)^3} = \mathbf{\frac{2s^3 - 54s}{(s^2+9)^3}}$$

Ans.

Prob.30. Find the Laplace transform of sin at sin bt.

Sol. Let, $L\{\sin at + \sin bt\} = \frac{1}{2} L\{2 \sin at \sin bt\}$

$$= \frac{1}{2} L\{\cos(a-b)t - \cos(a+b)t\}$$

$$= \frac{1}{2}\left[\frac{s}{s^2+(a-b)^2} - \frac{s}{s^2+(a+b)^2}\right]$$

$$= \frac{s}{2}\left[\frac{s^2+(a+b)^2-\{s^2+(a-b)^2\}}{\{s^2+(a-b)^2\}\{s^2+(a+b)^2\}}\right]$$

$$= \frac{s}{2}\left[\frac{s^2+a^2+b^2+2ab-s^2-a^2-b^2+2ab}{\{s^2+(a-b)^2\}\{s^2+(a+b)^2\}}\right]$$

$$= \left[\frac{\mathbf{2abs}}{\{\mathbf{s^2+(a-b)^2}\}\{\mathbf{s^2+(a+b)^2}\}}\right] \quad \textbf{Ans.}$$

Prob.31. Find the Laplace transform of –

$$\frac{\cos at - \cos bt}{t}.$$

Sol. Since, $L[\cos at - \cos bt] = \frac{s}{s^2+a^2} - \frac{s}{s^2+b^2}$

$$\therefore \quad L\left[\frac{\cos at - \cos bt}{t}\right] = \int_s^\infty \left(\frac{s}{s^2+a^2} - \frac{s}{s^2+b^2}\right) ds = \frac{1}{2}\left[\log\left(\frac{s^2+a^2}{s^2+b^2}\right)\right]_s^\infty$$

$$= \frac{1}{2}\lim_{s\to\infty} \log \frac{s^2\left(1+\frac{a^2}{s^2}\right)}{s^2\left(1+\frac{b^2}{s^2}\right)} - \frac{1}{2}\log\frac{s^2+a^2}{s^2+b^2}$$

$$= 0 - \frac{1}{2}\log\frac{s^2+a^2}{s^2+b^2} = \mathbf{\frac{1}{2}\log\frac{s^2+b^2}{s^2+a^2}} \quad \textbf{Ans.}$$

Prob.32. Find the L.T. of the triangular wave function of period 2k given by –

$$f(t) = \begin{cases} t, & 0 < t < k \\ 2k - t, & k < t < 2k \end{cases}$$

Sol. Here, f(t) is a periodic function with period T = 2k.

$$\therefore L\{f(t)\} = \int_0^{2k} \frac{e^{-st} f(t)\, dt}{1-e^{-2ks}}$$

$$= \frac{1}{1-e^{-2ks}}\left[\int_0^k e^{-st} f(t)\, dt + \int_k^{2k} e^{-st} f(t)\, dt\right]$$

$$= \frac{1}{1-e^{-2ks}}\left[\int_0^k e^{-st}.t\, dt + \int_k^{2k} e^{-st}(2k-t)\, dt\right]$$

$$= \frac{1}{1-e^{-2ks}}\left[\left\{t.\frac{e^{-st}}{(-s)} - \frac{e^{-st}}{s^2}\right\}_0^k + \left\{(2k-t).\frac{e^{-st}}{(-s)} + \frac{e^{-st}}{s^2}\right\}_k^{2k}\right]$$

$$= \frac{1}{1-e^{-2ks}}\left[-\frac{ke^{-ks}}{s} - \frac{e^{-ks}}{s^2} + 0 + \frac{1}{s^2} + 0 + \frac{e^{-2ks}}{s^2} + \frac{ke^{-ks}}{s} - \frac{e^{-ks}}{s^2}\right]$$

$$= \frac{1}{1-e^{-2ks}}\left[-\frac{2e^{-ks}}{s^2} + \frac{1}{s^2} + \frac{e^{-2ks}}{s^2}\right] = \frac{1}{1-e^{-2ks}}\left[\frac{-2e^{-ks}+1+e^{-2ks}}{s^2}\right]$$

$$= \frac{(1-e^{-ks})^2}{(1-e^{-ks})(1+e^{-ks})}.\frac{1}{s^2} = \frac{1}{s^2}.\frac{1-e^{-ks}}{1+e^{-ks}} = \frac{1}{s^2}.\tanh\left(\frac{ks}{2}\right) \quad \textbf{Ans.}$$

Prob.33. Find Laplace transform of the

$$f(t) = \begin{cases} 3t, & 0 < t < 2 \\ 6, & 2 < t < 4 \end{cases}, \text{ f(t) has a period 4.}$$

Sol. Here, f(t) is a periodic function with period T = 4.

$$\therefore \quad L\{f(t)\} = \int_0^T \frac{e^{-st} f(t)\, dt}{1-e^{-sT}} = \frac{\int_0^2 3t\, e^{-st} dt + \int_2^4 6.e^{-st} dt}{1-e^{-4s}}$$

$$= \frac{1}{1-e^{-4s}}\left[3t\left(\frac{e^{-st}}{-s}\right) - 3\left(\frac{e^{-st}}{s^2}\right)\right]_0^2 + \frac{1}{1-e^{-4s}}\left[\frac{6e^{-st}}{-s}\right]_2^4$$

$$= \frac{1}{1-e^{-4s}}\left[-\frac{6}{s}e^{-2s} - \frac{3e^{-2s}}{s^2} + \frac{3}{s^2} - \frac{6e^{-4s}}{s} + \frac{6}{s}e^{-2s}\right]$$

$$= \frac{1}{(1-e^{-4s})s^2}[-3e^{-2s} + 3 - 6se^{-4s}] = \frac{3-3e^{-2s}-6se^{-4s}}{s^2(1-e^{-4s})} \quad \textbf{Ans.}$$

FINDING INVERSE LAPLACE TRANSFORM BY DIFFERENT METHODS, CONVOLUTION THEOREM, SOLVING ODEs BY LAPLACE TRANSFORM METHOD

Inverse Laplace Transform –

Definition – If the Laplace transform of a function f(t) is F(s) or F(p) i.e., L [f(t)] = F(s), then f(t) is said to be an ***inverse Laplace transform of*** F(s).

We also write $f(t) = L^{-1}\{F(s)\}$ or $L^{-1}\{F(p)\}$, L^{-1} is said to be the ***inverse Laplace transformation operator***.

Null Function – Let N(t) be a function of t such that

$$\int_0^{t_0} N(t)\,dt = 0 \quad \forall\, t_0 > 0$$

Then N(t) is called a ***Null function.***

Lerch's Theorem – Let L{f(t)} = F(s). Let f(t) be piecewise continuous in every finite interval $0 \le t \le a$ and of exponential order for $t > a$, then the inverse Laplace transform f(t) of F(s) is unique.

Properties of Inverse Laplace Transform –

(i) Linear Property – If $F_1(s)$ and $F_2(s)$ are Laplace transforms of $f_1(t)$ and $f_2(t)$ respectively, then

$$L^{-1}\{c_1F_1(s) + c_2F_2(s)\} = c_1L^{-1}\{F_1(s)\} + c_2L^{-1}\{F_2(s)\}$$

where, c_1 and c_2 are any constants.

(ii) First Shifting Property –

If $L^{-1}\{F(s)\} = f(t)$, then

$L^{-1}\{F(s-a)\} = e^{at} f(t)$.

(iii) Second Shifting Property –

If $L^{-1}\{F(s)\} = f(t)$, then

$L^{-1}\{e^{-as} F(s)\} = g(t)$, where

$$g(t) = \begin{cases} f(t-a), & t > a \\ 0, & t < a \end{cases}$$

(iv) Change of Scale Property – If F(s) = L [f(t)] denotes the Laplace transform of the function f(t), prove that

$$L^{-1}[F(as)] = \frac{1}{a} f\left(\frac{t}{a}\right), a > 0,$$

Proof. $\because F(s) = \int_0^\infty e^{-st} f(t)dt, \quad \therefore F(as) = \int_0^\infty e^{-ast} f(t)dt$

$$= \int_0^\infty e^{-sx} f\left(\frac{x}{a}\right)\frac{dx}{a} \qquad \left(\text{put, } at = x,\ dt = \frac{dx}{a}\right)$$

$$= \frac{1}{a}\int_0^\infty e^{-sx} f\left(\frac{x}{a}\right)dx = \int_0^\infty e^{-sx} f\left(\frac{1}{a}\right)dt = \frac{1}{a} f\left(\frac{t}{a}\right)$$ **Proved**

Inverse Laplace Transform of Derivative –

If $L^{-1}\{F(s)\} = f(t)$

then $L^{-1}\{F^{n}(s)\} = (-1)^n t^n f(t)$.

Proof. By the Laplace transform, we know that

$$L\{t^n f(t)\} = (-1)^n F^n(s)$$

$$\therefore \quad t^n f(t) = L^{-1}\{(-1)^n F^n(s)\} = (-1)^n L^{-1}\{F^n(s)\}$$

or $$L^{-1}\{F^n(s)\} = (-1)^{-n} t^n f(t) = \mathbf{(-1)^n t^n f(t)}$$ **Proved**

Division by s – If $L^{-1}\{F(s)\} = f(t)$, then

$$L^{-1}\left\{\frac{F(s)}{s}\right\} = \int_0^t f(u)\,du$$

Convolution Theorem –

Statement – Suppose f(t) and g(t) are two functions of class A, and suppose

$$L^{-1}\{F(s)\} = f(t) \text{ and } L^{-1}\{G(s)\} = g(t)$$

Then $$L^{-1}\{F(s).G(s)\} = \int_0^t f(u)g(t-u)\,du \quad \text{....(i)}$$

Proof. Suppose $\psi(t) = \int_0^t f(u)g(t-u)\,du$

Taking Laplace transform on both sides of equation (i), we have

$$L[\psi(t)] = \int_0^\infty e^{-st}\left[\int_0^t f(u).g(t-u)\,du\right]dt$$

$$= \int_0^\infty \int_0^t e^{-st} f(u).g(t-u)\,du\,dt \quad \text{...(ii)}$$

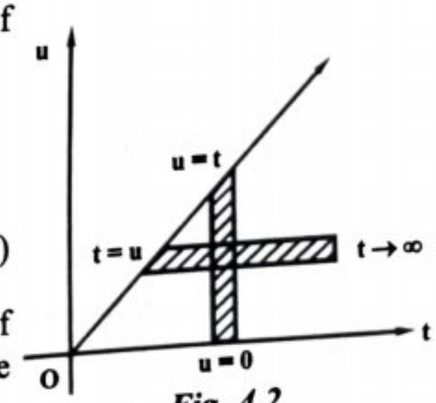

Fig. 4.2

For this double integral, the domain of integration is the entire area lying between the lines u = 0 and u = t.

On changing the order of integration, we get

$$L[\psi(t)] = \int_0^\infty \int_u^\infty e^{-st} f(u).g(t-u)\, dt\, du$$

$$= \int_0^\infty e^{-su} f(u) \left[\int_u^\infty e^{-s(t-u)}.g(t-u)\, dt \right] du$$

$$= \int_0^\infty e^{-su} f(u) \left[\int_0^\infty e^{-sv}.g(v)\, dv \right] du \quad [\text{Put } t-u = v, \text{ so that } dt = dv]$$

$$= \left[\int_0^\infty e^{-su} f(u)\, du\, G(s) \right] = F(s).G(s)$$

which gives $L^{-1}[F(s).G(s)] = \psi(t)$

Hence from equation (i), we get

$$\mathbf{L^{-1}[F(s).G(s)] = \int_0^t f(u).g(t-u)\, du}$$

Proved

Method of Finding Inverse Laplace Transform by Partial Fractions – A rational algebraic function is said to be a ***proper fraction*** if degree of numerator < degree of denominator.

Let us consider a fraction $\frac{f(x)}{g(x)}$.

First of all we shall divide f(x) by g(x), if possible, in order to change the given fraction into a proper fraction and let

$$\frac{f(x)}{g(x)} = \psi(x) + \frac{f_1(x)}{g(x)}.$$

Next step is to resolve $\frac{f_1(x)}{g(x)}$ into partial fractions. We first factorise g(x) into simplest real factors. These factors will be either linear or quadratic and some of the factors may be repeated.

(i) Suppose $g(x) = (x-a)^m (x^2 + ax + b)^n$. We resolve $\frac{f_1(x)}{g(x)}$ into partial fraction as follows –

$$\frac{f(x)}{(x-a)^m (x^2+ax+b)^n} = \frac{a_1}{x-a} + \frac{a_2}{(x-a)^2} + \ldots + \frac{a_m}{(x-a)^m} + \frac{b_1x+c_1}{x^2+ax+b} + \frac{b_2x+c_2}{(x^2+ax+b)^2} + \ldots + \frac{b_nx+c_n}{(x^2+ax+b)^n}$$

In order to determine the constants $a_1, a_2, \ldots b_1, b_2, \ldots c_1, c_2 \ldots$

We multiply both sides by g(x) and then equate the equations for $a_1, a_2, \ldots b_1, b_2, \ldots, c_1, c_2 \ldots$.

(ii) Suppose $g(x) = (x-a)(x-b)(x-c)(x-d)$. In this case $\frac{f_1(x)}{g(x)}$ is resolved as follows –

$$\frac{f_1(x)}{g(x)} = \frac{a_1}{x-a} + \frac{a_2}{x-b} + \frac{a_3}{x-c} + \frac{a_4}{x-d}$$

For determining the constants $a_1, a_2, \ldots$, we follow the method of case (i) However these constants can also be determined in very simple way as follows –

$$a_1 = \frac{f_1(a)}{(a-b)(a-c)(a-d)}, \quad a_2 = \frac{f_1(b)}{(b-a)(b-c)(b-d)},$$

$$a_3 = \frac{f_1(c)}{(c-a)(c-b)(c-d)}, \quad a_4 = \frac{f_1(d)}{(d-a)(d-b)(d-c)}.$$

Application to Differential Equations – The Laplace transform method of solving differential equations gives particular solution without the necessity of first finding the general solution and then solving linear differential equations with constant coefficients.

Working Procedure – To solve a linear differential equation with constant coefficient by transform method –

(i) Take the Laplace transform of both sides of the differential equation and using the given initial conditions.

(ii) Transpose the terms with minus signs to the right.

(iii) Divide by the coefficient of $\bar{y}$, getting $\bar{y}$ as a known function of s.

(iv) Resolve this function of s into partial fractions and take the inverse transforms of both sides. This gives y as a function of t which is the desired solution satisfied the given condition.

Some Important Result for Inverse Laplace Transforms –

(i) $L^{-1}\left(\frac{1}{s}\right) = 1$ (ii) $L^{-1}\left(\frac{1}{s-a}\right) = e^{at}$

(iii) $L^{-1}\left(\frac{1}{s^n}\right) = \frac{t^{n-1}}{(n-1)!}$, if n is positive integer. Otherwise $= \frac{t^{n-1}}{\Gamma(n)}$

(iv) $L^{-1}\left(\frac{1}{(s-a)^n}\right) = \frac{e^{at}t^{n-1}}{(n-1)!}$ (v) $L^{-1}\left(\frac{1}{s^2+a^2}\right) = \frac{1}{a}\sin at$

(vi) $L^{-1}\left(\frac{s}{s^2+a^2}\right) = \cos at$ (vii) $L^{-1}\left(\frac{1}{s^2-a^2}\right) = \frac{1}{a}\sinh at$

(viii) $L^{-1}\left(\frac{s}{s^2-a^2}\right) = \cosh at$ (ix) $L^{-1}\left(\frac{1}{(s-a)^2+b^2}\right) = \frac{1}{b}e^{at}\sin bt$

(x) $L^{-1}\left(\dfrac{s-a}{(s-a)^2+b^2}\right)=e^{at}\cos bt$

(xi) $L^{-1}\left(\dfrac{s}{(s^2+a^2)^2}\right)=\dfrac{1}{2a}t\sin at$

(xii) $L^{-1}\left(\dfrac{1}{(s^2+a^2)^2}\right)=\dfrac{1}{2a^3}(\sin at-at\cos at)$.

NUMERICAL PROBLEMS

Prob.34. Evaluate $L^{-1}\left\{\dfrac{1}{9s^2+25}\right\}$.

Sol. $L^{-1}\left\{\dfrac{1}{9s^2+25}\right\}=\dfrac{1}{9}L^{-1}\left\{\dfrac{1}{s^2+\frac{25}{9}}\right\}=\dfrac{1}{9}L^{-1}\left\{\dfrac{1}{s^2+\left(\frac{5}{3}\right)^2}\right\}$

$$=\frac{1}{9}\cdot\frac{3}{5}\sin\frac{5}{3}t=\mathbf{\frac{1}{15}\sin\frac{5}{3}t}$$ **Ans.**

Prob.35. Evaluate $L^{-1}\left\{\dfrac{s}{2s^2-1}\right\}$.

Sol. $L^{-1}\left\{\dfrac{s}{2s^2-1}\right\}=\dfrac{1}{2}L^{-1}\left\{\dfrac{s}{s^2-\frac{1}{2}}\right\}=\dfrac{1}{2}.L^{-1}\left\{\dfrac{s}{s^2-\left(\frac{1}{\sqrt{2}}\right)^2}\right\}$

$$=\mathbf{\frac{1}{2}.\cosh\frac{1}{\sqrt{2}}t}$$ **Ans.**

Prob.36. Evaluate $L^{-1}\left\{\dfrac{1}{(s-3)^2}\right\}$.

Sol. $L^{-1}\left\{\dfrac{1}{(s-3)^2}\right\}=\dfrac{e^{3t}t^{2-1}}{(2-1)!}$ $\qquad\left[\because L^{-1}\left(\dfrac{1}{(s-a)^n}\right)=\dfrac{e^{at}t^{n-1}}{(n-1)!}\right]$

$$=\frac{e^{3t}t}{1!}=\mathbf{e^{3t}.t}$$ **Ans.**

Prob.37. Compute the inverse Laplace transform of $\log\dfrac{s(s+1)}{(s^2+4)}$.

Sol. Let, $f(t)=L^{-1}\log\dfrac{s(s+1)}{(s^2+4)}$ then $\{t\,f(t)\}=L^{-1}\left\{-\dfrac{d}{ds}\log\dfrac{s(s+1)}{s^2+4}\right\}$

$$=-L^{-1}\left\{\frac{d}{ds}\log s\right\}-L^{-1}\left\{\frac{d}{ds}\log(s+1)\right\}+L^{-1}\left\{\frac{d}{ds}\log(s^2+4)\right\}$$

$$=-L^{-1}\left\{\frac{1}{s}\right\}-L^{-1}\left\{\frac{1}{s+1}\right\}+L^{-1}\left\{\frac{2s}{s^2+4}\right\}=-1-e^{-t}+2\cos 2t$$

or $\mathbf{f(t)=-\dfrac{1}{t}\left\{1+e^{-t}-2\cos 2t\right\}}$ **Ans.**

Prob.38. Find the inverse L.T. of $\log\left(1+\dfrac{1}{s^2}\right)$.

Sol. Let, $f(t)=L^{-1}\log\left(1+\dfrac{1}{s^2}\right)=L^{-1}\log\left(\dfrac{s^2+1}{s^2}\right)$

then $\quad t\,f(t)=L^{-1}\left\{-\dfrac{d}{ds}\log\left(\dfrac{s^2+1}{s^2}\right)\right\}$

$$=L^{-1}\left\{-\frac{d}{ds}\log(s^2+1)\right\}+L^{-1}\left\{\frac{d}{ds}\log s^2\right\}$$

$$=-L^{-1}\left\{\frac{2s}{s^2+1}\right\}+L^{-1}\left\{\frac{2s}{s^2}\right\}=-2L^{-1}\left\{\frac{s}{s^2+1}\right\}+2L^{-1}\left\{\frac{1}{s}\right\}$$

or $\quad t\,f(t)=-2\cos t+2$

Hence $\quad \mathbf{f(t)=\dfrac{2-2\cos t}{t}}$ **Ans.**

Prob.39. Find the inverse Laplace transform of $\dfrac{p}{(p^2-a^2)^2}$.

Sol. Let, $L^{-1}\left[\dfrac{p}{(p^2-a^2)^2}\right]=f(t)$

Then $\quad L\left[\dfrac{f(t)}{t}\right]=\displaystyle\int_p^\infty\frac{p\,dp}{(p^2-a^2)^2}=\frac{1}{2}\int_p^\infty\frac{2\,p\,dp}{(p^2-a^2)^2}$

$$=-\frac{1}{2}\left[\frac{1}{p^2-a^2}\right]_p^\infty=\frac{1}{2}\left[\frac{1}{p^2-a^2}\right]$$

$\therefore \quad \dfrac{f(t)}{t}=\dfrac{1}{2}L^{-1}\left[\dfrac{1}{p^2-a^2}\right]=\dfrac{1}{2a}\sinh at$

or $$f(t) = \frac{1}{2a} t \sinh at$$

Hence $$L^{-1}\left[\frac{p}{(p^2-a^2)^2}\right] = \frac{t}{2a}\sinh at$$ **Ans.**

Prob.40. Evaluate $L^{-1}\left\{\frac{s+1}{s^2+6s+25}\right\}$. ***[R.G.P.V., Nov. 2019 (III-Sem)]***

Sol. (ii) Here $$L^{-1}\left\{\frac{s+1}{s^2+6s+25}\right\} = L^{-1}\left\{\frac{(s+3)-2}{(s+3)^2+(4)^2}\right\}$$

$$= L^{-1}\left\{\frac{s+3}{(s+3)^2+(4)^2}\right\} - 2L^{-1}\left\{\frac{1}{(s+3)^2+4^2}\right\}$$

$$= e^{-3t}\cos 4t - 2.\frac{1}{4}.e^{-3t}\sin 4t$$

$$= e^{-3t}\left[\cos 4t - \frac{1}{2}\sin 4t\right]$$ **Ans.**

Prob.41. Find $L^{-1}\left\{\frac{(2s+1)}{(s-1)^2(s+2)^2}\right\}$.

Sol. We can express

$$\frac{2s+1}{(s-1)^2(s+2)^2} = \frac{1}{3}\left[\frac{(s+2)^2-(s-1)^2}{(s-1)^2(s+2)^2}\right] = \frac{1}{3}\left[\frac{1}{(s-1)^2} - \frac{1}{(s+2)^2}\right]$$

$$\therefore L^{-1}\left[\frac{(2s+1)}{(s-1)^2(s+2)^2}\right] = \frac{1}{3}\left[L^{-1}\left\{\frac{1}{(s-1)^2}\right\} - L^{-1}\left\{\frac{1}{(s+2)^2}\right\}\right]$$

$$= \frac{1}{3}\left[e^{t}L^{-1}\left\{\frac{1}{s^2}\right\} - e^{-2t}L^{-1}\left\{\frac{1}{s^2}\right\}\right]$$

(By first shifting property)

$$= \frac{1}{3}(e^{t}-e^{-2t})L^{-1}\left[\frac{1}{s^2}\right] = \frac{1}{3}(e^{t}-e^{-2t})\left[\frac{t}{1!}\right]$$

$$= \frac{1}{3}t(e^{t}-e^{-2t})$$ **Ans.**

Prob.42. Evaluate $L^{-1}\left\{\frac{6s^2+22s+18}{(s+1)(s+2)(s+3)}\right\}$.

[R.G.P.V., Nov. 2019 (IV-Sem)]

Sol. Let, $$\frac{6s^2+22s+18}{(s+1)(s+2)(s+3)} = \frac{A}{s+1}+\frac{B}{s+2}+\frac{C}{s+3} \quad ...(i)$$

$6s^2 + 22s + 18 = A(s+2)(s+3) + B(s+1)(s+3) + C(s+1)(s+2)$

Putting $s = -1$, $2 = 2A \Rightarrow A = 1$

Putting $s = -2$, $-2 = -B \Rightarrow B = 2$

Putting $s = -3$, $6 = 2C \Rightarrow C = 3$

Putting the values of A, B, C in equation (i), we get

$$\frac{6s^2+22s+18}{(s+1)(s+2)(s+3)} = \frac{1}{s+1}+\frac{2}{s+2}+\frac{3}{s+3}$$

$$L^{-1}\left\{\frac{6s^2+22s+18}{(s+1)(s+2)(s+3)}\right\} = L^{-1}\left\{\frac{1}{s+1}\right\}+2L^{-1}\left\{\frac{1}{s+2}\right\}+3L^{-1}\left\{\frac{1}{s+3}\right\}$$

$$= e^{-t} + 2e^{-2t} + 3e^{-3t}$$ **Ans.**

Prob.43. Find $L^{-1}\left(\frac{5s^2-15s-11}{(s+1)(s-2)^3}\right)$.

Sol.

Let, $$\frac{5s^2-15s-11}{(s+1)(s-2)^3} = \frac{A}{(s+1)}+\frac{B}{(s-2)}+\frac{C}{(s-2)^2}+\frac{D}{(s-2)^3} \quad ...(i)$$

$5s^2 - 15s - 11 = A(s-2)^3 + B(s+1)(s-2)^2 + C(s+1)(s-2) + D(s+1)$

Putting $s = -1$, $9 = -27A \Rightarrow A = -\frac{1}{3}$

Putting $s = 2$, $-21 = 3D \Rightarrow D = -7$

Equating coefficients of s^3 and s^2 on both sides,

$$0 = A + B \Rightarrow B = 1/3$$

$$5 = -6A - 3B + C$$

$$\Rightarrow \quad C = 5 + 6A + 3B$$

$$\Rightarrow \quad C = 5 - 2 + 1 = 4$$

Therefore, from equation (i), we have

$$L^{-1}\left\{\frac{5s^2-15s-11}{(s+1)(s-2)^3}\right\} = -\frac{1}{3}L^{-1}\left\{\frac{1}{(s+1)}\right\}+\frac{1}{3}L^{-1}\left\{\frac{1}{(s-2)}\right\}$$

$$+4L^{-1}\left\{\frac{1}{(s-2)^2}\right\}-7L^{-1}\left\{\frac{1}{(s-2)^3}\right\}$$

$$= -\frac{1}{3}e^{-t} + \frac{1}{3}e^{2t} + 4e^{2t}L^{-1}\left\{\frac{1}{s^2}\right\} - 7e^{2t}L^{-1}\left\{\frac{1}{s^3}\right\}$$

$$= -\frac{1}{3}e^{-t} + \frac{1}{3}e^{2t} + 4e^{2t}\left(\frac{t}{1!}\right) - 7e^{2t}\left\{\frac{t^2}{2!}\right\}$$

$$= -\frac{1}{3}e^{-t} + e^{2t}\left(\frac{1}{3} + 4t - \frac{7}{2}t^2\right)$$

Ans.

Prob.44. Find the inverse Laplace transform of $L^{-1}\left\{\log\frac{s^2-1}{s^2}\right\}$.

Sol. Let, $F(s) = \log\frac{(s^2-1)}{s^2}$

$$F(s) = \log(s^2-1) - \log s^2$$

$$F(s) = \log(s-1) + \log(s+1) - 2\log s$$

$$F'(s) = \frac{1}{s-1} + \frac{1}{s+1} - \frac{2}{s}$$

$$L^{-1}\{F'(s)\} = e^t + e^{-t} - 2$$

$$(-1)t\,L^{-1}\{F(s)\} = e^t + e^{-t} - 2$$

$$L^{-1}\{F(s)\} = 1/t\,(2 - e^t - e^{-t})$$

$$= \frac{1}{t}\left[2 - \frac{2(e^t + e^{-t})}{2}\right]$$

or $$L^{-1}\left\{\log\frac{s^2-1}{s^2}\right\} = \frac{2}{t}(1 - \cosh t) \qquad \left[\because \cosh t = \frac{e^t + e^{-t}}{2}\right]$$ Ans.

Prob.45. Find the inverse L.T. of $F(s) = \log\frac{s+2}{s+3}$.

Sol. Given, $F(s) = \log\frac{s+2}{s+3} = \log(s+2) - \log(s+3)$

then $$F'(s) = \frac{1}{s+2} - \frac{1}{s+3}$$

Taking L^{-1} of both sides

$$L^{-1}\{F'(s)\} = L^{-1}\left\{\frac{1}{s+2}\right\} - L^{-1}\left\{\frac{1}{s+3}\right\}$$

or $$(-1)\,t\,L^{-1}\{F(s)\} = e^{-2t} - e^{-3t}$$

(By inverse transformation of derivatives)

$$L^{-1}\{F(s)\} = \frac{e^{-2t} - e^{-3t}}{-t} = \frac{e^{-3t} - e^{-2t}}{t}$$

or $$L^{-1}\left\{\log\frac{s+2}{s+3}\right\} = \frac{1}{t}(e^{-3t} - e^{-2t})$$ **Ans.**

Prob.46. Evaluate –

$$L^{-1}\left\{\log\left(\frac{s+1}{s-1}\right)\right\}.$$

Sol. Let, $F(s) = \log\frac{(s+1)}{(s-1)} = \log(s+1) - \log(s-1)$

then $$f'(s) = \frac{1}{s+1} - \frac{1}{s-1}$$

Taking L^{-1} of both sides

$$L^{-1}\{F'(s)\} = L^{-1}\left\{\frac{1}{s+1}\right\} - L^{-1}\left\{\frac{1}{s-1}\right\}$$

$$(-1)\,t\,L^{-1}\{F(s)\} = e^{-t} - e^{t} \text{ (By inverse transformation of derivatives)}$$

$$L^{-1}\{F(s)\} = \frac{e^{-t} - e^{t}}{-t} = \frac{e^{t} - e^{-t}}{t}$$

$$L^{-1}\left\{\log\frac{(s+1)}{(s-1)}\right\} = \frac{2\sinh t}{t}$$ **Ans.**

Prob.47. Find the inverse Laplace transform of $F(s) = \frac{1}{s^3(s^2+1)}$.

Sol. Here, $F(s) = \frac{1}{s^3(s^2+1)}$

Taking inverse Laplace transform, we have

$$L^{-1}\left[\frac{1}{s^3(s^2+1)}\right] = \int_0^t\left[\int_0^t\left\{\int_0^t \sin t\,dt\right\}dt\right]dt$$

$$= \int_0^t\left[\int_0^t\{-\cos t\}_0^t\,dt\right]dt = \int_0^t\left[\int_0^t(1-\cos t)\,dt\right]dt$$

$$= \int_0^t[t - \sin t]_0^t\,dt = \int_0^t(t - \sin t)\,dt$$

$$= \left[\frac{t^2}{2} + \cos t\right]_0^t = \frac{t^2}{2} + \cos t - 1$$ **Ans.**

Prob.48. Find the inverse Laplace transform of the $\tan^{-1}\frac{2}{p}$.

Sol. Let, $$F(p) = \tan^{-1}\left(\frac{2}{p}\right)$$

then $$F'(p) = \frac{-2}{p^2+2^2}$$

Taking L^{-1} of both sides

$$L^{-1}\{F'(p)\} = L^{-1}\left\{\frac{-2}{p^2+2^2}\right\}$$

$(-1)^1\, t^1\, L^{-1}\{F(p)\} = -\sin 2t$ (By inverse transformation of derivatives)

$$L^{-1}\left\{\tan^{-1}\left(\frac{2}{p}\right)\right\} = \frac{\sin 2t}{t}$$ **Ans.**

Prob.49. Find $L^{-1}\left\{\frac{p+2}{p^2(p+3)}\right\}$.

Sol. $$L^{-1}\left\{\frac{p+2}{p^2(p+3)}\right\} = L^{-1}\left\{\frac{(p+3)-1}{p^2(p+3)}\right\} = L^{-1}\left\{\frac{1}{p^2} - \frac{1}{p^2(p+3)}\right\}$$

$$= L^{-1}\left\{\frac{1}{p^2}\right\} - L^{-1}\left\{\frac{1}{p^2(p+3)}\right\} \quad ...(i)$$

For first term

$$L^{-1}\left\{\frac{1}{p^2}\right\} = \frac{t}{1!} = t$$

For second term

$$L^{-1}\left\{\frac{1}{p+3}\right\} = e^{-3t} = f(t) \text{ say}$$

$$\therefore \quad L^{-1}\left\{\frac{1}{p(p+3)}\right\} = \int_0^t f(x)dx$$

$$= \int_0^t e^{-3x}\,dx = \left[\frac{e^{-3x}}{-3}\right]_0^t = \frac{1}{3}(1-e^{-3t})$$

Again $$L^{-1}\left\{\frac{1}{p^2(p+3)}\right\} = \int_0^t \frac{(1-e^{-3x})}{3}\,dx$$

$$= \frac{1}{3}\left[x - \frac{e^{-3x}}{-3}\right]_0^t = \frac{1}{3}\left[t + \frac{e^{-3t}}{3} - \frac{1}{3}\right]$$

Hence from equation (i), we have

$$L^{-1}\left\{\frac{p+2}{p^2(p+3)}\right\} = t - \frac{1}{3}\left[t + \frac{e^{-3t}}{3} - \frac{1}{3}\right]$$

$$= t - \frac{t}{3} - \frac{e^{-3t}}{9} + \frac{1}{9} = \frac{2}{3}t - \frac{1}{9}(e^{-3t}-1)$$ **Ans.**

Prob.50. Find the inverse Laplace transform of $\frac{1}{s^3-a^3}$.

Sol. Since, $A^3 - B^3 = (A-B)(A^2+AB+B^2)$

$$\therefore \quad s^3 - a^3 = (s-a)(s^2+as+a^2)$$

Let, $$\frac{1}{s^3-a^3} = \frac{A}{s-a} + \frac{Bs+C}{s^2+as+a^2}$$

$$1 = A(s^2+as+a^2) + Bs(s-a) + C(s-a) \quad ....(i)$$

On solving equation (i), we get $A = \frac{1}{3a^2}$, $B = -\frac{1}{3a^2}$ and $C = -\frac{2}{3a}$, then

$$\frac{1}{s^3-a^3} = \frac{1}{3a^2}\cdot\frac{1}{(s-a)} + \frac{\left(-\frac{1}{3a^2}.s - \frac{2}{3a}\right)}{s^2+as+a^2}$$

$$\therefore \quad L^{-1}\left[\frac{1}{s^3-a^3}\right] = \frac{1}{3a^2}L^{-1}\left(\frac{1}{s-a}\right) - \frac{1}{3a^2}L^{-1}\left(\frac{s}{s^2+as+a^2}\right)$$

$$-\frac{2}{3a}L^{-1}\left(\frac{1}{s^2+as+a^2}\right)$$

$$= \frac{1}{3a^2}e^{at} - \frac{1}{3a^2}\left[e^{-at/2}\cos\frac{\sqrt{3}at}{2} + e^{-at/2}.\sqrt{3}\sin\frac{\sqrt{3}at}{2}\right]$$

$$= \frac{1}{3a^2}\left[e^{at} - e^{-at/2}\left\{\cos\left(\frac{\sqrt{3}at}{2}\right) + \sqrt{3}\sin\left(\frac{\sqrt{3}at}{2}\right)\right\}\right]$$ **Ans.**

Prob.51. Find $L^{-1}\left\{\frac{s}{s^4+s^2+1}\right\}$ ***[R.G.P.V., May 2019 (IV-Sem)]***

Sol. Here,

$$\frac{s}{s^4+s^2+1} = \frac{s}{(s^2+s+1)(s^2-s+1)}$$

Suppose

$$\frac{s}{s^4+s^2+1} = \frac{As+B}{s^2+s+1} + \frac{Cs+D}{s^2-s+1}$$

Multiplying $(s^2+s+1)(s^2-s+1)$ both sides, we get

$$s = (As+B)(s^2-s+1) + (Cs+D)(s^2+s+1)$$

$$\Rightarrow s = As^3 + Bs^2 - As^2 - Bs + As + B + Cs^3 + Cs^2 + Cs + Ds^2 + Ds + D$$

Equating the coefficients of s^3, s^2, s and s^0, we have

$$A + C = 0 \quad \text{...(i)}$$

$$-A + B + C + D = 0 \quad \text{...(ii)}$$

$$A - B + C + D = 1 \quad \text{...(iii)}$$

and $$B + D = 0 \quad \text{...(iv)}$$

Using equations (i) and (iv) in equations (ii) and (iii), we get

$$2C = 0 \Rightarrow C = 0 \quad \text{...(v)}$$

and $$A = 0 \quad \text{...(vi)}$$

Now on solving equations (ii), (iii) and (iv), we get

$$2D = 1 \Rightarrow D = \frac{1}{2} \text{ and } B = -\frac{1}{2}$$

$$\therefore \quad L^{-1}\left[\frac{s}{s^4+s^2+1}\right] = -\frac{1}{2}L^{-1}\left[\frac{1}{s^2+s+1}\right] + \frac{1}{2}L^{-1}\left[\frac{1}{s^2-s+1}\right]$$

$$= -\frac{1}{2}L^{-1}\left[\frac{1}{s^2+2.s.\frac{1}{2}+\frac{1}{4}+\frac{3}{4}}\right] + \frac{1}{2}L^{-1}\left[\frac{1}{s^2-2.s.\frac{1}{2}+\frac{1}{4}+\frac{3}{4}}\right]$$

$$= -\frac{1}{2}L^{-1}\left[\frac{1}{\left(s+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}\right] + \frac{1}{2}L^{-1}\left[\frac{1}{\left(s-\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}\right]$$

$$= -\frac{1}{2}.e^{-\frac{1}{2}t}.\frac{2}{\sqrt{3}}\sin\frac{\sqrt{3}}{2}t + \frac{1}{2}.e^{\frac{1}{2}t}.\frac{2}{\sqrt{3}}\sin\frac{\sqrt{3}}{2}t$$

$$= \mathbf{\frac{2}{\sqrt{3}}\sinh\frac{t}{2}\sin\frac{\sqrt{3}}{2}t}$$

Ans.

Prob.52. Find –

$$L^{-1}\left[\frac{s+2}{(s^2+4s+5)^2}\right]$$

Sol. Here,

$$L^{-1}\left(\frac{1}{s^2+4s+5}\right) = L^{-1}\left\{\frac{1}{(s+2)^2+1}\right\} = e^{-2t}\sin t$$

By shifting property II, we have

$$L^{-1}\left\{\frac{d}{ds}\left(\frac{1}{s^2+4s+5}\right)\right\} = (-1)^1.t.e^{-2t}\sin t$$

i.e., $$L^{-1}\left\{\frac{-(2s+4)}{(s^2+4s+5)^2}\right\} = -t.e^{-2t}\sin t$$

or $$L^{-1}\left\{\frac{s+2}{(s^2+4s+5)^2}\right\} = \mathbf{\frac{1}{2}t.e^{-2t}\sin t}$$ **Ans.**

Prob.53. Find inverse Laplace transform of the followings –

(i) $\dfrac{e^{-4s}}{(s-2)^4}$ ***(ii)*** $\dfrac{1}{s^2-6s+18}$.

Sol. (i) $$L^{-1}\left\{\frac{1}{(s-2)^4}\right\} = e^{2t}L^{-1}\left\{\frac{1}{s^4}\right\} = e^{2t}\left(\frac{t^3}{3!}\right) = \frac{1}{6}t^3e^{2t}$$

Then by second shifting theorem, we get

$$L^{-1}\left\{\frac{e^{-4s}}{(s-2)^4}\right\} = \begin{cases}\frac{1}{6}(t-4)^3e^{2(t-4)}, & t>4\\ 0, & t<4\end{cases}$$

$$= \mathbf{\frac{1}{6}(t-4)^3e^{2(t-4)}H(t-4),}$$ **Ans.**

where, H(t – 4) is the Heaviside unit step function.

