Notes V1.2

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0.1 2-component

To begin with, consider

$$L_g(x) = g(x)\phi_{1_g}(x)\phi_{2_g}(x)$$

Equations of motion:

$$\begin{aligned}(\Box + m_1^2)\phi_{1_g}(x) &= g(x)\phi_{2_g}(x),\\ (\Box + m_2^2)\phi_{2_g}(x) &= g(x)\phi_{1_g}(x).\end{aligned}$$

Dyson expansion for the fields

$$\begin{aligned}\phi_{1_g}(x) &= \phi_1(x) + \int dy_1 g(y_1) G^{ret}_{m_1}(x-y_1)\phi_{1_g}(y_1)\\ \phi_{2_g}(x) &= \phi_2(x) + \int dy_1 g(y_1) G^{ret}_{m_2}(x-y_1)\phi_{2_g}(y_1)\end{aligned}$$

$$\begin{aligned}\phi_{1_g}(x) &= \phi_1(x) + \int dy_1 g(y_1) G^{ret}_{m_1}(x-y_1)\phi_2(y_1) + \int\int dy_1 dy_2\, g(y_1) G^{ret}_{m_1}(x-y_1) g(y_2) G^{ret}_{m_2}(y_1-y_2)\phi_1(y_2)\ldots O(g^3)\\ \phi_{2_g}(x) &= \phi_2(x) + \int dy_1 g(y_1) G^{ret}_{m_2}(x-y_1)\phi_1(y_1) + \int\int dy_1 dy_2\, g(y_1) G^{ret}_{m_2}(x-y_1) g(y_2) G^{ret}_{m_1}(y_1-y_2)\phi_2(y_2)\ldots O(g^3)\end{aligned}$$

In the Fourier representation,

$$\begin{aligned}\phi_{1_g}(x) = \int d^4k \Bigg[&\left(1 + \int dq_1 \frac{\hat{g}(q_1)e^{-iq_1x}}{m_1^2 - k_1^2 - i\varepsilon k_1^0}\int dq_2 \frac{\hat{g}(q_2)e^{-iq_2x}}{m_2^2 - k_2^2 - i\varepsilon k_2^0} + \ldots O(g^4)\right)\hat{\phi_1}(k)e^{-ikx} + \Bigg(\int dq_1 \frac{\hat{g}(q_1)e^{-iq_1x}}{m_1^2 - k_1^2 - i\varepsilon k_1^0} +\\ &\int dq_1 \frac{\hat{g}(q_1)e^{-iq_1x}}{m_1^2 - k_1^2 - i\varepsilon k_1^0}\int dq_2 \frac{\hat{g}(q_2)e^{-iq_2x}}{m_2^2 - k_2^2 - i\varepsilon k_2^0}\int dq_3 \frac{\hat{g}(q_3)e^{-iq_3x}}{m_1^2 - k_3^2 - i\varepsilon k_3^0} + \ldots O(g^5)\Bigg)\hat{\phi}_2(x)e^{-ikx}\Bigg]\end{aligned}$$

$$\begin{aligned}\phi_{2_g}(x) = \int d^4k \Bigg[&\left(1 + \int dq_1 \frac{\hat{g}(q_1)e^{-iq_1x}}{m_2^2 - k_1^2 - i\varepsilon k_1^0}\int dq_2 \frac{\hat{g}(q_2)e^{-iq_2x}}{m_1^2 - k_2^2 - i\varepsilon k_2^0} + \ldots O(g^4)\right)\hat{\phi_2}(k)e^{-ikx} + \Bigg(\int dq_1 \frac{\hat{g}(q_1)e^{-iq_1x}}{m_2^2 - k_1^2 - i\varepsilon k_1^0} +\\ &\int dq_1 \frac{\hat{g}(q_1)e^{-iq_1x}}{m_2^2 - k_1^2 - i\varepsilon k_1^0}\int dq_2 \frac{\hat{g}(q_2)e^{-iq_2x}}{m_1^2 - k_2^2 - i\varepsilon k_2^0}\int dq_3 \frac{\hat{g}(q_3)e^{-iq_3x}}{m_2^2 - k_3^2 - i\varepsilon k_3^0} + \ldots O(g^5)\Bigg)\hat{\phi}_1(x)e^{-ikx}\Bigg].\end{aligned}$$

