(*vii*) $\int_1^2 \frac{dx}{x\sqrt{x^2-1}}$ (*viii*) $\int_2^3 \frac{x-1}{\sqrt{x-2}}\,dx$ (*ix*) $\int_0^e \frac{dx}{x(\log x)^3}$

(*x*) $\int_1^2 \frac{dx}{2-x}$ (*xi*) $\int_0^1 \frac{dx}{x^2-1}$ (*xii*) $\int_0^{2a} \frac{dx}{(x-a)^2}$

(*xiii*) $\int_0^2 \frac{dx}{2x-x^2}$.

Answers

1. (*i*) Divergent (*ii*) Divergent (*iii*) Converges to $\frac{3}{8}$

(*iv*) Diverges to ∞ (*v*) Converges to $\frac{3\pi}{4}$ (*vi*) Converges to 2

(*vii*) Convergent (*viii*) Converges to $\frac{8}{3}$ (*ix*) Converges to $-\frac{1}{2}$

(*x*) Diverges to ∞ (*xi*) Diverges to $-\infty$ (*xii*) Diverges to ∞

(*xiii*) Diverges to ∞

2.6 GAMMA FUNCTION [*A.K.T.U. 2018*]

If n is positive, then the definite integral $\int_0^\infty e^{-x} x^{n-1}\,dx$, which is a function of n, is called the Gamma function (or Eulerian integral of second kind) and is denoted by $\Gamma(n)$. Thus

$$\boxed{\Gamma(n) = \int_0^\infty e^{-x} x^{n-1}\,dx,\ n > 0.}$$

In particular, $\Gamma(1) = \int_0^\infty e^{-x}\,dx = \left| -e^{-x} \right|_0^\infty = 1.$

2.7 REDUCTION FORMULA FOR g(n)

$$\Gamma(n+1) = \int_0^\infty e^{-x} x^n\,dx$$

Integrating by parts, we have

$$\Gamma(n+1) = \left| -x^n e^{-x} \right|_0^\infty + n\int_0^\infty x^{n-1} e^{-x}\,dx = n\int_0^\infty e^{-x} x^{n-1}\,dx \qquad \left[\because \ \underset{x\to\infty}{\text{Lt}} \frac{x^n}{e^x} = 0\right]$$

$$= n\Gamma(n)$$

$\therefore$ $\boxed{\Gamma(n+1) = n\Gamma(n)}$ which is the reduction or recurrence formula for $\Gamma(n)$.

Note 1. If n is a positive integer, then by repeated application of above formula, we get

$$\begin{aligned}\Gamma(n+1) &= n\Gamma(n)\\ &= n(n-1)\,\Gamma(n-1)\\ &= n(n-1)(n-2)\,\Gamma(n-2)\\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\ &= n(n-1)(n-2)\ldots\ldots 3\cdot 2\cdot 1\,\Gamma(1)\\ &= n\,!, \quad \text{since } \Gamma(1)=1\end{aligned}$$

Hence $\boxed{\Gamma(n+1) = n\,!}$ **when n is a positive integer.**

Note 2. If n is a positive fraction, then by repeated application of above formula, we get

$$\Gamma(n) = (n-1)(n-2) \times \text{go on decreasing by 1} \ldots\ldots$$

the series of factors being continued so long as the factors remain positive, multiplied by Γ (last factor).

Thus $\Gamma\left(\frac{11}{4}\right) = \frac{7}{4}\Gamma\left(\frac{7}{4}\right) = \frac{7}{4}\cdot\frac{3}{4}\Gamma\left(\frac{3}{4}\right)$

The value of $\Gamma\left(\frac{3}{4}\right)$ can be obtained from the table of gamma functions.

Note 3. $\Gamma(n+1) = n\,\Gamma(n)$

$$\begin{aligned}\Rightarrow\quad \Gamma(n) &= \frac{\Gamma(n+1)}{n}, n \neq 0\\ &= \frac{(n+1)\Gamma(n+1)}{n(n+1)} = \frac{\Gamma(n+2)}{n(n+1)}, n \neq 0, -1\\ &= \frac{(n+2)\Gamma(n+2)}{n(n+1)(n+2)} = \frac{\Gamma(n+3)}{n(n+1)(n+2)}, n \neq 0, -1, -2\\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\ &= \frac{\Gamma(n+k+1)}{n(n+1)(n+2)\ldots(n+k)}, n \neq 0, -1, -2\ldots, -k\end{aligned}$$

This result defines $\Gamma(n)$ for $n < 0$, k being the least positive integer such that $n + k + 1 > 0$.

For example, to evaluate $\Gamma(-3.4)$

$$n + k + 1 > 0 \quad \Rightarrow \quad -3.4 + k + 1 > 0 \quad \Rightarrow \quad k > 2.4$$

We choose $k = 3$

$$\therefore \quad \Gamma(-3.4) = \frac{\Gamma(-3.4+3+1)}{(-3.4)(-2.4)(-1.4)(-.4)} = \frac{\Gamma(.6)}{(3.4)(2.4)(1.4)(.4)}$$

The value of $\Gamma(.6)$ can be obtained from the table of gamma functions.

Also we observe that $\Gamma(n)$ is infinite when $n = 0$ or a negative integer.

2.7.1 Value of $\Gamma\left(\frac{1}{2}\right)$

$$\Gamma\left(\frac{1}{2}\right) = \int_0^\infty e^{-t}\, t^{-1/2}\, dt$$

Putting $t = x^2$ so that $dt = 2x dx$

$$\Gamma\left(\frac{1}{2}\right) = \int_0^\infty e^{-x^2}\cdot\frac{1}{x}\cdot 2x dx = 2\int_0^\infty e^{-x^2} dx \qquad \ldots(1)$$

Writing y for x, we have $\Gamma\left(\frac{1}{2}\right) = 2\int_0^\infty e^{-y^2}\,dy$ ...(2)

Now we use the following result from double integrals:

If $f(x)$ and $g(y)$ are functions of x and y only, and the limits of integration are constants, then the double integral can be represented as a product of two integrals. Thus

$$\int_c^d \int_a^b f(x)\,g(y)\,dx\,dy = \int_a^b f(x)\,dx \cdot \int_c^d g(y)\,dy$$

$\therefore$ From (1) and (2), we have

$$\left[\Gamma\left(\frac{1}{2}\right)\right]^2 = 4\int_0^\infty e^{-x^2}\,dx.\int_0^\infty e^{-y^2}\,dy = 4\int_0^\infty\int_0^\infty e^{-(x^2+y^2)}\,dx\,dy$$

Changing to polar co-ordinates with $x = r\cos\theta$, $y = r\sin\theta$, $dx\,dy = r\,dr\,d\theta$; the region of integration in this integral is the complete positive quadrant, to cover which, r must vary from 0 to ∞ and θ from 0 to $\frac{\pi}{2}$.

$$\therefore \quad \left[\Gamma\left(\frac{1}{2}\right)\right]^2 = 4\int_0^{\pi/2}\int_0^\infty e^{-r^2}\,r\,dr\,d\theta = 4\int_0^{\pi/2}\left[-\frac{1}{2}e^{-r^2}\right]_0^\infty d\theta = 2\int_0^{\pi/2} d\theta = \pi$$

Hence, $\boxed{\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}.}$

2.8 TRANSFORMATIONS OF GAMMA FUNCTIONS

(1) $$\boxed{\Gamma(n) = k^n\int_0^\infty e^{-kx}\,x^{n-1}\,dx}$$ [A.K.T.U. 2016]

Proof. We have, $\Gamma(n) = \int_0^\infty e^{-x}\,x^{n-1}\,dx$

$$= \int_0^\infty e^{-ky}\,k^{n-1}\,y^{n-1}\,k\,dy = k^n\int_0^\infty e^{-ky}\,y^{n-1}\,dy$$

Put $x = ky$

$\therefore$ $dx = k\,dy$

$$\Rightarrow \quad \Gamma(n) = k^n\int_0^\infty e^{-kx}\,x^{n-1}\,dx$$

(2) $$\boxed{\Gamma(n) = \int_0^1 \left[\log\left(\frac{1}{x}\right)\right]^{n-1} dx\ ; \quad n > 0}$$

Proof. We have, $\Gamma(n) = \int_0^\infty e^{-x}\cdot x^{n-1}\,dx$

Put $e^{-x} = y$

$\therefore$ $-x = \log y \Rightarrow x = \log\left(\frac{1}{y}\right)$ and $dx = -\frac{dy}{y}$

$$\therefore \quad \Gamma(n) = -\int_1^0 y\left(\log\frac{1}{y}\right)^{n-1}\frac{dy}{y} = \int_0^1\left(\log\frac{1}{y}\right)^{n-1} dy$$

Hence, $$\Gamma(n) = \int_0^1 \left[\log\left(\frac{1}{x}\right)\right]^{n-1} dx.$$

(3) $$\boxed{\Gamma(n) = \frac{1}{n}\int_0^\infty e^{-x^{1/n}}\, dx}$$

Proof. We have $\Gamma(n) = \int_0^\infty e^{-x} \cdot x^{n-1}\, dx$

Put $x^n = y \quad \Rightarrow \quad x = y^{1/n}$

$\therefore$ $$dx = \frac{1}{n} y^{(1/n)-1} dy$$

Now, $$\Gamma(n) = \int_0^\infty e^{-y^{1/n}}\, y^{\frac{n-1}{n}} \cdot \frac{1}{n} y^{\frac{1}{n}-1}\, dy = \frac{1}{n}\int_0^\infty e^{-y^{1/n}}\, dy$$

or $$\Gamma(n) = \frac{1}{n}\int_0^\infty e^{-x^{1/n}}\, dx.$$

2.8.1 Deduction. Put $n = \frac{1}{2}$ in (3), we get

$$\Gamma(1/2) = 2\int_0^\infty e^{-x^2} dx = 2\left(\frac{\sqrt{\pi}}{2}\right) = \sqrt{\pi}.$$

2.9 BETA FUNCTION

[A.K.T.U. 2018]

If m, n are positive, then the definite integral $\int_0^1 x^{m-1}(1-x)^{n-1}\, dx$, which is a function of m and n, is called the Beta Function (or Eulerian integral of first kind) and is denoted by $\beta(m, n)$. Thus,

$$\boxed{\beta(m, n) = \int_0^1 x^{m-1}(1-x)^{n-1} dx,\ m > 0, n > 0.}$$

2.10 SYMMETRY OF BETA FUNCTION i.e., $\beta(m, n) = \beta(n, m)$

$$\beta(m, n) = \int_0^1 x^{m-1}(1-x)^{n-1} dx,\ m > 0, n > 0$$

Since $$\int_0^a f(x)\, dx = \int_0^a f(a-x)\, dx$$

$\therefore$ $$\beta(m, n) = \int_0^1 (1-x)^{m-1}\,[1-(1-x)]^{n-1}\, dx = \int_0^1 x^{n-1}\,(1-x)^{m-1}\, dx = \beta(n, m)$$

Hence, $$\boxed{\beta(m, n) = \beta(n, m).}$$

2.11 TRANSFORMATIONS OF BETA FUNCTION

(1) $$\boxed{\beta(m, n) = \int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}}\, dx}$$

Proof. $\beta(m, n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx$ | Put $x = \frac{1}{1+y}$ $\therefore$ $dx = -\frac{1}{(1+y)^2}\,dy$

$$= \int_\infty^0 \frac{1}{(1+y)^{m-1}}\left(1-\frac{1}{1+y}\right)^{n-1}\left\{\frac{-1}{(1+y)^2}\right\}dy$$

$$= \int_0^\infty \frac{y^{n-1}}{(1+y)^{m-1}(1+y)^{n-1}(1+y)^2}\,dy = \int_0^\infty \frac{y^{n-1}}{(1+y)^{m+n}}\,dy$$

$$\beta(m, n) = \int_0^\infty \frac{x^{n-1}}{(1+x)^{m+n}}\,dx$$

$\therefore$ $$\beta(n, m) = \int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}}\,dx$$

But $$\beta(m, n) = \beta(n, m)$$

$\therefore$ $$\beta(m, n) = \int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}}\,dx$$

(2) $$\boxed{\beta(m, n) = 2\int_0^{\pi/2} \sin^{2m-1}\theta\,\cos^{2n-1}\theta\,d\theta}$$

Proof. $\beta(m, n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx$ | Put $x = \sin^2\theta$ $\therefore$ $dx = 2\sin\theta\cos\theta\,d\theta$

$$= \int_0^{\pi/2} \sin^{2m-2}\theta\,(1-\sin^2\theta)^{n-1}\cdot 2\sin\theta\cos\theta\,d\theta$$

$$= 2\int_0^{\pi/2} \sin^{2m-1}\theta\,\cos^{2n-1}\theta\,d\theta.$$

2.12 RELATION BETWEEN BETA AND GAMMA FUNCTIONS

$$\beta(m, n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$$ [A.K.T.U. 2018; U.K.T.U. 2010]

Proof. From (1) of Art. 2.8, we have

$$\Gamma(n) = k^n \int_0^\infty e^{-kx}\,x^{n-1}\,dx$$

$$= z^n \int_0^\infty e^{-zx}\,x^{n-1}\,dx$$ (Replace k by z)