(ii) Here $$L^{-1}\left\{\frac{1}{s^2-6s+18}\right\} = L^{-1}\left\{\frac{1}{(s-3)^2+9}\right\}$$

$$= L^{-1}\left\{\frac{1}{(s-3)^2+(3)^2}\right\} = \mathbf{\frac{1}{3}e^{3t}\sin 3t}$$ **Ans.**

Prob.54. Find inverse Laplace transform of the following functions –

(i) $\tan^{-1}(1/\beta)$ (ii) $\log\left(\frac{\beta+3}{\beta+2}\right)$.

Sol. (i) Let, $F(\beta) = \tan^{-1}(1/\beta)$

Then $$F'(\beta) = \frac{1}{1+\left(\frac{1}{\beta}\right)^2}.(-1)\frac{1}{\beta^2} = -\frac{1}{1+\beta^2}$$

Taking L^{-1} of both sides

$$L^{-1}\{F'(\beta)\} = L^{-1}\left\{-\frac{1}{1+\beta^2}\right\} = -\sin t$$

$$(-1)t\, L^{-1}\{F(\beta)\} = -\sin t$$

$$L^{-1}\{F(\beta)\} = \frac{\sin t}{t}$$ **Ans.**

(ii) Let, $F(\beta) = \log\frac{(\beta+3)}{(\beta+2)}$

$$F(\beta) = \log(\beta+3) - \log(\beta+2)$$

Then $$F'(\beta) = \frac{1}{\beta+3} - \frac{1}{\beta+2}$$

Taking L^{-1} of both sides

$$L^{-1}\{F'(\beta)\} = L^{-1}\left\{\frac{1}{\beta+3}\right\} - L^{-1}\left\{\frac{1}{\beta+2}\right\}$$

$$(-1)t\, L^{-1}\{F(\beta)\} = e^{-3t} - e^{-2t}$$

or $$L^{-1}\{F(\beta)\} = \frac{e^{-2t} - e^{-3t}}{t}$$ **Ans.**

Prob.55. Find the inverse Laplace transform of

$$\frac{s^2+6}{(s^2+1)(s^2+4)}.$$

Sol. Let, $$\frac{s^2+6}{(s^2+1)(s^2+4)} = \frac{As+B}{s^2+1} + \frac{Cs+D}{s^2+4}$$

Multiplying both sides by $(s^2+1)(s^2+4)$, we get

$$s^2+6 = (As+B)(s^2+4) + (Cs+D)(s^2+1)$$

or $$s^2+6 = As^3 + 4As + Bs^2 + 4B + Cs^3 + Cs + Ds^2 + D$$

Equating the coefficients of s^3, s^2, s, and s^0, we get

$$0 = A + C \quad \text{...(i)}$$
$$1 = B + D \quad \text{...(ii)}$$
$$0 = 4A + C \quad \text{...(iii)}$$
$$6 = 4B + D \quad \text{...(iv)}$$

and

On solving above equations, we get $A = C = 0$ and $B = \frac{5}{3}$, $D = -\frac{2}{3}$.

$$L^{-1}\left[\frac{s^2+6}{(s^2+1)(s^2+4)}\right] = \frac{5}{3}L^{-1}\left\{\frac{1}{s^2+1}\right\} - \frac{2}{3}L^{-1}\left\{\frac{1}{s^2+4}\right\}$$

or $$L^{-1}\left[\frac{s^2+6}{(s^2+1)(s^2+4)}\right] = \frac{5}{3}\sin t - \frac{1}{3}\sin 2t$$ **Ans.**

Prob.56. Applying convolution theorem find the inverse transform of

$$\frac{s^2}{(s^2+a^2)^2}.$$

Sol. Here, $$L^{-1}\left\{\frac{s^2}{(s^2+a^2)^2}\right\} = L^{-1}\left\{\frac{s}{(s^2+a^2)}.\frac{s}{(s^2+a^2)}\right\}$$

Since, $$L^{-1}\left\{\frac{s}{s^2+a^2}\right\} = \cos at$$

Hence, by convolution theorem

$$L^{-1}\left\{\frac{s^2}{(s^2+a^2)^2}\right\} = \int_0^t \cos au.\cos a(t-u)\,du$$

$$= \frac{1}{2}\int_0^t [\cos at + \cos(2au - at)]\,du$$

$$= \frac{1}{2}\left[u\cos at + \frac{1}{2a}\sin(2au - at)\right]_0^t$$

$$= \frac{1}{2}\left[t\cos at + \frac{1}{2a}\sin at - 0 - \frac{1}{2a}\sin(-at)\right]$$

$$= \frac{1}{2}\left[t\cos at + \frac{1}{a}\sin at\right]$$

$$= \frac{1}{2a}[at\cos at + \sin at]$$ **Ans.**

Prob.57. Using convolution theorem to evaluate –

$$L^{-1}\left(\frac{s}{(s^2+a^2)^2}\right).$$

Sol. Since, $L^{-1}\left(\frac{s}{s^2+a^2}\right) = \cos at$ and $L^{-1}\left(\frac{1}{s^2+a^2}\right) = \frac{1}{a}\sin at$

By convolution theorem, we get

$$L^{-1}\left(\frac{s}{s^2+a^2}\cdot\frac{1}{s^2+a^2}\right) = \int_0^t \cos a(t-u)\frac{\sin au}{a}du$$

$$= \frac{1}{2a}\int_0^t [(\sin at + \sin(2au - at)]du$$

$$= \frac{1}{2a}\left[u\sin at - \frac{1}{2a}\cos(2au - at)\right]_0^t$$

$$= \frac{1}{2a}\left[t\sin at - \frac{1}{2a}\cos at - 0 + \frac{1}{2a}\cos(-at)\right]$$

Hence $L^{-1}\left\{\frac{s}{(s^2+a^2)^2}\right\} = \mathbf{\frac{1}{2a}t\sin at}$ **Ans.**

Prob.58. Apply convolution theorem to evaluate –

$$L^{-1}\left(\frac{1}{(s^2+9)(s+3)}\right).$$

Sol. Since,

$$L^{-1}\left\{\frac{1}{s^2+9}\right\} = \frac{1}{3}\sin 3t \text{ and } L^{-1}\left\{\frac{1}{(s+3)}\right\} = e^{-3t}$$

By convolution theorem, we get

$$L^{-1}\left\{\frac{1}{(s^2+9)(s+3)}\right\} = \int_0^t \frac{1}{3}\sin 3u\, e^{-3(t-u)}du = \frac{1}{3}e^{-3t}\int_0^t e^{3u}\sin 3u\, du$$

$$= \frac{1}{3}e^{-3t}\left[\frac{e^{3u}}{9+9}(3\sin 3u - 3\cos 3u)\right]_0^t$$

$$\left[\because e^{ax}\sin bx\, dx = \frac{e^{ax}}{a^2+b^2}(a\sin bx - b\cos bx)\right]$$

$$= \frac{1}{18}e^{-3t}\left[e^{3t}(\sin 3t - \cos 3t) + 1\right]$$

$$= \mathbf{\frac{1}{18}[e^{-3t} + \sin 3t - \cos 3t]}$$ **Ans.**

Prob.59. Using convolution theorem find $L^{-1}\left\{\frac{1}{(s-1)(s+2)}\right\}$ ***(R.G.P.V., Nov. 2018)***

Sol. $L^{-1}\left\{\frac{1}{(s-1)(s+2)}\right\} = L^{-1}\left\{\frac{1}{(s-1)}\cdot\frac{1}{(s+2)}\right\}$

Since, $L^{-1}\left\{\frac{1}{(s-1)}\right\} = e^t$, $L^{-1}\left\{\frac{1}{(s+2)}\right\} = e^{-2t}$

Hence, by convolution theorem

$$L^{-1}\left\{\frac{1}{(s-1)(s+2)}\right\} = \int_0^t e^u.e^{-2(t-u)}du = e^{-2t}\int_0^t e^{3u}du$$

$$= e^{-2t}\left[\frac{e^{3u}}{3}\right]_0^t = \frac{1}{3}e^{-2t}(e^{3t}-1) = \frac{1}{3}(e^t - e^{-2t})$$ **Ans.**

Prob.60. Use convolution theorem, to find $L^{-1}\left\{\frac{1}{s^3(s^2+1)}\right\}$.

Sol. Here, $L^{-1}\left\{\frac{1}{s^3(s^2+1)}\right\} = L^{-1}\left\{\frac{1}{s^3}\cdot\frac{1}{(s^2+1)}\right\}$

$\because$ $L^{-1}\left[\frac{1}{s^3}\right] = \frac{t^2}{2}$ and $L^{-1}\left[\frac{1}{(s^2+1)}\right] = \sin t$

$\therefore$ By convolution theorem, we have

$$L^{-1}\left\{\frac{1}{s^3(s^2+1)}\right\} = \int_0^t \frac{u^2}{2}.\sin(t-u)\,du$$

$$= \frac{1}{2}\left[u^2\left(\frac{-\cos(t-u)}{(-1)}\right) - \int 2u.\left(\frac{-\cos(t-u)}{(-1)}\right)du\right]_0^t$$

$$= \frac{1}{2}\left[u^2\cos(t-u) - 2\left\{u\left(\frac{\sin(t-u)}{(-1)}\right) - \int 1.\left(\frac{\sin(t-u)}{(-1)}\right)du\right\}\right]_0^t$$

$$= \frac{1}{2}\left[u^2\cos(t-u) + 2u\sin(t-u) - 2\left(\frac{-\cos(t-u)}{(-1)}\right)\right]_0^t$$

$$= \frac{1}{2}\left[u^2\cos(t-u) + 2u\sin(t-u) - 2\cos(t-u)\right]_0^t$$

$$= \frac{1}{2}\left[t^2.\cos 0 + 2t\sin 0 - 2\cos 0 - 0 - 0 + 2\cos t\right]$$

$$= \frac{1}{2}\left[t^2.1 + 0 - 2.1 + 2\cos t\right] = \frac{1}{2}\left[t^2 - 2 + 2\cos t\right] = \mathbf{\frac{t^2}{2} - 1 + \cos t}$$ **Ans.**

Prob.61. Using convolution theorem, find the inverse Laplace transform of

$$F(s) = \frac{1}{(s+1)(s^2+1)}.$$

Sol. $L^{-1}\left\{\frac{1}{(s+1)(s^2+1)}\right\} = L^{-1}\left\{\frac{1}{(s+1)}\cdot\frac{1}{(s^2+1)}\right\}$

Since, $L^{-1}\left\{\frac{1}{(s+1)}\right\} = e^{-t}$, $L^{-1}\left\{\frac{1}{(s^2+1)}\right\} = \sin t$

Hence, by convolution theorem

$$L^{-1}\left\{\frac{1}{(s+1)(s^2+1)}\right\} = \int_0^t e^{-u}.\sin(t-u)\,du \quad ...(i)$$

Let,

$$I = \int_0^t e^{-u}\sin(t-u)\,du \quad ...(ii)$$

$$= \left[e^{-u}\cos(t-u)\right]_0^t + \int_0^t e^{-u}\cos(t-u)\,du$$

$$= \left[e^{-u}\cos(t-u)\right]_0^t + \left[e^{-u}\sin(t-u).(-1)\right]_0^t - \int_0^t e^{-u}\sin(t-u)\,du$$

or $$I = \left[e^{-u}\cos(t-u)\right]_0^t - \left[e^{-u}\sin(t-u)\right]_0^t - I$$

[From equation (ii)]

or $$2I = (e^{-t} - \cos t) - (0 - \sin t)$$

or $$I = \frac{1}{2}(e^{-t} - \cos t + \sin t)$$

∴ From equation (i)

$$L^{-1}\left\{\frac{1}{(s+1)(s^2+1)}\right\} = \frac{1}{2}(e^{-t} - \cos t + \sin t) \quad \text{Ans.}$$

Prob.62. Using the convolution theorem, find –

$$L^{-1}\left\{\frac{s}{(s^2+1)(s^2+4)}\right\}.$$

Sol. Here, $\frac{s}{(s^2+1)(s^2+4)} = \frac{As+B}{(s^2+1)} + \frac{Cs+D}{(s^2+4)}$

Multiplying $(s^2+1)(s^2+4)$ both sides, we get

$$s = (As+B)(s^2+4) + (Cs+D)(s^2+1)$$

$$\Rightarrow s = As^3 + 4As + Bs^2 + 4B + Cs^3 + Cs + Ds^2 + D$$

Equating the coefficients of s^3, s^2, s and s^0, we get

$$A + C = 0 \quad ...(i)$$

$$B + D = 0 \quad ...(ii)$$

$$4A + C = 1 \quad ...(iii)$$

and $$4B + D = 0 \quad ...(iv)$$

On solving above equations, we get $A = \frac{1}{3}$, $B = 0$, $C = -\frac{1}{3}$ and $D = 0$.

$$\therefore L^{-1}\left\{\frac{s}{(s^2+1)(s^2+4)}\right\} = \frac{1}{3}L^{-1}\left\{\frac{s}{s^2+1}\right\} - \frac{1}{3}L^{-1}\left\{\frac{s}{s^2+4}\right\}$$

$$= \frac{1}{3}\cos t - \frac{1}{3}\cos 2t = \frac{1}{3}(\cos t - \cos 2t) \quad \text{Ans.}$$

Prob.63. Using convolution theorem, evaluate –

$$L^{-1}\left\{\frac{s^2}{(s^2+a^2)(s^2+b^2)}\right\}.$$

Sol. Here, $L^{-1}\left\{\frac{s^2}{(s^2+a^2)(s^2+b^2)}\right\} = L^{-1}\left\{\frac{s}{(s^2+a^2)}\cdot\frac{s}{(s^2+b^2)}\right\}$

$\because L^{-1}\left[\frac{s}{s^2+a^2}\right] = \cos at$ and $L^{-1}\left[\frac{s}{s^2+b^2}\right] = \cos bt$

∴ By convolution theorem, we have

$$L^{-1}\left[\frac{s^2}{(s^2+a^2)(s^2+b^2)}\right] = L^{-1}\left[\left(\frac{s}{s^2+a^2}\right)\cdot\left(\frac{s}{s^2+b^2}\right)\right]$$

$$= \int_0^t \cos a(t-u)\cos bu\,du = \frac{1}{2}\int_0^t 2\cos a(t-u)\cos bu\,du$$

$$= \frac{1}{2}\int_0^t [\cos\{at-(a-b)u\} + \cos\{at-(a+b)u\}]\,du$$

$$= \frac{1}{2}\left[\frac{\sin\{at-(a-b)u\}}{-(a-b)} + \frac{\sin\{at-(a+b)u\}}{-(a+b)}\right]_0^t$$

$$= \frac{1}{2}\left[\frac{\sin\{at+(b-a)t\} - \sin at}{b-a} - \frac{\sin\{at-(b+a)t\} - \sin at}{b+a}\right]$$

$$= \frac{1}{2}\left[\frac{\sin bt - \sin at}{b-a} + \frac{\sin bt + \sin at}{b+a}\right]$$

$$= \frac{1}{2(b^2-a^2)}(2b \sin bt - 2a \sin at)$$

$$= \frac{1}{b^2-a^2}(b \sin bt - a \sin at)$$ **Ans.**

Prob.64. Using convolution theorem find the inverse Laplace transform $\left(\frac{1}{s^2+a^2}\right)^2$.

Sol. Here, $L^{-1}\left\{\frac{1}{(s^2+a^2)}\right\}^2 = L^{-1}\left\{\frac{1}{(s^2+a^2)}\cdot\frac{1}{(s^2+a^2)}\right\}$

Since, $L^{-1}\left\{\frac{1}{s^2+a^2}\right\} = \frac{\sin at}{a}$

Hence by convolution theorem

$$L^{-1}\left\{\frac{1}{(s^2+a^2)}\right\}^2 = \int_0^t \frac{\sin au}{a}\frac{1}{a}\sin a(t-u)\,du$$

$$= \frac{1}{2a^2}\int_0^t 2\sin au \sin(at-au)du$$

$$= \frac{1}{2a^2}\int_0^t [\cos(au-at+au) - \cos(au+at-au)]du$$

$$= \frac{1}{2a^2}\int_0^t [\cos(2au-at) - \cos at]\,du$$

$$= \frac{1}{2a^2}\left[\frac{\sin(2au-at)}{2a} - \cos at.u\right]_0^t$$

$$= \frac{1}{2a^2}\left[\frac{\sin at}{2a} - t\cos at + \frac{1}{2a}\sin at\right] = \frac{1}{2a^3}[\sin at - at\cos at]$$ **Ans.**

Prob.65. Solve the simultaneous equations using Laplace transform -

$$\frac{dx}{dt} - y = e^t, \frac{dy}{dt} + x = \sin t, \text{ given } x(0) = 1, y(0) = 0.$$

Sol. Given, $\frac{dx}{dt} - y = e^t$ ...(i)

$\frac{dy}{dt} + x = \sin t$ ...(ii)

Taking Laplace transform on both sides of equations (i) and (ii) respectively, we have

$$L\left\{\frac{dx}{dt} - y\right\} = L\{e^t\}$$

$$s\bar{x} - x(0) - \bar{y} = \frac{1}{s-1}, \quad [\text{where } L(x) = \bar{x}]$$

$$s\bar{x} - \bar{y} = \frac{s}{s-1} \quad [\because x(0) = 1] \quad ...(iii)$$

and

$$L\left\{\frac{dy}{dt} + x\right\} = L\{\sin t\}$$

$$s\bar{y} - y(0) + \bar{x} = \frac{1}{s^2+1}$$

$$s\bar{y} + \bar{x} = \frac{1}{s^2+1} \quad [\because y(0) = 0] \quad ...(iv)$$

On solving equations (iii) and (iv), we get

$$\bar{x} = \frac{s^4+s^2+s-1}{(s^2+1)^2(s-1)} \text{ and } \bar{y} = -\left[\frac{s^3-s^2+2s}{(s^2+1)^2(s-1)}\right]$$

Now $$\bar{x} = \frac{s^4+s^2+s-1}{(s^2+1)^2(s-1)} = \frac{As+B}{(s^2+1)} + \frac{(Cs+D)}{(s^2+1)^2} + \frac{E}{(s-1)} \quad ...(v)$$

(by partial fraction)

$$\frac{s^4+s^2+s-1}{(s^2+1)^2(s-1)} = \frac{(As+B)(s^2+1)(s-1)+(Cs+D)(s-1)+E(s^2+1)^2}{(s^2+1)^2(s-1)}$$

$$s^4+s^2+s-1 = (As+B)(s^2+1)(s-1)+(Cs+D)(s-1)+E(s^2+1)^2 \quad ...(vi)$$

By equating the coefficient of various power of s, we get

$$A = B = E = \frac{1}{2}, \; C = 0, D = 1$$

Putting these values in equation (v), we get

$$\bar{x} = \frac{\frac{1}{2}s + \frac{1}{2}}{s^2+1} + \frac{1}{(s^2+1)^2} + \frac{1}{2}\cdot\frac{1}{s-1}$$

Taking inverse Laplace transform, we get

$$x = L^{-1}\left\{\frac{1}{2}\frac{s}{s^2+1} + \frac{1}{2}\frac{1}{s^2+1} + \frac{1}{(s^2+1)^2} + \frac{1}{2}\frac{1}{s-1}\right\}$$

$$= \frac{1}{2}\left[\cos t + \sin t + (\sin t - t\cos t) + e^t\right]$$

$$= \frac{1}{2}\left[\cos t + 2\sin t - t\cos t + e^t\right]$$

Now $\bar{y} = \left[\dfrac{-s^3+s^2-2s}{(s^2+1)^2(s-1)}\right]$

Taking R.H.S.

$$\frac{-s^3+s^2-2s}{(s^2+1)^2(s-1)} = \frac{As+B}{s^2+1} + \frac{Cs+D}{(s^2+1)^2} + \frac{E}{(s-1)} \quad \text{...(vii)}$$

(by partial fraction)

or $-s^3+s^2-2s = (As+B)(s^2+1)(s-1) + (Cs+D)(s-1) + E(s^2+1)^2$

On equating coefficient of various powers of s, we get

$$A = \frac{1}{2},\ B = -\frac{1}{2},\ C = 1,\ D = 0,\ E = -\frac{1}{2}$$

Putting these values in equation (vii), we get

$$\frac{-s^3+s^2-2s}{(s^2+1)^2(s-1)} = \frac{\frac{1}{2}s-\frac{1}{2}}{(s^2+1)} + \frac{s}{(s^2+1)^2} - \frac{1}{2}\cdot\frac{1}{(s-1)}$$

or $$\bar{y} = \frac{\frac{1}{2}s-\frac{1}{2}}{(s^2+1)} + \frac{s}{(s^2+1)^2} - \frac{1}{2}\frac{1}{(s-1)}$$

Taking inverse Laplace transform, we have

$$L^{-1}\{\bar{y}\} = L^{-1}\left\{\frac{1}{2}\frac{s}{(s^2+1)} - \frac{1}{2}\frac{1}{(s^2+1)} + \frac{s}{(s^2+1)^2} - \frac{1}{2}\frac{1}{(s-1)}\right\}$$

$$y = \frac{1}{2}\left\{\cos t - \sin t + t\sin t - e^t\right\}$$

Thus, $x = \dfrac{1}{2}\left[\cos t + 2\sin t - t\cos t + e^t\right]$

and $y = \dfrac{1}{2}\left[\cos t - \sin t + t\sin t - e^t\right]$ **Ans.**

Prob.66. Solve the simultaneous equations $\dfrac{dx}{dt} + 5x - 2y = t,\ \dfrac{dy}{dt} + 2x + y = 0$ ***given x = y = 0 when t = 0.***

Sol. Taking $\dfrac{d}{dt} = D$, the given equations becomes,

$$(D+5)x - 2y = t$$
$$2x + (D+1)y = 0$$

Taking the Laplace transforms of the given equations, we get

$$[s\bar{x} - x(0)] + 5\bar{x} - 2\bar{y} = \frac{1}{s^2}$$

i.e. $$(s+5)\bar{x} - 2\bar{y} = \frac{1}{s^2} \quad [\because x(0) = 0] \quad \text{...(i)}$$

and $$s\bar{y} - y(0) + 2\bar{x} + \bar{y} = 0$$

i.e. $$2\bar{x} + (s+1)\bar{y} = 0 \quad [\because y(0) = 0] \quad \text{....(ii)}$$

On solving equations, we get

$$\bar{x} = \frac{s+1}{s^2(s+3)^2} = \frac{1}{27s} + \frac{1}{9s^2} - \frac{1}{27(s+3)} - \frac{2}{9(s+3)^2}$$

Putting the value of $\bar{x}$ in equation (ii), we get

$$\bar{y} = -\frac{2}{s^2(s+3)^2} = \frac{4}{27s} - \frac{2}{9s^2} - \frac{4}{27(s+3)} - \frac{2}{9(s+3)^2}$$

On taking inverse Laplace transform, we get

$$x = \frac{1}{27} + \frac{t}{9} - \frac{1}{27}e^{-3t} - \frac{2}{9}te^{-3t}$$

and $$y = \frac{4}{27} - \frac{2t}{9} - \frac{4}{27}e^{-3t} - \frac{2}{9}te^{-3t}$$ **Ans.**

Prob.67. Solve $y'' - 2y' + 2y = 0$, ***given*** $y(0) = y'(0) = 1$, ***by Laplace transformation method.*** ***[R.G.P.V., Nov. 2019 (IV-Sem)]***

Sol. Given, $y'' - 2y' + 2y = 0$

where $y' = \dfrac{dy}{dt}$ and $y'' = \dfrac{d^2y}{dt^2}$.

Taking Laplace transform on both sides, we get

$$s^2\bar{y} - sy(0) - y'(0) - 2[s\bar{y} - y(0)] + 2\bar{y} = 0$$

$\Rightarrow$ $$s^2\bar{y} - s - 1 - 2[s\bar{y} - 1] + 2\bar{y} = 0$$

$\Rightarrow$ $$(s^2 - 2s + 2)\bar{y} - s + 1 = 0$$

$\Rightarrow$ $$(s^2 - 2s + 2)\bar{y} = s - 1$$

$\Rightarrow$ $$\bar{y} = \frac{s-1}{s^2-2s+2}$$

$$\bar{y} = \frac{s-1}{(s-1)^2+1}$$

Taking inverse Laplace transform, we get

$$y = L^{-1}\left\{\frac{s-1}{(s-1)^2+1}\right\} = e^tL^{-1}\left\{\frac{s}{s^2+1}\right\} = e^t\cos t$$ **Ans.**

Prob.68. Solve the following differential equation $y''(t) - 3y'(t) + 2y(t) = 4t + e^{3t}$, when $y(0) = 1$ and $y'(0) = -1$. (By Laplace transformation). **(R.G.P.V., May 2019 (IV-Sem))**

Sol. Given, $y'' - 3y' + 2y = 4t + e^{3t}$

Taking Laplace transform on both sides, we get

$$s^2\bar{y} - sy(0) - y'(0) - 3[s\bar{y} - y(0)] + 2\bar{y} = \frac{4}{s^2} + \frac{1}{s-3}$$

$$\Rightarrow (s^2 - 3s + 2)\,\bar{y} - s + 1 + 3 = \frac{4}{s^2} + \frac{1}{s-3}$$

$$\Rightarrow (s^2 - 3s + 2)\,\bar{y} = \frac{4}{s^2} + \frac{1}{s-3} + (s-4)$$

$$\Rightarrow \bar{y} = \frac{4}{s^2(s-1)(s-2)} + \frac{1}{(s-1)(s-2)(s-3)} + \frac{s-4}{(s-1)(s-2)}$$

On inversion, we get

$$y = 4\int_0^t\left\{\int_0^t\left(e^{2t} - e^t\right)dt\right\}dt + \int_0^t e^{3(t-u)}\left(e^{2u} - e^u\right)du + L^{-1}\left[\frac{3}{s-1}\right] - L^{-1}\left[\frac{2}{s-2}\right]$$

$$= 4\int_0^t\left[\frac{e^{2t}}{2} - e^t\right]_0^t dt + e^{3t}\int_0^t e^{-u}du - e^{3t}\int_0^t e^{-2u}du + 3.e^t - 2e^{2t}$$

$$= 4\int_0^t\left(\frac{e^{2t}}{2} - e^t - \frac{1}{2} + 1\right)dt + e^{3t}\left[\frac{e^{-u}}{-1}\right]_0^t - e^{3t}\left[\frac{e^{-2u}}{-2}\right]_0^t + 3e^t - 2e^{2t}$$

$$= \frac{4}{2}\left[\frac{e^{2t}}{2}\right]_0^t - 4\left[e^t\right]_0^t + \frac{4}{2}\left[t\right]_0^t + e^{3t}\left[-e^{-t} + 1\right] - e^{3t}\left[\frac{-e^{-2t}}{2} + \frac{1}{2}\right] + 3e^t - 2e^{2t}$$

$$= \left(e^{2t} - 1\right) - 4\left(e^t - 1\right) + 2t - e^{2t} + e^{3t} + \frac{1}{2}e^t - \frac{1}{2}e^{3t} + 3e^t - 2e^{2t}$$

$$= \mathbf{\frac{1}{2}e^{3t} - 2e^{2t} - \frac{1}{2}e^t + 3 + 2t}$$ **Ans.**

Prob.69. Solve the differential equation $(D^2 + 2D + 1)\,y = 3te^{-t}$, $t > 0$, subject to the conditions $y = 4$, $Dy = 2$ when $t = 0$.

Sol. Given equation is

$$(D^2 + 2D + 1)\,y = 3te^{-t} \qquad ...(i)$$

and $\quad y(0) = 4,\ y'(0) = 2$

Equation (i) can be written as

$$y'' + 2y' + y = 3te^{-t}$$

Taking Laplace transform on both sides, we get

$$s^2\bar{y} - sy(0) - y'(0) + 2[s\bar{y} - y(0)] + \bar{y} = -3\frac{d}{ds}\{L(e^{-t})\}$$

$$(s^2 + 2s + 1)\,\bar{y} - 4s - 2 + 2\times(-4) = -3\frac{d}{ds}\left(\frac{1}{s+1}\right)$$

$$\Rightarrow (s+1)^2\bar{y} = 4s + 10 + \frac{3}{(s+1)^2}$$

$$\Rightarrow \bar{y} = \frac{4(s+1)+6}{(s+1)^2} + \frac{3}{(s+1)^4}$$

$$\Rightarrow \bar{y} = \frac{4}{(s+1)} + \frac{6}{(s+1)^2} + \frac{3}{(s+1)^4}$$

Taking inverse Laplace transform, we get

$$y = 4e^{-t} + 6\frac{e^{-t}t^{2-1}}{(2-1)!} + 3\frac{e^{-t}t^{4-1}}{(4-1)!}$$

$$= 4e^{-t} + 6e^{-t}.t + 3e^{-t}.\frac{t^3}{3!} = e^{-t}\left(4 + 6t + \frac{1}{2}t^3\right)$$

Ans.

Prob.70. Solve the following differential equation by Laplace transform –

$$\frac{d^2y}{dt^2} + 5\frac{dy}{dt} + 6y = 5e^t,\ \text{given } y(0) = 2,\ y'(0) = 1.$$

Sol. Here, given differential equation is

$$\frac{d^2y}{dt^2} + 5\frac{dy}{dt} + 6y = 5e^t \qquad ...(i)$$

and $\quad y(0) = 2,\ y'(0) = 1$

Equation (i) can be written as

$$y'' + 5y' + 6y = 5e^t$$

Taking Laplace transform of both sides, we get

$$s^2\bar{y} - sy(0) - y'(0) + 5[s\bar{y} - y(0)] + 6\bar{y} = 5.\frac{1}{s-1}$$

$$(s^2 + 5s + 6)\bar{y} - (s+5)2 - 1 = \frac{5}{s-1}$$

$$(s^2 + 5s + 6)\bar{y} - (2s + 10 + 1) = \frac{5}{s-1}$$

$$(s+2)(s+3)\bar{y} = \frac{5}{s-1} + (2s+11)$$

$$\bar{y} = \frac{5}{(s-1)(s+2)(s+3)} + \frac{2s+11}{(s+2)(s+3)}$$

$$= \frac{5 + (2s+11)(s-1)}{(s-1)(s+2)(s+3)} = \frac{2s^2 + 9s - 6}{(s-1)(s+2)(s+3)}$$

On solving by partial fraction, we get

$$\bar{y} = \frac{5}{12}.\frac{1}{(s-1)} + \frac{16}{3}.\frac{1}{(s+2)} - \frac{15}{4}.\frac{1}{s+3}$$

Taking inverse Laplace transform, we get

$$y = \frac{5}{12}e^{t} + \frac{16}{3}e^{-2t} - \frac{15}{4}e^{-3t}$$ **Ans.**

Prob.71. Solve the equation $\frac{d^2y}{dt^2} - 3\frac{dy}{dt} + 2y = 4e^{2t}$, $y(0) = -3, y'(0) = 5$ ***using Laplace transform.***

Sol. Here given differential equation is

$$\frac{d^2y}{dt^2} - 3\frac{dy}{dt} + 2y = 4e^{2t} \quad ...(i)$$

and $y(0) = -3$, $y'(0) = 5$

Equation (i) can be written as

$$y'' - 3y' + 2y = 4e^{2t}$$

Taking Laplace transform of both sides, we get

$$s^2\bar{y} - sy(0) - y'(0) - 3[s.\bar{y} - y(0)] + 2\bar{y} = 4.\frac{1}{s-2}$$

$$(s^2 - 3s + 2)\bar{y} - (s-3)(-3) - 5 = \frac{4}{s-2}$$

$$(s^2 - 3s + 2)\bar{y} + (3s - 9 - 5) = \frac{4}{s-2}$$

$$(s-1)(s-2)\bar{y} = \frac{4}{(s-2)} - (3s-14)$$

$$\bar{y} = \frac{4}{(s-2)^2(s-1)} - \frac{(3s-14)}{(s-1)(s-2)}$$

$$= \frac{4-(3s-14)(s-2)}{(s-2)^2(s-1)} = \frac{-3s^2+20s-24}{(s-2)^2(s-1)}$$

On solving by partial fraction, we get

$$\bar{y} = \frac{4}{(s-2)} + \frac{4}{(s-2)^2} - \frac{7}{(s-1)}$$

Taking inverse Laplace transform, we get

$$y = 4e^{2t} + 4te^{2t} - 7e^{t}$$

$$= \mathbf{4e^{2t}(1+t) - 7e^{t}}$$ **Ans.**

Prob.72. Using Laplace transform, solve –

$$\frac{d^3y}{dt^3} + \frac{2d^2y}{dt^2} - \frac{dy}{dt} - 2y = 0$$

given that

$$y = 1,\ \frac{dy}{dt} = 2, \frac{d^2y}{dt^2} = 2 \text{ at } t = 0.$$

Sol. Taking Laplace transform of both sides, we get

$$\left[s^3\bar{y} - s^2y(0) - sy'(0) - y''(0)\right] + 2\left[s^2\bar{y} - sy(0) - y'(0)\right] - \left[s\bar{y} - y(0)\right] - 2\bar{y} = 0$$

$$\Rightarrow (s^3 + 2s^2 - s - 2)\ \bar{y} - s^2 - 2s - 2 - 2s - 4 + 1 = 0$$

$$\Rightarrow \left(s^3 + 2s^2 - s - 2\right)\bar{y} - s^2 - 4s - 5 = 0$$

$$\Rightarrow (s-1)(s+1)(s+2)\ \bar{y} = s^2 + 4s + 5 \Rightarrow \bar{y} = \frac{s^2+4s+5}{(s-1)(s+1)(s+2)}$$

On solving by partial fraction, we get

$$\bar{y} = \frac{5}{3}.\frac{1}{(s-1)} - \frac{1}{s+1} + \frac{1}{3}.\frac{1}{(s+2)}$$

Taking inverse Laplace transform, we get

$$y = \frac{5}{3}e^{t} - e^{-t} + \frac{1}{3}e^{-2t}$$ **Ans.**

Prob.73. Solve the ordinary differential equation using Laplace transform method $(D^4 + 2D^2 + 1)\,y = 0$.

where $y(0) = 0$, $y'(0) = 1$, $y''(0) = 2$ ***and*** $y'''(0) = -3$.

Sol. Taking Laplace transform of both sides, we get

$$[s^4\bar{y} - s^3y(0) - s^2y'(0) - sy''(0) - y'''(0)] + 2[s^2\bar{y} - sy(0) - y'(0)] + \bar{y} = 0$$

$$[s^4 + 2s^2 + 1)\bar{y} - (s^3 + 2s)y(0) - (s^2 + 2)y'(0) - sy''(0) - y'''(0) = 0$$

Putting $y(0) = 0$, $y'(0) = 1$, $y''(0) = 2$, and $y'''(0) = -3$

$$(s^4 + 2s^2 + 1)\bar{y} - 0 - (s^2 + 2).1 - s.2 + 3 = 0$$

$$\bar{y} = \frac{(s^2 + 2s - 1)}{(s^4 + 2s^2 + 1)} = \frac{(s^2+1) + (2s-2)}{(s^2+1)^2}$$

$$\bar{y} = \frac{1}{(s^2+1)} + \frac{2s-2}{(s^2+1)^2} = \frac{1}{s^2+1} + \frac{2s}{(s^2+1)^2} - \frac{2}{(s^2+1)^2}$$

Taking L^{-1} of both sides

$$y = L^{-1}\left\{\frac{1}{s^2+1}\right\} + L^{-1}\left\{\frac{2s}{(s^2+1)^2}\right\} - L^{-1}\left\{\frac{2}{(s^2+1)^2}\right\}$$

$$y = \sin t + t\sin t - (\sin t - t\cos t)$$ **Ans.**

$$y = \mathbf{t(\sin t + \cos t)}$$

Prob.74. Solve the equation –

$$\frac{d^2x}{dt^2}+2\frac{dx}{dt}+5x=e^{-t}\sin t,$$

x(0) = 0, x'(0) = 1, using Laplace transform.

Sol. Here, $\frac{d^2x}{dt^2}+2\frac{dx}{dt}+5x=e^{-t}\sin t$

Taking the Laplace transform of both sides, we get

$$[s^2\bar{x}-sx(0)-x'(0)]+2[s\bar{x}-x(0)]+5\bar{x}=L(e^{-t}\sin t)$$

or $$[s^2\bar{x}-sx(0)-x'(0)]+2[s\bar{x}-x(0)]+5\bar{x}=\frac{1}{(s+1)^2+1} \quad ...(i)$$

On putting the values of x(0) and x'(0) in equation (i), we get

$$(s^2\bar{x}-1)+2s\bar{x}+5\bar{x}=\frac{1}{s^2+2s+2}$$

$$s^2\bar{x}+2s\bar{x}+5\bar{x}=1+\frac{1}{s^2+2s+2}$$

or $$(s^2+2s+5)\,\bar{x}=\frac{s^2+2s+3}{s^2+2s+2} \text{ or } \bar{x}=\frac{s^2+2s+3}{(s^2+2s+5)(s^2+2s+2)}$$

On resolving the R.H.S. into partial fraction, we get

$$\bar{x}=\frac{2}{3}\frac{1}{(s^2+2s+5)}+\frac{1}{3}\frac{1}{(s^2+2s+2)}$$

On inversion, we get

$$x=\frac{2}{3}L^{-1}\frac{1}{(s^2+2s+5)}+\frac{1}{3}L^{-1}\frac{1}{(s^2+2s+2)}$$

$$=\frac{1}{3}L^{-1}\frac{2}{(s+1)^2+2^2}+\frac{1}{3}L^{-1}\frac{1}{(s+1)^2+1}$$

or $$\mathbf{x=\frac{1}{3}e^{-t}\sin 2t+\frac{1}{3}e^{-t}\sin t}$$ **Ans.**

Prob.75. Solve the equation –

$$\frac{d^2y}{dt^2}+2\frac{dy}{dt}+y=\sin t,\ y(0)=1,\ y'(0)=0$$

using Laplace transform.

Sol. Taking Laplace transform, we have

$$\left[s^2\bar{y}-sy(0)-y'(0)\right]+2\left[s\bar{y}-y(0)\right]+\bar{y}=\frac{1}{s^2+1}$$

Using given conditions, we get

$$\left(s^2\bar{y}-s\right)+2\left(s\bar{y}-1\right)+\bar{y}=\frac{1}{s^2+1}$$

or $$(s^2+2s+1)\,\bar{y}-s-2=\frac{1}{s^2+1}$$

or $$\bar{y}=\frac{1}{(s^2+1)(s+1)^2}+\frac{(s+2)}{(s+1)^2}=\frac{1+(s+2)(s^2+1)}{(s+1)^2(s^2+1)}$$

or $$\bar{y}=\frac{s^3+2s^2+s+3}{(s+1)^2(s^2+1)}$$

$$\bar{y}=\frac{3}{2}\frac{1}{(s+1)}+\frac{3}{2}\frac{1}{(s+1)^2}-\frac{1}{2}\frac{s}{s^2+1} \quad \text{(by partial fraction)}$$

On inversion, we get

$$\mathbf{y=\frac{3}{2}e^{-t}+\frac{3}{2}te^{-t}-\frac{1}{2}\cos t}$$ **Ans.**

Prob.76. Solve $(D^2+9)\,y=\cos 2t$ if $y(0)=1$, $y\left(\frac{\pi}{2}\right)=-1$.

