Notes V1.3

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1 Strategy with Dyson series

1.1 Free scalar field, 1-component

Steering away from the Kyle's Hamiltonian approach to realize the Haag's theorem (describe it a bit), we turn our heads back to the Dyson series approach to understand massless-massive p-loc and s-loc couplings with an argument (what argument) at hand which circumvents Haag's theorem, and allows us to move forward with the Dyson series approach. One point that Prof. Rehren mentioned is that Haag's theorem is described for time-zero fields, and the Dyson series is written for general spacetime coordinate x.

To begin with, consider

$$L_g(x) = \frac{1}{2} g(x)\phi_m^2(x), \tag{1}$$

with the Dyson series

$$\phi_g(x) = \phi_m(x) + \sum_{n=1}^{\infty} \int \cdots \int \prod_{i=1}^{n} \left[d^4y_i g(y_i) G_m^{ret}(y_{i-1} - y_i) \right] \phi_m(y_n) \,, \quad \text{where } y_0 = x,$$

such that

$$(\Box + m^2)\phi_g(x) = g(x)\phi_g(x). \tag{2}$$

In the limit $g(x) \to \kappa$, this becomes

$$(\Box + m_\kappa^2)\phi_\kappa(x) = 0 \,, \quad \text{where } m_\kappa^2 = m^2 - \kappa \tag{3}$$

Claim: The two-point function of the Dyson series for $\phi_g(x)$, evaluated in the vacuum state of ϕ_m, converges to the two-point function of the field ϕ_{m_κ}. In other words, $\langle \phi_\kappa \phi_\kappa \rangle_m = \langle \phi_{m_\kappa} \phi_{m_\kappa} \rangle_{m_\kappa}$.

Proof: The Dyson series (written explicitly upto second order in g),

$$\phi_g(x) = \phi_m(x) + \int dy_1 g(y_1) G_m^{ret}(x - y_1)\phi_m(y_1) + \int\int dy_1 dy_2 g(y_1)g(y_2) G_m^{ret}(x-y_1) G_m^{ret}(y_1 - y_2)\phi_m(y_2) + \ldots$$

Inserting the Fourier representations,

$$\begin{aligned}
\phi_g(x) = &\int d^4k e^{-ikx}\hat{\phi}_m(k) + \int dy_1 \int d^4q_1 \hat{g}(q_1) e^{-iq_1y_1} \int d^4k_1 \frac{e^{-ik_1(x-y_1)}}{m^2 - k_1^2 - i\varepsilon k_1^0} \int d^4k e^{-iky_1}\hat{\phi}_m(k) \\
&+ \int dy_1 \int d^4q_1 \hat{g}(q_1) e^{-iq_1y_1} \int d^4k_1 \frac{e^{-ik_1(x-y_1)}}{m^2 - k_1^2 - i\varepsilon k_1^0} \\
&\times \int dy_2 \int d^4q_2 \hat{g}(q_2) e^{-iq_2y_2} \int d^4k_2 \frac{e^{-ik_2(y_1-y_2)}}{m^2 - k_2^2 - i\varepsilon k_2^0} \int d^4k e^{-iky_2}\hat{\phi}_m(k) + \ldots .
\end{aligned}$$

One can do now the y_1 integral

$$\begin{aligned}
\phi_g(x) = &\int d^4k e^{-ikx}\hat{\phi}_m(k) + \int d^4q_1 \hat{g}(q_1) \int d^4k_1 \frac{e^{-ik_1x}}{m^2 - k_1^2 - i\varepsilon k_1^0} \int d^4k \hat{\phi}_m(k) \int dy_1 e^{-i(q_1 - k_1 + k)y_1} \\
&+ \int d^4q_1 \hat{g}(q_1) \int d^4k_1 \frac{e^{-ik_1x}}{m^2 - k_1^2 - i\varepsilon k_1^0} \\
&\times \int d^4q_2 \hat{g}(q_2) \int d^4k_2 \frac{1}{m^2 - k_2^2 - i\varepsilon k_2^0} \int d^4k \hat{\phi}_m(k) \int\int dy_1 dy_2 e^{-i(q_1-k_1+k_2)y_1} e^{-i(q_2-k_2+k)y_2} + \ldots
\end{aligned}$$

$$
\begin{aligned}
\phi_g(x) = &\int d^4k e^{-ikx}\hat{\phi}_m(k) + \int d^4q_1 g(q_1)\int d^4k_1 \frac{e^{-ik_1x}}{m^2-k_1^2-i\varepsilon k_1^0}\int d^4k\hat{\phi}_m(k)\delta(q_1-k_1+k)\\
&+\int d^4q_1 g(q_1)\int d^4k_1 \frac{e^{-ik_1x}}{m^2-k_1^2-i\varepsilon k_1^0}\\
&\times\int d^4q_2 g(q_2)\int d^4k_2\frac{1}{m^2-k_2^2-i\varepsilon k_2^0}\int d^4k\hat{\phi}_m(k)\delta(q_1-k_1+k_2)\delta(q_2-k_2+k)+\dots
\end{aligned}
$$

For the term first order in g

$$k_1 = k+q_1$$

For the term second order in g

$$
\begin{aligned}
k_1 &= k+q_1+q_2\\
k_2 &= k+q_2
\end{aligned}
$$

$$
\begin{aligned}
\phi_g(x) = &\int d^4k e^{-ikx}\hat{\phi}_m(k) + \int d^4q_1\frac{\hat{g}(q_1)e^{-iq_1x}}{m^2-k_1^2-i\varepsilon k_1^0}\int d^4k e^{-ikx}\hat{\phi}_m(k)\\
&+\int d^4q_1\frac{\hat{g}(q_1)e^{-iq_1x}}{m^2-k_1^2-i\varepsilon k_1^0}\int d^4q_2\frac{\hat{g}(q_2)e^{-iq_2x}}{m^2-k_2^2-i\varepsilon k_2^0}\int d^4k e^{-ikx}\hat{\phi}_m(k)+\dots
\end{aligned}
$$

$$\phi_g(x) = \int d^4k\left[1+\int d^4q_1\frac{\hat{g}(q_1)e^{-iq_1x}}{m^2-k_1^2-i\varepsilon k_1^0}+\int d^4q_1\frac{\hat{g}(q_1)e^{-iq_1x}}{m^2-k_1^2-i\varepsilon k_1^0}\int d^4q_2\frac{\hat{g}(q_2)e^{-iq_2x}}{m^2-k_2^2-i\varepsilon k_2^0}+\dots\right]e^{-ikx}\hat{\phi}_m(k)$$

$$\phi_g(x) = \int d^4k\left[1+\sum_{n=1}^{\infty}\int\cdots\int\prod_{i=1}^{n}\left(dq_i\frac{\hat{g}(q_i)e^{-iq_ix}}{m^2-k_i^2-i\varepsilon k_i^0}\right)\right]e^{-ikx}\hat{\phi}_m(k)\,,\quad \text{where } k_i = k+\sum_{j=i}^{n}q_j$$

For the third order (in g) term

$$
\begin{aligned}
k_1 &= k+q_1+q_2+q_3\\
k_2 &= k+q_2+q_3\\
k_3 &= k+q_3.
\end{aligned}
$$

In the limit $g(x)\to\kappa$, $\hat{g}(q)\to\kappa\delta(q)$, one has $k_i\to k$ and

$$\int\cdots\int\prod_{i=1}^{n}\left(dq_i\frac{\hat{g}(q_i)e^{-iq_ix}}{m^2-k_i^2-i\varepsilon k_i^0}\right)\to\frac{\kappa^n}{(m^2-k^2-i\varepsilon k^0)^n}.$$

