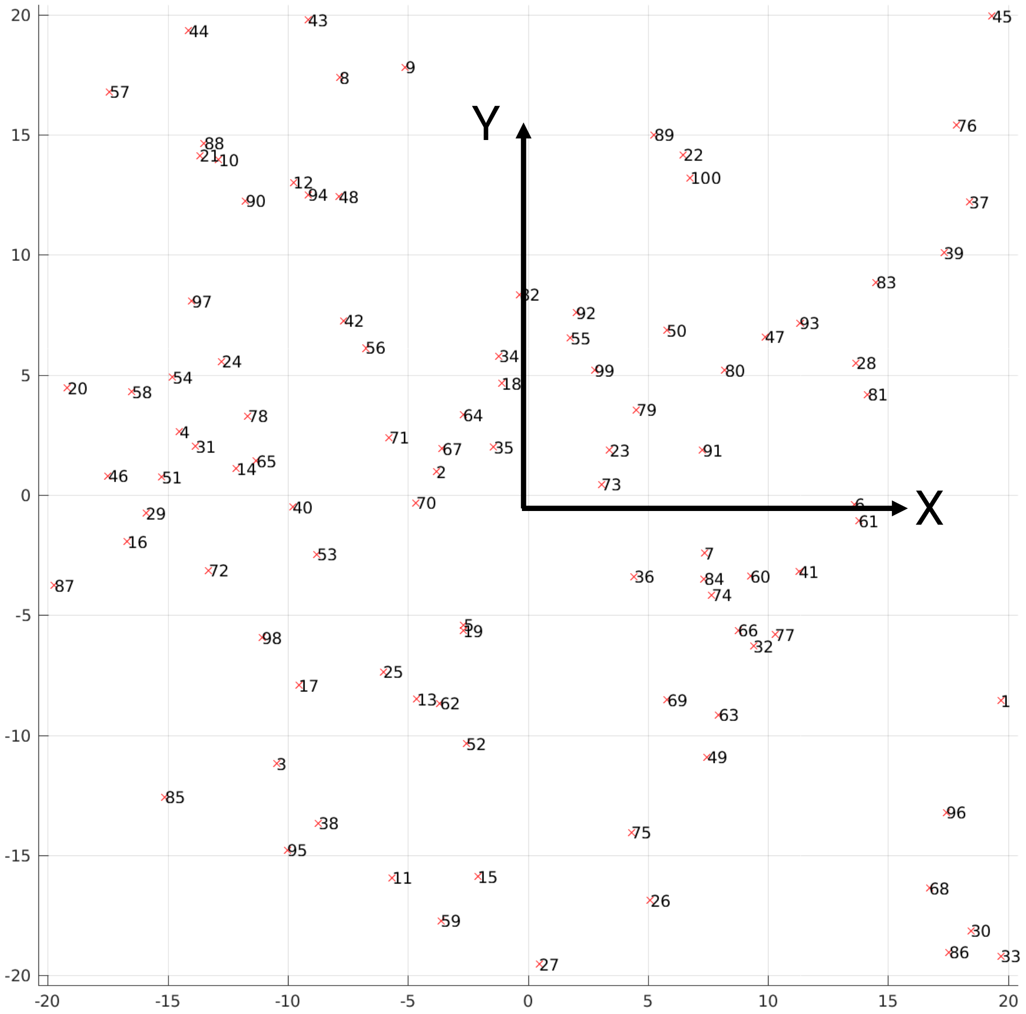
ENAE 788M
Hands-On Autonomous Aerial Robotics

FACTOR GRAPH BASED FILTERING USING GTSAM

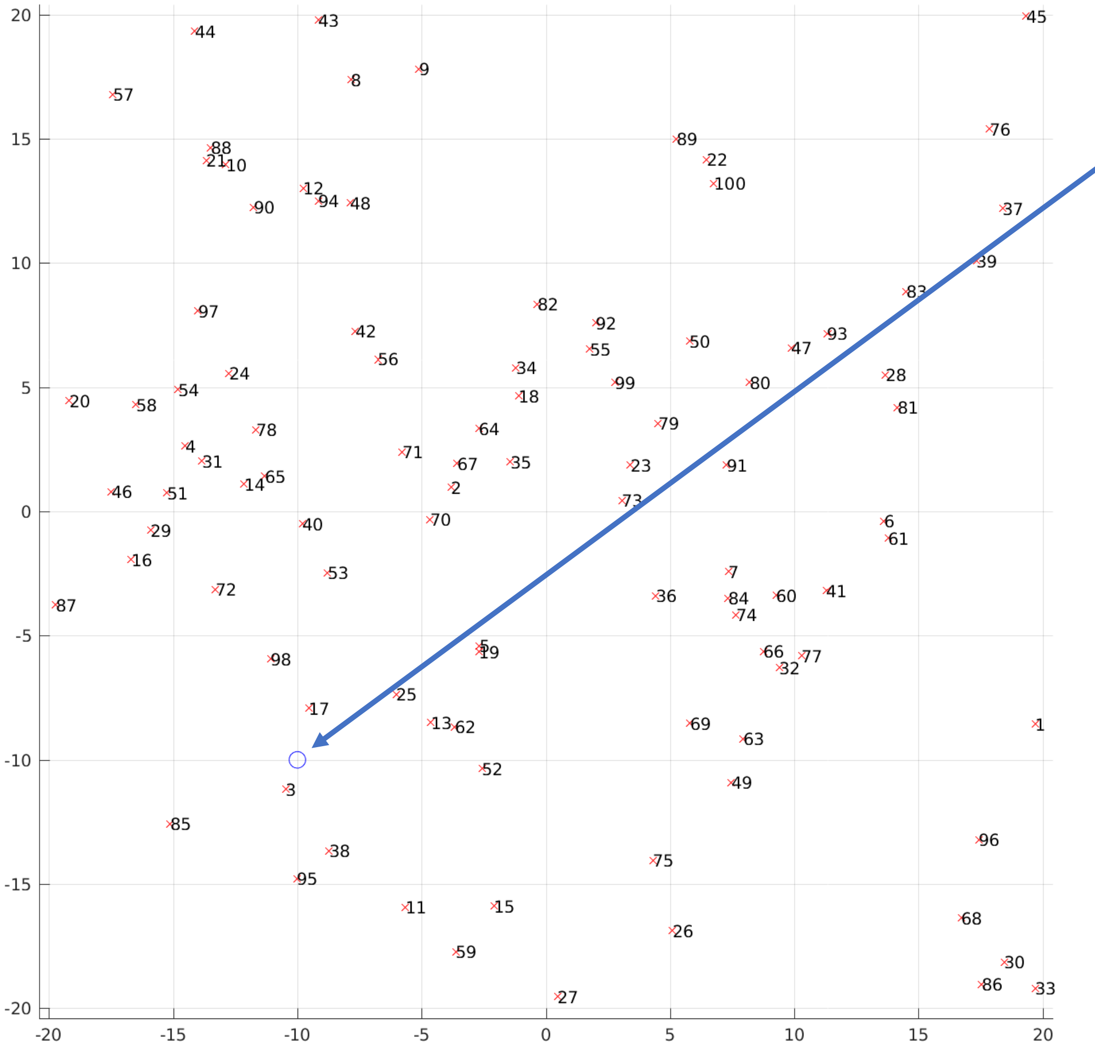
2D Robot World



- World with landmarks
- Landmarks are unique and numbered
- Each landmark l_k is denoted by,

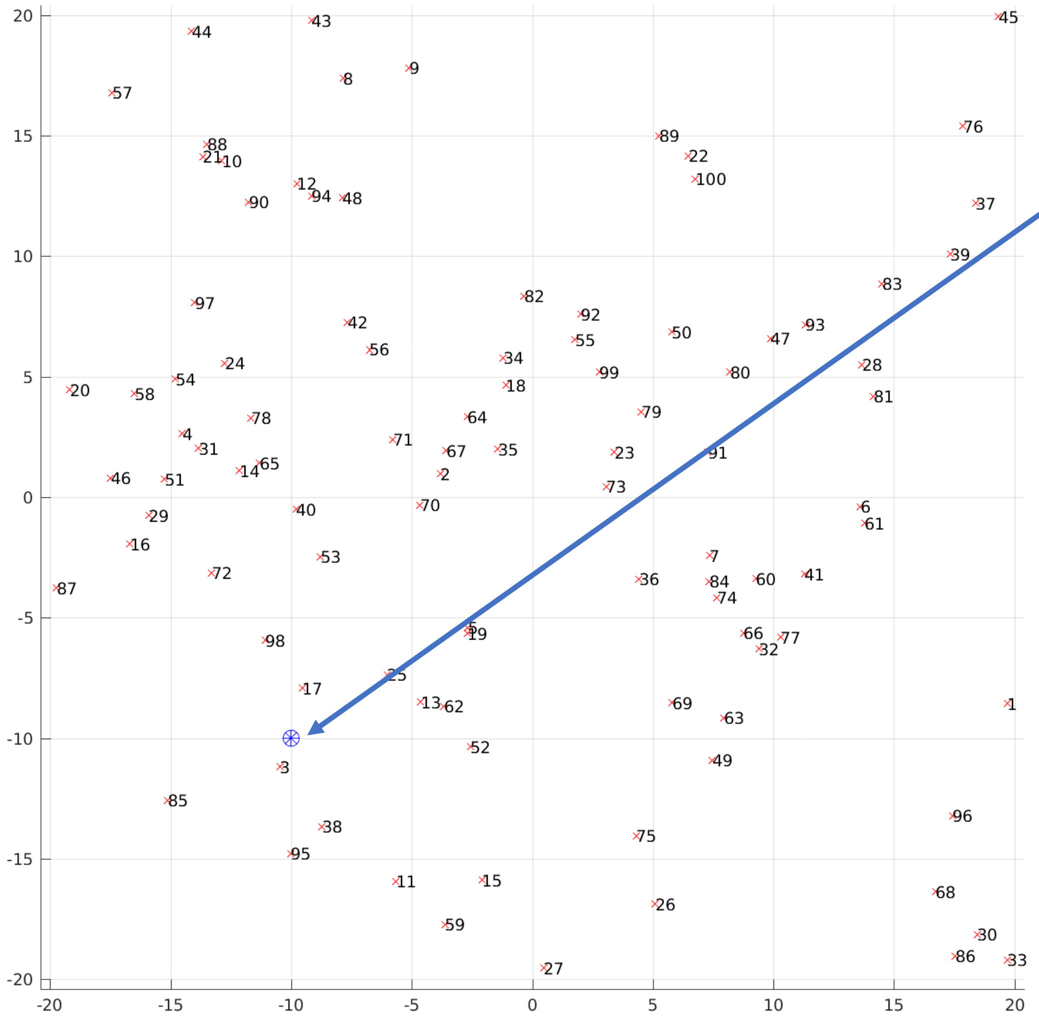
$$l_k = \begin{bmatrix} l_{k,x} \\ l_{k,y} \end{bmatrix}$$