Multiplying both sides by $e^{-z}\,z^{m-1}$, we get

$$\Gamma(n)\cdot e^{-z}z^{m-1} = \int_0^\infty z^n\cdot e^{-zx}\cdot x^{n-1}\cdot e^{-z}\cdot z^{m-1}\,dx = \int_0^\infty z^{n+m-1}e^{-z(1+x)}\,x^{n-1}\,dx$$

Integrating both sides w.r.t. z from 0 to ∞, we get

$$\Gamma(n)\int_0^\infty e^{-z} z^{m-1}\,dz = \int_0^\infty x^{n-1}\left\{\int_0^\infty e^{-z(1+x)}\, z^{m+n-1}\,dz\right\}dx$$

$$\Rightarrow \qquad \Gamma n\,\Gamma m = \int_0^\infty x^{n-1}\left\{\int_0^\infty e^{-y}\cdot\frac{y^{m+n-1}}{(1+x)^{m+n-1}}\frac{dy}{(1+x)}\right\}dx$$

where $z(1+x) = y$ so that $dz = \frac{dy}{1+x}$

$$= \int_0^\infty \frac{x^{n-1}}{(1+x)^{m+n}}\left\{\int_0^\infty e^{-y}\, y^{m+n-1}\,dy\right\}dx$$

$$= \int_0^\infty \frac{x^{n-1}}{(1+x)^{m+n}}\,dx\;\Gamma(m+n) = \Gamma(m+n)\,\beta(m,n)$$

$$\therefore \qquad \boxed{\beta(m,n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}}$$

Aliter:

We know that $\qquad \Gamma(m) = \int_0^\infty e^{-t}\, t^{m-1}\,dt$

Putting $t = x^2$ so that $dt = 2x\,dx$

$$\Gamma(m) = 2\int_0^\infty e^{-x^2} x^{2m-1}\,dx \qquad \ldots(1)$$

Similarly, $\qquad \Gamma(n) = 2\int_0^\infty e^{-y^2}\, y^{2n-1}\,dy$

$$\Gamma(m)\,\Gamma(n) = 4\int_0^\infty e^{-x^2} x^{2m-1}\,dx \cdot \int_0^\infty e^{-y^2}\, y^{2n-1}\,dy$$

$$= 4\int_0^\infty\int_0^\infty e^{-(x^2+y^2)} x^{2m-1} y^{2n-1}\,dx\,dy$$

Changing to polar co-ordinates, we have

$$\Gamma(m)\,\Gamma(n) = 4\int_0^{\pi/2}\int_0^\infty e^{-r^2}\, r^{2(m+n)-1}\cos^{2m-1}\theta\,\sin^{2n-1}\theta\,dr\,d\theta$$

$$= 4\int_0^\infty e^{-r^2}\, r^{2(m+n)-1}\,dr\cdot\int_0^{\pi/2}\cos^{2m-1}\theta\,\sin^{2n-1}\theta\,d\theta \qquad \ldots(2)$$

$$= \left[2\int_0^\infty e^{-r^2}\, r^{2(m+n)-1}\,dr\right]\cdot\left[2\int_0^{\pi/2}\cos^{2m-1}\theta\,\sin^{2n-1}\theta\,d\theta\right]$$

$$= \Gamma(m+n)\,\beta(m,n) \qquad | \text{ Using (2) of 2.11}$$

Hence, $\qquad \boxed{\beta(m,n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}}$

2.13 TO EVALUATE $\int_0^{\pi/2} \sin^m\theta \cos^n\theta \, d\theta$ (U.K.T.U. 2011)

From (2) of Art. 2.11, we have

$$\int_0^{\pi/2} \sin^{2m-1}\theta \cos^{2n-1}\theta \, d\theta = \frac{1}{2}\beta(m, n) \quad \text{...(1)}$$

Using the relation $\beta(m, n) = \dfrac{\Gamma(m)\,\Gamma(n)}{\Gamma(m+n)}$, we get from (1)

$$\int_0^{\pi/2} \sin^{2m-1}\theta \cos^{2n-1}\theta \, d\theta = \frac{\Gamma(m)\,\Gamma(n)}{2\,\Gamma(m+n)} \quad \text{...(2)}$$

Replacing m by $\dfrac{m+1}{2}$ and n by $\dfrac{n+1}{2}$ in (2), we get

$$\boxed{\int_0^{\pi/2} \sin^m\theta \cos^n\theta \, d\theta = \frac{\Gamma\left(\frac{m+1}{2}\right)\Gamma\left(\frac{n+1}{2}\right)}{2\Gamma\left(\frac{m+n+2}{2}\right)}}$$

Cor. 1 Putting $m = n = 0$, we have

$$\frac{(\Gamma\,\frac{1}{2})^2}{2\,\Gamma(1)} = \int_0^{\pi/2} d\theta = \frac{\pi}{2} \quad \Rightarrow \quad \Gamma(\tfrac{1}{2}) = \sqrt{\pi} \qquad \Big|\ \text{since } \Gamma 1 = 1$$

Cor. 2 Putting $n = 0$, we get

$$\int_0^{\pi/2} \sin^m\theta \, d\theta = \frac{\Gamma\left(\frac{m+1}{2}\right)}{\Gamma\left(\frac{m+2}{2}\right)} \cdot \frac{\sqrt{\pi}}{2} \qquad \Big|\ \because\ \Gamma(\tfrac{1}{2}) = \sqrt{\pi}$$

and similarly, putting $m = 0$, we get

$$\int_0^{\pi/2} \cos^n\theta \, d\theta = \frac{\Gamma\left(\frac{n+1}{2}\right)}{\Gamma\left(\frac{n+2}{2}\right)} \cdot \frac{\sqrt{\pi}}{2}$$

2.13.1 Deductions

Using $\int_0^{\infty} \dfrac{x^{n-1}}{1+x} dx = \dfrac{\pi}{\sin n\pi}$, where $0 < n < 1$, we can deduce the following important results.

(1) $$\boxed{\Gamma n \Gamma(1-n) = \frac{\pi}{\sin n\pi}} \quad \text{(G.B.T.U. 2010)}$$

Proof. We have,

$$\beta(m, n) = \frac{\Gamma(m)\,\Gamma(n)}{\Gamma(m+n)}$$

$$\Rightarrow \quad \int_0^{\infty} \frac{x^{n-1}}{(1+x)^{m+n}} dx = \frac{\Gamma(m)\,\Gamma(n)}{\Gamma(m+n)}$$

Setting $m + n = 1$ so that $m = 1 - n$, we get

$$\int_0^\infty \frac{x^{n-1}}{1+x}\,dx = \frac{\Gamma(1-n)\,\Gamma(n)}{\Gamma(1)}$$

$$\Rightarrow \qquad \frac{\pi}{\sin n\pi} = \Gamma(n)\,\Gamma(1-n) \qquad | \because \Gamma(1) = 1$$

(2) $\boxed{\Gamma(\tfrac{1}{2}) = \sqrt{\pi}}$

Proof. We have

$$\Gamma(n)\,\Gamma(1-n) = \frac{\pi}{\sin n\pi}$$

As a particular case, put $n = \frac{1}{2}$, we get

$$\Gamma(\tfrac{1}{2})\,\Gamma(\tfrac{1}{2}) = \frac{\pi}{\sin\frac{\pi}{2}}$$

$$[\Gamma(\tfrac{1}{2})]^2 = \pi \qquad \Rightarrow \qquad \Gamma(\tfrac{1}{2}) = \sqrt{\pi}$$

(3) $\boxed{\Gamma(\tfrac{1}{4})\,\Gamma(\tfrac{3}{4}) = \pi\sqrt{2}}$

Proof. Putting $n = \frac{1}{4}$ in result (1), we obtain

$$\Gamma(\tfrac{1}{4})\,\Gamma(1-\tfrac{1}{4}) = \frac{\pi}{\sin\frac{\pi}{4}}$$

$$\Rightarrow \qquad \Gamma(\tfrac{1}{4})\,\Gamma(\tfrac{3}{4}) = \frac{\pi}{\left(\frac{1}{\sqrt{2}}\right)} \quad \text{or} \quad \Gamma(\tfrac{1}{4})\,\Gamma(\tfrac{3}{4}) = \pi\sqrt{2}$$

Note. Similarly $\Gamma(1/3)\,\Gamma(2/3) = \dfrac{\pi}{\sin \pi/3} = \dfrac{2\pi}{\sqrt{3}}$.

2.14 DUPLICATION FORMULA

$\Gamma(m)\,\Gamma\left(m+\frac{1}{2}\right) = \frac{\sqrt{\pi}}{(2)^{2m-1}}\,\Gamma(2m)$ **where m is positive.** (*M.T.U. 2013; G.B.T.U. 2013*)

We have already established

$$\int_0^{\pi/2} \sin^{2m-1}\theta\,\cos^{2n-1}\theta\,d\theta = \frac{\Gamma(m)\,\Gamma(n)}{2\,\Gamma(m+n)} \qquad ...(1)$$

Putting $2n - 1 = 0$ or $n = \dfrac{1}{2}$ in (1), we obtain

$$\int_0^{\pi/2} \sin^{2m-1}\theta\,d\theta = \frac{\Gamma(m)\sqrt{\pi}}{2\,\Gamma\left(m+\frac{1}{2}\right)} \qquad ...(2) \; [\because \Gamma(1/2) = \sqrt{\pi}]$$

Again putting $n = m$ in equation (1), we obtain

$$\int_0^{\pi/2} \sin^{2m-1}\theta\,\cos^{2m-1}\theta\,d\theta = \frac{(\Gamma m)^2}{2\,\Gamma(2m)}$$

i.e., $$\frac{1}{2^{2m-1}}\int_0^{\pi/2}(2\sin\theta\cos\theta)^{2m-1}\,d\theta = \frac{(\Gamma m)^2}{2\,\Gamma(2m)}$$

i.e., $$\frac{1}{2^{2m}}\int_0^{\pi/2}(\sin 2\theta)^{2m-1}\cdot 2d\theta = \frac{(\Gamma m)^2}{2\,\Gamma(2m)}$$

Putting $2\theta = \phi$ so that $2\,d\theta = d\phi$, this reduces to

$$\frac{1}{2^{2m}}\int_0^{\pi}\sin^{2m-1}\phi\,d\phi = \frac{(\Gamma\, m)^2}{2\,\Gamma\,(2m)}$$

i.e., $$\frac{2}{2^{2m}}\int_0^{\pi/2}\sin^{2m-1}\phi\,d\phi = \frac{(\Gamma m)^2}{2\,\Gamma(2m)}$$

Replacing ϕ by θ, we finally obtain

$$\int_0^{\pi/2}\sin^{2m-1}\theta\,d\theta = \frac{2^{2m-1}\,(\Gamma m)^2}{2\,\Gamma(2m)} \qquad \text{...(3)}$$

From (2) and (3), we get

$$\frac{\Gamma(m)\sqrt{\pi}}{2\,\Gamma\left(m+\frac{1}{2}\right)} = \frac{2^{2m-1}\,(\Gamma m)^2}{2\,\Gamma(2m)}$$

$$\Rightarrow \quad \boxed{\Gamma(m)\,\Gamma\left(m+\frac{1}{2}\right) = \frac{\sqrt{\pi}}{2^{2m-1}}\,\Gamma(2m)}$$

2.15 TO SHOW THAT

$$\Gamma\left(\frac{1}{n}\right)\Gamma\left(\frac{2}{n}\right)\Gamma\left(\frac{3}{n}\right)\ldots\Gamma\left(\frac{n-1}{n}\right) = \frac{(2\pi)^{\left(\frac{n-1}{2}\right)}}{n^{1/2}}$$

where n is a positive integer greater than one.

