ME 202 - Strength of Materials

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1 Review

1.1 Strain Energy

$$u = \frac{1}{2}E\epsilon^2 = \frac{1}{2}\sigma\epsilon = \frac{1}{2}\frac{\sigma^2}{E}$$

1.2 Poison's Ratio

$\epsilon_{axial} = \frac{\delta}{L}$, δ is the change in longitudinal length
$\epsilon_{radial} = \frac{\delta'}{r}$, δ' is the change in radius

Poisson's ratio -

$$\nu = -\frac{\epsilon_{radial}}{\epsilon_{axial}}$$
$$0 \le \nu \le 0.5$$

Shear stress and strain

$$\tau = G\gamma$$

Relation between G, E and ν -

$$G = \frac{E}{2(1+\nu)}$$

For a axial load,

$$d\delta = \frac{N(x)dx}{A(x)E(x)}$$

2 Torsion

2.1 Uniform Torsion

- **Angle of twist** : It is the angle by which one end of the rod is displaced wrt the other end of the rod under the effect of some torsion (twisting).
- Consider a small cylidrical section which has length dx, one end has been displaced wrt another by an angle of $d\phi$. Consider a line ab on the circumference along the length of the cylinder which has be changed to ab'. Then shear strain

$$\gamma_{max} = \frac{bb'}{ab} \qquad \text{(bb' can be assumed to be a straight line)}$$

$$\boxed{\gamma_{max} = \frac{rd\phi}{dx}}$$

- Rate of twist or angle of twist per unit length

$$\theta = \frac{\mathrm{d}\phi}{\mathrm{d}x}$$
$$\gamma_{max} = r\theta$$

If θ is constant, then

$$\gamma_{max} = \frac{r\phi}{L}$$

This is called as γ_{max} because we are measuring the shear strain at the outer end i.e, with maximum radius and hence, maximum shear strain.

$$\boxed{\gamma = \rho\theta = \rho\frac{\mathrm{d}\phi}{\mathrm{d}x}}$$

where ρ is the perpendicular distance of the point from the axis (radius) we are considering.

$$\boxed{\gamma = \frac{\rho}{r}\gamma_{max}}$$

2.2 Hooke's Law

Hooke' law for shear stress and shear strain

$$\tau = G\gamma = G\frac{\rho}{r}\gamma_{max} = \frac{\rho}{r}\tau_{max}$$

$$\tau_{max} = G\gamma_{max}$$

2.3 Torsion Formula

- Polar moment of inertia (this is integral over area - double integral)-

$$\boxed{I_P = \int_A \rho^2 dA}$$

  where ρ is the distance at which area element dA is located.

- For a circle of radius r,

$$I_P = \int_A \rho^2 dA = \int_{\theta=0}^{2\pi}\int_{\rho=0}^{r} \rho^2(\rho d\theta d\rho) = \frac{\pi r^4}{2}$$

$$I_{Pcircle} = \frac{\pi r^4}{2}$$

- Consider a cross-section of any shape, we are trying to sum all the small torques and equate it to the torque applied on this

$$\begin{aligned} T = \int_A dM &= \int_A \rho\tau dA \\ &= \int_A \frac{\rho^2}{r}\tau_{max} dA \\ &= \frac{\tau_{max}}{r}\int_A \rho^2 dA \\ T &= \frac{\tau_{max}}{r} I_P \end{aligned}$$

  Or,

$$\tau_{max} = \frac{Tr}{I_P}$$

  item The shear stress at distance ρ from the center of the bar with polar moment of inertia I_P is

$$\boxed{\tau = \frac{T\rho}{I_P}}$$

  For a rod of circular cross section of radius r,

$$\boxed{\tau_{max} = \frac{2T}{\pi r^3} = \frac{16T}{\pi d^3}}$$

- Rate of twist

$$\boxed{\theta = \frac{T}{GI_P}}$$

  Hence, GI_P is also known as **Torsional Rigidity**

- For a bar in pure torsion ($\theta = const$), the total angle of twist

$$\boxed{\phi = \frac{TL}{GI_P}}$$

  The quantity $\frac{GI_P}{L}$ is also known as **torsional stiffness** of the bar.

