Parul University

Faculty of Engineering & Technology

Department of Applied Sciences and Humanities

1st Year B.Tech Programme

Mathematics – II (203191152)

Unit 3: Laplace Transform:(Lecture Notes)

Overview:

This unit is a mathematical section to establish a base for the theory of control systems. This is a tool and it is indispensable as most of linear systems dynamics are described in a mapped space that can only be understood when the main theorems of the Laplace transform are known. The module contains only the essential results, which are explained by several examples from the area of differential equations and their solutions. Some additional mathematical details can be found in the mathematical appendix module. The correspondences of the Laplace transform are given in tabular form to be simply used for the forward and back transformation. Special focus is put on the solution of differential equations using the Laplace transform and on special signals, e.g. impulse or step.

Objectives:

When you have completed this unit you should be able to:

1. Apply the Laplace transform to differential equations.
2. Solve linear differential equations.
3. Apply the main theorems of the Laplace transform.
4. Know how useful this techniques is to handle dynamical systems.

Prerequisites:

Mathematics: integrals, differential equations, complex numbers, rational and analytical functions.

Application:

- ➢ Solve vibrating spring problems with or without damping/external forces using standard methods and Laplace transform methods.
- ➢ Solve electric circuit problems with or without charge/inductance/resistance/capacitance/electromotive force, using standard methods and Laplace transform methods.
- ➢ Solve systems of differential equations by Laplace transforms.
- ➢ Solve electric networks using standard methods and Laplace transform methods.

Outline

- Laplace Transform,
- Linearity of Laplace Transform,
- Laplace transform of elementary function,
- Inverse Transform,
- First Shifting Theorem,
- Differentiation and Integration of Transform,
- Transform of Derivatives and Integrals,
- Evaluation of integrals by Laplace transform,
- Unit Step Function,
- Second Shifting Theorem,
- Dirac's Delta Function,
- Convolution theorem,
- Partial Fraction,
- Solution of Ordinary Differential Equations.

Weightage: 25%

Teaching Hrs. : 12 Hours

Reference Book:

1. Advanced Engineering Mathematics, Erwin Kreyszig, Willey India Edition.
2. Integral Transforms for Engineers, Larry S. Andrews,Bhimsen K. Shivamoggi
3. Integral Transforms and their applications, Lokenath Debnath, Dambaru Bhatta, Chapman and Hall/CRC (Unit 4)

Laplace Transform:

Let $f(t)$ be a function of $t \geq 0$, then the Laplace Transformation of $f(t)$ is defined as

$$L\{f(t)\} = \int_0^\infty f(t)e^{-St}dt = F(s)$$

provided that integral exists. s is a parameter which may be real or complex number.

Laplace transform of elementary function:

***Ex*: Find the Laplace transform of 1, where $s > 0$.**

***Sol*. We know that,**

$$L\{f(t)\} = \int_0^\infty f(t)e^{-St}dt = F(s)$$

$$\Rightarrow L\{1\} = \int_0^\infty 1.e^{-St}dt = \left[\frac{e^{-St}}{-s}\right]_0^\infty = \left[\frac{e^{-S(\infty)}}{-s} - \frac{e^{-S(0)}}{-s}\right] = \left[0 - \frac{1}{-s}\right]$$

$$= \frac{1}{s} \quad (\because \quad e^{-\infty} = 0)$$

$$\therefore \quad L\{1\} = \frac{1}{s}$$

***Ex*: Find the Laplace transform of e^{-at}, where $s > -a$.**

***Sol*. We know that,**

$$L\{f(t)\} = \int_0^\infty f(t)e^{-St}dt = F(s)$$

$$\Rightarrow L\{e^{-at}\} = \int_0^\infty e^{-at}.e^{-St}dt = \left[\frac{e^{-(S+a)t}}{-(S+a)}\right]_0^\infty = \left[\frac{e^{-(S+a)(\infty)}}{-(S+a)} - \frac{e^{-(S+a)(0)}}{-(S+a)}\right]$$

$$= \left[0 - \frac{1}{-(S+a)}\right] \quad (\because \quad e^{-\infty} = 0) = \frac{1}{(S+a)}$$

$$\therefore \quad L\{ e^{-at}\} = \frac{1}{(S+a)}$$

Also

$$\therefore \quad L\{ e^{at}\} = \frac{1}{(S-a)}, s > a$$

***Ex*: Show that** $\mathrm{L}\{\mathrm{t}^{\mathrm{n}}\} = \frac{\Gamma_{\mathrm{n}+1}}{\mathrm{s}^{\mathrm{n}+1}}$ for n > -1

$= \frac{\mathrm{n}!}{\mathrm{s}^{\mathrm{n}+1}}$ for n is a positive Integer and $s > 0$

Sol. We know that,

$$L\{f(t)\} = \int_0^\infty f(t)e^{-St}dt$$

Here $f(t) = \mathrm{t}^{\mathrm{n}}$

$$\Rightarrow L\{\mathrm{t}^{\mathrm{n}}\} = \int_0^\infty \mathrm{t}^{\mathrm{n}}.e^{-St}dt$$

Putting $st = x \Rightarrow dt = \frac{1}{s}dx$

Also when $t = 0, x = 0$ and $t = \infty, x = \infty$

$$L\{\mathrm{t}^{\mathrm{n}}\} = \int_0^\infty \mathrm{t}^{\mathrm{n}}.e^{-St}dt = \int_0^\infty \left(\frac{x}{s}\right)^n.e^{-x}\frac{dx}{s} = \frac{1}{\mathrm{s}^{\mathrm{n}+1}}\int_0^\infty x^n.e^{-x}dx = \frac{1}{\mathrm{s}^{\mathrm{n}+1}}\int_0^\infty x^{(n+1)-1}.e^{-x}dx$$

We know that

$$\Gamma_n = \int_0^\infty x^{n-1}.e^{-x}dx$$

Therefore,

$$L\{t^n\} = \frac{\Gamma_{n+1}}{s^{n+1}}$$

Also, if n is a positive integer, then $\Gamma_{\mathrm{n}+1} = \mathrm{n}!$

So,

$$L\{t^n\} = \frac{n!}{s^{n+1}}$$

***Ex*: Show that** $\mathbf{L\{\sin at\} = \frac{a}{s^2+a^2}}$

Sol. Here $f(t) = \ sin\ at = \ \frac{e^{iat}-e^{-iat}}{2i}$

We know that,

$$L\{f(t)\} = \int_0^\infty f(t)e^{-St}dt$$

$$L\ (sin\ at) = L\ \left\{\frac{e^{iat} - e^{-iat}}{2i}\right\}$$

$$= \ \frac{1}{2i}[L(e^{iat}) - \ L(e^{-iat})]$$

$$= \frac{1}{2i}\left[\frac{1}{s - ia} - \frac{1}{s + ia}\right]$$

$$= \ \frac{1}{2i}\left[\frac{s + ia - s + ia}{(s - ia)(s + ia)}\right]$$

$$= \frac{1}{2}\left[\frac{2ia}{s^2 - i^2a^2}\right] = \frac{a}{s^2 + a^2}$$

$$\therefore L\ (sin\ hat) = \ \frac{a}{s^2 + a^2}$$

Exercise 1: Show that $L\ \{cos\ at\} = \ \frac{\boldsymbol{s}}{\boldsymbol{s^2+a^2}}$ (Hint: $cos\ at\ = \frac{e^{iat}+e^{-iat}}{2}$)