Two-point function of the Dyson field $\phi_g(x)$, evaluated in the vacuum state of $\phi_m(x)$,

$$
\begin{aligned}
&\langle\phi_g(x)\phi_g(x')\rangle_m =\\
&\int\int d^4k\,d^4k'\left[\sum_{n=0}^{\infty}\int\cdots\int\prod_{i=1}^{n}\left(dq_i\frac{\hat{g}(q_i)e^{-iq_ix}}{m^2-k_i^2-i\varepsilon k_i^0}\right)\right]\left[\sum_{n'=0}^{\infty}\int\cdots\int\prod_{i=1}^{n'}\left(dq_i'\frac{\hat{g}(-q_i')e^{iq_i'x}}{m^2-k_i'^2+i\varepsilon k_i'^0}\right)\right]e^{-ikx+ik'x'}\\
&\times\langle\hat{\phi}_m(k)\hat{\phi}_m(-k')\rangle_m\\
&\int d\mu_m(k)e^{-ik(x-x')}\left[\sum_{n=0}^{\infty}\int\cdots\int\prod_{i=1}^{n}\left(dq_i\frac{\hat{g}(q_i)e^{-iq_ix}}{m^2-k_i^2-i\varepsilon k_i^0}\right)\right]\left[\sum_{n'=0}^{\infty}\int\cdots\int\prod_{i=1}^{n'}\left(dq_i'\frac{\hat{g}(-q_i')e^{iq_i'x}}{m^2-k_i'^2+i\varepsilon k_i'^0}\right)\right].
\end{aligned}
\tag{4}
$$

The annihilation part of $\phi_g(x)$ participates, and hence the negative sign. The creation part of $\phi_g(x')$ participates, and thus giving the positive sign in the denominator[1]. In the limit, the two-point function becomes

$$\langle \phi_\kappa(x)\phi_\kappa(x')\rangle_m = \int d\mu_m(k) e^{-ik(x-x')} \left[\sum_{n=0}^{\infty} \frac{\kappa^n}{(m^2-k^2-i\varepsilon k^0)^n}\right]\left[\sum_{n'=0}^{\infty} \frac{\kappa^{n'}}{(m^2-k^2+i\varepsilon k^0)^{n'}}\right].$$

The Lorentz invariant measure

$$d\mu_m(k) = \frac{d^4k}{(2\pi)^3}\theta(k^0)\delta(m^2-k^2) = -i\frac{d^4k}{(2\pi)^4}\theta(k^0)\left[\frac{1}{m^2-k^2-i\varepsilon} - \frac{1}{m^2-k^2+i\varepsilon}\right]$$

Writing more, (Why are we able to set $k^0=1$?)

$$\langle \phi_\kappa(x)\phi_\kappa(x')\rangle_m =$$
$$-i\int \frac{d^4k}{(2\pi)^4}\theta(k^0)e^{-ik(x-x')}\left[\sum_{n=0}^{\infty} \frac{\kappa^n}{(m^2-k^2-i\varepsilon)^n}\right]\left[\frac{1}{m^2-k^2-i\varepsilon} - \frac{1}{m^2-k^2+i\varepsilon}\right]\left[\sum_{n'=0}^{\infty} \frac{\kappa^{n'}}{(m^2-k^2+i\varepsilon)^{n'}}\right].$$

Collecting terms order by order

$$\left[\sum_{n=0}^{\infty} \frac{\kappa^n}{(m^2-k^2-i\varepsilon)^n}\right]\left[\frac{1}{m^2-k^2-i\varepsilon} - \frac{1}{m^2-k^2+i\varepsilon}\right]\left[\sum_{n'=0}^{\infty} \frac{\kappa^{n'}}{(m^2-k^2+i\varepsilon)^{n'}}\right] =$$
$$\left[1+\frac{\kappa}{m^2-k^2-i\varepsilon}+\frac{\kappa^2}{(m^2-k^2-i\varepsilon)^2}+\dots\right]\left[\frac{1}{m^2-k^2-i\varepsilon} - \frac{1}{m^2-k^2+i\varepsilon}\right]$$
$$\times\left[1+\frac{\kappa}{m^2-k^2+i\varepsilon}+\frac{\kappa^2}{(m^2-k^2+i\varepsilon)^2}+\dots\right]$$

$$\frac{1}{m^2-k^2\pm i\varepsilon} = \frac{1}{X^\pm}$$

$$\left[1+\frac{\kappa}{X^-}+\frac{\kappa^2}{(X^-)^2}+\dots\right]\left[\frac{1}{X^-}-\frac{1}{X^+}\right]\left[1+\frac{\kappa}{X^+}+\frac{\kappa^2}{(X^+)^2}\dots\right] =$$
$$\left[\frac{1}{X^-}-\frac{1}{X^+}\right] + \left[\cancel{\frac{1}{X^-}}-\frac{1}{X^+}\right]\frac{\kappa}{X^+} + \frac{\kappa}{X^-}\left[\frac{1}{X^-}-\cancel{\frac{1}{X^+}}\right] + \frac{\kappa}{X^-}\left[\cancel{\frac{1}{X^-}}-\cancel{\frac{1}{X^+}}\right]\frac{\kappa}{X^+} + \left[\frac{1}{X^-}-\cancel{\frac{1}{X^+}}\right]\frac{\kappa^2}{(X^-)^2}$$
$$+\frac{\kappa^2}{(X^+)^2}\left[\cancel{\frac{1}{X^-}}-\frac{1}{X^+}\right]+\dots O(\kappa^3)$$

$$\left[1+\frac{\kappa}{X^-}+\frac{\kappa^2}{(X^-)^2}+\dots\right]\left[\frac{1}{X^-}-\frac{1}{X^+}\right]\left[1+\frac{\kappa}{X^+}+\frac{\kappa^2}{(X^+)^2}\dots\right] = \left[\frac{1}{X^-}-\frac{1}{X^+}\right]+\left[\frac{\kappa}{(X^-)^2}-\frac{\kappa}{(X^+)^2}\right]+$$
$$\left[\frac{\kappa^2}{(X^-)^3}-\frac{\kappa^2}{(X^+)^3}\right]+\dots$$

$$\langle \phi_\kappa(x)\phi_\kappa(x')\rangle_m = -i\int \frac{d^4k}{(2\pi)^4}\theta(k^0)e^{-ik(x-x')}\left[\sum_{n=0}^{\infty}\left(\frac{\kappa^n}{(m^2-k^2-i\varepsilon)^{n+1}} - \frac{\kappa^n}{(m^2-k^2+i\varepsilon)^{n+1}}\right)\right]$$

This is just the Taylor expansion of the mass shell Dirac delta about κ. In an sense, we have a formula which is only true order by order in perturbation theory

$$\frac{a}{X}\delta(X)\frac{a}{X} = \delta(X-a).$$

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[1]
$$\hat{\phi}(k) \sim a(k) + a^*(-k)\,, \quad \langle\hat{\phi}(k)\hat{\phi}(-k')\rangle \sim \delta(k-k')$$

$\sim$ because haven't talked about relativistic normalization of creation and annihilation operators. Everything will fit later into the Lorentz invariant measure $\int d\mu_m$.

$$\langle\phi_\kappa(x)\phi_\kappa(x')\rangle_m = \int \frac{d^4k}{(2\pi)^3}\theta(k^0)e^{-ik(x-x')}\left[\sum_{n=0}^{\infty}\frac{(-\kappa\partial_{m^2})^n}{n!}\delta(m^2-k^2)\right]$$
$$= \int \frac{d^4k}{(2\pi)^3}\theta(k^0)e^{-ik(x-x')}\delta(m_\kappa^2-k^2)\,, \quad \text{where } m_\kappa^2 = m^2-\kappa^2 \text{ in the limit.}$$
$$= \int d\mu_{m_\kappa}(k)e^{-ik(x-x')}$$

