

MA 214 - Introduction to Numerical Analysis

Vishal Neeli

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1 Interpolation Theory

1.1 Introduction

- Given finite set of points, reconstructing the original curve is interpolation.
- There will be obviously infinitely many curve.

- Interpolation problem

Given $n+1$ real distinct points: x_0, x_1, \dots, x_n and real numbers: y_0, y_1, \dots, y_n

Find a function $f : \mathbb{R} \rightarrow \mathbb{R}$ such that

$$f(x_i) = y_i \quad \text{for } i = 0, 1, \dots, n$$

Such a function is called **interpolant** and points x_i are called **interpolation points**.

We attempt to rebuild original function using polynomial functions. This is called polynomial interpolation and function is polynomial interpolant.

- A polynomial is function of the form

$$p(x) = a_0 + a_1x + \dots + a_nx^n$$

- \mathbb{P}_n is the set of polynomials consisting of all polynomials of degree $\leq n$

1.2 Polynomial Interpolation

Theorem (Joseph-Louis Lagrange Theorem). *Given $n + 1$ data points with unique x_i s, then there exists a unique polynomial $p_n \in \mathbb{P}_n$ such that*

$$p(x_i) = y_i \quad \text{for } i = 0, 1, \dots, n$$

Proof. (1) This can be shown by linear algebra. In a n degree polynomial, we substitute the points and get $n + 1$ equations in $n+1$ variables (coeff) and all the rows are unique (since x_0, x_1, \dots, x_n are unique), hence in $AX = b$, $|A| \neq 0$. □

Proof. (2) Part 1: Uniqueness : If there is an interpolant, then the interpolant is unique
Let there be 2 interpolants, p_n and q_n and let $r(x) = p(x) - q(x)$,

$$r(x) = 0 \quad \text{for } i = 0, 1, \dots, n$$

This contradicts the fundamental theorem of Algebra. (A polynomial of degree n can have at most n real roots). Therefore

$$r(x) = 0 \quad \forall x \in \mathbb{R}$$

$$p(x) = q(x) \quad \forall x \in \mathbb{R}$$

Part 2: Existence (construction):

Given $n+1$ data points, build $n+1$ Lagrange polynomials

$$L_k^n(x_i) = \begin{cases} 0 & \text{for } i \neq k \\ 1 & \text{for } i = k \end{cases}$$

$$L_k^n(x) = \frac{(x - x_0) \dots (x - x_{k-1})(x - x_{k+1}) \dots (x - x_n)}{(x_k - x_0) \dots (x_k - x_{k-1})(x_k - x_{k+1}) \dots (x_k - x_n)}$$

$$p(x) = \sum_{k=0}^n y_k L_k^n(x)$$

□

1.3 Closeness between functions

Given two continuous functions $f, g : [a, b] \rightarrow \mathbb{R}$, to evaluate how close the functions are consider the following

$$\max_{x \in [a, b]} |f(x) - g(x)|$$

1.4 Set of continuous Functions

$C[a, b]$ is the set of all continuous functions on $[a, b]$

$C[a, b]$ is a infinite dimensional vector space

$$f, g \in C \implies f + g \in C \text{ and } \lambda f \in C$$

We define norm on $C[a, b]$ as

$$\|f\| = \max_{x \in [a, b]} |f(x)|$$

$C^k[a, b]$ denotes the set of all functions which are continuously k-times differentiable

1.5 Polynomial Approximation and Error

Theorem (Weierstrass approximation Theorem). *Given a function $f \in C[a, b]$ and given $\epsilon > 0$, there exists a polynomial $p(x)$ such that,*

$$\|f(x) - p\| < \epsilon$$

Using Langrange's recipe to approximate

Take $n + 1$ interpolation points in the $[a, b]$ and collect the function values at all the points. We have $n + 1$ data points. Using Lagrange polynomials, find the interpolant

Theorem (Error equation). *Let $f \in C^k[a, b]$, $x_0, x_1, \dots, x_n \in [a, b]$ and $p \in \mathbb{P}_n$ be the interpolant using these points, then for all x , there exists a $\zeta = \zeta(x) \in (a, b)$ such that*

$$f(x) - p(x) = \frac{1}{(n+1)!} f^{(n+1)}(\zeta) \prod_{k=0}^n (x - x_k)$$

Note: Here ζ is dependent on the x , i.e, for every x you choose, ζ generally changes.