Exercise 2: Prove that $L\ \{sin\ hat\} = \ \frac{a}{s^2-a^2}$ (Hint: $sin\ hat\ = \ \frac{e^{at}-e^{-at}}{2}$)

Exercise 3: Find $L\{cos\ hat\}$ (Hint: $cos\ hat = \frac{e^{at}+e^{-at}}{2}$) **Ans**: $\frac{s}{s^2-a^2}$

Linearity of Laplace Transform:

If $L\{f(t)\} = \bar{f}(s)$ and $\{g(t)\} = \bar{g}(s)$, then for any constants a and b $L\{af(t) + b\ g(t)\} = a\ L\{f(t)\} + bL\{g(t)\} = a\bar{f}(s) + b\bar{g}(s)$.

Proof: We know that,

$$L\{f(t)\} = \int_0^\infty f(t)e^{-St}dt$$

$$L\{af(t) + b\ g(t)\} = \int_0^\infty \{af(t) + b\ g(t)\}e^{-St}dt = \int_0^\infty af(t)e^{-St}dt + \int_0^\infty b\ g(t)e^{-St}dt$$

$$= a\int_0^\infty f(t)e^{-St}dt + b\int_0^\infty g(t)e^{-St}dt = a\ L\{f(t)\} + bL\{g(t)\} = a\bar{f}(s) + b\bar{g}(s)$$

Ex: Find the Laplace transformation of $\boldsymbol{f(t) = t^2 + sin3t - 2e^{-t}}$.

Sol: Given that $f(t) = t^2 + sin3t - 2e^{-t}$

By using formula and linearity property,

$$L\{f(t)\} = L\{t^2 + sin3t - 2e^{-t}\} = L\{t^2\} + L\{sin3t\} - L\{2e^{-t}\} = \frac{2!}{s^3} + \frac{3}{s^2+3^2} - \frac{2}{s+1}$$

Ex: Evaluate $\boldsymbol{L\{cos^2 t\}}$ $= L\left(\frac{1+cos\ 2t}{2}\right)$

Sol: $L\{cos^2 t\} = L\left(\frac{1+cos\ 2t}{2}\right) = L\left(\frac{1}{2}\right) + \frac{1}{2}L\ (cos\ 2t) = \frac{1}{2S} + \frac{1}{2}\frac{S}{S^2+4}$

Ex: Find $\boldsymbol{L\{cos^3 2t\}}$

Sol: we know that $cos^3\theta = \frac{1}{4}(3\ cos\ \theta + cos\ 3\theta)$

$$L\{cos^3 2t\} = \frac{1}{4}L\ (3\ cos\ 2t + cos\ 6t) = \frac{1}{4}\left(3\frac{S}{S^2+4} + \frac{S}{S^2+36}\right)$$

Ex: Evaluate $L\{\boldsymbol{sin2tcos2t}\}$**.**

Sol: $L\{sin2tcos2t\} = L\left\{\frac{1}{2}(2sin2tcos2t)\right\} = \frac{1}{2}L\{sin4t\} = \frac{1}{2}\frac{4}{s^2+16} = \frac{2}{s^2+16}$

Ex: Find the Laplace transformation of $\boldsymbol{f(t)} = \begin{cases} \boldsymbol{-1} & \boldsymbol{0 < t \leq 4} \\ \boldsymbol{1} & \boldsymbol{t} > 4 \end{cases}$**.**

Sol: We have,

$$L\{f(t)\} = \int_0^\infty f(t)e^{-St}dt$$

Here $f(t) = \begin{cases} -1 & 0 < t \leq 4 \\ 1 & t > 4 \end{cases}$

$$L\{f(t)\} = \int_0^\infty f(t)e^{-St}dt = \int_0^4 (-1)e^{-St}dt + \int_4^\infty (1)e^{-St}dt = -\left[\frac{e^{-St}}{-s}\right]_0^4 + \left[\frac{e^{-St}}{-s}\right]_4^\infty$$

$$= -\left(\frac{e^{-4S}}{-s} - \frac{e^0}{-s}\right) + \left(\frac{e^{-\infty}}{-s} - \frac{e^{-4S}}{-s}\right) = 2\frac{e^{-4S}}{s} - \frac{1}{s} = \frac{2e^{-4S} - 1}{s}$$

Exercise: Find the Laplace transformation $\boldsymbol{f(t) = 2cos\,5t\,sin2t}$ **Ans:** $\frac{4s^2-84}{(s^2+49)(s^2+9)}$

Exercise: Evaluate $\boldsymbol{L\{f(t)\}}$ **where** $\boldsymbol{f(t)} = \begin{cases} \boldsymbol{sin\,at}, \boldsymbol{0} < t < \frac{\boldsymbol{\pi}}{\boldsymbol{a}} \\ \boldsymbol{0}, \quad \boldsymbol{t} > \frac{\boldsymbol{\pi}}{\boldsymbol{a}} \end{cases}$**.** **Ans:** $\frac{\boldsymbol{a\left(1+e^{-\frac{s\pi}{a}}\right)}}{\boldsymbol{s^2+a^2}}$

Exercise: Find $\boldsymbol{L\{cos^2 2t\}}$ **Ans:**$\frac{\boldsymbol{s^2+8}}{\boldsymbol{s\,(s^2+16)}}$

- **First Shifting Theorem:**

If $\mathrm{L\{f(t)\} = \bar{f}(s)}$, then $\mathrm{L\{e^{at}f(t)\} = \bar{f}(s-a) = [L\{f(t)\}]_{s \to s-a}}$

Proof: By, definition of L.T.

$$L\left(e^{at}f(t)\right) = \int_0^\infty e^{-st}e^{at}f(t)dt$$

$$= \int_0^\infty e^{-st+at}f(t)dt$$

$$= \int_0^\infty e^{-(s-a)t} f(t)dt$$

$$L\left(e^{at} f(t)\right) = F(s-a)$$

$$(\because \int_0^\infty e^{-st} f(t)dt = F(s))$$

Some important formula of First shifting theorem:

$L\{e^{at}.1\} = \frac{1}{(s-a)}$

$L\{e^{at}.t^n\} = \frac{\Gamma_{n+1}}{(s-a)^{n+1}} (n = \frac{p}{q}) = \frac{n!}{(s-a)^{n+1}} (n = \text{Integer})(n > -1)$

$L\{e^{at}.\sin bt\} = \dfrac{b}{(s-a)^2 + b^2}$

$L\{e^{at}.\cos bt\} = \dfrac{(s-a)}{(s-a)^2 + b^2}$

$L\{e^{at}.\sin h\, bt\} = \dfrac{b}{(s-a)^2 - b^2}$

$L\{e^{at}.\cos h\, bt\} = \dfrac{(s-a)}{(s-a)^2 - b^2}$

$L\{\sin h\, at\, f(t)\} = \dfrac{1}{2}\left[\bar{f}(s-a) - \bar{f}(s+a)\right]$

and

$L\{\cos h\, at\, f(t)\} = \dfrac{1}{2}[\bar{f}(s-a) + \bar{f}(s+a)]$

Ex: Evaluate $\boldsymbol{L\{e^{-3t}(\cos 4t + 3\sin\ 4t\}}$

Sol: $L\{e^{-3t}(\cos 4t + 3\sin\ 4t\} = L\ (e^{-3t}\cos\ \ 4t) + L\ (e^{-3t}\, 3\sin 4t)$

First, $L\ (e^{-3t}\cos\ \ 4t)$

$L\ (\cos 4t) = \dfrac{s}{s^2 + 16}$

$L\ (e^{-3t}\cos\ \ 4t) = \dfrac{s+3}{(s+3)^2 + 16}$

Second, $L\ (e^{-3t}\,3\sin 4t)$

$$L\ (\sin 4t) = \frac{4}{s^2+16}$$

$$L\ (e^{-3t}\cos\ \ 4t) = \frac{4}{(s+3)^2+16}$$

$$\therefore L\{e^{-3t}(\cos 4t + 3\sin\ 4t\} = \frac{s+3}{(s+3)^2+16} + \frac{3(4)}{(s+3)^2+16} = \frac{s+15}{(s+3)^2+16}$$

Ex: Find $\boldsymbol{L\ (e^{-3t} sin^2 t)}$

***Sol*:**

$$L\ (sin^2 t) = L\ \left\{\frac{(1-\cos 2t\ )}{2}\right\} = \frac{1}{2s} - \frac{s}{2\ (S^2+4)}$$

$$L\ (e^{-3t} sin^2 t) = \frac{1}{2}\left[\frac{1}{s+3} - \frac{s+3}{(s+3)^2+4}\right]$$

Exercise: Find $\boldsymbol{L\{e^{2t} sin^2 3t\}}$ Ans: $\frac{18}{(s-2)(s^2-4s+40)}$

Exercise: Evaluate $\boldsymbol{L\{(t+1)^3 e^t\}}$ Ans: $\frac{s\ (s^2-2s+7)}{(s-1)^4}$

Exercise: Find the Laplace transformation $\boldsymbol{g(t) = (t+1)^2 e^t}$

Ans: $\frac{2}{(s-1)^3} + \frac{2}{(s-1)^2} + \frac{1}{s-1}$

- **Differentiation and Integration of Transform:**

If $\mathrm{L\{f(t)\} = F(s)}$, then $\mathrm{L\{t^n f(t)\} = (-1)^n \frac{d^n}{ds^n}\{\bar{f}(s)\} = (-1)^n \frac{d^n}{ds^n}\{L\{f(t)\}\}}$.

Proof: Given $L\{f(t)\} = F(s) = \int_0^\infty e^{-st}\, f(t)dt$

Differentiating both sides w.r.t. s , we get

$$\frac{d}{ds}[F(s)] = \frac{d}{ds}\left[\int_0^\infty e^{-st}\, f(t)dt\right]$$

$$= \int_0^\infty \frac{\partial}{\partial s} e^{-st} f(t)dt$$

$$= \int_0^\infty (-t)e^{-st} f(t)dt$$

$$= -\int_0^\infty e^{-st} (t\, f(t))dt$$

$$L\{t\, f(t)\} = (-1)^n \frac{d}{ds}[F(s)]$$

Similarly, differentiating both the side, w.r.t.s (times), one can find

$$L\{t^n f(t)\} = (-1)^n \frac{d^n}{ds^n}[F(s)]$$

Ex: Find the Laplace Transform of $\boldsymbol{h(t) = t^2 \sin \pi\, t}$

Sol: We know that, $L\,(\sin \pi t) = \frac{\pi}{s^2+\pi^2}$

$$L\,(t^2 \sin \pi\, t) = (-1)^2 \frac{d^2}{ds^2}[F(s)]$$

$$= \frac{d^2}{ds^2}\left(\frac{\pi}{s^2+\pi^2}\right)$$

$$= \frac{d}{ds}\left(\frac{-2s\pi}{(s^2+\pi^2)^2}\right)$$

$$= -2\,\pi\, \frac{d}{ds}\left(\frac{s}{(s^2+\pi^2)^2}\right)$$

Using $\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\,u' - u\,v\prime}{v^2}$

$$\frac{d}{ds}\left(\frac{s}{(s^2+\pi^2)^2}\right) = \frac{(s^2+\pi^2)^2 - 2s(s^2+\pi^2)(2s)}{(s^2+\pi^2)^4}$$

$$= \frac{(s^2+\pi^2)\,[s^2+\pi^2-4s^2]}{(s^2+\pi^2)^4}$$

$$= \frac{\pi^2-3s^2}{(s^2+\pi^2)^3}$$

$$= -2\,\pi\,\left[\frac{\pi^2-3s^2}{(s^2+\pi^2)^3}\right]$$

$$= \frac{(3s^2-\pi^2)2\,\pi}{(s^2+\pi^2)^3}$$

Ex: Evaluate $\boldsymbol{L\{t\,e^{-2t}\sin t\}}$

Sol: We know that $L\,(\sin t) = \frac{1}{s^2+1}$

$$L\{t\,e^{-2t}\sin t\} = (-1)\frac{d}{ds}\left(\frac{1}{s^2+1}\right) = \frac{1}{(s^2+1)^2}(2s) = \frac{2s}{(s^2+1)^2}$$

$$\therefore L\{t\,e^{-2t}\sin t\} = \frac{2(s+2)}{((s^2+1)^2+1)^2}$$

Exercise: Find $\boldsymbol{L\,[t\,e^{-t}\cos t]}$ **Ans**: $\frac{s\,(s+2)}{(s^2+2s+2)^2}$

Exercise: Evaluate $\boldsymbol{L\,[t\sin\omega t]}$ **Ans**:$\frac{2\omega s}{(s^2+1)^2}$

Change of scale Property:

If $L\{f(t)\} = F(s)$ then $L\{f(at)\} = \frac{1}{a}F\left(\frac{s}{a}\right) = \frac{1}{a}[\,L\{f(t)\}]_{s\to\frac{s}{a}}$.

