ME 202 - Strength of Materials

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1 Review

1.1 Strain Energy

$$u = \frac{1}{2}E\epsilon^2 = \frac{1}{2}\sigma\epsilon = \frac{1}{2}\frac{\sigma^2}{E}$$

1.2 Poison's Ratio

$\epsilon_{axial} = \frac{\delta}{L}$, δ is the change in longitudinal length
$\epsilon_{radial} = \frac{\delta'}{r}$, δ' is the change in radius

Poisson's ratio -

$$\nu = -\frac{\epsilon_{radial}}{\epsilon_{axial}}$$
$$0 \leq \nu \leq 0.5$$

Shear stress and strain

$$\tau = G\gamma$$

Relation between G, E and ν -

$$G = \frac{E}{2(1+\nu)}$$

For a axial load,

$$d\delta = \frac{N(x)dx}{A(x)E(x)}$$

2 Torsion

2.1 Uniform Torsion

- **Angle of twist** : It is the angle by which one end of the rod is displaced wrt the other end of the rod under the effect of some torsion (twisting).
- Consider a small cylidrical section which has length dx, one end has been displaced wrt another by an angle of $d\phi$. Consider a line ab on the circumference along the length of the cylinder which has be changed to ab'. Then shear strain

$$\gamma_{max} = \frac{bb'}{ab} \qquad \text{(bb' can be assumed to be a straight line)}$$

$$\boxed{\gamma_{max} = \frac{rd\phi}{dx}}$$

- Rate of twist or angle of twist per unit length

$$\theta = \frac{\mathrm{d}\phi}{\mathrm{d}x}$$
$$\gamma_{max} = r\theta$$

If θ is constant, then

$$\gamma_{max} = \frac{r\phi}{L}$$

This is called as γ_{max} because we are measuring the shear strain at the outer end i.e, with maximum radius and hence, maximum shear strain.

$$\boxed{\gamma = \rho\theta = \rho\frac{\mathrm{d}\phi}{\mathrm{d}x}}$$

where ρ is the perpendicular distance of the point from the axis (radius) we are considering.

$$\boxed{\gamma = \frac{\rho}{r}\gamma_{max}}$$

2.2 Hooke's Law

Hooke' law for shear stress and shear strain

$$\tau = G\gamma = G\frac{\rho}{r}\gamma_{max} = \frac{\rho}{r}\tau_{max}$$

$$\tau_{max} = G\gamma_{max}$$

2.3 Torsion Formula

- Polar moment of inertia (this is integral over area - double integral)-

$$\boxed{I_P = \int_A \rho^2 dA}$$

  where ρ is the distance at which area element dA is located.

- For a circle of radius r,

$$I_P = \int_A \rho^2 dA = \int_{\theta=0}^{2\pi}\int_{\rho=0}^{r} \rho^2(\rho d\theta d\rho) = \frac{\pi r^4}{2}$$

$$I_{Pcircle} = \frac{\pi r^4}{2}$$

- Consider a cross-section of any shape, we are trying to sum all the small torques and equate it to the torque applied on this

$$\begin{aligned} T = \int_A dM &= \int_A \rho\tau dA \\ &= \int_A \frac{\rho^2}{r}\tau_{max} dA \\ &= \frac{\tau_{max}}{r}\int_A \rho^2 dA \\ T &= \frac{\tau_{max}}{r} I_P \end{aligned}$$

  Or,

$$\tau_{max} = \frac{Tr}{I_P}$$

  item The shear stress at distance ρ from the center of the bar with polar moment of inertia I_P is

$$\boxed{\tau = \frac{T\rho}{I_P}}$$

  For a rod of circular cross section of radius r,

$$\boxed{\tau_{max} = \frac{2T}{\pi r^3} = \frac{16T}{\pi d^3}}$$

- Rate of twist

$$\boxed{\theta = \frac{T}{GI_P}}$$

  Hence, GI_P is also known as **Torsional Rigidity**

- For a bar in pure torsion ($\theta = const$), the total angle of twist

$$\boxed{\phi = \frac{TL}{GI_P}}$$

  The quantity $\frac{GI_P}{L}$ is also known as **torsional stiffness** of the bar.

- For a thin circular tube, $I_P = 2\pi r^3 t$

2.4 Non-uniform Torsion

For a bar with non-uniform cross-section, tension :

$$\boxed{\phi = \int_0^L \frac{T(x)dx}{GI_P(x)}}$$

2.5 Stresses in a Plane and Mohr's Circle

Using physical equations or using cauchy's stress tensor with rotation matrix, we get

$$\sigma_{x'} = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2} cos2\theta + \tau_{xy} sin2\theta \tag{1}$$

$$\tau_{xy'} = -\frac{\sigma_x - \sigma_y}{2} sin2\theta + \tau_{xy} cos2\theta \tag{2}$$

$$\sigma_{y'} = \frac{\sigma_x + \sigma_y}{2} - \frac{\sigma_x - \sigma_y}{2} cos2\theta - \tau_{xy} sin2\theta \tag{3}$$

The above equations can be viewed as parametric equations to a circle.
On squaring and adding, we get Mohr's circle

$$\left(\sigma_{x'} - \frac{\sigma_x + \sigma_y}{2}\right)^2 + \tau_{xy'}^2 = \left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2$$

Center = $(\frac{\sigma_x+\sigma_y}{2}, 0)$ and Radius = $\sqrt{\left(\frac{\sigma_x-\sigma_y}{2}\right)^2 + \tau_{xy}^2}$

- This is plotted by taking $\sigma_{x'}$ on x axis and $\tau xy'$ on **negative** y-axis and angle 2θ anticlockwise as positive.

- τxy is taken as positive if it tends to rotate in the anticlockwise direction, negative otherwise.
- σ_x is taken as positive if it is tensile and negative for compressive.
- A rotation of angle θ in the plane corresponds to a rotation of 2θ on the Mohr's circle.