[R.G.P.V., Nov. 2019 (III-Sem)]

Sol. Given $(D^2+9)y=\cos 2t$

Taking Laplace transform of both sides of given equation, we have

$$[s^2\bar{y}-sy(0)-y'(0)]+9\bar{y}=\frac{s}{s^2+4}$$

$\therefore$ $$(s^2+9)\bar{y}-sy(0)-y'(0)=\frac{s}{s^2+4}$$

Since $y'(0)$ is not given, let $y'(0)=A$

$\therefore$ $$(s^2+9)\bar{y}-s-A=\frac{s}{s^2+4}$$

or $$(s^2+9)\bar{y}=\frac{s}{s^2+4}+s+A$$

or $$\bar{y}=\frac{s}{(s^2+4)(s^2+9)}+\frac{s}{s^2+9}+\frac{A}{s^2+9}$$

or $$\bar{y}=s\left[\frac{1}{5}\left(\frac{1}{s^2+4}-\frac{1}{s^2+9}\right)\right]+\frac{s}{s^2+9}+\frac{A}{s^2+9}$$

or $$\bar{y}=\frac{s}{5(s^2+4)}-\frac{s}{5(s^2+9)}+\frac{s}{s^2+9}+\frac{A}{s^2+9}$$

On inversion, we get

$$y = \frac{1}{5}L^{-1}\left\{\frac{s}{s^2+4}\right\} - \frac{1}{5}L^{-1}\left\{\frac{s}{s^2+9}\right\} + L^{-1}\left\{\frac{s}{s^2+9}\right\} + AL^{-1}\left\{\frac{1}{s^2+9}\right\}$$

$$y = \frac{1}{5}\cos 2t - \frac{1}{5}\cos 3t + \cos 3t + \frac{1}{3}A\sin 3t$$

$$y = \frac{1}{5}\cos 2t + \frac{4}{5}\cos 3t + \frac{1}{3}A\sin 3t \quad \text{...(i)}$$

But we are given that $y\left(\frac{\pi}{2}\right) = -1$, so putting $t = \frac{\pi}{2}$ and $y = -1$, we get

$$-1 = \frac{1}{5}\cos\pi + \frac{4}{5}\cos\frac{3\pi}{2} + \frac{1}{3}A\sin\frac{3\pi}{2}$$

$$-1 = -\frac{1}{5} + 0 - \frac{A}{3} \text{ or } \frac{A}{3} = 1 - \frac{1}{5} = \frac{4}{5} \text{ or } A = \frac{12}{5}$$

Putting the value $A = \frac{12}{5}$ in equation (i), we get

$$y = \frac{1}{5}\cos 2t + \frac{4}{5}\cos 3t + \frac{1}{3}.\frac{12}{5}\sin 3t$$

or $$y = \frac{1}{5}\cos 2t + \frac{4}{5}\cos 3t + \frac{4}{5}\sin 3t$$ **Ans.**

Prob. 77. Solve $(D^2 + 9)\, y = 18t$, $y(0) = 0$, $y\left(\frac{\pi}{2}\right) = 1$.

Sol. Taking Laplace transform of both sides of given equation, we have

$$[s^2\bar{y} - sy(0) - y'(0)] + 9\bar{y} = 18.\frac{1}{s^2}$$

$$\therefore \quad (s^2+9)\bar{y} - sy(0) - y'(0) = \frac{18}{s^2}$$

Since $y'(0)$ is not given, let $y'(0) = A$

$$\therefore \quad (s^2+9)\bar{y} - A = \frac{18}{s^2} \text{ or } (s^2+9)\bar{y} = \frac{18}{s^2} + A$$

or $$\bar{y} = \frac{18}{s^2(s^2+9)} + \frac{A}{s^2+9}$$

or $$\bar{y} = 18\left[\frac{1}{9}\left(\frac{1}{s^2} - \frac{1}{s^2+9}\right)\right] + \frac{A}{s^2+9}$$

or $$\bar{y} = \frac{2}{s^2} - \frac{2}{s^2+9} + \frac{A}{s^2+9}$$

or $$\bar{y} = \frac{2}{s^2} + \frac{A-2}{s^2+9}$$

On inversion, we get

$$y = 2t + (A-2).\frac{1}{3}\sin 3t$$

But we are given that $y\left(\frac{\pi}{2}\right) = 1$

$$1 = 2\left(\frac{\pi}{2}\right) + (A-2).\frac{1}{3}\sin\left(\frac{3\pi}{2}\right)$$

$$1 = \pi + (A-2)\frac{1}{3}(-1) \text{ or } \frac{1}{3}(A-2) = \pi - 1$$

$$A - 2 = 3\pi - 3$$

$$A = 3\pi - 1$$

Therefore, $$y = 2t + (3\pi - 3).\frac{1}{3}\sin 3t$$

$$= \mathbf{2t + (\pi - 1)\sin 3t}$$ **Ans.**

Prob. 78. Using Laplace transform, solve equation $(D^2 + 2D + 1)\, y = t$. Given that $y(0) = -3$, $y(1) = -1$. ***(R.G.P.V., Nov. 2018)***

Sol. Taking Laplace transform of both sides of given equation, we have

$$[s^2\bar{y} - s\,y(0) - y'(0)] + 2[s\bar{y} - y(0)] + \bar{y} = \frac{1}{s^2}$$

or $$(s^2 + 2s + 1)\bar{y} - (s+2)\,y(0) - y'(0) = \frac{1}{s^2}$$

Since $y'(0)$ is not given, let $y'(0) = A$

$$\therefore \quad (s^2 + 2s + 1)\bar{y} - (s+2)(-3) - A = \frac{1}{s^2}$$

or $$(s^2 + 2s + 1)\bar{y} = \frac{1}{s^2} - 3(s+2) + A$$

or $$\bar{y} = \frac{1}{s^2(s^2+2s+1)} - \frac{3(s+2)}{(s^2+2s+1)} + \frac{A}{(s^2+2s+1)}$$

or $$\bar{y} = \frac{1}{s^2(s+1)^2} - 3\frac{(s+1+1)}{(s+1)^2} + \frac{A}{(s+1)^2}$$

or $$\bar{y} = \left[-\frac{2}{s} + \frac{1}{s^2} + \frac{2}{(s+1)} + \frac{1}{(s+1)^2}\right] - 3\left[\frac{1}{(s+1)} + \frac{1}{(s+1)^2}\right] + \frac{A}{(s+1)^2}$$

(By partial fraction)

or $$\bar{y} = -\frac{2}{s} + \frac{1}{s^2} - \frac{1}{(s+1)} + \frac{A-2}{(s+1)^2}$$

On inversion, we get

$$y = -2 + t - e^{-t} + (A-2)te^{-t}$$

But we are given that $y(1) = -1$

$$-1 = -2 + 1 - e^{-1} + (A-2).1.e^{-1}$$

$$e^{-1} = (A-2)\,e^{-1}$$

or $$A - 2 = 1$$

or $$A = 3$$

or

Therefore, $$y = -2 + t - e^{-t} + (3-2)\,t\,e^{-t}$$

$$= \mathbf{-2 + t - e^{-t} + te^{-t}}$$ **Ans.**

Prob.79. Solve the equation by the transform method –

$$\frac{d^2x}{dt^2} - 2\frac{dx}{dt} + x = e^t$$

$$x(0) = 2,\ x'(0) = -1.$$

Sol. Here, $$\frac{d^2x}{dt^2} - 2\frac{dx}{dt} + x = e^t \quad ...(i)$$

Taking the Laplace transform of both sides, we get

$$[s^2\bar{x} - sx(0) - x'(0)] - 2[s\bar{x} - x(0)] + \bar{x} = \frac{1}{s-1}$$

Using the given conditions, it reduces to

$$(s^2 - 2s + 1)\bar{x} = \frac{1}{s-1} + 2s - 5 = \frac{2s^2 - 7s + 6}{s-1}$$

or $$\bar{x} = \frac{2s^2 - 7s + 6}{(s-1)^3} = \frac{2}{s-1} - \frac{3}{(s-1)^2} + \frac{1}{(s-1)^3} \quad ...(ii)$$

(on braking into partial fraction)

Taking inverse Laplace transform on both sides of equation (ii), we get

$$x = 2L^{-1}\left(\frac{1}{s-1}\right) - 3L^{-1}\left[\frac{1}{(s-1)^2}\right] + L^{-1}\left[\frac{1}{(s-1)^3}\right]$$

$$= 2e^t - \frac{3e^t.t}{1!} + \frac{e^t.t^2}{2!}$$

$$x = \mathbf{2e^t - 3te^t + \frac{1}{2}t^2e^t}$$ **Ans.**

Prob.80. Solve $(D^3 - 3D^2 + 3D - 1)\,y = t^2e^t$, given that $y(0) = 1,\ y'(0) = 0,\ y''(0) = -2$ using Laplace transform.

Sol. Taking Laplace transform of both sides of given equation, we have

$$[s^3\bar{y} - s^2y(0) - sy'(0) - y''(0)] - 3[s^2\bar{y} - sy(0) - y'(0)]$$

$$+ 3[s\bar{y} - y(0)] - \bar{y} = \frac{2}{(s-1)^3} \quad ...(i)$$

Putting $y(0) = 1,\ y'(0) = 0,\ y''(0) = -2$ in equation (i), we get

$$s^3\bar{y} - s^2 - 0 + 2 - 3s^2\bar{y} + 3s + 0 + 3s\bar{y} - 3 - \bar{y} = \frac{2}{(s-1)^3}$$

$$(s^3 - 3s^2 + 3s - 1)\bar{y} - s^2 + 3s - 1 = \frac{2}{(s-1)^3}$$

or $$(s-1)^3\bar{y} = s^2 - 3s + 1 + \frac{2}{(s-1)^3}$$

or $$\bar{y} = \frac{s^2 - 3s + 1}{(s-1)^3} + \frac{2}{(s-1)^6}$$

On resolving the R.H.S. into partial fraction, we get

$$\bar{y} = \frac{1}{(s-1)} - \frac{1}{(s-1)^2} - \frac{1}{(s-1)^3} + \frac{2}{(s-1)^6}$$

On inversion, we obtain

$$y = L^{-1}\frac{1}{(s-1)} - L^{-1}\frac{1}{(s-1)^2} - L^{-1}\frac{1}{(s-1)^3} + 2L^{-1}\frac{1}{(s-1)^6}$$

$$y = e^t - te^t - \frac{1}{2}t^2e^t + \frac{1}{60}t^5e^t = \mathbf{e^t\left(1 - t - \frac{1}{2}t^2 + \frac{1}{60}t^5\right)}$$ **Ans.**

FOURIER TRANSFORMS

Fourier Transforms –

Definition – The Fourier transform of f(x) is denoted by F(s) and it defined as follows –

$$F(s) = \int_{-\infty}^{\infty} f(t)\,e^{ist}dt$$

and the inverse Fourier transform of F(s) is given below and is denoted by f(x).

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(s)e^{-isx}ds$$

Properties of Fourier Transform – The basic properties of Fourier transform are as follows –

(i) Linearity/Linear Property – If complex Fourier transform of two functions f(x) and g(x) are F(s) and G(s) respectively then

$$F\{a\,f(x) + b\,g(x)\} = a\,F(s) + b\,G(s) \quad ...(i)$$

Here a and b are constants.

Proof – We know that

$$F(s) = \int_{-\infty}^{\infty} e^{isx} f(x)\, dx$$

and

$$G(s) = \int_{-\infty}^{\infty} e^{isx} g(x)\, dx$$

$\therefore$ Taking R.H.S. of equation (i)

$$a\,F(s) + b\,G(s) = a\int_{-\infty}^{\infty} e^{isx} f(x)\, dx + b\int_{-\infty}^{\infty} e^{isx} g(x)\, dx$$

$$= \int_{-\infty}^{\infty} e^{isx}\{a\,f(x) + b\,g(x)\}\, dx$$

$$= F\{a\,f(x) + b\,g(x)\} = \textbf{L.H.S.}$$

Proved

(ii) Change of Scale Property – If the complex Fourier transform of f(x) is F(s), then

$$F\{f(ax)\} = \frac{1}{a}F\left(\frac{s}{a}\right), \quad a \neq 0$$

Proof – We know that

$$F(s) = \int_{-\infty}^{\infty} f(x) e^{isx} dx \qquad ...(i)$$

$$\therefore \qquad F\{f(ax)\} = \int_{-\infty}^{\infty} e^{isx}\, f(ax)\, dx \qquad ...(ii)$$

Putting ax = t and dx = dt/a in equation (ii)

$$F\{f(ax)\} = \int_{-\infty}^{\infty} e^{ist/a}\, f(t)\frac{dt}{a} = \frac{1}{a}\int_{-\infty}^{\infty} e^{i(s/a)t} f(t)\, dt$$

$$= \frac{1}{a}\mathbf{F(s/a)} \quad \{\text{From equation (i)}\}$$

Proved

(iii) Shifting Property – If the complex Fourier transform of f(x) is F(s), then

$$F\{f(x-a)\} = e^{isa}\, F(s)$$

Proof – We know that

$$F(s) = \int_{-\infty}^{\infty} e^{isx} f(x)\, dx \qquad ...(i)$$

then

$$F\{f(x-a)\} = \int_{-\infty}^{\infty} e^{isx} f(x-a)\, dx \qquad ...(ii)$$

Putting (x – a) = t and dx = dt in equation (ii)

$$F\{f(x-a)\} = \int_{-\infty}^{\infty} e^{is(t+a)} f(t)\, dt$$

$$= \int_{-\infty}^{\infty} e^{isa}.e^{ist} f(t)\, dt = e^{isa}\int_{-\infty}^{\infty}.e^{ist} f(t)dt$$

$$= \mathbf{e^{isa}F(s)} \qquad \{\text{From equation (i)}\}$$

Proved

(iv) Modulation Theorem – If the complex Fourier transform of f(x) is F(s), then

$$F\{f(x)\cos ax\} = \frac{1}{2}\{F(s+a) + F(s-a)\}$$

Proof – We know that

$$F(s) = \int_{-\infty}^{\infty} e^{isx} f(x)\, dx \qquad ...(i)$$

Then

$$F\{f(x)\cos ax\} = \int_{-\infty}^{\infty} e^{isx} f(x)\cos ax\, dx$$

$$= \int_{-\infty}^{\infty} e^{isx} f(x)\frac{e^{iax} + e^{-iax}}{2} dx$$

$$= \frac{1}{2}\int_{-\infty}^{\infty} e^{i(s+a)x} f(x)\, dx + \frac{1}{2}\int_{-\infty}^{\infty} e^{i(s-a)x} f(x)\, dx$$

$$= \frac{1}{2}F(s+a) + \frac{1}{2}F(s-a)$$

$$= \mathbf{\frac{1}{2}\{F(s+a) + F(s-a)\}}$$

Proved

Fourier Sine Transform – The Fourier sine transform is denoted by $F_s(s)$ and is defined in interval $(0, \infty)$ as follows –

$$F_s(s) = \int_0^{\infty} f(x)\sin sx\, dx$$

and the inverse Fourier sine transform of $F_s(s)$ is denoted by f(x) is defined as follows –

$$f(x) = \frac{2}{\pi}\int_0^{\infty} F_s(s)\sin sx\, ds$$

Fourier Cosine Transform – The Fourier cosine transform of f(x) denoted by $F_c(s)$ and is defined in the interval $(0, \infty)$ as follows –

$$F_c(s) = \int_0^{\infty} f(x)\cos sx\, dx$$

and the inverse Fourier cosine transform of $F_c(s)$ is denoted by f(x) is defined as

$$f(x) = \frac{2}{\pi}\int_0^{\infty} F_c(s)\cos sx\, ds$$

Finite Fourier Sine Transform – This transform is used in boundary value problem in which at least two boundaries are separated by a finite distance.

The finite Fourier sine transform in the interval (0, c) is defined as

$$F_s(n) = \bar{f}_s(n) = \int_0^{c} f(x)\sin\frac{n\pi x}{c} dx$$

Here n is an integer and the ***inverse finite Fourier sine transform*** is defined as

$$f(x) = \frac{2}{c}\sum_{n=1}^{\infty} F_s(n)\sin\frac{n\pi x}{c}$$

Finite Fourier Cosine Transform – This transform is also used in boundary value problem in which at least two boundaries are separated by a finite distance.

The finite Fourier cosine transform in the interval (0, c) is defined as

$$F_c(n) = \bar{f}_c(n) = \int_0^c f(x)\cos\frac{n\pi x}{c}dx$$

Here n is an integer and the ***inverse finite Fourier cosine transform*** is defined as

$$F_c(n) = \bar{f}_c(n) = \frac{1}{c}F_c(0) + \frac{2}{c}\sum_{n=1}^{\infty} F_c(n)\cos\frac{n\pi x}{c}$$

Finite Fourier Sine and Cosine Transforms of Derivatives of a Function – Let f(x) and f'(x) be continuous and f''(x) be sectionally continuous over the interval $0 \le x \le c$. Then

(i) $F_s\{f'(x)\} = -\frac{n\pi}{c}F_c(n)$

(ii) $F_c\{f'(x)\} = (-1)^n f(c) - f(0) + \frac{n\pi}{c}F_s(n)$

(iii) $F_s\{f''(x)\} = -\frac{n^2\pi^2}{c^2}F_s(n) + \frac{n\pi}{c}\left[f(0) - (-1)^n f(c)\right]$

(iv) $F_c\{f''(x)\} = -\frac{n^2\pi^2}{c^2}F_c(n) + (-1)^n f'(c) - f'(0)$

Convolution – The convolution of two functions f(x) and g(x) over the interval $(-\infty, \infty)$ is defined as

$$f(x) * g(x) = (f * g)x = \int_{-\infty}^{\infty} f(u)\, g(x-u)\, du$$

Convolution Theorem on Fourier Transform – The Fourier transform of the convolution of f(x) and g(x) is the product of their Fourier transform i.e.,

$$F\{f(x) * g(x)\} = F\{f(x).F\{g(x)\} = F(s).G(s)$$

Proof – We know that,

$$f(x) * g(x) = \int_{-\infty}^{\infty} f(u)\, g(x-u)\, du \qquad ...(i)$$

Taking Fourier transform of both sides of equation (i), we have

$$F\{f(x) * g(x)\} = F\left\{\int_{-\infty}^{\infty} f(u).g(x-u)du\right\}$$

$$= \int_{-\infty}^{\infty}\left\{\int_{-\infty}^{\infty} f(u)\, g(x-u)du\right\}e^{isx}dx$$

$$= \int_{-\infty}^{\infty} f(u)\, du\left\{\int_{-\infty}^{\infty} g(x-u)e^{isx}dx\right\}$$

(on changing the order of integration)

$$= \int_{-\infty}^{\infty} f(u)\left\{\int_{-\infty}^{\infty} g(x-u)e^{is(x-u)}dx\right\}e^{isu}du$$

$$= \int_{-\infty}^{\infty} f(u)\, e^{isu}du\left\{\int_{-\infty}^{\infty} g(t)e^{ist}dt\right\}$$

[on putting $x - u = t$, so that $dx = dt$]

$$= \int_{-\infty}^{\infty} f(u)\, e^{isu}du.G(s),$$

where, $G(s) = F\{g(t)\} = F\{g(x)\}$

$= F(s).G(s)$, where $F(s) = F\{f(u)\} = F\{f(x)\}$

i.e., $F\{f(x) * g(x)\} = F(s).G(s) = F\{f(x)\}.F\{g(x)\}$

By inversion

$$F^{-1}\{F(s).G(s)\} = f(x) * g(x) = F^{-1}\{F(s)\}\, F^{-1}\{G(s)\}$$

Parseval's Identity for Fourier Transforms – If the Fourier transform of f(x) and g(x) be F(s) and G(s) respectively, then

(i) $\frac{1}{2\pi}\int_{-\infty}^{\infty} F(s).\overline{G}(s)ds = \int_{-\infty}^{\infty} f(x)\,\overline{g}(x)dx$

where $\overline{G}(s)$ is the complex conjugate of G(s) and $\overline{g}(x)$ is the complex conjugate of g(x).

(ii) $\frac{1}{2\pi}\int_{-\infty}^{\infty}[F(s)]^2 ds = \int_{-\infty}^{\infty}|f(x)|^2 dx$

Proof. (i) We have $F(s) = \int_{-\infty}^{\infty} f(x)\, e^{isx}dx$ and $f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(s)\, e^{-isx}ds$

$$\therefore \quad \int_{-\infty}^{\infty} f(x)\,\overline{g}(x)\, dx = \int_{-\infty}^{\infty} f(x)\left[\frac{1}{2\pi}\int_{-\infty}^{\infty}\overline{G}(s)e^{isx}ds\right]dx$$

$$\left[\because \overline{g}(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\overline{G}(s)\, e^{isx}ds\right]$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} \overline{G}(s)\left[\int_{-\infty}^{\infty} f(x)e^{isx}dx\right]ds$$

(on changing the order of integration)

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} \overline{G}(s).F(s)ds$$

i.e., $$\frac{1}{2\pi}\int_{-\infty}^{\infty} F(s).\overline{G}(s)ds = \int_{-\infty}^{\infty} f(x)\,\overline{g}(x)dx$$

(ii) Putting $g(x) = f(x)$, we get

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} F(s).\overline{F}(s)ds = \int_{-\infty}^{\infty} f(x)\,\overline{f}(x)dx$$

or $$\frac{1}{2\pi}\int_{-\infty}^{\infty} [F(s)]^2 ds = \int_{-\infty}^{\infty} |f(x)|^2\, dx$$ **Proved**

Parseval's Identity for Cosine and Sine Transforms –

Cosine Transform –

(i) $$\frac{2}{\pi}\int_0^{\infty} F_c(s).G_c(s)ds = \int_0^{\infty} f(x).g(x)dx$$

(ii) $$\frac{2}{\pi}\int_0^{\infty} [F_c(s)]^2 ds = \int_0^{\infty} |f(x)|^2\, dx$$

Sine Transform –

(i) $$\frac{2}{\pi}\int_0^{\infty} F_s(s).G_s(s)ds = \int_0^{\infty} f(x), g(x)dx$$

(ii) $$\frac{2}{\pi}\int_0^{\infty} [F_s(s)]^2 ds = \int_0^{\infty} |f(x)|^2\, dx$$

Applications of Fourier Transform in Solving the Ordinary Differential Equation – Some integral equations can be solved very easily with the help of Fourier transforms. Different Fourier transforms can be applied in different initial and boundary value problems. In one dimensional boundary value problem, the partial differential equation can easily be transformed into an ordinary differential equation by applying a suitable transform. The required solution is then obtained by solving this equation and inverting by means of inversion formula or by any other method. In two dimensional problems, it is sometimes required to apply the transforms twice and the desired solution is obtained by double inversion.

NUMERICAL PROBLEMS

Prob.81. Find the Fourier transform of e^{-ax^2}, where $a > 0$.

Sol. Let, $f(x) = e^{-ax^2}$

Here we have, by definition,

$$F(s) = F\{f(x)\} = \int_{-\infty}^{\infty} f(x).e^{isx}dx$$

$$= \int_{-\infty}^{\infty} e^{-ax^2} e^{isx}dx = \int_{-\infty}^{\infty} e^{-ax^2+isx}dx$$

$$= \int_{-\infty}^{\infty} \exp\left[-a\left(x-\frac{is}{2a}\right)^2 - \frac{s^2}{4a}\right]dx = e^{(-s^2/4a)}\int_{-\infty}^{\infty} e^{\left[-a\left(x-\frac{is}{2a}\right)^2\right]}dx$$

$$= e^{(-s^2/4a)}\int_{-\infty}^{\infty} e^{-ay^2}dy \quad \left[\text{on putting } \left(x-\frac{is}{2a}\right) = y, \text{ so that } dx = dy\right]$$

$$= e^{(-s^2/4a)}\left[2.\int_0^{\infty} e^{-ay^2}dy\right] \quad [\because e^{-ay^2} \text{ is even function of } y]$$

$$= 2e^{(-s^2/4a)}\left(\frac{1}{2}\sqrt{\frac{\pi}{a}}\right) \quad \left[\because \int_0^{\infty} e^{-ay^2}dy = \frac{1}{2}\sqrt{\frac{\pi}{a}}\right]$$

$$= \sqrt{\frac{\pi}{a}}e^{(-s^2/4a)}$$ **Ans.**

in which the change of variable $y = x - \frac{is}{2a}$ is used. The above result is correct. **Ans.**

Prob.82. Find the Fourier transform of

$$f(x) = \begin{cases} 1, & |x| < a \\ 0, & |x| > a \end{cases}$$

Hence find the value of $\int_0^{\infty} \frac{\sin x}{x}dx$

Sol. Given that

$$f(x) = \begin{cases} 1, & -a < x < a \\ 0, & x < -a \text{ and } x > a \end{cases}$$

The Fourier transform of f(x) is given as

$$F\{f(x)\} = \int_{-\infty}^{\infty} e^{isx} f(x)\, dx$$

$$= \int_{-\infty}^{-a} e^{isx}.0\, dx + \int_{-a}^{a} e^{isx}.1\, dx + \int_{a}^{\infty} e^{isx}.0\, dx$$

$$= \int_{-a}^{a} e^{isx}\,dx = \left[\frac{e^{isx}}{is}\right]_{-a}^{a}$$

$$= \frac{e^{isa} - e^{-isa}}{is} = \frac{2}{s}\sin sa\,,\ s \neq 0$$

Ans.

By inversion formulae

$$f(x) = F^{-1}\{F(s)\} = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(s)e^{-isx}ds$$

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{2\sin sa}{s} e^{-isx}\,ds$$

$$\therefore \int_{-\infty}^{\infty} \frac{\sin sa}{s} e^{-isx}\,ds = \pi f(x) = \begin{cases} \pi, & |x| < a \\ 0, & |x| > a \end{cases}$$ {By def. of f(x)}

Putting x = 0, a = 1, we get f(0) = 1

$$\therefore \quad \int_{-\infty}^{\infty} \frac{\sin s}{s} ds = \pi$$

$$\therefore \quad \int_{0}^{\infty} \frac{\sin s}{s} ds = \frac{\pi}{2} \quad \text{i.e.} \quad \int_{0}^{\infty} \frac{\sin x}{x} dx = \frac{\pi}{2}$$

Ans.

Prob.83. Find the Fourier transform of

$$f(x) = \begin{cases} 1 & \text{for } |x| < a \\ 0 & \text{for } |x| > a \end{cases}$$

[R.G.P.V., May 2019 (III-Sem)]

Sol. Refer to Prob.82.

Prob.84. Find the Fourier transform of

$$f(t) = \begin{cases} t, & |t| \le a \\ 0, & |t| \ge a \end{cases}$$

Sol. Given that

$$f(t) = \begin{cases} t, & -a \le t \le a \\ 0, & t \ge a \text{ and } t \le -a \end{cases}$$

$\therefore$ The Fourier transform of f(t) is

$$F\{f(t)\} = \int_{-a}^{a} e^{ist} f(t)dt = \int_{-a}^{a} e^{ist}.t\,dt = \left[t\frac{e^{ist}}{is} - 1.\frac{e^{ist}}{i^2s^2}\right]_{-a}^{a}$$

$$= \left[t\frac{e^{ist}}{is} - \frac{e^{ist}}{(-1)s^2}\right]_{-a}^{a} = \left[t\frac{e^{ist}}{is} + \frac{e^{ist}}{s^2}\right]_{-a}^{a}$$

$$= \left[a\frac{e^{isa}}{is} + \frac{e^{isa}}{s^2} - (-a)\frac{e^{-isa}}{is} - \frac{e^{-isa}}{s^2}\right]$$

$$= \frac{a}{is}[e^{isa} + e^{-isa}] + \frac{1}{s^2}[e^{isa} - e^{-isa}]$$

$$= \frac{2a}{is}[\cos as] + \frac{2i}{s^2}[\sin as]$$

Ans.

Prob.85. Find Fourier sine transform of $f(t) = te^{-at}$.

Sol. We know that

$$F_s[t\,f(t)] = -\frac{d}{ds}F_c[f(t)]$$

Given $f(t) = e^{-at}$ then

$$F_c[f(t)] = \int_0^{\infty} f(t)\cos st\,dt$$

$$= \int_0^{\infty} e^{-at}\cos st\,dt = \frac{a}{s^2 + a^2}$$

Now, we have

$$F_s[t.e^{-at}] = F_s[t.f(t)] = -\frac{d}{ds}[F_c\{F(t)\}]$$

$$= -\frac{d}{ds}\left[\frac{a}{s^2 + a^2}\right] = \frac{2as}{(s^2 + a^2)^2}$$

Ans.

which is the required Fourier sine transform of te^{-at}.

Prob.86. Find the Fourier sine transform of

$$f(x) = \begin{cases} \sin x, & 0 < x < a \\ 0, & x > a \end{cases}$$

Sol. The Fourier sine transform $F_s(s)$ of f(x) is given by

$$F_s\{f(x)\} = F_s(s) = \int_0^{\infty} f(x).\sin sx\,dx$$

$$F_s(s) = \int_0^{a} f(x)\sin sx\,dx + \int_a^{\infty} f(x)\sin sx\,dx$$

$$= \int_0^{a} \sin x \sin sx\,dx + \int_a^{\infty} 0.\sin sx\,dx \quad \text{[Substituting given values of f(x)]}$$

$$= \frac{1}{2}\int_0^{a} 2\sin x \sin sx\,dx = \frac{1}{2}\int_0^{a}[\cos(1-s)x - \cos(1+s)x]dx$$

$$= \frac{1}{2}\left[\frac{\sin(1-s)x}{(1-s)} - \frac{\sin(1+s)x}{(1+s)}\right]_0^{a}$$

Ans.

$$= \frac{1}{2}\left[\frac{\sin\{a(1-s)\}}{(1-s)} - \frac{\sin\{a(1+s)\}}{(1+s)}\right]$$

Prob.87. Find the Fourier cosine transform of

$$f(x) = e^{-2x} + 4e^{-3x}.$$

Sol. The Fourier cosine transform $F_c(s)$ of f(x) is given by

$$F_c\{f(x)\} = F_c(s) = \int_0^\infty f(x)\cos sx\,dx$$

Putting the value of f(x), we get

$$F_c(s) = \int_0^\infty (e^{-2x} + 4e^{-3x})\cos sx\,dx$$

$$= \int_0^\infty e^{-2x}\cos sx\,dx + 4\int_0^\infty e^{-3x}\cos sx\,dx$$

$$= \left[\frac{e^{-2x}}{4+s^2}(-2\cos sx + s\sin sx)\right]_0^\infty + 4\left[\frac{e^{-3x}}{9+s^2}(-3\cos sx + s\sin sx)\right]_0^\infty$$

$$\left[\because \int e^{ax}\cos bx\,dx = \frac{e^{ax}}{a^2+b^2}(a\cos bx + b\sin bx)\right]$$

$$= \left[0 + \frac{2}{4+s^2}\right] + 4\left[0 + \frac{3}{9+s^2}\right] = \frac{2}{s^2+4} + \frac{12}{s^2+9}$$ **Ans.**

Prob.88. Find the Fourier sine transform of $f(x) = \begin{cases} x, & 0 < x < 1 \\ 2-x & 1 < x < 2 \\ 0, & x > 2 \end{cases}$

Sol. The Fourier sine transform of f(x) is given by

$$F_s\{f(x)\} = F_s(s) = \int_0^\infty f(x).\sin sx\,dx$$

$$F_s(s) = \int_0^1 f(x)\sin sx\,dx + \int_1^2 f(x)\sin sx\,dx + \int_2^\infty f(x)\sin sx\,dx$$

$$= \int_0^1 x.\sin sx\,dx + \int_1^2 (2-x)\sin sx\,dx + \int_2^\infty 0.\sin sx\,dx$$

[Substituting given values of f(x)]

$$= \left[\left\{x.\left(\frac{-\cos sx}{s}\right)\right\}_0^1 - \int_0^1 1.\left(\frac{-\cos sx}{s}\right)dx\right]$$

$$+ \left[\left\{(2-x).\left(\frac{-\cos sx}{s}\right)\right\}_1^2 - \int_1^2 (-1).\left(\frac{-\cos sx}{s}\right)dx\right] + 0$$

$$= \left[\frac{-\cos s}{s} + 0\right] + \left[\frac{\sin sx}{s^2}\right]_0^1 + \left[0 + \frac{\cos s}{s}\right] - \left[\frac{\sin sx}{s^2}\right]_1^2$$

$$= \frac{-\cos s}{s} + \frac{\sin s}{s^2} + \frac{\cos s}{s} - \frac{\sin 2s}{s^2} + \frac{\sin s}{s^2}$$

$$= \frac{2\sin s}{s^2} - \frac{\sin 2s}{s^2} = \frac{2\sin s - 2\sin s\cos s}{s^2}$$

$$= \frac{2\sin s(1-\cos s)}{s^2}$$ **Ans.**

Prob.89. Find the Fourier cosine transform of

$$f(x) = \begin{cases} x & 0 < x < 1 \\ 2-x & 1 < x < 2 \\ 0 & x > 2 \end{cases}$$

Sol. The Fourier cosine transform of f(x) is given by

$$F_c\{f(x)\} = F_c(s) = \int_0^\infty f(x)\cos sx\,dx$$

$$F_c(s) = \int_0^1 f(x)\cos sx\,dx + \int_1^2 f(x)\cos sx\,dx + \int_2^\infty f(x)\cos sx\,dx$$

$$F_c(s) = \int_0^1 x.\cos sx\,dx + \int_1^2 (2-x).\cos sx\,dx + \int_2^\infty 0.\cos sx\,dx$$

[Substituting given values of f(x)]

$$= \left[\frac{x\sin sx}{s} + \frac{\cos sx}{s^2}\right]_0^1 + \left[\frac{(2-x)\sin sx}{s} - \frac{\cos sx}{s^2}\right]_1^2$$

$$= \left(\frac{\sin s}{s} + \frac{\cos s}{s^2} - \frac{1}{s^2}\right) + \left(-\frac{\cos 2s}{s^2} - \frac{\sin s}{s} + \frac{\cos s}{s^2}\right)$$

$$= \frac{2\cos s - \cos 2s - 1}{s^2} = \frac{2\cos s - (2\cos^2 s - 1) - 1}{s^2}$$

$$= \frac{2\cos s - 2\cos^2 s}{s^2} = \frac{2\cos s(1-\cos s)}{s^2}$$ **Ans.**

Prob.90. Find the function whose sine transform is $\frac{e^{-s}}{s}$.

Sol. Given,

$$F_s(s) = \frac{e^{-s}}{s}$$

The inverse formula for Fourier transform is given by

$$f(x) = \frac{2}{\pi}\int_0^\infty F_s(s)\sin sx\,ds$$

$$f(x) = \frac{2}{\pi}\int_0^\infty \left(\frac{e^{-s}}{s}\right)\sin sx\,ds = I \quad \text{...(i)}$$

Differentiate w.r.t. x, on both sides, we get

$$\frac{dI}{dx} = \frac{2}{\pi}\frac{d}{dx}\left[\int_0^\infty \left(\frac{e^{-s}}{s}\right)\sin sx\, ds\right] = \frac{2}{\pi}\int_0^\infty \left(\frac{e^{-s}}{s}\right)\frac{\partial}{\partial x}(\sin sx)ds$$

$$= \frac{2}{\pi}\int_0^\infty \left(\frac{e^{-s}}{s}\right)(s\cos sx)ds = \frac{2}{\pi}\int_0^\infty e^{-s}\cos sx\, ds$$

$$= \frac{2}{\pi}\left[\frac{e^{-s}}{(-1)^2+x^2}\{-1\cos sx + x\sin sx\}\right]_0^\infty$$

$$\left[\because \int e^{ax}\cos bx\, dx = \frac{e^{ax}}{a^2+b^2}(a\cos bx + b\sin bx)\right]$$

$$= \frac{2}{\pi}\left(\frac{1}{1+x^2}\right)[\{0\} - \{-1+0\}]$$

$$\frac{dI}{dx} = \frac{2}{\pi}\left(\frac{1}{1+x^2}\right)$$

Integrating both sides, w.r.t. x, we get

$$I = \frac{2}{\pi}\tan^{-1}x + c \qquad ...(ii)$$

For the initial condition, putting $x = 0$ then $c = 0$

$\therefore$ From equation (ii), we get

$$I = \frac{2}{\pi}\tan^{-1}x$$

From equation (i), we get

$$f(x) = \frac{2}{\pi}\tan^{-1}x \qquad \textbf{Ans.}$$

Prob.91. Find the Fourier transform of Dirac-delta function $\delta(t-a)$.

Sol. The Dirac-delta function $\delta(t-a)$ is defined as

$$\delta(t-a) = \lim_{h\to 0} I(h, t-a)$$

where, $$(h, t-a) = \begin{cases} \frac{1}{h}; & a < t < a+h \\ 0; & t < a \text{ and } t > a+h \end{cases}$$

The Fourier transform of f(t) is given as

$$F\{f(t)\} = \int_{-\infty}^{\infty} f(t)e^{ist}dt$$

$$F\{\delta(t-a)\} = \int_{-\infty}^{\infty}\delta(t-a)e^{ist}dt$$

$$= \lim_{h\to 0}\left[\int_{-\infty}^{a} 0.e^{ist}dt + \int_a^{a+h}\frac{1}{h}e^{ist}dt + \int_{a+h}^{-\infty}0.e^{ist}dt\right]$$

$$= 0 + \lim_{h\to 0}\int_a^{a+h}\left(\frac{1}{h}\right)e^{ist}dt + 0 = \lim_{h\to 0}\int_a^{a+h}\left(\frac{1}{h}\right)e^{ist}dt$$

$$= \lim_{h\to 0}\frac{1}{h}\left[\frac{e^{ist}}{is}\right]_a^{a+h} = \lim_{h\to 0}\frac{1}{ish}\left[e^{is(a+h)} - e^{isa}\right]$$

$$= e^{isa}\lim_{h\to 0}\left[\frac{e^{ish}-1}{ish}\right] = e^{isa}.1 \qquad \left[\because \lim_{x\to 0}\frac{e^x - 1}{x} = 1\right]$$

$$= e^{isa}$$

Hence $F\{\delta(t-a)\} = \mathbf{e^{isa}}$ **Ans.**

Prob.92. Find the Fourier transform of $f(x) = \begin{cases} 1-x^2, & |x| \le 1 \\ 0, & |x| > 1 \end{cases}$

Hence evaluate

$$\int_0^\infty \frac{(x\cos x - \sin x)}{x^3}\cos\frac{x}{2}dx.$$

Sol. Given, $f(x) = \begin{cases} 1-x^2, & |x| \le 1 \\ 0, & |x| > 1 \end{cases}$

The Fourier transform of f(x) is given as

$$F\{f(x)\} = \int_{-\infty}^{\infty} f(x)e^{isx}dx = F(s), \text{ say}$$

$$= \int_{-\infty}^{\infty}(0)e^{isx}dx + \int_{-1}^{1}(1-x^2)e^{isx}dx + \int_1^\infty (0)e^{isx}dx$$

$$= \int_{-1}^{1}(1-x^2)e^{isx}dx$$

Integrating by parts, we get

$$F(s) = \left[(1-x^2)\frac{e^{isx}}{is} - (-2x)\frac{e^{isx}}{(is)^2} + (-2)\frac{e^{isx}}{(is)^3}\right]_{-1}^{1}$$

$$= -2\frac{e^{is}}{s^2} + 2\frac{e^{is}}{is^3} - 2\frac{e^{-is}}{s^2} - 2\frac{e^{-is}}{is^3}$$

$$= -\frac{2}{s^2}(e^{is} + e^{-is}) + \frac{2}{is^3}(e^{is} - e^{-is})$$

$$= -\frac{2}{s^2}(2\cos s) + \frac{2}{is^3}(2i\sin s) = -\frac{4}{s^3}(s\cos s - \sin s) \qquad ...(i) \quad \textbf{Ans.}$$

Now by inversion formula, we have

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(s)e^{-isx}ds \quad \text{...(ii)}$$

From equation (i), we get

$$f(x) = -\frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{4}{s^3}(s\cos s - \sin s)e^{-isx}ds$$

or $$-\frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{4}{s^3}(s\cos s - \sin s)e^{-isx}ds = \begin{cases} 1-x^2, & |x|\le 1 \\ 0, & |x|>1 \end{cases}$$

Putting $x = \frac{1}{2}$, we obtain

$$-\frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{4}{s^3}(s\cos s - \sin s)e^{\frac{-is}{2}}ds = \frac{3}{4}$$

or $$\int_{-\infty}^{\infty} \frac{(s\cos s - \sin s)}{s^3}\left(\cos\frac{s}{2} - i\sin\frac{s}{2}\right)ds = -\frac{3\pi}{8}$$

or $$\int_{-\infty}^{\infty} \frac{(s\cos s - \sin s)}{s^3}\cos\frac{s}{2}ds = -\frac{3\pi}{8}$$

or $$\int_{0}^{\infty} \frac{(x\cos x - \sin x)}{x^3}\cos\frac{x}{2}dx = -\frac{3\pi}{16} \quad (\because \text{ The integral is even})$$ **Ans.**

Prob.93. Find the Fourier transform of f(x) defined by

$$f(x) = \begin{cases} 1, & |x|<1 \\ 0, & |x|>1 \end{cases}$$

Hence evaluate $\int_{-\infty}^{\infty} \frac{\sin s \cos sx}{s}ds$.

