MA 214 - Introduction to Numerical Analysis

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1 Interpolation Theory

1.1 Introduction

- Given finite set of points, reconstructing the original curve is interpolation.
- There will be obviously infinitely many curve.
- Interpolation problem
  Given n+1 real distinct points: $x_0, x_1, \ldots, x_n$ and real numbers: $y_0, y_1, \ldots, y_n$
  Find a function $f : \mathbb{R} \to \mathbb{R}$ such that
  $$f(x_i) = y_i \qquad \text{for } i = 0, 1, \ldots, n$$
  Such a function is called **interpolant** and points x_i are called **interpolation points**.
  We attempt to rebuild original function using polynomial functions. This is called polynomial interpolation and function is polynomial interpolant.
- A polynomial is function of the form
  $$p(x) = a_0 + a_1 x + \ldots + a_n x^n$$
- $\mathbb{P}_n$ is the set of polynomials consisting of all polynomials of degree $\leq n$

1.2 Polynomial Interpolation

Theorem (Joseph-Louis Lagrange Theorem). *Given $n+1$ data points with unique x_is, then there exists a unique polynomial $p_n \in \mathbb{P}_n$ such that*

$$p(x_i) = y_i \qquad \textit{for } i = 0, 1, \ldots, n$$

Proof. (1) This can be shown by linear algebra. In a n degree polynomial, we substitute the points and get $n+1$ equations in n+1 variables (coeff) and all the rows are unique (since $x_0, x_1, \ldots, x_n$ are unique), hence in $AX = b$, $|A| \neq 0$.

□

Proof. (2) Part 1: Uniqueness : If there is an interpolant, then the interpolant is unique
Let there be 2 interpolants, p_n and q_n and let $r(x) = p(x) - q(n)$,

$$r(x) = 0 \text{ for } i = 0, 1, \ldots, n$$

This contradicts the fundamental theorem of Algebra. (A polynomial of degree n can have at most n real roots). Therefore

$$r(x) = 0 \quad \forall x \in \mathbb{R}$$
$$p(x) = q(x) \quad \forall x \in \mathbb{R}$$

Part 2: Existence (construction):
Given n+1 data points, build n+1 Langrange polynomials

$$L_k^n(x_i) = \begin{cases} 0 & \text{for } i \neq k \\ 1 & \text{for } i = k \end{cases}$$

$$L_k^n(x) = \frac{(x - x_0)...(x - x_{k-1})(x - x_{k+1})...(x - x_n)}{(x_k - x_0)...(x_k - x_{k-1})(x_k - x_{k+1})...(x_k - x_n)}$$

$$p(x) = \sum_{k=0}^{n} y_k L_k^n(x)$$

□

1.3 Closeness between functions

Given two continuous functions $f, g : [a, b] \to \mathbb{R}$, to evaluate how close the functions are consider the following

$$max_{x\in[a,b]}|f(x) - g(x)|$$

1.4 Set of continuous Functions

$C[a, b]$ is the set of all continuous functions on [a,b]
$C[a, b]$ is a infinite dimensional vector space

$$f, g \in C \implies f + g \in C \text{ and } \lambda f \in C$$

We define norm on $C[a, b]$ as

$$||f|| = max_{x\in[a,b]}|f(x)|$$

$C^k[a, b]$ denotes the set of all functions which are continuously k-times differentiable

1.5 Polynomial Approximation and Error

Theorem (Weierstrass approximation Theorem)**.** *Given a function $f \in C[a, b]$ and given $\epsilon > 0$, there exists a polynomial $p(x)$ such that,*

$$||f(x) - p|| < \epsilon$$

Using Langrange's recipe to approximate
Take $n+1$ interpolation points in the [a,b] and collect the function values at all the points. We have $n+1$ data points. Using Lagrange polynomials, find the interpolant

Theorem (Error equation)**.** *Let $f \in C^k[a, b]$, $x_0, x_1, \ldots, x_n \in [a, b]$ and $p \in \mathbb{P}_n$ be the interpolant using these points, then for all x, there exists a $\zeta = \zeta(x) \in (a, b)$ such that*

$$f(x) - p(x) = \frac{1}{(n+1)!} f^{(n+1)}(\zeta) \prod_{k=0}^{n} (x - x_k)$$

Note: *Here ζ is dependent on the x, i.e, for every x you choose, ζ generally changes.*

Proof. Consider the function,

$$\psi(t) = (f(t) - p(t)) \prod_{k=0}^{n} (x - x_k) - (f(x) - p(x)) \prod_{k=0}^{n} (t - x_k)$$