Therefore, order by order in perturbation theory,

$$\langle\phi_g(x)\phi_g(x')\rangle_m \xrightarrow{g(x)\to\kappa} \langle\phi_\kappa(x)\phi_\kappa(x')\rangle_m = \int d\mu_{m_\kappa}e^{-ik(x-x')} = \langle\phi_{m_\kappa}(x)\phi_{m_\kappa}(x')\rangle_{m_\kappa} \tag{5}$$

What does this imply? (In which state of the field of mass m should the two-point function of the field of mass m_κ (expressed via the Dyson expansion in terms of the field of mass m) be evaluated, so that it converges to the vacuum two-point function of the field of mass m_κ?)

$$\langle \textit{Unknown state of } m|\,\phi_\kappa(x)\phi_\kappa(x')\,|\textit{Unknown state of } m\rangle \longrightarrow \langle\Omega_{m_\kappa}|\,\phi_{m_\kappa}(x)\phi_{m_\kappa}(x')\,|\Omega_{m_\kappa}\rangle \tag{6}$$

The Dyson expansion is (in the limit) a poincare-invariant expression of one field in terms of the other, hence the unknown state of one field, that upon insertion of the Dyson expansion is to produce the invariant state on the other field, should be poincare-invariant as well – and there is only the vacuum with these properties!

By the way, the Nth term in the Taylor expansion of $\int d\mu_{m_\kappa}e^{-ik(x-x')}$,

$$-i\int\frac{d^4k}{(2\pi)^4}\theta(k^0)e^{-ik(x-x')}\left[\frac{\kappa^N}{(m^2-k^2-i\varepsilon)^N}-\frac{\kappa^N}{(m^2-k^2+i\varepsilon)^N}\right] = \frac{(-\kappa\partial_{m^2})^N}{N!}\int d\mu_m(k)e^{-ik(x-x')}.$$

Looking at the step of taking the limit inside the integral,

$$\lim_{g(x)\to\kappa}\int d^4k\left[\sum_{n=0}^{\infty}\int\cdots\int\prod_{i=1}^{n}\left(dq_i\frac{\hat g(q_i)e^{-iq_ix}}{m^2-k_i^2-i\varepsilon k_i^0}\right)\right] \to \int d^4k\left[\sum_{n=0}^{\infty}\frac{\kappa^n}{(m^2-k^2-i\varepsilon k^0)^n}\right].$$

Proof: According to the Lebesgue's Dominated Convergence Theorem (DCT), if $h_g(x)$ is a sequence of functions which converges as

$$\lim_{g\to1}h_g(x) = h(x) \quad \forall x,$$

such that $\int dx\, h(x)$ converges and

$$h(x) \ge |h_g(x)| \quad \forall x, g$$

then

$$\lim_{g\to1}\int dx\, h_g(x) = \int dx\, h(x).$$

With the Dyson expansion

$$\phi_g(x) = \sum_{n=0}^{\infty}\int\cdots\int\prod_{i=1}^{n}\left[dy_i\left(g(y_i)G_m^{ret}(y_{i-1}-y_i)\right)\right]\phi_m(y_n)\,, \quad \text{where } y_0 = x$$

So the problem at hand (?)

$$\lim_{g\to\kappa}\int dy_i\, g(y_i)G_m^{ret}(y_{i-1}-y_i) \overset{?}{=} \int dy_i\,\kappa\, G_m^{ret}(y_{i-1}-y_i)$$

Since the integrand is an integrable quantity ($i\varepsilon$ prescription makes G_m^{ret} integrable), define

$$h_g = g(y_i)G_m^{ret}(y_{i-1} - y_i)$$

$$\lim_{g\to 1} h_g = \kappa G_m^{ret}(y_{i-1} - y_i) = h \quad \forall y_i \in \text{supp(g)}$$

$$\int dy_i\, h(y_i) = \int dy_i\, \kappa\, G_m^{ret}(y_{i-1} - y_i) \text{ converges, and}$$

$$h = \kappa\, G_m^{ret}(y_{i-1} - y_i) \geq |g(y_i)||G_m^{ret}(y_{i-1} - y_i)| = h_g, \quad \text{with } |g(x)| \leq \kappa.$$

Therefore,

$$\lim_{g\to\kappa} \int dy_i\, g(y_i)G_m^{ret}(y_{i-1} - y_i) = \int dy_i\, \kappa\, G_m^{ret}(y_{i-1} - y_i). \qquad \blacksquare$$

One can write out finite two-point functions using 'smeared' field operators

$$\phi(f) = \int dt\, d^3x\, f(t, \mathbf{x}))\phi(t, \mathbf{x})$$

$$\begin{aligned}
\langle \phi_\kappa^2(f)\rangle_m &= \int \frac{d^4k}{(2\pi)^3}\theta(k^0)\left[\sum_{n=0}^{\infty} \frac{(-\kappa\partial_{m^2})^n}{n!}\left(\delta(m^2 - k^2)|\hat{f}(k)|^2\right)\right] \\
&= \int \frac{d^4k}{(2\pi)^3}\theta(k^0)\delta(m_\kappa^2 - k^2)|\hat{f}(k)|^2 = \int d\mu_{m_\kappa}(k)|\hat{f}(k)|^2
\end{aligned}$$

The smearing function f is smooth and decays rapidly.

1.2 2-component

To begin with, consider

$$L_g(x) = g(x)\phi_{1_g}(x)\phi_{2_g}(x)$$

Equations of motion:

$$\begin{aligned}
(\Box + m_1^2)\phi_{1_g}(x) &= g(x)\phi_{2_g}(x), \\
(\Box + m_2^2)\phi_{2_g}(x) &= g(x)\phi_{1_g}(x).
\end{aligned}$$

Dyson expansion for the fields

$$\begin{aligned}
\phi_{1_g}(x) &= \phi_1(x) + \int dy_1 g(y_1)G_{m_1}^{ret}(x - y_1)\phi_{1_g}(y_1) \\
\phi_{2_g}(x) &= \phi_2(x) + \int dy_1 g(y_1)G_{m_2}^{ret}(x - y_1)\phi_{2_g}(y_1)
\end{aligned}$$

$$\phi_{1_g}(x) = \phi_1(x) + \int dy_1 g(y_1)G_{m_1}^{ret}(x - y_1)\phi_2(y_1) + \int\int dy_1 dy_2\, g(y_1)G_{m_1}^{ret}(x - y_1)g(y_2)G_{m_2}^{ret}(y_1 - y_2)\phi_1(y_2)\ldots O(g^3)$$

$$\phi_{2_g}(x) = \phi_2(x) + \int dy_1 g(y_1)G_{m_2}^{ret}(x - y_1)\phi_1(y_1) + \int\int dy_1 dy_2\, g(y_1)G_{m_2}^{ret}(x - y_1)g(y_2)G_{m_1}^{ret}(y_1 - y_2)\phi_2(y_2)\ldots O(g^3)$$