Robot Start



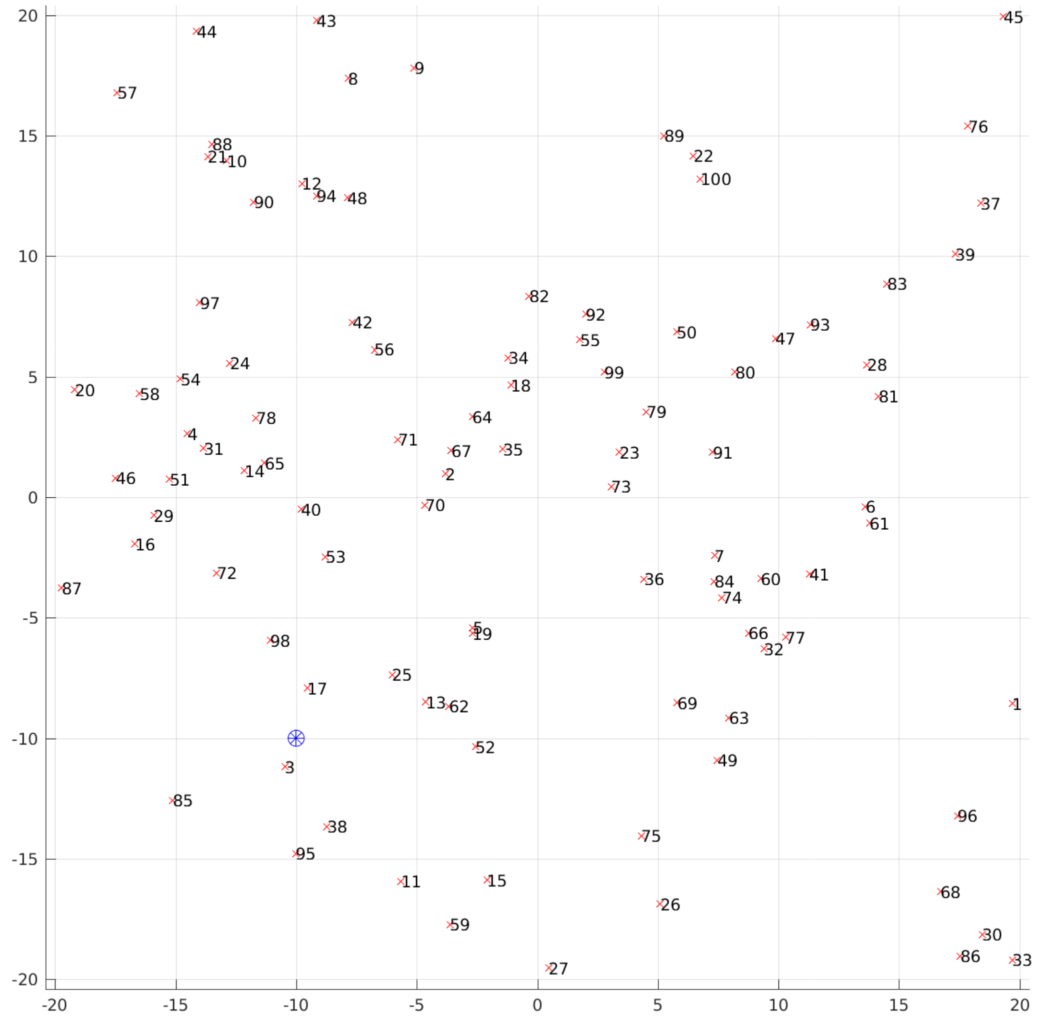
Robot Starts here

Robot Start



Robot Odometry says this is origin

Robot State

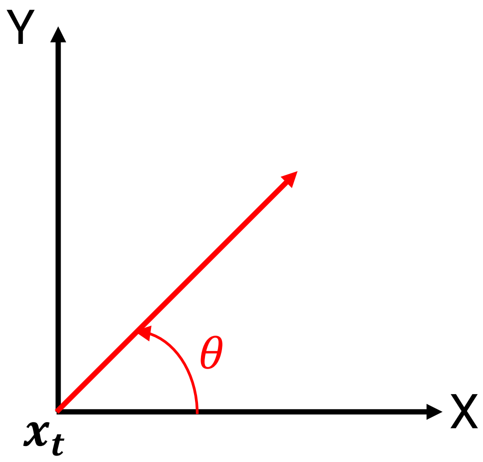


Robot State is given by,

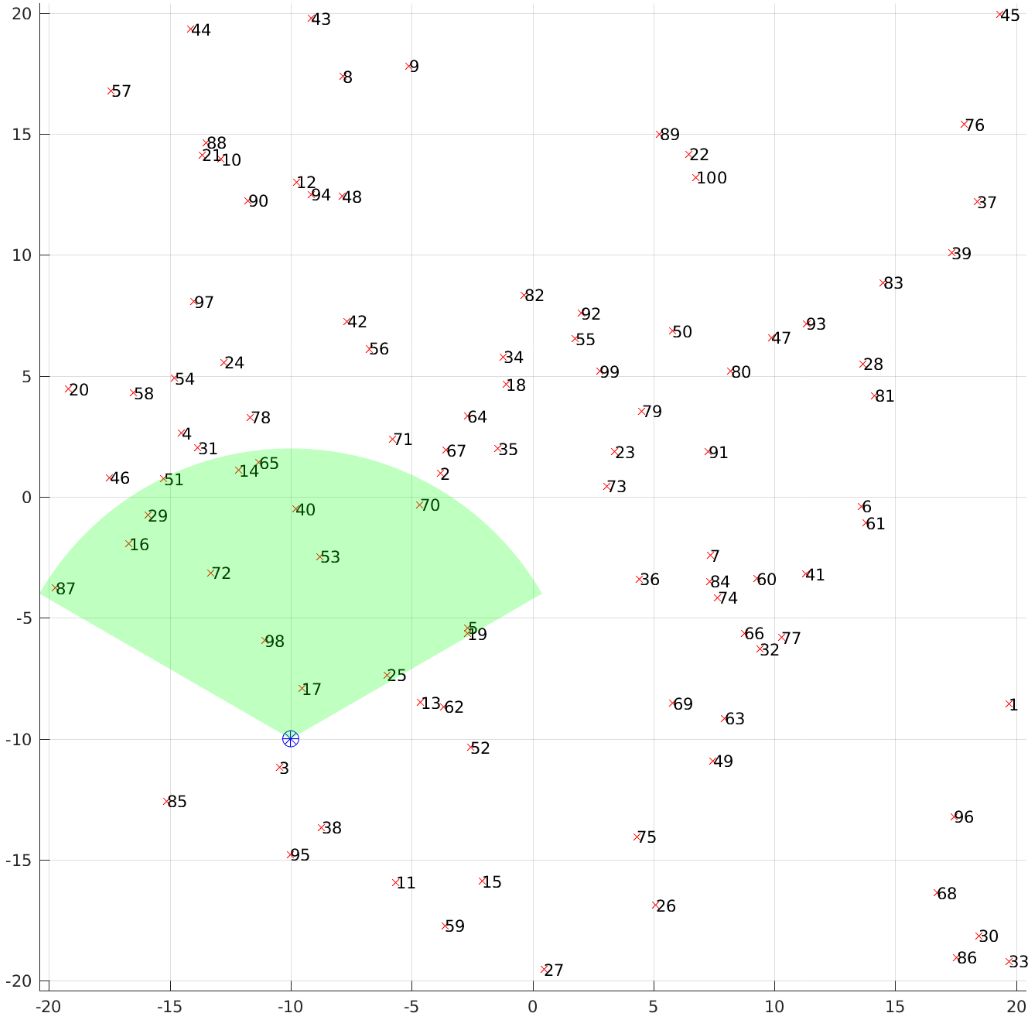
$$\boldsymbol{x} = \begin{bmatrix} x \\ y \\ \theta \end{bmatrix}$$

Because this varies with respect to time,
We will write the state as,

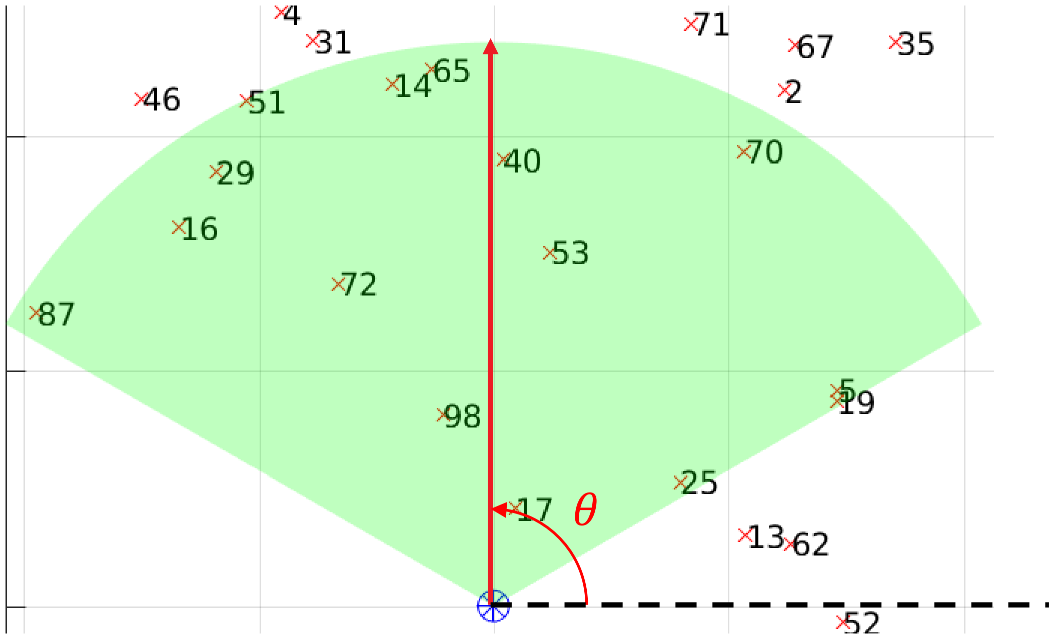
$$\boldsymbol{x_t} = \begin{bmatrix} x_t \\ y_t \\ \theta_t \end{bmatrix} = \begin{bmatrix} x \\ y \\ \theta \end{bmatrix}_t$$