- For a thin circular tube, $I_P = 2\pi r^3 t$

2.4 Non-uniform Torsion

For a bar with non-uniform cross-section, tension :

$$\boxed{\phi = \int_0^L \frac{T(x)dx}{GI_P(x)}}$$

3 Hooke's Law

3.1 Stresses in a Plane and Mohr's Circle

Using physical equations or using cauchy's stress tensor with rotation matrix, we get

$$\sigma_{x'} = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2} cos2\theta + \tau_{xy} sin2\theta \tag{1}$$

$$\tau_{xy'} = -\frac{\sigma_x - \sigma_y}{2} sin2\theta + \tau_{xy} cos2\theta \tag{2}$$

$$\sigma_{y'} = \frac{\sigma_x + \sigma_y}{2} - \frac{\sigma_x - \sigma_y}{2} cos2\theta - \tau_{xy} sin2\theta \tag{3}$$

The above equations can be viewed as parametric equations to a circle.
On squaring and adding, we get Mohr's circle

$$\left(\sigma_{x'} - \frac{\sigma_x + \sigma_y}{2}\right)^2 + \tau_{xy'}^2 = \left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2$$

Center $= (\frac{\sigma_x+\sigma_y}{2}, 0)$ and Radius $= \sqrt{\left(\frac{\sigma_x-\sigma_y}{2}\right)^2 + \tau_{xy}^2}$

- This is plotted by taking $\sigma_{x'}$ on x axis and $\tau xy'$ on **negative** y-axis and angle 2θ anticlockwise as positive.

- τxy is taken as positive if it tends to rotate in the anticlockwise direction, negative otherwise.
- σ_x is taken as positive if it is tensile and negative for compressive.
- A rotation of angle θ in the plane corresponds to a rotation of 2θ on the Mohr's circle.

3.2 Hooke's law for plane stresses

For a point (isotropic material) under planar stresses ($\sigma_{xz} = \sigma_{yz} = \sigma_{zz} = 0$), we have

$$\epsilon_x = \frac{1}{E}(\sigma_x - \nu\sigma_y)$$
$$\epsilon_y = \frac{1}{E}(\sigma_y - \nu\sigma_x)$$
$$\gamma_{xy} = \frac{\tau_{xy}}{G}$$
$$\sigma_x = \frac{E}{1-\nu^2}(1 + \nu\epsilon_x)$$
$$\sigma_y = \frac{E}{1-\nu^2}(1 + \nu\epsilon_y)$$
$$\tau_{xy} = G\gamma_{xy}$$

3.3 Volume change and Strain-energy density

Let a cuboid of sides a,b and c be under stresses.

$$V_0 = abc$$

$$\begin{aligned} V &= (a + a\epsilon_x)(b + b\epsilon_y)(c + c\epsilon_z) \\ &= abc(1 + \epsilon_x)(1 + \epsilon_y)(1 + \epsilon_z) \\ &= V_0(1 + \epsilon_x + \epsilon_y + \epsilon_z) && \text{ignoring the } \epsilon_x\epsilon_y \text{ terms} \end{aligned}$$

Unit volume change e, also known as **dialatation**

$$e = \frac{\Delta V}{V_0} = (1 + \epsilon_x)(1 + \epsilon_y)(1 + \epsilon_z)$$

Strain-energy density in plane stress,

$$u = \frac{1}{2}(\sigma_x\epsilon_x + \sigma_y\sigma_y + \tau_{xy}\gamma_{xy})$$
$$u = \frac{1}{2E}(\sigma_x^2 + \sigma_y^2 - 2\nu\sigma_x\sigma_y + \frac{\tau_{xy}^2}{2G})$$
$$u = \frac{E}{2(1-\nu^2)}(\epsilon_x^2 + \epsilon_y^2 + 2\nu\epsilon_x\epsilon_y) + \frac{G\gamma_{xy}^2}{2}$$