Let $$P = \Gamma\left(\frac{1}{n}\right)\Gamma\left(\frac{2}{n}\right)\Gamma\left(\frac{3}{n}\right)\ldots\Gamma\left(\frac{n-2}{n}\right)\Gamma\left(\frac{n-1}{n}\right)$$

$$= \Gamma\left(\frac{1}{n}\right)\Gamma\left(\frac{2}{n}\right)\Gamma\left(\frac{3}{n}\right)\ldots\Gamma\left(1-\frac{2}{n}\right)\Gamma\left(1-\frac{1}{n}\right) \qquad \text{...(1)}$$

Writing the value of P in the reverse order, we have

$$P = \Gamma\left(1-\frac{1}{n}\right)\Gamma\left(1-\frac{2}{n}\right)\ldots\Gamma\left(\frac{3}{n}\right)\Gamma\left(\frac{2}{n}\right)\Gamma\left(\frac{1}{n}\right) \qquad \text{...(2)}$$

Multiplying (1) and (2), we get

$$P^2 = \left(\Gamma(1/n)\,\Gamma\left(1-\frac{1}{n}\right)\right)\left(\Gamma(2/n)\,\Gamma\left(1-\frac{2}{n}\right)\right)\ldots\left(\Gamma\left(1-\frac{2}{n}\right)\Gamma\,(2/n)\right)\left(\Gamma\left(1-\frac{1}{n}\right)\Gamma\,(1/n)\right)$$

$$P^2 = \frac{\pi}{\sin\left(\frac{\pi}{n}\right)}\cdot\frac{\pi}{\sin\left(\frac{2\pi}{n}\right)}\cdot\frac{\pi}{\sin\left(\frac{3\pi}{n}\right)}\cdots\frac{\pi}{\sin\frac{(n-1)\,\pi}{n}} \qquad \Big|\ \because\ \Gamma(n)\,\Gamma(1-n) = \frac{\pi}{\sin n\pi}$$

$$\Rightarrow \quad P^2 = \frac{\pi^{n-1}}{\sin\left(\frac{\pi}{n}\right)\sin\left(\frac{2\pi}{n}\right)\sin\left(\frac{3\pi}{n}\right)\ldots\sin\left\{\frac{(n-1)\,\pi}{n}\right\}} \qquad \text{...(3)}$$

But from Trigonometry, we know that

$$\frac{\sin n\theta}{\sin\theta} = 2^{n-1}\sin\left(\theta+\frac{\pi}{n}\right)\sin\left(\theta+\frac{2\pi}{n}\right)\ldots\sin\left\{\theta+\frac{(n-1)\pi}{n}\right\} \quad \ldots(4)$$

Take Limit as $\theta \to 0$,

$$\underset{\theta\to 0}{\text{Lt}}\frac{\sin n\theta}{\sin\theta} = \underset{\theta\to 0}{\text{Lt}}\left(n\cdot\frac{\sin n\theta}{n\theta}\cdot\frac{\theta}{\sin\theta}\right) = n$$

$\therefore$

$$n = 2^{n-1}\sin\frac{\pi}{n}\sin\frac{2\pi}{n}\sin\frac{3\pi}{n}\ldots\sin\frac{(n-1)\pi}{n} \quad | \text{ From (4)}$$

Substituting this in equation (3), we obtain

$$P^2 = \frac{\pi^{n-1}}{\left(\frac{n}{2^{n-1}}\right)} = \frac{(2\pi)^{n-1}}{n}$$

$\therefore$

$$P = \frac{(2\pi)^{(n-1)/2}}{n^{1/2}}$$

or

$$\boxed{\Gamma\left(\frac{1}{n}\right)\Gamma\left(\frac{2}{n}\right)\Gamma\left(\frac{3}{n}\right)\ldots\Gamma\left(\frac{n-1}{n}\right) = \frac{(2\pi)^{(n-1)/2}}{n^{1/2}}}$$

2.16 TO SHOW THAT

(i) $\displaystyle\int_0^\infty e^{-ax}x^{n-1}\cos bx\,dx = \frac{\Gamma(n)\cos n\theta}{(a^2+b^2)^{n/2}}$

(ii) $\displaystyle\int_0^\infty e^{-ax}x^{n-1}\sin bx\,dx = \frac{\Gamma(n)\sin n\theta}{(a^2+b^2)^{n/2}}$, where $\theta = \tan^{-1}\left(\frac{b}{a}\right)$

We know that $\displaystyle\int_0^\infty e^{-ax}\cdot x^{n-1}\,dx = \frac{\Gamma(n)}{a^n}$, where a, n are (+)ve.

Put $ax = z$ so that $dx = \frac{dz}{a}$

$\therefore$

$$\int_0^\infty e^{-ax}x^{n-1}dx = \int_0^\infty e^{-z}\left(\frac{z}{a}\right)^{n-1}\cdot\frac{dz}{a} = \frac{1}{a^n}\int_0^\infty e^{-z}z^{n-1}\,dz = \frac{\Gamma(n)}{a^n}.$$

2.16.1 Deduction

Replacing a by $(a + ib)$, we have

$$\int_0^\infty e^{-(a+ib)x}x^{n-1}\,dx = \frac{\Gamma(n)}{(a+ib)^n}$$

Now $e^{-(a+ib)x} = e^{-ax}\cdot e^{-ibx} = e^{-ax}(\cos bx - i\sin bx)$ ...(1)

Putting $a = r\cos\theta$ and $b = r\sin\theta$ so that $r^2 = a^2 + b^2$ and $\theta = \tan^{-1}\frac{b}{a}$

$$(a+ib)^n = (r\cos\theta + ir\sin\theta)^n = r^n(\cos\theta + i\sin\theta)^n$$
$$= r^n(\cos n\theta + i\sin n\theta)$$

[De Moivre's Theorem]

$\therefore$ From (1), we have

$$\int_0^\infty e^{-ax} (\cos bx - i \sin bx)\, x^{n-1}\, dx = \frac{\Gamma(n)}{r^n (\cos n\theta + i \sin n\theta)}$$

$$= \frac{\Gamma(n)}{r^n} (\cos n\theta + i \sin n\theta)^{-1} = \frac{\Gamma(n)}{r^n} (\cos n\theta - i \sin n\theta)$$

Now equating real and imaginary parts on the two sides, we get

$$\int_0^\infty e^{-ax}\, x^{n-1} \cos bx\, dx = \frac{\Gamma(n)}{r^n} \cos n\theta$$

and $$\int_0^\infty e^{-ax}\, x^{n-1} \sin bx\, dx = \frac{\Gamma(n)}{r^n} \sin n\theta$$

where $r^2 = a^2 + b^2$ and $\theta = \tan^{-1} \frac{b}{a}$.

ILLUSTRATIVE EXAMPLES

Example 1. *Using Beta and Gamma functions, evaluate:*

(i) $\int_0^\infty x^{1/4}\, e^{-\sqrt{x}}\, dx$ (ii) $\int_0^1 \left(\frac{x^3}{1-x^3}\right)^{1/2} dx$ (A.K.T.U. 2014, 2018)

(iii) $\int_0^1 x^5 (1-x^3)^{10}\, dx$

Sol. (*i*) Let $$I = \int_0^\infty x^{1/4}\, e^{-\sqrt{x}}\, dx \quad \text{...(1)}$$

Put $\sqrt{x} = y \Rightarrow x = y^2$ so that $dx = 2y\, dy$ then equation (1) becomes

$$I = \int_0^\infty y^{1/2} e^{-y} \cdot 2y\, dy = 2\int_0^\infty e^{-y}\, y^{3/2}\, dy$$

$$= 2\int_0^\infty e^{-y}\, y^{(5/2)-1} dy = 2\,\Gamma(5/2) \qquad | \text{ By definition}$$

$$= 2 \cdot \frac{3}{2} \cdot \frac{1}{2} \sqrt{\pi} = \frac{3}{2}\sqrt{\pi} \qquad | \because \Gamma(n+1) = n\,\Gamma(n)$$

(*ii*) Let $$I = \int_0^1 x^{3/2} (1-x^3)^{-1/2}\, dx \quad \text{...(1)}$$

Put $x^3 = y \Rightarrow x = y^{1/3}$ so that $dx = \frac{1}{3} y^{-2/3}\, dy$ then equation (1) becomes

$$I = \int_0^1 y^{1/2} (1-y)^{-1/2} \cdot \frac{1}{3} y^{-2/3}\, dy = \frac{1}{3}\int_0^1 y^{-1/6} (1-y)^{-1/2} dy$$

$$= \frac{1}{3}\int_0^1 y^{\left(\frac{5}{6}\right)-1} (1-y)^{\left(\frac{1}{2}\right)-1} dy = \frac{1}{3}\beta\left(\frac{5}{6}, \frac{1}{2}\right)$$

$$= \frac{1}{3} \frac{\Gamma(5/6)\,\Gamma(1/2)}{\Gamma(4/3)} \qquad \Big| \because \beta(m, n) = \frac{\Gamma(m)\,\Gamma(n)}{\Gamma(m+n)}$$

$$= \frac{\sqrt{\pi}}{3} \cdot \frac{\Gamma(5/6)}{\frac{1}{3}\Gamma(1/3)} = \sqrt{\pi} \cdot \frac{\Gamma(5/6)\,\Gamma(1/6)\,\Gamma(2/3)}{\Gamma(1/6)\,\Gamma(1/3)\,\Gamma(2/3)} \qquad \left| \because \Gamma(n+1) = n\Gamma n \right.$$

$$= \sqrt{\pi} \cdot \frac{\Gamma(2/3)}{\Gamma(1/6)} \cdot \frac{\pi}{\sin\frac{\pi}{6}} \cdot \frac{\sin\frac{\pi}{3}}{\pi} \qquad \left| \because \Gamma n\,\Gamma(1-n) = \frac{\pi}{\sin n\pi} \right.$$

$$= \sqrt{3\pi}\,\frac{\Gamma(2/3)}{\Gamma(1/6)}$$

(*iii*) Let $\qquad I = \int_0^1 x^5 (1 - x^3)^{10}\, dx \qquad$ ...(1)

Put $x^3 = y \quad \Rightarrow \quad x = y^{1/3}$ so that $dx = \frac{1}{3} y^{-2/3} dy$ then equation (1) becomes

$$I = \int_0^1 y^{5/3} (1 - y)^{10} \cdot \frac{1}{3} y^{-2/3} dy$$

$$= \frac{1}{3}\int_0^1 y(1-y)^{10}\, dy = \frac{1}{3}\beta(2, 11) = \frac{1}{3}\,\frac{\Gamma 2\,\Gamma(11)}{\Gamma(13)} = \frac{1}{3} \cdot \frac{1}{12 \cdot 11} = \frac{1}{396}$$

Example 2. *Prove that:*

(*i*) $\beta(l, m) \cdot \beta(l + m, n) \cdot \beta(l + m + n, p) = \dfrac{\Gamma l\, \Gamma m\, \Gamma n\, \Gamma p}{\Gamma(l + m + n + p)}$

(*ii*) $\displaystyle\int_0^\infty \frac{x^c}{c^x}\, dx = \frac{\Gamma(c+1)}{(\log c)^{c+1}}\,;\ c > 1$

(*iii*) $\beta(m, m) = 2^{1-2m}\, \beta(m, 1/2)$

(*iv*) $\displaystyle\int_0^{\pi/2} \tan^n x\, dx = \frac{\pi}{2} \sec \frac{n\pi}{2}$

Sol. (*i*) $\qquad$ LHS $= \beta(l, m) \cdot \beta(l + m, n) \cdot \beta(l + m + n, p)$

$$= \frac{\Gamma l\, \Gamma m}{\Gamma(l+m)} \cdot \frac{\Gamma(l+m)\,.\,\Gamma n}{\Gamma(l+m+n)} \cdot \frac{\Gamma l+m+n\ \Gamma p}{\Gamma(l+m+n+p)}$$

$$= \frac{\Gamma l\, \Gamma m\, \Gamma n\, \Gamma p}{\Gamma(l+m+n+p)} = \text{RHS}$$

(*ii*) Let $\qquad I = \int_0^\infty \frac{x^c}{c^x}\, dx = \int_0^\infty e^{-x \log c}\, x^c\, dx \qquad$ ...(1)

Put $x \log c = y \quad \Rightarrow \quad x = \frac{y}{\log c}$ so that $dx = \frac{dy}{\log c}$ then equation (1) becomes

$$I = \int_0^\infty e^{-y} \left(\frac{y}{\log c}\right)^c \frac{dy}{\log c} = \frac{1}{(\log c)^{c+1}} \int_0^\infty e^{-y}\, y^c\, dy$$

$$= \frac{\Gamma(c+1)}{(\log c)^{c+1}}\,;\ c > 1$$

(iii) $\beta(m, 1/2) = \dfrac{\Gamma m\,\Gamma(1/2)}{\Gamma(m+1/2)}$ $\quad\Big|\ \because\ \beta(m,n) = \dfrac{\Gamma(m)\,\Gamma(n)}{\Gamma(m+n)}$

$$= \frac{(\Gamma m)^2\sqrt{\pi}}{\Gamma m\,\Gamma\left(m+\frac{1}{2}\right)} = \frac{\Gamma(m)^2\sqrt{\pi}}{\left(\frac{\sqrt{\pi}}{2^{2m-1}}\right)\Gamma 2m}$$ | By Duplication formula

$$= 2^{2m-1}\frac{\Gamma m\,\Gamma m}{\Gamma(2m)} = 2^{2m-1}\beta(m, m)$$

$\Rightarrow$ $\beta(m, m) = 2^{1-2m}\,\beta(m, 1/2)$

(iv) Let $I = \int_0^{\pi/2}\tan^n x\,dx = \int_0^{\pi/2}\sin^n x\cos^{-n} x\,dx$

$$= \frac{\Gamma\left(\frac{n+1}{2}\right)\Gamma\left(\frac{-n+1}{2}\right)}{2\,\Gamma\left(\frac{n-n+2}{2}\right)} = \frac{1}{2}\Gamma\left(\frac{n+1}{2}\right)\Gamma\left(1-\left(\frac{n+1}{2}\right)\right)$$

$$= \frac{1}{2}\cdot\frac{\pi}{\sin\left(\frac{n+1}{2}\right)\pi} = \frac{\pi}{2\cos\frac{n\pi}{2}} = \frac{\pi}{2}\sec\frac{n\pi}{2}$$

Example 3. *Evaluate:*

(i) $\int_0^\infty \dfrac{x^8(1-x^6)}{(1+x)^{24}}\,dx$ (M.T.U. 2013) (ii) $\int_0^\infty \dfrac{x^4(1+x^5)}{(1+x)^{15}}\,dx$.