In the Fourier representation,

$$\begin{aligned}
\phi_{1_g}(x) = \int d^4k \Bigg[\left(1 + \int dq_1 \frac{\hat{g}(q_1)e^{-iq_1x}}{m_1^2 - k_1^2 - i\varepsilon k_1^0}\int dq_2 \frac{\hat{g}(q_2)e^{-iq_2x}}{m_2^2 - k_2^2 - i\varepsilon k_2^0} + \ldots O(g^4)\right)\hat{\phi_1}(k)e^{-ikx} + \Bigg(\int dq_1 \frac{\hat{g}(q_1)e^{-iq_1x}}{m_1^2 - k_1^2 - i\varepsilon k_1^0} + \\
\int dq_1 \frac{\hat{g}(q_1)e^{-iq_1x}}{m_1^2 - k_1^2 - i\varepsilon k_1^0}\int dq_2 \frac{\hat{g}(q_2)e^{-iq_2x}}{m_2^2 - k_2^2 - i\varepsilon k_2^0}\int dq_3 \frac{\hat{g}(q_3)e^{-iq_3x}}{m_1^2 - k_3^2 - i\varepsilon k_3^0} + \ldots O(g^5)\Bigg)\hat{\phi}_2(x)e^{-ikx}\Bigg]
\end{aligned}$$

$$\phi_{2_g}(x)=\int d^4k\left[\left(1+\int dq_1\frac{\hat g(q_1)e^{-iq_1x}}{m_2^2-k_1^2-i\varepsilon k_1^0}\int dq_2\frac{\hat g(q_2)e^{-iq_2x}}{m_1^2-k_2^2-i\varepsilon k_2^0}+\dots O(g^4)\right)\hat{\phi_2}(k)e^{-ikx}+\left(\int dq_1\frac{\hat g(q_1)e^{-iq_1x}}{m_2^2-k_1^2-i\varepsilon k_1^0}+\right.\right.$$
$$\left.\left.\int dq_1\frac{\hat g(q_1)e^{-iq_1x}}{m_2^2-k_1^2-i\varepsilon k_1^0}\int dq_2\frac{\hat g(q_2)e^{-iq_2x}}{m_1^2-k_2^2-i\varepsilon k_2^0}\int dq_3\frac{\hat g(q_3)e^{-iq_3x}}{m_2^2-k_3^2-i\varepsilon k_3^0}+\dots O(g^5)\right)\hat\phi_1(x)e^{-ikx}\right].$$

For the ease of typing, define

$$\frac{1}{m_{1,2}^2-k^2\pm i\varepsilon k^0}=\frac{1}{X_{1,2}^{\pm}}$$

and take the limit

$$\lim_{\hat g(q)\to\kappa\delta(q)}\int dq\,\frac{\hat g(q)e^{-iqx}}{X_{1,2}^{\pm}}=\frac{\kappa}{X_{1,2}^{\pm}}.$$

Set them in matrices

$$\varphi_\kappa(x)=\begin{pmatrix}\phi_{1_\kappa}(x)\\ \phi_{2_\kappa}(x)\end{pmatrix}=\int d^4k\,\underbrace{\begin{pmatrix}1+\frac{\kappa}{X_1^-}\frac{\kappa}{X_2^-}+\dots O(\kappa^4) & \frac{\kappa}{X_1^-}+\frac{\kappa}{X_1^-}\frac{\kappa}{X_2^-}\frac{\kappa}{X_1^-}+\dots O(g^5)\\ \frac{\kappa}{X_2^-}+\frac{\kappa}{X_2^-}\frac{\kappa}{X_1^-}\frac{\kappa}{X_2^-}+\dots O(g^5) & 1+\frac{\kappa}{X_2^-}\frac{\kappa}{X_1^-}+\dots O(g^4)\end{pmatrix}}_{L_\kappa}\begin{pmatrix}\hat{\phi}_1(k)e^{-ikx}\\ \hat{\phi}_2(k)e^{-ikx}\end{pmatrix}.$$

The two-point function

$$\lim_{g(x)\to\kappa}\langle\varphi_g(x)\varphi_g(x')\rangle_?=\langle\begin{pmatrix}\phi_{1_\kappa}(x)\\ \phi_{2_\kappa}(x)\end{pmatrix}\begin{pmatrix}\phi_{1_\kappa}(x') & \phi_{2_\kappa}(x')\end{pmatrix}\rangle_?=\int\int d^4k\,d^4k'\,\langle L_\kappa\begin{pmatrix}\hat\phi_1(k)\\ \hat\phi_2(k)\end{pmatrix}\begin{pmatrix}\hat\phi_1(-k') & \hat\phi_2(-k')\end{pmatrix}L_\kappa^\dagger\rangle_?e^{-ikx+ik'x'}$$

$$\int\int d^4k\,d^4k'\langle L_\kappa\begin{pmatrix}\hat\phi_1(k)\\ \hat\phi_2(k)\end{pmatrix}\begin{pmatrix}\hat\phi_1(-k') & \hat\phi_2(-k')\end{pmatrix}L_\kappa^\dagger\rangle_?e^{-ikx+ik'x'}=\int\int d^4k\,d^4k'$$
$$\langle\begin{pmatrix}1+\frac{\kappa}{X_1^-}\frac{\kappa}{X_2^-}+\dots O(\kappa^4) & \frac{\kappa}{X_1^-}+\frac{\kappa}{X_1^-}\frac{\kappa}{X_2^-}\frac{\kappa}{X_1^-}+\dots O(\kappa^5)\\ \frac{\kappa}{X_2^-}+\frac{\kappa}{X_2^-}\frac{\kappa}{X_1^-}\frac{\kappa}{X_2^-}+\dots O(\kappa^5) & 1+\frac{\kappa}{X_2^-}\frac{\kappa}{X_1^-}+\dots O(\kappa^4)\end{pmatrix}\begin{pmatrix}\hat\phi_1(k)\hat\phi_1(-k') & \hat\phi_1(k)\hat\phi_2(-k')\\ \hat\phi_2(k)\hat\phi_1(-k') & \hat\phi_2(k)\hat\phi_2(-k')\end{pmatrix}$$
$$\times\underbrace{\begin{pmatrix}1+\frac{\kappa}{X_1^+}\frac{\kappa}{X_2^+}+\dots O(\kappa^4) & \frac{\kappa}{X_2^+}+\frac{\kappa}{X_2^+}\frac{\kappa}{X_1^+}\frac{\kappa}{X_2^+}+\dots O(\kappa^5)\\ \frac{\kappa}{X_1^+}+\frac{\kappa}{X_1^+}\frac{\kappa}{X_2^+}\frac{\kappa}{X_1^+}+\dots O(\kappa^5) & 1+\frac{\kappa}{X_2^+}\frac{\kappa}{X_1^+}+\dots O(\kappa^4)\end{pmatrix}}_{\textbf{At this step, these X's have k' inside them.}}\rangle_?e^{-ikx+ik'x'}$$

Writing

$$L_\kappa=\begin{pmatrix}A & B\\ C & D\end{pmatrix}$$

$$\int\int d^4k\,d^4k'\langle L_\kappa\begin{pmatrix}\hat\phi_1(k)\\ \hat\phi_2(k)\end{pmatrix}\begin{pmatrix}\hat\phi_1(-k') & \hat\phi_2(-k')\end{pmatrix}L_\kappa^\dagger\rangle_?e^{-ikx+ik'x')}=$$
$$\begin{pmatrix}\int d\mu_{m_1}\left(AA^T\right)+\int d\mu_{m_2}\left(BB^T\right) & \int d\mu_{m_1}\left(AC^T\right)+\int d\mu_{m_2}\left(BD^T\right)\\ \int d\mu_{m_1}\left(CA^T\right)+\int d\mu_{m_2}\left(DB^T\right) & \int d\mu_{m_1}\left(CC^T\right)+\int d\mu_{m_2}\left(DD^T\right)\end{pmatrix}_\star e^{-ik(x-x')}$$