For the ease of typing, define

$$\frac{1}{m_{1,2}^2 - k^2 \pm i\varepsilon k^0} = \frac{1}{X^{\pm}_{1,2}}$$

and take the limit

$$\lim_{\hat{g}(q)\to\kappa\delta(q)}\int dq\,\frac{\hat{g}(q)e^{-iqx}}{X^{\pm}_{1,2}} = \frac{\kappa}{X^{\pm}_{1,2}}.$$

Set them in matrices

$$\varphi_\kappa(x) = \begin{pmatrix}\phi_{1_\kappa}(x)\\ \phi_{2_\kappa}(x)\end{pmatrix} = \int d^4k \underbrace{\begin{pmatrix} 1+\frac{\kappa}{X_1^-}\frac{\kappa}{X_2^-}+\dots O(\kappa^4) & \frac{\kappa}{X_1^-}+\frac{\kappa}{X_1^-}\frac{\kappa}{X_2^-}\frac{\kappa}{X_1^-}+\dots O(g^5)\\ \frac{\kappa}{X_2^-}+\frac{\kappa}{X_2^-}\frac{\kappa}{X_1^-}\frac{\kappa}{X_2^-}+\dots O(g^5) & 1+\frac{\kappa}{X_2^-}\frac{\kappa}{X_1^-}+\dots O(g^4)\end{pmatrix}}_{L_\kappa} \begin{pmatrix}\hat\phi_1(k)e^{-ikx}\\ \hat\phi_2(k)e^{-ikx}\end{pmatrix}.$$

The two-point function

$$\lim_{g(x)\to\kappa}\langle\varphi_g(x)\varphi_g(x')\rangle_? = \langle\begin{pmatrix}\phi_{1_\kappa}(x)\\ \phi_{2_\kappa}(x)\end{pmatrix}\begin{pmatrix}\phi_{1_\kappa}(x') & \phi_{2_\kappa}(x')\end{pmatrix}\rangle_? = \int\int d^4k\, d^4k'\,\langle L_\kappa \begin{pmatrix}\hat\phi_1(k)\\ \hat\phi_2(k)\end{pmatrix}\begin{pmatrix}\hat\phi_1(-k') & \hat\phi_2(-k')\end{pmatrix} L_\kappa^\dagger\rangle_? e^{-ikx+ik'x'}$$

$$\int\int d^4k\, d^4k'\langle L_\kappa \begin{pmatrix}\hat\phi_1(k)\\ \hat\phi_2(k)\end{pmatrix}\begin{pmatrix}\hat\phi_1(-k') & \hat\phi_2(-k')\end{pmatrix} L_\kappa^\dagger\rangle_? e^{-ikx+ik'x'} = \int\int d^4k\, d^4k'$$

$$\langle\begin{pmatrix} 1+\frac{\kappa}{X_1^-}\frac{\kappa}{X_2^-}+\dots O(\kappa^4) & \frac{\kappa}{X_1^-}+\frac{\kappa}{X_1^-}\frac{\kappa}{X_2^-}\frac{\kappa}{X_1^-}+\dots O(\kappa^5)\\ \frac{\kappa}{X_2^-}+\frac{\kappa}{X_2^-}\frac{\kappa}{X_1^-}\frac{\kappa}{X_2^-}+\dots O(\kappa^5) & 1+\frac{\kappa}{X_2^-}\frac{\kappa}{X_1^-}+\dots O(\kappa^4)\end{pmatrix}\begin{pmatrix}\hat\phi_1(k)\hat\phi_1(-k') & \hat\phi_1(k)\hat\phi_2(-k')\\ \hat\phi_2(k)\hat\phi_1(-k') & \hat\phi_2(k)\hat\phi_2(-k')\end{pmatrix}$$
$$\times\underbrace{\begin{pmatrix} 1+\frac{\kappa}{X_1^+}\frac{\kappa}{X_2^+}+\dots O(\kappa^4) & \frac{\kappa}{X_2^+}+\frac{\kappa}{X_2^+}\frac{\kappa}{X_1^+}\frac{\kappa}{X_2^+}+\dots O(\kappa^5)\\ \frac{\kappa}{X_1^+}+\frac{\kappa}{X_1^+}\frac{\kappa}{X_2^+}\frac{\kappa}{X_1^+}+\dots O(\kappa^5) & 1+\frac{\kappa}{X_2^+}\frac{\kappa}{X_1^+}+\dots O(\kappa^4)\end{pmatrix}}_{\textbf{At this step, these X's have k' inside them.}}\rangle_? e^{-ikx+ik'x'}$$