Division by $\boldsymbol{t}$ Property:

If $L\{f(t)\} = F(s)$ then $L\left\{\frac{1}{t}f(t)\right\} = \int_s^\infty F(s)\,ds = \int_s^\infty L\{f(t)\}\,ds$ provided the integral exists.

Proof: We have $L\{f(t)\} = F(s) = \int_0^\infty e^{-st}\,f(t)dt$

Integrating both side w.r.t. s to ∞,

$$\int_s^\infty F(s)\,ds = \int_s^\infty \left[\int_0^\infty e^{-st}\,f(t)dt\right]ds = \int_0^\infty \left[\int_s^\infty e^{-st}ds\right]f(t)dt$$
$$= \int_0^\infty \left[\frac{e^{-st}}{-t}\right]_s^\infty f(t)dt$$

$$= \int_0^\infty \frac{e^{-st}}{-t}f(t)dt = L\left\{\frac{f\,(t)}{t}\right\}$$

$$\therefore L\left\{\frac{f\,(t)}{t}\right\} = \int_s^\infty F(s)\,ds$$

Ex: Evaluate $\boldsymbol{L\left\{\frac{t-\sin h\, 5t}{t}\right\}}$

Sol: We know that,

$$L\left\{\frac{f\ (t)}{t}\right\} = \int_s^\infty F(s)\ ds$$

By comparison we get,

$f(t) = (t - \sin h\ 5t)$

$$L\{f(t)\} = L(t - \sin h\ 5t) = \frac{1}{s^2} - \frac{5}{s^2 - 25}$$

$$L\left\{\frac{f\ (t)}{t}\right\} = \int_s^\infty \left(\frac{1}{s^2} - \frac{5}{s^2 - 25}\right)\ ds = \left[-\ \frac{1}{s} - \frac{5}{10}\log\left(\frac{s-5}{s+5}\right)_s^\infty\right]$$

$$\therefore L\left\{\frac{f\ (t)}{t}\right\} = \frac{1}{s} - \frac{1}{2}\log\left(\frac{s-5}{s+5}\right)$$

Ex: Find $\boldsymbol{L\left\{\frac{1-\cos t}{t}\right\}}$

Sol: We know that,

$$L\left\{\frac{f\ (t)}{t}\right\} = \int_s^\infty F(s)\ ds$$

$$L\{f(t)\} = L\{1 - \cos t\} = \frac{1}{s} - \frac{s}{s^2+1}$$

$$L\left\{\frac{f\ (t)}{t}\right\} = \int_s^\infty \frac{1}{s} - \frac{s}{s^2+1}\ ds = \int_s^\infty \frac{1}{s} - \frac{1}{2}\left(\frac{2s}{s^2+1}\right)\ ds = \left[\log s - \frac{1}{2}\log(s^2+1)\right]_s^\infty$$

$$= \left[\log\left(\frac{s}{\sqrt{s^2+1}}\right)\right]_s^\infty = -\log\left(\frac{s}{\sqrt{s^2+1}}\right)$$

Exercise: Find $L\left[\frac{e^{-3t}\sin 2t}{t}\right]$ Ans: $cot^{-1}\frac{s+3}{2}$

Exercise: Evaluate $L\left[\frac{sin^3 t}{t}\right]$ Ans: $\frac{3}{4}\left[cot^{-1}s-\frac{1}{3}cot^{-1}\frac{s}{3}\right]$

- **Transform of Derivatives and Integrals:**

- **Laplace transform of the derivative:**

$L\{f'(t)\} = s\, L\{f(t)\} - f(0),$

$L\{f''(t)\} = s^2 L\{f(t)\} - sf(0) - f'(0)$,

Similarly, in general

$L\{f^n(t)\} = s^n L\{f(t)\} - s^{n-1}f(0) - s^{n-2}f'(0) \dots - f^{(n-1)}(0).$

- **Laplace transform of the integral of a function:**

If $L\{f(t)\} = F(s)$, then $L\left[\int_0^t f(u)du\right] = \frac{1}{s}F(s)$

Proof: The function $f(t)$ should be integrable in such a way that

$$g(t) = \int_0^t f(u)du$$

Is of exponential order. Then $g(0) = 0$ and $g'(t) = f(t)$. Therefore,

$L\{g'(t)\} = s\, L\{g(t)\} - g(0) = s\, L\{g(t)\}$

and so

$$L\left[\int_0^t f(u)du\right] = L\{g(t)\} = \frac{L\{g'(t)\}}{s} = \frac{L\{f(t)\}}{s} = \frac{1}{s}F(s)$$

Ex: Find the Laplace Transform of $\int_0^t e^{-t}\, dt$

Sol: $L\{e^{-t}\} = \frac{1}{s+1}$

$L\left\{\int_0^t e^{-t}\, dt\right\} = \frac{1}{s}L\{e^{-t}\} = \frac{1}{s(s+1)}$

Ex: Find $L\left\{\int_0^t e^{-t}\cos t\ \ dt\right\}$

Sol: We know that $L\,\{cos\ t\} = \frac{s}{s^2+1}$

$$\therefore L\,\{e^{-t}cos\ t\} = \frac{(s+1)}{(s+1)^2+1} = \frac{(s+1)}{s^2+2s+2}$$

$$\therefore L\left\{\int_0^t e^{-t}\cos t\ \ dt\right\} = \frac{1}{s}\left(\frac{s+1}{s^2+2s+2}\right)$$

Exercise: Evaluate $L\left\{\int_0^t te^{-4t}sin3t\ \ dt\right\}$

Ans: $\frac{1}{s}\left(\frac{6(s+4)}{(s^2+8s+25)^2}\right)$

- **Evaluation of integrals by Laplace transform**

Ex: Evaluate $\int_0^\infty e^{-3t}t^5\ \ dt$.

Sol: $\int_0^\infty e^{-st}t^5\ \ dt = L\{t^5\} = \frac{\mathbf{5!}}{\boldsymbol{s^6}}$

Putting $s = 3$, we have

$$\int_0^\infty e^{-3t}t^5\ \ dt = \frac{120}{3^6} = \frac{40}{243}$$

Ex: Evaluate $\boldsymbol{\int_0^\infty e^{-2t}sin^3t\ \ dt}$ **.**

Sol: $\int_0^\infty \mathrm{e^{-st}sin^3t\ dt = L\{sin^3t\}}$

$= \mathrm{L\{\frac{3sint-sin3t}{4}\}}$

$=\frac{3}{4}\frac{1}{s^2+1} - \frac{1}{4}\frac{3}{s^2+9}$

$=\frac{3}{4}\left[\frac{s^2+9-s^2-1}{(s^2+1)(s^2+9)}\right]$

$=\frac{6}{(s^2+1)(s^2+9)}$ …(1)

Putting $\mathrm{s} = 2$ in eq (1)

$\int_0^\infty \mathrm{e^{-2t}sin^3t\ dt} = \frac{6}{(4+1)(4+9)} = \frac{6}{65}$

Ex: Evaluate $\int_0^\infty te^{-2t}cost\,dt$.