Sol. Given, $f(x) = \begin{cases} 1, & -1<x<1 \\ 0, & x<-1 \text{ and } x>1 \end{cases}$

The Fourier transform of f(x) is given as

$$F\{f(x)\} = \int_{-\infty}^{\infty} e^{-isx} f(x)\,dx$$

$$= \int_{-\infty}^{-1} e^{-isx} f(x)\,dx + \int_{-1}^{1} e^{-isx} f(x)\,dx + \int_{1}^{\infty} e^{-isx} f(x)\,dx$$

$$= \int_{\infty}^{1} e^{isy} f(-y)(-dy) + \int_{-1}^{1} e^{-isx}.1\,dx + \int_{1}^{\infty} e^{-isx}.0\,dx$$

$$= \int_{1}^{\infty} e^{isy} f(-y)\,dy + \int_{-1}^{1} e^{-isx}\,dx + 0$$

$$= \int_{1}^{\infty} e^{isy}.0\,dy + \left[\frac{e^{-isx}}{-is}\right]_{-1}^{1} = \frac{e^{-is} - e^{is}}{-is} = \frac{e^{is} - e^{-is}}{is}$$

or $$F\{f(x)\} = \frac{2\sin s}{s}$$

Let, $F\{f(x)\} = F(s)$

We know that if $F(s) = F\{f(x)\} = \int_{-\infty}^{\infty} f(x)\,e^{-isx}dx$

then $$f(x) = F^{-1}\{F(s)\} = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(s)e^{-isx}ds$$

$$\therefore \int_{-\infty}^{\infty} F(s)e^{-isx}dx = 2\pi f(x) = \begin{cases} 2\pi, & \text{if } |x|<1 \\ 0, & \text{if } |x|>1 \end{cases} \quad \text{[by definition of f(x)]}$$

But $F(s) = \frac{2\sin s}{s}$

$$\therefore \int_{-\infty}^{\infty} \frac{2\sin s}{s}(\cos sx + i\sin sx)dx = \begin{cases} 2\pi, & \text{if } |x|<1 \\ 0, & \text{if } |x|>1 \end{cases}$$

or $$\int_{-\infty}^{\infty} \frac{\sin s.\cos sx}{s}ds + i\int_{-\infty}^{\infty} \frac{\sin s.\sin sx}{s}ds = \begin{cases} \pi, & |x|<1 \\ 0, & |x|>1 \end{cases}$$

Equating the real parts on both sides, we get

$$\int_{-\infty}^{\infty} \frac{\sin s \cos sx}{s}ds = \begin{cases} \pi, & |x|<1 \\ 0, & |x|>1 \end{cases}$$ **Ans.**

Prob.94. Find the Fourier sine transform of $\frac{e^{-ax}}{x}$.

Sol. The Fourier sine transform $F_s(s)$ of f(x) is given by

$$F_s\{f(x)\} = F_s(s) = \int_{0}^{\infty} f(x).\sin sx\,dx$$

$$F_s(s) = \int_{0}^{\infty} \left(\frac{e^{-ax}}{x}\right)\sin sx\,dx \quad \text{...(i)}$$

Differentiating both sides with respect to s, we have

$$\frac{dF_s(s)}{ds} = \int_{0}^{\infty} \left(\frac{e^{-ax}}{x}\right)x\cos sx\,dx = \int_{0}^{\infty} e^{-ax}\cos sx\,dx$$

$$= \left[\frac{e^{-ax}}{a^2+s^2}(-a\cos sx + s\sin sx)\right]_0^{\infty}$$

$$\left[\text{By using} \int e^{ax}\cos bx\,dx = \frac{e^{ax}}{a^2+b^2}(a\cos bx + b\sin bx)\right]$$

$$\frac{d}{ds}F_s(s) = \frac{a}{a^2+s^2} \quad ...(ii)$$

Integrating with respect to s, we get

$$F_s(s) = \int \frac{a}{a^2+s^2}ds$$

$$F_s(s) = \tan^{-1}\left(\frac{s}{a}\right) + c \quad ...(iii)$$

But, from equation (i), we note that $F_s(s) = 0$ when $s = 0$

Hence $c = 0$

$$\therefore \quad F_s(s) = F_s\{f(x)\} = \tan^{-1}\left(\frac{s}{a}\right) \quad \text{Ans.}$$

Prob.95. Find Fourier sine and cosine transform of $f(x) = e^{-ax}$.

Sol. The Fourier sine transform of f(x) is given by

$$F_s\{f(x)\} = F_s(s) = \int_0^\infty f(x)\sin sx\,dx$$

Putting the value of f(x), we get

$$F_s(s) = \int_0^\infty e^{-ax}\sin sx\,dx = \left[\frac{e^{-ax}}{a^2+s^2}(-a\sin sx - s\cos sx)\right]_0^\infty$$

$$\left[\because \int e^{ax}\sin bx\,dx = \frac{e^{ax}}{a^2+b^2}(a\sin bx - b\cos bx)\right]$$

$$= 0 + \frac{s}{a^2+s^2} = \frac{s}{a^2+s^2} \quad \text{Ans.}$$

The Fourier cosine transform is

$$F_c\{f(x)\} = F_c(s) = \int_0^\infty f(x)\cos sx\,dx$$

Putting the value of f(x), we get

$$F_c(s) = \int_0^\infty e^{-ax}\cos sx\,dx$$

$$= \left[\frac{e^{-ax}}{a^2+s^2}(-a\cos sx + s\sin sx)\right]_0^\infty$$

$$\left[\because e^{ax}\cos bx\,dx = \frac{e^{ax}}{a^2+b^2}(a\cos bx + b\sin bx)\right]$$

$$= \left[0 + \frac{a}{a^2+s^2}\right] = \frac{a}{a^2+s^2} \quad \text{Ans.}$$

Prob.96. Find the Fourier cosine transform of $f(x) = \frac{1}{1+x^2}$ and hence find Fourier sine transform of $\phi(x) = \frac{x}{1+x^2}$.

Sol. The Fourier cosine transform of f(x) is given by

$$F_c\{f(x)\} = \int_0^\infty \frac{\cos sx}{1+x^2}dx = I, \text{ say} \quad ...(i)$$

$$\therefore \quad \frac{dI}{ds} = \int_0^\infty \frac{-x\sin sx}{1+x^2}dx \quad ...(ii)$$

or $$\frac{dI}{ds} = -\int_0^\infty \frac{x^2\sin sx}{x(1+x^2)}dx = -\int_0^\infty \frac{[(1+x^2)-1]\sin sx}{x(1+x^2)}dx$$

or $$\frac{dI}{ds} = -\int_0^\infty \frac{\sin sx}{x}dx + \int_0^\infty \frac{\sin sx}{x(1+x^2)}dx$$

or $$\frac{dI}{ds} = -\frac{\pi}{2} + \int_0^\infty \frac{\sin sx}{x(1+x^2)}dx \quad ...(iii)$$

Now $$\frac{d^2I}{ds^2} = \int_0^\infty \frac{x\cos sx}{x(1+x^2)}dx$$

or $$\frac{d^2I}{ds^2} = \int_0^\infty \frac{\cos sx}{1+x^2}dx = I$$

or $$\frac{d^2I}{ds^2} - I = 0$$

or $$(D^2 - 1)I = 0, \text{ where } D = \frac{d}{ds}$$

Its solution is $$I = c_1e^s + c_2e^{-s} \quad ...(iv)$$

$$\frac{dI}{ds} = c_1e^s - c_2e^{-s} \quad ...(v)$$

When s = 0, equations (i) and (iv) give

$$c_1 + c_2 = \int_0^\infty \frac{1}{1+x^2}dx$$

or $$c_1 + c_2 = \frac{\pi}{2} \quad ...(vi)$$

Also when s = 0, equations (iii) and (v) give

$$c_1 - c_2 = -\frac{\pi}{2} \quad ...(vii)$$

Solving equations (vi) and (vii), we get

$$c_1 = 0,\ c_2 = \frac{\pi}{2}$$

Hence from equations (i) and (iv), we have

$$F_c\{f(x)\} = I = \frac{\pi}{2}e^{-s}$$ **Ans.**

Now $$F_s\{\phi(x)\} = \int_0^\infty \frac{x\sin sx}{1+x^2}dx$$

$$= -\frac{dI}{ds}$$ [From equation (ii)]

$$= \frac{\pi}{2}e^{-s},$$ from equation (v), with $c_1 = 0,\ c_2 = \frac{\pi}{2}$

Ans.

Prob.97. Evaluate $\int_0^\infty \frac{dx}{(x^2+a^2)(x^2+b^2)}$ ***by using Parseval's identity for Fourier cosine transform.***

Sol. Let, $f(x) = e^{-ax}$ and $g(x) = e^{-bx}$, then

$$F_c(s) = \frac{a}{s^2+a^2} \text{ and } G_c(s) = \frac{b}{s^2+b^2}$$

By Parseval's identity for cosine transform

$$\frac{2}{\pi}\int_0^\infty F_c(s).G_c(s)ds = \int_0^\infty f(x).g(x)dx$$

or $$\frac{2}{\pi}\int_0^\infty \frac{ab}{(s^2+a^2)(s^2+b^2)}ds = \int_0^\infty e^{-ax}e^{-bx}dx$$

$$= \int_0^\infty e^{-(a+b)x}dx = \left[\frac{e^{-(a+b)x}}{-(a+b)}\right]_0^\infty$$

or $$\frac{2ab}{\pi}\int_0^\infty \frac{ds}{(s^2+a^2)(s^2+b^2)} = \frac{1}{a+b}$$

Hence $$\int_0^\infty \frac{dx}{(x^2+a^2)(x^2+b^2)} = \frac{\pi}{2ab(a+b)}$$ **Ans.**

Prob.98. Using Parseval's identity, prove that

$$\int_0^\infty \frac{x^2}{(x^2+1)^2}dx = \frac{\pi}{4}$$

Sol. Let, $f(x) = \frac{x}{x^2+1}$, then

$$F_s(s) = \frac{\pi}{2}e^{-s}$$

Now using Parseval's identity for sine transform, we have

$$\frac{2}{\pi}\int_0^\infty [F_s(s)]^2 ds = \int_0^\infty |f(x)|^2 dx$$

or $$\int_0^\infty \left|\frac{x}{x^2+1}\right|^2 dx = \frac{2}{\pi}\int_0^\infty \left[\frac{\pi}{2}e^{-s}\right]^2 ds = \frac{2}{\pi}\left(\frac{\pi}{2}\right)^2\int_0^\infty e^{-2s}ds$$

$$= \frac{\pi}{2}\left[\frac{e^{-2s}}{-2}\right]_0^\infty = \frac{\pi}{2}\left[0+\frac{1}{2}\right] = \frac{\pi}{4}$$ **Proved**

Prob.99. Using Parseval's identity, show that

$$\int_0^\infty \frac{\sin ax}{x(a^2+x^2)}dx = \frac{\pi}{2}\cdot\frac{1-e^{-a^2}}{a^2}$$

Sol. Let, $f(x) = e^{-ax}$ and $g(x) = \begin{cases}1, & 0<x<a\\ 0, & x>a\end{cases}$

Then $F_c(s) = \frac{a}{a^2+s^2}$ and $G_c(s) = \frac{\sin as}{s}$

By Parseval's identity for cosine transform

$$\frac{2}{\pi}\int_0^\infty F_c(s)\,G_c(s)\,ds = \int_0^\infty f(x)\,g(x)\,dx$$

$$\frac{2}{\pi}\int_0^\infty \frac{a\sin as}{s(a^2+s^2)}ds = \int_0^a e^{-ax}.1\,dx = \left[\frac{e^{-ax}}{-a}\right]_0^a$$

or $$\frac{2a}{\pi}\int_0^\infty \frac{\sin as}{s(a^2+s^2)}ds = \frac{1-e^{-a^2}}{a}$$

Hence $$\int_0^\infty \frac{\sin ax}{x(a^2+x^2)}dx = \frac{\pi}{2}\cdot\frac{1-e^{-a^2}}{a^2}$$ **Proved**

Prob.100. Evaluate $\int_0^\infty \frac{dx}{(x^2+1)^2}$ ***by using Parseval's identity for Fourier cosine transform.***

Sol. Let, $f(x) = e^{-x}$, then

$$F_c(s) = \frac{1}{1+s^2}$$

Now using Parseval's identity for cosine transform, we have

$$\frac{2}{\pi}\int_0^\infty [F_c(s)]^2 ds = \int_0^\infty |f(x)|^2 dx$$

or $$\frac{2}{\pi}\int_0^\infty \left[\frac{1}{(1+s^2)}\right]^2 ds = \int_0^\infty |e^{-x}|^2\,dx = \int_0^\infty e^{-2x}dx$$

$$= \left[\frac{e^{-2x}}{-2}\right]_0^\infty = 0+\frac{1}{2}=\frac{1}{2}$$

or $$\int_0^\infty \frac{1}{(1+s^2)^2}ds = \frac{\pi}{4}$$

Hence $$\int_0^\infty \frac{dx}{(x^2+1)^2} = \frac{\pi}{4}$$ **Ans.**

Prob.101. Find the Fourier transform of

$$f(x) = \begin{cases} 1-|x|, & \text{if } |x|<1 \\ 0, & \text{for } |x|>1 \end{cases}$$

Hence evaluate $\int_0^\infty \left(\frac{\sin t}{t}\right)^4 dt$.

Sol. Given that

$$f(x) = \begin{cases} 1-|x|, & |x|<1 \\ 0, & |x|>1 \end{cases}$$

The Fourier transform of f(x) is

$$F\{f(x)\} = \int_{-1}^1 f(x)e^{isx}dx = \int_{-1}^1 (1-|x|)(\cos sx + i\sin sx)dx$$

$$= \underbrace{\int_{-1}^1 (1-|x|)\cos sx\,dx}_{\text{(even function)}} + i\underbrace{\int_{-1}^1 (1-|x|)\sin sx\,dx}_{\text{(odd function)}}$$

$$= 2\int_0^1 (1-x)\cos sx\,dx + 0$$

$$= 2\left[\left\{(1-x)\frac{\sin sx}{s}\right\}_0^1 + \int_0^1 \frac{\sin sx}{s}dx\right]$$

$$= 2\left[0+\left\{\frac{-\cos sx}{s^2}\right\}_0^1\right] = 2\left(\frac{1-\cos s}{s^2}\right)$$

Using Parseval's identity, we get

$$\frac{1}{2\pi}\int_{-\infty}^\infty [F(s)]^2 ds = \int_{-\infty}^\infty |f(t)|^2 dt$$

$$\frac{4}{2\pi}\int_{-\infty}^\infty \frac{(1-\cos s)^2}{s^4}ds = \int_{-1}^1 (1-|x|)^2 dx$$

$$\frac{2}{\pi}.2\int_0^\infty \frac{\left(1-1+2\sin^2\frac{s}{2}\right)^2}{s^4}ds = \int_{-1}^1 (1+x^2-2x)dx \text{ (odd function)}$$

$$\frac{4}{\pi}.4\int_0^\infty \frac{\sin^4\frac{s}{2}}{s^4}ds = 2\int_0^1 (1+x^2)dx$$

$$\frac{16}{\pi}\int_0^\infty \frac{\sin^4\frac{s}{2}}{s^4}ds = 2\left[x+\frac{x^3}{3}\right]_0^1 = 2\left(1+\frac{1}{3}\right) \text{ or } \frac{16}{\pi}\int_0^\infty \frac{\sin^4\frac{s}{2}}{s^4}ds = \frac{8}{3}$$

Putting $\frac{s}{2} = x$, we get

$$\frac{16}{\pi}\int_0^\infty \frac{\sin^4 x}{16x^4}.2\,dx = \frac{8}{3} \text{ or } \int_0^\infty \frac{\sin^4 x}{x^4}dx = \frac{4}{3}\pi$$ **Ans.**

Prob.102. Using finite Fourier transform, solve

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} \text{ given } u(0, t) = 0 \text{ and } u(4, t) = 0$$

and $u(x, 0) = 2x$ ***where*** $0 < x < 4,\ t > 0$.

Sol. Since u(0, t) given, take finite Fourier sine transform.

$$\int_0^4 \frac{\partial u}{\partial t}\sin\frac{n\pi x}{4}dx = \int_0^4 \frac{\partial^2 u}{\partial x^2}\sin\frac{n\pi x}{4}dx$$

$$\frac{d}{dt}\bar{u}_s = F_s\left(\frac{\partial^2 u}{\partial x^2}\right) = -\frac{n^2\pi^2}{16}\bar{u}_s + \frac{n\pi}{4}\left[u(0,t)-(-1)^n u(4,t)\right]$$

$$= -\frac{n^2\pi^2}{16}\bar{u}_s \qquad \text{[using } u(0, t) = 0,\ u(4, t) = 0]$$

or $$\frac{d\bar{u}_s}{\bar{u}_s} = -\frac{n^2\pi^2}{16}dt$$

Integrating

$$\log \bar{u}_s = -\frac{n^2\pi^2}{16}t + c \qquad \text{...(i)}$$

or $$\bar{u}_s = A\,e^{-\frac{n^2\pi^2}{16}t}$$

Since u(x, 0) = 2x

$$\bar{u}_s(n, 0) = \int_0^4 (2x)\sin\left(\frac{n\pi x}{4}\right)dx$$

$$\bar{u}_s(n, 0) = -\frac{32}{n\pi}\cos n\pi$$

Using equation (ii) in equation (i), we get ...(ii)

$$\bar{u}_s(n, 0) = A = -\frac{32}{n\pi}\cos n\pi$$

Substituting in equation (i), we get

$$\bar{u}_s = -\frac{32}{n\pi}(-1)^n e^{-\frac{n^2\pi^2}{16}t}$$

By inversion theorem,

$$u(x, t) = \frac{2}{4}\sum_{n=1}^{\infty}\frac{32}{n\pi}(-1)^{n+1}e^{-\frac{n^2\pi^2}{16}t}\sin\frac{n\pi x}{4}$$

$$= \frac{16}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}e^{-\frac{n^2\pi^2}{16}t}\sin\frac{n\pi x}{4}$$ **Ans.**

OO

MODULE 5

CONCEPT OF PROBABILITY

PROBABILITY MASS FUNCTION, PROBABILITY DENSITY FUNCTION

Concept of Probability – The concept of probability is extremely important. It has found a very extensive application in the development of all physical sciences. The chance of happening of an event when expressed quantitatively is called probability.

Theorem of Total Probability or Addition Law of Probability –

If the probability of an event happening as a result of a trial is P(A) and the probability of a mutually exclusive event B happening is P(B), then the probability of either of the events happening as a result of the trial is

$$P(A + B) \text{ or } P(A \cup B) = P(A) + P(B)$$

Proof. Let n be the total number of equally likely cases and let m_1 be favourable to the event A and m_2 be favourable to the event B. Then the number of cases favourable to A or B is $m_1 + m_2$.

Hence the probability of A or B happening as a result of the trial

$$= \frac{m_1 + m_2}{n} = \frac{m_1}{n} + \frac{m_2}{n} = P(A) + P(B)$$

If the events A and B are not mutually exclusive, then, there are some outcomes which favour both A and B. If m_3 be their number, then these are included in both m_1 and m_2.

Hence the total number of outcomes favouring either A or B or both is

$$m_1 + m_2 - m_3$$

Thus the probability P(A + B) or P(A ∪ B) of occurrence of A and B or Both

$$= \frac{m_1 + m_2 - m_3}{n} = \frac{m_1}{n} + \frac{m_2}{n} - \frac{m_3}{n}$$

$$P(A + B) = P(A) + P(B) - P(AB)$$

or $$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

or when A and B are mutually exclusive $P(A \cap B)$ or $P(AB) = 0$ and we have

$$P(A + B) \text{ or } P(A \cup B) = P(A) + P(B)$$ **Proved**

Particular Cases –

(i) If A and B are defined on the sample space S, then

$$n(A \cup B) = n(A) + n(B)$$

$$\therefore \quad P(A \cup B) = P(A) + P(B)$$

(ii) Since S and ϕ are mutually exclusive events and $S \cup \phi = S$

$$P(S \cup \phi) = P(S)$$

$$P(S) + P(\phi) = P(S)$$

$$P(\phi) = 0$$

(iii) Let A and $\overline{A}$ be complementary events. Then A and $\overline{A}$ are mutually exclusive by definition. Hence

$$P(A \cup \overline{A}) = P(S)$$

$$P(A) + P(\overline{A}) = 1 \qquad [\because P(S) = 1]$$

$$P(\overline{A}) = 1 - P(A)$$

or $$P(A) = 1 - P(\overline{A})$$

(iv) We know that

$$A = (A \cap \overline{B}) \cup (A \cap B)$$

That A is the union of two mutually exclusive events

$$\therefore \quad P(A) = P(A \cap \overline{B}) + P(A \cap B)$$

or $$P(A \cap \overline{B}) = P(A) - P(A \cap B)$$

or $$P(A \cap B) = P(A) - P(A \cap \overline{B})$$

Similarly,

$$P(\overline{A} \cap B) = P(B) - P(A \cap B)$$

(v) If $B \subset A$, then

(a) $P(A \cap \overline{B}) = P(A) - P(B)$

(b) $P(B) \le P(A)$.

Probability Function – In terms of symbols of a variable X can assume discrete set of values $X_1, X_2, \ldots, X_k$ with respective probabilities $p_1, p_2, \ldots, p_k$ where $p_1 + p_2 + \ldots + p_k = 1$, we say that a discrete probability distribution for X has been defined. The function P(X) which has the respective values $p_1, p_2, \ldots$ for $X = X_1, X_2, \ldots, X_k$ is called the ***probability function*** or frequency function of X.

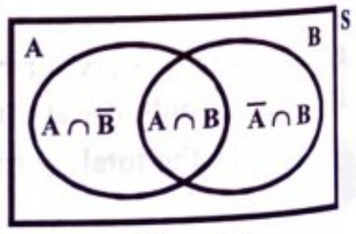

Fig. 5.1

Probability Mass Function – In probability theory and statistics, a probability mass function (p.m.f.) is a function that gives the probability that a discrete random variable is exactly equal to some value. The probability mass function is often the primary means of defining a discrete probability distribution, and such functions exist for either scalar or multivariate random variables, given that the distribution is discrete.

A probability mass function differs from a probability density function (p.d.f.) in that the latter is associated with continuous rather than discrete random variables; the values of the latter are not probabilities as such : a p.d.f. must be integrated over an interval to yield a probability.

Distribution Function – The distribution function F(x) of the discrete variate X is defined by

$$F(x) = P(X \le x) = \sum_{i=1}^{x} p(x_i) \text{ where x is any integer.}$$

F(x) is called the cumulative distribution function of X or simply the distributive function.

The cumulative distribution function has the following properties –

(i) $F'(x) = \frac{d}{dx} F(x) = f(x) \ge 0$, so that F(x) is a non-decreasing function. This means that $dF(x) = f(x)\,dx$. This is called probability differential of X.

(ii) $F(-\infty) = 0$

(iii) $F(\infty) = 1$

(iv) $P(a \le x \le b) = \int_a^b f(x)dx = \int_{-\infty}^b f(x)dx - \int_{-\infty}^a f(x)dx = F(b) - F(a)$

Probability Density Function – The range of possible values is uncountably infinite for continuous random variables. So in this case, the distribution is defined by the ***probability density function f(x)*** for the given range of random variable X.

The ***probability density function f(x),*** is a function which, when integrated a and b, gives the probability that the random variable will assume a value between a and b.

The function f(x) is said to be ***probability density function,*** if

(i) $f(x) \ge 0$, for every value of x.

(ii) $\int_{-\infty}^{\infty} f(x)\,dx = 1$. (iii) $\int_a^b f(x)\,dx = P(a < x < b)$.

Q.1. Define the following terms –

Probability function, probability mass function and probability density function. ***(R.G.P.V., June 2012)***

Or

Define the following –

(i) Probability function

(ii) Probability mass function

(iii) Probability density function. ***(R.G.P.V., June 2016)***

Ans. Refer to the matter given on page 266 and 267.

Q.2. What do you mean by probability density function ? ***[R.G.P.V., May 2019 (III-Sem)]***

Ans. Refer to the matter given on page 267.

NUMERICAL PROBLEMS

Prob.1. Given A and B two events with $P(A \cup B) = \frac{7}{8}$, $P(A \cap B) = \frac{1}{4}$ and $P(A \cap \overline{B}) = \frac{5}{8}$. Find P(A) and P(B). ***(R.G.P.V., Dec. 2014)***

Sol. Given, $P(A \cup B) = \frac{7}{8}$, $P(A \cap B) = \frac{1}{4}$ and $P(A \cap \overline{B}) = \frac{5}{8}$.

We know that

$$P(A) + P(B) = P(A \cup B) + P(A \cap B) \quad ...(i)$$

and

$$P(A \cap \overline{B}) = P(A) - P(A \cap B) \quad ...(ii)$$

Putting the given values in equations (i) and (ii), we get

$$P(A) + P(B) = \frac{7}{8} + \frac{1}{4} = \frac{7+2}{8} = \frac{9}{8} \text{ and } \frac{5}{8} = P(A) - \frac{1}{4}$$

or

$$P(A) = \frac{5}{8} + \frac{1}{4} = \frac{5+2}{8} = \frac{7}{8}$$

$\because$

$$P(A) + P(B) = \frac{9}{8}$$

$\therefore$

$$P(B) = \frac{9}{8} - P(A) = \frac{9}{8} - \frac{7}{8} = \frac{9-7}{8} = \frac{2}{8} = \frac{1}{4}$$

Hence $P(A) = \frac{7}{8}$ and $P(B) = \frac{1}{4}$. **Ans.**

Prob.2. The life time of a certain kind of battery is a random variable, which as an exponential distribution with a mean of 200 hrs. Find the probability that such a battery will last. (i) At most 100 hrs and (ii) last anywhere from 400 to 600 hrs. ***(R.G.P.V., Dec. 2014)***

Sol. If the random variable X denotes the life time of batteries then X follows exponential distribution with parameter, $\sigma = 200$ hrs.

$\therefore$ Probability density function,

$$f(x) = \frac{1}{\sigma} e^{-\frac{x}{\sigma}} = \frac{1}{200} e^{-\frac{x}{200}}$$

(i) $$P(X \le 100) = \int_0^{100} \frac{1}{200} e^{-\frac{x}{200}} dx = \frac{1}{200} \int_0^{100} e^{-\frac{x}{200}} dx$$

$$= \frac{1}{200}\left[e^{-\frac{x}{200}}.(-200)\right]_0^{100} = \frac{-200}{200}\left[e^{-\frac{100}{200}} - 1\right]$$

$$= -1(e^{-0.5} - 1) = \mathbf{1 - e^{-0.5}}$$ **Ans.**

(ii) $$P(400 \le x \le 600) = \int_{400}^{600} \frac{1}{200} e^{-\frac{x}{200}} dx$$

$$= \frac{1}{200}\left[e^{-\frac{x}{200}}.(-200)\right]_{400}^{600} = -\frac{200}{200}\left[e^{-\frac{600}{200}} - e^{-\frac{400}{200}}\right]$$

$$= -1(e^{-3} - e^{-2}) = \mathbf{e^{-2} - e^{-3}}$$ **Ans.**

Prob.3. Thirteen cards are drawn simultaneously from a deck of 52. If aces count 1, face card 10 and others according to denomination, find the expectation of the total score on the 13 cards.

Sol. Let x_i be the number corresponding to the ith cards, then x_i takes the value 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 10, 10, 10, each having the probability $\frac{1}{13}$. Hence X = Number of the cards, $P(X) = \frac{1}{13}$ for i = 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 10, 10, 10.

$$\therefore E(x_i) = \frac{1}{13}.1 + \frac{1}{13}.2 + \frac{1}{13}.3 + ... + \frac{1}{13}.9 + \frac{1}{13}.10 + \frac{1}{13}.10 + \frac{1}{13}.10 + \frac{1}{13}.10$$

$$= \frac{1}{13}(1 + 2 + 3 + ... + 9 + 10 + 10 + 10 + 10) = \mathbf{\frac{85}{13}}$$ **Ans.**

Prob.4. The probability density function p(x) of a random variable is given by $p(x) = y_0 e^{-|x|}$, $-\infty \le x \le \infty$. Find the value of y_0.

Sol. Let us find the value of y_0, so that $y_0 \int_{-\infty}^{\infty} e^{-|x|} dx = 1$

Since $e^{-|x|}$ is an even function, we have

$$2y_0 \int_0^{\infty} e^{-x} dx = 1, \quad \text{as } 0 \le x \le \infty, |x| = x$$

$$2y_0 = 1 \text{ or } y_0 = \mathbf{\frac{1}{2}}$$ **Ans.**

Prob.5. The probability mass function of a variable X is given by

$$P(X = x) = \begin{cases} k, & \text{if } x = 0 \\ 2k, & \text{if } x = 1 \\ 3k, & \text{if } x = 2 \\ 0, & \text{otherwise} \end{cases}$$

Then find k and also evaluate P(X < 2).

Sol. Since $P(x_{ki}) > 0$ for i = 0, 1, 2, we have $k < 0$

Further since $\sum_{i=0}^{2} p(x_i) = 1$, we have

$$k + 2k + 3k = 1$$

or $$6k = 1 \text{ or } k = \frac{1}{6}$$ **Ans.**

$$P(X < 2) = P(x = 0) + P(x = 1)$$
$$= k + 2k$$
$$= 3k = 3 \times \frac{1}{6} = \frac{1}{2}$$ **Ans.**

Prob.6. The probability density function of a variate X is

X:	0	1	2	3	4	5	6
P(X):	k	3k	5k	7k	9k	11k	13k

(i) Find $P(X < 4)$, $P(X \geq 5)$, $P(3 < X \leq 6)$

(ii) What will be the minimum value of k so that $P(X \leq 2) > 0.3$.

Sol. (i) If X is a random variable, then

$$\sum_{i=0}^{6} p(x_i) = 1$$

i.e. $$k + 3k + 5k + 7k + 9k + 11k + 13k = 1$$

or $$k = \frac{1}{49}$$

$\therefore$ $$P(X < 4) = k + 3k + 5k + 7k = 16k = \frac{16}{49}$$ **Ans.**

$$P(X \geq 5) = 11k + 13k = 24k = \frac{24}{49}$$ **Ans.**

$$P(3 < X \leq 6) = 9k + 11k + 13k = 33k = \frac{33}{49}$$ **Ans.**

(ii) $P(X \leq 2) = k + 3k + 5k = 9k > 0.3$

or $$k > \frac{1}{30}$$

Thus minimum value of $k = \frac{1}{30}$ **Ans.**

Prob.7. Prove that the given function –

$$f(x) = \begin{cases} 3x^2, & 0 < x < 1 \\ 0, & \text{otherwise} \end{cases}$$

is a valid density function. Also, find the probability that $\frac{1}{3} < x < \frac{1}{2}$.

Sol. For a density function,

$$\int_{-\infty}^{\infty} f(x)\,dx = 1$$

$\therefore$ $$\int_0^1 3x^2 dx = 1 \text{ or } 3\left[\frac{x^3}{3}\right]_0^1 = 1 \text{ or } 3\left[\frac{1}{3} - 0\right] = 1$$

$$1 = 1$$

or Hence function defined is a density function. **Proved**

Required probability $$= \int_{1/3}^{1/2} 3x^2\,dx$$

$$= 3\left[\frac{x^3}{3}\right]_{1/3}^{1/2} = \left[x^3\right]_{1/3}^{1/2} = \left(\frac{1}{2}\right)^3 - \left(\frac{1}{3}\right)^3$$

$$= \frac{1}{8} - \frac{1}{27} = \frac{27-8}{216} = \frac{19}{216}$$ **Ans.**

DISCRETE DISTRIBUTION – BINOMIAL, POISSON'S

Discrete Distribution – If a real variable X be associated with the outcome of a random experiment then since the values which X takes depend on chance it is said to be a ***random variable*** or **a *stochastic variable*** or ***variate.***

If a random variable takes a finite set of values, it is called a ***discrete random variable,*** and if it assumes an infinite number of uncountable values, it is called a ***continuous random variable.***

Binomial Distribution $P(r) = {}^nC_r\, p^r q^{n-r}$ –

To obtain the probability of the happening of an event once, twice, thrice…r times. ..exactly in n trials. Suppose the probability of the happening of an event A in one trial is p and not happening is $1 - p = q$. We suppose that there are n trials and the happening of the event A is r times and its not is $n - r$ times.

This may be shown as follows.

$$\underbrace{AA\ldots A}_{r \text{ times}} \quad \underbrace{\bar{A}.\bar{A}\ldots\bar{A}}_{n-r \text{ times}} \qquad \ldots(i)$$

Here A indicates its happening, $\bar{A}$ its failure then $P(A) = p$ and $P(\bar{A}) = q$

We see that relation (i) has the probability

$$\underbrace{p\,p\ldots p}_{r \text{ times}},\ \underbrace{q.q\ldots q}_{n-r \text{ times}} = p^r q^{n-r} \qquad \ldots(ii)$$

Clearly relation (i) is merely one order of arranging rA's

The probability of relation (i) $= p^r q^{n-r} \times$ number of different arrangements of rA's and $(n - r)$ $\bar{A}$'s

$\because$ The number of different arrangements of rA's and $(n-r)\bar{A}$'s $= {}^nC_r$

$\therefore$ Probability of happening of an event r times $= {}^nC_r p^r q^{n-r}$

$$P(r) = {}^nC_r p^r q^{n-r}, \ (r = 0, 1, 2, \ldots n)$$
$$= (r+1) \text{ term of } (q+p)^n$$

If r = 0, the probability of happening of an event 0 times $= {}^nC_0 p^0 q^n = q^n$

If r = 1, the probability of happening of an event 1 times $= {}^nC_1 q^{n-1} p$.

If r = 2, the probability of happening of an event 2 times

$= {}^nC_2 q^{n-2} p^2$ and so on.

These terms are clearly the successive terms in the expansion of $(q+p)^n$ Hence it is called ***Binomial distribution.***

Mean of Binomial Distribution –

Here $(q+p)^n = q^n + {}^nC_1 q^{n-1} p + {}^nC_2 q^{n-2} p^2 + \ldots + {}^nC_r q^{n-r} p^r + \ldots + p^n$

Successes r	*Frequency f*	*Product rf*
0	q^n	0
1	$nq^{n-1}p$	$n q^{n-1} p$
2	$\frac{n(n-1)}{2} q^{n-2} p^2$	$n(n-1) q^{n-2} p^2$
3	$\frac{n(n-1)(n-2)}{6} q^{n-3} p^3$	$\frac{n(n-1)(n-2)}{2} q^{n-3} p^3$
....	.....	.....
n	p^n	np^n

$$\Sigma fr = nq^{n-1}p + n(n-1)q^{n-2}p^2 + \frac{n(n-1)(n-2)}{2} q^{n-3}p^3 + \ldots\ldots + np^n$$

$$= np\left[q^{n-1} + \frac{(n-1)}{1!} q^{n-2} p + \frac{(n-1)(n-2)}{2!} q^{n-3} p^2 + \ldots\ldots + p^{n-1}\right]$$

$$= np\,(q+p)^{n-1} = np \qquad [\because q + p = 1]$$

$$\Sigma f = q^n + nq^{n-1}p + \frac{n(n-1)}{2} q^{n-2}p^2 + \ldots\ldots + p^n = (q+p)^n = 1$$

$$\textbf{Mean} = \frac{\Sigma fr}{\Sigma f} = \mathbf{np}$$

Standard Deviation of Binomial Distribution –

Successes r	*Frequency f*	*Product $r^2 f$*
0	q^n	0
1	$nq^{n-1}p$	$nq^{n-1}p$
2	$\frac{n(n-1)}{2} q^{n-2} p^2$	$2n(n-1) q^{n-2} p^2$
.....	.....	.....
n	p^n	$n^2 p^n$

We know that $\sigma^2 = \frac{\Sigma fr^2}{\Sigma f} - \left(\frac{\Sigma fr}{\Sigma f}\right)^2$ ....(i)

r is the deviation of items (successes) from 0.

$$\Sigma f = 1, \ \Sigma fr = np$$

$$\Sigma fr^2 = 0 + nq^{n-1}p + 2n(n-1)q^{n-2}p^2 + \frac{3n(n-1)(n-2)}{2} q^{n-3}p^3 + \ldots + n^2p^n$$

$$= np\left[q^{n-1} + \frac{2(n-1)}{1!} q^{n-2} p + \frac{3(n-1)(n-2)}{2!} q^{n-3} p^2 + \ldots + np^{n-1}\right]$$

$$= np\left[q^{n-1} + \frac{(n-1)q^{n-2}p}{1!} + \frac{(n-1)(n-2)}{2!} q^{n-3}p^2 + \ldots + p^{n-1}\right.$$
$$\left. + \frac{(n-1)q^{n-2}p}{1!} + \frac{2(n-1)(n-2)}{2!} q^{n-3}p^2 + \ldots + (n-1)p^{n-1}\right]$$

$$= np\left[q^{n-1} + (n-1)q^{n-2}p + \frac{(n-1)(n-2)}{2!} q^{n-3}p^2 + \ldots + p^{n-1}\right.$$
$$\left. + (n-1)p\left\{q^{n-2} + (n-2)q^{n-3}p + \frac{(n-2)(n-3)}{2!} q^{n-4}p^2 + \ldots + p^{n-2}\right\}\right]$$

$$= np\,[(q+p)^{n-1} + (n-1)\,p\,(q+p)^{n-2}]$$
$$= np\,[1 + (n-1)\,p] = np\,[np + (1-p)]$$
$$= np\,[np + q] = n^2p^2 + npq.$$

Putting these values in equation (i), we have

$$\text{Variance} = \sigma^2 = \frac{n^2p^2 + npq}{1} - \left(\frac{np}{1}\right)^2 = npq$$

$$\text{S.D.} = \sigma = \sqrt{npq}$$

Hence for the binomial distribution

Mean = np, $\mu_2 = \sigma^2 = npq$

Central Moments (Moment About the Mean) –

Now the first four central moments are found as follows –

Second Central Moment –

$$\mu_2 = \mu_2' - \mu_1'^2 = \left[n(n-1)p^2 + np\right] - n^2p^2$$
$$= np\,[(n-1)\,p + 1 - np] = np\,(1-p) = npq$$

Variance of Binomial distribution is npq.