Writing

$$L_\kappa = \begin{pmatrix}A & B\\ C & D\end{pmatrix}$$

$$\int\int d^4k\, d^4k'\langle L_\kappa \begin{pmatrix}\hat\phi_1(k)\\ \hat\phi_2(k)\end{pmatrix}\begin{pmatrix}\hat\phi_1(-k') & \hat\phi_2(-k')\end{pmatrix} L_\kappa^\dagger\rangle_? e^{-ikx+ik'x')} =$$
$$\begin{pmatrix}\int d\mu_{m_1}\left(AA^T\right)+\int d\mu_{m_2}\left(BB^T\right) & \int d\mu_{m_1}\left(AC^T\right)+\int d\mu_{m_2}\left(BD^T\right)\\ \int d\mu_{m_1}\left(CA^T\right)+\int d\mu_{m_2}\left(DB^T\right) & \int d\mu_{m_1}\left(CC^T\right)+\int d\mu_{m_2}\left(DD^T\right)\end{pmatrix}_\star e^{-ik(x-x')}$$

Recalling the formula (which makes sense only order by order in perturbation theory)

$$\frac{a}{X}\delta(X)\frac{a}{X} = \delta(X-a),$$

$$A\delta(m_1^2-k^2)A^T = \left(1+\frac{\kappa}{X_1^-}\frac{\kappa}{X_2^-}+\dots O(\kappa^4)\right)\left[\frac{1}{X_1^-}-\frac{1}{X_1^+}\right]\left(1+\frac{\kappa}{X_1^+}\frac{\kappa}{X_2^+}+\dots O(\kappa^4)\right)$$

$$B\delta(m_2^2-k^2)B^T = \left(\frac{\kappa}{X_1^-}+\frac{\kappa}{X_1^-}\frac{\kappa}{X_2^-}\frac{\kappa}{X_1^-}+\dots O(\kappa^5)\right)\left[\frac{1}{X_2^-}-\frac{1}{X_2^+}\right]\left(\frac{\kappa}{X_1^+}+\frac{\kappa}{X_1^+}\frac{\kappa}{X_2^+}\frac{\kappa}{X_1^+}+\dots O(\kappa^5)\right)$$

$$C\delta(m_1^2-k^2)C^T = \left(\frac{\kappa}{X_2^-}+\frac{\kappa}{X_2^-}\frac{\kappa}{X_1^-}\frac{\kappa}{X_2^-}+\dots O(\kappa^5)\right)\left[\frac{1}{X_1^-}-\frac{1}{X_1^+}\right]\left(\frac{\kappa}{X_2^+}+\frac{\kappa}{X_2^+}\frac{\kappa}{X_1^+}\frac{\kappa}{X_2^+}+\dots O(\kappa^5)\right)$$

$$D\delta(m_2^2-k^2)D^T = \left(1+\frac{\kappa}{X_2^-}\frac{\kappa}{X_1^-}+\dots O(\kappa^4)\right)\left[\frac{1}{X_2^-}-\frac{1}{X_2^+}\right]\left(1+\frac{\kappa}{X_2^+}\frac{\kappa}{X_1^+}+\dots O(\kappa^4)\right)$$

$$A\delta(m_1^2-k^2)C^T = \left(1+\frac{\kappa}{X_1^-}\frac{\kappa}{X_2^-}+\dots O(\kappa^4)\right)\left[\frac{1}{X_1^-}-\frac{1}{X_1^+}\right]\left(\frac{\kappa}{X_2^+}+\frac{\kappa}{X_2^+}\frac{\kappa}{X_1^+}\frac{\kappa}{X_2^+}+\dots O(\kappa^5)\right)$$

$$B\delta(m_2^2-k^2)D^T = \left(\frac{\kappa}{X_1^-}+\frac{\kappa}{X_1^-}\frac{\kappa}{X_2^-}\frac{\kappa}{X_1^-}+\dots O(\kappa^5)\right)\left[\frac{1}{X_2^-}-\frac{1}{X_2^+}\right]\left(1+\frac{\kappa}{X_2^+}\frac{\kappa}{X_1^+}+\dots O(\kappa^4)\right)$$

$$C\delta(m_1^2-k^2)A^T = \left(\frac{\kappa}{X_2^-} + \frac{\kappa}{X_2^-}\frac{\kappa}{X_1^-}\frac{\kappa}{X_2^-} + \ldots O(\kappa^5)\right)\left[\frac{1}{X_1^-} - \frac{1}{X_1^+}\right]\left(1 + \frac{\kappa}{X_1^+}\frac{\kappa}{X_2^+} + \ldots O(\kappa^4)\right)$$

$$D\delta(m_2^2-k^2)B^T = \left(1 + \frac{\kappa}{X_2^-}\frac{\kappa}{X_1^-} + \ldots O(\kappa^4)\right)\left[\frac{1}{X_2^-} - \frac{1}{X_2^+}\right]\left(\frac{\kappa}{X_1^+} + \frac{\kappa}{X_1^+}\frac{\kappa}{X_2^+}\frac{\kappa}{X_1^+} + \ldots O(\kappa^5)\right)$$