Recalling the formula (which makes sense only order by order in perturbation theory)

$$\frac{a}{X}\delta(X)\frac{a}{X}=\delta(X-a),$$

$$A\delta(m_1^2-k^2)A^T=\left(1+\frac{\kappa}{X_1^-}\frac{\kappa}{X_2^-}+\dots O(\kappa^4)\right)\left[\frac{1}{X_1^-}-\frac{1}{X_1^+}\right]\left(1+\frac{\kappa}{X_1^+}\frac{\kappa}{X_2^+}+\dots O(\kappa^4)\right)$$

$$B\delta(m_2^2-k^2)B^T=\left(\frac{\kappa}{X_1^-}+\frac{\kappa}{X_1^-}\frac{\kappa}{X_2^-}\frac{\kappa}{X_1^-}+\dots O(\kappa^5)\right)\left[\frac{1}{X_2^-}-\frac{1}{X_2^+}\right]\left(\frac{\kappa}{X_1^+}+\frac{\kappa}{X_1^+}\frac{\kappa}{X_2^+}\frac{\kappa}{X_1^+}+\dots O(\kappa^5)\right)$$

$$C\delta(m_1^2-k^2)C^T = \left(\frac{\kappa}{X_2^-}+\frac{\kappa}{X_2^-}\frac{\kappa}{X_1^-}\frac{\kappa}{X_2^-}+\ldots O(\kappa^5)\right)\left[\frac{1}{X_1^-}-\frac{1}{X_1^+}\right]\left(\frac{\kappa}{X_2^+}+\frac{\kappa}{X_2^+}\frac{\kappa}{X_1^+}\frac{\kappa}{X_2^+}+\ldots O(\kappa^5)\right)$$

$$D\delta(m_2^2-k^2)D^T = \left(1+\frac{\kappa}{X_2^-}\frac{\kappa}{X_1^-}+\ldots O(\kappa^4)\right)\left[\frac{1}{X_2^-}-\frac{1}{X_2^+}\right]\left(1+\frac{\kappa}{X_2^+}\frac{\kappa}{X_1^+}+\ldots O(\kappa^4)\right)$$

$$A\delta(m_1^2-k^2)C^T = \left(1+\frac{\kappa}{X_1^-}\frac{\kappa}{X_2^-}+\ldots O(\kappa^4)\right)\left[\frac{1}{X_1^-}-\frac{1}{X_1^+}\right]\left(\frac{\kappa}{X_2^+}+\frac{\kappa}{X_2^+}\frac{\kappa}{X_1^+}\frac{\kappa}{X_2^+}+\ldots O(\kappa^5)\right)$$

$$B\delta(m_2^2-k^2)D^T = \left(\frac{\kappa}{X_1^-}+\frac{\kappa}{X_1^-}\frac{\kappa}{X_2^-}\frac{\kappa}{X_1^-}+\ldots O(\kappa^5)\right)\left[\frac{1}{X_2^-}-\frac{1}{X_2^+}\right]\left(1+\frac{\kappa}{X_2^+}\frac{\kappa}{X_1^+}+\ldots O(\kappa^4)\right)$$

$$C\delta(m_1^2-k^2)A^T = \left(\frac{\kappa}{X_2^-}+\frac{\kappa}{X_2^-}\frac{\kappa}{X_1^-}\frac{\kappa}{X_2^-}+\ldots O(\kappa^5)\right)\left[\frac{1}{X_1^-}-\frac{1}{X_1^+}\right]\left(1+\frac{\kappa}{X_1^+}\frac{\kappa}{X_2^+}+\ldots O(\kappa^4)\right)$$

$$D\delta(m_2^2-k^2)B^T = \left(1+\frac{\kappa}{X_2^-}\frac{\kappa}{X_1^-}+\ldots O(\kappa^4)\right)\left[\frac{1}{X_2^-}-\frac{1}{X_2^+}\right]\left(\frac{\kappa}{X_1^+}+\frac{\kappa}{X_1^+}\frac{\kappa}{X_2^+}\frac{\kappa}{X_1^+}+\ldots O(\kappa^5)\right)$$

The **first entry** of the matrix:

$$\begin{aligned}
&\kappa^0: \left(\frac{1}{X_1^-}-\frac{1}{X_1^+}\right)\\
&\kappa^2: \frac{\kappa^2}{X_1^-X_2^-}\left(\frac{1}{X_1^-}-\cancel{\frac{1}{X_1^+}}\right)+\left(\cancel{\frac{1}{X_1^-}}-\frac{1}{X_1^+}\right)\frac{\kappa^2}{X_1^+X_2^+}+\frac{\kappa}{X_1^-}\left(\cancel{\frac{1}{X_2^-}}-\cancel{\frac{1}{X_2^+}}\right)\frac{\kappa}{X_1^+}\\
&\to \quad \left(\frac{1}{X_1^-}-\frac{1}{X_1^+}\right)+\kappa^2\left(\frac{1}{X_1^-X_2^-X_1^-}-\frac{1}{X_1^+X_2^+X_1^+}\right)+O(\kappa^4)
\end{aligned}$$

The **second entry** of the matrix:

$$\begin{aligned}
&\kappa: \left(\cancel{\frac{1}{X_1^-}}-\frac{1}{X_1^+}\right)\frac{\kappa}{X_2^+}+\frac{\kappa}{X_1^-}\left(\frac{1}{X_2^-}-\cancel{\frac{1}{X_2^+}}\right)\\
&\kappa^3: \frac{\kappa^2}{X_1^-X_2^-}\left(\cancel{\frac{1}{X_1^-}}-\cancel{\frac{1}{X_1^+}}\right)\frac{\kappa}{X_2^+}+\left(\cancel{\frac{1}{X_1^-}}-\frac{1}{X_1^+}\right)\frac{\kappa^3}{X_2^+X_1^+X_2^+}+\frac{\kappa}{X_1^-}\left(\cancel{\frac{1}{X_2^-}}-\cancel{\frac{1}{X_2^+}}\right)\frac{\kappa^2}{X_2^+X_1^+}+\frac{\kappa^3}{X_1^-X_2^-X_1^-}\left(\frac{1}{X_1^-}-\cancel{\frac{1}{X_1^+}}\right)\\
&\to \quad \kappa\left(\frac{1}{X_1^-X_2^-}-\frac{1}{X_1^+X_2^+}\right)+\kappa^3\left(\frac{1}{X_1^-X_2^-X_1^-X_2^-}-\frac{1}{X_1^+X_2^+X_1^+X_2^+}\right)+O(\kappa^5)
\end{aligned}$$

Similarly, the **third entry**

$$\kappa\left(\frac{1}{X_2^-X_1^-}-\frac{1}{X_2^+X_1^+}\right)+\kappa^3\left(\frac{1}{X_2^-X_1^-X_2^-X_1^-}-\frac{1}{X_2^+X_1^+X_2^+X_1^+}\right)+O(\kappa^5),$$

and the **fourth entry**

$$\left(\frac{1}{X_2^-}-\frac{1}{X_2^+}\right)+\kappa^2\left(\frac{1}{X_2^-X_1^-X_2^-}-\frac{1}{X_2^+X_1^+X_2^+}\right)+O(\kappa^4).$$

Setting everything in matrix

$$\langle\varphi_\kappa(x)\varphi_\kappa(x')\rangle_? = -i\int\frac{d^4k}{(2\pi)^4}\theta(k^0)\begin{pmatrix} s & t\\ u & v\end{pmatrix}e^{-ik(x-x')}$$

$$\begin{aligned}
s &= \left(\frac{1}{X_1^-}-\frac{1}{X_1^+}\right)+\kappa^2\left(\frac{1}{X_1^-X_2^-X_1^-}-\frac{1}{X_1^+X_2^+X_1^+}\right)+O(\kappa^4)\\
t &= \kappa\left(\frac{1}{X_1^-X_2^-}-\frac{1}{X_1^+X_2^+}\right)+\kappa^3\left(\frac{1}{X_1^-X_2^-X_1^-X_2^-}-\frac{1}{X_1^+X_2^+X_1^+X_2^+}\right)+O(\kappa^5)\\
u &= \kappa\left(\frac{1}{X_2^-X_1^-}-\frac{1}{X_2^+X_1^+}\right)+\kappa^3\left(\frac{1}{X_2^-X_1^-X_2^-X_1^-}-\frac{1}{X_2^+X_1^+X_2^+X_1^+}\right)+O(\kappa^5)\\
v &= \left(\frac{1}{X_2^-}-\frac{1}{X_2^+}\right)+\kappa^2\left(\frac{1}{X_2^-X_1^-X_2^-}-\frac{1}{X_2^+X_1^+X_2^+}\right)+O(\kappa^4)
\end{aligned}$$

The trace s+v matches with the result below $\delta A+\delta D$.