Sol. (i) $I = \int_0^\infty \dfrac{x^8}{(1+x)^{24}}\,dx - \int_0^\infty \dfrac{x^{14}}{(1+x)^{24}}\,dx = \int_0^\infty \dfrac{x^{9-1}}{(1+x)^{9+15}}\,dx - \int_0^\infty \dfrac{x^{15-1}}{(1+x)^{15+9}}\,dx$

$= \beta(9, 15) - \beta(15, 9) = 0$ $\quad\Big|\ \because\ \beta(m,n) = \beta(n,m)$

(ii) $I = \int_0^\infty \dfrac{x^4}{(1+x)^{15}}\,dx + \int_0^\infty \dfrac{x^9}{(1+x)^{15}}\,dx = \int_0^\infty \dfrac{x^{5-1}}{(1+x)^{5+10}}\,dx + \int_0^\infty \dfrac{x^{10-1}}{(1+x)^{10+5}}\,dx$

$= \beta(5, 10) + \beta(10, 5) = 2\beta(5, 10) = 2\,\dfrac{\Gamma(5)\,\Gamma(10)}{\Gamma(15)}$ $\quad\Big|\ \because\ \beta(m,n) = \dfrac{\Gamma(m)\,\Gamma(n)}{\Gamma(m+n)}$

$= \dfrac{1}{5005}$. | On simplification

Example 4. *Evaluate:*

(i) $\int_0^2 x(8-x^3)^{1/3}\,dx$ (ii) $\int_0^\infty \dfrac{dx}{1+x^4}$ (G.B.T.U. 2012) (iii) $\int_0^1 \dfrac{dx}{\sqrt{1+x^4}}$ (G.B.T.U. 2011)

Sol. (i) Putting $x^3 = 8y$ or $x = 2y^{1/3}$ so that $dx = \dfrac{2}{3}y^{-2/3}\,dy$, we get

$$I = \int_0^1 2y^{1/3}(8-8y)^{1/3} \cdot \frac{2}{3} y^{-2/3}\, dy$$

$$= \frac{8}{3}\int_0^1 y^{-1/3}(1-y)^{1/3} dy = \frac{8}{3}\beta\left(\frac{2}{3}, \frac{4}{3}\right)$$

$$= \frac{8}{3}\frac{\Gamma(\frac{2}{3})\Gamma(\frac{4}{3})}{\Gamma(2)} = \frac{8}{9}\Gamma(\tfrac{2}{3})\Gamma(\tfrac{1}{3}) = \frac{8}{9}\cdot\frac{\pi}{\sin\frac{\pi}{3}} = \frac{16\pi}{9\sqrt{3}}.$$

(*ii*) Putting $x^4 = y$ so that $x = y^{1/4}$ and $dx = \frac{1}{4}y^{-3/4}\, dy$, we get

$$I = \int_0^\infty \frac{\frac{1}{4}y^{-3/4}}{1+y} dy = \frac{1}{4}\int_0^\infty \frac{y^{\frac{1}{4}-1}}{1+y} dy = \frac{1}{4}\frac{\pi}{\sin\frac{\pi}{4}} = \frac{\pi\sqrt{2}}{4}.$$

$$\because \int_0^\infty \frac{x^{n-1}}{1+x} dx = \frac{\pi}{\sin n\pi}$$

(*iii*) $I = \int_0^1 \frac{dx}{\sqrt{1+x^4}}$

$$\Rightarrow\quad I = \int_0^{\pi/4} \frac{1}{\sec\theta}\cdot\frac{\sec^2\theta}{2\sqrt{\tan\theta}} d\theta$$

Put $x^2 = \tan\theta \;\Rightarrow\; x = \sqrt{\tan\theta}$

$$= \frac{1}{2}\int_0^{\pi/4} \frac{d\theta}{\sqrt{\sin\theta\cos\theta}} = \frac{1}{2}\int_0^{\pi/4}\frac{d\theta}{\sqrt{\sin 2\theta}}$$

$\therefore\; dx = \frac{1}{2\sqrt{\tan\theta}}\sec^2\theta\, d\theta$

$$\Rightarrow\quad I = \frac{1}{2\sqrt{2}}\int_0^{\pi/2} \sin^{-1/2} t\, dt$$

Put $2\theta = t \;\therefore\; d\theta = \frac{dt}{2}$

$$= \frac{1}{2\sqrt{2}}\,\frac{\Gamma\frac{(-1/2)+1}{2}\,\Gamma\frac{0+1}{2}}{2\Gamma\frac{(-1/2)+0+2}{2}} = \frac{1}{4\sqrt{2}}\cdot\frac{\Gamma(1/4)\,\Gamma(1/2)}{\Gamma(3/4)}$$

$$= \frac{\sqrt{\pi}}{4\sqrt{2}}\cdot\frac{\Gamma(1/4)^2}{\Gamma(1/4)\,\Gamma(3/4)} = \frac{\sqrt{\pi}}{4\sqrt{2}}\cdot\frac{\Gamma(1/4)^2}{\left(\dfrac{\pi}{\sin\pi/4}\right)}$$

$$\because\; \Gamma(n)\,\Gamma(1-n) = \frac{\pi}{\sin n\pi}; 0 < n < 1$$

$$= \frac{1}{8\sqrt{\pi}}\left(\Gamma\frac{1}{4}\right)^2.$$

Example 5. *Prove that:*

(*i*) $\int_0^1 y^{q-1}\left(\log\frac{1}{y}\right)^{p-1} dy = \frac{\Gamma(p)}{q^p}$, *where* $p > 0$, $q > 0$.

(*ii*) $\int_0^\infty \frac{x^{m-1}}{(a+bx)^{m+n}} dx = \frac{\beta(m,n)}{a^n b^m}$, *where* m, n, a, b *are positive.*

Sol. (*i*) Put $\log \dfrac{1}{y} = x$ so that $\dfrac{1}{y} = e^x$ or $y = e^{-x}$ and $dy = -e^{-x}\,dx$

$$\therefore \quad \int_0^1 y^{q-1}\left(\log \frac{1}{y}\right)^{p-1} dy = \int_\infty^0 e^{-(q-1)x} \,.\, x^{p-1}(-e^{-x})\,dx = \int_0^\infty e^{-qx} x^{p-1}\,dx$$

$$= \int_0^\infty e^{-t} \,.\left(\frac{t}{q}\right)^{p-1} . \frac{dt}{q}, \quad \text{where } qx = t$$

$$= \frac{1}{q^p}\int_0^\infty e^{-t} t^{p-1}\,dt = \frac{\Gamma(p)}{q^p}.$$

(*ii*) Put $bx = at$ *i.e.* $x = \dfrac{at}{b}$ so that $dx = \dfrac{a}{b}dt$

$$\therefore \quad \int_0^\infty \frac{x^{m-1}}{(a+bx)^{m+n}}dx = \int_0^\infty \frac{\left(\frac{at}{b}\right)^{m-1}}{(a+at)^{m+n}} \cdot \frac{a}{b}\,dt = \frac{1}{a^n b^m}\int_0^\infty \frac{t^{m-1}}{(1+t)^{m+n}}dt = \frac{\beta(m, n)}{a^n b^m}.$$

Example 6. *Show that* $\beta(p, q) = \displaystyle\int_0^1 \frac{x^{p-1} + x^{q-1}}{(1+x)^{p+q}}\,dx.$ (M.T.U. 2012)

Sol. $\beta(p, q) = \displaystyle\int_0^1 x^{p-1}(1-x)^{q-1}\,dx$

Putting $x = \dfrac{1}{1+y}$ so that $dx = -\dfrac{1}{(1+y)^2}dy$

$$\beta(p, q) = \int_\infty^0 \left(\frac{1}{1+y}\right)^{p-1}\left(\frac{y}{1+y}\right)^{q-1} \cdot \frac{-1}{(1+y)^2}dy = \int_0^\infty \frac{y^{q-1}}{(1+y)^{p+q}}dy$$

$$= \int_0^1 \frac{y^{q-1}}{(1+y)^{p+q}}dy + \int_1^\infty \frac{y^{q-1}}{(1+y)^{p+q}}dy \qquad ...(1)$$

Now, putting $y = \dfrac{1}{z}$ in the second integral, we have

$$\int_1^\infty \frac{y^{q-1}}{(1+y)^{p+q}}dy = \int_1^0 \frac{\left(\frac{1}{z}\right)^{q-1}}{\left(1+\frac{1}{z}\right)^{p+q}}\left(-\frac{1}{z^2}\right)dz = \int_0^1 \frac{z^{p-1}}{(1+z)^{p+q}}dz$$

$\therefore$ From (1), we have

$$\beta(p, q) = \int_0^1 \frac{y^{q-1}}{(1+y)^{p+q}}dy + \int_0^1 \frac{z^{p-1}}{(1+z)^{p+q}}dz = \int_0^1 \frac{x^{p-1} + x^{q-1}}{(1+x)^{p+q}}dx.$$

Example 7. *Prove that* $\beta(m, n) = \beta(m + 1, n) + \beta(m, n + 1).$ (U.P.T.U. 2009)

Sol. $\beta\,(m + 1, n) + \beta\,(m, n + 1)$

$$= \frac{\Gamma(m+1)\,\Gamma n}{\Gamma(m+n+1)} + \frac{\Gamma m\,\Gamma(n+1)}{\Gamma(m+n+1)} \qquad \because\ \beta(m, n) = \frac{\Gamma m\,\Gamma n}{\Gamma(m+n)}$$

$$= \frac{1}{\Gamma(m+n+1)}[m\,\Gamma m\,\Gamma n + \Gamma m \cdot n\,\Gamma(n)] \qquad \Big|\ \because\ \Gamma(n+1) = n\,\Gamma(n)$$

$$= \frac{\Gamma m\,\Gamma n}{\Gamma(m+n)(m+n)}(m+n) = \frac{\Gamma m\,\Gamma n}{\Gamma(m+n)} = \beta(m, n).$$

Example 8. *Prove that* $\int_a^b (x-a)^m (b-x)^n\, dx = (b-a)^{m+n+1}\, \beta(m+1, n+1)$

Sol. Let $$\mathrm{I} = \int_a^b (x-a)^m (b-x)^n\, dx \qquad ...(1)$$

Put $x = a + (b-a)z \Rightarrow x - a = (b-a)z$ and $b - x = (b-a)(1-z)$ so that $dx = (b-a)\,dz$ then (1) becomes

$$\mathrm{I} = \int_0^1 (b-a)^m z^m \cdot (b-a)^n (1-z)^n (b-a)\, dz$$

$$= (b-a)^{m+n+1} \int_0^1 z^m (1-z)^n\, dz$$

$$= (b-a)^{m+n+1}\, \beta(m+1, n+1)$$

Example 9. *Prove the following results:*

(i) $\int_0^{\pi/2} \sqrt{\tan\theta}\, d\theta = \int_0^{\pi/2} \sqrt{\cot\theta}\, d\theta = \frac{\pi}{\sqrt{2}}$ (U.K.T.U. 2011)

(ii) $\int_0^{\pi/2} \frac{d\theta}{\sqrt{\sin\theta}} \times \int_0^{\pi/2} \sqrt{\sin\theta}\, d\theta = \pi.$ (U.P.T.U. 2014)

Sol. (i) $\int_0^{\pi/2} \sqrt{\tan\theta}\, d\theta$

$$= \int_0^{\pi/2} \sqrt{\cot\theta}\, d\theta \qquad ...(1) \ \Big|\ \because\ \int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$$

$$= \int_0^{\pi/2} \cos^{1/2}\theta \sin^{-1/2}\theta\, d\theta$$

$$= \frac{\Gamma\left(\frac{-\frac{1}{2}+1}{2}\right)\Gamma\left(\frac{\frac{1}{2}+1}{2}\right)}{2\,\Gamma\left(\frac{-\frac{1}{2}+\frac{1}{2}+2}{2}\right)} = \frac{\Gamma(\frac{1}{4})\,\Gamma(\frac{3}{4})}{2\,\Gamma(1)} = \frac{1}{2}\Gamma(\tfrac{1}{4})\,\Gamma(1-\tfrac{1}{4})$$

$$= \frac{1}{2}\,\frac{\pi}{\sin\frac{\pi}{4}} = \frac{\pi}{\sqrt{2}} \qquad ...(2) \ \Big|\ \because\ \Gamma(n)\,\Gamma(1-n) = \frac{\pi}{\sin n\pi}$$

(ii) LHS
$$= \int_0^{\pi/2} \sin^{-1/2}\theta \cos^0\theta\, d\theta \times \int_0^{\pi/2} \sin^{1/2}\theta \cos^0\theta\, d\theta$$

$$= \frac{\Gamma\left(\frac{-\frac{1}{2}+1}{2}\right)\Gamma\left(\frac{0+1}{2}\right)}{2\,\Gamma\left(\frac{-\frac{1}{2}+0+2}{2}\right)} \times \frac{\Gamma\left(\frac{\frac{1}{2}+1}{2}\right)\Gamma\left(\frac{0+1}{2}\right)}{2\,\Gamma\left(\frac{\frac{1}{2}+0+2}{2}\right)} = \frac{\Gamma(\frac{1}{4})\,\Gamma(\frac{1}{2})}{2\,\Gamma(\frac{3}{4})} \times \frac{\Gamma(\frac{3}{4})\,\Gamma(\frac{1}{2})}{2\,\Gamma(\frac{5}{4})}$$

$$= \frac{\Gamma(\frac{1}{4})\sqrt{\pi}}{4} \times \frac{\sqrt{\pi}}{\frac{1}{4}\Gamma(\frac{1}{4})} = \pi = \text{RHS} \qquad \Big|\ \because\ \Gamma(n+1) = n\,\Gamma(n) \text{ and } \Gamma(\tfrac{1}{2}) = \sqrt{\pi}$$

Example 10. *Evaluate:*

(*i*) $\int_0^\infty \cos x^2 \, dx$ (*ii*) $\int_{-\infty}^\infty \cos \frac{\pi}{2} x^2 \, dx$ (*iii*) $\int_0^1 \log \Gamma(x) \, dx$.

