Pose graph SLAM: Foundations & 1D Solved Example

Comments	
Dates Taught	@September 22, 2020 → September 25, 2020
Lecture No.	L12 L13
Links of Videos	L12, L13 (Udit and Shubodh's demo)
Module	SLAM: Smoothing
Related to All Questions (Property)	

> Please read this note first.

Resources

▼ Resources / Future reading

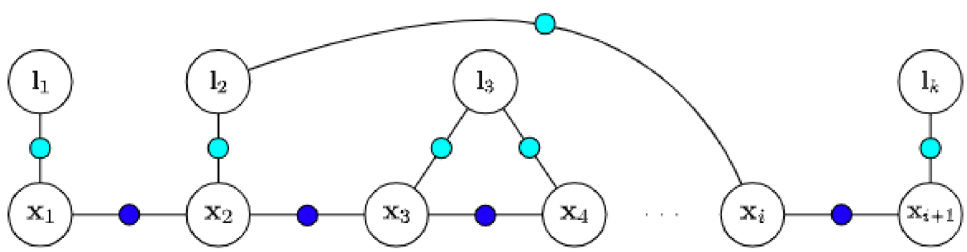
- (this lecture is based on) Niko Sunderhauf Thesis: Robust Optimization for Simultaneous Localization and Mapping Chapter: 2 and 3
- A Tutorial on Graph-Based SLAM

- Cyrill Stachniss recent video on Pose graphs
- A micro Lie theory for state estimation in robotics
- Factor Graphs for Robot Perception: Dellaert & Kaess 2017
- Statistical Methods in AI course
- Intoduction to Applied Linear Algebra by Stephen Boyd, Chapter: 12 and 18

The Complete SLAM Problem

$$P(X_t, M|U_{t-1}, Z_t) \qquad \begin{array}{l} X_t \rightarrow \text{Poses}; M \rightarrow \text{Map}; \\ U_{t-1} \rightarrow \text{Control Inputs}; Z_t \rightarrow \text{Observations} \end{array}$$

Given robot's control inputs and observations, we want to estimate the *probability distribution* of robot's path and the map.



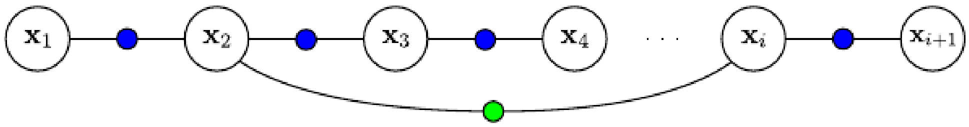
Over here, dark blue dots refer to the control inputs and light blue dots refer to the observations. $\mathbf{l}$ stands for landmarks and $\mathbf{x}$ stands for poses.

The Pose Graph SLAM problem

$$P(X_t|U_{t-1})$$

- **Pose Graph:**

Two different kinds of constraints are necessary for pose graph SLAM. The first is odometric constraints that connect two successive states $\mathbf{x}_i$ *and* $\mathbf{x}_{i+1}$ via a motion model. Furthermore, in order to perform loop closing, the robot has to recognize places it already visited before. This place recognition is also a part of the front-end and provides the second type of constraint, the loop closure constraints. These constraints connect two not necessarily successive poses $\mathbf{x}_i$ *and* $\mathbf{x}_j$.



Pose Graph Representation

Odometry constraints/edges u_i between x_i and x_{i+1}: As denoted by blue 🔵 circle

$$x_{i+1} \sim \mathcal{N}(f(x_i, u_i), \Sigma_i)$$

Loop closure constraints/edges u_{ij} between x_i and x_j : As denoted by green 🎾 circle

$$x_j \sim \mathcal{N}(f(x_i, u_{ij}), \Lambda_{ij})$$

💡 Both of these are actually what makes it an optimization problem!

The *Optimal Set* of Robot Poses

> What is MAP estimate?

The MAP estimate of the random variable X, given that we have observed $Y = y$, is given by the value of x that maximizes $P(X \mid Y = y)$ i.e.

> In our case,

$$X^* = \underset{X}{argmax} \;\; P(X|U)$$

We can factor the joint probability distribution as

$$P(X \mid U) \propto \prod_i \underbrace{P\left(\mathbf{x}_{i+1} \mid \mathbf{x}_i, \mathbf{u}_i\right)}_{\text{Odometry Constraints}} \cdot \underbrace{\prod_{ij} P\left(\mathbf{x}_j \mid \mathbf{x}_i, \mathbf{u}_{ij}\right)}_{\text{Loop Closure Constraints}}$$

💡 This is true for non-gaussian distributions as well, but in robotics, Gaussians are considered for simplification.

$$\underset{X}{argmin} \underbrace{\sum_i \| f(x_i, u_i) - x_{i+1} \|^2_{\Sigma_i}}_{Odometry\ Constraints} + \underbrace{\sum_{ij} \| f(x_i, u_{ij}) - x_j \|^2_{\Lambda_{ij}}}_{Loop\ Closure\ Constraints}$$

Here, error function is $e_{ij}(\mathbf{x}_i, \mathbf{x}_j) = f(\mathbf{x}_i, \mathbf{u}_{ij}) - \mathbf{x}_j$.

> Therefore, we ended up with a ***weighted* non-linear least-squares** optimization problem.