Proof. Consider the function,

$$\psi(t) = (f(t) - p(t)) \prod_{k=0}^n (t - x_k) - (f(x) - p(x)) \prod_{k=0}^n (x - x_k)$$

This $n + 2$ roots ($n+1$ data points and x), applying rolle theorem's gives us that $\psi^{(1)}(t)$ has at least $n+1$ roots. Applying like this repeatedly on its derivatives, we get that $\psi^{(n+1)}$ has at least 1 root in $[a, b]$. Assuming the root to be ζ . We have,

$$f(x) - p(x) = \frac{1}{(n+1)!} f^{(n+1)}(\zeta) \prod_{k=0}^n (x - x_k)$$

□

Approximating the error:

Taking norm on both the sides of error equation, we have,

$$\max_{x \in [a, b]} |f(x) - p(x)| = \frac{1}{(n+1)!} \|f^{(n+1)}(\zeta)\| \prod_{k=0}^n (x - x_k) \quad (1)$$

$$\max_{x \in [a, b]} |f(x) - p(x)| \leq \frac{1}{(n+1)!} \|f^{(n+1)}\| \max_{x \in [a, b]} \prod_{k=0}^n (x - x_k) \quad (2)$$

Chebyshev interpolation points:

$$x_k = \frac{a+b}{2} + \frac{b-a}{2} \cos\left(\frac{j\pi}{n}\right)$$

These points minimise $\max_{x \in [a, b]} \prod_{k=0}^n (x - x_k)$ and therefore preferred over equally spaced points on real line. These points can be visualised as projections of equally spaced points on the arc of the semicircle with $\frac{a+b}{2}$ as center and $\frac{b-a}{2}$ as radius.

1.6 Another method for calculating interpolant

This is similar to the linear algebra method (given as proof(1) to Joseph-Louis Lagrange Theorem) for finding the interpolant.

Consider the polynomial

$$p(x) = a_0 + a_1(x - x_0) + a_2(x - x_0)(x - x_1) + \dots + a_n(x - x_0)\dots(x - x_{n-1})$$

Find the coefficients a_0, a_1, \dots, a_n by substituting the data points. On substituting x_0 , we get a_0 , again on substituting x_1 and using a_0 , we get a_1 and so on.

1.7 Divided difference - recursion relation

- **Divided difference** : It is the coefficient of x_n in the interpolant $p \in \mathbb{P}_n$ and denoted by $f[x_0, x_1, \dots, x_n]$.

Using Lagrange polynomials, we have

$$p(x) = \sum_{k=0}^n f(x_k) \prod_{j=0, j \neq k}^n \frac{x - x_j}{x_k - x_j}$$

So the divided difference is

$$f[x_0, x_1, \dots, x_n] = \sum_{k=0}^n f(x_k) \prod_{j=0, j \neq k}^n \frac{1}{x_k - x_j}$$

- interpolant $p(x)$ of x_0, x_1, \dots, x_n can be calculated using interpolants $a(x)$ and $b(x)$ of x_1, x_2, \dots, x_n and x_0, x_1, \dots, x_{n-1} respectively as

$$p(x) = \frac{(x - x_0)a(x) - (x - x_n)b(x)}{x_n - x_0}$$

Theorem (Divided difference recursion theorem).

$$f[x_0, x_1, \dots, x_{m+1}] = \frac{f[x_1, x_2, \dots, x_{m+1}] - f[x_0, x_1, \dots, x_m]}{x_{m+1} - x_0}$$

Proof. Let $p(x)$ be the interpolant for x_0, x_1, \dots, x_m and $q(x)$ be the interpolant for x_1, x_2, \dots, x_{m+1} . Then,

$$L(x) = \frac{(x - x_0)q(x) + (x_{m+1} - x)p(x)}{x_{m+1} - x_0}$$

is an interpolant. Since, interpolant is unique, considering coeff of x_m we have,

$$f[x_0, x_1, \dots, x_m] = \frac{f[x_1, x_2, \dots, x_{m+1}] - f[x_0, x_1, \dots, x_m]}{x_{m+1} - x_0}$$

□

Theorem (Interpolant using divided differences). Suppose x_0, x_1, \dots, x_n be the data points. Then interpolant $p \in \mathbb{P}_n$ is

$$p(x) = f[x_0] + f[x_0, x_1](x - x_0) + \dots + f[x_0, x_1, \dots, x_n] \prod_{j=0}^{n-1} (x - x_j)$$

Proof. We prove this by induction. Base case $n = 0$ is trivially satisfied. Assume that this is satisfied for p_k ,

$$p_k(x) = f[x_0] + f[x_0, x_1](x - x_0) + \dots + f[x_0, x_1, \dots, x_k] \prod_{j=0}^{k-1} (x - x_j)$$

Consider the polynomial $p_{k+1}(x) - p_k(x) \in \mathbb{P}_{k+1}$ which has x_0, x_1, \dots, x_k as roots. Hence,

$$p_{k+1}(x) - p_k(x) = c \prod_{j=0}^k (x - x_j)$$

Comparing leading coefficient on both sides, we have $c = f[x_0, x_1, \dots, x_k]$. Hence,

$$p_{k+1}(x) = f[x_0] + f[x_0, x_1](x - x_0) + \dots + f[x_0, x_1, \dots, x_{k+1}] \prod_{j=0}^k (x - x_j)$$