Well, one can form geometric progressions in the expressions for s, t, u, and v. (If we don't worry about series convergence and the ratio of terms being ≤ 1.)

$$
\begin{aligned}
s &= \frac{X_2^-}{X_2^- X_1^- - \kappa^2} - \frac{X_2^+}{X_2^+ X_1^+ - \kappa^2} \\
t &= \kappa \left(\frac{1}{X_1^- X_2^- - \kappa^2} - \frac{1}{X_1^+ X_2^+ - \kappa^2} \right) \\
u &= \kappa \left(\frac{1}{X_2^- X_1^- - \kappa^2} - \frac{1}{X_2^+ X_1^+ - \kappa^2} \right) \\
t &= \frac{X_1^-}{X_1^- X_2^- - \kappa^2} - \frac{X_1^+}{X_1^+ X_2^+ - \kappa^2}
\end{aligned}
$$

The matrix $\begin{pmatrix} s & t \\ u & v \end{pmatrix}$ is symmetric, and for an ensured non-zero determinant, this matrix can be diagonalized. The eigenvalues read

$$
\lambda_{1/2} = \frac{1}{2} \left[\frac{1}{X_1^- X_2^- - \kappa^2} \left(X_1^- + X_2^- \pm \sqrt{(m_2^2 - m_1^2) - 4\kappa^2} \right) - \frac{1}{X_1^+ X_2^+ - \kappa^2} \left(X_1^+ + X_2^+ \pm \sqrt{(m_2^2 - m_1^2) - 4\kappa^2} \right) \right]
$$

Diagonalizing the mass matrix:

$$
\begin{pmatrix} c_\kappa & -s_\kappa \\ s_\kappa & c_\kappa \end{pmatrix} \underbrace{\begin{pmatrix} m_1^2 & -\kappa \\ -\kappa & m_2^2 \end{pmatrix}}_{M^2} \underbrace{\begin{pmatrix} c_\kappa & s_\kappa \\ -s_\kappa & c_\kappa \end{pmatrix}}_{A_\kappa} = \underbrace{\begin{pmatrix} \frac{m_1^2+m_2^2}{2} + \frac{m_1^2-m_2^2}{2} \sqrt{1 + \frac{4\kappa}{(m_1^2-m_2^2)^2}} & 0 \\ 0 & \frac{m_1^2+m_2^2}{2} - \frac{m_1^2-m_2^2}{2} \sqrt{1 + \frac{4\kappa}{(m_1^2-m_2^2)^2}} \end{pmatrix}}_{M_D^2}
$$

$$
\begin{aligned}
c_\kappa^2 m_1^2 + 2c_\kappa s_\kappa \kappa + s_\kappa^2 m_2^2 &= \frac{m_1^2 + m_2^2}{2} + \frac{m_1^2 - m_2^2}{2} \sqrt{1 + \frac{4\kappa}{(m_1^2 - m_2^2)^2}}, \\
c_\kappa s_\kappa (m_1^2 - m_2^2) &= \kappa (c_\kappa^2 - s_\kappa^2), \\
s_\kappa^2 m_1^2 - 2c_\kappa s_\kappa \kappa + c_\kappa^2 m_2^2 &= \frac{m_1^2 + m_2^2}{2} - \frac{m_1^2 - m_2^2}{2} \sqrt{1 + \frac{4\kappa}{(m_1^2 - m_2^2)^2}}
\end{aligned}
$$

Solving gives

$$
\begin{aligned}
c_\kappa^2 &= \frac{1}{2} \left(1 + \frac{1}{\sqrt{1 + \frac{4\kappa^2}{(m_1^2 - m_2^2)^2}}} \right) \\
s_\kappa^2 &= \frac{1}{2} \left(1 - \frac{1}{\sqrt{1 + \frac{4\kappa^2}{(m_1^2 - m_2^2)^2}}} \right)
\end{aligned}
$$

Realizing that

$$
\varphi_\kappa = A_\kappa \varphi_\kappa^D, \quad \text{D for diagonalized,}
$$

we write the two point function

$$
\begin{aligned}
\langle \varphi_\kappa(x) \varphi_\kappa(x') \rangle_M &= A_\kappa \langle \varphi_\kappa^D(x) \varphi_\kappa^D(x') \rangle_{M_D} A_\kappa^T \\
&= -i \int \frac{d^4k}{(2\pi)^4} \theta(k^0) e^{-ik(x-x')} \begin{pmatrix} c_\kappa & s_\kappa \\ -s_\kappa & c_\kappa \end{pmatrix} \begin{pmatrix} \frac{1}{X_1^-} - \frac{1}{X_1^+} & 0 \\ 0 & \frac{1}{X_2^-} - \frac{1}{X_2^+} \end{pmatrix} \begin{pmatrix} c_\kappa & -s_\kappa \\ s_\kappa & c_\kappa \end{pmatrix}
\end{aligned}
$$

$$\begin{pmatrix} s & t \\ u & v \end{pmatrix} = \begin{pmatrix} c_\kappa & s_\kappa \\ -s_\kappa & c_\kappa \end{pmatrix} \begin{pmatrix} \frac{1}{X_1^-} - \frac{1}{X_1^+} & 0 \\ 0 & \frac{1}{X_2^-} - \frac{1}{X_2^+} \end{pmatrix} \begin{pmatrix} c_\kappa & -s_\kappa \\ s_\kappa & c_\kappa \end{pmatrix}$$

$$\begin{pmatrix} s & t \\ u & v \end{pmatrix} = \begin{pmatrix} c_\kappa \left(\frac{1}{X_1^-} - \frac{1}{X_1^+}\right) c_\kappa + s_\kappa \left(\frac{1}{X_2^-} - \frac{1}{X_2^+}\right) s_\kappa & -c_\kappa \left(\frac{1}{X_1^-} - \frac{1}{X_1^+}\right) s_\kappa + s_\kappa \left(\frac{1}{X_2^-} - \frac{1}{X_2^+}\right) c_\kappa \\ -s_\kappa \left(\frac{1}{X_1^-} - \frac{1}{X_1^+}\right) c_\kappa + c_\kappa \left(\frac{1}{X_2^-} - \frac{1}{X_2^+}\right) s_\kappa & s_\kappa \left(\frac{1}{X_1^-} - \frac{1}{X_1^+}\right) s_\kappa + c_\kappa \left(\frac{1}{X_2^-} - \frac{1}{X_2^+}\right) c_\kappa \end{pmatrix}$$