Sol: $\int_0^\infty te^{-st}cost\,dt\ = L\{tcost\}$

$=-\frac{\boldsymbol{d}}{\boldsymbol{ds}}\ L\{cost\}$

$= -\frac{d}{ds}\left(\frac{s}{s^2+1}\right)$

$= \frac{s^2-1}{(s^2+1)^2}$ ...(1)

Putting s=2 in eq 1, we have

$\int_0^\infty te^{-2t} cost\, dt = \frac{4-1}{(4+1)^2} = \frac{3}{25}$

Exercise: Evaluate $\int_0^\infty t^2 e^{-2t} sin3t\, dt$. Ans $= \frac{18}{2197}$

- **Laplace Transform of periodic function:**

If $f(t)$ is a periodic function with period p, i.e.$f(t+p) = f(t), then$

$$L\left(f(t)\right) = \frac{1}{1-e^{-sp}} \int_0^p e^{-st} f(t) dt$$

Ex: Find L.T. of half wave rectification of $Sin\omega t$ defined by

$$f(t) = \begin{cases} Sin\omega t \;\; if\; 0 < t < \frac{\pi}{\omega} \\ 0 \quad if \;\; \frac{\pi}{\omega} < t < \frac{2\pi}{\omega} \end{cases}$$

Where $f\left(t + \frac{2\pi}{\omega}\right) = f(t)$

Sol: The given function is a periodic function with period $\frac{2\pi}{\omega}$

$$L\{f(t)\} = \frac{1}{1-e^{-\frac{2\pi s}{\omega}}} \int_0^{\frac{2\pi}{\omega}} f(t) e^{-st}\, dt$$

$$= \frac{1}{1-e^{-\frac{2\pi s}{\omega}}} \left[\int_0^{\frac{\pi}{\omega}} e^{-st} \sin \omega t\, dt + \int_{\frac{\pi}{\omega}}^{\frac{2\pi}{\omega}} e^{-st}(0)\, dt\right]$$

$$= \frac{1}{1-e^{-\frac{2\pi s}{\omega}}} \left[\frac{e^{-st}}{s^2+\omega^2} \left(-s \sin \omega t - \omega \cos \omega t\right.\right]_0^{\pi/\omega}$$

$$= \frac{1}{1-e^{-\frac{2\pi s}{\omega}}}\left[\frac{e^{-s\pi/\omega}\omega}{s^2+\omega^2} + \frac{\omega}{s^2+\omega^2}\right] = \frac{1}{1-e^{-\frac{2\pi s}{\omega}}}\left[\frac{\omega e^{-s\pi/\omega}+\omega}{s^2+\omega^2}\right]$$

Ex: Find the Laplace transform of the function $\boldsymbol{f(t) = |\sin \omega t|; t \geq 0}$

Sol. The given function is a periodic function of period $\frac{\pi}{\omega}$

$$L\{f(t)\} = \frac{1}{1-e^{-ps}}\int_0^p e^{-st} f(t)dt = \frac{1}{1-e^{-\frac{\pi s}{\omega}}}\int_0^{\frac{\pi}{\omega}} e^{-st}|\sin \omega t|dt$$

$$= \frac{1}{1-e^{-\frac{\pi s}{\omega}}}\left[\frac{e^{-st}}{s^2+\omega^2}(-s \sin \omega t - \omega \cos \omega t)\right]_0^{\frac{\pi}{\omega}}$$

$$= \frac{1}{1-e^{-\frac{\pi s}{\omega}}}\left[\frac{\omega e^{-\frac{\pi s}{\omega}}}{s^2+\omega^2} + \frac{\omega}{s^2+\omega^2}\right]$$

Exercise: Find the Laplace transform of $\boldsymbol{f(t)} = \begin{cases} \boldsymbol{k} & \boldsymbol{0} < t < a \\ -\boldsymbol{k} & \boldsymbol{a} < t < 2\boldsymbol{a} \end{cases}$ **and**

$\boldsymbol{f(t+2a) = f(t)}$ **.**

Ans: $\frac{k}{s} \tan h \left(\frac{as}{2}\right)$

Inverse Laplace Transform:

If $L\{f(t)\} = \int_0^\infty f(t)e^{-St}dt = F(s)$ then $L^{-1}\{F(s)\} = f(t)$

Some basic Inverse Laplace Transform Formula:

$$\mathrm{L}^{-1}\left\{\frac{1}{\mathrm{s}}\right\} = 1$$

$\mathrm{L}^{-1}\left\{\frac{1}{\mathrm{s}^2}\right\} = \mathrm{t}$ and $\mathrm{L}^{-1}\left\{\frac{1}{\mathrm{s}^{n+1}}\right\} = \frac{\mathrm{t}^n}{n!}$

$\mathrm{L}^{-1}\left\{\frac{1}{\mathrm{s-a}}\right\} = \mathrm{e}^{\mathrm{at}}$ and $\mathrm{L}^{-1}\left\{\frac{1}{\mathrm{s+a}}\right\} = \mathrm{e}^{-\mathrm{at}}$

$$\mathrm{L}^{-1}\left\{\frac{1}{\mathrm{s}^2+\mathrm{a}^2}\right\} = \frac{\mathrm{sinat}}{\mathrm{a}}$$

$$L^{-1}\left\{\frac{s}{s^2+a^2}\right\} = \cos at$$

$$L^{-1}\left\{\frac{1}{s^2-a^2}\right\} = \frac{\sin h\, at}{a}$$

$$L^{-1}\left\{\frac{s}{s^2-a^2}\right\} = \cos h\, at$$

$$L^{-1}\left\{\frac{1}{(s-a)^2+b^2}\right\} = \frac{e^{at}\sin bt}{b}$$

$$L^{-1}\left\{\frac{s-a}{(s-a)^2+b^2}\right\} = e^{at}\cos bt$$

$$L^{-1}\left\{\frac{1}{(s-a)^2-b^2}\right\} = \frac{e^{at}\sin h\, bt}{b}$$

$$L^{-1}\left\{\frac{s-a}{(s-a)^2-b^2}\right\} = e^{at}\cos h\, bt$$

Partial Fractions:

Mainly four types of partial fraction are use repeatedly.