The **first entry** of the matrix:

$$\begin{aligned}
&\kappa^0 : \left(\frac{1}{X_1^-} - \frac{1}{X_1^+}\right)\\
&\kappa^2 : \frac{\kappa^2}{X_1^-X_2^-}\left(\frac{1}{X_1^-} - \cancel{\frac{1}{X_1^+}}\right) + \left(\cancel{\frac{1}{X_1^-}} - \frac{1}{X_1^+}\right)\frac{\kappa^2}{X_1^+X_2^+} + \frac{\kappa}{X_1^-}\left(\cancel{\frac{1}{X_2^-}} - \cancel{\frac{1}{X_2^+}}\right)\frac{\kappa}{X_1^+}\\
&\to \quad \left(\frac{1}{X_1^-} - \frac{1}{X_1^+}\right) + \kappa^2\left(\frac{1}{X_1^-X_2^-X_1^-} - \frac{1}{X_1^+X_2^+X_1^+}\right) + O(\kappa^4)
\end{aligned}$$

The **second entry** of the matrix:

$$\begin{aligned}
&\kappa : \left(\cancel{\frac{1}{X_1^-}} - \frac{1}{X_1^+}\right)\frac{\kappa}{X_2^+} + \frac{\kappa}{X_1^-}\left(\frac{1}{X_2^-} - \cancel{\frac{1}{X_2^+}}\right)\\
&\kappa^3 : \frac{\kappa^2}{X_1^-X_2^-}\left(\cancel{\frac{1}{X_1^-}} - \cancel{\frac{1}{X_1^+}}\right)\frac{\kappa}{X_2^+} + \left(\cancel{\frac{1}{X_1^-}} - \frac{1}{X_1^+}\right)\frac{\kappa^3}{X_2^+X_1^+X_2^+} + \frac{\kappa}{X_1^-}\left(\cancel{\frac{1}{X_2^-}} - \cancel{\frac{1}{X_2^+}}\right)\frac{\kappa^2}{X_2^+X_1^+} + \frac{\kappa^3}{X_1^-X_2^-X_1^-}\left(\frac{1}{X_1^-} - \cancel{\frac{1}{X_1^+}}\right)\\
&\to \quad \kappa\left(\frac{1}{X_1^-X_2^-} - \frac{1}{X_1^+X_2^+}\right) + \kappa^3\left(\frac{1}{X_1^-X_2^-X_1^-X_2^-} - \frac{1}{X_1^+X_2^+X_1^+X_2^+}\right) + O(\kappa^5)
\end{aligned}$$

Similarly, the **third entry**

$$\kappa\left(\frac{1}{X_2^-X_1^-} - \frac{1}{X_2^+X_1^+}\right) + \kappa^3\left(\frac{1}{X_2^-X_1^-X_2^-X_1^-} - \frac{1}{X_2^+X_1^+X_2^+X_1^+}\right) + O(\kappa^5),$$

and the **fourth entry**

$$\left(\frac{1}{X_2^-} - \frac{1}{X_2^+}\right) + \kappa^2\left(\frac{1}{X_2^-X_1^-X_2^-} - \frac{1}{X_2^+X_1^+X_2^+}\right) + O(\kappa^4).$$

Setting everything in matrix

$$\langle\varphi_\kappa(x)\varphi_\kappa(x')\rangle_? = -i\int\frac{d^4k}{(2\pi)^4}\theta(k^0)\begin{pmatrix} s & t\\ u & v\end{pmatrix}e^{-ik(x-x')}$$