Sol. (*i*) We know that

$$\int_0^\infty e^{-ax} \cdot x^{n-1} \cos bx \, dx = \frac{\Gamma(n) \cos n\theta}{(a^2+b^2)^{n/2}} \text{ where } \theta = \tan^{-1}\left(\frac{b}{a}\right)$$

Put $a = 0$, $\int_0^\infty x^{n-1} \cos bx \, dx = \frac{\Gamma(n)}{b^n} \cos \frac{n\pi}{2}$

Put $x^n = z$ so that $x^{n-1} dx = \frac{dz}{n}$ and $x = z^{1/n}$

then, $$\int_0^\infty \cos bz^{1/n} \, dz = \frac{n\,\Gamma(n)}{b^n} \cos \frac{n\pi}{2}$$

or $$\int_0^\infty \cos (bx^{1/n}) \, dx = \frac{\Gamma(n+1)}{b^n} \cos \frac{n\pi}{2} \quad \text{...(1)}$$

Here $b = 1$, $n = \frac{1}{2}$

$$\therefore \quad \int_0^\infty \cos x^2 \, dx = \Gamma(\tfrac{3}{2}) \cos \frac{\pi}{4} = \frac{\sqrt{\pi}}{2} \cdot \frac{1}{\sqrt{2}} = \frac{1}{2}\sqrt{\frac{\pi}{2}}.$$

(*ii*) $$I = \int_{-\infty}^\infty \cos \frac{\pi x^2}{2} \, dx = 2\int_0^\infty \cos \frac{\pi x^2}{2} \, dx \quad \text{...(2)}$$

Putting $b = \frac{\pi}{2}$ and $n = \frac{1}{2}$ in equation (1), we get

$$\int_0^\infty \cos\left(\frac{\pi}{2} x^2\right) dx = \frac{\Gamma\left(\frac{3}{2}\right)}{\left(\frac{\pi}{2}\right)^{1/2}} \cos \frac{\pi}{4}$$

$$\therefore \text{ From (2), } \quad I = 2\,\frac{\Gamma\left(\frac{3}{2}\right)}{\left(\frac{\pi}{2}\right)^{1/2}} \cos \frac{\pi}{4} = 2 \cdot \frac{1}{2}\sqrt{\pi} \cdot \frac{\sqrt{2}}{\sqrt{\pi}} \cdot \frac{1}{\sqrt{2}} = 1.$$

(*iii*) Let $$I = \int_0^1 \log \Gamma(x) \, dx \quad \text{...(1)}$$

$$= \int_0^1 \log \Gamma(1-x) \, dx \quad \text{...(2)}$$

Adding (1) and (2),

$$2I = \int_0^1 (\log \Gamma(x) + \log \Gamma(1-x)) \, dx$$

$$= \int_0^1 \log (\Gamma(x)\,\Gamma(1-x)) \, dx = \int_0^1 \log\left(\frac{\pi}{\sin \pi x}\right) dx$$

$$= \int_0^1 (\log \pi - \log \sin \pi x)\, dx = \int_0^1 \log \pi\, dx - \int_0^1 \log \sin \pi x\, dx$$

$$= I_1 - I_2 \qquad \ldots(3)$$

where $\quad I_1 = \int_0^1 \log \pi\, dx = \log \pi$

$$I_2 = \int_0^1 \log \sin \pi x\, dx = \int_0^\pi \log \sin t \left(\frac{dt}{\pi}\right) \qquad \left|\ \text{Put } \pi x = t \Rightarrow dx = \frac{1}{\pi} dt\right.$$

$$= \frac{1}{\pi} \cdot 2 \int_0^{\pi/2} \log \sin t\, dt = \frac{2}{\pi}\left(-\frac{\pi}{2} \log 2\right) = -\log 2$$

From (3), $\quad 2I = \log \pi + \log 2 = \log 2\pi$

$$I = \frac{1}{2} \log 2\pi.$$

Example 11. *Prove that:*

(*a*) $\iint_D x^{l-1} y^{m-1} dxdy = \dfrac{\Gamma(l)\,\Gamma(m)}{\Gamma(l+m+1)} a^{l+m}$, *where D is the domain* $x \geq 0, y \geq 0$ *and* $x + y \leq a$.

(*b*) *Establish Dirichlet's integral:* $\iiint_V x^{l-1} y^{m-1} z^{n-1}\, dx\, dy\, dz = \dfrac{\Gamma(l)\,\Gamma(m)\,\Gamma(n)}{\Gamma(l+m+n+1)}$

where V is the region $x \geq 0, y \geq 0, z \geq 0$ *and* $x + y + z \leq 1$.

Sol. (*a*) Putting $x = a\mathrm{X}$ and $y = a\mathrm{Y}$, the given integral reduces to

$$I = \iint_{D'} (a\mathrm{X})^{l-1} (a\mathrm{Y})^{m-1} a^2\, d\mathrm{X} d\mathrm{Y}$$

where D′ is the domain $\mathrm{X} \geq 0$, $\mathrm{Y} \geq 0$ and $\mathrm{X} + \mathrm{Y} \leq 1$

$$I = a^{l+m} \int_0^1 \int_0^{1-\mathrm{X}} \mathrm{X}^{l-1}\, \mathrm{Y}^{m-1}\, d\mathrm{Y} d\mathrm{X}$$

$$= a^{l+m} \int_0^1 \mathrm{X}^{l-1} \left| \frac{\mathrm{Y}^m}{m} \right|_0^{1-\mathrm{X}} d\mathrm{X} = \frac{a^{l+m}}{m} \int_0^1 \mathrm{X}^{l-1} (1-\mathrm{X})^m\, d\mathrm{X}$$

$$= \frac{a^{l+m}}{m} \beta(l, m+1) = \frac{a^{l+m}}{m} \cdot \frac{\Gamma(l)\,\Gamma(m+1)}{\Gamma(l+m+1)} = a^{l+m} \frac{\Gamma(l)\,\Gamma(m)}{\Gamma(l+m+1)}$$

(*b*) Taking $y + z \leq 1 - x = a$ (say), the given integral

$$I = \int_0^1 \int_0^{1-x} \int_0^{1-x-y} x^{l-1} y^{m-1} z^{n-1}\, dzdydx$$

$$= \int_0^1 x^{l-1} \left[\int_0^a \int_0^{a-y} y^{m-1} z^{n-1}\, dz\, dy \right] dx$$

$$= \int_0^1 x^{l-1} \frac{\Gamma(m)\,\Gamma(n)}{\Gamma(m+n+1)} a^{m+n}\, dx \qquad [\text{by } (a)]$$

$$= \frac{\Gamma(m)\,\Gamma(n)}{\Gamma(m+n+1)} \int_0^1 x^{l-1} (1-x)^{m+n}\, dx, \text{ since } a = 1 - x$$

$$= \frac{\Gamma(m)\,\Gamma(n)}{\Gamma(m+n+1)} B(l, m+n+1) = \frac{\Gamma(m)\,\Gamma(n)}{\Gamma(m+n+1)} \cdot \frac{\Gamma(l)\,\Gamma(m+n+1)}{\Gamma(l+m+n+1)}$$

$$= \frac{\Gamma(l)\,\Gamma(m)\,\Gamma(n)}{\Gamma(l+m+n+1)}.$$

Example 12. *The plane* $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$ *meets the axes in A, B and C. Apply Dirichlet's integral to find the volume of the tetrahedron OABC. Also find its mass if the density at any point is kxyz.* (A.K.T.U. 2012, 2018)

Sol. Put $\frac{x}{a} = u, \frac{y}{b} = v, \frac{z}{c} = w$, then $u \ge 0, v \ge 0, w \ge 0$ and $u + v + w \le 1$.

Also, $dx = a\,du$, $dy = b\,dv$, $dz = c\,dw$.

$$\text{Volume OABC} = \iiint_D dx\,dy\,dz$$

$$= \iiint_{D'} abc\,du\,dv\,dw, \quad \text{where } u + v + w \le 1$$

$$= abc \iiint_{D'} u^{1-1}\, v^{1-1}\, w^{1-1}\, du\,dv\,dw$$

$$= abc \frac{\Gamma(1)\,\Gamma(1)\,\Gamma(1)}{\Gamma(1+1+1+1)} = \frac{abc}{3!} = \frac{abc}{6}.$$

$$\text{Mass} = \iiint_D kxyz\,dx\,dy\,dz = \iiint_{D'} k(au)(bv)(cw)\,abc\,du\,dv\,dw$$

$$= ka^2b^2c^2 \iiint_{D'} u^{2-1}v^{2-1}w^{2-1}\,du\,dv\,dw$$

$$= ka^2b^2c^2 \frac{\Gamma(2)\,\Gamma(2)\,\Gamma(2)}{\Gamma(2+2+2+1)} = ka^2b^2c^2 \frac{1!\,1!\,1!}{6!} = \frac{ka^2b^2c^2}{720}.$$

Example 13. *Evaluate* $I = \iiint_V x^{\alpha-1}\, y^{\beta-1}\, z^{\gamma-1}\, dx\,dy\,dz$ *where v is the region in the first octant bounded by sphere* $x^2 + y^2 + z^2 = 1$ *and the coordinate planes.*

Sol. Let $x^2 = u \Rightarrow x = \sqrt{u} \quad \therefore \quad dx = \frac{1}{2\sqrt{u}} du$

$y^2 = v \Rightarrow y = \sqrt{v} \quad \therefore \quad dy = \frac{1}{2\sqrt{v}} dv$

$z^2 = w \Rightarrow z = \sqrt{w} \quad \therefore \quad dz = \frac{1}{2\sqrt{w}} dw$

Then, $u + v + w = 1$. Also, $u \ge 0, v \ge 0, w \ge 0$.

$$I = \iiint_V (\sqrt{u})^{\alpha-1} (\sqrt{v})^{\beta-1} (\sqrt{w})^{\gamma-1} \frac{du}{2\sqrt{u}} \cdot \frac{dv}{2\sqrt{v}} \cdot \frac{dw}{2\sqrt{w}}$$

$$= \frac{1}{8} \iiint u^{(\alpha/2)-1}\, v^{(\beta/2)-1}\, w^{(\gamma/2)-1}\, du\,dv\,dw$$

$$= \frac{1}{8} \frac{\Gamma(\alpha/2)\,\Gamma(\beta/2)\,\Gamma(\gamma/2)}{\Gamma((\alpha/2) + (\beta/2) + (\gamma/2) + 1)}.$$

Example 14. *Show that if l, m, n are all positive,*

$$\iiint x^{l-1}y^{m-1}z^{n-1}dx\,dy\,dz = \frac{a^l b^m c^n}{8} \cdot \frac{\Gamma(l/2)\,\Gamma(m/2)\,\Gamma(n/2)}{\Gamma(l/2+m/2+n/2+1)},$$ *where the triple integral is taken throughout the part of ellipsoid* $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ *which lies in the positive octant.*

Sol. Put $\left(\frac{x}{a}\right)^2 = u, \left(\frac{y}{b}\right)^2 = v, \left(\frac{z}{c}\right)^2 = w$ so that $dx = \frac{a\,du}{2\sqrt{u}}$ etc.

$$\therefore \quad \iiint x^{l-1}y^{m-1}z^{n-1}dx\,dy\,dz$$

$$= \iiint a^{l-1}(\sqrt{u})^{l-1}b^{m-1}(\sqrt{v})^{m-1}c^{n-1}(\sqrt{w})^{n-1}\frac{abc\,du\,dv\,dw}{8\sqrt{u}\sqrt{v}\sqrt{w}}$$

$$= \frac{a^l b^m c^n}{8}\iiint u^{\frac{l}{2}-1}\,v^{\frac{m}{2}-1}\,w^{\frac{n}{2}-1}\,du\,dv\,dw \qquad \text{subject to } u+v+w=1$$

$$= \frac{a^l b^m c^n}{8}\,\frac{\Gamma(l/2)\,\Gamma(m/2)\,\Gamma(n/2)}{\Gamma\frac{l}{2}+\frac{m}{2}+\frac{n}{2}+1}.$$

Example 15. *Evaluate the integral* $\iiint x^{l-1}y^{m-1}z^{n-1}\,dx\,dy\,dz$ *where x, y, z are all positive but limited by the condition* $\left(\frac{x}{a}\right)^p+\left(\frac{y}{b}\right)^q+\left(\frac{z}{c}\right)^r \le 1.$

[G.B.T.U. 2011]

Sol. Let $\left(\frac{x}{a}\right)^p = u \quad \Rightarrow \quad x = au^{1/p} \quad \therefore \quad dx = \frac{a}{p}u^{\frac{1}{p}-1}\,du$

$\left(\frac{y}{b}\right)^q = v \quad \Rightarrow \quad y = bv^{1/q} \quad \therefore \quad dy = \frac{b}{q}v^{\frac{1}{q}-1}\,dv$

$\left(\frac{z}{c}\right)^r = w \quad \Rightarrow \quad z = cw^{1/r} \quad \therefore \quad dz = \frac{c}{r}w^{\frac{1}{r}-1}\,dw$

Then, $u+v+w \le 1$. Also $u \ge 0, v \ge 0, w \ge 0$ since x, y, z are all positive.

$$I = \iiint a^{l-1}u^{\frac{l-1}{p}}\,b^{m-1}v^{\frac{m-1}{q}}\,c^{n-1}w^{\frac{n-1}{r}}\,\frac{abc}{pqr}\,u^{\frac{1}{p}-1}\,v^{\frac{1}{q}-1}\,w^{\frac{1}{r}-1}\,du\,dv\,dw$$

$$= \frac{a^l b^m c^n}{pqr}\iiint u^{\frac{l}{p}-1}\,v^{\frac{m}{q}-1}\,w^{\frac{n}{r}-1}\,du\,dv\,dw$$

$$= \frac{a^l b^m c^n}{pqr}\,\frac{\Gamma\left(\frac{l}{p}\right)\Gamma\left(\frac{m}{q}\right)\Gamma\left(\frac{n}{r}\right)}{\Gamma\left(\frac{l}{p}+\frac{m}{q}+\frac{n}{r}+1\right)}$$

Example 16. *Find the volume of the solid bounded by the co-ordinate planes and the surface* $\sqrt{\frac{x}{a}}+\sqrt{\frac{y}{b}}+\sqrt{\frac{z}{c}}=1$.