This $n+2$ roots (n+1 data points and x), applying rolle theorem's gives us that $\phi^{(1)}(t)$ has at least n+1 roots. Applying like this repeatedly on its derivatives, we get that $f^{(n+1)}$ has at least 1 root in $[a, b]$. Assuming the root to be ζ. We have,

$$f(x) - p(x) = \frac{1}{(n+1)!} f^{(n+1)}(\zeta) \prod_{k=0}^{n} (x - x_k)$$

□

Approximating the error:
Taking norm on both the sides of error equation, we have,

$$\max_{x\in[a,b]} |f(x) - p(x)| = \frac{1}{(n+1)!} ||f^{(n+1)}(\zeta) \prod_{k=0}^{n} (x - x_k)|| \tag{1}$$

$$\max_{x\in[a,b]} |f(x) - p(x)| \le \frac{1}{(n+1)!} ||f^{(n+1)}|| \; max_{x\in[a,b]} \prod_{k=0}^{n} (x - x_k) \tag{2}$$

Chebyshef interpolation points:

$$x_k = \frac{a+b}{2} + \frac{b-a}{2} cos\left(\frac{j\pi}{n}\right)$$

These points minimise $max_{x\in[a,b]} \prod_{k=0}^{n}(x - x_k)$ and therefore prefered over equally spaced points on real line. These points can be visualised as projections of equally spaced points on the arc of the semicircle with $\frac{a+b}{2}$ as center and $\frac{(b-a)}{2}$ as radius.

1.6 Another method for calculating interpolant

This is similar to the linear algebra method (given as proof(1) to Joseph-Louis Lagrange Theorem) for finding the interpolant.
Consider the polynomial

$$p(x) = a_0 + a_1(x - x_0) + a_2(x - x_0)(x - x_1) + \ldots + a_n(x - x_0)...(x - x_{n-1})$$

Find the coefficients $a_0, a_1, \ldots, a_n$ by substituting the data points. On substituting x_0, we get a_0, again on substituting x_1 and using a_0, we get a_1 and so on.

1.7 Divided difference - recursion relation

- **Divided difference** : It is the coefficient of x_n in the interpolant $p \in \mathbb{P}_n$ and denoted by $f[x_0, x_1 \ldots, x_n]$.
  Using Langrange polynomials, we have

  $$p(x) = \sum_{k=0}^{n} f(x_k) \prod_{j=0, j \neq k}^{n} \frac{x - x_j}{x_k - x_j}$$

  So the divided difference is

  $$f[x_0, x_1, \ldots, x_n] = \sum_{k=0}^{n} f(x_k) \prod_{j=0, j \neq k}^{n} \frac{1}{x_k - x_j}$$

- interpolant $p(x)$ of $x_0, x_1, \ldots, x_n$ can be calculated using interpolants $a(x)$ and $b(x)$ of $x_1, x_2, \ldots, x_n$ and $x_0, x_1, \ldots, x_{n-1}$ respectively as

  $$p(x) = \frac{(x - x_0)a(x) - (x - x_n)b(x)}{x_n - x_0}$$

Theorem (Divided difference recursion theorem)**.**

$$f[x_0, x_1, \ldots, x_{m+1}] = \frac{[f[x_1, x_2, \ldots, x_{m+1}] - f[x_0, x_1, \ldots, x_m]}{x_{m+1} - x_0}$$

Proof. Let p(x) be the interpolant for $x_0, x_1, \ldots, x_m$ and q(x) be the interpolant for $x_1, x_2 \ldots, x_{m+1}$. Then,

$$L(x) = \frac{(x - x_0)q(x) + (x_{m+1} - x)p(x)}{x_{m+1} - x_0}$$

is an interpolant. Since, interpolant is unique, considering coeff of x_m we have,

$$f[x_0, x_1, \ldots, x_m] = \frac{f[x_1, x_2, \ldots, x_{m+1}] - f[x_0, x_1, \ldots, x_m]}{x_{m+1} - x_0}$$

□

Theorem (Interpolant using divided differences)**.** *Suppose $x_0, x_1, \ldots, x_n$ be the data points. Then interpolant $p \in \mathbb{P}_n$ is*

$$p(x) = f[x_0] + f[x_0, x_1](x - x_0) + \ldots + f[x_0, x_1, \ldots, x_n] \prod_{j=0}^{n-1} (x - x_j)$$

Proof. We prove this by induction. Base case $n = 0$ is trivially satisfied. Assume that this is satisfied for p_k,

$$p_k(x) = f[x_0] + f[x_0, x_1](x - x_0) + \ldots + f[x_0, x_1, \ldots, x_k] \prod_{j=0}^{k-1} (x - x_j)$$

Consider the polynomial $p_{k+1}(x) - p_k(x) \in \mathbb{P}_{k+1}$ which has $x_0, x_1, \ldots, x_k$ as roots. Hence,

$$p_{k+1}(x) - p_k(x) = c \prod_{j=0}^{k} (x - x_j)$$

Comparing leading coefficient on both sides, we have $c = f[x_0, x_1, \ldots, x_k]$. Hence,

$$p_{k+1}(x) = f[x_0] + f[x_0, x_1](x - x_0) + \ldots + f[x_0, x_1, \ldots, x_{k+1}] \prod_{j=0}^{k} (x - x_j)$$