$$\begin{aligned}
s &= \left(\frac{1}{X_1^-} - \frac{1}{X_1^+}\right) + \kappa^2\left(\frac{1}{X_1^-X_2^-X_1^-} - \frac{1}{X_1^+X_2^+X_1^+}\right) + O(\kappa^4)\\
t &= \kappa\left(\frac{1}{X_1^-X_2^-} - \frac{1}{X_1^+X_2^+}\right) + \kappa^3\left(\frac{1}{X_1^-X_2^-X_1^-X_2^-} - \frac{1}{X_1^+X_2^+X_1^+X_2^+}\right) + O(\kappa^5)\\
u &= \kappa\left(\frac{1}{X_2^-X_1^-} - \frac{1}{X_2^+X_1^+}\right) + \kappa^3\left(\frac{1}{X_2^-X_1^-X_2^-X_1^-} - \frac{1}{X_2^+X_1^+X_2^+X_1^+}\right) + O(\kappa^5)\\
v &= \left(\frac{1}{X_2^-} - \frac{1}{X_2^+}\right) + \kappa^2\left(\frac{1}{X_2^-X_1^-X_2^-} - \frac{1}{X_2^+X_1^+X_2^+}\right) + O(\kappa^4)
\end{aligned}$$

The trace s+v matches with the result below $\delta A + \delta D$.

Well, one can form geometric progressions in the expressions for s, t, u, and v. (If we don't worry about series convergence and the ratio of terms being ≤ 1.)

$$
\begin{aligned}
s &= \frac{X_2^-}{X_2^- X_1^- - \kappa^2} - \frac{X_2^+}{X_2^+ X_1^+ - \kappa^2} \\
t &= \kappa\left(\frac{1}{X_1^- X_2^- - \kappa^2} - \frac{1}{X_1^+ X_2^+ - \kappa^2}\right) \\
u &= \kappa\left(\frac{1}{X_2^- X_1^- - \kappa^2} - \frac{1}{X_2^+ X_1^+ - \kappa^2}\right) \\
t &= \frac{X_1^-}{X_1^- X_2^- - \kappa^2} - \frac{X_1^+}{X_1^+ X_2^+ - \kappa^2}
\end{aligned}
$$

The matrix $\begin{pmatrix} s & t \\ u & v \end{pmatrix}$ is symmetric, and for an ensured non-zero determinant, this matrix can be diagonalized. The eigenvalues read

$$
\lambda_{1/2} = \frac{1}{2}\left[\frac{1}{X_1^- X_2^- - \kappa^2}\left(X_1^- + X_2^- \pm \sqrt{(m_2^2 - m_1^2) - 4\kappa^2}\right) - \frac{1}{X_1^+ X_2^+ - \kappa^2}\left(X_1^+ + X_2^+ \pm \sqrt{(m_2^2 - m_1^2) - 4\kappa^2}\right)\right]
$$

Exercise: Look at

$$\int f_1(x_1) f_2(x_2) g_\epsilon(x_1 - x_2)$$

for $\epsilon \to 0$ and f's being smooth functions.

$$\begin{aligned}
&\int\int dq_1 e^{iq_1x_1} \hat{f}_1(q_1) \int dq_2\, e^{iq_2x_2} \hat{f}_2(q_2) \int dk\, e^{ik(x_1-x_2)} \left[2\pi\left(\frac{1}{k+i\epsilon} - \frac{1}{k-i\epsilon}\right)\right] \\
&\int dk\, \phi(k) \left(\frac{1}{k+i\epsilon} - \frac{1}{k-i\epsilon}\right), \quad \phi(k) = 2\pi \hat{f}_1(-k)\hat{f}_2(k) \\
&\lim_{\epsilon\to 0} \int dk\, \phi(k) \left(\frac{1}{k+i\epsilon} - \frac{1}{k-i\epsilon}\right)
\end{aligned}$$

$$\frac{1}{k \mp i\epsilon} = P\frac{1}{k} \pm i\pi \underbrace{\frac{\epsilon}{\pi(k^2+\epsilon^2)}}_{\delta_\epsilon(k)}, \quad P\frac{1}{k} = \frac{E}{E^2+\epsilon^2}$$

$$\lim_{\epsilon\to 0} \int dk\, \phi(k) \left(\frac{1}{k+i\epsilon} - \frac{1}{k-i\epsilon}\right) = \lim_{\epsilon\to 0} \int dk\, \phi(k)(-2i\pi\delta_\epsilon(k)) \stackrel{?}{=} -2i\pi\phi(0)$$

The principle value function P/k avoids k=0 pole.

$$\delta_\epsilon(k) = \frac{\epsilon}{\pi(k^2+\epsilon^2)}$$

Let

$$\phi(k) = e^{-k^2}$$

$$F(k) = \phi(k).\delta_\epsilon(k) = e^{-k^2}.\frac{\epsilon}{\pi(k^2+\epsilon^2)}$$