Case-1: If the denominator has non – repeated linear factors $(s-a),(s-b),(s-c)$then $Partial\ fraction = \frac{A}{(s-a)} + \frac{B}{(s-b)} + \frac{C}{(s-c)}$.

Case-2: If the denominator has repeated linear factors $(s-a)(\text{n times})$, then $Partial\ fraction = \frac{A_1}{(s-a)} + \frac{A_2}{(s-a)^2} + \frac{A_3}{(s-a)^3} + \cdots + \frac{A_n}{(s-a)^n}$.

Case-3: If the denominator has non-repeated quadratic factors

$(s^2+as+b),(s^2+cs+d)$ then

$Partial\ fraction = \frac{As+B}{(s^2+as+b)} + \frac{Cs+D}{(s^2+cs+d)}$.

Case-4: If the denominator has repeated quadratic factors $(s^2+as+b)(\text{n times})$then

$Partial\ fraction = \frac{As+B}{(s^2+as+b)} + \frac{Cs+D}{(s^2+as+b)^2} + \cdots (n\ times)$.

Ex: $\boldsymbol{L^{-1}\left(\frac{1}{s-7}\right) = e^{7t}}$

Ex: $\boldsymbol{L^{-1}\left(\frac{4}{s^2-121}\right) = \frac{4}{11}\ sin\,h\ 11t}$

Ex: Find $\boldsymbol{L^{-1}\left\{\frac{6s-8}{s^2-s-6}\right\}}$

Sol: we have

$$\frac{6s-8}{s^2-s-6} = \frac{6s-8}{(s+2)(s-3)} = \frac{A}{(s+2)} + \frac{B}{(s-3)} = \frac{A(s-3)+B(s+2)}{(s+2)(s-3)}$$

$\therefore 6s-8 = A(s-3)+B(s+2)$ ... (1)

Put $s = 3$ in above equation, we can find

$18-8 = A(0)+B(5)$

$10 = 5B$

$\therefore \boldsymbol{B = 2}$

Put $s = -2$ in equation (1), one can get

$6(-2)-8 = A\,(-2-3)+B(0)$

$-12-8 = A(-5)$

$-20 = -5A$

$\therefore \boldsymbol{A = 4}$

By using Partial Fraction,

$$L^{-1}\left\{\frac{6s-8}{s^2-s-6}\right\} = L^{-1}\left(\frac{4}{(s+2)} + \frac{2}{(s+3)}\right) = 4\,e^{-2t} + 3e^{3t}$$

Ex: Evaluate $\boldsymbol{L^{-1}\left\{\log\frac{1}{s}\right\}}$

Sol: Here

$$F(s) = \log\frac{1}{s} = \log 1 - \log s = -\log s$$

Differentiating with respect to s

$$-\frac{d}{ds}\,F(s) = -\frac{d}{ds}\,(-\log s) = \frac{1}{s}$$

We know that,

$$L^{-1}\left\{-\frac{d}{ds}\,F(s)\right\} = t\,f(t)$$

$$\Rightarrow 1 = L^{-1}\left\{\frac{1}{s}\right\} = t\,f(t)$$

$$\Rightarrow f(t) = \frac{1}{t}$$

Exercise: Show that $L^{-1}\left(\frac{s}{s^2+4}\right) = \boldsymbol{\cos 2t}$

Exercise: Prove that $L^{-1}\left(\frac{3}{s^2+6s+18}\right) = \boldsymbol{e^{-3t}\sin 3t}$

Exercise: Show that $L^{-1}\left(\frac{5s+3}{(s^2+2s+5)(s-1)}\right) = \boldsymbol{-e^{-t}\cos 2t + \frac{3}{2}e^{-t}\sin 2t + e^{t}}$

Exercise: Prove that $\mathbf{L^{-1}}\left\{\mathbf{\log}\left(\frac{s+1}{s}\right)\right\} = \frac{1-e^{-t}}{t}$

Exercise: Show that $\mathbf{L^{-1}}\left\{\int_s^\infty \cot^{-1}(s+1)ds\right\} = \frac{-e^{t}\sin t}{t^2}$

Definition of convolution:

The $(f*g)$ is known as convolution and it is defined as

$(f*g)(t) = \int_0^t f(u)\,g(t-u)\,du$. Also $f*g = g*f$

Convolution Theorem: If $L^{-1}\{F(s)\} = f(t)$ and $L^{-1}\{G(s)\} = g(t)$ then

$$L^{-1}\{F(s).G(s)\} = L^{-1}\{F(s)\} * L^{-1}\{G(s)\} = f(t)*g(t) = \int_0^t f(u)\,g(t-u)\,du$$

Ex: Find $\mathbf{1 * 1}$

Sol: Let $f(t) = 1$ and $g(t) = 1$

Then by definition of convolution,

$$f(t) * g(t) = \int_0^t f(u)g(t-u)du = \int_0^t 1 \cdot 1\, du = \int_0^t 1\, du = [u]_1^t = t$$

$$\therefore 1 * 1 = t$$

Ex: Using convolution theorem, evaluate $\boldsymbol{L^{-1}\left\{\frac{1}{s\,(s^2+4)}\right\}}$

Sol: $f(s) = \frac{1}{s}$ and $g(s) = \frac{1}{(s^2+4)}$

We know that

$$g(t) = L^{-1}\left\{\frac{1}{s}\right\} = 1$$

and

$$f(t) = L^{-1}\left\{\frac{1}{(s^2+4)}\right\} = \frac{1}{2}\sin 2t$$

Therefore, we get

$$g(t) = 1 \quad ; \quad f(t) = \frac{1}{2}\sin 2t$$

By convolution theorem,

$$L^{-1}\left\{\frac{1}{s\,(s^2+4)}\right\} = \int_0^t f(u)g(t-u)du = \int_0^t \frac{1}{2}\sin 2u \cdot (1)du = \frac{1}{2}\left[-\frac{\cos 2u}{2}\right]_0^t$$

$$= \frac{1}{4}[-\cos 2t + \cos 0] = \frac{1}{4}[1 - \cos 2t]$$

Exercise: Find $\boldsymbol{\cos \omega t * \sin \omega t}$ **Ans:** $\frac{t \sin \omega t}{2}$

Exercise: Apply Convolution Theorem to find $\boldsymbol{L^{-1}\left\{\frac{s^2}{s^4-a^4}\right\}}$ **Ans:** $\frac{\sin at}{2a} + \frac{e^{at}-e^{at}}{2a}$

Exercise: Find the convolution of $\boldsymbol{e^t}$ and $\boldsymbol{e^{-t}}$ **Ans**: $\sin ht$

- **Unit Step Function (or Heaviside's unit function):**

Find the Laplace transform of unit step function

Or

Show that $\boldsymbol{L\{u(t-a)\} = \frac{e^{-as}}{s}}$ **where** $\boldsymbol{u(t-a) = \begin{cases} 0\,, t < a \\ 1\,, t \geq a \end{cases}}$ *(This function is known as unit step function).*