Third Central Moment –

$$\mu_3 = \mu_3' - 3\,\mu_2'\,\mu_1' + 2\mu_1'^3$$
$$= \{\,n\,(n-1)\,(n-2)\,p^3 + 3n\,(n-1)\,p^2 + np\}$$
$$- 3\,\{(n^2p^2 + npq)\,np\} + 2n^3p^3$$

$$= np\,[-3np^2 + 3np + 2p^2 - 3p + 1 - 3npq]$$
$$= np\,[3np\,(1-p) + 2p^2 - 3p + 1 - 3npq]$$
$$= np\,[3npq + 2p^2 - 3p + 1 - 3npq]$$
$$= np\,[2p^2 - 3p + 1] = np\,[2p^2 - 2p + q]$$
$$= np\,[-2p\,(1-p) + q] = np\,(-2pq + q]$$
$$= npq\,(1-2p) = npq\,(q-p)$$

Fourth Central Moment –

$$\mu_4 = \mu'_4 - 4\mu'_3\mu'_1 + 6\mu'_2\mu'^2_1 - 3\mu'^4_1$$
$$= n(n-1)(n-2)(n-3)p^4 + 6n(n-1)(n-2)p^3 + 7n(n-1)p^2 + np - 4[n(n-1)(n-2)p^3 + 3n(n-1)p^2 + np]np + 6[n(n-1)p^2 + np]n^2p^2 - 3n^4p^4$$
$$= np\,[(n-1)(n-2)(n-3)p^3 + 6(n-1)(n-2)p^2 + 7(n-1)p + 1 - 4\{n(n-1)(n-2)p^3 + 3n(n-1)p^2 + np\} + 6\{n(n-1)p^2 + np\}np - 3n^3p^3]$$
$$= np\,[\{(n^3 - 6n^2 + 11n - 6)p^3 + (6n^2 - 18n + 12)p^2 + 7np - 7p + 1\} + \{(-4n^3 + 12n^2 - 8n)p^3 - 4(3n^2 - 3n)p^2 - 4np\} + \{(6n^3 - 6n^2)p^3 + 6n^2p^2 - 3n^3p^3]$$
$$= np\,[(n^3 - 6n^2 + 11n - 6 - 4n^3 + 12n^2 - 8n + 6n^3 - 6n^2 - 3n^3)p^3 + (6n^2 - 18n + 12 - 12n^2 + 12n + 6n^2)p^2 + (7n - 7 - 4n)p + 1]$$
$$= np\left[(3n-6)p^3 + (-6n+12)p^2 + (3n-7)p + 1\right]$$
$$= np\left[3np^3 - 6p^3 - 6np^2 + 12p^2 + 3np - 7p + 1\right]$$
$$= np\left[3np^3 - 3np^2 - 6p^3 + 6p^2 - 3np^2 + 3np + 6p^2 - 6p - p + 1\right]$$
$$= np\left[-3np^2(1-p) + 6p^2(1-p) + 3np(1-p) - 6p(1-p) + (1-p)\right]$$
$$= np\left[-3np^2q + 6p^2q + 3npq - 6pq + q\right]$$
$$= npq\left[-3np^2 + 6p^2 + 3np - 6p + 1\right] = npq\left[3np(1-p) - 6p(1-p) + 1\right]$$
$$= npq\left[3npq - 6pq + 1\right] = \mathbf{npq\left[1 + 3(n-2)pq\right]}$$

Moment Generating Functions of Binomial Distribution About Origin -

The moment generating function about the origin is

$$M_0(t) = \Sigma\left(e^{xt}\right) = \Sigma\,{}^nC_x\,p^x\,q^{n-x}e^{tx} \qquad ...(i)$$
$$= \Sigma\,{}^nC_x\left(pe^t\right)^x q^{n-x} = \left(q + pe^t\right)^n$$

Differentiating with respect to 't' and substituting t = 0, we get

$$\mu'_1 = np$$

Since $M_a(t) = e^{-at}M_0(t)$, the moment generating function of the Binomial distribution about its mean, m = np is given by

$$M_m(t) = e^{-npt}(q + pe^t)^n = (q\,e^{-pt} + pe^{qt})^n$$
$$= \left[1 + pq\frac{t^2}{2!} + pq\left(q^2 - p^2\right)\frac{t^3}{3!} + pq\left(q^3 + p^3\right)\frac{t^4}{4!} + ...\right]$$
$$= 1 + \mu_1 t + \mu_2\frac{t^2}{2!} + \mu_3\frac{t^3}{3!} + \mu_4\frac{t^4}{4!} + ...$$
$$= 1 + npq\frac{t^2}{2!} + npq(q-p)\frac{t^3}{3!} + npq\left[1 + 3(n-2)pq\right]\frac{t^4}{4!} + ...$$

Equating the coefficients of like powers of t on both sides we have

$$\mu_2 = npq, \mu_3 = npq(q-p), \mu_4 = npq\left[1 + 3(n-2)pq\right]$$

Also, $\beta_1 = \dfrac{\mu_3^2}{\mu_2^3} = \dfrac{(q-p)^2}{npq} = \dfrac{(1-2p)^2}{npq}$ and $\beta_2 = \dfrac{\mu_4}{\mu_2^2} = 3 + \dfrac{1-6pq}{npq}$

Skewness, $\gamma_1 = \sqrt{\beta_1} = \dfrac{1-2p}{\sqrt{(npq)}}$, Kurtosis $= \beta_2$ and $\gamma_2 = \beta_2 - 3 = \dfrac{1-6pq}{npq}$

Note – The skewness is positive for $p < \frac{1}{2}$ and negative for $p > \frac{1}{2}$. When $p = \frac{1}{2}$, the skewness is zero i.e., the probability curve of the Binomial distribution will be symmetrical.

Recurrence Relation for the Binomial Distribution –

By Binomial distribution, we have

$$P(r) = {}^nC_r p^r q^{n-r} \qquad ...(i)$$
$$P(r+1) = {}^nC_{r+1} p^{r+1} q^{n-r-1} \qquad ...(ii)$$

On dividing relation (ii), by relation (i), we get

$$\frac{P(r+1)}{P(r)} = \frac{{}^nC_{r+1}}{{}^nC_r}\,\frac{p^{r+1}q^{n-r-1}}{p^r q^{n-r}}$$
$$= \frac{n(n-1)(n-2).....(n-r)}{(r+1)!}\,\frac{r!}{n(n-1)(n-2)....(n-r+1)}\,\frac{p}{q}$$
$$= \frac{P(r+1)}{P(r)} = \frac{n-r}{r+1}.\frac{p}{q} \quad \text{or} \quad P(r+1) = \frac{n-r}{r+1}\frac{p}{q}P(r)$$

Binomial Frequency Distribution – If n dependent trials constitutes one experiment and this experiment be repeated N times, then the frequency of r successes is $N\,{}^nC_r\,p^r\,q^{n-r}$. The possible number of successes together with these expected frequencies constitute the ***Binomial frequency distribution***.

Applications of Binomial Distribution – Bionomial distribution is applied to problems concerning –

(i) Number of defectives in a sample from production line.
(ii) Estimation of reliability of systems.
(iii) Number of rounds fired from a gun hitting a target.
(iv) Radar detection.

Poisson Distribution – It is a distribution related to the probabilities of events which are extremely rare, but which have a large number of independent opportunities for occurrence.

Poisson distribution is a particular limiting form of binomial distribution by making n very large and p very small, keeping np fixed (= m say).

The probability of r successes in a Binomial distribution is

$$P(r) = {}^nC_r p^r q^{n-r} = \frac{n(n-1)(n-2).....(n-r+1)}{r!} p^r q^{n-r}$$

$$= \frac{np(np-p)(np-2p).....(np-\overline{r-1}p)}{r!}(1-p)^{n-r}$$

As $n \to \infty, p \to 0\,(np = m)$, we have

$$P(r) = \frac{m^r}{r!} \lim_{n \to \infty} \frac{\left(1-\frac{m}{n}\right)^n}{\left(1-\frac{m}{n}\right)^r} = \frac{m^r}{r!} e^{-m}$$

So that the probabilities of 0, 1, 2, ... r... successes in a Poisson distribution are given by

$$e^{-m}, me^{-m}, \frac{m^2 e^{-m}}{2!} ..... \frac{m^r e^{-m}}{r!} .....$$

The sum of these probabilities is unity as it should be

Mean of Poisson Distribution –

Here $$P(r) = \frac{e^{-m} m^r}{r!}$$

Successes, r	*Frequency, f*	*f.r*
0	$\frac{e^{-m}m^0}{0!}$	0
1	$\frac{e^{-m}m^1}{1!}$	$e^{-m}.m$
2	$\frac{e^{-m}m^2}{2!}$	$e^{-m}m^2$
3	$\frac{e^{-m}m^3}{3!}$	$\frac{e^{-m}m^3}{2!}$
.....	.....	.....
r	$\frac{e^{-m}m^r}{r!}$	$\frac{e^{-m}m^r}{(r-1)!}$
.....	.....	.....

$$\Sigma fr = 0 + e^{-m}.m + e^{-m}.m^2 + \frac{e^{-m}.m^3}{2!} + ... + \frac{e^{-m}m^r}{(r-1)!} + ...$$

$$= e^{-m}.m\left[1 + \frac{m}{1!} + \frac{m^2}{2!} + ... + \frac{m^{r-1}}{(r-1)!} + ...\right]$$

$$= m.e^{-m}.e^m = m$$

$$\text{Mean} = \frac{\Sigma fr}{\Sigma f} = \frac{m}{1},$$

i.e., **Mean = m**

Standard Deviation of Poisson Distribution –

We know that $$P(r) = \frac{e^{-m}m^r}{r!}$$

Successes, r	*Frequency, f*	*Product, rf*	*Product, $r^2 f$*
0	$\frac{e^{-m}m^0}{0!}$	0	0
1	$\frac{e^{-m}m^1}{1!}$	$e^{-m}.m$	$e^{-m}.m$
2	$\frac{e^{-m}m^2}{2!}$	$e^{-m}.m^2$	$2e^{-m}.m^2$
3	$\frac{e^{-m}m^3}{3!}$	$e^{-m}\frac{m^3}{2!}$	$3e^{-m}\frac{m^3}{2!}$
.....	$\frac{e^{-m}m^r}{r!}$	$\frac{e^{-m}m^r}{(r-1)!}$	.....
r	.....	.....	$\frac{re^{-m}m^r}{(r-1)!}$
.....			

$\Sigma f = 1, \Sigma fr = m$

and $$\Sigma fr^2 = 0 + m.e^{-m} + 2m^2e^{-m} + \frac{3m^3}{2!}e^{-m} + ... + \frac{rm^re^{-m}}{(r-1)!} + ...$$

$$= me^{-m}\left[1+2m+\frac{3m^2}{2!}+\frac{4m^3}{3!}+.....+\frac{r\,m^{r-1}}{(r-1)!}+.....\right]$$

$$= me^{-m}\left[\begin{array}{l}1+m+\frac{m^2}{2!}+\frac{m^3}{3!}+.....+\frac{m^{r-1}}{(r-1)!}+.....+m+\frac{2m^2}{2!}\\ \qquad\qquad +\frac{3m^3}{3!}+.....+\frac{(r-1)m^{r-1}}{(r-1)!}+....\end{array}\right]$$

$$= me^{-m}\left[\begin{array}{l}\left\{1+m+\frac{m^2}{2!}+\frac{m^3}{3!}+.....+\frac{m^{r-1}}{(r-1)!}+.....\right\}\\ +m\left\{1+\frac{m}{1!}+\frac{m^2}{2!}+.....+\frac{m^{r-2}}{(r-2)!}+.....\right\}\end{array}\right]=me^{-m}\left[e^m+me^m\right]=m+m^2$$

$$\because \quad \sigma^2=\frac{\Sigma fr^2}{\Sigma f}-\left(\frac{\Sigma fr}{\Sigma f}\right)^2 \qquad \therefore\ \sigma^2=\frac{m+m^2}{1}-(m)^2=m \quad \text{or} \quad \sigma=\sqrt{m}$$

Hence mean and variance of a Poisson distribution are each equal to m.

Similarly we can find

$$\mu_3 = m,\ \ \mu_4 = 3\,m^2 + m$$

$$\beta_1=\frac{1}{m},\ \beta_2=3+\frac{1}{m}$$

$$\gamma_1=\frac{1}{\sqrt{m}} \text{ and } \gamma_2=\frac{1}{m}.$$

Moment Generating Function of Poisson Distribution –

Here, $\quad P(r)=\dfrac{e^{-m}m^r}{r!}$

Let $M_x(t)$ be the moment generating function, then

$$M_x(t)=\sum_{r=0}^{\infty}e^{tr}.\frac{e^{-m}m^r}{r!}=\sum_{r=0}^{\infty}e^{-m}.\frac{\left(me^t\right)^r}{r!}$$

$$=e^{-m}\left[1+me^t+\frac{\left(me^t\right)^2}{2!}+\frac{\left(me^t\right)^3}{3!}+.....\right]=e^{-m}e^{me^t}=e^{m\left(e^t-1\right)}$$

Recurrence Formula for Poisson Distribution –

Here, $$P(r)=\frac{e^{-m}.m^r}{r!} \qquad ....(i)$$

$$P(r+1)=\frac{e^{-m}.m^{r+1}}{(r+1)!} \qquad ....(ii)$$

Dividing relation (ii) by relation (i), we get

$$\frac{P(r+1)}{P(r)}=\frac{e^{-m}m^{r+1}}{(r+1)!}\times\frac{r!}{e^{-m}m^r}=\frac{m}{r+1}$$

Hence $$\mathbf{P(r+1)=\frac{m}{r+1}P(r)}$$ **Ans.**

Applications of Poisson Distribution – Poisson distribution is applied to problems concerning –

(i) Arrival pattern of defective vehicles in a workshop 'patients in a hospital' or 'telephone calls'.

(ii) Demand pattern for certain spare parts.

(iii) Numbers of fragments from a shall hitting a target.

(iv) Spatial distribution of bomb hits.

Q.3. Write short note on discrete and continuous random variable. ***(R.G.P.V., June 2012)***

Or

Define discrete and continuous random variable. (R.G.P.V., June 2015)

Ans. Refer to matter given on page 271, under heading 'Discrete Distribution'.

Q.4. Explain the Binomial theorem in brief. ***[R.G.P.V., May 2019 (III-Sem)]***

Ans. If p and q are real numbers and n is a positive integer, then

$$(q+p)^n={}^nC_0q^n+{}^nC_1q^{n-1}p^1+{}^nC_2q^{n-2}p^2+.....$$
$$.....+{}^nC_rq^{n-r}p^r+.....+{}^nC_np^n$$

where $\quad {}^nC_r=\dfrac{n!}{r!\,(n-r)!}$ for $0\le r\le n$

The general term or $(r+1)^{th}$ term in the expansion is given by

$$T_{r+1}={}^nC_r\,q^{n-r}p^r$$

NUMERICAL PROBLEMS

Prob.8. Find the mean of the Binomial distribution. (R.G.P.V., Dec. 2014)

Sol. Refer to the matter given on page 272.

Prob.9. Find the mean and variance of the Binomial distribution. ***[R.G.P.V., Dec. 2016, Nov. 2018, 2019 (IV-Sem)]***

Or

Derive the expression for mean and variance of Binomial distribution. ***(R.G.P.V., June 2012, 2015)***

Or

Write the mean and variance of the Binomial distribution. ***(R.G.P.V., Dec. 2015)***

Sol. Refer to the matter given on page 272.

Prob.10. Find the probability of getting 4 heads in 6 tosses of a fair coin.

Sol. Here, $p = \frac{1}{2}$, $q = 1 - p = 1 - \frac{1}{2} = \frac{1}{2}$, $n = 6, r = 4$.

We know that

$$P(r) = {}^nC_r p^r q^{n-r} = {}^6C_4\left(\frac{1}{2}\right)^4\left(\frac{1}{2}\right)^{6-4}$$

$$= {}^6C_4\left(\frac{1}{2}\right)^4\left(\frac{1}{2}\right)^2 = \frac{6\times5}{1\times2}\times\left(\frac{1}{2}\right)^6 = \mathbf{\frac{15}{64}}$$ **Ans.**

Prob.11. Six dices are thrown 729 times. How many times do you expect at least three dices to show a five or six ? [(R.G.P.V., Nov. 2019 (III-Sem)]

Sol. Here, $p = \frac{2}{6} = \frac{1}{3}, q = 1 - p = 1 - \frac{1}{3} = \frac{2}{3}, n = 6, N = 729$

Hence the required number

$$= 729\left[{}^6C_3\left(\frac{1}{3}\right)^3\left(\frac{2}{3}\right)^3 + {}^6C_4\left(\frac{1}{3}\right)^4\left(\frac{2}{3}\right)^2 + {}^6C_5\left(\frac{1}{3}\right)^5\left(\frac{2}{3}\right) + \left(\frac{1}{3}\right)^6\right]$$

$$= 729\left[\frac{160}{729} + \frac{60}{729} + \frac{12}{729} + \frac{1}{729}\right]$$

$$= 160 + 60 + 12 + 1 = \mathbf{233}$$ **Ans.**

Prob.12. If the 10 percent of the bolts produced by a machine are defective, find the probability that out of 5 bolts chosen at random at least two will be defective.

Sol. Here, $p = \frac{10}{100} = \frac{1}{10}$, $q = 1 - p = 1 - \frac{1}{10} = \frac{9}{10}$

$n = 5, r = 2$

$$P(r) = {}^nC_r p^r q^{n-r}$$

$$= {}^5C_2\left(\frac{1}{10}\right)^2\left(\frac{9}{10}\right)^{5-2} = {}^5C_2\left(\frac{1}{10}\right)^2\left(\frac{9}{10}\right)^3$$

$$= \frac{5\times4}{2\times1}\cdot\frac{1}{10\times10}\cdot\frac{9\times9\times9}{10\times10\times10} = \frac{729}{10000} = \mathbf{0.0729}$$ **Ans.**

Prob.13. A lot has 10% defective items. Ten items are chosen randomly from this lot. Find the probability that exactly 2 of the chosen items are defective. ***(R.G.P.V., Dec. 2016)***

Sol. Here $p = \frac{10}{100} = \frac{1}{10}, q = 1 - p = 1 - \frac{1}{10} = \frac{9}{10}$, $n = 10, r = 2$

$$p(r) = {}^nC_r p^r q^{n-r}$$

$$= {}^{10}C_2\left(\frac{1}{10}\right)^2\left(\frac{9}{10}\right)^{10-2} = {}^{10}C_2\left(\frac{1}{10}\right)^2\left(\frac{9}{10}\right)^8$$

$$= \frac{10\times9}{2\times1}.(0.01)(0.4305) = \mathbf{0.1937}$$ **Ans.**

Prob.14. A perfect cubical die is thrown a large number of times in sets of 8. The occurrence of 5 or 6 is called a success. In what proportion of the sets you expect 3 success.

Sol. Here, $n = 8$, $p = \frac{1}{3}, q = \frac{2}{3}$

Thus the Binomial distribution is given by $N\left(\frac{2}{3}+\frac{1}{3}\right)^8$

The number of sets in which 3 success are expected

$$= N\left\{{}^8C_3\left(\frac{1}{3}\right)^3\left(\frac{2}{3}\right)^5\right\} = N\,\frac{56\times32}{27\times243}$$

$\therefore$ Percentage $= N \times \frac{56\times32}{27\times243} \times \frac{100}{N} = \frac{179200}{6561} = \mathbf{27.31\%}$ **Ans.**

Prob.15. The probability that a bomb dropped from a plane will strike the target is 1/5. If six bombs are dropped find the probability that

(i) exactly two will strike the target

(ii) at least two will strike the target.

(R.G.P.V., June 2013)

Sol. Given the probability of bomb that will strike the target is $p = \frac{1}{5}$

$\therefore$ $q = 1 - \frac{1}{5} = \frac{4}{5}$

(i) Therefore the probability of exactly two will strike the target out of 6 bombs

$$= {}^6C_2\, p^2\, q^{6-2}$$

$$= {}^6C_2\, p^2\, q^4 = \frac{6\times5}{2\times1}\cdot\left(\frac{1}{5}\right)^2\cdot\left(\frac{4}{5}\right)^4 = \frac{7680}{31250} = \mathbf{0.246}$$ **Ans.**

(ii) The probability at least two will strike the target out of 6 bombs.

= 1 – (Probability that either none, or one are not strike the target)

$$= 1 - \left[{}^6C_0\left(\frac{4}{5}\right)^6 + {}^6C_1\left(\frac{1}{5}\right)\left(\frac{4}{5}\right)^5\right] = 1 - \left[\left(\frac{4}{5}\right)^6 + \frac{6}{5}\left(\frac{4}{5}\right)^5\right]$$

$$=1-\left(\frac{4}{5}\right)^5\left[\frac{4}{5}+\frac{6}{5}\right]=1-\frac{1024}{3125}\left[\frac{10}{5}\right]$$

$$=1-\frac{2048}{3125}=\frac{1077}{3125}=\mathbf{0.345}$$

Ans.

Prob.16. The probability that an evening college student will graduate is 0.4. Determine the probability that out of 5 students –

(i) None (ii) One and (iii) At least one will graduate.

(R.G.P.V., Dec. 2015, 2016)

Sol. Given, the probability that an evening college student will graduate is $p = 0.4$.

The probability that an evening college student will no graduate is $q = 1 - 0.4 = 0.6$.

$\therefore$ (i) The probability of none will graduate out of 5 student

$$= {}^5C_0\, p^0\, q^5$$
$$= {}^5C_0 \times (0.4)^0 \times (0.6)^5$$
$$= 1 \times 1 \times 0.07776 \simeq \mathbf{0.078}$$

Ans.

(ii) The probability of exactly one student will graduate out of 5 students

$$= {}^5C_1\,(0.4)^1\,(0.6)^4$$
$$= 5 \times 0.4 \times 0.1296 = \mathbf{0.2592}$$

Ans.

(iii) The probability of at least one will graduate out of 5 students

= 1 – (Probability of none will graduate)

$= 1 - 0.078 = \mathbf{0.922}$

Ans.

Prob.17. If 10% of bolts produced by a machine are defective. Determine the probability that out of 12 bolts, chosen at random (i) one (ii) none (iii) at most two bolts will be defective.

Sol. Probability of defective bolts, $p = 10\% = 0.1$

Probability of not defective bolts, $q = 1 - p = 1 - 0.1 = 0.9$

Total number of bolts, $n = 12$

(i) Probability of one defective bolt ${}^{12}C_1(0.1)^1(0.9)^{11}$

$$= 12 \times 0.1 \times 0.3138 = \mathbf{0.3766}$$

Ans.

(ii) Probability that none is defective

= Probability of 0 defective bolt = P(0)

$$= {}^{12}C_0(0.1)^0(0.9)^{12}$$
$$= 1 \times 1 \times 0.2824 = \mathbf{0.2824}$$

Ans.

(iii) Probability of two defective bolts

$$= {}^{12}C_2(0.1)^2(0.9)^{10} = \frac{12\times 11}{1\times 2}\times 0.01\times 0.3487 = 0.2301$$

Probability of at most two defective bolts

= P(0 or 1 or 2)

= P(0) + P(1) + P(2)

$= 0.2824 + 0.3766 + 0.2301 = \mathbf{0.8891}$ **Ans.**

Prob.18. An irregular six faced die is thrown and the expectation that in 10 throws it will give five even numbers is twice the expectation that it will give four even numbers. How many times in 10,000 sets of 10 throws would you expect it to give no even number ?

Sol. Let p be the probability of getting an even number.

The probability of 5 even numbers in 10 throws $= {}^{10}C_5\, p^5\, q^5$

The probability of 4 even numbers in 10 throws $= {}^{10}C_4\, p^4\, q^6$

Now according to the given condition

$${}^{10}C_5\, p^5\, q^5 = 2 \times {}^{10}C_4\, p^4\, q^6$$

$$\Rightarrow \quad \frac{10\times 9\times 8\times 7\times 6}{5\times 4\times 3\times 2\times 1}p^5 q^5 = 2 \times \frac{10\times 9\times 8\times 7}{4\times 3\times 2\times 1}\, p^4 q^6$$

$$\Rightarrow \quad \frac{6}{5}p = 2q \Rightarrow \frac{3}{5}(1-q) = q \Rightarrow 3 - 3q = 5q$$

$$\Rightarrow \quad 8q = 3 \quad \text{or} \quad q = \frac{3}{8}$$

Hence the number of times in 10,000 throws, where we get no even number

$= 10{,}000\,(3/8)^{10} = \mathbf{0.55}$ **Ans.**

Prob.19. If 10% of bolt's produced by a machine are defective. Determine the probability that out of 10 bolts, chosen at random (i) 1 (ii) none (iii) at most 2 bolts will be defective. ***[R.G.P.V., May 2019 (III-Sem)]***

Sol. Here the probability of a bolt being defective is $p = \frac{10}{100} = 0.1$

$\therefore \quad q = 1 - p = 1 - 0.1 = 0.9$

(i) The probability of one defective bolt out of 10

$= {}^{10}C_1\,(0.1)^1\,(0.9)^9 = \mathbf{0.3874}$ **Ans.**

(ii) The probability that none is defective

$= {}^{10}C_0\,(0.1)^0\,(0.9)^{10} = (0.9)^{10} = \mathbf{0.3487}$ **Ans.**

(iii) Probability of 2 defective $= {}^{10}C_2\,(0.1)^2\,(0.9)^8 = 0.1937$

$\therefore$ Probability of at most 2 defective

= Probability of none defective

+ Probability of one defective

+ Probability of two defective

$= 0.3487 + 0.3874 + 0.1937 = \mathbf{0.9298}$ **Ans.**

Prob.20. If in a lot of 500 solenoids 25 are defective, find the probability of 0, 1, 2, 3 defective solenoids in a random sample of 20 solenoids.

Sol. Here, $p = \frac{25}{500} = \frac{1}{20}$

$\therefore \quad q = 1 - p = 1 - \frac{1}{20} = \frac{19}{20}$

(i) The probability that none is defective out of 20

$$= {}^{20}C_0 \left(\frac{1}{20}\right)^0 \left(\frac{19}{20}\right)^{20} = \mathbf{0.3585}$$ **Ans.**

(ii) The probability that one is defective out of 20

$$= {}^{20}C_1 \left(\frac{1}{20}\right)^1 \left(\frac{19}{20}\right)^{19} = 20 \cdot \frac{1}{20} \cdot \left(\frac{19}{20}\right)^{19} = \mathbf{0.3773}$$ **Ans.**

(iii) The probability that two is defective out of 20

$$= {}^{20}C_2 \left(\frac{1}{20}\right)^2 \left(\frac{19}{20}\right)^{18} = 190 \frac{1}{400} \cdot \left(\frac{19}{20}\right)^{18} = \mathbf{0.1887}$$ **Ans.**

(iv) The probability that three is defective out of 20

$$= {}^{20}C_3 \left(\frac{1}{20}\right)^3 \left(\frac{19}{20}\right)^{17}$$

$$= \frac{20 \times 19 \times 18}{3 \times 2 \times 1} \times \frac{1}{8000} \times \left(\frac{19}{20}\right)^{17} = \mathbf{0.0596}$$ **Ans.**

Prob.21. The following data are the number of seeds germinating out of 10 on damp filter for 80 sets of seeds. Fit a Binomial distribution to these data –

x	0	1	2	3	4	5	6	7	8	9	10
f	6	20	28	12	8	6	0	0	0	0	0

Sol. Here, n = 10, N = 80 and total frequency = 80

$$\therefore \quad A.M = \frac{\Sigma fx}{\Sigma f} = \frac{1\times 20 + 2\times 28 + 3\times 12 + 4\times 8 + 5\times 6}{80} = \frac{174}{80} = \frac{87}{40}$$

$$\therefore \quad \text{Mean} = np = \frac{87}{40}$$

or $\quad p = \frac{87}{400}$

or $\quad p = 0.2175 \qquad [\because n = 10]$

$\therefore \quad q = 1 - p = 1 - 0.2175 = 0.7825$

Hence the Binomial distribution to be fitted to the data is

$$= 80\,(0.7825 + 0.2175)^{10}$$

From this expansion the successive frequencies of 0, 1, 2, ..., 10 successes are respectively **6.9, 19.1, 24.0, 17.7, 8.6, 2.9, 0.7, 0.1, 0, 0, 0** **Ans.**

Prob.22. Show that for a Binomial distribution $(q + p)^n$,

$$\mu_{r+1} = pq\left(nr\mu_{r-1} + \frac{d\mu_r}{dp}\right)$$

where μ_r is the rth moment about the mean.

Sol. Mean of the Binomial distribution is np,

$$\therefore \quad \mu_r = \sum_{x=0}^{n} (x - np)^r \, {}^nC_x \, p^x q^{n-x}$$

Differentiating with respect to p, we get

$$\frac{d\mu_r}{dp} = \sum_{x=0}^{n} - rn\,(x - np)^{r-1} \, {}^nC_x \, p^x q^{n-x}$$

$$+ \sum_{x=0}^{n} [(x - np)^r . \, {}^nC_x \, \{x\, p^{x-1} q^{n-x} - (n - x)\, p^x q^{n-x-1}\}]$$

$$\frac{d\mu_r}{dp} = -nr\,\mu_{r-1} + \frac{1}{pq}\sum_{x=0}^{n} {}^nC_x \, p^x q^{n-x} . (x - np)^r . (xq - np + xp)$$

$$\frac{d\mu_r}{dp} = -nr\,\mu_{r-1} + \frac{1}{pq}\sum_{x=0}^{n} {}^nC_x p^x q^{n-x} (x - np)^{r+1}$$

or $\quad nr\,\mu_{r-1} + \frac{d\mu_r}{dp} = \frac{1}{pq}\sum_{x=0}^{n} {}^nC_x p^x q^{n-x} (x - np)^{r+1}$

$$\therefore \quad pq\left(nr\mu_{r-1} + \frac{d\mu_r}{dp}\right) = \sum_{x=0}^{n} {}^nC_x p^x q^{n-x} . (x - np)^{r+1} = \mu_{r+1}$$

Hence $\quad \mu_{r+1} = \mathbf{pq}\left(\mathbf{nr\mu_{r-1}} + \frac{\mathbf{d\mu_r}}{\mathbf{dp}}\right)$ **Proved**

Prob.23. Show that if two symmetrical Binomial distributions of degree n (the some number of observations) are so superposed that the first term of the one coincides with the $(r + 1)^{th}$ term of the other, the distribution formed by adding superposed term is a symmetrical Binomial distribution of degree (n + 1).

Sol. We know that, for symmetrical distribution p = q = 1/2

Then the Binomial distribution $= N.(q + p)^n = N.\left(\frac{1}{2} + \frac{1}{2}\right)^n$

$$= N\left[{}^nC_0\left(\frac{1}{2}\right)^n + {}^nC_1\left(\frac{1}{2}\right)^{n-1} . \frac{1}{2} + \ldots\ldots + {}^nC_r\left(\frac{1}{2}\right)^{n-r}\left(\frac{1}{2}\right)^r + \ldots\ldots + {}^nC_n\left(\frac{1}{2}\right)^n\right]$$

Hence, the total frequencies can be represented by

$$N = \left(\frac{1}{2}\right)^n . N\left[{}^nC_0 + {}^nC_1 + {}^nC_2 + \ldots + {}^nC_{r-1} + {}^nC_r + \ldots + {}^nC_n\right] \quad \ldots(i)$$

On this distribution a similar distribution has been superposed so that $(r+1)^{th}$ term of this coincides with r^{th} of the other i.e.

$$N = \left(\frac{1}{2}\right)^n . N\left[{}^nC_0 + {}^nC_1 + {}^nC_2 + \ldots {}^nC_{r-1} + {}^nC_r + \ldots + {}^nC_n\right] \quad \ldots(ii)$$

Adding equations (i) and (ii), we get

$$2N = \left(\frac{1}{2}\right)^n N\left[{}^nC_0 + ({}^nC_1 + {}^nC_0) + ({}^nC_2 + {}^nC_1) + ({}^nC_r + {}^nC_{r-1}) + \ldots + ({}^nC_n + {}^nC_{n-1}) + {}^nC_n\right]$$

$$2N = \left(\frac{1}{2}\right)^n . N\left[{}^{n+1}C_0 + {}^{n+1}C_1 + {}^{n+1}C_2 + \ldots + {}^{n+1}C_r + \ldots + {}^{n+1}C_n + {}^{n+1}C_{n+1}\right]$$

$$\left(\because {}^nC_0 = {}^{n+1}C_0,\ {}^nC_r + {}^nC_{r+1} = {}^{n+1}C_{r+1},\ {}^nC_n = {}^{n+1}C_{n+1}\right)$$

So,

$$N = \left(\frac{1}{2}\right)^{n+1} . N\left[{}^{n+1}C_0 + {}^{n+1}C_1 + {}^{n+1}C_2 + \ldots + {}^{n+1}C_r + \ldots + {}^{n+1}C_{n+1}\right]$$

$$= N.\left(\frac{1}{2} + \frac{1}{2}\right)^{n+1}$$

Hence, the resulting distribution is of degree $(n + 1)$. **Proved**

Prob.24. The sum and the product of the mean and variance of a binomial distribution are 24 and 128. Find the distribution. ***(R.G.P.V., Dec. 2013)***

Sol. Let $X \sim B(n, p)$ then, we have

$$np + npq = 24$$

or

$$np(1 + q) = 24 \quad \ldots(i)$$

or

$$np \times npq = 128$$

$$n^2p^2q = 128 \quad \ldots(ii)$$

Squaring equation (i) and dividing by equation (ii), we get

$$\frac{n^2p^2(1+q)^2}{n^2p^2q} = \frac{576}{128}$$

$$\frac{(1+q)^2}{q} = \frac{9}{2} \text{ or } \frac{1+q^2+2q}{q} = \frac{9}{2}$$

$$2 + 2q^2 + 4q = 9q$$

$$2q^2 - 5q + 2 = 0$$

$$(q - 2)(2q - 1) = 0$$

$$q = \frac{1}{2} \qquad (\because q \neq 2)$$

Now

$$p = 1 - q = 1 - \frac{1}{2} = \frac{1}{2}$$

Putting the values of p and q in equation (i), we get

$$n \times \frac{1}{2}\left(1 + \frac{1}{2}\right) = 24$$

$$3n = 96 \text{ or } n = 32$$

Hence, the distribution is

$${}^{32}C_r\left(\frac{1}{2}\right)^r\left(\frac{1}{2}\right)^{32-r}, \qquad r = 0, 1, 2, \ldots, 32 \qquad \textbf{Ans.}$$

Prob.25. If mean and variance of a Binomial distribution are 4 and 2 respectively. Find the probability of (i) exactly 2 successes (ii) less than 2 successes (iii) at least 2 successes.

Sol. It is given that mean = np = 4 and variance = npq = 2, then

$$\frac{npq}{np} = \frac{2}{4} = \frac{1}{2} \Rightarrow q = \frac{1}{2} \therefore p = \frac{1}{2} \qquad (\because q = 1 - p)$$

$$\because \quad np = 4 \Rightarrow n\left(\frac{1}{2}\right) = 4 \Rightarrow n = 8$$

The Binomial distribution giving the probability of r successes in 8 trials is

$$P(r) = {}^8C_r\left(\frac{1}{2}\right)^{8-r}\left(\frac{1}{2}\right)^r = {}^8C_r\left(\frac{1}{2}\right)^8 \quad \ldots(i)$$

where, $r = 0, 1, 2, \ldots, 8$

(i) The probability of exactly 2 successes is given by $r = 2$ in relation (i)

$$P(2) = {}^8C_2\left(\frac{1}{2}\right)^8 = 28 \times \frac{1}{256} = \mathbf{\frac{7}{64}} \qquad \textbf{Ans.}$$

(ii) The probability of less than two successes is given by,

$$P(r < 2) = P(r = 0) + P(r = 1) = P(0) + P(1)$$

$$= {}^8C_0\left(\frac{1}{2}\right)^8 + {}^8C_1\left(\frac{1}{2}\right)^8 = \frac{1}{256} + \frac{8}{256} = \mathbf{\frac{9}{256}}$$

Ans.

(iii) The probability of at least two successes is given by

$$= 1 - [P(0) + P(1)] \qquad \left[\because \sum_{r=0}^{\infty} P(r) = 1\right]$$

$$= 1 - \frac{9}{256} = \mathbf{\frac{247}{256}} \qquad \textbf{Ans.}$$

Prob.26. In a Binomial distribution the mean and standard deviations are 12 and 2 respectively. Find n and p. ***[R.G.P.V., May 2019 (IV-Sem)]***

Sol. Given,

$$\text{Mean} = np = 12$$

and $\quad$ Standard deviation $= \sigma = \sqrt{npq} = 2$

$\therefore \quad npq = 4$

Then $\quad \dfrac{npq}{np} = \dfrac{4}{12} = \dfrac{1}{3}$ or $q = \dfrac{1}{3}$

$\therefore \quad p = 1 - \dfrac{1}{3} = \dfrac{2}{3} \qquad (\because q = 1 - p)$

Now $\quad np = 12$

$$n \times \frac{2}{3} = 12 \text{ or } n = \frac{12 \times 3}{2} = 18$$

Hence n = **18** and p = $\mathbf{\dfrac{2}{3}}$ **Ans.**

Prob.27. 10% of tools produced in a certain manufacturing process turn out to be defective find the probability that in a sample of 10 tools chosen at random (i) exactly two will be defective (ii) more than one will be the defective.

Sol. Given the probability of defective tools is p = 0.1

$\therefore \quad m = np = 10\,(0.1) = 1$

(i) Here $\quad r = 2,\ m = 1$

$\therefore$ Probability of 2 defective tools $= \dfrac{e^{-1}.(1)^2}{2!} = \dfrac{e^{-1}}{2} = \dfrac{1}{2e}$

$= \mathbf{0.18}$ **(Approx.)** **Ans.**

(ii) p (more than one defective) = 1 – [p (zero defective) + p (one defective)]

$= 1 - (e^{-1} + e^{-1}) = 1 - \dfrac{2}{e} = \mathbf{0.26}$ **(Approx.)** **Ans.**

Prob.28. Find the probability that at most 5 defective fuses will be found in a box of 200 fuses if experience shows that 2 % of such fuses are defective. ***(R.G.P.V., Dec. 2012, Nov. 2018)***

Sol. Here, $\quad m = np = 200 \times \dfrac{2}{100} = 4$

$$\therefore \quad e^{-4} = 1 - 4 + \frac{1}{2}.4^2 - \frac{1}{6}.4^3 + \frac{1}{24}.4^4 + \ldots\ldots = 0.0183$$

Hence $P(r \le 5) = \displaystyle\sum_{r=0}^{5} e^{-4}.\frac{4^r}{r!} = e^{-4}\left(1 + 4 + \frac{4^2}{2} + \frac{4^3}{6} + \frac{4^4}{24} + \frac{4^5}{120}\right)$

$= \mathbf{0.7845}$ **Ans.**

Prob.29. The mortality rate for a certain disease is 7 per 1000. What is the probability for just 5 deaths from this disease is a group of 400 ?

Sol. Here $\quad m = np = 400 \times \dfrac{7}{1000} = 2.8$ and r = 5

$\therefore$ Probability required $= \dfrac{e^{-2.8}(2.8)^5}{5!} = \mathbf{0.0872}$ **Ans.**

Prob.30. A Poisson distribution has a double mode at x = 3 and x = 4, what is the probability that x will have one or the other of these two values.

Sol. We know that 2 modes are given when m is an integer and in that case modes are m – 1 and m,

$\therefore \quad m - 1 = 3$, i.e. $m = 4$

$\therefore$ Probability when x is 3 $= P(X = x = 3) = \dfrac{e^{-4}4^3}{3!}$

$\therefore$ Probability when x is 4 $= P(X = x = 4) = \dfrac{e^{-4}4^4}{4!} = \dfrac{e^{-4}4^3}{3!}$

Required probability $= \dfrac{e^{-4}4^3}{3!} + \dfrac{e^{-4}4^3}{3!} = \mathbf{0.391}$ **Ans.**

Prob.31. Show that for the Poisson distribution with mean m

$$\mu_{r+1} = r\,m\,\mu_{r-1} + m\,\frac{d\mu_r}{dm}$$

where $\mu_r = \displaystyle\sum_{x=0}^{\infty} (x - m)^r \frac{e^{-m}m^x}{x!}$.