By PMI,

$$p(x) = f[x_0] + f[x_0, x_1](x - x_0) + \dots + f[x_0, x_1, \dots, x_n] \prod_{j=0}^{n-1} (x - x_j)$$

□

1.8 Time complexity of the algorithms

(Slide is useless)

- Lagrange's method - $O(n^2)$: Computing each Lagrange polynomial can be done in $O(n)$ (Finding the coefficients given roots can be done in $O(n \log^2(n))$). We need to do this n times. So, $O(n^2)$.
- Divided differences - $O(n^2)$: Summing operations in each stage - $n + (n-1) + (n-2) + \dots + 1$. Hence, $O(n^2)$.
- Divided difference can be considered better because we can extend from n to $n+1$ and so on without discarding previous computation.

1.9 Weierstrass theorem consequences

- For Weierstrass theorem, take $\epsilon_n = \frac{1}{n}$. Then Weierstrass theorem proves the existence of sequence of polynomials $p^{(1)}, p^{(2)}, \dots$ such that

$$\lim_{n \rightarrow \infty} \|f - p^{(n)}\| = 0$$

- If f is not a polynomial, then

$$\lim_{n \rightarrow \infty} \text{degree of } p(n) = \infty$$

To prove (crude) this, assume that there exist a non-polynomial function f such that $\|f - p^{(n)}\| = 0$, then $f = p^{(n)}$ which contradicts the assumption that f is not a polynomial.

1.10 Spline Interpolation

- **Piece wise polynomial:** $\phi \in C[a, b]$ is a piecewise polynomial function, if there exists $a = x_0 < x_1 < \dots < x_n = b$ such that $\phi \in \mathbb{P}_m$ when $x \in [x_i, x_{i+1}]$ for all $i = 0, 1, \dots, n$ and some $m > 0$.
- Piece wise polynomial ϕ need not be polynomial in whole domain.
- Splines interpolation for $f \in C[a, b]$
 - Pick some data points x_0, x_1, \dots, x_n such that $a = x_0 < x_1 < \dots < x_n = b$
 - Fix $m \leq n$
 - Build ϕ in each subinterval $[x_i, x_{i+1}]$ using the following conditions:

$$\phi(x_i) = f_i \quad \text{for } i = 0, 1, \dots, n$$

$$\lim_{h \rightarrow 0+} \phi(x_i - h) = \lim_{h \rightarrow 0+} \phi(x_i + h) \quad \text{for } i = 1, 2, \dots, n-1$$

$$\lim_{h \rightarrow 0+} \frac{d\phi(x_i - h)}{dx} = \lim_{h \rightarrow 0+} \frac{d\phi(x_i + h)}{dx} \quad \text{for } i = 1, 2, \dots, n-1$$

$$\lim_{h \rightarrow 0+} \frac{d^2\phi(x_i - h)}{dx^2} = \lim_{h \rightarrow 0+} \frac{d^2\phi(x_i + h)}{dx^2} \quad \text{for } i = 1, 2, \dots, n-1$$

\vdots

$$\lim_{h \rightarrow 0+} \frac{d^{m-1}\phi(x_i - h)}{dx^{m-1}} = \lim_{h \rightarrow 0+} \frac{d^{m-1}\phi(x_i + h)}{dx^{m-1}} \quad \text{for } i = 1, 2, \dots, n-1$$

- We have $(n+1) + m(n-1) = n(m+1) - (m-1)$ conditions. We need $m-1$ more conditions.

1.11 Linear Splines

Constructing linear splines:

Since the degree is only 1, we can construct the splines using the equation of straight lines between the knots (data points).

$$\begin{aligned}
s_0 &= \frac{f_1 - f_0}{x_1 - x_0}x + \frac{x_1 f_0 - x_0 f_1}{x_1 - x_0} & \text{for } x \in [x_0, x_1] \\
s_1 &= \frac{f_2 - f_1}{x_2 - x_1}x + \frac{x_2 f_1 - x_1 f_2}{x_2 - x_1} & \text{for } x \in [x_1, x_2] \\
&\vdots \\
s_{n-1} &= \frac{f_n - f_{n-1}}{x_n - x_{n-1}}x + \frac{x_n f_{n-1} - x_{n-1} f_n}{x_n - x_{n-1}} & \text{for } x \in [x_{n-1}, x_n]
\end{aligned}$$