From previous calculation,

$$\begin{aligned}
s &= \left(\frac{1}{X_1^-} - \frac{1}{X_1^+}\right) + \kappa^2 \left(\frac{1}{X_1^- X_2^- X_1^-} - \frac{1}{X_1^+ X_2^+ X_1^+}\right) + O(\kappa^4) \\
t &= \kappa \left(\frac{1}{X_1^- X_2^-} - \frac{1}{X_1^+ X_2^+}\right) + \kappa^3 \left(\frac{1}{X_1^- X_2^- X_1^- X_2^-} - \frac{1}{X_1^+ X_2^+ X_1^+ X_2^+}\right) + O(\kappa^5) \\
u &= \kappa \left(\frac{1}{X_2^- X_1^-} - \frac{1}{X_2^+ X_1^+}\right) + \kappa^3 \left(\frac{1}{X_2^- X_1^- X_2^- X_1^-} - \frac{1}{X_2^+ X_1^+ X_2^+ X_1^+}\right) + O(\kappa^5) \\
v &= \left(\frac{1}{X_2^-} - \frac{1}{X_2^+}\right) + \kappa^2 \left(\frac{1}{X_2^- X_1^- X_2^-} - \frac{1}{X_2^+ X_1^+ X_2^+}\right) + O(\kappa^4)
\end{aligned}$$

Now one can check the closed form order by order (using the expressions for c_κ and s_κ obtained above). For s

$$\kappa^0 : \left(\frac{1}{X_1^-} - \frac{1}{X_1^+}\right)$$

$$c_\kappa \left(\frac{1}{X_1^-} - \frac{1}{X_1^+}\right) c_\kappa + s_\kappa \left(\frac{1}{X_2^-} - \frac{1}{X_2^+}\right) s_\kappa \sim \left(\frac{1}{X_1^-} - \frac{1}{X_1^+}\right)$$

$$\begin{aligned}
\kappa^2 : &\left(\frac{1}{X_1^- X_2^- X_1^-} - \frac{1}{X_1^+ X_2^+ X_1^+}\right) = \left(\frac{1}{X_1^- X_2^-} - \frac{1}{X_1^+ X_2^+}\right)\left(\frac{1}{X_1^+} + \frac{1}{X_1^+}\right) - \frac{1}{X_1^- X_1^+}\left(\frac{1}{X_2^-} - \frac{1}{X_2^+}\right) \\
&= \left[\frac{1}{2}\left(\frac{1}{X_2^-} + \frac{1}{X_2^+}\right)\left(\frac{1}{X_1^-} - \frac{1}{X_1^+}\right)\left(\frac{1}{X_1^-} + \frac{1}{X_1^+}\right) + \frac{1}{2}\left(\frac{1}{X_1^-} + \frac{1}{X_1^+}\right)^2\left(\frac{1}{X_2^-} - \frac{1}{X_2^+}\right)\right] - \frac{1}{X_1^- X_1^+}\left(\frac{1}{X_2^-} - \frac{1}{X_2^+}\right) \\
&= \underbrace{\left(\frac{1}{X_2^-} - \frac{1}{X_2^+}\right)\left[\frac{1}{2}\left(\frac{1}{X_1^-} + \frac{1}{X_1^+}\right)^2 - \frac{1}{X_1^- X_1^+}\right]}_{\text{This term is fine}} + \underbrace{\frac{1}{2}\left(\frac{1}{X_1^-} - \frac{1}{X_1^+}\right)\left[\left(\frac{1}{X_1^-} + \frac{1}{X_1^+}\right)\left(\frac{1}{X_2^-} + \frac{1}{X_2^+}\right)\right]}_{?} \\
&\stackrel{?}{=} \frac{\kappa^2}{(m_1^2 - m_2^2)^2}\left[\frac{1}{X_2^-} - \frac{1}{X_2^+} - \left(\frac{1}{X_1^-} - \frac{1}{X_1^+}\right)\right]
\end{aligned}$$

$$c_\kappa \left(\frac{1}{X_1^-} - \frac{1}{X_1^+}\right) c_\kappa + s_\kappa \left(\frac{1}{X_2^-} - \frac{1}{X_2^+}\right) s_\kappa \sim \frac{\kappa^2}{(m_1^2 - m_2^2)^2}\left[\frac{1}{X_2^-} - \frac{1}{X_2^+} - \left(\frac{1}{X_1^-} - \frac{1}{X_1^+}\right)\right]$$

For u,

$$\begin{aligned}
\kappa : \left(\frac{1}{X_2^- X_1^-} - \frac{1}{X_2^+ X_1^+}\right) &= \frac{1}{2}\left(\frac{1}{X_2^-} + \frac{1}{X_2^+}\right)\left(\underbrace{\frac{1}{X_1^-} - \frac{1}{X_1^+}}_{\sim\delta(m_1^2 - k^2)}\right) + \frac{1}{2}\left(\frac{1}{X_1^-} + \frac{1}{X_1^+}\right)\left(\underbrace{\frac{1}{X_2^-} - \frac{1}{X_2^+}}_{\sim\delta(m_2^2 - k^2)}\right) \\
&= \frac{1}{m_1^2 - m_2^2}\left[\frac{1}{X_2^-} - \frac{1}{X_2^+} - \left(\frac{1}{X_1^-} - \frac{1}{X_1^+}\right)\right]
\end{aligned}$$

$$-s_\kappa \left(\frac{1}{X_1^-} - \frac{1}{X_1^+}\right) c_\kappa + c_\kappa \left(\frac{1}{X_2^-} - \frac{1}{X_2^+}\right) s_\kappa \sim \frac{\kappa}{m_1^2 - m_2^2}\left[\frac{1}{X_2^-} - \frac{1}{X_2^+} - \left(\frac{1}{X_1^-} - \frac{1}{X_1^+}\right)\right]$$

Test of convergence: Write $\phi_g(x) = \phi(x) + \underbrace{\int dy R(x-y)g(y)\phi(y)}_{(R(g\phi))(x)} + \ldots$ Then

$$\phi_g(f) = \int dx f(x)\phi(x) + \int dy g(y) \underbrace{\int dx R(x-y) f(x)}_{=:(R^t f)(y)} \phi(y) = \phi(\underbrace{f + g\cdot R^t f + g \cdot R^t \left(g\cdot R^t f\right) + \ldots}_{F_g})$$

First order:

$$\left\langle \phi_g(\bar{f})\phi_g(f)\right\rangle^{(1)} = \left\langle \phi(\bar{f})\phi\left(gR^t f\right)\right\rangle + \left\langle \phi\left(g\overline{R^t f}\right)\phi(f)\right\rangle = \int d\mu_m(k)\left(\overline{\hat{f}(k)}\widehat{gR^t f}(k) + \overline{\widehat{gR^t f}(k)}\hat{f}(k)\right)$$

- Compute $\widehat{gR^t}f(k)$ as a convolution integral over $f(k')$
- Write out $d\mu_m(k)$ and split the δ-fn $\delta\left(m^2-k^2\right)$.
- See how the formal cancellations that we have made with $g(y) = \kappa$ are only approximative cancellations when g is not constant.

$$\phi_g(f) = \int d^4k \hat{f}(-k)\hat{\phi}(k) + \int d^4k \int dq_1 \frac{\hat{g}(-q_1)\hat{f}(-k+q_1)}{m^2 - k_1^2 + i\varepsilon k_1^0}\hat{\phi}(k) + \ldots \quad (k_1 = k - q_1)$$

$$\phi_g(\bar{f}) = \int d^4k \overline{\hat{f}(-k)}\hat{\phi}(k) + \int d^4k \int dq_1 \frac{\hat{g}(q_1)\overline{\hat{f}(-k-q_1)}}{m^2 - k_1^2 - i\varepsilon k_1^0}\hat{\phi}(k) + \ldots \quad (k_1 = k + q_1)$$

$$\langle\phi_g(\bar{f})\phi_g(f)\rangle^{(1)} = \int d\mu_m(k)\overline{\hat{f}(-k)}\hat{f}(k) + \int d\mu_m(k)\overline{\hat{f}(-k)}\int dq_1 \frac{\hat{g}(-q_1)\hat{f}(k+q_1)}{m^2-k_1^2+i\varepsilon k_1^0} + \int d\mu_m(k)\int dq_1 \frac{\hat{g}(q_1)\overline{\hat{f}(-k-q_1)}}{m^2-k_1^2-i\varepsilon k_1^0}\hat{f}(k) + \ldots O(g^2)$$

Using convolution theorem,

$$= -i\int \frac{d^4k}{(2\pi)^4}\theta(k^0)\left(\overline{\hat{f}(-k)}\hat{f}(k)\left(\frac{1}{X_k^-} - \frac{1}{X_k^+}\right) + \int dq_1 \hat{g}(q_1)\overline{\hat{f}(-k-q_1)}\hat{f}(k)\left[\left(\frac{1}{X_k^-} - \frac{1}{X_k^+}\right)\frac{1}{X_k^+} + \frac{1}{X_{k_1}^-}\left(\frac{1}{X_k^-} - \frac{1}{X_k^+}\right)\right]\right]$$

where $\frac{1}{X_k^\pm} = \frac{1}{m^2-k^2\pm i\varepsilon k^0}$.