Sol. Here

$$\frac{d\mu_r}{dm} = -r\sum_{x=0}^{\infty} (x - m)^{r-1} \frac{e^{-m}m^x}{x!} + \sum_{x=0}^{\infty} (x - m)^r \left[\frac{xm^{x-1}e^{-m} - m^x e^{-m}}{x!}\right]$$

$$\therefore \quad \frac{d\mu_r}{dm} = -r\,\mu_{r-1} + \sum_{x=0}^{\infty} \frac{(x - m)^r e^{-m} m^{x-1}(x - m)}{x!}$$

$$\therefore \quad m\frac{d\mu_r}{dm} + mr\,\mu_{r-1} = \sum_{x=0}^{\infty} \frac{(x - m)^{r+1} e^{-m} m^x}{x!} = \mu_{r+1}$$ **Proved**

Prob.32. Find the mean and variance of Poisson's distribution. ***[R.G.P.V., June 2017, May 2019 (IV-Sem), Nov. 2019 (III-Sem)]***

Sol. Refer to the matter given on page 276 and 277.

Prob.33. An insurance company found that only 0.01% of the population is involved in a certain type of accident each year. If its 1000 policy holders were randomly selected from the population, what is the probability that not more than two of its clients are involved in such an accident next year ? (Given that $e^{-0.1} = 0.9048$).

Sol. Given the probability of an accident is

$$p = 0.01\% \text{ or } 0.0001$$

Number of people selected from population n = 1000

then, mean m = np

$$m = 1000 \times 0.0001 = 0.1$$

Probability that not more than 2 clients involved in such an accident next year

= Probability of 0 client involved

+ Probability of 1 client involved

+ Probability of 2 clients involved

$$= \frac{e^{-m}.m^0}{0!} + \frac{e^{-m}(m)^1}{1!} + \frac{e^{-m}(m)^2}{2!}$$

$$= \frac{e^{-0.1}.(0.1)^0}{0!} + \frac{e^{-0.1}(0.1)^1}{1!} + \frac{e^{-0.1}(0.1)^2}{2!}$$

$$= 0.9048 + 0.9048 \times 0.1 + \frac{0.9048 \times 0.01}{2}$$

$$= 0.9048 + 0.0905 + 0.0045 = \mathbf{0.9998} \quad \textbf{Ans.}$$

Prob.34. A manufacturer who produces medicine bottles, finds that 0.1% of the bottles are defective. The bottles are packed in boxes containing 500 bottles. A drug manufacturer buys 100 boxes from the producer of bottles. Using Poisson distribution, find how many boxes will contain no defectives. (Given $e^{-0.5} = 0.6065$). ***(R.G.P.V., Dec. 2015)***

Sol. Let p = Probability of the bottles to be defective = 0.1% = 0.001

Number of bottles in box, n = 500

Number of boxes, N = 100

$\therefore$ m = np = 500 × 0.001 = 0.5

Let X denotes the number of defective bottles in a box of 500 bottles. Then X is a poisson variate with parameter m = 0.5.

$$\therefore \quad P(X = r) = \frac{e^{-0.5}(0.5)^r}{r!}, \; r = 0, 1, 2, \ldots, 500$$

Hence, probability of no defective bottle = P(X = 0)

$$\frac{e^{-m}(m)^0}{0!} = \frac{e^{-0.5}(0.5)^0}{0!} = e^{-0.5} = 0.6065 \quad (\text{Given } e^{-0.5} = 0.6065)$$

Then the number of boxes in which no bottles are defective

$$= Ne^{-m}$$

$$= 100 \times e^{-0.5} = 100 \times 0.6065 = 60.65$$

$$= \mathbf{60 \text{ (Approx.)}} \quad \textbf{Ans.}$$

Prob.35. A car hire firm has two cars which it hires out day by day. The number of demands for a car on each day is distributed as Poisson distribution with mean 1.5. Calculate the proportion of days on which neither car is used and the proportion of days on which same demand is refused ($e^{-1.5} = 0.2231$).

Sol. The proportion of days when no car will be required is

$$= e^{-m}.\frac{m^0}{0!} = 0.2231$$

The probability when no car, one car, two cars will required is

$$= e^{-m}\frac{m^0}{0!} + e^{-m}\frac{m^1}{1!} + e^{-m}\frac{m^2}{2!}$$

$$= e^{-1.5}\left(1 + 1.5 + \frac{1.5^2}{2}\right)$$

$$= 0.2231\,(1 + 1.5 + 1.125)$$

$$= 0.2231(3.625) = 0.8087$$

The proportion of days on which some demand is refused

$$= 1 - 0.8087 = \mathbf{0.1913} \quad \textbf{Ans.}$$

Prob.36. Using Poisson distribution, find the probability that the ace of spades will be drawn from a pack of well-shuffled cards at least once in 104 consecutive trials.

Sol. Probability of the ace of spades, $P = \frac{1}{52}$, n = 104

$$\therefore \quad m = np = 104 \times \frac{1}{52} = 2$$

$$\therefore \quad P(r) = \frac{e^{-m}.m^r}{r!} = \frac{e^{-2}.2^r}{r!} = \frac{1}{e^2}.\frac{2^r}{r!}$$

Thus, P(at least once) = P(1) + P(2) + P(3) + ...+ P(104)

$$= 1 - P(0)$$

$$= 1 - \frac{1}{e^2} \times \frac{2^0}{0!} = 1 - \frac{1}{e^2} = 1 - 0.135 = \mathbf{0.865} \textbf{ Ans.}$$

Prob.37. If the probability of a bad reaction from certain injection is 0.001. Determine the chance that out of 2000 individuals more than two will get a bad reaction. ***(R.G.P.V., Dec. 2014)***

Sol. Given, the probability of a bad reaction is p = 0.001 (very small)

n = 2000 (very large)

$\therefore$ Mean, $m = np = 2000 \times 0.001 = 2$

$\therefore$ $$P(X = r) = \frac{e^{-m} m^r}{r!}, \; r = 0, 1, 2$$

Thus required probability,

$$P(X > 2) = 1 - P(X \le 2)$$
$$= 1 - [P(X=0) + P(X=1) + P(X=2)]$$
$$= 1 - \left[e^{-2}\frac{2^0}{0!} + e^{-2}\frac{2^1}{1!} + e^{-2}\frac{2^2}{2!}\right] = 1 - e^{-2}(1+2+2)$$
$$= 1 - \frac{5}{e^2} = 1 - \frac{5}{7.389} = 1 - 0.6767 = \mathbf{0.3233} \text{ Ans.}$$

Prob.38. If X_1 and X_2 be two dependent random variables with Poisson distribution with parameters m_1 and m_2 respectively then the sum $X_1 + X_2$ is a random variable with Poisson distribution with parameter $m_1 + m_2$.

Sol. Poisson's distribution for

$$P(X_1 = r_1) = \frac{m_1^{r_1}}{r_1!} e^{-m_1} \text{ and } P(X_2 = r_2) = \frac{m_2^{r_2}}{r_2!} e^{-m_2}$$

Then,

$$P(X_1 = r_1, X_2 = r_2) = \frac{m_1^{r_1}}{r_1!} e^{-m_1} \frac{m_2^{r_2}}{r_2!} e^{-m_2} = e^{-(m_1+m_2)} \frac{m_1^{r_1}.m_2^{r_2}}{r_1!r_2!}$$

$$P[X_1 + X_2 = (r_1 + r_2) = r] = \sum_{r_1+r_2=r} e^{-(m_1+m_2)} \frac{m_1^{r_1}.m_2^{r_2}}{r_1!r_2!}$$

$$= e^{-(m_1+m_2)} \sum_{r_1+r_2=r} \frac{m_1^{r_1}.m_2^{r_2}}{r_1!r_2!} = e^{-(m_1+m_2)} . \frac{(m_1+m_2)^r}{r_1!} \quad \textbf{Proved}$$

Prob.39. The random variable X has a Poisson distribution if P(X = 1) = 0.01487, P(X = 2) = 0.04461. Then find P(X = 3).

[R.G.P.V., May 2019 (III-Sem)]

Sol. Given $P(X = 1) = 0.01487$ and $P(X = 2) = 0.04461$.

We know that by Poisson distribution

$$P(X = r) = \frac{e^{-m}.m^r}{r!}$$

$\therefore$ $$P(X = 1) = \frac{e^{-m}.m^1}{1!} = 0.01487$$

or $$e^{-m}.m = 0.01487 \quad ...(i)$$

and $$P(X = 2) = \frac{e^{-m}.m^2}{2!} = 0.04461$$

or $$e^{-m}.m^2 = 2 \times 0.04461 = 0.08922 \quad ...(ii)$$

Dividing equation (ii) by equation (i), we get

$$\frac{e^{-m}.m^2}{e^{-m}.m} = \frac{0.08922}{0.01487}$$

or $$m = 6$$

Hence $$P(X = 3) = \frac{e^{-m}.m^3}{3!} = \frac{e^{-6}.6^3}{3 \times 2 \times 1} = 36e^{-6} = \mathbf{0.08923} \quad \textbf{Ans.}$$

Prob.40. Fit a Poisson's distribution and calculate theoretical frequencies for the following data –

x:	0	1	2	3	4
y:	46	38	22	9	1

[R.G.P.V., Nov. 2019 (IV-Sem)]

Or

Fit Poisson distribution to the following data –

x:	0	1	2	3	4
y:	46	38	22	9	1

(R.G.P.V., June 2013)

Sol. Here,

$$\text{Mean, } m = \frac{\Sigma fx}{N} = \frac{\Sigma fx}{\Sigma f} = \frac{0 \times 46 + 1 \times 38 + 2 \times 22 + 3 \times 9 + 4 \times 1}{46 + 38 + 22 + 9 + 1}$$
$$= \frac{113}{116} = 0.97$$

$\therefore$ $$e^{-m} = e^{-0.97} = \left[1 - 0.97 + \frac{(0.97)^2}{2!} - \frac{(0.97)^3}{3!} + \frac{(0.97)^4}{4!} - .....\right]$$
$$= [1 - 0.97 + 0.47 - 0.152 + 0.0369 .....] = 0.38 \text{ (approx.)}$$

Hence, the theoretical frequency for r successes is

$$Ne^{-m}\frac{m^r}{r!} = 116 \times 0.38 \times \frac{(0.97)^r}{r!}, \text{ where } r = 0, 1, 2, 3, 4$$

$$NP(0) = 116 \times 0.38 \times \frac{(0.97)^0}{0!} = 116 \times 0.38 = 44.08 = 44 \text{ (say)}$$

$$NP(1) = 116 \times 0.38 \times \frac{(0.97)^1}{1!} = 116 \times 0.369 = 42.80 = 43 \text{ (say)}$$

$$NP(2) = 116 \times 0.38 \times \frac{(0.97)^2}{2!} = 116 \times 0.179 = 20.76 = 21 \text{ (say)}$$

$$NP(3) = 116 \times 0.38 \times \frac{(0.97)^3}{3!} = 116 \times 0.058 = 6.73 = 7 \text{ (say)}$$

$$NP(4) = 116 \times 0.38 \times \frac{(0.97)^4}{4!} = 116 \times 0.014 = 1.62 = 2 \text{ (say)}$$

Thus it gives frequencies **44, 43, 21, 7** and **2** respectively for r = 0, 1, 2, 3 and 4.

Ans.

Prob.41. Fit a Poisson's distribution to the following calculate theoretical frequencies.

Deaths :	*0*	*1*	*2*	*3*	*4*
Frequencies :	*122*	*60*	*15*	*2*	*1*

Sol. Here,

$$\text{Mean} = m = \frac{\Sigma fx}{\Sigma f} = \frac{0 \times 122 + 1 \times 60 + 2 \times 15 + 3 \times 2 + 4 \times 1}{200}$$

$$= \frac{60+30+6+4}{200} = \frac{100}{200} = \frac{1}{2} = 0.5$$

$$\therefore \quad e^{-m} = e^{-0.5} = \left[1 - \frac{0.5}{1!} + \frac{(0.5)^2}{2!} - \frac{(0.5)^3}{3!} + \ldots\ldots\ldots\ldots\right]$$

$$= [1 - 0.5 + 0.125 - 0.0208 + \ldots\ldots\ldots] = 0.61 \text{ (Approx.)}$$

The theoretical frequency of r death is

$$= N.\ e^{-m}\frac{m^r}{r!} = 200 \times (0.61) \times \frac{(0.5)^r}{r!}, \text{ where } r = 0, 1, 2, 3, 4$$

$$NP(0) = 200 \times 0.61 \times \frac{(0.5)^0}{0!} = 200 \times 0.61 = 122$$

$$NP(1) = 200 \times 0.61 \times \frac{(0.5)^1}{1!} = 200 \times 0.305 = 61$$

$$NP(2) = 200 \times 0.61 \times \frac{(0.5)^2}{2!} = 200 \times 0.076 = 15.2 = 15 \text{ (say)}$$

$$NP(3) = 200 \times 0.61 \times \frac{(0.5)^3}{3!} = 200 \times 0.0127 = 2.54 = 2 \text{ (say)}$$

$$NP(4) = 200 \times 0.61 \times \frac{(0.5)^4}{4!} = 200 \times 0.0016 = 0.32 = 0 \text{ (say)}$$

Thus it gives frequencies **122, 61, 15, 2** and **0** respectively for r = 0, 1, 2, 3 and 4.

Ans.

Prob.42. If the probability that an individual suffers a bad reaction from a certain injection is 0.001, determine the probability that out of 2000 individuals,

(i) exactly 3 (ii) more then 2 individuals (iii) none (iv) more than 1 individual will suffer a bad reaction.

Sol. Here p = 0.001, n = 2000

$$\therefore \quad m = np = 2000 \times 0.001 = 2 \text{ then } P(r) = \frac{e^{-m}.m^r}{r!} = \frac{e^{-2}2^r}{r!}$$

(i) The probability that 3 individuals suffer from bad reaction

$$= \frac{e^{-2}2^3}{3!} = \frac{4}{3e^2} = \mathbf{0.18}$$

Ans.

(ii) P (more than 2) = P(3) + P(4) + P(5) + ...+ P(2000)

$$= 1 - [P(0) + P(1) + P(2)]$$

$$= 1 - \left[\frac{e^{-2}(2)^0}{0!} + \frac{e^{-2}(2)^1}{1!} + \frac{e^{-2}(2)^2}{2!}\right]$$

$$= 1 - e^{-2}[1 + 2 + 2] = 1 - \frac{5}{e^2} = 1 - 5 \times 0.1353 = \mathbf{0.323}$$

Ans.

(iii) The probability that none suffers from bad reaction

$$= \frac{e^{-2}.2^0}{0!} = e^{-2} = \mathbf{0.1353}$$

Ans.

(iv) The probability of more then 1 suffer from bad reaction

= 1 – (Probability of 0 + Probability of 1)

$$= 1 - \left(\frac{1}{e^2} + \frac{2}{e^2}\right) = 1 - \frac{3}{e^2} = \mathbf{0.6}$$

Ans.

Prob.43. Show that in a Poisson distribution with unit mean, the mean deviation about the mean is 2/e times the standard deviation.

Sol. If P(r) denote probability of r successes, then

$$P(r) = \frac{m^r e^{-m}}{r!} \quad \text{....(i)}$$

Given, $m = 1$

Putting the above value in equation (i), then we get

$$P(r) = \frac{(1)^r e^{-1}}{r!} = \frac{e^{-1}}{r!}$$

We know that mean deviation about the mean.

$$= \frac{\sum_{r=0}^{\infty} f|r - m|}{\Sigma f} = \frac{\Sigma f|r - 1|}{\Sigma f} \quad [\because m = 1]$$

Now, $f = P(r) = \frac{e^{-1}}{r!} \Rightarrow \Sigma f = \sum_{r=0}^{\infty} \frac{e^{-1}}{r!} = e^{-1}\sum_{r=0}^{\infty}\frac{1}{r!} = 1$

Thus required mean deviation $= \frac{\sum_{r=0}^{\infty} f|r-1|}{\Sigma f} = \sum_{r=0}^{\infty} f|r-1|$ $\quad [\because \Sigma f = 1]$

$$= \sum_{r=0}^{\infty}\frac{e^{-1}}{r!}|r-1| = e^{-1}\left[\frac{|0-1|}{0!}+\frac{|1-1|}{1!}+\frac{|2-1|}{2!}+\frac{|3-1|}{3!}+\ldots\ldots\ldots\right]$$

$$= e^{-1}\left[1+\frac{2-1}{2!}+\frac{3-1}{3!}+\ldots\ldots\ldots+\infty\right] = e^{-1}\left[1+\left(1-\frac{1}{2!}\right)+\left(\frac{1}{2!}-\frac{1}{3!}\right)+\ldots\ldots+\infty\right]$$

$$= e^{-1}\left[1+\frac{1}{1!}+\frac{1}{2!}+\frac{1}{3!}+\ldots\ldots\ldots+\infty-\left(\frac{1}{2!}+\frac{1}{3!}+\ldots\ldots\infty\right)\right] = e^{-1}[e-(e-2)] = \frac{2}{e}$$

Since, S.D $= \sqrt{m} = 1$, we have mean deviation about the mean $= \frac{2}{e}$ times S.D. **Proved**

CONTINUOUS DISTRIBUTION – NORMAL DISTRIBUTION, EXPONENTIAL DISTRIBUTION

Continuous Distribution – So far we have dealt with discrete distributions where the variate takes only the integral values. But the variates like temperatures, heights and weights can take all values in a given intervals. Such variable are said to be ***continuous variables***.

Suppose f(x) is a continuous function, then

$$\text{Mean} = \int_{-\infty}^{+\infty} x.f(x)\,dx$$

and $$\text{Variance} = \int_{-\infty}^{+\infty}(x-\bar{x})^2.f(x)\,dx \qquad (\because \bar{x} = \text{mean})$$

Normal Distribution – Normal distribution is a continuous distribution. It is derived as the limiting form of the Binomial distribution for large values of n and, p and q are not very small.

The normal distribution is given by the equation

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}}e^{-\frac{(x-\mu)^2}{2\sigma^2}} \qquad \ldots(i)$$

where, μ = Mean, σ = Standard deviation.
$e = 2.71828\ldots\ldots$, $\pi = 3.14159\ldots$

$$P(x_1 < x < x_2) = \int_{x_1}^{x_2}\frac{1}{\sigma\sqrt{2\pi}}e^{-\frac{(x-\mu)^2}{2\sigma^2}}\,dx$$

On putting $z = \frac{x-\mu}{\sigma}$ in relation (i), we get $f(z) = \frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}z^2}$ ....(ii)

Here, mean = 0, standard deviation = 1.

Relation (ii) is known as standard form of normal distribution.

Note – Moment generating function of the continuous probability distribution about x = a is given by

$$M_a(t) = \int_{-\infty}^{\infty} e^{t(x-a)} f(x)\,dx \qquad \text{where, f(x) is p.d.f.}$$

Normal Curve – Let us show Binomial distribution graphically. The probabilities of heads in 1 tosses are ${}^{10}C_0q^{10}p^0$, ${}^{10}C_1q^9p^1$, ${}^{10}C_2q^8p^2$, ${}^{10}C_3q^7p^3$, ${}^{10}C_4q^6p^4$, ${}^{10}C_5q^5p^5$, ${}^{10}C_6q^4p^6$, ${}^{10}C_7q^3p^7$, ${}^{10}C_8q^2p^8$, ${}^{10}C_9q^1p^9$, ${}^{10}C_{10}q^0p^{10}$. $p = \frac{1}{2}, q = \frac{1}{2}$. It is shown in the fig. 5.2.

If the variate (heads hear) are treated as if they were continuous, the required probability curve will be a ***normal curve*** as shown in above fig. 5.2 by dotted line.

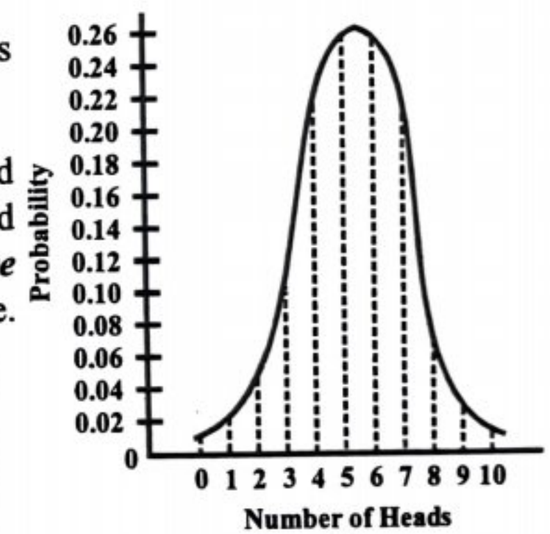

Fig. 5.2

Properties of the Normal Curve

$$\left(y = y_0 e^{-\frac{x^2}{2\sigma^2}}.\right)$$

(i) The mean, median and mode concide at origin. The curve is symmetrical about the x-axis.

(ii) If mean (origin of x) and standard deviation are given, the curve is drawn. The value of y_0 can be evaluated from the fact that the area of the curve must be equal to the total number of observations.

(iii) As x increases numerically, y decrease rapidly. The curve extends to infinity on either side of the origin.

(iv) (a) $P(\mu-\sigma < x < \mu+\sigma) = 68\%$

(b) $P(\mu-2\sigma < x < \mu+2\sigma) = 95.5\%$

(c) $P(\mu-3\sigma < x < \mu+3\sigma) = 99.7\%$

Mean and Standard Deviation for Normal Distribution –

Here,
$$\text{Mean} = \int_{-\infty}^{+\infty} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{x^2}{2\sigma^2}} x\,dx \quad \text{...(i)}$$

Substituting $\frac{x}{\sigma} = t$ so that $dx = \sigma\,dt$ in relation (i), we have

$$\text{Mean} = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{-t^2/2}(t\sigma)(\sigma\,dt) = \frac{\sigma}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} t.e^{-t^2/2}dt$$

$$= \frac{\sigma}{\sqrt{2\pi}}\left[e^{-t^2/2}\right]_{-\infty}^{+\infty} = \frac{\sigma}{\sqrt{2\pi}}[0] = 0$$

i.e., **Mean = 0**

Now we know that

$$\mu_2' = \int x^2 f(x)\,dx = \int_{-\infty}^{+\infty} x^2 \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{x^2}{2\sigma^2}} dx$$

Substituting $\frac{x^2}{2\sigma^2} = t$ or $x = \sqrt{2}\,\sigma\, t^{1/2}$ so that $dx = \frac{\sqrt{2}\,\sigma\,dt}{2t^{1/2}}$

Then

$$\mu_2' = \int_{-\infty}^{+\infty} (2\sigma^2 t).\frac{1}{\sigma\sqrt{2\pi}} e^{-t}\left(\frac{\sqrt{2}\sigma}{2t^{1/2}}dt\right)$$

$$= \frac{2\sigma^2}{\sigma\sqrt{2\pi}}.\frac{\sqrt{2}\sigma}{2}\int_{-\infty}^{+\infty} t^{3/2-1}e^{-t}dt$$

$$= \frac{\sigma^2}{\sqrt{\pi}}.2\Gamma\frac{3}{2} = 2\frac{\sigma^2}{\sqrt{\pi}}.\frac{1}{2}\Gamma\frac{1}{2} = \frac{\sigma^2}{\sqrt{\pi}}\sqrt{\pi} = \sigma^2$$

$\because \quad \mu_2 = \mu_2' - (\mu_1)^2 = \sigma^2 - 0 = \sigma^2$

Hence, **Standard deviation (i.e., S.D.) = σ**

Moment for a Normal Curve –

About the origin

$$\mu_n' = \int_{-\infty}^{\infty} \frac{1}{\sigma\sqrt{2\pi}} e^{-x^2/2\sigma^2}.x^n dx$$

Substituting $\frac{x}{\sigma} = t$, we get

$$\mu_n' = \frac{1}{\sigma\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\frac{1}{2}t^2}\sigma^{n+1}t^n dt = \frac{\sigma^n}{\sqrt{2\pi}}\int_{-\infty}^{\infty} t^n e^{-\frac{1}{2}t^2} dt = 0$$

when n is odd by the properties of definite integrals.

Hence, all the moments of odd order about the origin vanish in a normal distribution.

$$\mu_3' = 0 = \mu_5' = \ldots\ldots$$

Let the mean be the origin, then all odd order moments about the mean vanish.

Let n be even say 2p, then

$$\mu_{2p}' = \frac{1}{\sigma\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-x^2/2\sigma^2} x^{2p} dx$$

On putting $\frac{x^2}{2\sigma^2} = t$ and simplifying, we get

$$\mu_{2p}' = \frac{2^p\sigma^{2p}}{\sqrt{\pi}}\int_0^{\infty} e^{-t}t^{p+\frac{1}{2}-1}dt = \frac{2^p\sigma^{2p}}{\sqrt{\pi}}\Gamma\left(p+\frac{1}{2}\right)$$

$$= (2p-1)(2p-3)\ldots\ldots 3.1.\sigma^{2p}\mu_0$$

$$= (2p-1)(2p-3)\ldots\ldots 3.1.\sigma^{2p} \text{ as } \mu_0 = 1$$

$$= \frac{2^{n/2}\sigma^n}{\sqrt{\pi}}\Gamma\left(\frac{n+1}{2}\right) = \frac{\sigma^n(\sqrt{2})^n}{\sqrt{\pi}}\Gamma\left(\frac{n+1}{2}\right)$$

Hence, when n = 2

$$\mu_2' = \frac{\sigma^2.2\Gamma\left(\frac{3}{2}\right)}{\sqrt{\pi}} = \sigma^2$$

$$\mu_4' = 3\sigma^4, \quad \mu_{2p}' = \frac{\sigma^{2p}2^p}{\sqrt{\pi}}\Gamma\frac{(2p+1)}{2}$$

Also $\quad \mu_0' = \frac{\sigma^0.1}{\sqrt{\pi}}\Gamma\left(\frac{1}{2}\right) = 1$

Moments about the Mean, m –

Since $\mu_{2n+1} = \int_{-\infty}^{\infty} (x-m)^{2n+1}\frac{1}{\sigma\sqrt{2\pi}}e^{-\frac{1}{2}\left(\frac{x-m}{\sigma}\right)^2}dx$

$$= \frac{\sigma^{2n+1}}{\sqrt{2\pi}}\int_{-\infty}^{\infty} t^{2n+1}e^{-\frac{1}{2}t^2}dt \quad \text{as, } t = \frac{x-m}{\sigma}$$

$= 0$, since the integral is an odd function of t.

$\therefore \quad \mu_{2n+1} = 0.$

Here in a normal distribution, all the moments of odd number about the mean vanish.

$$\mu_{2n} = \frac{1}{\sigma\sqrt{2\pi}}\int_{-\infty}^{\infty}(x-m)^{2n}e^{-\frac{1}{2}\left(\frac{x-m}{\sigma}\right)^2}dx = \frac{\sigma^{2n}}{\sqrt{2\pi}}\int_{-\infty}^{\infty} t^{2n-1}.t.e^{-\frac{1}{2}t^2}dt$$

$$= \frac{\sigma^{2n}}{\sqrt{2\pi}}\left[-t^{2n-1}.e^{-\frac{1}{2}t^2}\right]_{-\infty}^{\infty} + \frac{\sigma^{2n}}{\sqrt{2\pi}}(2n-1)\int_{-\infty}^{\infty} t^{2n-2}e^{-\frac{1}{2}t^2}dt$$

$$= (2n-1)\sigma^2\mu_{2n-2}$$

Hence $\quad \mu_{2n} = (2n-1)\sigma^2\mu_{2n-2}$

Other Constants –

$$\beta_1 = \frac{\mu_3^2}{\mu_2^3} = 0, \text{ and so}$$

$$\gamma_1 = \sqrt{\beta_1} = 0; \quad \beta_2 = \frac{\mu_4}{\mu_2^2} = \frac{3\sigma^4}{\sigma^4} = 3$$

and $\gamma_2 = 0$

i.e., the normal curve has zero kurtosis.

Probable Error – Any lot of articles manufactured to certain specification is subject to small errors. In fact, measurement of any physical quantity shows slight error. These errors of experiment or manufacture are of random nature and therefore, follow a normal distribution. We usually mention the probable error (λ), while quoting a specification of an experimental result. It is such that the probability of an error failing within the limits $m - \lambda$ and $m + \lambda$ is exactly equal to the chance of an error falling outside these limits, i.e. the chance of an error lying within $m - \lambda$ and $m + \lambda$ is 1/2

Therefore $$\frac{1}{\sigma\sqrt{2\pi}}\int_{m-\lambda}^{m+\lambda} e^{-\frac{(x-m)^2}{2\sigma^2}}\,dx = \frac{1}{2}$$

or $$\frac{1}{\sqrt{2\pi}}\int_0^{\lambda/\sigma} e^{-z^2/2}\,dz = \frac{1}{4} \qquad \left[\because\ z = \frac{x-m}{\sigma}\right]$$

From the value of $\frac{\lambda}{\sigma} = 0.6745$

Hence the probable error $\lambda = 0.6745\sigma \sim \frac{2}{3}\sigma$.

Obs. Quartile deviation $= \frac{1}{2}(Q_3 - Q_1) \sim \frac{2}{3}\sigma$.

Mean deviation $= \frac{4}{5}\sigma$

$\therefore$ Q.D : M.D : S.D. $= \frac{2}{3} : \frac{4}{5} : 1$.

Fitting of a Normal Curve –

Let total frequencies be N, the normal curve is

$$y = \frac{N}{\sigma\sqrt{2\pi}} e^{-(x-m)^2/2\sigma^2}$$

Sometimes it is necessary to fit a normal curve to a distribution which is reasonably symmetrical. Suppose it is required to fit to the given data the above curve. First of all we suppose that,

(i) the area under the curve is equal to N, which is the total frequency of the given frequency distribution.

(ii) from the given frequency distribution the mean m and standard deviation σ calculated are equal to the mean and standard deviation of the theoretical normal distribution which is to be fitted.

In order to fit a curve we obtain the theoretical frequencies corresponding to the given values of x. We first calculate the values of y corresponding to the given values of x by the relation

$$y = \frac{(x-m)}{\sigma}$$

where m and σ are the mean and the standard deviation of the frequency distribution. After this write the value of $\phi(y)$ corresponding to the values of y, from the tables. The ordinate y for any value of $\phi(y)$ is found by the relation.

$$y = \frac{N}{\sigma}\phi(y)$$

Thus we get a number of values of y corresponding to a number of the values of x. They can plotted on a graph paper and smooth frequency can be drawn through these points.

Table 5.1 Area under Standard Normal Curve from 0 to $\frac{x-m}{\sigma}$

$\frac{x-m}{\sigma}$	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	.0000	.0040	.0080	.0120	.0159	.0199	.0239	.0279	.0319	.0359
0.1	.0398	.0438	.0478	.0517	.0557	.0596	.0636	.0675	.0714	.0753
0.2	.0793	.0832	.0871	.0910	.0948	.0987	.1026	.1084	.1103	.1141
0.3	.1179	.1217	.1255	.1293	.1331	.1368	.1406	.1443	.1480	.1517
0.4	.1554	.1591	.1628	.1664	.1700	.1736	.1772	.1808	.1844	.1879
0.5	.1915	.1950	.1985	.2019	.2054	.2088	.2123	.2157	.2190	.2224
0.6	.2257	.2291	.2324	.2357	.2389	.2422	.2454	.2486	.2518	.2549
0.7	.2580	.2611	.2642	.2671	.2704	.2734	.2764	.2794	.2823	.2852
0.8	.2881	.2910	.2939	.2967	.2995	.3023	.3051	.3078	.3106	.3133
0.9	.3159	.3186	.3212	.3238	.3264	.3289	.3315	.3340	.3365	.3389
1.0	.3413	.3438	.3461	.3485	.3508	.3531	.3554	.3577	.3599	.3621
1.1	.3643	.3665	.3686	.3708	.3729	.3749	.3770	.3790	.3810	.3830
1.2	.3849	.3869	.3888	.3907	.3925	.3944	.3962	.3980	.3997	.4015
1.3	.4032	.4049	.4066	.4082	.4099	.4115	.4131	.4147	.4162	.4177
1.4	.4192	.4207	.4222	.4236	.4251	.4265	.4279	.4292	.4306	.4319
1.5	.4332	.4345	.4357	.4370	.4382	.4394	.4406	.4418	.4430	.4441
1.6	.4452	.4463	.4474	.4485	.4495	.4505	.4515	.4525	.4535	.4545
1.7	.4554	.4564	.4573	.4582	.4591	.4599	.4608	.4616	.4625	.4633
1.8	.4641	.4649	.4656	.4664	.4671	.4678	.4686	.4693	.4699	.4706
1.9	.4713	.4719	.4726	.4732	.4738	.4744	.4750	.4756	.4762	.4767
2.0	.4772	.4778	.4783	.4788	.4793	.4798	.4803	.4808	.4812	.4817
2.1	.4821	.4826	.4830	.4834	.4838	.4842	.4846	.4850	.4854	.4857
2.2	.4861	.4865	.4868	.4871	.4875	.4878	.4881	.4884	.4887	.4890
2.3	.4893	.4896	.4898	.4901	.4904	.4906	.4909	.4911	.4913	.4916

2.4	.4918	.4920	.4922	.4925	.4927	.4929	.4931	.4932	.4934	.4936
2.5	.4938	.4940	.4941	.4943	.4945	.4946	.4948	.4949	.4951	.4952
2.6	.4953	.4955	.4956	.4957	.4959	.4960	.4961	.4962	.4963	.4964
2.7	.4965	.4966	.4967	.4968	.4969	.4970	.4971	.4972	.4973	.4974
2.8	.4974	.4975	.4976	.4977	.4977	.4978	.4979	.4980	.4980	.4981
2.9	.4981	.4982	.4983	.4983	.4984	.4984	.4985	.4985	.4986	.4986
3.0	.49865	.4987	.4987	.4988	.4988	.4989	.4989	.4989	.4990	.4990
3.1	.49903	.4991	.4991	.4991	.4992	.4992	.4992	.4992	.4993	.4993

Application of Normal Distribution – Normal distribution is applied to problem concerning –

(i) Calculation of errors made by chance in experimental measurements.
(ii) Computation of hit probability of a shot.
(iii) Statistical inference in almost every branch of science.

Exponential Distribution – The distribution of a random variable whose natural logarithm follows a normal distribution is called as the exponential distribution.

The exponential density function is given by

$$f(x) = \frac{1}{\sigma x\sqrt{2\pi}} e^{-(\log x-\mu)^2/2\sigma^2}$$

where the range of random variable is $x > 0$. The parameters μ and σ are defined as

$$\mu = E(\log X)$$

and

$$\sigma^2 = V(\log X)$$

Mean and Variance – The mean and variance of exponential distribution are

$$E(X) = e^{\mu+(1/2\sigma 2)}$$

$$V(X) = e^{2\mu + \sigma 2}(e^{\sigma 2} - 1)$$

respectively.

Application of Exponential Distribution – The exponential distribution has been used to describe lifetimes of electrical and mechanical systems, abundance of species of animals, incubation period of infectious diseases, concentration of chemical elements in geological materials, and many other random phenomena occurring in both the social and natural sciences.

Q.5. Define normal distribution. Give the basic properties and the standard form of normal distribution. ***(R.G.P.V., June 2012)***

Ans. Refer matter given on page 296, 297 and 298, under same headings.

NUMERICAL PROBLEMS

Prob.44. Assume the mean height of soldiers to be 68.22 inches with a variance of 10.8 (inches)^2. How many soldiers in a regiment of 1000 would you expect to be over 6 feet tall ? Given that the area under the standard normal curve between $t = 0$ and $t = 0.35$ is 0.1368 and between $t = 0$ and $t = 1.15$ is 0.3746.

Sol. No. of soldiers = 1000, Mean height = 68.22 inches
Variance = 10.8 (inches)^2 or Standard deviation = 3.286 inches
Soldiers more than 6 feet or 72 inches = ?

$$\phi\ (0 \le t \le 0.35) = 0.1368$$

$$\phi(0 \le t \le 1.15) = 0.3746$$

$$\therefore \qquad z = \frac{x - \bar{x}}{\sigma} = \frac{72 - 68.22}{3.286} = 1.15$$

$$\phi(1.15) = 0.3746$$

For more than 1.15 = 1 – 0.3746 = 0.6254
Soldiers more than 72 inches = 1000 × 0.6254 = **625.4** **Ans.**

Prob.45. Find the mean deviation from mean for normal distribution. ***(R.G.P.V., June 2016)***

Sol. Refer to the matter given on page 298.

Prob.46. The probability density function p(x) of a random variable is given by $p(x) = y_0 e^{-|x|}$, $-\infty \le x \le \infty$. Find the value of y_0. ***(R.G.P.V., June 2014)***

Sol. Let us find the value of y_0, so that $y_0\int_{-\infty}^{\infty} e^{-|x|}dx = 1$

Since $e^{-|x|}$ is an even function, we have

$$2y_0\int_0^{\infty} e^{-x}dx = 1, \quad \text{as } 0 \le x \le \infty, |x| = x$$

$$2y_0 = 1$$

or

$$y_0 = \frac{1}{2}$$ **Ans.**

Prob.47. The life of army shoes is normally distributed with mean 8 months standard deviation 2 months. It 5000 pairs are issued how many pairs would be expected to need replacement after 12 months ? [Given that $P(z \ge 2) = 0.0228$ and $z = (x - m)/\sigma$].

Sol. Here, Mean, (m) = 8, Standard deviation (σ) = 2, Number of pairs of shoes = 5000, Total months (x) = 12.

When, $z = \dfrac{x - m}{\sigma} = \dfrac{12 - 8}{2} = 2$

Area ($z \ge 2$) = 0.0228

Number of pairs whose life is more than 12 months ($z > 2$)

= 5000 × 0.0228 = 114.

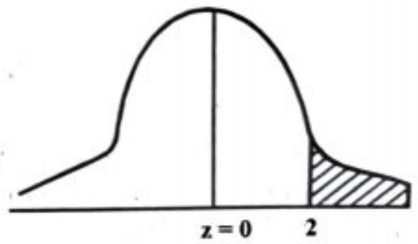

Fig. 5.3

Replacement after 12 months

= 5000 – 114 = **4886 pairs of shoes** **Ans.**

Prob.48. Prove that for the normal distribution, mean = median.

Sol. If a is the median then

$$\int_{-\infty}^{a} f(x)\,dx = \frac{1}{2} = \int_{a}^{\infty} f(x)\,dx,$$

where $f(x) = \dfrac{1}{\sigma\sqrt{2\pi}} e^{\frac{-(x-m)^2}{2\sigma^2}}$

Since $\displaystyle\int_{-\infty}^{m} f(x)\,dx = \frac{1}{2} = \int_{m}^{\infty} f(x)\,dx$

$\therefore$ $a = m$

i.e. **Median = Mean**

Proved

Prob.49. Prove that in a normal distribution, all the moments of odd order about the mean vanishes. ***(R.G.P.V., Dec. 2013)***

Sol. Refer to the matter given on page 299, under heading "Moments about the Mean, m".

Prob.50. Prove that the mean deviation from mean of a normal distribution is approximately $\frac{4}{5}$ times of standard deviation.

(R.G.P.V., June 2014)

Sol. We have, the mean deviation from the mean

$$= \int_{-\infty}^{\infty} |x-m| \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-m}{\sigma}\right)^2} dx$$

$$= \frac{\sigma}{\sqrt{2\pi}} \int_{-\infty}^{\infty} |z| e^{-\frac{1}{2}z^2} dz, \qquad \text{where } z = \frac{x-m}{\sigma}$$

$$= \frac{\sigma}{\sqrt{2\pi}} \left[\int_{-\infty}^{0} -z e^{-\frac{1}{2}z^2} dz + \int_{0}^{\infty} z e^{-\frac{1}{2}z^2} dz \right]$$

$$= \frac{2\sigma}{\sqrt{2\pi}} \int_{0}^{\infty} z e^{-\frac{1}{2}z^2} dz = \sigma\sqrt{\frac{2}{\pi}} = 0.7979\sigma = \frac{4}{5}\sigma \text{ (Approx.)}$$ **Proved**

Prob.51. Show that, for the normal distribution, the quartile deviation, the mean deviation from mean and the standard deviation are approximately in the ratio 10:12:15.