Sol. Put $\sqrt{\frac{x}{a}}=u, \sqrt{\frac{y}{b}}=v, \sqrt{\frac{z}{c}}=w$ then $u \ge 0, v \ge 0, w \ge 0$ and $u+v+w=1$

Also, $dx = 2au\, du,\quad dy = 2bv\, dv,\ dz = 2cw\, dw$

$$\text{Required volume} = \iiint_D dx\, dy\, dz$$

$$= \iiint_{D'} 8\,abc\, uvw\, du\, dv\, dw, \qquad \text{where } u+v+w=1$$

$$= 8\,abc \iiint_{D'} u^{2-1}\, v^{2-1}\, w^{2-1}\, du\, dv\, dw$$

$$= 8\,abc \frac{\Gamma 2\, \Gamma 2\, \Gamma 2}{\Gamma(2+2+2+1)} = 8\,abc \cdot \frac{1\cdot 1\cdot 1}{\Gamma(7)} = \frac{abc}{90}.$$

Example 17. *Apply Dirichlet's integral to find the volume and the mass contained in the solid region in the first octant of the ellipsoid* $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ *if the density at any point is* $\rho(x, y, z) = kxyz$.

(U.P.T.U. 2015)

Sol. Put $\frac{x^2}{a^2}=u, \frac{y^2}{b^2}=v, \frac{z^2}{c^2}=w$ then $u \ge 0, v \ge 0, w \ge 0$ and $u+v+w=1$

Also, $dx = \frac{a}{2\sqrt{u}}du,\ dy = \frac{b}{2\sqrt{v}}dv,\ dz = \frac{c}{2\sqrt{w}}dw$

$$\text{Required Volume} = \iiint_D dx\, dy\, dz = \iiint_{D'} \frac{abc}{8\sqrt{u}\sqrt{v}\sqrt{w}}\, du\, dv\, dw$$

$$= \frac{abc}{8}\iiint_{D'} u^{\frac{1}{2}-1}\, v^{\frac{1}{2}-1}\, w^{\frac{1}{2}-1}\, du\, dv\, dw$$

$$= \frac{abc}{8}\frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{5}{2}\right)} = \frac{abc}{8}\cdot\frac{\pi\sqrt{\pi}}{\frac{3}{2}\cdot\frac{1}{2}\sqrt{\pi}}$$

$$= \frac{\pi\, abc}{6}$$

and
$$\text{Required Mass} = \iiint_D k\, xyz\, dx\, dy\, dz$$

$$= \iiint_{D'} k\cdot a\sqrt{u}\cdot b\sqrt{v}\cdot c\sqrt{w}\cdot \frac{abc}{8\sqrt{u}\sqrt{v}\sqrt{w}}\, du\, dv\, dw$$

$$= k\,\frac{a^2b^2c^2}{8}\iiint_{D'} du\,dv\,dw = k\,\frac{a^2b^2c^2}{8}\iiint_{D'} u^{1-1}\,v^{1-1}\,w^{1-1}\,du\,dv\,dw$$

$$= k\,\frac{a^2b^2c^2}{8}\,\frac{\Gamma(1)\,\Gamma(1)\,\Gamma(1)}{\Gamma(4)} = k\,\frac{a^2b^2c^2}{48}.$$

Example 18. *Find the volume of the solid surrounded by the surface*

$$\left(\frac{x}{a}\right)^{2/3} + \left(\frac{y}{b}\right)^{2/3} + \left(\frac{z}{c}\right)^{2/3} = 1.$$

Sol. Let $\left(\frac{x}{a}\right)^{2/3} = u \Rightarrow x = au^{3/2} \quad \therefore \quad dx = \frac{3a}{2}u^{1/2}\,du$

$\left(\frac{y}{b}\right)^{2/3} = v \Rightarrow y = bv^{3/2} \quad \therefore \quad dy = \frac{3b}{2}v^{1/2}\,dv$

$\left(\frac{z}{c}\right)^{2/3} = w \Rightarrow z = cw^{3/2} \quad \therefore \quad dz = \frac{3c}{2}w^{1/2}\,dw$

For the positive octant,

$$x \geq 0 \Rightarrow au^{3/2} \geq 0 \Rightarrow u \geq 0$$
$$y \geq 0 \Rightarrow bv^{3/2} \geq 0 \Rightarrow v \geq 0$$
$$z \geq 0 \Rightarrow cw^{3/2} \geq 0 \Rightarrow w \geq 0$$

Then, we have $u + v + w = 1,\ u \geq 0,\ v \geq 0,\ w \geq 0$.

Required volume $= 8\iiint dx\,dy\,dz$

$$= 8\iiint \frac{3a}{2}u^{1/2}\cdot\frac{3b}{2}v^{1/2}\cdot\frac{3c}{2}w^{1/2}\,du\,dv\,dw$$

$$= 27\,abc\iiint u^{\frac{3}{2}-1}\,v^{\frac{3}{2}-1}\,w^{\frac{3}{2}-1}\,du\,dv\,dw$$

$$= 27\,abc\cdot\frac{\Gamma\left(\frac{3}{2}\right)\Gamma\left(\frac{3}{2}\right)\Gamma\left(\frac{3}{2}\right)}{\Gamma\left(\frac{11}{2}\right)} = \frac{4\pi abc}{35}$$

Example 19. *Show that the area bounded by the curve $x^n + y^n = a^n$ and the co-ordinate axes in the first quadrant is* $\dfrac{a^2\,\Gamma\left(\frac{1}{n}\right)^2}{2n\,\Gamma\left(\frac{2}{n}\right)}$.

Sol. Required area $\quad A = \iint_D dx\,dy \quad$ ...(1)

Let, $\left(\frac{x}{a}\right)^n = u$ so that $x = au^{1/n} \quad \therefore \quad dx = \frac{a}{n}u^{\frac{1}{n}-1}\,du$

and $\left(\frac{y}{a}\right)^n = v$ so that $y = av^{1/n} \quad \therefore \quad dy = \frac{a}{n}v^{\frac{1}{n}-1}\,dv$

Then, $u = 0, v = 0$ and $u + v = 1$

From (1), $$A = \frac{a^2}{n^2} \iint_{D'} u^{\frac{1}{n}-1} v^{\frac{1}{n}-1} \, du \, dv = \frac{a^2}{n^2} \frac{\Gamma\left(\frac{1}{n}\right)\Gamma\left(\frac{1}{n}\right)}{\Gamma\left(\frac{2}{n}+1\right)}$$

$$\Rightarrow \qquad A = \frac{a^2}{n^2} \cdot \frac{1}{\left(\frac{2}{n}\right)} \cdot \frac{\Gamma\left(\frac{1}{n}\right)^2}{\Gamma\left(\frac{2}{n}\right)} = \frac{a^2}{2n} \frac{\Gamma\left(\frac{1}{n}\right)^2}{\Gamma\left(\frac{2}{n}\right)}.$$

Example 20. *Find the area and the mass contained in the first quadrant enclosed by the curve* $\left(\frac{x}{a}\right)^{\alpha} + \left(\frac{y}{b}\right)^{\beta} = 1$ *where* $\alpha > 0$, $\beta > 0$ *given that density at any point* $\rho(x, y)$ *is* $k\sqrt{xy}$.

Sol. The area A of the plane region is $A = \iint_D dx \, dy$ ...(1)

Let $\left(\frac{x}{a}\right)^{\alpha} = u$ so that $x = au^{1/\alpha}$ $\quad \therefore \quad dx = \frac{a}{\alpha} u^{\frac{1}{\alpha}-1} du$

and $\left(\frac{y}{b}\right)^{\beta} = v$ so that $y = bv^{1/\beta}$ $\quad \therefore \quad dy = \frac{b}{\beta} v^{\frac{1}{\beta}-1} dv$

Then, $u > 0, v > 0$ and $u + v = 1$.

From (1), $$A = \frac{ab}{\alpha\beta} \iint_{D'} u^{\frac{1}{\alpha}-1} v^{\frac{1}{\beta}-1} \, du \, dv = \frac{ab}{\alpha\beta} \frac{\Gamma\left(\frac{1}{\alpha}\right)\Gamma\left(\frac{1}{\beta}\right)}{\Gamma\left(\frac{1}{\alpha}+\frac{1}{\beta}+1\right)}$$

Now, the total mass M contained in the plane region A is

$$M = \iint_D \rho(x, y) \, dx \, dy = k \iint_D \sqrt{xy} \, dx \, dy$$

$$= k \iint_{D'} \sqrt{a}\, u^{\frac{1}{2\alpha}} \sqrt{b}\, v^{\frac{1}{2\beta}} \cdot \frac{a}{\alpha} u^{\frac{1}{\alpha}-1} \cdot \frac{b}{\beta} v^{\frac{1}{\beta}-1} \, du \, dv$$

$$= k \frac{(ab)^{3/2}}{\alpha\beta} \iint_{D'} u^{\frac{3}{2\alpha}-1} v^{\frac{3}{2\beta}-1} \, du \, dv$$

$$= k \frac{(ab)^{3/2}}{\alpha\beta} \frac{\Gamma(3/2\alpha)\,\Gamma(3/2\beta)}{\Gamma\left(\frac{3}{2\alpha}+\frac{3}{2\beta}+1\right)}$$

TEST YOUR KNOWLEDGE

Prove that (1–22):

1. (i) $\int_0^\infty x^2 e^{-x^2}\, dx = \frac{\sqrt{\pi}}{4}$. (ii) $\int_0^\infty \sqrt{x}\, e^{-x^3}\, dx = \frac{\sqrt{\pi}}{3}$. (iii) $\int_0^\infty e^{-x^2} x^{2n-1} dx = \frac{1}{2}\Gamma(n)$.

2. (i) $\int_0^1 x^5(1-x^3)^3 dx = \frac{1}{60}$ (ii) $\int_0^1 x^3(1-x)^{4/3}\, dx = \frac{243}{7280}$

3. $\int_0^2 (8-x^3)^{-1/3}\, dx = \frac{1}{3}\Gamma\left(\tfrac{1}{3}\right)\Gamma\left(\tfrac{2}{3}\right) = \frac{2\pi}{3\sqrt{3}}$.

4. (i) $\int_0^\infty 4x^4 e^{-x^4} dx = \Gamma\left(\tfrac{5}{4}\right)$ (ii) $\int_0^\infty x^6 e^{-2x} dx = \frac{45}{8}$ (iii) $\int_0^1 \sqrt{\frac{1-x}{x}}\, dx = \frac{\pi}{2}$

5. (i) $\int_0^\infty \frac{e^{-st}}{\sqrt{t}}\, dt = \sqrt{\frac{\pi}{s}}, s > 0$ (ii) $\int_0^1 \frac{dx}{\sqrt{-\log x}} = \sqrt{\pi}$.

   (iii) $\int_0^1 \left(\sqrt{\frac{x^3}{1-x^3}}\right)^{1/2} dx = \frac{\Gamma\frac{7}{12}\Gamma\frac{3}{4}}{\Gamma\frac{1}{3}}$

6. (i) $\Gamma(.1)\,\Gamma(.2)\,\Gamma(.3)\ldots\Gamma(.9) = \frac{(2\pi)^{9/2}}{\sqrt{10}}$

   (ii) $\Gamma\left(\frac{3}{2}-p\right)\Gamma\left(\frac{3}{2}+p\right) = \left(\frac{1}{4}-p^2\right)\pi \sec p\pi,\ -1 < 2p < 1$

7. (i) $\int_0^1 \frac{dx}{\sqrt{1-x^n}} = \frac{\sqrt{\pi}}{n}\frac{\Gamma\left(\frac{1}{n}\right)}{\Gamma\left(\frac{1}{n}+\frac{1}{2}\right)}$ (ii) $\int_0^1 \frac{dx}{\sqrt{1-x^4}} = \frac{1}{4\sqrt{2\pi}}\left(\Gamma\tfrac{1}{4}\right)^2$.

8. (i) $\int_0^\infty \frac{x\,dx}{1+x^6} = \frac{\pi}{3\sqrt{3}}$ (ii) $\int_0^\infty \frac{x^2\,dx}{1+x^4} = \frac{\pi}{2\sqrt{2}}$

   (iii) $\int_0^2 \frac{x^2}{\sqrt{2-x}}\, dx = \frac{64\sqrt{2}}{15}$ (iv) $\int_0^3 \frac{dx}{\sqrt{3x-x^2}} = \pi$.