3.4 Hooke's law for triaxial stresses

$(\sigma_z = \tau_{xz} = \tau_{yz} = 0)$

$$\epsilon_x = \frac{1}{E}(\sigma_x - \nu\sigma_y - \nu\sigma_z)$$
$$\sigma_x = \frac{E}{(1+\nu)(1-2\nu)}[(1-\nu)\epsilon_x + \nu(\epsilon_y + \epsilon_z)]$$

3.5 Strain Energy in Torsion

$$d\phi = \frac{Tdx}{GJ}$$

$$\begin{aligned} \text{Energy stored due to torsion, } U &= \int_0^L \frac{1}{2} T d\phi \\ &= \int_0^L \frac{T^2 dx}{2GJ} \end{aligned}$$

4 Torsion formula

4.1 Torsion Formula for non-prismatic bars

- For elliptical cross section, maximum shear stress

$$\tau_{max} = \frac{2T}{\pi ab^2}$$
$$\phi = \frac{TL}{GJ_e}$$
$$J_e = \frac{\pi a^3 b^3}{a^2 + b^2}$$

- For triangular cross section

$$\tau_{max} = \frac{T\frac{h}{2}}{J_t}$$
$$\phi = \frac{TL}{GJ_t}$$
$$J_e = \frac{h_t}{15\sqrt{3}}$$

- For rectangular cross section

$$\tau_{max} = \frac{T}{k_1 bt^2}$$
$$\phi = \frac{TL}{(k_2 bt^3)G} = \frac{TL}{J_r G}$$
$$J_r = k_2 bt^3$$

  k_1 and k_2 are emphirically determined and are dependent on $\frac{b}{t}$

- Thin walled open cross sections We treat flange and web as seperate rectangles.

$$J = J_w + 2J_f$$

$$J_f = k_1 b_f t_f^3$$
$$J_w = k_1(b_w - 2t_f)t_w^3$$
$$\tau_m ax = \frac{2T(\frac{t}{2})}{J}$$
$$\phi = \frac{TL}{GJ}$$

4.2 Thin walled tubes

Consider a small element of length of dx, then shear stresses are τ_a,τ_b, τ_c and τ_d. Then the shear stresses on opposite wall should be equal (to satisfy Newton's second law). Hence,

$$\begin{aligned} F_b &= F_c \\ \tau_b t_b dx &= \tau_c t_c dx \\ \tau_b t_b &= \tau_c t_c \end{aligned}$$

Shear flow f,

$$\boxed{f = \tau t = const}$$

Deriving torsion formula for the thin walled tubes-

$$\begin{aligned} T &= \int_0^{L_m} \tau dA\rho \\ &= \int_0^{L_m} frds \\ &= 2fA_m \qquad\qquad \text{where } A_m \text{ is the area enclosed by the median line.} \end{aligned}$$

$$\boxed{\tau = \frac{T}{2tA_m}}$$

Using shear stress, we get J -
Shear energy density of a solid under pure shear stress is $\frac{\tau^2}{2G}$.

$$\begin{aligned} \text{Total shear energy } U &= \int\int \frac{\tau^2}{2G} t dx ds \\ &= \int\int \frac{f^2}{2Gt} dx ds \\ &= \frac{f^2}{2G} \int_0^L dx \int_0^{L_m} ds \\ &= \frac{T^2 L}{2GA_m^2} \int_0^{L_m} ds \end{aligned}$$

Also, $U = \frac{T^2 L}{2GJ}$. Hence,

$$\boxed{J = \frac{4A_m^2}{\int_0^{L_m} \frac{dS}{t}}}$$

For tube with **constant thickness**,

$$\boxed{J = \frac{4A_m^2 t}{L_m}}$$

For circular tube,

$$J = 2\pi r^3 t$$

For rectangular tube,

$$J = \frac{2b^2h^2t_1t_2}{bt_1 + ht_2}$$

Angle of twist,

$$\begin{aligned} \phi &= \frac{TL}{GJ} \\ &= \frac{TL}{4GA_m^2} \int_0^{L_m} \frac{ds}{t} \end{aligned}$$

$$\boxed{\phi = \frac{TL}{4GA_m^2} \int_0^{L_m} \frac{ds}{t}}$$