By PMI,

$$p(x) = f[x_0] + f[x_0, x_1](x - x_0) + \ldots + f[x_0, x_1, \ldots, x_n] \prod_{j=0}^{n-1} (x - x_j)$$

□

1.8 Time complexity of the algorithms

(Slide is useless)

- Langrange's method - $O(n^2)$: Computing each Langrange polynomial can be done in $O(n)$ (Finding the coefficients given roots can be done in $O(nlog^2(n))$). We need to do to this n times. So, $O(n)$.
- Divided differences - $O(n^2)$: Summing operations in each stage - $n + (n-1) + (n-2) + \ldots + 1$. Hence, $O(n^2)$.
- Divided difference can be considered better because we can extend from n to n+1 and so on without discarding previous computation.

1.9 Weierstrass theorem consequences

- For Weierstrass theorem, take $\epsilon_n = \frac{1}{n}$. Then weirstrass theorem proves the existence of sequence of polynomials $p^{(1)}, p^{(2)}, \ldots$ such that

$$\lim_{n\to\infty} ||f - p^{(n)}|| = 0$$

- If **f in not a polynomial**, then

$$\lim_{n\to\infty} \text{degree of } p(n) = 0$$

To prove (crude) this, assume that there exist a non-polynomial function f such that $||f - p^{(n)}|| = 0$, then $f = p^(n)$ which contradicts the assumtion that f is not a polynomial.

1.10 Spline Interpolation

- **Piece wise polynomial**: $\phi \in C[a,b]$ is a piecewise polynomial function, if there exists $a = x_0 < x_1 < \ldots < x_n = b$ such that $\phi \in \mathbb{P}_m$ when $x \in [x_i, x_{i+1}]$ for all $i = 0, 1, \ldots, n$ and some $m > 0$.
- Piece wise polynomial ϕ need not be polynomial in whole domain.
- Splines interpolation for $f \in C[a,b]$
  - Pick some data points $x_0, x_1, \ldots, x_n$ such that $a = x_0 < x_1 < \ldots < x_n = b$
  - Fix $m \leq n$
  - Build ϕ in each subinterval $[x_i, x_{i+1}]$ using the following conditions:

$$\phi(x_i) = f_i \qquad \text{for } i = 0, 1, \ldots, n$$

$$\lim_{h\to 0+} \phi(x_i - h) = \lim_{h\to 0+} \phi(x_i + h) \qquad \text{for } i = 1, 2, \ldots, n-1$$

$$\lim_{h\to 0+} \frac{\mathrm{d}\phi(x_i - h)}{\mathrm{d}x} = \lim_{h\to 0+} \frac{\mathrm{d}\phi(x_i + h)}{\mathrm{d}x} \qquad \text{for } i = 1, 2, \ldots, n-1$$

$$\lim_{h\to 0+} \frac{\mathrm{d}^2\phi(x_i - h)}{\mathrm{d}x^2} = \lim_{h\to 0+} \frac{\mathrm{d}^2\phi(x_i + h)}{\mathrm{d}x^2} \qquad \text{for } i = 1, 2, \ldots, n-1$$

$$\vdots$$

$$\lim_{h\to 0+} \frac{\mathrm{d}^{n-1}\phi(x_i - h)}{\mathrm{d}x^{n-1}} = \lim_{h\to 0+} \frac{\mathrm{d}^{n-1}\phi(x_i + h)}{\mathrm{d}x^{n-1}} \qquad \text{for } i = 1, 2, \ldots, n-1$$

  - We have $(n+1) + m(n-1) = n(m+1) - (m-1)$ conditions. We need $m - 1$ more conditions.

Theorem (Error - Linear Splines)**.** *Let $f \in C^2[a,b]$ and $s_L(x)$ be the interpolating **linear spline** at (n+1) knots $a = x_0 < x_1 < \ldots < x_n = b$ and let h be the maximum subinterval length, then*

$$||f - s_L|| \leq \frac{h^2}{8} ||f''||$$

Proof. Consider the interval $[x_i, x_{i+1}]$, $\phi(x)$ is the interpolating polynomial in this interval, using the error equation for interpolating polynomials,

$$f(x) - s_L(x) = \frac{1}{2} f''(\zeta)(x - x_i)(x - x_{i+1})$$

Taking absolute value on both the sides,

$$\begin{aligned} |f(x) - s_L(x)| &= \frac{1}{2} |f''(\zeta)||(x - x_i)(x - x_{i+1})| \\ &\leq \frac{1}{2} ||f''|| \frac{h_i^2}{4} \qquad \text{where } h_i = \frac{x_{i+1} - x_i}{2} \\ &\leq \frac{h_i^2}{8} ||f''|| \end{aligned}$$

Considering $h = max(h_i)$, then for $x \in [a, b]$

$$max_{x \in [a,b]} |f(x) - s_L(x)| \leq \frac{1}{8} h^2 ||f''||$$

$\square$