Using Levenberg-Marquard's algorithm:

$$(\mathbf{J}^\top \mathbf{\Omega} \mathbf{J} \ + \lambda \mathbf{I}) \Delta \mathbf{x} = -\mathbf{J}^\top \mathbf{\Omega}^\top \mathbf{f}(\mathbf{x})$$

💡 **Let's first go through the 1D solved example and come back!**

> Simple 1D motion model:

$$\mathbf{x}_j = f(\mathbf{x}_i, \mathbf{u}_{ij}) = \mathbf{x}_i + \mathbf{u}_{ij}$$

> For 2D/3D SLAM, the error function is

$$\mathbf{e}_{ij}(\mathbf{x}_i, \mathbf{x}_j) = \mathrm{t2v}\left(\mathbf{U}_{ij}^{-1}\left(\mathbf{X}_i^{-1}\mathbf{X}_j\right)\right)$$

where

- X, U are transformation matrices
- $\left(\mathrm{X}_i^{-1}\mathrm{X}_j\right)$ is $\mathrm{X}_j \ w.r.t. \ \mathrm{X}_i$
- U_{ij} is $\mathrm{U}_j \ w.r.t. \ \mathrm{U}_i$.
- ▼ More details about X and t2v: transformation to vector —
  - In 2D case, X would look like $\begin{bmatrix} c\theta & -s\theta & x \\ s\theta & c\theta & y \\ 0 & 0 & 1 \end{bmatrix}$
  - In 2D case,

    > transform matrix (3×3) expressed in homogenous coordinates $\to$ vector (x, y, θ)

  - $t2v$ will give 0,0,0 when matrix is Identity.

> Final objective to be minimized is:

$$\mathbf{x}^* = \underset{\mathbf{x}}{\operatorname{argmin}} \sum_k \mathbf{e}_k^T(\mathrm{x}) \Sigma_k^{-1} \mathbf{e}_k(\mathrm{x})$$

Sparsity in SLAM

Here

Optimization In Detail

$$P(X \mid U) \propto \prod_i \underbrace{P\left(\mathbf{x}_{i+1} \mid \mathbf{x}_i, \mathbf{u}_i\right)}_{\text{Odometry Constraints}} \cdot \underbrace{\prod_{ij} P\left(\mathbf{x}_j \mid \mathbf{x}_i, \mathbf{u}_{ij}\right)}_{\text{Loop Closure Constraints}}$$

$$x_{i+1} \sim \mathcal{N}(f(x_i, u_i), \Sigma_i)$$

$$P\left(\mathbf{x}_{i+1} | \mathbf{x}_i, \mathbf{u}_i\right) = \frac{1}{\sqrt{2\pi\left|\boldsymbol{\Sigma}_i\right|}} \exp\left(-\frac{1}{2}\left(f\left(\mathbf{x}_i, \mathbf{u}_i\right) - \mathbf{x}_{i+1}\right)^\top \boldsymbol{\Sigma}_i^{-1}\left(f\left(\mathbf{x}_i, \mathbf{u}_i\right) - \mathbf{x}_{i+1}\right)\right)$$

$$f\left(x | \mu, \sigma^2\right) = -\sqrt{}$$

$$P\left(\mathbf{x}_{i+1} | \mathbf{x}_i, \mathbf{u}_i\right) = \eta \exp -\frac{1}{2}\left\|f\left(\mathbf{x}_i, \mathbf{u}_i\right) - \mathbf{x}_{i+1}\right\|_{\boldsymbol{\Sigma}_i}^2$$

$$\left\|\mathbf{x}_i - \mathbf{y}_i\right\|_{\boldsymbol{\Sigma}_i}^2 = \left(\mathbf{x}_i - \mathbf{y}_i\right)^\top \boldsymbol{\Sigma}$$

$$P(X|U) \propto \prod_i \exp -\frac{1}{2}\left\|f\left(\mathbf{x}_i, \mathbf{u}_i\right) - \mathbf{x}_{i+1}\right\|_{\Sigma_i}^2 \cdot \prod_{ij} \exp -\frac{1}{2}\left\|f\left(\mathbf{x}_i, \mathbf{u}_{ij}\right) - \mathbf{x}_j\right\|_{\boldsymbol{\Lambda}_{ij}}^2$$

Taking negative $\log$ on both the sides,

$$-\log P(X|U) \propto \sum_i \|f(x_i, u_i) - x_{i+1}\|_{\sum_i}^2 + \sum_{ij} \|f(x_i, u_{ij}) - x_j\|_{\Lambda_{ij}}^2$$

$$X^* = \underset{X}{argmax}\ P(X|U)$$

$$= \underset{X}{argmin}\ -log\, P(X|U)$$

$$= \underset{X}{argmin} \underbrace{\sum_i \|f(x_i, u_i) - x_{i+1}\|_{\sum_i}^2}_{Odometry\ Constraints} + \underbrace{\sum_{ij} \|f(x_i, u_{ij}) - x_j\|_{\Lambda_{ij}}^2}_{Loop\ Closure\ Constraints}$$

Solved Example: Weighted Least Squares & Sparsity in SLAM briefly

🔬 Solved Example: 1D SLAM (weighted LS) & Illustrating Sparsity in SLAM

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