Proof:

$$L\left(u\,(t-a)\right) = \int_0^\infty e^{-st}\,u(t-a)dt = \int_0^a e^{-st}(0)dt + \int_a^\infty e^{-st}(1)dt = \left[\frac{e^{-st}}{-s}\right]_a^\infty$$

$$= \frac{1}{-s}\left[e^{-\infty} - e^{-as}\right] = \frac{e^{-as}}{s}$$

$$\therefore L\left(u\,(t-a)\right) = \frac{e^{-as}}{s}$$

Dirac's Delta Function:

Definition: Consider the *Dirac Delta Function* $f_\varepsilon(where\ \varepsilon > 0)$ which is defined by,

$$f_\varepsilon(t) = \delta\,(t-\varepsilon) = \begin{cases} \frac{1}{\varepsilon}\,, a \leq t \leq a+\varepsilon \\ 0\,, \quad t > \varepsilon \end{cases}.$$

Exercise: Find the Laplace transform of Dirac – delta function $\boldsymbol{\delta\,(t-a)}$

Ans: $L\,\{\delta\,(t-a)\} = e^{-as}$

Second Shifting Theorem:

If $L\{f(t)\} = F(s)$, then $L\{f(t-a)\,.\,u(t-a)\} = e^{-as}F(s) = e^{-as}L\{f(t)\}$

Proof: We know that $u(t-a) = \begin{cases} 0 & for\ t < a \\ 1 & t \geq a \end{cases}$

$L\{f(t-a)u\,(t-a)\}$

$$= \int_0^\infty e^{st}\,f(t-a)u\,(t-a)dt$$

$$= \int_0^a e^{st}(0)dt + \int_a^\infty e^{-st} f\,(t-a)dt$$

$$= \int_a^\infty e^{-st} f\,(t-a)dt$$

Let $u = t - a \Rightarrow du = dt$

Also $if\ t \rightarrow a\ then\ u \rightarrow 0$ and $t \rightarrow \infty\ \ then\, u \rightarrow \infty$

$$\int_a^\infty e^{-st} f\,(t-a)dt = \int_0^\infty e^{-s\,(u+a)} f(u)du = \int_0^\infty e^{-su-sa} f(u)du$$

$$= e^{-sa}\int_0^\infty e^{-su} f(u)du = e^{-as} F(s)$$

$$\therefore L\,\{f\,(t-a)u\,(t-a)\} = e^{-as} F(s)$$

Note: If a function $f(t)$ is defined as follows

$f(t) = \begin{cases} f_1(t); 0 < t < a \\ f_2(t); a < t < b \\ f_3(t); \qquad t > b \end{cases}$ then $f(t)$ can be written as,

$$f(t) = [f_1(t)u(t) - f_1(t)u\,(t-a)] + [f_2(t)u(t-a) - f_2(t)u\,(t-b)] + f_3(t)u\,(t - b)$$

Corollary 1: $L\,\{f\,(t)u\,(t-a)\} = e^{-as}\,L\{f\,(t+a)\}$

Corollary 2: $L\,\{u\,(t-a)\} = L\,\{1.u\,(t-a)\} = \frac{e^{-as}}{s}$

Corollary 3: $L\,\{u\,(t-a) - u\,(t-b)\} = \frac{e^{-as} - e^{-bs}}{s}$

Corollary 4:

$$L\,\{f(t)[u\,(t-a) - u\,(t-b)]\} = [e^{-as}\,L\{f\,(t+a)\} - e^{-bs}\,L\{f\,(t+b)\}]$$

Ex: Evaluate $\boldsymbol{L\,\{(t-3)^2 u\,(t-3)\}}$

Sol. Here $f(t-a) = (t-3)^2$

$f(t+a) = (t)^2$

$\therefore L\{(t-3)^2u\,(t-3)\} = e^{-3s}L\{f(t+a)\} = e^{-3s}L\{(t)^2\} = \dfrac{2e^{-3s}}{s^3}$

Ex: Find $\boldsymbol{L\{(t)^2u\,(t-3)\}}$

Sol. Here, $f(t) = t^2$

$L\{f(t)u\,(t-a)\} = e^{-as}\,L\{f\,(t+a)\}$

$L\{(t)^2u\,(t-3)\} = e^{-3s}\,L\{f\,(t+3)\} = e^{-3s}\,L\{(t+3)^2\} = e^{-3s}\,L\{t^2+6t+9)\}$

$= e^{-3s}\,[L\,(t^2) + 6\,L(t) + 9\,L\,(1)]$

$\therefore L\{(t)^2u\,(t-3)\} = e^{-3s}\left[\dfrac{2}{s^3} + \dfrac{6}{s^2} + \dfrac{9}{s}\right]$

Ex: Evaluate $\boldsymbol{L\{e^tu\,(t-2)\}}$

Sol: $L\{e^tu\,(t-2)\} = e^{-2s}L\{e^{t+2}\} = e^{-2s}e^2L\{e^t\} = \frac{e^{-2(s-1)}}{s-1}$

Exercise: Find $\boldsymbol{L\{e^{-3t}\,u\,(t-2)\}}$ **Ans:** $\boldsymbol{\frac{e^{-2s-6}}{s+3}}$

Exercise: Evaluate $\boldsymbol{L\left\{Sin\,t\,u\,(t-\frac{\pi}{2})\right\}}$ **Ans:** $e^{-\frac{\pi s}{2}}\frac{s}{s^2+1}$

Inverse Laplace by using second shifting theorem:

$$\boldsymbol{L^{-1}\{e^{-as}\,F(s)\} = f(t-a)u\,(t-a)}$$

Ex: Find $\boldsymbol{L^{-1}\left\{\frac{2e^{-s}}{s^3}\right\}}$

Sol: Here $L^{-1}\left\{\frac{2e^{-s}}{s^3}\right\} = L^{-1}\left\{e^{-s}\,\frac{2}{s^3}\right\}$

By comparison, we get $a = 1$

We know that,

$L^{-1}\left\{\frac{2}{s^3}\right\} = 2\,\frac{t^2}{2!} = t^2$

$\therefore L^{-1}\left\{e^{-s}\,\frac{2}{s^3}\right\} = (t-1)^2u\,(t-1)$

Application of Laplace Transform:

Solution of Ordinary Differential Equations:

$L\{f'(t)\} = s\, L\{f(t)\} - f(0)$,

$L\{f''(t)\} = s^2 L\{f(t)\} - sf(0) - f'(0)$,

Similarly, in general Laplace transform of the n^{th} derivatives of $f\ (t)$

$L\left(f^n(t)\right) = s^n L\left(f(t)\right) - s^{n-1}f(0) - s^{n-2}f'(0) \ldots - f^{\,n-1}\ (0)$