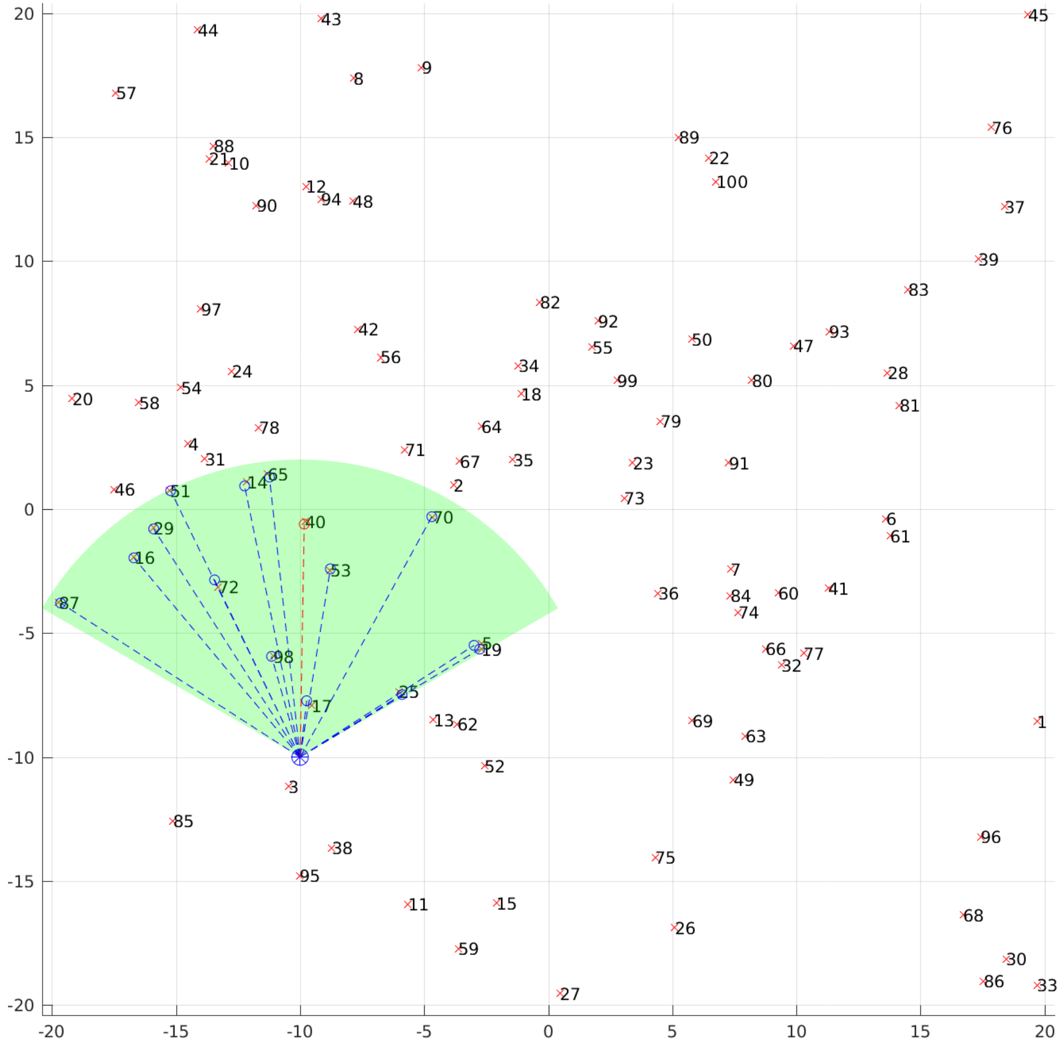
Sensor



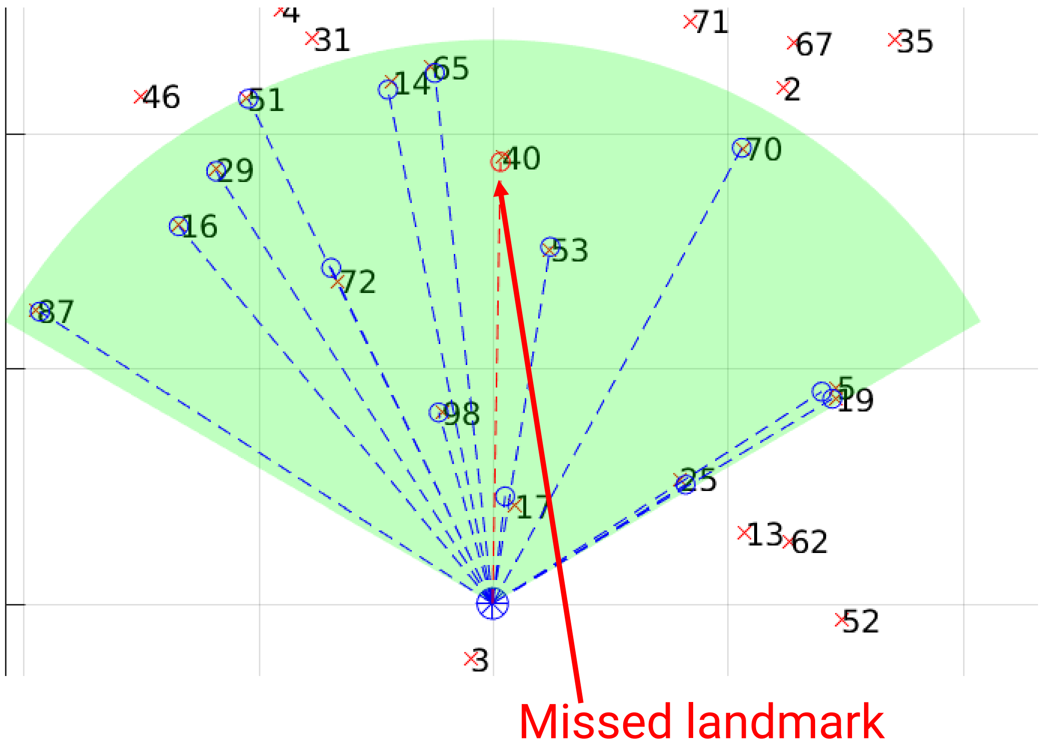
Robot has a Wide FOV Camera/Lidar of 120°



Measurement Model



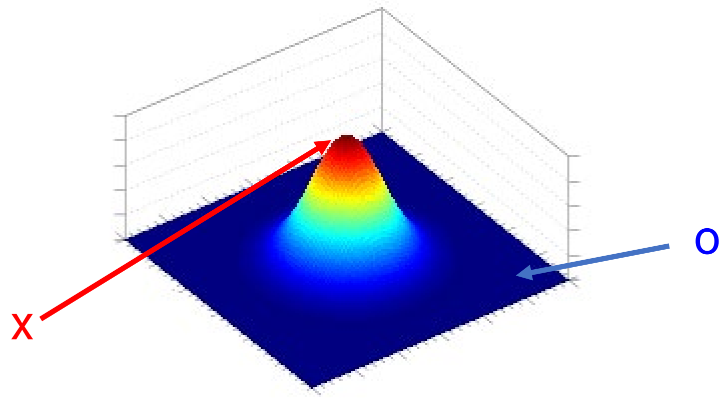
Measurement Model



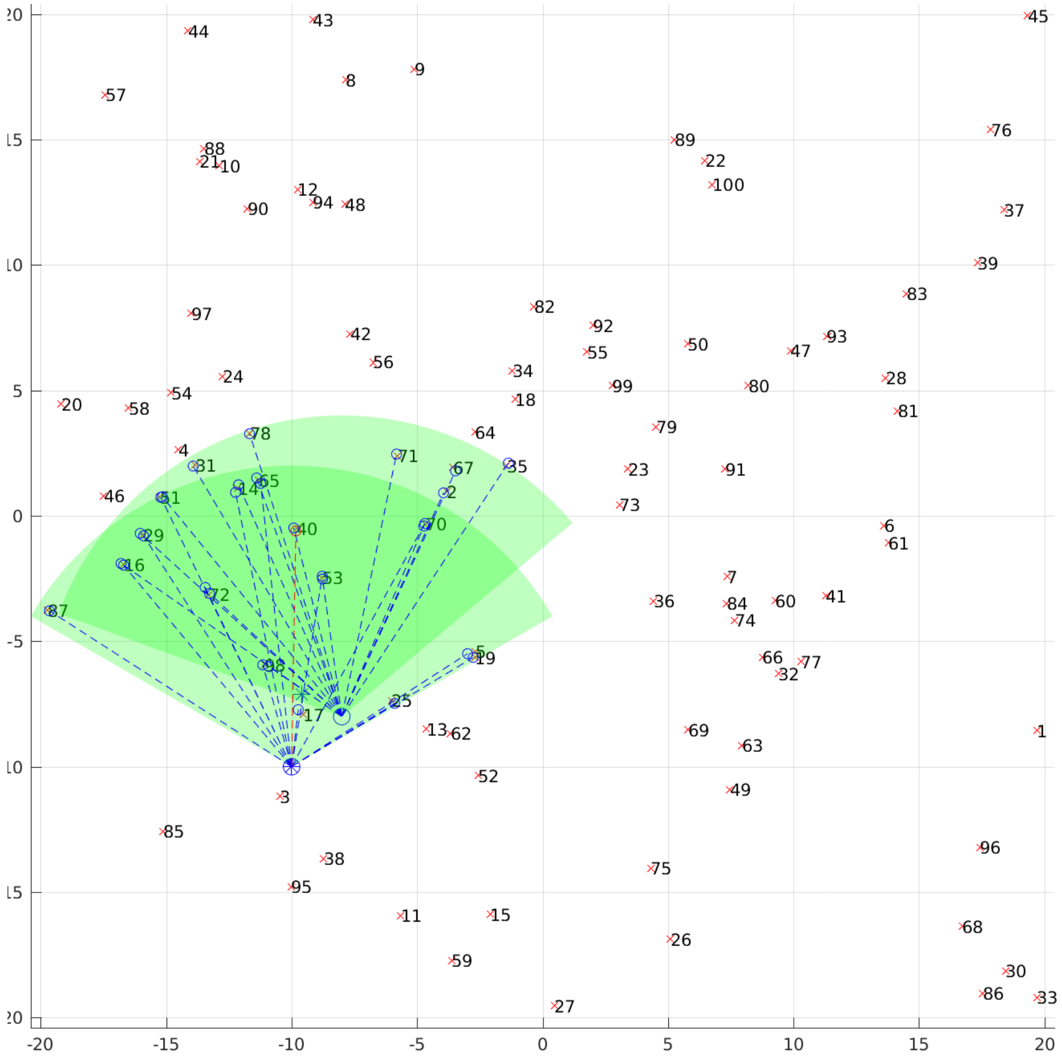
- At $\boldsymbol{x_t}$ the robot observes l_k landmarks where k denotes individual landmark IDs
- Sensor is not perfect hence you see the landmark with a probability of $p_{obs} = 0.95$
- The measurement at time t with respect to landmark l_k obtained by the robot is given by

$$m_{t,k} = \begin{bmatrix} m_{t,k,x} \\ m_{t,k,y} \end{bmatrix}$$

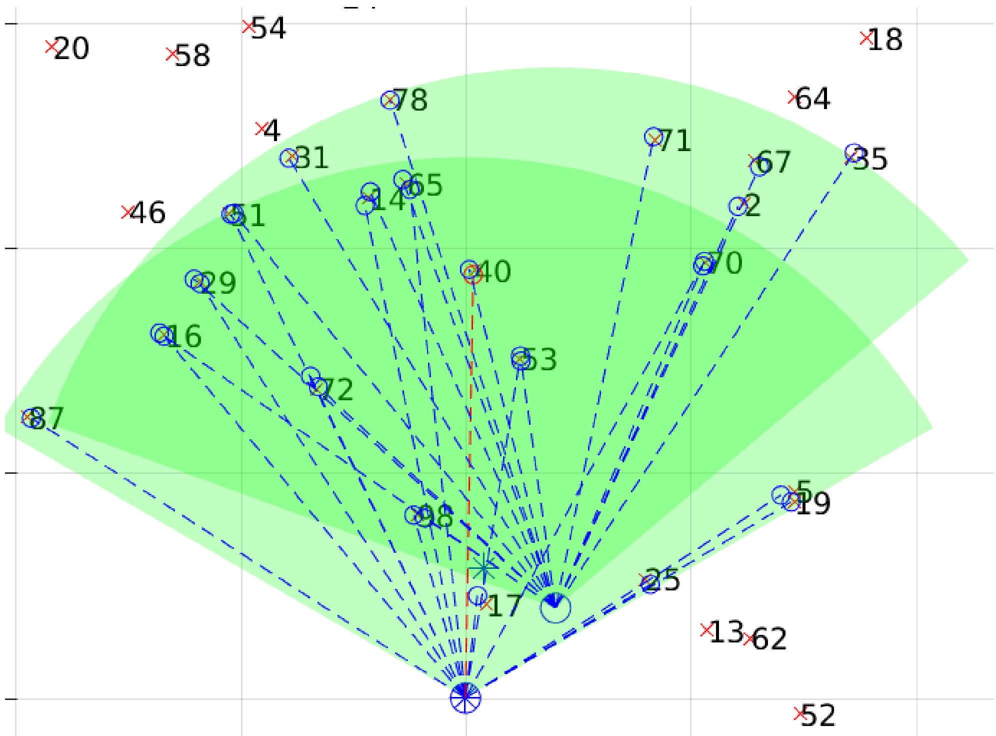
- $m_{t,k}$ is noisy and can be thought of as drawn from $\mathcal{N}(\widehat{m}_{t,k}, \Sigma_m)$