9. (i) $\int_0^\infty \sqrt{y}\, e^{-y^2} dy \times \int_0^\infty \frac{e^{-y^2}}{\sqrt{y}} dx = \frac{\pi}{2\sqrt{2}}$. (ii) $\int_0^\infty \frac{e^{-x^2}}{\sqrt{x}} dx \times \int_0^\infty x^2 e^{-x^4} dx = \frac{\pi}{4\sqrt{2}}$.

10. (i) $\int_0^\infty x^2 e^{-x^4} dx \cdot \int_0^\infty e^{-x^4} dx = \frac{\pi}{8\sqrt{2}}$ (ii) $\int_0^1 \frac{x^2 dx}{\sqrt{1-x^4}} \cdot \int_0^1 \frac{dx}{\sqrt{1+x^4}} = \frac{\pi}{4\sqrt{2}}$.

11. (i) $\int_0^{\pi/2} \sin^3 x \cos^{5/2} x\, dx = \frac{8}{77}$ (ii) $\int_0^a x^{n-1}(a-x)^{m-1}\, dx = a^{m+n-1}\beta(m, n)$

12. (i) $\int_0^1 \frac{dx}{\sqrt{1-x^6}} = \frac{\sqrt{3}}{2}\int_0^1 \frac{dx}{\sqrt{1-x^3}} = \frac{(\Gamma 1/3)^3}{(2)^{7/3}\pi}$ (ii) $\int_0^1 \frac{x\,dx}{\sqrt{1-x^5}} = \frac{1}{5}\beta\left(\frac{2}{5}, \frac{1}{2}\right)$.

13. (i) $\frac{\Gamma\left(\frac{1}{3}\right)\Gamma\left(\frac{5}{6}\right)}{\Gamma\left(\frac{2}{3}\right)} = (2)^{1/3}\sqrt{\pi}$ (A.K.T.U. 2017) (ii) $\Gamma\left(-\tfrac{3}{2}\right) = \frac{4}{3}\sqrt{\pi}$ (G.B.T.U. 2013)

   (iii) $\Gamma\left(\tfrac{1}{6}\right) = 2^{-1/3}\pi^{-1/2}\sqrt{3}\left(\Gamma\tfrac{1}{3}\right)^2$ (iv) $\Gamma\left(-\tfrac{1}{2}\right) = -2\sqrt{\pi}$.

14. $\int_0^1 x^m (\log x)^n\, dx = \frac{(-1)^n n!}{(m+1)^{n+1}}$, where n is a positive integer and $m > -1$.

15. $\int_0^\infty \int_0^\infty e^{-(ax^2+by^2)} x^{2m-1} y^{2n-1}\, dx\, dy = \frac{\Gamma(m)\Gamma(n)}{4a^m b^n}$, where a, b, m, n are positive.

16. (i) $\beta(m, m) \cdot \beta\left(m+\frac{1}{2}, m+\frac{1}{2}\right) = \frac{\pi m^{-1}}{(2)^{4m-1}}$ (ii) $\beta(m, n) = a^m b^n \int_0^\infty \frac{x^{m-1}}{(ax+b)^{m+n}}\, dx$ (U.P.T.U. 2015)

17. $\frac{B(p, q+1)}{q} = \frac{B(p+1, q)}{p} = \frac{B(p, q)}{p+q}$, $(p > 0, q > 0)$ (U.P.T.U. 2015; U.K.T.U. 2012)

18. $\frac{B(m+2, n-2)}{B(m, n)} = \frac{m(m+1)}{(n-1)(n-2)}$.

19. $2^n \Gamma\left(n+\frac{1}{2}\right) = 1 \cdot 3 \cdot 5 \ldots\ldots (2n-1)\sqrt{\pi}$

20. $\int_0^1 \frac{x^{m-1}(1-x)^{n-1}}{(a+bx)^{m+n}}\, dx = \frac{B(m, n)}{a^n (a+b)^m}$. $\left[\textbf{Hint.} \text{ Put } \frac{x}{a+bx} = \frac{z}{a+b \cdot 1}.\right]$

21. $\int_0^{\pi/2} \frac{d\theta}{(a\cos^4\theta + b\sin^4\theta)^{1/2}} = \frac{(\Gamma\frac{1}{4})^2}{4(ab)^{1/4}\sqrt{\pi}}$ [**Hint.** Put $\tan\theta = t$, then $bt^4 = az$.]

22. $\int_0^\infty x^n e^{-a^2x^2}\, dx = \frac{1}{2a^{n+1}} \Gamma\left(\frac{n+1}{2}\right)$, $n > 1$. Deduce that $\int_{-\infty}^\infty e^{-a^2x^2}\, dx = \frac{\sqrt{\pi}}{a}$.

23. Assuming $\Gamma n\, \Gamma(1-n) = \pi \operatorname{cosec} n\pi$, $0 < n < 1$, show that $\int_0^\infty \frac{x^{p-1}}{1+x}\, dx = \frac{\pi}{\sin p\pi}$; $0 < p < 1$. (G.B.T.U. 2010)

24. Show that $\iint x^{m-1} y^{n-1}\, dx\, dy$ over the positive octant of the ellipse $\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1$ is $\frac{a^m b^n}{2n} \beta\left(\frac{m}{2}, \frac{n}{2}+1\right)$.

25. Find the volume of the solid bounded by co-ordinate planes and the surface

$$\left(\frac{x}{a}\right)^{2n} + \left(\frac{y}{b}\right)^{2n} + \left(\frac{z}{c}\right)^{2n} = 1, n \text{ being a positive integer.}$$

26. (i) Find the volume of the solid bounded by the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$.

(ii) Find the volume contained in the solid region in the first octant of the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$. (U.P.T.U. 2014)

(iii) Evaluate $\iiint xyz\, dx\, dy\, dz$ for all positive value of variables of ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$.

27. (i) Find the mass of the region bounded by ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ if the density varies as the square of the distance from the centre. [**Hint.** $\rho = k(x^2 + y^2 + z^2)$] (G.B.T.U. 2010)

(ii) Find the mass of a solid $\left(\frac{x}{a}\right)^p + \left(\frac{y}{b}\right)^q + \left(\frac{z}{c}\right)^r = 1$, the density at any point being $\rho = kx^{l-1} y^{m-1} z^{n-1}$ where x, y, z are all positive. (A.K.T.U. 2016)

28. Evaluate $\iiint_V (ax^2 + by^2 + cz^2)\, dx\, dy\, dz$ where V is the region bounded by $x^2 + y^2 + z^2 \le 1$.

(G.B.T.U. 2013)

29. Compute $\iiint_v x^2\, dx\, dy\, dz$ over valume of tetraheron bounded by $x = 0$, $y = 0$, $z = 0$ and $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$.

(A.K.T.U. 2017)

30. Evaluate $\iiint_v x^2yz\, dx\, dy\, dz$ throughout the volume bounded by planes $x = 0$, $y = 0$, $z = 0$ and $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$.

(A.K.T.U. 2017)

Answers

25. $\dfrac{abc}{12n^2} \dfrac{(\Gamma\frac{1}{2n})^3}{\Gamma(\frac{3}{2n})}$ **26.** (i) $\frac{4}{3}\pi\, abc$ (ii) $\frac{\pi\, abc}{6}$ (iii) $\frac{a^2b^2c^2}{48}$

27. (i) $\dfrac{k\pi\, abc}{16}(a^2 + b^2 + c^2)$ (ii) $\dfrac{ka^l b^m c^n}{pqr} \dfrac{(l/p)!\,(m/q)!\,(n/r)!}{\left(\frac{l}{p} + \frac{m}{q} + \frac{n}{r} + 1\right)!}$

28. $\dfrac{\pi(a + b + c)}{30}$ **29.** $\dfrac{a^3bc}{60}$ **30.** $\dfrac{a^3b^2c^2}{2520}$

2.17 LIOUVILLE'S EXTENSION OF DIRICHLET THEOREM

If the variables x, y, z are all positive such that $h_1 < (x + y + z) < h_2$ then

$$\iiint f(x + y + z) x^{l-1} y^{m-1} z^{n-1} dx\, dy\, dz = \frac{\Gamma(l)\,\Gamma(m)\,\Gamma(n)}{\Gamma(l + m + n)} \int_{h_1}^{h_2} f(u)\, u^{l+m+n-1}\, du.$$

Proof. Let, $I = \iiint x^{l-1} y^{m-1} z^{n-1}\, dx\, dy\, dz$ under the condition $x + y + z \le u$ then

$$I = \left(\frac{\Gamma(l)\,\Gamma(m)\,\Gamma(n)}{\Gamma(l + m + n + 1)}\right) u^{l+m+n} \quad \text{...(1)} \;|\; \text{By Dirichlet's Theorem}$$

If $x + y + z \le u + \delta u$, then

$$I = (u + \delta u)^{l+m+n} \frac{\Gamma(l)\,\Gamma(m)\,\Gamma(n)}{\Gamma(l + m + n + 1)} \quad \text{...(2)}$$

Now, if $u < x + y + z < (u + \delta u)$, then

$$\iiint x^{l-1} y^{m-1} z^{n-1}\, dx\, dy\, dz = \frac{\Gamma(l)\,\Gamma(m)\,\Gamma(n)}{\Gamma(l + m + n + 1)}\left[(u + \delta u)^{l+m+n} - (u)^{l+m+n}\right]$$

$$= \frac{\Gamma(l)\,\Gamma(m)\,\Gamma(n)}{\Gamma(l + m + n + 1)} u^{l+m+n}\left[1 + \left(\frac{\delta u}{u}\right)^{l+m+n} - 1\right]$$

$$= \frac{\Gamma(l)\,\Gamma(m)\,\Gamma(n)}{\Gamma(l + m + n + 1)} u^{l+m+n}\left[1 + (l + m + n)\frac{\delta u}{u} + \ldots - 1\right]$$

$$= \frac{\Gamma(l)\,\Gamma(m)\,\Gamma(n)}{\Gamma(l+m+n+1)} u^{l+m+n}(l+m+n)\frac{\delta u}{u}$$

$$= \frac{\Gamma(l)\,\Gamma(m)\,\Gamma(n)}{\Gamma(l+m+n)} u^{l+m+n-1}\,\delta u$$

Now, consider $\iiint f(x+y+z)x^{l-1}y^{m-1}z^{n-1}\,dx\,dy\,dz$ under the condition $h_1 \le (x+y+z) \le h_2$. When $x+y+z$ lies between u and $u+\delta u$, the value of $f(x+y+z)$ can only differ from $f(u)$ by a small quantity of the same order as δu. Hence,

$$\iiint f(x+y+z)x^{l-1}y^{m-1}z^{n-1}\,dx\,dy\,dz = \frac{\Gamma(l)\,\Gamma(m)\,\Gamma(n)}{\Gamma(l+m+n)}\int f(u)\,u^{l+m+n-1}\,\delta u$$

where $x+y+z$ lies between u and $u+\delta u$.

Therefore, $$\boxed{\iiint f(x+y+z)x^{l-1}\,y^{m-1}\,z^{n-1}\,dx\,dy\,dz = \frac{\Gamma(l)\,\Gamma(m)\,\Gamma(n)}{\Gamma(l+m+n)}\int_{h_1}^{h_2} f(u)\,u^{l+m+n-1}\,du.}$$

ILLUSTRATIVE EXAMPLES

Example 1. *Evaluate $\iiint \log(x+y+z)\,dx\,dy\,dz$, the integral extending over all positive and zero values of x, y, z subject to $x+y+z<1$.*

Sol. $$0 \le x+y+z < 1$$

$$\therefore \iiint \log(x+y+z)\,dx\,dy\,dz = \iiint x^{1-1}y^{1-1}z^{1-1}\log(x+y+z)\,dx\,dy\,dz$$

$$= \frac{\Gamma(1)\,\Gamma(1)\,\Gamma(1)}{\Gamma(3)}\int_0^1 t^{1+1+1-1}\log t\,dt \qquad \text{| By Liouville's extension}$$

$$= \frac{1}{2}\int_0^1 t^2\log t\,dt = \frac{1}{2}\left[\left(\frac{t^3}{3}\log t\right)_0^1 - \int_0^1 \frac{t^3}{3}\cdot\frac{1}{t}\,dt\right] = \frac{1}{2}\left[-\frac{1}{3}\left(\frac{t^3}{3}\right)_0^1\right] = -\frac{1}{18}.$$

Example 2. *Evaluate $\iiint\ldots\int_n \frac{dx_1dx_2\ldots dx_n}{\sqrt{1-x_1^2-x_2^2\ldots-x_n^2}}$, integral being extended to all positive values of the variables for which the expression is real.*

Sol. The expression will be real, if

$$1-x_1^2-x_2^2-\ldots-x_n^2>0$$

or $$x_1^2+x_2^2+\ldots+x_n^2<1$$

Hence the given integral is extended for all positive value of the variables $x_1, x_2, \ldots, x_n$ such that $0<x_1^2+x_2^2+\ldots x_n^2<1$.

Let us put $x_1^2 = u_1$ i.e., $x_1 = \sqrt{u_1}$ so that, $dx_1 = \frac{1}{2\sqrt{u_1}}du_1$ etc.