Ex: Solve the differential equation

$$\boldsymbol{y'' + 6y = 1}\ , \qquad \boldsymbol{y(0) = 2}, \qquad \boldsymbol{y'(0) = 0}$$

Sol: Here $\quad y'' + 6y = 1$

Taking Laplace Transform on both sides, we get

$$L(y'' + 6y\ ) = L(1)$$

$$L(y'') + 6\ L(y) = \frac{1}{s}$$

$$s^2 L\left(y(t)\right) - sy(0) + 6L\left(y(t)\right) = \frac{1}{s}$$

Here, given conditions are $y(0) = 2,\ y'(0) = 0$

$$s^2 L\left(y(t)\right) - 2s + 6L\left(y(t)\right) = \frac{1}{s}$$

$$(s^2 + 6)L\left(y(t)\right) = \frac{1}{s} + 2s = \frac{1 + 2s^2}{s}$$

$$L\left(y(t)\right) = \frac{1+2s^2}{s\,(s^2+6)}$$

Apply Inverse Laplace Transformation on both sides, we get

$$y(t) = L^{-1}\left(\frac{1+2s^2}{s\,(s^2+6)}\right)$$

By using Partial fraction,

$$\frac{1+2s^2}{s\,(s^2+6)} = \frac{A}{s} + \frac{Bs+c}{(s^2+6)} \Rightarrow 1+2s^2 = A(s^2+6) + (Bs+c)s$$

$$\Rightarrow 1+2s^2 = A(s^2+6) + Bs^2 + cs$$

Put $s = 0,\, 2(0)+1 = A(0+6) \Rightarrow 1 = 6A \Rightarrow \boldsymbol{A} = \frac{\mathbf{1}}{\mathbf{6}}$

Substitute $s = 1,\, 2(1)+1 = A(1+6) + B + C \Rightarrow 3 = 7\left(\frac{1}{6}\right) + B + C \Rightarrow 3 = \frac{7+6B+6c}{6}$

$$\Rightarrow 18 = 7 + 6B + 6C$$

$$\Rightarrow 6B + 6C = 11 \qquad (1)$$

Put $S = -1,\, 2(1)+1 = A(1+6) + B - C \Rightarrow 3 = 7A + B - C \Rightarrow 3 = 7\left(\frac{1}{6}\right) + B - C$

$$\Rightarrow 3 = \frac{7+6B-6c}{6} \Rightarrow 18 = 7 + 6B - 6c$$

$$\Rightarrow 6B - 6c = 11 \qquad (2)$$

Solving equation (1) and (2), we get $\boldsymbol{B} = \frac{\mathbf{11}}{\mathbf{6}}$ and $\boldsymbol{C} = \mathbf{0}$

$$\therefore L^{-1}\left\{\frac{1+2s^2}{s\,(s^2+6)}\right\} = L^{-1}\left\{\frac{1}{6}\left(\frac{1}{s}\right) + \frac{11}{6}\left(\frac{s}{s^2+6}\right)\right\} = \frac{1}{6} + \frac{11}{6}\cos\sqrt{6}\,t$$

Therefore, the required solution of given differential equation is,

$$y(t) = \frac{1}{6} + \frac{11}{6}\cos\sqrt{6}\,t$$

Ex: Solve the differential equation

$$\boldsymbol{y'' + 2y' + 5y = e^{-t}\sin t}\,, \quad \boldsymbol{y(0) = 0\; y'(0) = 1}$$

Sol: Apply Laplace Transformation on both sides, we get

$$L\,(y'' + 2y' + 5y) = L(e^{-t}\sin t\,)$$

$$L\,(y'') + 2\,L(y') + 5L(y) = \frac{1}{(s+1)^2+1}$$

$$s^2L\big(y(t)\big) - sy(0) - y'(0) + 2sL\big(y(t)\big) - 2s\,y(0) + 5\,L\big(y(t)\big) = \frac{1}{s^2+2s+2}$$

Now substitute boundary conditions Immediately before solving in above equation, we find

$$s^2L\big(y(t)\big) - s(0) - 1 + 2sL\big(y(t)\big) - 2s(0) + 5\,L\big(y(t)\big) = \frac{1}{s^2+2s+2}$$

$$(s^2+2s+5)L\big(y(t)\big) - 1 = \frac{1}{s^2+2s+2}$$

$$(s^2+2s+5)L\big(y(t)\big) = \frac{1}{s^2+2s+2} + 1 = \frac{1+s^2+2s+2}{s^2+2s+2}$$

$$L\big(y(t)\big) = \frac{s^2+2s+3}{(s^2+2s+2)(s^2+2s+5)}$$

Taking Laplace Transform on both sides, we get

$$y(t) = L^{-1}\left\{\frac{s^2+2s+3}{(s^2+2s+2)(s^2+2s+5)}\right\} = L^{-1}\left\{\frac{2}{3(s^2+2s+5)} + \frac{1}{3(s^2+2s+2)}\right\}$$

$$\therefore\ y(t) = L^{-1}\left\{\frac{s^2+2s+3}{(s^2+2s+2)(s^2+2s+5)}\right\}$$

$$= L^{-1}\left\{\frac{2}{3}\left(\frac{1}{(s+1)^2+4}\right) + \frac{1}{3}\left(\frac{1}{(s+1)^2+1}\right)\right\}$$

Therefore, the required solution of given differential equation is

$$y(t) = \frac{e^{-t}}{3}[\sin 2t\ +\ \sin t\,]$$

Exercise: Using Laplace transformation solve the initial value Problem

$$\boldsymbol{y'' + y = \sin 2t\,;\ \ y(0) = 2;\, y'(0) = 1}$$

Ans: $y(t) = \frac{5}{3}\sin t - \frac{1}{3}\sin 2t + 2\cos t$

Exercise: Using the method of Laplace transform, solve the IVP

$$\boldsymbol{x'' + 2x' + x = e^{-t}, x(0) = -1\,, x'(0) = 1}$$

Ans: $x(t) = e^{-t}\left(\frac{t^2}{2} - 1\right)$

Exercise: Using the method of Laplace transform, solve the differential equation

$$\boldsymbol{y'' + y' - 6y = 1; \qquad y(0) = 0\,; \quad y'(0) = 1}$$

Ans: $y(t) = -\frac{1}{6} + \frac{3}{10}e^{2t} - \frac{2}{15}e^{-3t}$

Exercise: Solve $\boldsymbol{y'' + 5y' + 4y = 3\delta(t-2),\ y(0) = 2\,,\ y'(0) = -2}$

Ans: $y(t) = e^{-(t-2)} - e^{-4(t-2)}\,u(t-2) + 2\,e^{t}$