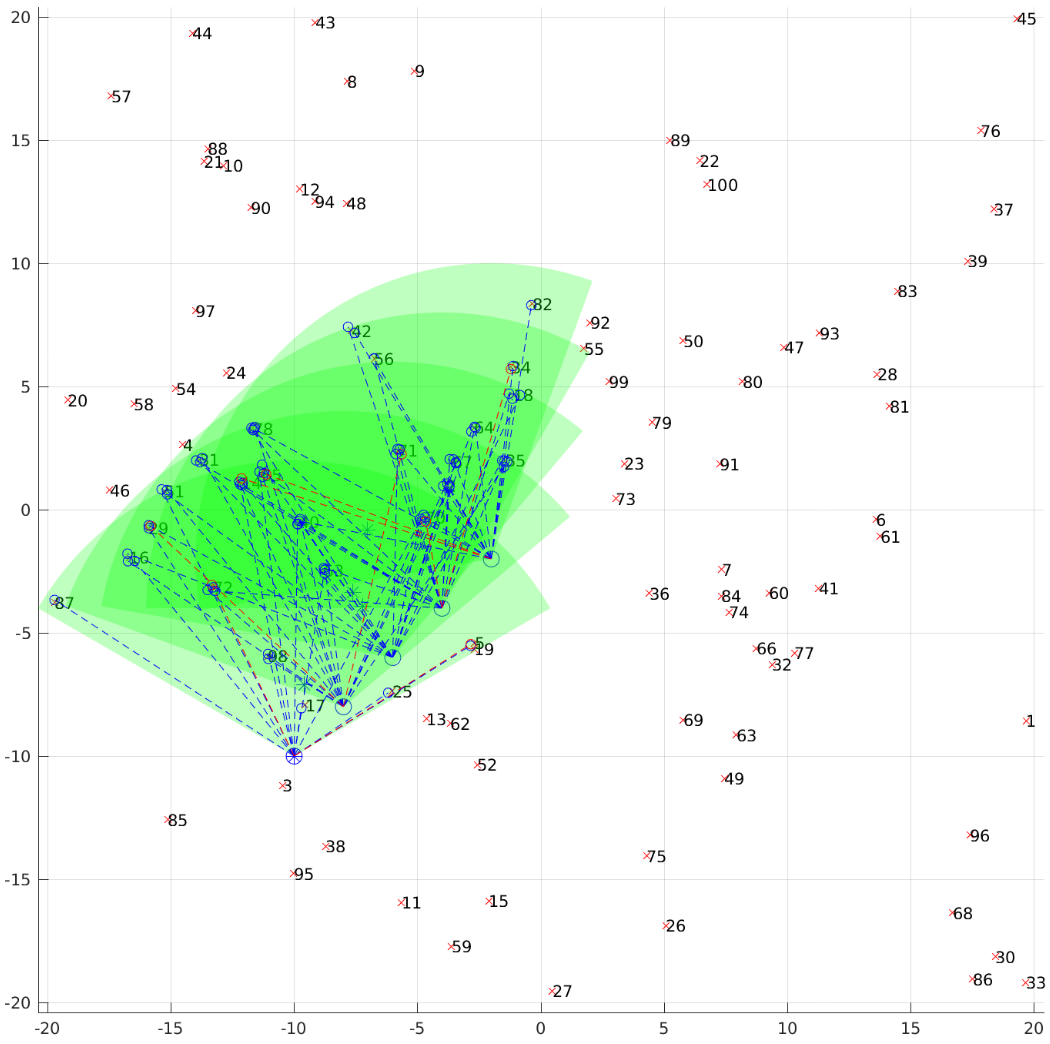
Robot at $t = 1$



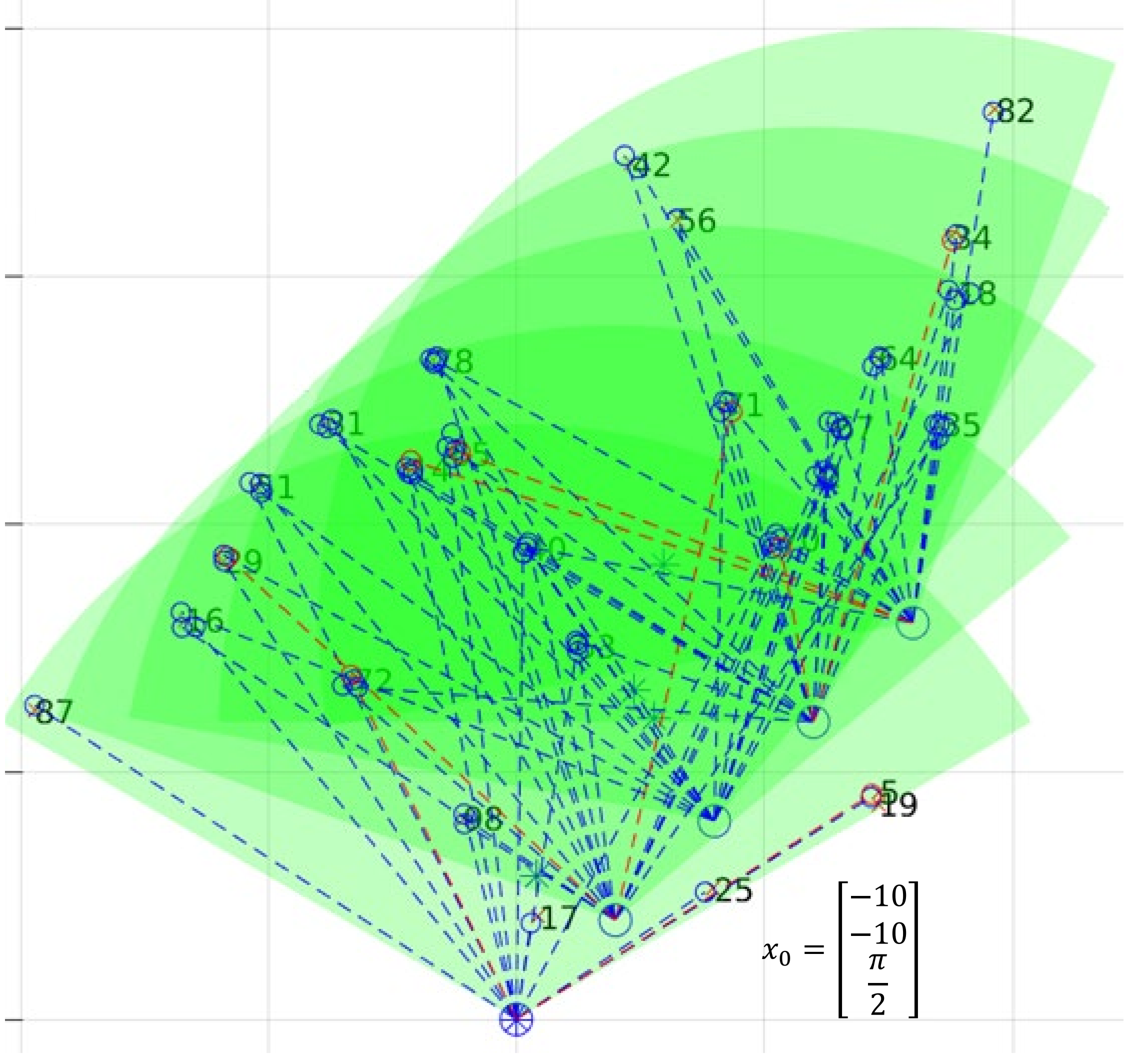
- Odometry is denoted by o_t^{t+1} between times t and $t+1$
- It is obtained by some wheel encoder and can be thought of as drawn from $\mathcal{N}(\hat{o}_t^{t+1}, \Sigma_o)$