Sol. Let, $f(x) = \dfrac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-m}{\sigma}\right)^2}, -\infty < x < \infty$

Then we have

Quartile deviation, Q.D. $= \frac{2}{3}\sigma$

Mean deviation from mean, M.D. $= \frac{4}{5}\sigma$

where, σ = Standard deviation = S.D.

$\therefore$ Q.D. : M.D. : S.D. $= \frac{2}{3}\sigma : \frac{4}{5}\sigma : \sigma$

$= 10\,\sigma : 12\,\sigma : 15\,\sigma =$ **10 : 12 : 15** **Proved**

Prob.52. For a certain normal distribution the first moment about 10 is 40 and the fourth moment about 50 is 48. What is the arithmetic mean and variance of normal distribution ?

Sol. Suppose m is the arithmetic mean and σ^2 is the variance for the normal distribution.

$$\mu'(0) = E(x - 10)$$
$$= E(x) - 10 = m - 10 = 40 \text{ (given)}$$

Thus $m = 40 + 10 = 50.$

Again since, mean = 50, we have

$$\mu_4'(50) = \mu_4 = 3\sigma^4 = 48 \text{ (given)}$$

$\therefore$ $\sigma^4 = 16 \Rightarrow \sigma^2 = 4$

Hence, **mean = 50** and $\boldsymbol{\sigma^2 = 4}$ **Ans.**

Prob.53. If the probability of committing an error of magnitude x is given by

$$y = \frac{h}{\sqrt{\pi}} e^{-h^2 x^2}$$

Compute the probable error from the following data –

$m_1 = 1.305,\ m_2 = 1.301,\ m_3 = 1.295,\ m_4 = 1.286,$
$m_5 = 1.318,\ m_6 = 1.321,\ m_7 = 1.283,\ m_8 = 1.289,$
$m_9 = 1.300,\ m_{10} = 1.286.$

Sol. From the given data which is normally distributed, we have

$$\text{Mean} = \frac{1}{10}\sum m_i = \frac{12.984}{10} = 1.2984 \text{ and } \sigma^2 = \frac{1}{10}\sum (m_i - \text{Mean})^2$$

$$= \frac{1}{10}\,[(0.007)^2 + (0.003)^2 + (0.003)^2 + (0.012)^2 + (0.02)^2 + (0.023)^2 + (0.015)^2 + (0.009)^2 + (0.002)^2 + (0.012)^2]$$

or $\sigma^2 = 0.0001594$ where $\sigma = 0.0126$

$\therefore$ Probable error $= \frac{2}{3}\sigma =$ **0.0084 (Approx.)** **Ans.**

Prob.54. The marks of 500 candidates appearing in an english paper have a mean 41.6 and standard deviation 11.2. Assuming a normal distribution, find how many students obtain first class marks (for which a minimum of 60 marks is necessary).

Sol. The number of students getting first class marks

$$= \frac{500}{11.2\sqrt{2\pi}} \int_{60}^{\infty} e^{-(x-41.6)^2/[2(11.2)^2]} dx \qquad [\text{i.e., } 500 \times P(x \geq 60)]$$

Putting $\frac{x-41.6}{11.2} = z$, so that when $x = 60$, $z = 1.643$,

$$\text{The above integral} = \frac{500}{\sqrt{2\pi}} \int_{1.643}^{\infty} e^{-z^2/2} dz$$

$$= 500 \left[0.5 - \frac{1}{\sqrt{2\pi}} \int_{0}^{1.643} e^{-z^2/2} dz \right]$$

$$= 500\,[0.5 - 0.4495] \qquad \text{[by using table]}$$

$$= 500\,[0.5 - 0.45] = 500\,[0.05] = \mathbf{25\ (Approx.)} \qquad \textbf{Ans.}$$

Prob.55. The mean height of 500 students in 151 cm and the standard deviation is 15 cm. Assuming that the heights are normally distributed. Find how many students have heights between 120 and 155 cm.

(R.G.P.V., Dec. 2012, June 2015)

Sol. Here $m = 151$cm, $\sigma = 15$ cm, $N = 500$, we have

$$z = \frac{x-m}{\sigma} = \frac{x-151}{15}$$

When $x = 120$,

$$z = \frac{120-151}{15} = -2.1$$

and when $x = 155$,

$$z = \frac{155-151}{15} = 0.3$$

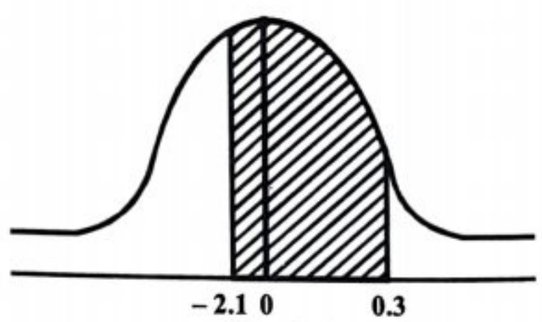

Fig. 5.4

Now,

$$P(120 < x < 155) = P(-2.1 < z < 0.3)$$
$$= (\text{Area between } z = -2.1 \text{ and } z = 0) + (\text{Area between } z = 0 \text{ and } z = 0.3)$$
$$= 0.4821 + 0.1179 = 0.6$$

Now the required no. of students $= 500 \times 0.6 = \mathbf{300}$ **Ans.**

Prob.56. Find the points of inflexion of the normal curve –

$$y = \frac{1}{\sigma\sqrt{2\pi}} \cdot e^{-\frac{1}{2}\left(\frac{x-m}{\sigma}\right)^2}$$

Sol. Here

$$y = \frac{1}{\sigma\sqrt{2\pi}} \cdot e^{-\frac{1}{2}\left(\frac{x-m}{\sigma}\right)^2} \qquad \text{...(i)}$$

At the point of inflexion,

$$\frac{d^2y}{dx^2} = 0 \text{ and } \frac{d^3y}{dx^3} \neq 0$$

Differentiating equation (i) w. r. t. x, we get

$$\frac{dy}{dx} = \frac{1}{\sigma\sqrt{2\pi}} \cdot e^{-\frac{1}{2}\left(\frac{x-m}{\sigma}\right)^2} \times -\left(\frac{x-m}{\sigma^2}\right)$$

and

$$\frac{d^2y}{dx^2} = -\frac{1}{\sigma^3\sqrt{2\pi}} \left[e^{-\frac{1}{2}\left(\frac{x-m}{\sigma}\right)^2} + (x-m) e^{-\frac{1}{2}\left(\frac{x-m}{\sigma}\right)^2} \times -\frac{(x-m)}{\sigma^2} \right]$$

$$\because \frac{d^2y}{dx^2} = 0 \quad \therefore e^{-\frac{1}{2}\left(\frac{x-m}{\sigma}\right)^2} \left[1 - \frac{(x-m)^2}{\sigma^2} \right] = 0$$

or

$$\left(\frac{x-m}{\sigma}\right)^2 = 1 \text{ or } (x-m)^2 = \sigma^2 \quad \therefore x = m \pm \sigma$$

Again,

$$\frac{d^3y}{dx^3} = \frac{1}{\sigma\sqrt{2\pi}} \left[\left\{ -\frac{(x-m)}{\sigma^2} \right\}^3 e^{-\frac{1}{2}\left(\frac{x-m}{\sigma}\right)^2} + \frac{3(x-m)}{\sigma^4} e^{-\frac{1}{2}\left(\frac{x-m}{\sigma}\right)^2} \right]$$

$$= \pm \frac{2}{\sigma^4\sqrt{(2\pi)}} e^{-\frac{1}{2}} \text{ on writing } m = x + \sigma \neq 0$$

When $x = m \pm \sigma$, $y = \frac{1}{\sigma\sqrt{2\pi}} \cdot e^{-\frac{1}{2}}$

The points of inflexion are $\left(m \pm \sigma, \frac{1}{\sigma\sqrt{2\pi}} \cdot e^{-\frac{1}{2}} \right)$ **Ans.**

Prob.57. Find the point of inflexion of the normal curve $y = y_0 e^{\frac{-x^2}{2\sigma^2}}$ and show that it occurs at a distance σ from the mean ordinate.

Sol. The normal curve is

$$y = y_0 e^{-\frac{x^2}{2\sigma^2}}$$

For a point of inflexion $\frac{d^2y}{dx^2} = 0$ and $\frac{d^3y}{dx^3} \neq 0$

$$\frac{dy}{dx} = y_0 e^{-\frac{x^2}{2\sigma^2}}\left(-\frac{2x}{2\sigma^2}\right) = -\frac{y_0}{\sigma^2}(x\, e^{-x^2/2\sigma^2})$$

$$\frac{d^2y}{dx^2} = -\frac{y_0}{\sigma^2}\left[x\, e^{-\frac{x^2}{2\sigma^2}}\left(-\frac{2x}{2\sigma^2}\right) + e^{-\frac{x^2}{2\sigma^2}}\right]$$

or
$$\frac{d^2y}{dx^2} = \frac{y_0}{\sigma^2} e^{-\frac{x^2}{2\sigma^2}}\left[\frac{x^2}{\sigma^2} - 1\right]$$

$$\because \quad \frac{d^2y}{dx^2} = 0$$

$$\therefore \quad \frac{x^2}{\sigma^2} - 1 = 0$$

or
$$\frac{x^2}{\sigma^2} = 1 \text{ or } x^2 = \sigma^2 \text{ or } x = \pm\sigma$$

Again

$$\frac{d^3y}{dx^3} = \frac{y_0}{\sigma^2}\left[e^{-x^2/2\sigma^2}\left(\frac{2x}{\sigma^2}\right) + \left(\frac{x^2}{\sigma^2} - 1\right).e^{-\frac{x^2}{2\sigma^2}}\left(\frac{-2x}{2\sigma^2}\right)\right]$$

$$\frac{d^3y}{dx^3} = \frac{y_0}{\sigma^2}\frac{x}{\sigma^2}e^{-x^2/2\sigma^2}\left[2 - \left(\frac{x^2}{\sigma^2} - 1\right)\right] = \frac{y_0}{\sigma^4}x\, e^{-x^2/2\sigma^2}\left(3 - \frac{x^2}{\sigma^2}\right)$$

$$= \pm 2\frac{y_0}{\sigma^3}e^{-1/2} \quad \text{on writing } x = \pm\sigma$$

$$\neq 0$$

Hence the result.

Proved

Prob.58. In a test on 2000 electric bulbs, it was found that the life of a particular make, was normally distributed with an average life of 2040 hours and S.D. of 60 hours. Estimate the number of bulbs likely to burn for
(i) more than 2150 hours, (ii) less than 1950 hours and
(iii) more than 1920 hours and but less than 2160 hours.

Sol. Here m = 2040 hours and σ = 60 hours

(i) For $x = 2150,\ z = \frac{x-m}{\sigma} = 1.833$

$\therefore$ Area against, $z = 1.83$ in the table = 0.4664

We, however require the area to the right of the ordinate at $z = 1.83$.

This area $= 0.5 - 0.4664 = 0.0336$

Thus the number of bulbs expected to burn for more than 2150 hours

$= 0.0336 \times 2000 =$ **67 (Approx.)** **Ans.**

(ii) For $x = 1950,\ z = \frac{x-m}{\sigma} = -1.5$

The area required in this case is to the left of

$z = -1.33$

i.e., $= 0.5 - 0.4082$ (table value for $z = 1.33$) $= 0.0918$

$\therefore$ The number of bulbs expected to burn for less than 1950 hours

$= 0.0918 \times 2000 =$ **184 (Approx.)** **Ans.**

(iii) When $x = 1920,\ z = \frac{1920-2040}{60} = -2$

When $x = 2160,\ z = \frac{2160-2040}{60} = 2$

The number of bulbs expected to burn for more than 1920 hours but less than 2160 hours will be represented by the area between $z = -2$ and $z = 2$. This is twice the area from the table for $z = 2$, i.e., $z = 2 \times 0.4772 = 0.9544$.

Thus the required number of bulbs $= 0.9544 \times 2000 =$ **1909 (Approx.)**

Ans.

Prob.59. If two normal universes have the same total frequency but the standard deviation of one is k times that of other, show that the maximum frequency of the first (1/k) times that of other.

Sol. Let the total number of frequencies be N and σ and $k\sigma$ be the standard deviations of the two universe. Hence the equations to the two normal curves are

$$y = y_0 e^{-x^2/2\sigma^2} \text{ and } y = y_0' e^{-x^2/2k^2\sigma^2}$$

If the total frequencies for both are the same, then

$$\int_{-\infty}^{\infty} y_0 e^{-x^2/2\sigma^2}dx = \int_{-\infty}^{\infty} y_0' e^{-x^2/2k^2\sigma^2}dx$$

or
$$\int_{0}^{\infty} y_0 e^{-x^2/2\sigma^2}dx = \int_{0}^{\infty} y_0' e^{-x^2/2k^2\sigma^2}dx \quad \text{...(i)}$$

In the L.H.S., we put $\frac{x}{\sqrt{2}\sigma} = t$ so that

$dx = \sqrt{2}\sigma\, dt$, therefore it becomes.

$$\text{L.H.S.} = y_0 \int_0^\infty e^{-t^2} \sqrt{2}\sigma\, dt = \sqrt{2}\sigma y_0 \frac{1}{2}\sqrt{\pi} \quad \text{....(ii)}$$

Similarly for the R.H.S., we put

$\frac{x}{\sqrt{2}k\sigma} = t$, so that $dx = \sqrt{2}k\sigma\, dt$, then it becomes

$$\text{R.H.S.} = \sqrt{2}k\sigma\, y_0' \frac{1}{2}\sqrt{\pi} \quad \text{....(iii)}$$

Equating equations (ii) and (iii), we get

$$y_0 = k y_0'$$

But y_0 or y_0' are the frequencies corresponding to the mean or median or mode. Hence the maximum frequency of first is 1/k that of the other.

Proved

Prob.60. A manufacturer of envelopes knows that the weight of the envelopes is normally distributed with mean 1.9 gm and variance 0.01 gm. Find how many envelopes weighing

(i) 2 gm or more

(ii) 2.1 gm or more, can be expacted in a given packet of 1000 envelopes.

[Given : if t is the normal variable, then $\phi(0 \le t \le 1) = 0.3413$ and $\phi(0 \le t \le 2) = 0.4772$].

Sol. Given m = 1.9 gm, Variance = 0.01 gm $\Rightarrow \sigma = 0.1$ gm.

(i) x = 2 gm or more

$$\therefore z = \frac{x-m}{\sigma} = \frac{2-1.9}{0.1} = \frac{0.1}{0.1} = 1$$

$P(z > 1)$ = Area right to z = 1

$= 0.5 - 0.3413 = 0.1587$

Number of envelopes heavier than 2 gm in a lot of 1000

$= 1000 \times 0.1587 = 158.7$

= 159 (Approx.) **Ans.**

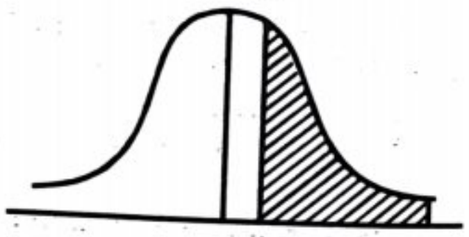

Fig. 5.5

(ii) When x = 2.1

$$\therefore \quad z = \frac{2.1-1.9}{0.1} = \frac{0.2}{0.1} = 2$$

$P(z > 2)$ = Area right to z = 2

$= 0.5 - 0.4772 = 0.0228$

Number of envelopes heavier than 2.1 gm in a lot of 1000

$= 1000 \times 0.0228 = 22.8$ **= 23 (Approx.)**

Ans.

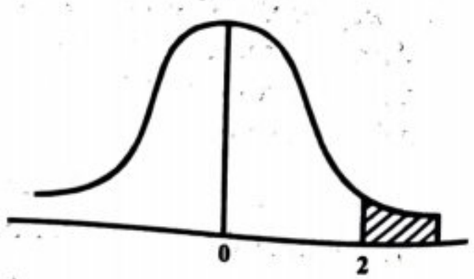

Fig. 5.6

Prob.61. Assuming that the diameters of 1000 brass plugs taken consecutively from a machine from a normal distribution with mean 0.7515 cm, and standard deviation 0.0020 cm, how many of the plugs are likely to be rejected, if the approved diameter is 0.752 ± 0.004 cm.

Sol. The approved range of diameter are

$0.752 - 0.004 = 0.748$ cm and

$0.752 + 0.004 = 0.756$ cm.

The corresponding values of the normal variation are

$$z_1 = \frac{0.748 - 0.7515}{0.002} = -1.75, \text{ where } x_1 = 0.748 \quad \left(\because z = \frac{x-m}{\sigma}\right)$$

If $x_2 = 0.756$, $\quad z_2 = \frac{0.756 - 0.7515}{0.002} = 2.25$

Area under, $z_1 = -1.75$ to $z_2 = 2.25$ = (Area from z = 0 to $z_1 = -1.75$)

+ (Area from z = 0 to $z_2 = 2.25$)

$= 0.4599 + 0.4878 = 0.9477$

Number of plugs likely to be rejected = 1000 (1 – 0.9477)

$= 1000 \times 0.0523 = 52.3$

Approximately **52 plugs** are likely to be rejected. **Ans.**

Prob.62. A sample of 100 dry battery cells tested to find the length of life produced the following results –

Mean ($\bar{x}$) = 12 hours, S.D. (σ) = 3 hours.

Assuming data to be normally distributed. What percentage of battery cells are expected to have life

(i) More than 15 hours

(ii) Less than 6 hours

(iii) Between 10 and 14 hours.

Sol. Suppose x denotes the length of the life of dry battery cells.

$$z = \frac{x-m}{\sigma} = \frac{x-12}{3}.$$

(i) When $x = 15, z = 1$

$\because \quad P(x > 15) = P(z > 1)$

= (Area to the right of z = 0)

– (Area between z = 0 and z = 1)

$= 0.5 - 0.3413 = 0.1587 =$ **15.87%** **Ans.**

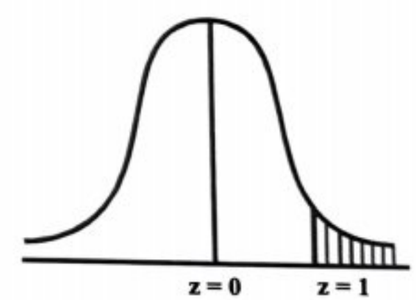

Fig. 5.7

(ii) When $x = 6, z = -2$

$\therefore \quad P(x < 6) = P(z < -2)$

$= $ (Area to the left of $z = 0$) $-$ (Area between $z = 0$ and $z = 2$)

$= 0.5 - 0.4772$

$= 0.0228$

$= \mathbf{2.28\ \%}$ **Ans.**

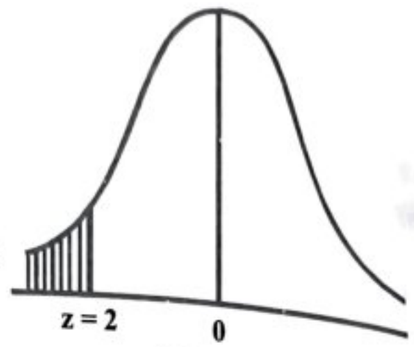

Fig. 5.8

(iii) When $x = 10$,

$$z = \frac{10-12}{3} = -0.67$$

When $x = 14$,

$$z = \frac{14-12}{3} = 0.67$$

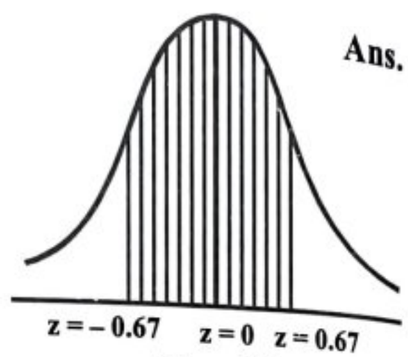

Fig. 5.9

$\therefore \quad P(10 < x < 14) = P(-0.67 < z < 0.67)$

$= 2 \times P(0 < z < 0.67)$

$= 2 \times 0.2486$

$= 0.4972$

$= \mathbf{49.72\ \%}$ **Ans.**

OOO

RGPV

B.E. (Fourth Semester) EXAMINATION, Dec. 2012
(Common for all Branches)
ENGINEERING MATHEMATICS-III

Note : (i) Attempt all questions.
(ii) All question carry equal marks.

Unit-I

1. (a) If f(z) is a regular function of z, prove that **

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)|f(z)|^2 = 4|f'(z)|^2$$

(b) Using Cauchy's Integral formula, prove that **

$$\int_C \frac{(\sin \pi z^2 + \cos \pi z^2)}{(z-1)(z-2)} dz = 4\pi i, \text{ where C is the circle } |z| = 3.$$

Or

2. (a) Apply the calculus of residue to prove that **

$$\int_0^{2\pi} e^{\cos\theta} \cos(\sin\theta - n\theta)\, d\theta = \frac{2\pi}{n!}, \text{ where n is positive integer.}$$

(b) Find a bilinear transformation which maps the points i, –i, 1 of the z-plane into 0, 1, ∞ of the w-plane respectively. **

Unit-II

3. (a) Find the root of the equation $xe^x = \cos x$ using the secant method correct to four decimal places. **

(b) Evaluate $\sqrt{12}$ correct to four decimal places by Newton-Raphson method. **(See Unit-I, Page-13, Prob.10)**

Or

4. (a) Apply Crout's method to solve the equations
$3x + 2y + 7z = 4,\ 2x + 3y + z = 5,\ 3x + 4y + z = 7$
(See Unit-II, Page-104, Prob.39)

(b) Solve, by Jacobi's iteration method, the equations
$20x + y - 2z = 17,\ 3x + 20y - z = -18,\ 2x - 3y + 20z = 25$
(See Unit-II, Page-109, Prob.43)

****Now, according to new revised syllabus of R.G.P.V., it is not included in syllabus**

Unit-III

5. (a) From the following table, estimate the number of students who obtained marks between 40 and 45.

Marks	30-40	40-50	50-60	60-70	70-80
No. of students	31	42	51	35	31

(See Unit-I, Page-45, Prob.41)

(b) A rod is rotating in a plane. The following table gives the angle θ (radians) through which the rod has turned for various values of the time t second.

t	0.0	0.2	0.4	0.6	0.8	1.0	1.2
θ	0	0.12	0.49	1.12	2.02	3.20	4.67

Calculate the angular velocity and acceleration of the rod when $t = 0.6$ second. **(See Unit-II, Page-65, Prob.1)**

Or

6. (a) Using Newton's divided difference formula, evaluate f(8) and f(15) from the following data :

x	4	5	7	10	11	13
f(x)	48	100	294	900	1210	2028

(See Unit-I, Page-53, Prob.50)

(b) A river is 80 ft. wide. The depth d in feet at a distance x ft from one bank is given by the following table :

x	0	10	20	30	40	50	60	70	80
y	0	4	7	9	12	15	14	8	3

Find approximately the area of cross-section.

(See Unit-II, Page-87, Prob.26)

Unit-IV

7. (a) Using Picard's method, find an approximate value of y when $x = 0.1$ if $\frac{dy}{dx} = x - y^2$ and $y = 1$ at $x = 0$. **(See Unit-III, Page-133, Prob.4)**

(b) Apply Milne's method to find the solution of $\frac{dy}{dx} = x + y$ given $y(0) = 0$ for $0.4 < x \le 1.0$ with $h = 0.1$. **(See Unit-III, Page-157, Prob.31)**

Or

8. (a) Fit a second degree parabola to the following data, using method of least squares : **

x	1.0	1.5	2.0	2.5	3.0	3.5	4.0
y	1.1	1.3	1.6	2.0	2.7	3.4	4.1

(b) Apply Runge-Kutta method to find approximate value of y for $x = 0.2$ in steps of 0.1 if $\frac{dy}{dx} = x + y^2$, given that $y = 1$ when $x = 0$.

(See Unit-III, Page-151, Prob.25)

Unit-V

9. (a) The mean height of 500 students is 151 cm. and the standard deviation is 15 cm. Assuming that the heights are normally distributed, find how many students have heights between 120 and 155 cm.

(See Unit-V, Page-306, Prob.55)

(b) Find the probability that at most 5 defective fuses will be found in a box of 200 fuses, if experience shows that 2% of such fuses are defective. **(See Unit-V, Page-288, Prob.28)**

Or

10.(a) Find the student's t-statistic for the following variable values in a sample : **

$$-4, -2, -2, 0, 2, 2, 3, 3$$

taking the mean of the universe to be zero.

(b) A dice is tossed 120 times with the following result : **

Number turned up	1	2	3	4	5	6	Total
Frequency	30	25	18	10	22	15	120

Test the hypothesis that the dice is unbiased.

$$\left(\chi^2_{0.05}\, 5 = 11.07\right)$$

RGPV

B.E. (Fourth Semester) EXAMINATION, June 2013
(Common for all Branches)
ENGINEERING MATHEMATICS-III

Note : (i) Answer any five questions.

****Now, according to new revised syllabus of R.G.P.V., it is not included in syllabus**

(ii) All question carry equal marks.

Unit-I

1. (a) If $u = x^2 - y^2$, find a corresponding analytic function by using Milne-Thomson method. **

(b) Evaluate $\int_C \frac{e^z}{e^{z-2}} dz$ where C is the circle **

(i) $|z| = 3$ and (ii) $|z| = 1$.

2. (a) Find the residue of $f(z) = \frac{1-e^{2z}}{z^4}$ at its poles. **

(b) Use calculus of residues to show that **

$$\int_0^{2x} \frac{\cos 2\theta}{5+4\cos\theta} d\theta = \frac{\pi}{6}$$

3. (a) If $n = 10\, x^3y^2z^2$ and error in x, y, z are respectively 0.03. 0.01, 0.02 at $x = 3, y = 1, z = 2$. Calculate the absolute error and percent relative error in the calculation of it. **

(b) Using Newton-Raphson method, find a root of the equation $f(x) = x \sin x + \cos x = 0$ correct to three decimal places, assuming that the root is near to $x = \pi$. **(See Unit-I, Page-11, Prob.7)**

4. (a) Solve by Gauss-Seidal method the equations –

$54x + y + z = 110$

$2x + 15y + 6z = 72$

$-x + 6y + 27z = 85$

(See Unit-II, Page-112, Prob.45)

(b) Express $f(x) = x^3 - 2x^2 + x - 1$ into factorial notation and show that $\Delta^4 f(x) = 0$. **(See Unit-I, Page-35, Prob.31)**

5. (a) Find a cubic polynomial in x for the following data –

x :	0	1	2	3	4	5
y :	– 3	3	11	27	57	107

(See Unit-I, Page-48, Prob.44)

(b) Given that :

x :	1.0	1.1	1.2	1.3	1.4	1.5
y :	7.989	8.403	8.781	9.129	9.451	9.750

Find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ at $x = 1.6$ **(See Unit-II, Page-69, Prob.5)**

6. (a) Use Euler's modified form to obtain y (0.2), y(0.4) and y (0.6) correct

****Now, according to new revised syllabus of R.G.P.V., it is not included in syllabus**

to three decimal places, given that $\frac{dy}{dx} = y - x^2$ with initial condition $y(0) = 1$. (See Unit-III, Page-145, Prob.19)

(b) Find the least square fit $y = a + bx + cx^2$ for the data **

x :	– 3	– 1	1	3
y :	15	5	1	3

7. (a) The probability that a bomb dropped from a plane will strike the target is $\frac{1}{5}$. If six bombs are dropped, find the probability that

(i) Exactly two will strike the target and

(ii) At least two will strike the target.

(See Unit-V, Page-281, Prob.15)

(b) Fit Poisson distribution to the following data –

x :	0	1	2	3	4
f :	46	38	22	9	1

(See Unit-V, Page-293, Prob.40)

8. (a) Calculate $\int_1^2 \frac{dx}{x}$ by,

(i) Simpson's $\frac{1}{3}$ rule with $h = 0.50$

(ii) Simpson's $\frac{1}{3}$ rule with $h = 0.25$. (See Unit-II, Page-79, Prob.14)

(b) Find the equation of the lines of regression based on the following data : **

x :	4	2	3	4	2
y :	2	3	2	4	4

RGPV

B.E. (Fourth Semester) EXAMINATION, Dec. 2013

(Common for all Branches)

ENGINEERING MATHEMATICS-III

Note : (i) Attempt all questions.

(ii) Each question carry equal marks.

Unit-I

1. (a) Show that function $f(z) = \sqrt{|xy|}$ is not regular at $z = 0$, although C-R equations are satisfied at this point. **

****Now, according to new revised syllabus of R.G.P.V., it is not included in syllabus**

(b) Using Cauchy integral formula, prove that – **

$$\int_C \frac{e^{2z}}{(z+1)^4}dz = \frac{8\pi i}{3e^2},$$ where C is the circle $|z| = 3$.

Or

2. (a) Apply the calculus of residue to show that – **

$$\int_0^{\pi} \frac{(1+2\cos\theta)}{(5+4\cos\theta)}d\theta = 0$$

(b) Show that the transformation $w = z^2$ transforms the circle $|z - a| = C$ in the z-plane. **

Unit-II

3. (a) Find a real root of the equation $x^3 - 2x - 5 = 0$ by the method of Regula-falsi correct to three decimal places. **(See Unit-I, Page-20, Prob.18)**

(b) Find the negative root of the equation $x^3 - 21x + 3500 = 0$ correct to 2 decimal places by Newton's-Raphson method.

(See Unit-I, Page-16, Prob.14)

Or

4. (a) Solve the following equations by Crout's method –

$10x + y + z = 12,\ 2x + 10y + z = 13,\ 2x + 2y + 10z = 14$

(See Unit-II, Page-106, Prob.41)

(b) Solve the following equations by Gauss-Seidel method –

$20x + y - 2z = 17,\ 3x + 20y - z = -18,\ 2x - 3y + 20z = 25.$

(See Unit-II, Page-122, Prob.52)

Unit-III

5. (a) Find the number of men getting wages between Rs. 10 and Rs. 15 from the following data –

Wages in (Rs.) :	0 – 10	10 – 20	20 – 30	30 – 40
Frequency :	9	30	35	42

(See Unit-I, Page-45, Prob.40)

(b) A slider in a machine moves along a fixed straight rod. Its distance x cm along the rod is given below for various values of the time t seconds. Find the velocity of the slider and its acceleration when t = 0.3 second.

t	0.0	0.1	0.2	0.3	0.4	0.5	0.6
x	30.13	31.62	32.87	33.64	33.95	33.81	33.24

(See Unit-II, Page-66, Prob.2)

Or

6. (a) Using Newton's divided difference formula, evaluate f(9) from the following values of x and y –

****Now, according to new revised syllabus of R.G.P.V., it is not included in syllabus**

x	5	7	11	13	17
y	150	392	1452	2366	5202

(See Unit-I, Page-54, Prob.51)

(b) Calculate the value of $\int_0^{\pi/2} \sin x\, dx$ by Simpson's 1/3 rule using 11 ordinates.

(See Unit-II, Page-85, Prob.22)

Unit-IV

7. (a) Find by Taylor's series method, the values of y at x = 0.1 and x = 0.2 to five places of decimal from $\frac{dy}{dx} = x^2y - 1,\ y(0) = 1.$

(See Unit-III, Page-139, Prob.11)

(b) Apply Milne's method to find a solution of the differential equation $dy/dx = x - y^2$ in the range $0 \le x \le 1$ for the boundary condition $y = 0$ at $x = 0$.

(See Unit-III, Page-154, Prob.29)

Or

8. (a) Find the parabola of the form $y = a + bx + cx^2$ which fits most closely with the observations given below – **

x	–3	–2	–1	0	1	2	3
y	4.63	2.11	0.67	0.09	0.63	2.15	4.58

(b) Apply Runge-Kutta method to find an approximate value of y when x = 0.2 given that $\frac{dy}{dx} = x + y$ and y = 1 when x = 0.

(See Unit-III, Page-146, Prob.20)

Unit-V

9. (a) The sum and the product of the mean and variance of a binomial distribution are 24 and 128. Find the distribution.

(See Unit-V, Page-286, Prob.24)

(b) Prove that in a normal distribution, all the moments of odd order about the mean vanishes.

(See Unit-V, Page-304, Prob.49)

Or

10. (a) Find the student's t-statistic for the following variable values in a sample –

– 4, – 2, – 2, 0, 2, 2, 3, 3 **

taking the mean of the universe to be zero.

(b) In a bank 100 accounts are selected at random and examined for errors. The following results have been obtained.

No. of errors	0	1	2	3	4	5	6
No. of accounts	36	40	19	2	0	2	1

****Now, according to new revised syllabus of R.G.P.V., it is not included in syllabus**

Does this information verify that the errors are distributed according to Poisson probability law ? ($\chi^2_{0.05,5} = 11.07$) **

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B.E. (Fourth Semester) EXAMINATION, June 2014
(Common for all Branches)
ENGINEERING MATHEMATICS-III

Note : (i) Answer five questions. In each question part A, B, C is compulsory and D part has internal choice.
(ii) All parts of each question are to be attempted at one place.
(iii) All questions carry equal marks, out of which part A and B (Max. 50 words) carry 2 marks, part C (Max. 100 words) carry 3 marks, part D (Max. 400 words) carry 7 marks.
(iv) Except numericals, Derivation. Design and Drawing etc.

1. (a) Show that the function $\frac{1}{2}\log(x^2+y^2)$ is harmonic. **2

(b) Evaluate the integral $\int_C \frac{z^2-z+1}{z-1}dz$, where C is the circle $|z| = 1$. **2

(c) Find Cauchy-Riemann equations in polar form. **3

(d) Find the analytic function
$f(z) = u + iv$ if $u - v = (x-y)(x^2+4xy+y^2)$ **7

Or

Evaluate the integral $\int_0^\infty \frac{\cos ax}{x^2+1}dx$. ** 7

2. (a) Determine the Newton-Raphson iterative formula to find the kth root of N. ** 2

(b) Find a real root of the equation $x \log_{10} x = 1.2$ by Regula-falsi method correct to one decimal place. **(See Unit-I, Page-18, Prob.16)** 2

(c) Find a real root of the equation $3x = \cos x + 1$ by iterative method correct to two decimal places. ** 3

(d) Apply Crout's factorization method to solve the system of equations – 7

$x - y = 0$
$-2x + 4y - 2z = -1$
$-y + 2z = 1.5$ **(See Unit-II, Page-108, Prob.42)**

Or

Apply Gauss-Seidel iteration method to solve the system of equations – 7

$20x + y - 2z = 17$
$3x + 20y - z = -18$
$2x - 3y + 20z = 1.5$

**Now, according to new revised syllabus of R.G.P.V., it is not included in syllabus

3. (a) Prove that – (See Unit-II, Page-116, Prob.48)

$$e^x = \left(\frac{\Delta^2}{E}\right)e^x \cdot \frac{Ee^x}{\Delta^2 e^x}.$$ 2

(b) Derive Newton's forward interpolation formula. (See Unit-I, Page-31, Prob.22) 2

(c) Evaluate the integral $\int_0^{0.6} e^{-x^2}dx$ by Simpson 1/3 rule. (See Unit-I, Page-41, Q.3) 3

(d) Apply Newton's divided difference formula to find the value of f(9) from the following table – 7

x	5	7	11	13	17
f(x)	150	392	1452	2368	5202

(See Unit-I, Page-55, Prob.52)

Or

Find $\frac{dy}{dx}$ at x = 1.1 from the following table –

x	1.0	1.2	1.4	1.6	1.8	2.0
y	0	0.128	0.544	1.296	2.432	4.000

(See Unit-II, Page-70, Prob.7)

4. (a) Find by Taylor's series method the value y(0.1) correct to three decimal places from the differential equation – 2

$$\frac{dy}{dx} = x^2y - y,\ y(0) = 1.$$ (See Unit-III, Page-137, Prob.8)

(b) Write the working rule of Runge-Kutta method of fourth order for the numerical solution of differential equation. (See Unit-III, Page-132, Q.1) 2

(c) If θ is the angle between the two regression lines show that – **3

$$\tan\theta = \frac{1-r^2}{r}\cdot\frac{\sigma_x\sigma_y}{\sigma_x^2+\sigma_y^2}$$

(d) Using modified Euler's method, find the value of y(0.3) from the equation – 7

$$\frac{dy}{dx} = x + y,\ y(0) = 1$$ (See Unit-III, Page-142, Prob.16)

Or

Find the lines of regression and coefficient of correlation from the following table – **7

**Now, according to new revised syllabus of R.G.P.V., it is not included in syllabus

X	3	4	6	7	10
Y	9	11	14	15	16

5. (a) The probability density function p(x) of a random variable is given by $p(x) = y_0 e^{-|x|}, -\infty \le x \le \infty$. Find the value of y_0. 2
(See Unit-V, Page-303, Prob.46)

(b) If the 10 percent of the bolts produced by a machine are defective, find the probability that out of 5 bolts chosen at random at least two will be defective. **2

(c) Write the procedure to test significance and goodness of fit using χ^2 method. **3

(d) Prove that the mean deviation from mean of a normal distribution is approximately 4/5 times of standard deviation. 7
(See Unit-V, Page-304, Prob.50)

Or

The nine items of a sample have the values – **7

45, 47, 50, 52, 48, 47, 49, 53, 51

Does the mean of these differ significant from the assumed mean 47.5? ($t_{0.05} = 2.31$ for v = 8)

RGPV

B.E. (Fourth Semester) EXAMINATION, Dec. 2014
(Common for all Branches)
MATHEMATICS-III

Note : (i) Answer five questions. In each question part A, B, C is compulsory and D part has internal choice.

(ii) All parts of each question are to be attempted at one place.

(iii) All questions carry equal marks, out of which part A and B (Max. 50 words) carry 2 marks, part C (Max. 100 words) carry 3 marks, part D (Max. 400 words) carry 7 marks.

(iv) Except numericals, Derivation. Design and Drawing etc.

1. (a) Show that $w = e^z$ is an analytic function and determine f'(z). **

(b) Evaluate $\int_C \frac{z^2 - z + 1}{z - 1} dz$, where C is the circle $|z| = 1$. **

(c) Evaluate $\int_C \frac{e^{2z}}{(z+1)^4} dz$, where C is the circle $|z| = 2$. **

(d) If f(z) is a regular function of z prove that $\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)|f(z)|^2 = 4|f'(z)|^2$. **

Or

If $f(\xi) = \int_C \frac{3z^2 + 7z + 1}{z - \xi} dz$, where C is the circle $x^2 + y^2 = 4$, find the values of f(3), f'(1 – i) and f''(1 – i).

**Now, according to new revised syllabus of R.G.P.V., it is not included in syllabus

2. (a) Define algebraic and transcendental equations. (See Unit-I, Page-6, Q.1) **

(b) Find the smallest positive root of the equation $x^3 - 2x + 0.5 = 0$ by Newton-Raphson method.
(See Unit-I, Page-10, Prob.5)

(c) The equation $x^6 - x^4 - x^3 - 1 = 0$, has one real root between 1.4 and 1.5. Find the root to four decimal places by the method of False-Position.
(See Unit-I, Page-21, Prob.19)

(d) Solve the following system of equations $8x - y + z = 18$; $2x + 5y - 2z = 3$; $x + 2y - 3z = -6$ using Gauss-Seidel iterative method.
(See Unit-II, Page-118, Prob.49)

Or

Apply Gauss-Jordan method to find the solution of the following system of equations $-10x + y + z = 12$; $2x + 10y + z = 13$, $x + y + 5z = 7$.
(See Unit-II, Page-102, Prob.38)

3. (a) Define interpolation and write the Newton's forward and backward interpolation formula.
(See Unit-I, Page-41, Q.4)

(b) Find the cubic polynomial which takes the following values –

x	:	0	1	2	3
f(x)	:	1	2	1	10

(See Unit-I, Page-46, Prob.42)

(c) Evaluate $\int_0^6 \frac{dx}{1+x^2}$ by using Weddle's rule. **

(d) The following table gives the normal weights of babies during the first 12 months of life.