Then the condition becomes, $0 < u_1 + u_2 + \ldots + u_n < 1$.

$\therefore$ Required integral $= \dfrac{1}{2^n} \iiint \ldots \int_n \dfrac{u_1^{-1/2} u_2^{-1/2} \ldots u_n^{-1/2}}{\sqrt{1 - u_1 - u_2 - \ldots - u_n}} du_1\, du_2 \ldots du_n$

$$= \frac{1}{2^n} \iiint \ldots \int_n \frac{u_1^{1/2-1} u_2^{1/2-1} \ldots u_n^{1/2-1}}{\sqrt{1 - u_1 - u_2 - \ldots - u_n}} du_1\, du_2 \ldots du_n$$

$$= \frac{1}{2^n} \frac{\Gamma(1/2)\,\Gamma(1/2) \ldots \Gamma(1/2)}{\Gamma(1/2 + 1/2 + \ldots + 1/2)} \int_0^1 \frac{1}{\sqrt{1-u}} u^{(1/2+1/2+\ldots\ldots+1/2)-1} du$$

| By Liouville's Extension

$$= \frac{1}{2^n} \cdot \frac{[\Gamma(1/2)]^n}{\Gamma(n/2)} \int_0^1 \frac{1}{\sqrt{1-u}} u^{\frac{n}{2}-1} du$$

$$= \frac{1}{2^n} \frac{(\sqrt{\pi})^n}{\Gamma(n/2)} \int_0^{\pi/2} \frac{1}{\sqrt{1 - \sin^2\theta}} (\sin^2\theta)^{\frac{n}{2}-1} \cdot 2 \sin\theta \cos\theta \, d\theta$$

(Putting $u = \sin^2\theta$)

$$= \frac{1}{2^{n-1}} \cdot \frac{(\sqrt{\pi})^n}{\Gamma(n/2)} \int_0^{\pi/2} \sin^{n-1}\theta \, d\theta = \frac{1}{2^{n-1}} \cdot \frac{(\pi)^{n/2}}{\Gamma(n/2)} \cdot \frac{\Gamma(n/2)\,\Gamma(1/2)}{2\,\Gamma\left(\frac{n+1}{2}\right)}$$

$$= \frac{1}{2^n} \frac{(\pi)^{\frac{n+1}{2}}}{\Gamma\left(\frac{n+1}{2}\right)}.$$

Example 3. *Show that* $\iiint \dfrac{dx\, dy\, dz}{(x+y+z+1)^3} = \dfrac{1}{2} \log 2 - \dfrac{5}{16}$, *the integral being taken throughout the volume bounded by planes* $x = 0$, $y = 0$, $z = 0$ *and* $x + y + z = 1$.

Sol. $0 \le x + y + z \le 1$

$$\therefore \quad \iiint \frac{dx\, dy\, dz}{(x+y+z+1)^3} = \iiint \frac{x^{1-1} y^{1-1} z^{1-1}}{(x+y+z+1)^3} dx\, dy\, dz$$

$$= \frac{\Gamma(1)\,\Gamma(1)\,\Gamma(1)}{\Gamma(1+1+1)} \int_0^1 \frac{1}{(u+1)^3} u^{1+1+1-1} du$$

$$= \frac{1}{2} \int_0^1 \frac{u^2}{(u+1)^3} du$$

| By Liouville's extension

Put $u + 1 = t$ so that $du = dt$

$\therefore$ Required integral $= \dfrac{1}{2} \displaystyle\int_1^2 \frac{(t-1)^2}{t^3} dt = \frac{1}{2} \int_1^2 \left(\frac{t^2 - 2t + 1}{t^3} \right) dt$

$$= \frac{1}{2} \int_1^2 \left(\frac{1}{t} - \frac{2}{t^2} + \frac{1}{t^3} \right) dt = \frac{1}{2} \left[\log t + \frac{2}{t} - \frac{1}{2t^2} \right]_1^2$$

$$= \frac{1}{2}\left[\log 2 + 1 - \frac{1}{8} - 2 + \frac{1}{2}\right] = \frac{1}{2}\left[\log 2 - \frac{5}{8}\right] = \frac{1}{2}\log 2 - \frac{5}{16}.$$

Example 4. *Evaluate* $\iiint \sqrt{\frac{1-x^2-y^2-z^2}{1+x^2+y^2+z^2}}\, dx\, dy\, dz$, *integral being taken over all positive values of x, y, z such that* $x^2 + y^2 + z^2 \le 1$.

Sol. Putting $x^2 = u$, $y^2 = v$, $z^2 = w$ so that $u + v + w \le 1$

Also, $x = \sqrt{u} \Rightarrow dx = \frac{1}{2\sqrt{u}} du$

$y = \sqrt{v} \Rightarrow dy = \frac{1}{2\sqrt{v}} dv$

$z = \sqrt{w} \Rightarrow dz = \frac{1}{2\sqrt{w}} dw$

$\therefore$ The given integral

$$= \iiint \sqrt{\frac{1-(u+v+w)}{1+(u+v+w)}} \frac{du\, dv\, dw}{8\sqrt{uvw}}$$

$$= \frac{1}{8}\iiint u^{1/2-1} v^{1/2-1} w^{1/2-1} \sqrt{\frac{1-(u+v+w)}{1+(u+v+w)}}\, du\, dv\, dw$$

$$= \frac{1}{8}\frac{\Gamma(\frac{1}{2})\Gamma(\frac{1}{2})\Gamma(\frac{1}{2})}{\Gamma(\frac{1}{2}+\frac{1}{2}+\frac{1}{2})}\int_0^1 \sqrt{\frac{1-u}{1+u}}\, u^{\frac{1}{2}+\frac{1}{2}+\frac{1}{2}-1} du \quad | \text{ Using Liouville's extension}$$

$$= \frac{1}{8}\frac{(\Gamma\frac{1}{2})^3}{\frac{1}{2}\Gamma(\frac{1}{2})}\int_0^1 \frac{(1-u)}{\sqrt{1-u^2}} u^{1/2} du = \frac{\pi}{4}\int_0^1 \frac{(1-\sqrt{t})}{\sqrt{1-t}} t^{1/4} \frac{dt}{2\sqrt{t}} \quad \text{where } u^2 = t$$

$$= \frac{\pi}{8}\int_0^1 \frac{(1-\sqrt{t})\, t^{-1/4}}{\sqrt{1-t}} dt = \frac{\pi}{8}\left[\int_0^1 t^{\frac{3}{4}-1}(1-t)^{1/2-1} dt - \int_0^1 t^{5/4-1}(1-t)^{1/2-1} dt\right]$$

$$= \frac{\pi}{8}\left[\beta\left(\frac{3}{4}, \frac{1}{2}\right) - \beta\left(\frac{5}{4}, \frac{1}{2}\right)\right].$$

TEST YOUR KNOWLEDGE

1. Evaluate $\iiint e^{x+y+z} dx\, dy\, dz$, taken over positive octant such that $x + y + z \le 1$.

2. (*i*) Evaluate the integral $\iiint \frac{dx\, dy\, dz}{\sqrt{a^2 - x^2 - y^2 - z^2}}$ the integral being extended to all positive values of the variables for which the expression is real. *[U.K.T.U. 2010]*

(*ii*) Prove that $\iiint \frac{dx\, dy\, dz}{\sqrt{1 - x^2 - y^2 - z^2}} = \frac{\pi^2}{8}$, the integral being extended to all positive values of the variables for which the expression is real. *(A.K.T.U. 2016; U.P.T.U. 2014)*

3. Apply Dirichlet's integral to find the moment of inertia about z-axis of an octant of the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$.

4. Show that $\iiint \frac{dx\,dy\,dz}{(x+y+z+1)^2}=\frac{3}{4}-\log 2$, the integral being taken throughout the volume bounded by the planes $x=0, y=0, z=0$ and $x+y+z=1$.

5. Evaluate $\iiint (x+y+z)^9\, dx\,dy\,dz$ over the region defined by $x \ge 0, y \ge 0, z \ge 0$ and $x+y+z \le 1$.

6. Evaluate $\iiint \sqrt{\frac{1-x-y-z}{xyz}}\, dx\,dy\,dz$, extended to all positive values of the variables subject to the condition $x+y+z<1$.

7. Evaluate $\iint x^{l-1}y^{-l}e^{x+y}dx\,dy$, extended to all positive values subject to $x+y<h$.

8. Evaluate $\iiint xyz \sin(x+y+z)\, dx\,dy\,dz$, the integral being extended to all positive values of the variables subject to the condition $x+y+z \le \frac{\pi}{2}$. (U.K.T.U. 2011)

9. Evaluate $\iint \frac{\sqrt{1-(x^2/a^2)-(y^2/b^2)}}{\sqrt{1+(x^2/a^2)+(y^2/b^2)}}dx\,dy$, where $\frac{x^2}{a^2}+\frac{y^2}{b^2}\le 1$

10. Evaluate $\iiint \sqrt{a^2b^2c^2-b^2c^2x^2-c^2a^2y^2-a^2b^2z^2}\, dx\,dy\,dz$ taken throughout the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$.

11. Evaluate $\iiint_V e^{-(x+y+z)}\, dx\,dy\,dz$, where the region of integration is bounded by planes $x=0, y=0, z=0$ and $x+y+z=a, a>0$. [U.P.T.U. (SUM) 2008]

12. For all values of the variables for which $x^2+y^2+z^2+w^2$ is not less than a^2 and not greater than b^2, prove that:

$$\iiiint dx\,dy\,dz\,dw=\frac{\pi^2}{32}(b^4-a^4).$$

Answers

1. $\frac{e}{2}-1$
2. (i) $\frac{\pi^2 a^2}{8}$
3. $\frac{\rho abc}{30}(a^2+b^2)\pi$
5. $\frac{1}{24}$
6. $\frac{1}{4}\pi^2$
7. $\frac{\pi}{\sin \pi l}(e^h-1)$
8. $\frac{1}{5!}\left[\frac{5}{16}\pi^4-15\pi^2+120\right]$
9. $\frac{\pi ab}{8}(\pi-2)$
10. $\frac{1}{4}\pi^2a^2b^2c^2$
11. $1-e^{-a}\left[1+a+\frac{a^2}{2}\right]$.

2.18 APPLICATION OF DEFINITE INTEGRALS TO VOLUMES AND SURFACE AREAS

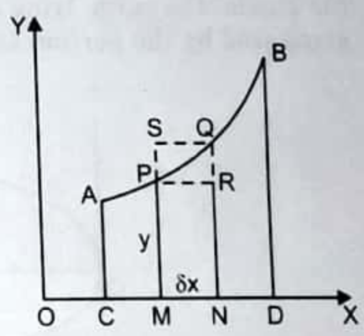
If a plane area is revolved about a fixed straight line lying in its own plane then the body so generated by the plane area is called the *solid of revolution* and the surface generated by the boundary of the plane area is called the *surface of revolution*. The fixed line about which the plane area rotates is called the axis of *revolution*.

For example, a right angled triangle when revolved about one of its sides (forming the right angle) generates a right circular cone.

Note: The section of a solid of revolution by a plane perpendicular to the axis of revolution is a circle having its centre on the axis of revolution.

2.19 VOLUME FORMULAE FOR CARTESIAN EQUATIONS

Prove that volume of the solid generated by revolution about x-axis, of the area bounded by the curve $y = f(x)$, the solid axis and ordinates $x = a, x = b$ is $\int_a^b \pi y^2\, dx$, where $y = f(x)$ is a finite, continuous and single-valued function of x in the interval $a \le x \le b$.



Let AB be the curve $y = f(x)$ and CA, DB be the ordinates $x = a, x = b$ respectively. Let $P(x, y)$ be any point on the curve AB. Draw PM ⊥ OX.

$\therefore \quad OM = x$ and $PM = y$

Let V denote the volume of the solid generated by the revolution about X-axis of the area $ACMP$.

As x increases *i.e.*, MP moves towards the right, V also increases.

Thus the volume V depends on x and is therefore a function of x.

Let $Q(x + \delta x, y + \delta y)$ be a point on the curve in the neighbourhood of P. Then the volume of the solid generated by the revolution about x-axis of the area ACNQ will be $V + \delta V$, so that the volume of the solid generated by the revolution about x-axis of the area PMNQ is δV.

Complete the rectangle PRQS.

Then the volume of the solid generated by the revolution about the x-axis of the area $PMNQ$ lies between the right circular cylinders generated by the rectangles PMNR and SMNQ *i.e.*, δV lies between $\pi y^2\, \delta x$ and $\pi(y + \delta y)^2\, \delta x$

or $\dfrac{\delta V}{\delta x}$ lies between πy^2 and $\pi(y + \delta y)^2$

In the limiting position as $Q \to P$, $\delta x \to 0$ and $\therefore \ \delta y \to 0$

$\therefore \quad \underset{\delta x \to 0}{\text{Lt}} \dfrac{\delta V}{\delta x}$ lies between πy^2 and $\underset{\delta y \to 0}{\text{Lt}} \pi(y + dy)^2$

or $\dfrac{dV}{dx}$ lies between πy^2 and a quantity which approaches πy^2.

$$\therefore \quad \frac{dV}{dx} = \pi y^2$$

$$\int_c^b \pi y^2\, dx = \int_a^b \frac{dV}{dx}\, dx = [V]_a^b$$