For multiple steps



SLAM Problem

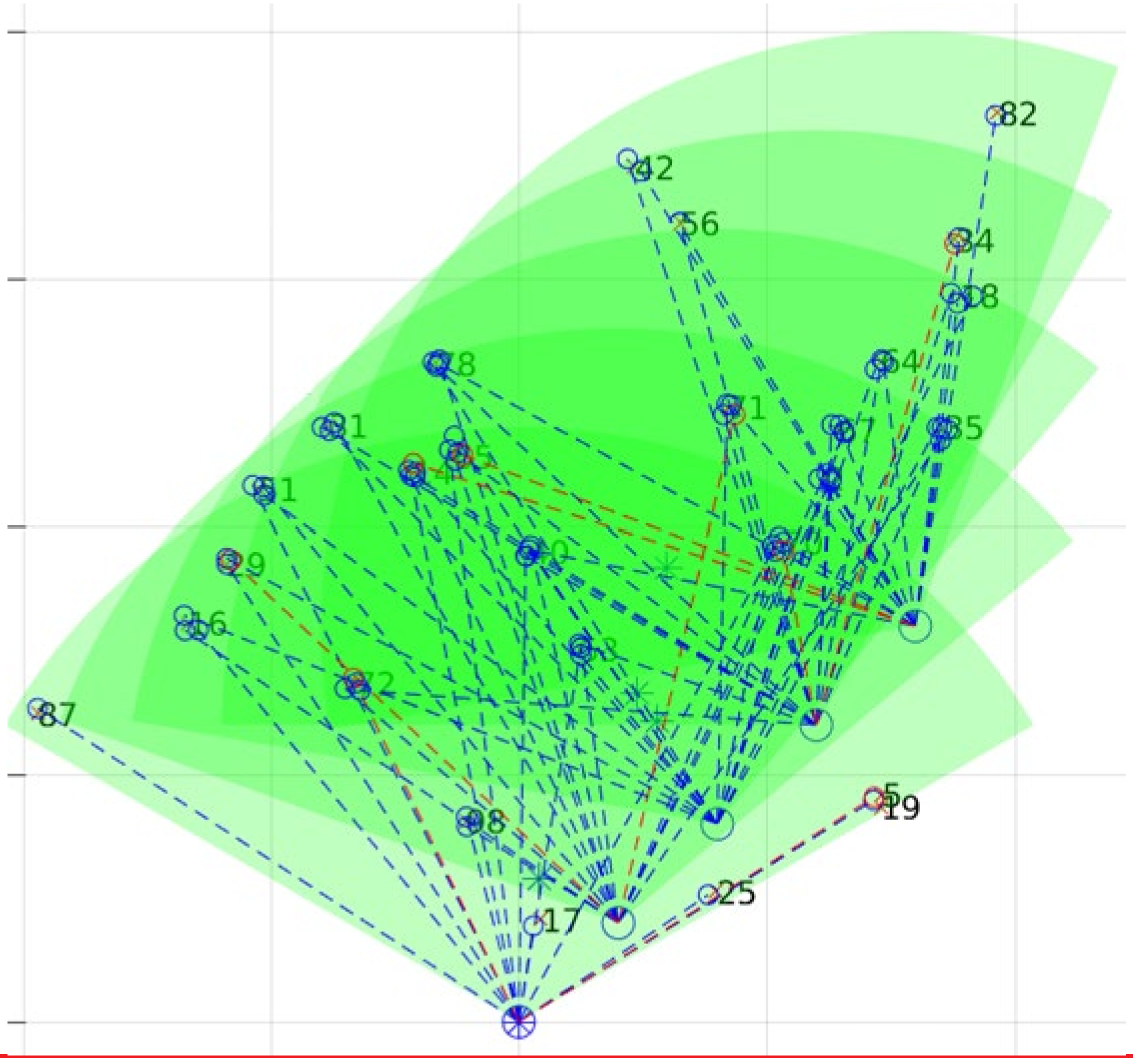


Given Initial pose x_1, odometry o_t^{t+1} and landmark measurements $m_{t,k}$

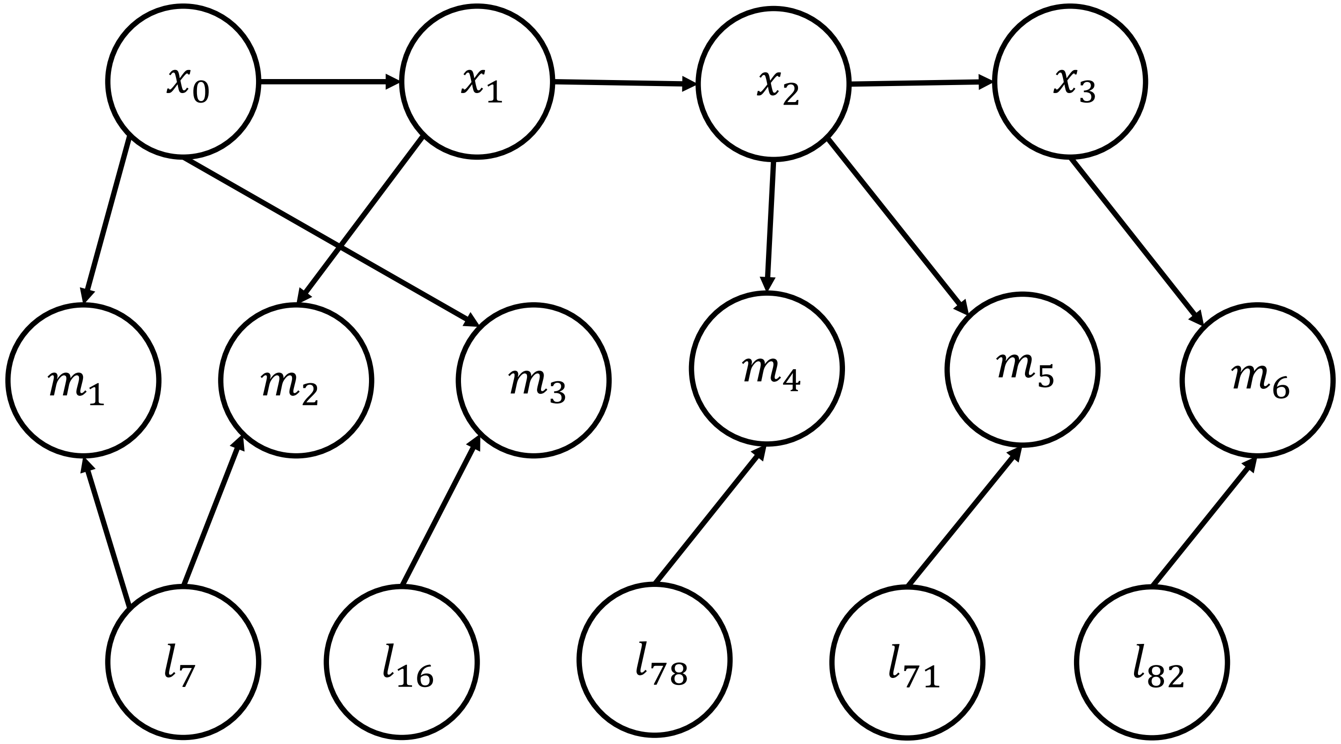
Obtain landmark locations l_k and robot pose x_t

SLAM stands for "Simultaneous Localization and Mapping"

SLAM as a Bayes Net



SLAM as a Bayes Net: Graph



SLAM as a Bayes Net: Math

Motion Model:

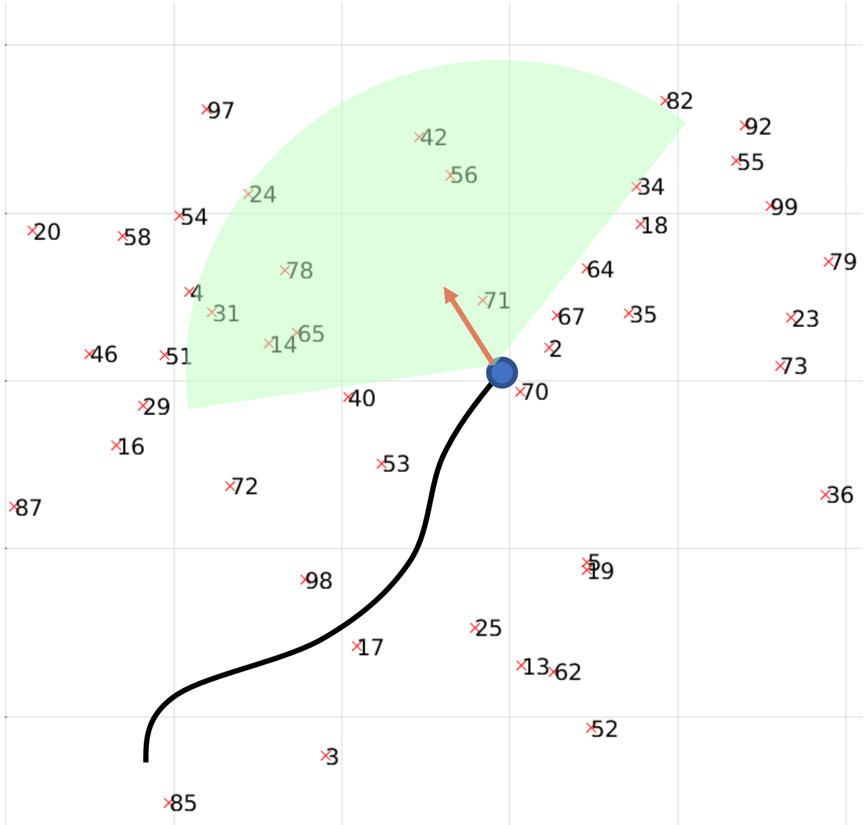
$$x_t = f_t(x_{t-1}, u_t) + w_i \Leftrightarrow P(x_t | x_{t-1}, u_t) \propto \mathrm{e}^{-\frac{1}{2}\| f_t(x_{t-1}, u_t) - x_t \|^2_{\Lambda_t}}$$

Measurement Model:

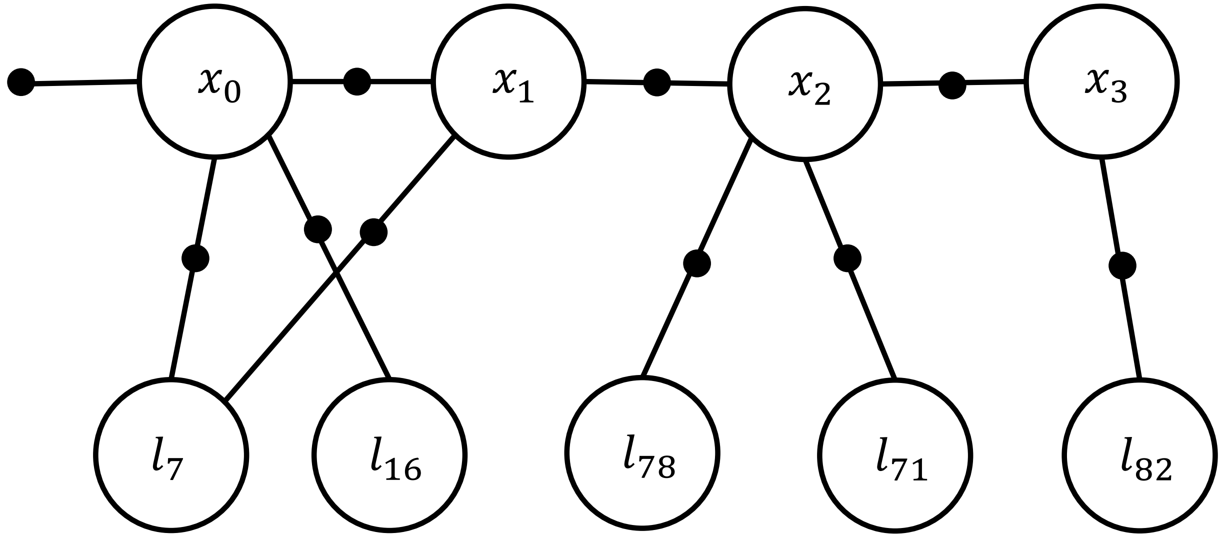
$$m_i = h_i(x_{t,i}, l_{k,i}) + v_i \Leftrightarrow P(m_i | x_{t,i}, l_{k,i}) \propto \mathrm{e}^{-\frac{1}{2}\| h_i(x_{t,i}, l_{k,i}) - m_i \|^2_{\Sigma_i}}$$

MAP to maximize:

$$P(X, L, M) = P(x_0) \prod_{t=1}^{T} P(x_t | x_{t-1}, u_t) \prod_{i=1}^{M} P(m_i | x_{t,i}, l_{k,i})$$



SLAM as a Factor Graph: Graph



SLAM as a Factor Graph: Math

Prior:

$$\phi_0(x_0) \propto P(x_0)$$

Motion Model:

$$\psi_{t-1,t}(x_{t-1}, x_t) \propto P(x_t | x_{t-1}, u_t)$$

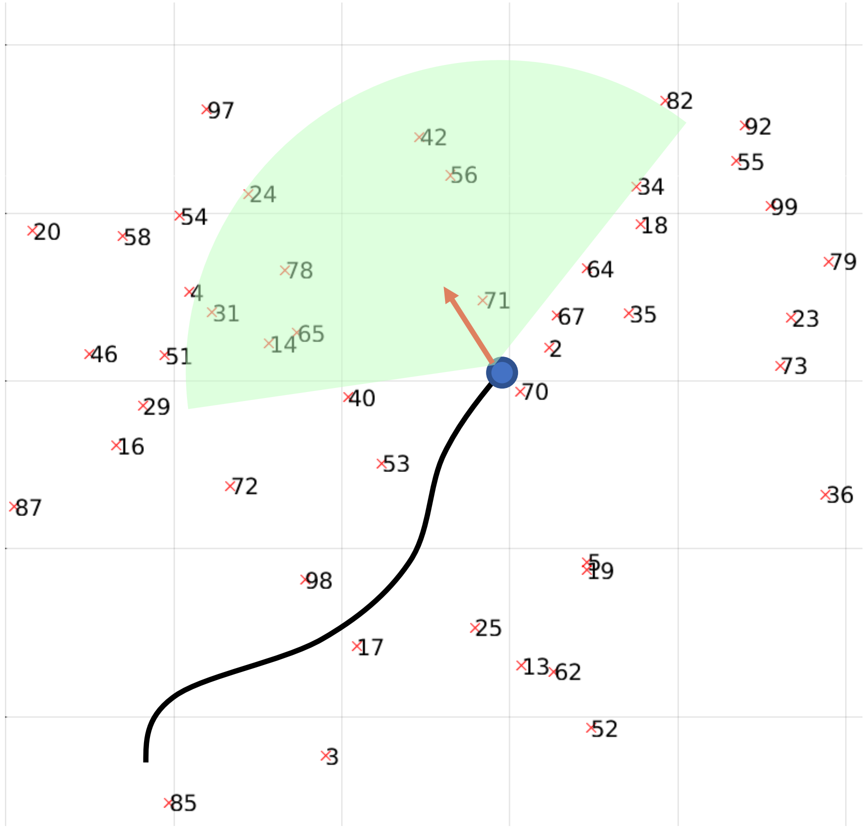
Measurement Model:

$$\psi_{t,k}(x_t, l_k) \propto P\left(m_{t,k} \middle| x_t, l_k\right)$$

Value of the graph to maximize:

$$P(\Theta) \propto \prod_{t=0}^{T} \phi_t(\theta_t) \prod_{\{i,j\},i<j} \psi_{ij}(\theta_i, \theta_j)$$

$$\Theta \triangleq (X, L)$$



SLAM as Non-Linear Least Squares

- Maximum A Posteriori (MAP) estimation

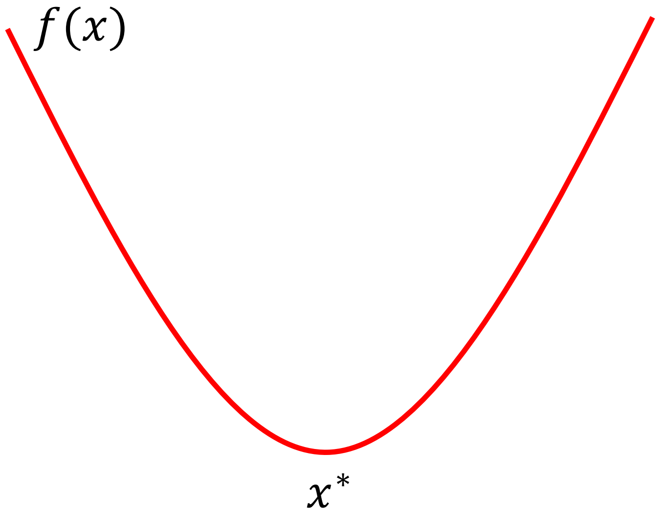
$$f(\Theta) = \prod_i f_i(\Theta_i)\,, \Theta \triangleq (X, L)\ \forall\ f_i(\Theta_i) \propto \mathrm{e}^{-\frac{1}{2}\|h_i(\Theta_i) - m_i\|_{\Sigma_i}^2}$$

$$\Theta^* = \underset{\Theta}{\operatorname{argmax}}\, f(\Theta)$$

- Negative Log Likelihood (NLL)

$$\underset{\Theta}{\operatorname{argmin}}(-\log f(\Theta)) = \underset{\Theta}{\operatorname{argmin}}\left(\frac{1}{2}\sum_i \|h_i(\Theta_i) - m_i\|_{\Sigma_i}^2\right)$$

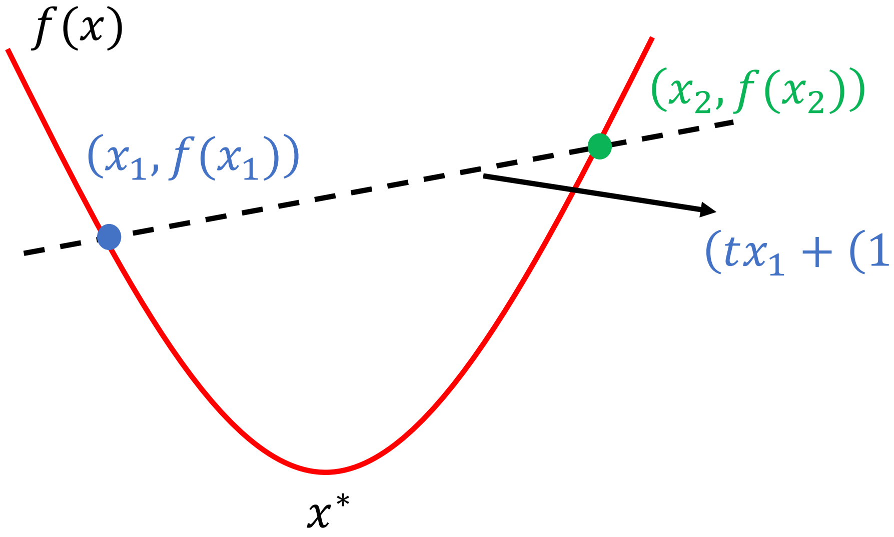
Numerical Optimization 101



Convex function

$$x^* = \underset{x}{\operatorname{argmin}}\, f(x)$$

Convex Function

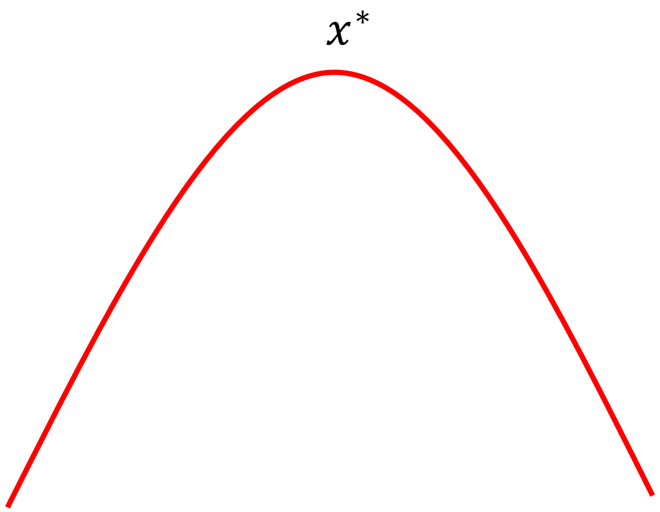


Convex function

Known as "Jensen's Inequality"

$(tx_1 + (1-t)x_2, f(tx_1 + (1-t)x_2)), t \in [0,1]$

Concave Function

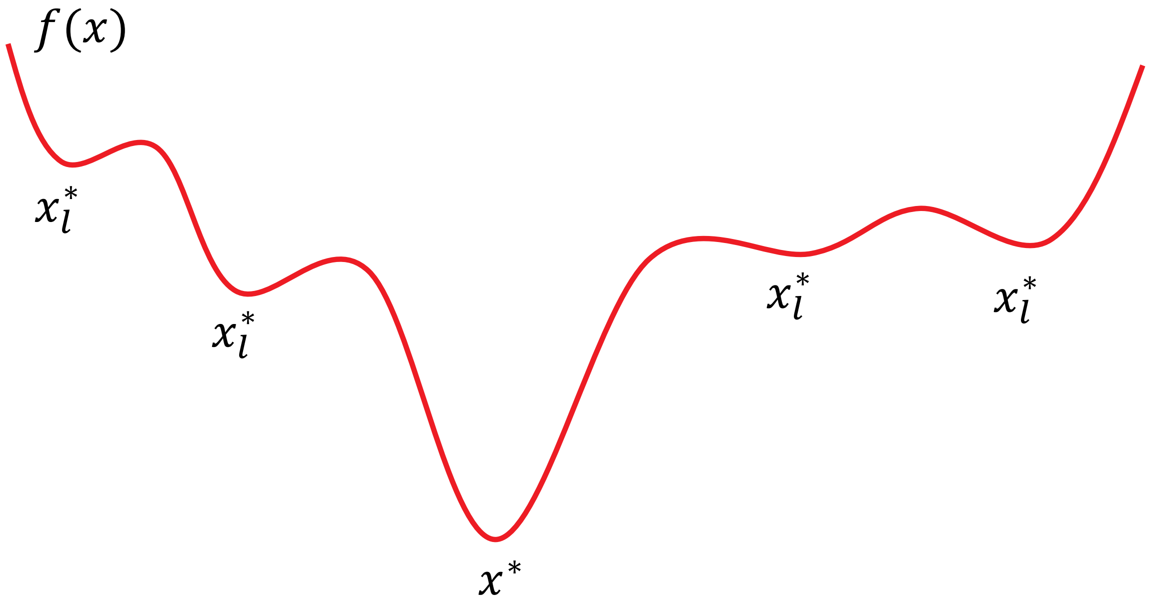


Concave function

$$x^* = \underset{x}{\operatorname{argmax}} f(x)$$

Intuitively:
Concave = $-$Convex or vice versa

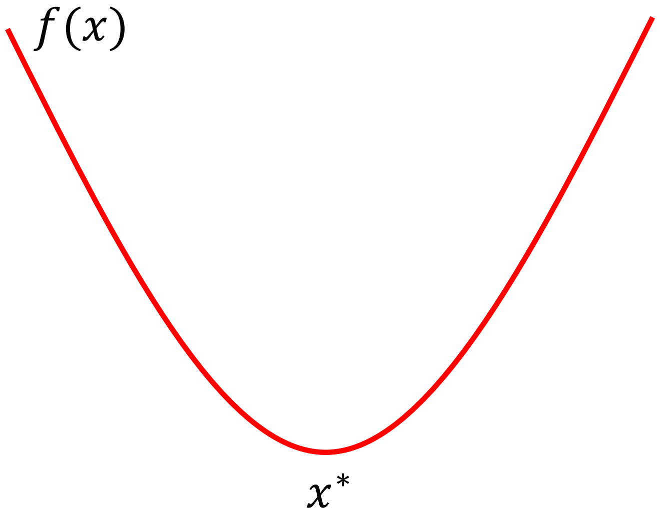
Non-Convex Function



Non-convex function

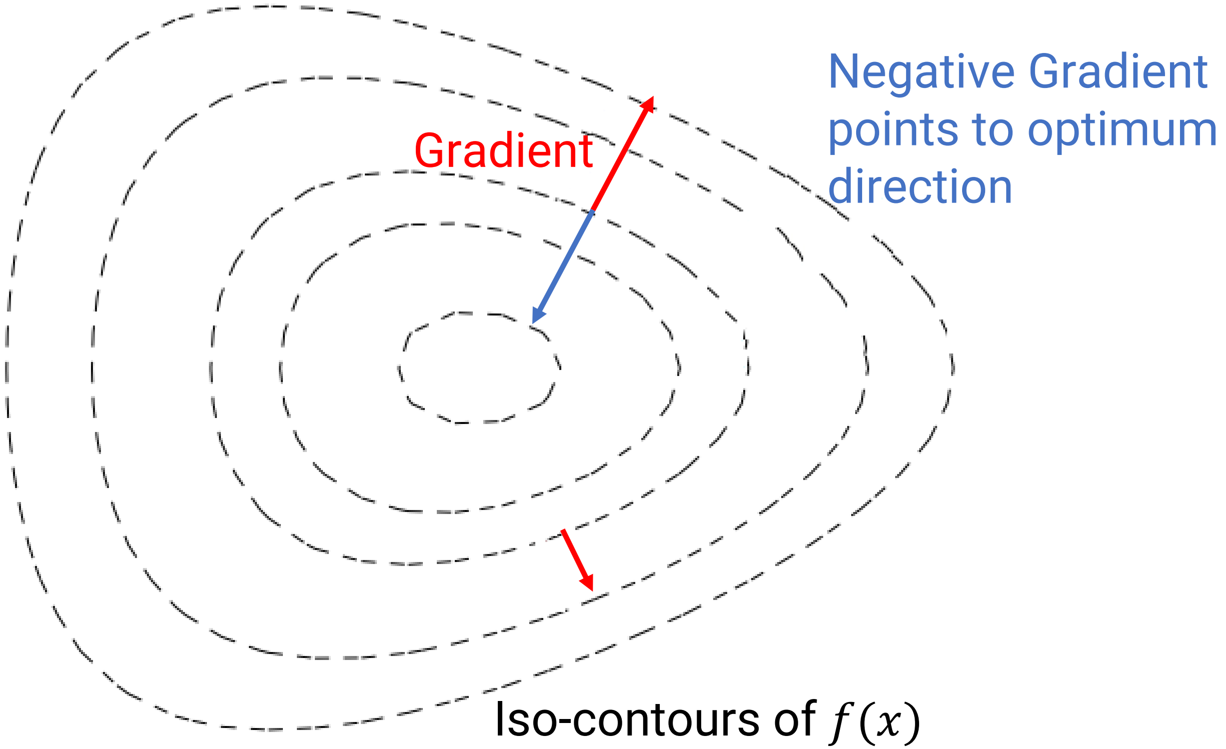
Neither convex nor concave
Intuitively has a lot of local optimum