The cancellations are not exact this time due to the presence of the k_1, which in the limit is k. Looking at the 'would be' cancelled terms

$$\begin{aligned}
\frac{1}{X_k^-}\frac{1}{X_k^+} - \frac{1}{X_{k_1}^-}\frac{1}{X_k^+} &= \frac{1}{(m^2-k^2-i\varepsilon)(m^2-k^2+i\varepsilon)} - \underbrace{\frac{1}{(m^2-(k+q_1)^2-i\varepsilon)}}_{\text{obvious from here as well}}\frac{1}{(m^2-k^2+i\varepsilon)} \\
&= \left[\frac{k^2-(k+q_1)^2}{(m^2-k^2-i\varepsilon)(m^2-(k+q_1)^2-i\varepsilon)}\right]\frac{1}{m^2-k^2+i\varepsilon} \\
&= \frac{-q_1(2k+q_1)}{(m^2-k^2-i\varepsilon)(m^2-(k+q_1)^2-i\varepsilon)}\cdot\frac{1}{m^2-k^2+i\varepsilon}
\end{aligned}$$

In the limit as $g \to \delta$, q's $\to 0$ and term above is tending to 0, and hence towards cancellation. The 'would be' true terms,

$$\begin{aligned}
\frac{1}{X_{k_1}^-}\frac{1}{X_k^-} - \frac{1}{X_k^+}\frac{1}{X_k^+} &= \frac{1}{(m^2-(k+q_1)^2-i\varepsilon)}\frac{1}{(m^2-k^2-i\varepsilon)} - \frac{1}{(m^2-k^2+i\varepsilon)}\frac{1}{(m^2-k^2+i\varepsilon)} \\
&= \frac{m^2-k^2-i\varepsilon}{(m^2-(k+q_1)^2-i\varepsilon)}\frac{1}{(m^2-k^2-i\varepsilon)}\frac{1}{(m^2-k^2-i\varepsilon)} - \frac{1}{(m^2-k^2+i\varepsilon)}\frac{1}{(m^2-k^2+i\varepsilon)} \\
&= \left[\frac{(k+q_1)^2-k^2}{(m^2-(k+q_1)^2-i\varepsilon)} + 1\right]\frac{1}{(m^2-k^2-i\varepsilon)}\frac{1}{(m^2-k^2-i\varepsilon)} - \frac{1}{(m^2-k^2+i\varepsilon)}\frac{1}{(m^2-k^2+i\varepsilon)} \\
&= \frac{q_1(2k+q_1)}{(m^2-(k+q_1)^2-i\varepsilon)}\frac{1}{(m^2-k^2-i\varepsilon)}\frac{1}{(m^2-k^2-i\varepsilon)} + \underbrace{\frac{1}{X^-}\frac{1}{X^-} - \frac{1}{X^+}\frac{1}{X^+}}_{\text{True terms}}
\end{aligned}$$

Again, in the limit $g \to \delta$, q's $\to 0$, and therefore the expression above tend towards the true expression found before where the limit was taken inside the integral first and then the cancellations were observed. Here we observe how terms actually tend towards cancellations and true terms, without taking the limit first.

But, but, but....I am getting the true terms already, without even having to take the limit, because the term $\frac{-q_1(2k+q_1)}{(m^2-k^2-i\varepsilon)(m^2-(k+q_1)^2-i\varepsilon)} \cdot \frac{1}{m^2-k^2+i\varepsilon}$ appears in both the expressions with opposite signs, and hence cancel on addition.
Therefore,

$$
\begin{aligned}
&\langle \phi_g(\bar{f})\phi_g(f)\rangle^{(1)} = \\
&-i \int \frac{d^4k}{(2\pi)^4}\theta(k^0)\left(\overline{\hat{f}(-k)}\hat{f}(k)\left(\frac{1}{X_k^-} - \frac{1}{X_k^+}\right) + \int dq_1 \hat{g}(q_1)\overline{\hat{f}(-k-q_1)}\hat{f}(k)\left[\frac{1}{X^-X^-} - \frac{1}{X^+X^+}\right]\right)
\end{aligned}
$$

Exercise: Look at

$$\int f_1(x_1)f_2(x_2)g_\epsilon(x_1-x_2)$$

for $\epsilon \to 0$ and f's being smooth functions.

$$\int\int dq_1 e^{iq_1x_1}\hat{f}_1(q_1)\int dq_2\, e^{iq_2x_2}\hat{f}_2(q_2)\int dk\, e^{ik(x_1-x_2)}\left[2\pi\left(\frac{1}{k+i\epsilon}-\frac{1}{k-i\epsilon}\right)\right]$$
$$\int dk\,\phi(k)\left(\frac{1}{k+i\epsilon}-\frac{1}{k-i\epsilon}\right), \quad \phi(k)=2\pi\hat{f}_1(-k)\hat{f}_2(k)$$
$$\lim_{\epsilon\to 0}\int dk\,\phi(k)\left(\frac{1}{k+i\epsilon}-\frac{1}{k-i\epsilon}\right)$$

$$\frac{1}{k\mp i\epsilon}=P\frac{1}{k}\pm i\pi\underbrace{\frac{\epsilon}{\pi(k^2+\epsilon^2)}}_{\delta_\epsilon(k)}, \quad P\frac{1}{k}=\frac{E}{E^2+\epsilon^2}$$

$$\lim_{\epsilon\to 0}\int dk\,\phi(k)\left(\frac{1}{k+i\epsilon}-\frac{1}{k-i\epsilon}\right)=\lim_{\epsilon\to 0}\int dk\,\phi(k)(-2i\pi\delta_\epsilon(k))\stackrel{?}{=}-2i\pi\phi(0)$$

The principle value function P/k avoids k=0 pole.

$$\delta_\epsilon(k)=\frac{\epsilon}{\pi(k^2+\epsilon^2)}$$

Let

$$\phi(k)=e^{-k^2}$$

$$F(k)=\phi(k).\delta_\epsilon(k)=e^{-k^2}.\frac{\epsilon}{\pi(k^2+\epsilon^2)}$$

Doubts and things to do:

- All scalar fields of different masses have the same vacuum state (by argument of Poincare invariance)?
- What happens if the poles are in the compact support of g?
- Can we prove the moving of limit inside the integral in real space and claim it for Fourier space?
- Generalize it?
- What kind of vacuum state do we have in the 2-component case?
- Write about delta formula which makes sense only order by order in Perturbation theory.
- Write out the smeared two-point functions?
- $\tilde{\phi}_\kappa = A_\kappa \phi_\kappa$, prove this till 2nd/3rd order in κ by diagonalizing the mass matrix using the matrix A_κ.
- The question of convergence.