Age of months:	0	2	5	8	10	12
Weights in lbs:	7½	10¼	15	16	18	21

Estimate the weight of the baby at the age of 7 months.
(See Unit-I, Page-59, Prob.57)

Or

Find f'(x) and f''(x) at x = 6, given that

x	:	4.5	5.0	5.5	6.0	6.5	7.0	7.5
y = f(x)	:	9.69	12.90	16.71	21.18	26.37	32.34	39.15

(See Unit-II, Page-73, Prob.9)

4. (a) Use Picard's method to approximate y when x = 0.1 given that y = 1, when x = 0 and $\frac{dy}{dx} = \frac{y - x}{y + x}$. (See Unit-III, Page-134, Prob.5)

(b) Write the steps of Runge-Kutta method. (See Unit-III, Page-132, Q.1)

**Now, according to new revised syllabus of R.G.P.V., it is not included in syllabus

(c) From the following data, obtain the value of the correlation coefficient: $n = 10$, $\Sigma x = 140$, $\Sigma y = 150$, $\Sigma(x-10)^2 = 180$, $\Sigma(y-15)^2 = 215$ and $\Sigma(x-10)(y-15) = 60$. **

(d) By the method of least squares, find the curve $y = ax + bx^2$, that best fits the following data – **

x :	1	2	3	4	5
y :	1.8	5.1	8.9	14.1	19.8

Or

Given $\frac{dy}{dx} = 1 + \frac{y}{x}$, $y = 2$ at $x = 1$. Find approximate value of y at $x = 1.4$ by taking step size $h = 0.2$, apply modified Euler's method. **(See Unit-III, Page-143, Prob.17)**

5. (a) Find the mean of the binomial distribution. **(See Unit-V, Page-279, Prob.8)**

(b) Given A and B two events with $P(A \cup B) = \frac{7}{8}$, $P(A \cap B) = \frac{1}{4}$ and $P(A \cap \overline{B}) = \frac{5}{8}$. Find P(A) and P(B). **(See Unit-V, Page-268, Prob.1)**

(c) If the probability of a bad reaction from certain injection is 0.001. Determine the chance that out of 2000 individuals more than two will get a bad reaction. **(See Unit-V, Page-291, Prob.37)**

(d) Fit a Poisson distribution for the following data and test the goodness of fit, given that $\chi^2_{0.05} = 7.815$ for 3 d.f. **

x :	0	1	2	3	4
f :	122	60	15	2	1

Or

The life time of a certain kind of battery is a random variable, which as an exponential distribution with a mean of 200 hrs. Find the probability that such a battery will last. (i) At most 100 hrs and (ii) last any-where from 400 to 600 hrs. **(See Unit-V, Page-268, Prob.2)**

RGPV

B.E. (Fourth Semester) EXAMINATION, June 2015
(Common for all Branches)
MATHEMATICS-III

Note : (i) Answer five questions. In each question part A, B, C is compulsory and D part has internal choice.
(ii) All parts of each questions are to be attempted at one place.
(iii) All questions carry equal marks, out of which part A and B (Max. 50 words) carry 2 marks, part C (Max. 100 words) carry 3 marks, part D (Max. 400 words) carry 7 marks.
(iv) Except numericals, Derivation. Design and Drawing etc.

****Now, according to new revised syllabus of R.G.P.V., it is not included in syllabus**

Unit-I

1. (a) Show that $u = 2x - x^3 + 3xy^2$ is harmonic. **

(b) Use Cauchy-Riemann equation to find v, where $u = 3x^2 y - y^3$. **

(c) Find the order of each pole and residue at it of $\frac{1-2z}{z(z-1)(z-2)}$. **

(d) Expand $f(z) = \frac{1}{(z-1)(z-2)}$ in the region $|z| > 2$. **

Or

Evaluate $\int_0^{2\pi} \frac{d\theta}{(a + b\cos\theta)^2}$, $a > b > 0$. **

Unit-II

2. (a) Find the relative error if 2/3 is approximate to 0.667. **

(b) Give convergence values of Regula falsi and Newton Raphson method. **(See Unit-I, Page-7, Q.2)**

(c) Find the real root of the equation $(48)^{1/3}$ by Iteration method upto two decimal places. **

(d) Solve by Gauss elimination method –
$10x + y + 2z = 13$
$3x + 10y + z = 14$
$2x + 3y + 10z = 15$
(See Unit-II, Page-101, Prob.36)

Or

Apply Crout's method to solve the equations –
$3x + 2y + 7z = 4$
$2x + 3y + z = 5$
$3x + 4y + z = 7$
(See Unit-II, Page-104, Prob.39)

Unit-III

3. (a) Find the function whose first difference is $9x^2 + 11x + 5$. **(See Unit-I, Page-34, Prob.29)**

(b) Evaluate $\int_0^6 \frac{1}{1+x^2} dx$ by using Simpson's 1/3 rule. **(See Unit-II, Page-90, Prob.29)**

(c) Find the missing values of the following data –

x:	0	5	10	15	20	25
y:	6	10	–	17	–	31

(See Unit-I, Page-36, Prob.33)

****Now, according to new revised syllabus of R.G.P.V., it is not included in syllabus**

(d) Use Newton's divided difference formula to find the form of f(x), given –

x	0	2	3	6
f(x)	648	704	729	792

(See Unit-I, Page-50, Prob.46)

Or

Estimate the number of students who obtained marks between 40 and 45

Marks	30 – 40	40 – 50	50 – 60	60 – 70	70 – 80
No. of students	31	42	51	35	31

(See Unit-I, Page-45, Prob.41)

Unit-IV

4. (a) Using Picard's method to obtain y for x = 0.1, if $\frac{dy}{dx} = 1 + xy$, with y(0) = 1. **(See Unit-III, Page-133, Prob.3)**

(b) Prove if one of the regression coefficient is greater than unity, then the other is less than unity. **

(c) From the following data obtain the value of the correlation coefficient: n = 10, $\Sigma x = 55$, $\Sigma y = 40$, $\Sigma x^2 = 385$, $\Sigma y^2 = 192$ and $\Sigma(x + y)^2 = 947$. **

(d) Apply Runge-Kutta method to find a approximate value of y when x = 0.1. If $\frac{dy}{dx} = x + y^2$ given that y(0) = 1. **(See Unit-III, Page-152, Prob.26)**

Or

Calculate the coefficient of regression lines and find the two lines of regression from the following data – **

x :	78	89	97	69	59	79	68	61
y :	125	137	156	112	107	136	123	108

Unit-V

5. (a) Give conditions for the validity of χ^2 test. **

(b) Define –

(i) Discrete and continuous random variable **(See Unit-V, Page-279, Q.3)** **

(ii) Student's T-test

(c) Derive the expression for mean and variance of Binomial distribution. **(See Unit-V, Page-279, Prob.9)**

(d) The mean height of 500 students in 151 cm and the standard deviation is 15 cm. Assuming that the heights are normally distributed. Find how many students have heights between 120 and 155 cm. **(See Unit-V, Page-306, Prob.55)**

Or

In a laboratory experiment, two samples gave the following results –

**Now, according to new revised syllabus of R.G.P.V., it is not included in syllabus

Sample	Size	Sample Mean	Sum of squares of deviation from the mean
I	10	15	90
II	12	14	108

Test the equality of sample variance at 05% level of significance ($F_{9,\ 11(0.05)} = 2.90$). **

RGPV

B.E. (Fourth Semester) EXAMINATION, Dec. 2015
(Common for all Branches)
MATHEMATICS-III

Note : (i) Answer five questions. In each question part A, B, C is compulsory and D part has internal choice.
(ii) All parts of each questions are to be attempted at one place.
(iii) All questions carry equal marks, out of which part A and B (Max. 50 words) carry 2 marks, part C (Max. 100 words) carry 3 marks, part D (Max. 400 words) carry 7 marks.
(iv) Except numericals, Derivation, Design and Drawing etc.

Unit-I

1. (a) Write Cauchy-Riemann equations in polar form. **

(b) If $f(z) = \frac{z^2}{(z-1)^2(z+2)}$ then find Res f(1). **

(c) Find the value of $\int_c \frac{z^2+3z+1}{z-1}dz$ where C being $|z| = \frac{1}{2}$. **

(d) Apply calculus of Residue to prove that – **

$$\int_0^{2\pi} \frac{\cos 2\theta\, d\theta}{1 - 2a\cos\theta + a^2} = \frac{2\pi a^2}{1-a^2}, (a^2 < 1).$$

Or

Show that $e^x(x \cos y - y \sin y)$ is a harmonic function. Find the analytic function for which $e^x(x \cos y - y \sin y)$ is imaginary part. **

2. (a) Write a short note on errors. **

(b) Write the procedure of Gauss-elimination method for solving the system of simultaneous linear equations. **(See Unit-II, Page-97, Q.1)**

(c) Find a real root of the equation $f(x) = x^3 - 4x - 9 = 0$, using bisection method in four stages. **(See Unit-I, Page-9, Prob.3)**

(d) Find a real root of the equation $x^4 - x - 13 = 0$ by Newton Raphson method correct to three decimal places. **(See Unit-I, Page-14, Prob.11)**

Or

**Now, according to new revised syllabus of R.G.P.V., it is not included in syllabus

Solve the following system by Gauss-Seidel method –

$10x + 2y + z = 9$
$2x + 20y - 2z = -44$
$2x + 3y + 10z = 22$

(See Unit-II, Page-113, Prob.46)

3. (a) Find the value of $\left(\frac{\Delta^2}{E}\right)e^x$. **(See Unit-I, Page-33, Prob.25)**

(b) Prepare the difference table for the following data –

x :	10	20	30	40	50
y :	12	15	20	27	39

(See Unit-I, Page-41, Prob.36)

(c) Evaluate $\int_0^1 \frac{dx}{1+x}$ by using Simpson's $\frac{1^{rd}}{3}$ rule, take n = 6.

(See Unit-II, Page-84, Prob.20)

(d) Find a polynomial satisfied by (–4, 1245), (–1, 33), (0, 5), (2, 9) and (5, 1335) by Newton's divided difference formula.

(See Unit-I, Page-49, Prob.45)

Or

Calculate the first and second derivatives of the function tabulated below, at the point x = 1.1.

x :	1.0	1.2	1.4	1.6	1.8	2.0
f(x) :	0	0.128	0.544	1.296	2.432	4.0

(See Unit-II, Page-72, Prob.8)

4. (a) If $\frac{dy}{dx} = f(x, y)$ with initial condition $y = y_0$ at $x = x_0$, then write first two approximations using Picard's method.

(See Unit-III, Page-132, Prob.1)

(b) Write the normal equations for equation of second degree parabola to be fitted to given set of data. **

(c) Use Euler's method to find y(0.2) from the differential equation $\frac{dy}{dx} = xy$, $y(0) = 1$ take h = 0.1. **(See Unit-III, Page-141, Prob.14)**

(d) Use Runge-Kutta method to solve the equation $\frac{dy}{dx} = 1 + y^2$ for x = 0.2 to x = 0.4 with h = 0.2, given that y(0) = 0.5.

(See Unit-III, Page-150, Prob.24)

Or

Calculate the coefficient of correlation between the marks in Mathematics and Physics for 8 students – **

Mathematics :	76	90	98	69	54	82	67	52
Physics :	25	37	56	12	17	36	27	11

5. (a) Write the mean and variance of the Binomial distribution.

(See Unit-V, Page-279, Prob.9)

(b) Define types of sampling. **

(c) A manufacturer who produces medicine bottles, finds that 0.1% of the bottles are defective. The bottles are packed in boxes containing 500 bottles. A drug manufacturer buys 100 boxes from the producer of bottles. Using Poisson distribution, find how many boxes will contain no defectives. (Given $e^{-0.5} = 0.6065$).

(See Unit-V, Page-290, Prob.34)

(d) A random sample of size 16 has 53 as mean. The sum of the squares of the deviations take from mean is 150. Can this sample be regarded as taken from the population having 56 as mean ? Obtain 95% and 99% confidence limits of the mean of the population. [For v = 15, $t_{0.01} = 2.95$ and $t_{0.05} = 2.13$] **

Or

The probability that an evening college student will graduate is 0.4. Determine the probability that out of 5 students.

(i) None (ii) one and (iii) at least one will graduate.

(See Unit-V, Page-282, Prob.16)

RGPV

BE-401

B.E. (Fourth Semester) EXAMINATION, June 2016

(Common for all Branches)

MATHEMATICS-III

Note : **(i)** Answer five questions. In each question part A, B, C is compulsory and D part has internal choice.

(ii) All parts of each questions are to be attempted at one place.

(iii) All questions carry equal marks, out of which part A and B (Max. 50 words) carry 2 marks, part C (Max. 100 words) carry 3 marks, part D (Max. 400 words) carry 7 marks.

(iv) Except numericals, Derivation, Design and Drawing etc.

Unit-I **

1. (a) Find the kind of singularity of the function

$$f(z) = \frac{z-2}{z^2}\sin\left(\frac{1}{(z-1)}\right).$$

**

(b) If f(z) is a regular function of z, prove that

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)|f(z)|^2 = 4|f'(z)|^2$$

**Now, according to new revised syllabus of R.G.P.V., it is not included in syllabus

(c) Show that the function $u = e^{-2xy\sin(x^2-y^2)}$ harmonic and find the conjugate function v. **

(d) Evaluate $\int_0^{\pi}\left(\frac{1+2\cos\theta}{5+4\cos\theta}\right)d\theta$ **

Or

Using Cauchy's Integral formula prove that **

$$\int_c \frac{e^{2z}}{(z+1)^4}dz = \frac{8\pi e^{-2}}{3} - i$$

whre c is the circle $|z| = 3$.

Unit-II

2. (a) If 0.333 is the approximate value of 1/3, find the absolute, relative and percentage errors. **

(b) Find the real root of the equation $x \log_{10}x = 1.2$ by Regula-falsi method correct to two decimal places. **(See Unit-I, Page-18, Prob.16)**

(c) Using Newton-Raphson method, find a real root of the equation $3x = \cos x + 1$. **(See Unit-I, Page-10, Prob.6)**

(d) Solve the equations –

$10x + 2y + z = 9$

$-2x + 3y + 10z = 22$

$x + 10y - z = -22$

By Gauss-Seidel method. **(See Unit-II, Page-114, Prob.47)**

Or

Solve by Crout's method

$10x + y + z = 12$

$2x + 10y + z = 13$

$2x + 2y + 10z = 14$

(See Unit-II, Page-106, Prob.41)

UNIT-III

3. (a) Express $y = 2x^3 - 3x^2 + 3x - 10$ in factorial notation. **(See Unit-I, Page-34, Prob.30)**

(b) Write Stirling's formula for equal interval. **

(c) Find the first term of the series whose second and sub sequent terms are 8, 3, 0, – 1, 0. **(See Unit-I, Page-35, Prob.32)**

(d) Apply Lagrange's method to find the value of x, when f(x) = 15, from the given data.

x	5	6	9	11
y	12	13	14	16

(See Unit-I, Page-61, Prob.59)

Or

Apply Bessel's formula to evaluate y_{25}, given $y_{20} = 2854$, $y_{24} = 3162$, $y_{28} = 3544$, $y_{32} = 3992$. **

****Now, according to new revised syllabus of R.G.P.V., it is not included in syllabus**

UNIT-IV

4. (a) Solve $\frac{dy}{dx} = 1 - 2xy$ given that y(0) = 0, by Taylor's method. **(See Unit-III, Page-138, Prob.10)**

(b) From the following data, find the number of items n. Where r = 0.5, $\Sigma xy = 120$, $\sigma_y = 8$, $\Sigma x^2 = 90$, where x and y are deviations of arithmetic mean. **

(c) Find y(2.2) using Euler's method for the equation $\frac{dy}{dx} = -xy^2$ with y(2) = 1.

(See Unit-III, Page-142, Prob.15)

(d) Using Runge-Kutta method of fourth order solve $\frac{dy}{dx} = \frac{y^2 - x^2}{y^2 + x^2}$ with y(0) = 1, at x = 0.2 and 0.4. **(See Unit-III, Page-152, Prob.27)**

Or

Fit a straight line to the following data : **

x	1	2	3	4	5	6	7	8	9
y	9	8	10	12	11	13	14	16	15

5. (a) Find the mean deviation from mean for normal distribution. **(See Unit-V, Page-303, Prob.45)**

(b) Prove that total probability of Beta distribution of second kind is 1. **

(c) Define the following –

(i) Probability function

(ii) Probability mass function

(iii) Probability density function.

(See Unit-V, Page-267, Q.1)

(d) Find the students t-statistic for the following varaible values in a sample –4, – 2, – 2, 0, 2, 2, 3, 3 taking the mean of the universe to be zero. **

Or

A dice is tossed 120 times with the following results :

No. of turned up :	1	2	3	4	5	6	Total
Frequency :	30	25	18	10	22	15	120

Test the hypothesis that the dice is unbiased $\left(\chi^2_{0.05,5} = 11.07\right)$ **

****Now, according to new revised syllabus of R.G.P.V., it is not included in syllabus**

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BE-401

B.E. (Fourth Semester) EXAMINATION, Dec. 2016

(Common for all Branches)

MATHEMATICS-III

Note : (i) Answer any five questions.
(iii) All questions carry equal marks.

1. **(a)** Show that the function $e^x\{\cos y + i\sin y\}$ in an analytic function. **

(b) Determine the analytic function $w = u + iv$ if **

$v = \log(x^2 + y^2) + (x - 2y)$

2. **(a)** Evaluate $\int_0^{1+i}(x^2 - iy)dz$ along the path $y = x$. **

(b) Determine the poles of the following function and residue at each **

pole $f(z) = \dfrac{z^2}{(z-1)(z-2)}$

3. **(a)** Find a real root of the equation $x^3 - 2x - 5 = 0$, by the method of false position, correct to three decimal places.
(See Unit-I, Page-20, Prob.18)

(b) Solve by Gauss-Seidel method the equations
$10x + y + z = 12$
$x + 10y + z = 12$
$x + y + 10z = 12$
(See Unit-II, Page-121, Prob.51)

4. **(a)** Solve the following system of euations by Gauss-elimination method the equations are
$2x + y + z = 12$
$3x + 2y + 3z = 18$
$x + 4y + 9z = 16$
(See Unit-II, Page-98, Prob.32)

(b) If 0.333 is the approximate value of 1/3, find the absolute, relative and percentage errors. **

5. **(a)** Given that – $\sin 45° = 0.7071$, $\sin 50° = 0.7660$, $\sin 55° = 0.8192$, $\sin 60° = 0.8660$, find the value of $\sin 52°$. **

(b) The probability that an evening college student will graduate is 0.4. Determine the probability that out of 5 students.
(i) None
(ii) One and
(iii) At least one will graduate.
(See Unit-V, Page-282, Prob.16)

**Now, according to new revised syllabus of R.G.P.V., it is not included in syllabus

6. **(a)** Use Trapezoidal rule to evaluate $\int_0^1 x^3 dx$ considering give sub-intervals.
(See Unit-II, Page-79, Prob.13)

(b) Find $\dfrac{dy}{dx}$ at $x = 0.1$ from the following table –

x	0.1	0.2	0.3	0.4
y	0.9975	0.9900	0.9776	0.9604

(See Unit-II, Page-69, Prob.6)

7. **(a)** Using Picards method of successive approximations, obtain a solution upto fifth approximation of the equation $\dfrac{dy}{dx} = y + x$, such that $y = 1$, when $x = 0$. **(See Unit-III, Page-135, Prob.7)**

(b) Use Runge-Kutta method to fourth order to aproximate y, when $x = 0.1$, given tha $y = 1$ at $x = 0$ and $\dfrac{dy}{dx} = 3x + y^2$.
(See Unit-III, Page-149, Prob.23)

8. **(a)** Find the mean and variance of the Binomial distribution.
(See Unit-V, Page-279, Prob.9)

(b) A lot has 10% defective items. Ten items are chosen randomly from this lot. Find the probability that exactly 2 of the chosen items are defective. **(See Unit-V, Page-280, Prob.13)**

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BE-401

B.E. (Fourth Semester) EXAMINATION, June 2017

(Common for all Branches)

MATHEMATICS-III

Note : (i) Attempt any five questions out of eight questions.
(iii) All questions carry equal marks.

1. **(a)** Show that e^z is every where analytic. **

(b) Show that $u = 2x - x^3 + 3xy^2$ is harmonic and determine its conjugate. **

2. **(a)** Find the order of each pole and residue at it of the function **

$$f(z) = \frac{1-2z}{z(z-1)(z-2)}$$

**Now, according to new revised syllabus of R.G.P.V., it is not included in syllabus

(b) Evaluate $\int_c \frac{e^{2z}}{(z-1)(z-2)}$ where c is the circle $|z| = 3$. **

3. **(a)** Find the real root of the equation $x \log_{10} x = 1.2$ by the method of false position. **(See Unit-I, Page-18, Prob.16)**

(b) Solve $x^3 - 3x + 1 = 0$ by Newton's method. **(See Unit-I, Page-17, Prob.15)**

4. **(a)** Using Gauss-elimination method solve

$2x - y + 3z = 9$
$x + y + z = 6$
$x - y + z = 2$

(See Unit-II, Page-99, Prob.33)

(b) Solve the following system by Gauss-Seidel method.

$27x + 6y - z = 85$
$6x + 15y + 2z = 72$
$x + y + 54z = 110$

(See Unit-II, Page-120, Prob.50)

5. **(a)** Evaluate $\left(\frac{\Delta^2}{E}\right)e^x . \frac{Ee^x}{\Delta^2 e^x}$ **(See Unit-I, Page-31, Prob.22)**

(b) Find the Cubic polynomial which takes the following values –

x	0	1	2	3
f(x)	1	2	1	10

(See Unit-I, Page-46, Prob.42)

6. **(a)** Find the missing term using Newton's divided difference formula

x	0	1	2	3	4
f(x)	1	3	9	–	81

(b) Evaluate $\int_0^6 \frac{1}{1+x^2} dx$ using Simpson's $\frac{1}{3}$ rule.

(See Unit-II, Page-90, Prob.29)

7. **(a)** Use Runge-Kutta method of fourth order to solve :

$\frac{dy}{dx} = xy$ for $x = 1.2$

Initially $x = 1$, $y = 2$ (take $h = 0.1$)

(See Unit-III, Page-147, Prob.22)

****Now, according to new revised syllabus of R.G.P.V., it is not included in syllabus**

(b) Using Taylor's series method, solve $\frac{dy}{dx} = 1 - 2xy$ given that $y(0) = 0$. **(See Unit-III, Page-138, Prob.10)**

8. **(a)** Find mean and variance of Poisson distribution. **(See Unit-V, Page-289, Prob.32)**

(b) Fit a straight line to the following data **

x	3	5	6	7	9
y	2	3	4	6	5

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BT-301 (AU/CE/CM/EC/IP/ME) CBGS
B.Tech., III Semester, Examination, November 2018
Choice Based Grading System (CBGS)
MATHEMATICS-III

Note : **(i)** Attempt any five questions out of eight.
(ii) All questions carry equal marks.
(iii) In case of any doubt or dispute the english version question should be treated as final.

1. **(a)** Find a root of the equation $x^3 - 4x - 9 = 0$ using bisection method correct to three decimal places. **(See Unit-I, Page-8, Prob.2)**

(b) Using Netwon-Raphson method, find the real root of the equation $3x = \cos x + 1$. **(See Unit-I, Page-10, Prob.6)**

2. **(a)** Find the cubic polynomical, which takes the following values –

x	0	1	2	3
f(x)	1	2	1	10

(See Unit-I, Page-46, Prob.42)

(b) By means of Newton's divided difference formula, find $f(8)$ an $f(9)$ from the following table –

x	4	5	7	10	11	13
$f(x)$	48	100	294	900	1210	2028

(See Unit-I, Page-52, Prob.48)

3. **(a)** Using Simpon's 1/3 rule, obtain $\int_{-3}^{3} x^4 dx$ by taking seven ordinates. Compare it with exact values. **(See Unit-II, Page-82, Prob.18)**

****Now, according to new revised syllabus of R.G.P.V., it is not included in syllabus**

(b) Solve the following system by Gauss elimination method.

$6x + 3y + 2z = 6$
$6x + 4y + 3z = 0$
$20x + 15y + 12z = 0$ **(See Unit-II, Page-100, Prob.35)**

4. (a) Solve the following system of equation by Gauss-Seidel method.

$27x + 6y - z = 85$
$6x + 15y + 2z = 72$
$x + y + 54z = 110$ **(See Unit-II, Page-120, Prob.50)**

(b) Using Lagrange's interpolation formula, find the cubic polynomial that takes the following values –

x	0	1	2	4
y	1	1	2	5

(See Unit-I, Page-60, Prob.58)

5. (a) Using Runge-Kutta method of fourth order solve the differenrtial equation –

$$\frac{dy}{dx} = xy \text{ for } x = 1.2$$

Given that y (1) = 2 (Take h = 0.1)**(See Unit-III, Page-147, Prob.22)**

(b) Using Picard's method find y for x = 0.1. Given that

$$\frac{dy}{dx} = \frac{y-x}{y+x}; y(0) = 1$$ **(See Unit-III, Page-134, Prob.5)**

6. (a) Solve the given equation for y (1.1) using Taylor series method –

$$\frac{dy}{dx} = 2y + 3e^x; y(0) = 0$$ **(See Unit-III, Page-137, Prob.9)**

(b) Find the following –

(i) $L\{t^2 \sin at\}$

(ii) $L\left\{\int_0^t e^{-t} \cos t \, dt\right\}$

(See Unit-IV, Page-204, Prob.23)

7. (a) Using convolution theorem find $L^{-1}\left\{\frac{1}{(s-1)(s+2)}\right\}$

(See Unit-IV, Page-227, Prob.59)

(b) Using Laplace transform, solve equation $(D^2 + 2D + 1)\, y = t$. Given that y (0) = – 3, y (1) = – 1. **(See Unit-IV, Page-241, Prob.78)**

8. (a) Find the probability that at most 5 defective fuses will be found in a box of 200 fuses of experience shows that 2 percent of such fuses are defective. **(See Unit-V, Page-288, Prob.28)**

(b) Find mean and variance of Binomial distribution.

(See Unit-V, Page-279, Prob.9)

RGPV

BT-301 (AU/CE/CM/EC/IP/ME) (CBGS)
B.E. III Semester
EXAMINATION, May 2019
Choice Based Grading System (CBGS)
MATHEMATICS-III

Note : **(i)** Attempt any five questions.
(ii) All questions carry equal marks.
(iii) Draw neat sketch if required.

1. (a) Find the real root of the equation $x\log_{10}x = 1.2$ by bisection method correct to four decimal places. **(See Unit-I, Page-9, Prob.4)**

(b) Find a real root of the equation $x = e^{-x}$ using the Newton-Raphson method. **(See Unit-I, Page-12, Prob.8)**

2. (a) Solve the following system of equations by Crout's method.

$x + y + z = 3$
$2x - y + 3z = 16$
$3x + y - z = -3$ **(See Unit-II, Page-105, Prob.40)**

(b) Find $\frac{dy}{dx}$ at x = 0.1 from the following table –

x :	0.1	0.2	0.3	0.4
y :	0.9975	0.9900	0.9776	0.9604

(See Unit-II, Page-69, Prob.6)

3. (a) Using Euler's method, compute y(0.04) for the differential equation –

$$\frac{dy}{dx} = -y, \; y(0) = 1. \text{ Take } h = 0.01.$$ **(See Unit-III, Page-140, Prob.12)**

(b) Given that $\frac{dy}{dx} = \log_{10}(x + y)$, y(0) = 1, find y(0.2) using modified Euler's method. **(See Unit-III, Page-144, Prob.18)**

4. (a) Find the Laplace transform of

$$f(t) = \begin{cases} 1 & 0 \le t < 2 \\ t-2 & 2 \le t \end{cases}$$ **(See Unit-IV, Page-200, Prob.14)**

(b) Write three properties of Laplace transform.

(See Unit-IV, Page-194, Q.1)

5. (a) What do you mean by probability density function ? **(See Unit-V, Page-268, Q.2)**

(b) Explain the binomial theorem in brief. **(See Unit-V, Page-279, Q.4)**

6. (a) The following table is given –

x :	0	1	2	5
y :	2	3	12	147

What is the form of the function ? **(See Unit-I, Page-51, Prob.47)**

(b) Evaluate $\int_0^1 \frac{dx}{1+x^2}$ using Simpson's $\frac{1}{3}$ rule taking $h = \frac{1}{4}$.

(See Unit-II, Page-80, Prob.15)

7. (a) If 10% of bolt's produced by a macine are defective. Determine the probability that out of 10 bolts, chosen at random (i) 1 (ii) none (iii) at most 2 bolts will be defective. **(See Unit-V, Page-283, Prob.19)**

(b) Evaluate $\int_0^6 \frac{dx}{1+x^2}$ by using Simpson's $\frac{3}{8}$ rule.

(See Unit-II, Page-90, Prob.30)

8. (a) The random variable X has a Poisson distribution if P(X = 1) = 0.01487, P(X = 2) = 0.04461. Then find P(X = 3).

(See Unit-V, Page-292, Prob.39)

(b) Find the Fourier transform of

$$f(x) = \begin{cases} 1 \text{ for } |x| < a \\ 0 \text{ for } |x| > a \end{cases}$$

(See Unit-IV, Page-250, Prob.83)

RGPV

BT-401 (AT/CS/EE/EX/EI/FT/IT/MI) (CBGS)
B.Tech., IV Semester
EXAMINATION, May 2019
Choice Based Grading System (CBGS)
MATHEMATICS-III

Note : (i) Attempt any five questions.
(ii) All questions carry equal marks.

1. (a) Find a root of the following equation using bisection method correct to three places of decimal –
$f(x) = x^3 - 3x - 5$. **(See Unit-I, Page-7, Prob.1)**

(b) Solve the algebraic equation $x^3 + 2x^2 + 10x - 20 = 0$ by Newton-Raphson metnod. **(See Unit-I, Page-15, Prob.13)**

2. (a) Find a polynomial which takes the following values –

x :	1	3	5	7	9	11
f(x) :	3	14	19	21	23	28

and hence compute f(x) at x = 2. **(See Unit-I, Page-47, Prob.43)**

(b) Given $\log_{10} 654 = 2.8156$, $\log_{10} 658 = 2.8182$, $\log_{10} 659 = 2.8189$, $\log_{10} 661 = 2.8202$. Find by using Newton's divided difference formula, the value of $\log_{10} 656$. **(See Unit-I, Page-55, Prob.53)**

3. (a) Find the first and second order derivative of f(x) at x = 1.5 if –

x :	1.5	2.0	2.5	3.0	3.5	4.0
f(x) :	3.375	7.000	13.625	24.000	38.875	59.000

(See Unit-II, Page-75, Prob.11)

(b) Apply Simpson's $\frac{1}{3}$ rule, to evaluate the integration $\int_0^1 \frac{dx}{1+x}$.

(See Unit-II, Page-84, Prob.20)

4. (a) Solve the following equations by Gauss-Seidel iteration method –
$10x + y + z = 12$; $x + 10y + z = 12$; $x + y + 10z = 12$.

(See Unit-II, Page-121, Prob.51)

(b) Solve the equations $10x + y + z = 12$; $2x + 10 + y + z = 13$ and $x + y + 5z = 7$ by Gauss-Jordan method. **(See Unit-II, Page-102, Prob.38)**

5. (a) Use Picard's method to approximate the value of y, when x = 0.1, 0.2 and 0.3, given that y = 1 at x = 0 and $\frac{dy}{dx} = 1 + xy$, correct to three decimal places. **(See Unit-III, Page-132, Prob.2)**

(b) Using modified Euler's method, find an approximate vlaue of y, when x = 0.3, given that $\frac{dy}{dx} = x + y$ and y = 1, when x = 0.

(See Unit-III, Page-142, Prob.16)

6. (a) Find $L\{\sin t + 3\cos 2t + t^3 + t^{3/2} + 3e^{-2t} - e^{3t}\}$.

(See Unit-IV, Page-195, Prob.3)

(b) Find $L\left\{\frac{\sin t}{t}\right\}$. **(See Unit-IV, Page-203, Prob.20)**

7. (a) Find $L^{-1}\left\{\frac{s}{s^4+s^2+1}\right\}$. **(See Unit-IV, Page-221, Prob.51)**

(b) Solve the following differential equation –
$y''(t) - 3y'(t) + 2y(t) = 4t + e^{3t}$, when $y'(0) = 1$ and $y''(0) = -1$. (By Laplace transformation). **(See Unit-IV, Page-234, Prob.68)**

8. (a) Find the mean and variance of Poisson's distribution. **(See Unit-V, Page-289, Prob.32)**

(b) In a Binomial distribution the mean and standard deviations are 12 and 2 respectively. Find n and p. **(See Unit-V, Page-288, Prob.26)**

RGPV

BT-301 (AU/CE/CM/EC/IP/ME) (CBGS)
B.Tech., III Semester
EXAMINATION, November 2019
Choice Based Grading System (CBGS)
MATHEMATICS-III

Note : (i) Attempt any five questions.
(ii) All questions carry equal marks.

1. (a) Solve the equation $3x = \cos x + 1$ by Newton-Raphson method. **(See Unit-I, Page-10, Prob.6)**

(b) Prove that $\Delta \log f(x) = \log\left\{1+\frac{\Delta f(x)}{f(x)}\right\}$. **(See Unit-I, Page-32, Prob.24)**

2. (a) Given $\sin 45° = 0.7071$, $\sin 50° = 0.7660$, $\sin 55° = 0.8192$, $\sin 60° = 0.8660$, find $\sin 52°$, by using Newton's forward difference interpolation formula. **(See Unit-I, Page-44, Prob.39)**

(b) Find f(8) by using Newton's divided difference formula from the following data –

x :	4	5	7	10	11	13
f(x) :	48	100	294	900	1210	2028

(See Unit-I, Page-53, Prob.49)

3. (a) Find the first and second order derivative at the point x = 1.5 for the following data –

x :	1.5	2.0	2.5	3.0	3.5	4.0
f(x) :	3.375	7.000	13.625	24.000	38.875	59.000

(See Unit-II, Page-75, Prob.11)

(b) Find the value of log 2 from $\int_0^1 \frac{x^2}{1+x^3}dx$, using Simpson's $\frac{1}{3}$ rule by dividing it, into four equal parts. **(See Unit-II, Page-84, Prob.21)**

4. (a) Solve the following equations by Jacobi's method –
$10x + 2y + 3z = 15$
$x + 7y + 2z = 10$
$4x + 2y + 10z = 16$. **(See Unit-II, Page-110, Prob.44)**

(b) Solve the equation by Gauss-Elimination method –
$2x + y + 4z = 2$
$8x - 3y + 2z = 23$
$4x + 11y - z = 33$. **(See Unit-II, Page-100, Prob.34)**

5. (a) Use Euler's method to find y (0.4) from the differential equation $\frac{dy}{dx} = xy, y(0) = 1$ where step size is h = 0.1. **(See Unit-III, Page-140, Prob.13)**

(b) Use Runge-Kutta method to approximate y, when x = 0.1, given that x = 0, when y = 1 and $\frac{dy}{dx} = x + y$. **(See Unit-III, Page-146, Prob.21)**

6. (a) Find $L\left\{e^{3t}\sin t + \frac{\sin t}{t}\right\}$. **(See Unit-IV, Page-205, Prob.25)**

(b) Evaluate $L\left\{\int_0^t \frac{\sin t}{t}dt\right\}$. **(See Unit-IV, Page-204, Prob.22)**

7. (a) Evaluate $L^{-1}\left\{\frac{s+1}{s^2+6s+25}\right\}$. **(See Unit-IV, Page-216, Prob.40)**

(b) Solve $\left(D^2+9\right)y=\cos 2t$ if $y(0)=1, y\left(\frac{\pi}{2}\right)=-1$.

(See Unit-IV, Page-239, Prob.76)

8. (a) Find the mean and variance of the Poisson's distribution.

(See Unit-V, Page-282, Prob.32)

(b) Six dice are thrown 729 times. How many times do you expect at least three dice to show a five or six. (See Unit-V, Page-280, Prob.11)

RGPV

BT-401 (AT/CS/EE/EX/EI/FT/IT/MI) (CBGS)
B.Tech., IV Semester
EXAMINATION, November 2019
Choice Based Grading System (CBGS)
MATHEMATICS-III

Note : (i) Attempt any five questions.

(ii) All questions carry equal marks.

1. (a) Find a real root of the equation $x^3 - 9x + 1 = 0$ by method of false position. (See Unit-I, Page-19, Prob.17)

(b) Prove that $e^x = \left\{\frac{\Delta^2}{E}\right\} e^x . \frac{Ee^x}{\Delta^2 e^x}$, the interval of differencing being h.

(See Unit-I, Page-31, Prob.22)

2. (a) Using Newton's forward interpolation formula, find the value of f (1.6) if –

x :	1	1.4	1.8	2.2
y :	3.49	4.82	5.96	6.5

(See Unit-I, Page-43, Prob.38)

(b) The values of x and y are given as below –

x :	5	6	9	11
y :	12	13	14	16

Find the value of y, when x = 10 by using Lagranges interpolation formula.

x :	5	6	9	11
y :	12	13	14	16

(See Unit-I, Page-57, Prob.55)

3. (a) Evaluate the integration $\int_0^1 \frac{dx}{1+x^2}$ by using Simpson's $\frac{1}{3}$ and $\frac{3}{8}$ rule. (See Unit-II, Page-88, Prob.27)

(b) Given

x :	0.1	0.2	0.3	0.4
y = f (x) :	1.10517	1.22140	1.34986	1.49182

find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ at $x = 0.4$. (See Unit-II, Page-67, Prob.3)

4. (a) Solve the following equations by Gauss-Seidal method –

$10x + y + z = 12;\ x + 10y + z = 12;\ x + y + 10z = 12.$

(See Unit-II, Page-121, Prob.51)

(b) Solve the following equations by Relaxation method –

$10x - 2y - 2z = 6;\ -x + 10y - 2z = 7;\ -x - y + 10z = 8.$

(See Unit-II, Page-123, Prob.53)

5. (a) Solve $\frac{dy}{dx} = 1 - 2xy$, given that y (0) = 0, by Taylor's series method.

(See Unit-III, Page-138, Prob.10)

(b) Solve the differential equation $\frac{dy}{dx} = x + y$ by the Milne's method for x = 0.2, given that –

x :	0	0.05	0.10	0.15
y :	1	1.0525	1.1103	1.1737

(See Unit-III, Page-154, Prob.28)

6. (a) Find $L\left\{e^{-t}\sin^2 t + \cos t\right\}$. (See Unit-IV, Page-202, Prob.17)

(b) Find $L\left\{\frac{1-\cos 2t}{t}\right\}$. (See Unit-IV, Page-197, Prob.9)

7. (a) Evaluate $L^{-t}\left\{\frac{6s^2+22s+18}{(s+1)(s+2)(s+3)}\right\}$. (See Unit-IV, Page-217, Prob.42)

(b) Solve $y'' - 2y' + 2y = 0$ given $y(0) = y'(0) = 1$, by Laplace transformation method. **(See Unit-IV, Page-233, Prob.67)**

8. **(a)** Find the mean and variance of Binomial distribution.

(See Unit-V, Page-279, Prob.9)

(b) Fit a Poisson's distribution and calculate theoretical frequencies for the following data –

x :	0	1	2	3	4
y :	46	38	22	9	1

(See Unit-V, Page-293, Prob.40)