Convex Optimization 101



Convex function

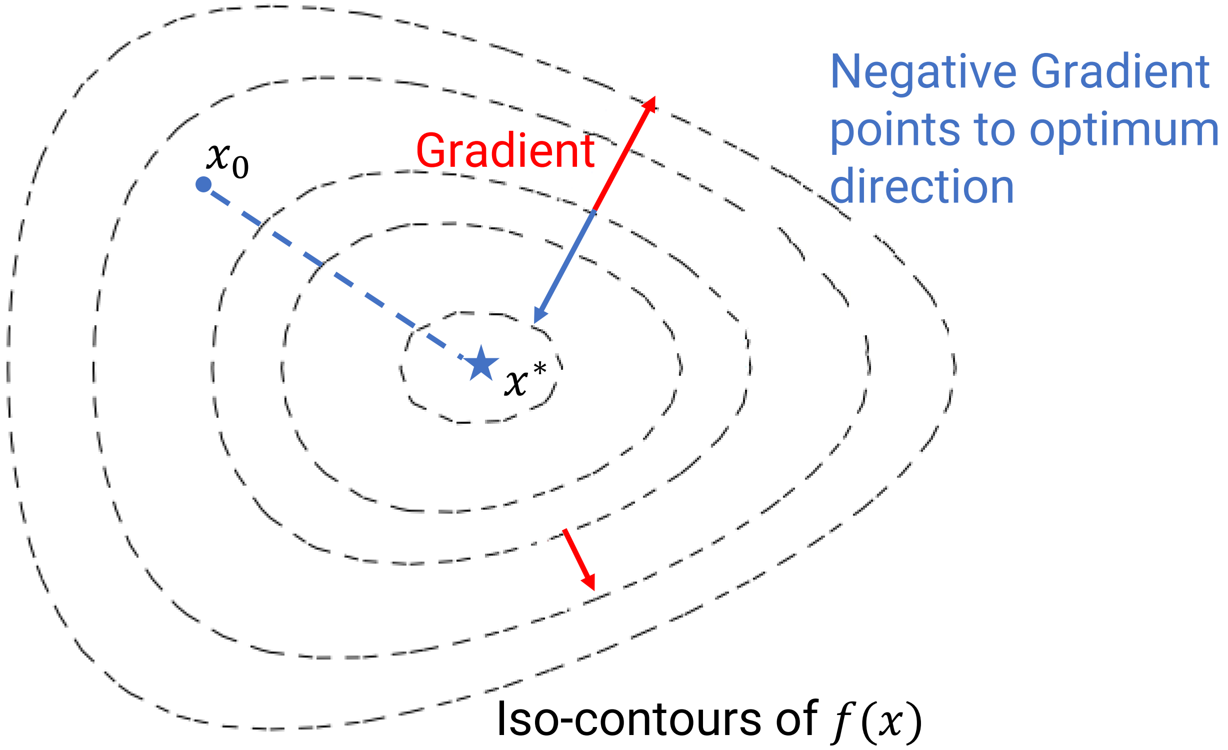
Gradient Direction is the direction of steepest descent



Iso-contours of $f(x)$

Steepest Descent

$x^{t+1} = x^t - \tau \nabla f(x^t)$
τ is called the step-size
$\nabla f(x)$ is the local gradient at x



Step-size Restrictions

$\tau < \frac{2}{\alpha}$ for $f(x) = \frac{\alpha}{2}x^2$

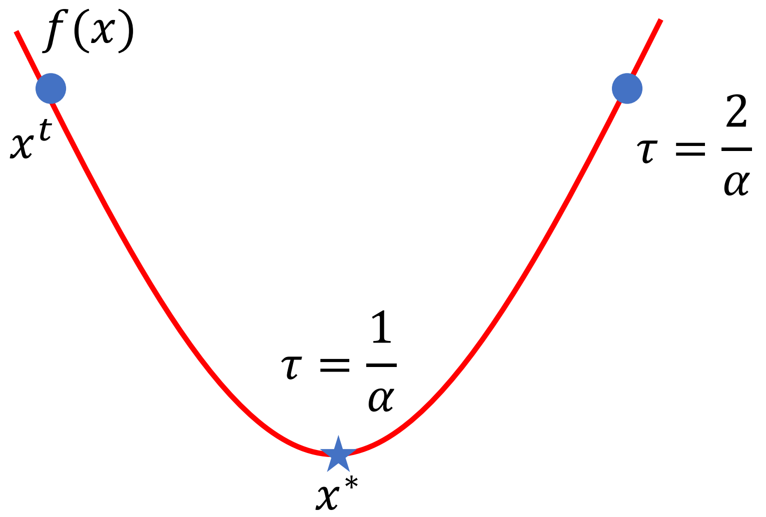
$\nabla f(x) = \alpha x$

$x^{t+1} = x^t - \tau\alpha x^t$

Sort of like PD controller

Too high τ will cause you to diverge

Too low τ will take forever to converge



Convex function

Lipschitz Constant

$\|\nabla f(x) - \nabla f(y)\| \leq M\|x - y\|$

Here M is the Lipschitz constant or intuitively M represents a function of maximum curvature

If a Hessian exists: $M \geq \left\|\nabla^2 f(x)\right\|$

The step-size restriction becomes

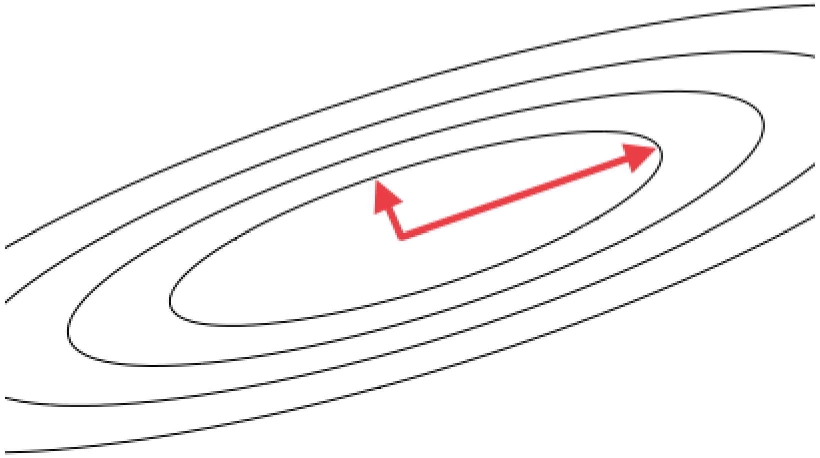
$$\tau < \frac{2}{M}$$

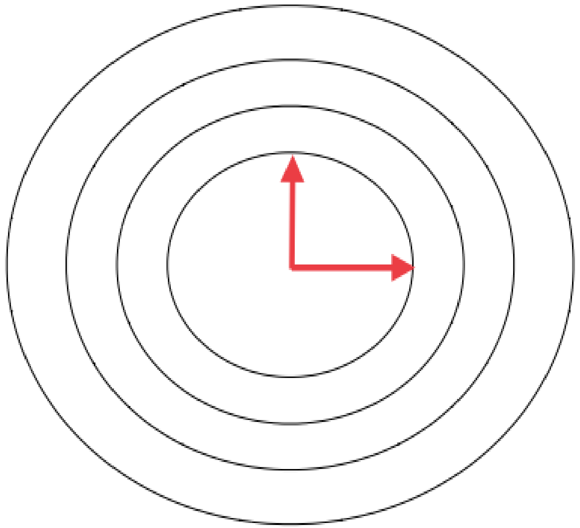
It is generally hard to obtain a value of M

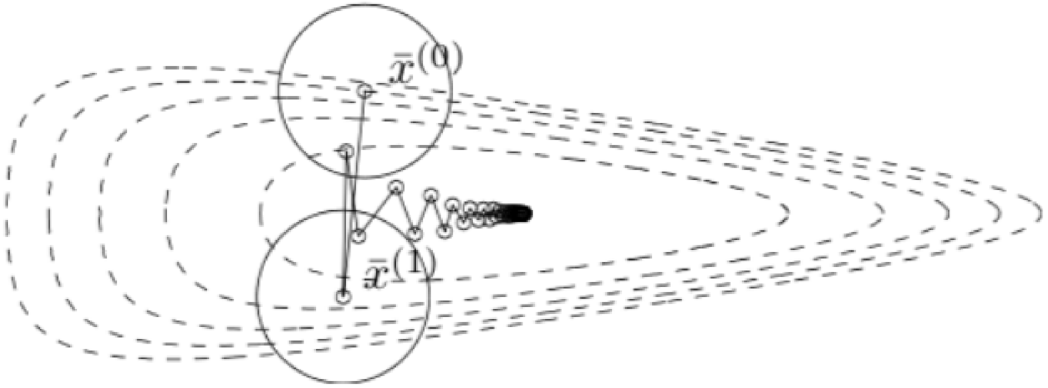
There are methods to find “best” τ for each step and are called Line Search Methods

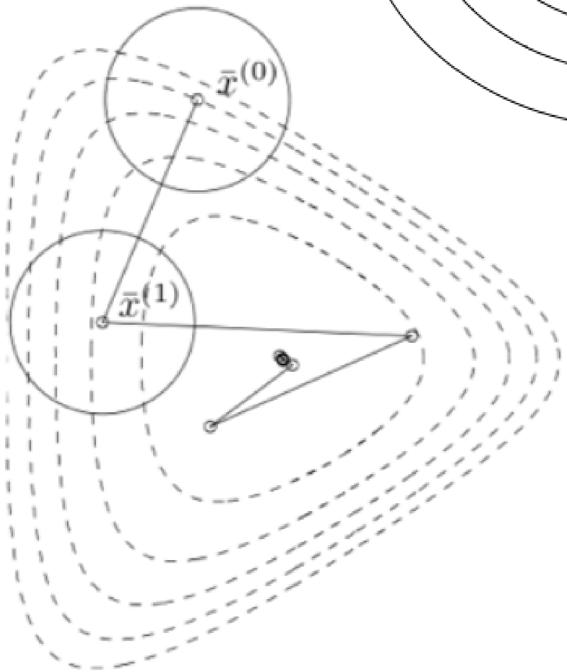
Recall Condition Number

κ denotes how sensitive the function is to noise or in other words how circular are the iso-contours