Theorem (Linear Splines error). *Let $f \in C^2[a, b]$ and $s_L(x)$ be the interpolating **linear spline** at $(n+1)$ knots $a = x_0 < x_1 < \dots < x_n = b$ and let h be the maximum subinterval length, then*

$$\|f - s_L\| \leq \frac{h^2}{8} \|f''\|$$

Proof. Consider the interval $[x_i, x_{i+1}]$, $\phi(x)$ is the interpolating polynomial in this interval, using the error equation for interpolating polynomials,

$$f(x) - s_L(x) = \frac{1}{2} f''(\zeta)(x - x_i)(x - x_{i+1})$$

Taking absolute value on both the sides,

$$\begin{aligned}
|f(x) - s_L(x)| &= \frac{1}{2} |f''(\zeta)| |(x - x_i)(x - x_{i+1})| \\
&\leq \frac{1}{2} \|f''\| \frac{h_i^2}{4} \quad \text{where } h_i = \frac{x_{i+1} - x_i}{2} \\
&\leq \frac{h_i^2}{8} \|f''\|
\end{aligned}$$

Considering $h = \max(h_i)$, then for $x \in [a, b]$

$$\max_{x \in [a, b]} |f(x) - s_L(x)| \leq \frac{1}{8} h^2 \|f''\|$$

□

1.12 Cubic Splines

Constructing cubic splines:

$$s_i(x) = a_i x^3 + b_i x^2 + c_i x + d_i$$

Using the conditions given in the section 1.10, we have $4n-2$ equations to for $4n$ coefficients. We choose the other two conditions as

$$s_0''(x_0) = s_{n-1}''(x_n) = 0$$

We have $4n$ variables (coefficients) and $4n$ equations, i.e, we have a $4n \times 4n$ matrix which can be solved to get the coefficients of the spline.

We can simplify this by choosing the form of spline as

$$s_i(x) = a_i(x - x_i)^3 + b_i(x - x_i)^2 + c_i(x - x_i) + d_i \quad \text{for } i = 0, 1, \dots, n-1$$

We will work only with **equally spaced knots**, i.e, $x_{i+1} - x_i = \text{const}$ for $i = 0, 1, \dots, n-1$. We also define

$$\sigma_i = s''(x_i) \quad \text{for } i = 0, 1, \dots, n$$

After doing a lot of simplification, we get

$$\begin{aligned}
a_i &= \frac{\sigma_{i+1} - \sigma_i}{6h} \\
b_i &= \frac{\sigma_i}{2} \\
c_i &= \frac{f_{i+1} - f_i}{h} - \frac{h}{6}(2\sigma_i + \sigma_{i+1}) \\
d_i &= f_i
\end{aligned}$$

σ_i s can be obtained by solving these equations

$$\sigma_{i-1} + 4\sigma_i + \sigma_{i+1} = \frac{6}{h^2}(f_{i-1} - 2f_i + f_{i+1})$$

This can be put in matrix form as

$$\begin{bmatrix} 4 & 1 & 0 & \dots & 0 & 0 & 0 \\ 1 & 4 & 1 & \dots & 0 & 0 & 0 \\ 0 & 1 & 4 & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 4 & 1 & 0 \\ 0 & 0 & 0 & \dots & 1 & 4 & 1 \\ 0 & 0 & 0 & \dots & 0 & 1 & 4 \end{bmatrix} \begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \vdots \\ \sigma_{n-3} \\ \sigma_{n-2} \\ \sigma_{n-1} \end{bmatrix} = \frac{6}{h^2} \begin{bmatrix} f_0 - 2f_1 + f_2 \\ f_1 - 2f_2 + f_3 \\ f_2 - 2f_3 + f_4 \\ \vdots \\ f_{n-4} - 2f_{n-3} + f_{n-2} \\ f_{n-3} - 2f_{n-2} + f_{n-1} \\ f_{n-2} - 2f_{n-1} + f_n \end{bmatrix}$$

Theorem (Error - Cubic Splines (equispaced knots)). *Let $f \in C^4[a, b]$ and $s(x)$ be the interpolating **cubic spline** at $(n+1)$ **equispaced knots** $a = x_0 < x_1 < \dots < x_n = b$ and let h be the subinterval length ($h = x_{i+1} - x_i$), then*

$$\|f - s\| \leq C \|f^{(4)}\| h^4 \quad \text{for some } C > 0$$

Proof. Consider the function g which is defined as $g = f - s_i$ on each subinterval $[x_i, x_{i+1}]$. Then $g(x_i) = 0$ for $i = 0, 1, \dots, n-1$.

We can see that the zero polynomial is the linear spline of $g(x)$ and using the theorem [Linear Splines error](#), we have

$$\begin{aligned} \|g - 0\| &\leq \frac{h^2}{8} \|g''\| \\ \|f - s\| &\leq \frac{h^2}{8} \|f'' - s''\| \end{aligned}$$

Assuming that $\|f'' - s''\| \leq Ch^2 \|f^{(4)}\|$, we have

$$\|f - s\| \leq Ch^4 \|f^{(4)}\| \quad \text{for some } C > 0$$

□