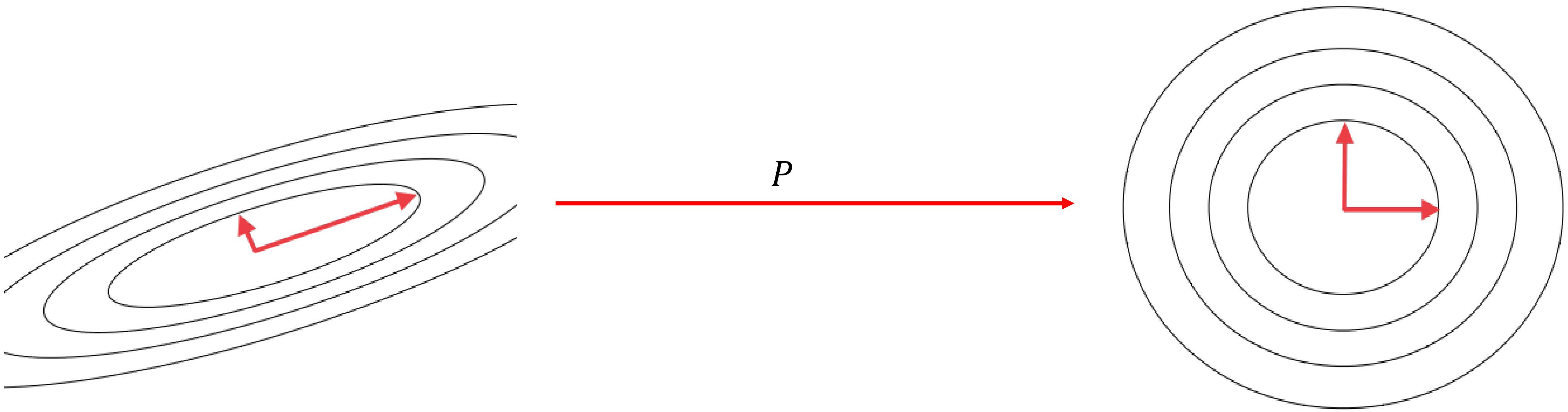


The “Best” Pre-conditioner

$$\operatorname*{argmin}_{x} f(x) \xrightarrow{x = \mathbf{P}y} \operatorname*{argmin}_{y} f(\mathbf{P}y)$$

$\mathbf{H}$ Hessians $\mathbf{P}^{\mathbf{T}}\mathbf{H}\mathbf{P}$

When $\mathbf{P} = \mathbf{H}^{-\frac{1}{2}}, \mathbf{P}^{\mathbf{T}}\mathbf{H}\mathbf{P} = \mathbf{H}^{-\frac{1}{2}}\mathbf{H}\mathbf{H}^{-\frac{1}{2}} = \mathbf{I}$



Newton's Method

$$\underset{x}{\operatorname{argmin}} f(x) \xrightarrow{x = \mathbf{H}^{-\frac{1}{2}} y} \underset{y}{\operatorname{argmin}} f\left(\mathbf{H}^{-\frac{1}{2}} y\right)$$

Gradient Step becomes $y^{k+1} = y^k - \mathbf{H}^{-\frac{1}{2}} \nabla f\left(\mathbf{H}^{-\frac{1}{2}} y^k\right)$

Changing back variables to x we get

$x^{t+1} = x^t - \mathbf{H}^{-1} \nabla f(x^t)$

$-\mathbf{H}^{-1} \nabla f(x^t)$ is called the Newton direction

Gauss Newton Method

Modification of Newton's method to find minimum of a sum of squared function values
Let the function we are minimizing be

$$F(x) = \frac{1}{2}\sum_{i=1}^{m} f_i(x)^2 = \frac{1}{2}\|f(x)\|^2 = \frac{1}{2} f(x)^T f(x)$$

Our problem setup is as follows: $\underset{x}{\operatorname{argmin}}\, F(x)$

The gradient vector g is obtained as follows,

$$g_j = \sum_{i=1}^{m} f_i \frac{\partial f_i}{\partial x_j}$$

To obtain the Hessian we need to differentiate the gradient elements with respect to x_k

$$H_{jk} = \sum_{i=1}^{m} \left(\frac{\partial f_i}{\partial x_j} \frac{\partial f_i}{\partial x_k} + f_i \frac{\partial^2 f_i}{\partial x_j \partial x_k} \right)$$

Gauss Newton Method

$$H_{jk} = \sum_{i=1}^{m} \left(\frac{\partial f_i}{\partial x_j} \frac{\partial f_i}{\partial x_k} + f_i \frac{\partial^2 f_i}{\partial x_j \partial x_k} \right)$$

Now, ignore all the second-order derivative terms (second term) in the above expression

$$H_{jk} \approx \sum_{i=1}^{m} J_{ij} J_{ik}$$

Where $J_{ij} = \frac{\partial f_i}{\partial x_j}$ are the components of the Jacobian Matrix $\mathbf{J}$

Now, $g = \mathbf{J}^T f$ and $\mathbf{H} \approx \mathbf{J}^T \mathbf{J}$

The update equations for Gauss Newton method become

$$x^{t+1} = x^t - \left(\mathbf{J}^T \mathbf{J}\right)^{-1} \mathbf{J}^T f$$

Why is this better than Newton's Method with the update rule $x^{t+1} = x^t - \mathbf{H}^{-1} \nabla f(x^t) = x^t - \mathbf{H}^{-1} \mathbf{J}^T f$

Complicated Hessian computation is avoided

Levenberg–Marquardt (LM) Method

Also called "Damped Least Squares"
The update rule is

$$x^{t+1} = x^t - \widehat{\mathbf{H}}^{-1} \nabla f(x^t)$$

Here $\widehat{\mathbf{H}}$ is modified Hessian

$$\widehat{\mathbf{H}} = \boldsymbol{H} + \lambda \mathrm{diag}(\mathbf{H})$$

LM method blends the Steepest Descent method and Newton's method
Recall steepest descent method update rule is

$$x^{t+1} = x^t - \tau \nabla f(x^t)$$

And Newton's method update rule is

$$x^{t+1} = x^t - \mathbf{H}^{-1} \nabla f(x^t)$$

- Steepest descent works well when we are far from minima and Newton's method which assumes local quadratic approximation works well near the minima as the quadratic approximation is good
- In the LM update rule when λ gets small the rule approaches Newton's method and when λ is large LM approaches steepest descent

Dogleg Method

Chooses between steepest descent (Cauchy) step and Gauss-Newton step

Let $h = x^{t+1} - x^t$ be the update rule such that $x^{t+1} = x^t + h$

Cauchy step is given by $h_C = -\tau \mathbf{J}^T f$

Gauss-Newton step is given by $h_{GN} = -\left(\mathbf{J}^T \mathbf{J}\right)^{-1} \mathbf{J}^T f$

Dogleg uses a region of trust Δ around the linearization point to choose between

Cauchy and GN steps

$$h_{dl} = h_C + \lambda(d_{GN} - d_C)$$

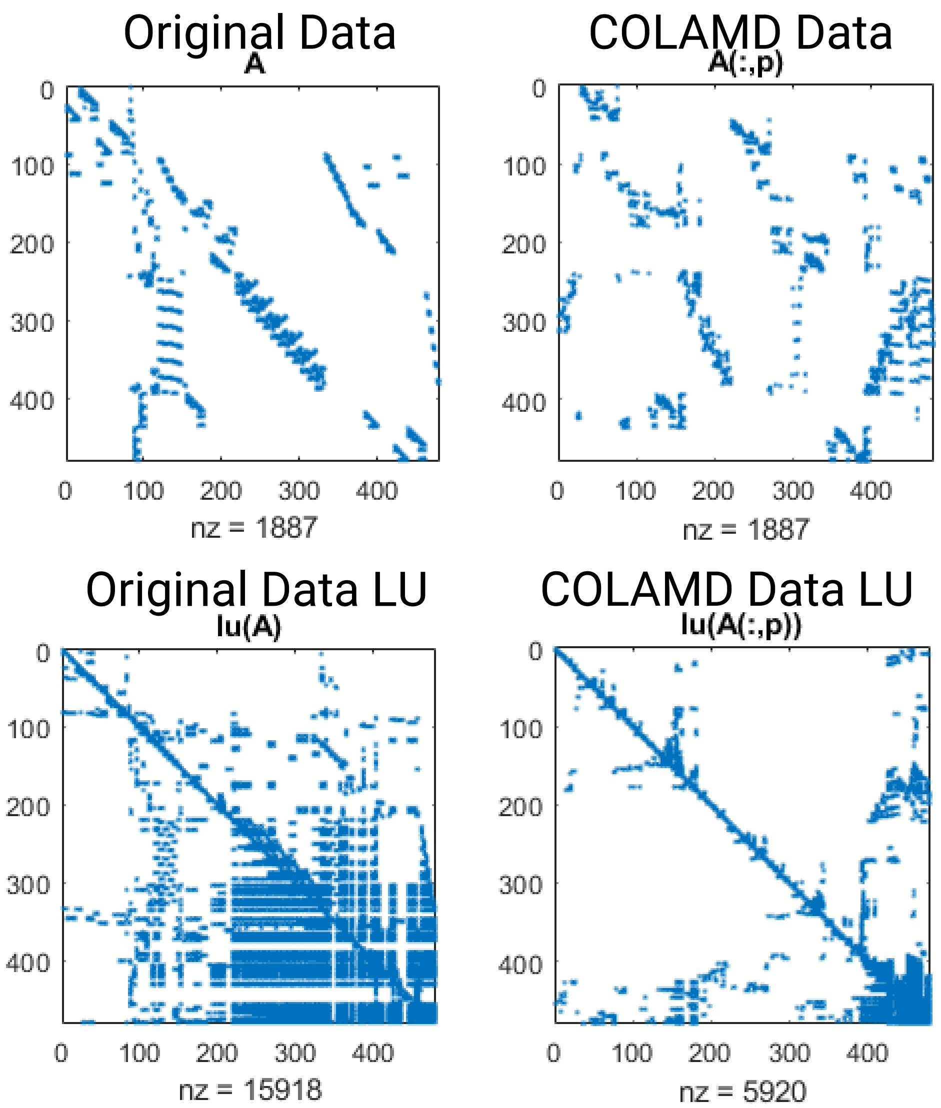
Here $\lambda \in [0,1]$ is the largest value in such that $\|h_{dl}\| \leq \Delta$

If $\mathbf{J}$ is nearly singular then $h_{dl} = h_C$

Update rule is: $x^{t+1} = x^t + h_{dl}$

Ordering

- Selecting the correct column ordering matters since it decides the sparsity of information matrix
- Use COLAMD to find the best ordering just based on information matrix
- COLAMD stands for "COLumn Approximate Minimum Degree permutation"



Sparsity patterns

References

- GTSAM4 Tutorial Slides: https://www.cc.gatech.edu/grads/j/jdong37/files/gtsam-tutorial.pdf
- Frank Dallert's Hands-On GTSAM Tutorial: https://research.cc.gatech.edu/borg/sites/edu.borg/files/downloads/gtsam.pdf
- Tom Goldstein's amazing optimization slides: https://www.cs.umd.edu/~tomg/course/764_2017/L7_grad_descent.pdf
- Boyd's Optimization book: https://web.stanford.edu/~boyd/cvxbook/
- Simple Optimization: https://www.neuraldesigner.com/blog/5_algorithms_to_train_a_neural_network
- Matlab's Optimization: https://www.mathworks.com/help/optim/ug/equation-solving-algorithms.html#f51887
- LM Optimizer: https://www.cs.nyu.edu/~roweis/notes/lm.pdf
- Dogleg Optimizer: http://ceres-solver.org/nnls_solving.html
- COLAMD: https://www.mathworks.com/help/matlab/ref/colamd.html