Unit Test 1: Probability and Statistics
2024-2025 (Odd Semester): Department of Computer Science and Engineering

1. A consignment of 15 record players contains 4 defectives. The record players are selected at random, one by one, and examined. Those examined are not put back. What is the probability that the 9^{th} one examined is the last defective?

2. Die A has four red and two white faces whereas die B has two red and four white faces. A coin is flipped once. If it falls heads the game continues by throwing die A, if it falls tails die B is to be used.
   a) Show that the probability of getting a red face at any throw is $\frac{1}{2}$.
   b) If the first two throws resulted in red faces, What is the probability of red face at the 3^{rd} throw?
   c) If red face turns up at the first n throws, what is the probability that die A is being used?

3. A random variable X has the following probability function:

Value of $X = x_i$	1	2	3	4	5	6
$p(x_i)$	k	$3k$	$5k$	$7k$	$9k$	$11k$

   a) Find k,
   b) Evaluate $P(1 \leq X \leq 5)$,
   c) If $P(X > 3)$.

4. Suppose that X is a continuous random variable has probability density function given by:

$$f(x) = \begin{cases} C\,(4x - 2x^2), & 0 < x < 2, \\ 0, & elsewhere. \end{cases}$$

   a) What is the value of C?
   b) Find $P(X > 1)$.

5. Suppose that two-dimensional continuous random variable (X,Y) has joint probability density function given by:

$$f(x) = \frac{9\,(1 + x + y)}{2\,(1 + x)^4\,(1 + y)^4}; 0 \leq x < \infty, 0 \leq y < \infty.$$

   Find the marginal distribution of X and Y, and the conditional distribution of Y for $X = x$.

Example 3·66. *A consignment of 15 record players contains 4 defectives. The record players are selected at random, one by one, and examined. Those examined are not put back. What is the probability that the 9th one examined is the last defective ?*

Solution. Let A be the event of getting exactly 3 defectives in examination of 8 record players and let B denote the event that the 9th piece examined is a defective one.

Since it is a problem of sampling without replacement and since there are 4 defectives (and 11 non-defectives) out of 15 record players, $P(A) = {}^4C_3 \times {}^{11}C_5 / {}^{15}C_8$.

$P(B \mid A)$ = Probability that the 9th examined record player is defective given that there were 3 defectives in the first 8 pieces examined $= \dfrac{4-3}{15-8} = \dfrac{1}{7}$,

since there is only one defective piece left among the remaining $15 - 8 = 7$ record players.

Hence, required probability $= P(A \cap B) = P(A)\, P(B \mid A)$

$$= \frac{{}^4C_3 \times {}^{11}C_5}{{}^{15}C_8} \times \frac{1}{7} = \frac{8}{195}.$$

Example 4·20. *Die A has four red and two white faces whereas die B has two red and four white faces. A coin is flipped once. If it falls heads the game continues by throwing die A, if it falls tails die B is to be used.*

(i) Show that the probability of getting a red face at any throw is $\frac{1}{2}$.

(ii) If the first two throws resulted in red faces, what is the probability of red face at the 3rd throw ?

(iii) If red face turns up at the first n throws, what is the probability that die A is being used ?

Solution. Let us define the events E_1, E_2, E_3 and A as follows :

E_1 : Die A is used, E_2 : Die B is used

H_3 : Getting a red face in any throw.

and A : Getting a red face in each of the first n throws.

(i) $P(E_3 \mid E_1) = \frac{4}{6} = \frac{2}{3}$, $P(E_3 \mid E_2) = \frac{2}{6} = \frac{1}{3}$,

Since $P(E_1) = P(E_2) = \frac{1}{2}$, the probability of getting a red face in any throw is given by :

$$\begin{aligned} P(E_3) &= P(E_1).\, P(E_3 \mid E_1) + P(E_2).\, P(E_3 \mid E_2) \\ &= \frac{1}{2} \times \frac{2}{3} + \frac{1}{2} \times \frac{1}{3} = \frac{1}{2} \end{aligned}$$

(ii) Clearly by the '*Law of Succession*', the probability of getting a red face at the third throw when the first two gave red faces is :

$$\frac{\frac{1}{2} + 1}{\frac{1}{2} + 2} = \frac{3}{5}.$$

Aliter. Let $R_3 \equiv$ Third throw results in red face.

$$P(R_3 \mid R_1 \cap R_2) = \frac{P(R_1 \cap R_2 \cap R_3)}{P(R_1 \cap R_2)} = \frac{\frac{1}{2}\left[\left(\frac{4}{6}\right)^3 + \left(\frac{2}{6}\right)^3\right]}{\frac{1}{2}\left[\left(\frac{4}{6}\right)^2 + \left(\frac{2}{6}\right)^2\right]} = \frac{72}{120} = \frac{3}{5}$$

(iii) $P(A \mid E_1) = (2/3)^n$, $P(A \mid E_2) = (1/3)^n$.

Using Baye's Theorem, the required probability is given by :

$$P(E_1 \mid A) = \frac{P(E_1) \cdot P(A \mid E_1)}{\sum_{i=1}^{2} P(E_i) \cdot P(A \mid E_i)} = \frac{\frac{1}{2} \cdot \left(\frac{9}{3}\right)^n}{\frac{1}{2} \cdot \left(\frac{2}{3}\right)^n + \frac{1}{2}\left(\frac{1}{3}\right)^n} = \frac{2^n}{2^n + 1}.$$

6. A random variable X has the following probability distribution

x	1	2	3	4	5	6
P(x)	k	3k	5k	7k	9k	11k

Determine i) k ii) P $(1 \leq x \leq 5)$ iii) P(x>3)

Sol: Given probability distribution of a random variable X is

x	1	2	3	4	5	6
P(x)	k	3k	5k	7k	9k	11k

(i)Since total probability of the distribution is unity i.e, $\sum_{i=1}^{n} P_i = 1$

We have, $k + 3k + 5k + 7k + 9k + 11k = 1 \Rightarrow k = \frac{1}{36}$

ii)P $(1 \leq x \leq 5) = P(1) + P(2) + P(3) + P(4) + P(5)$

$= k + 3k + 5k + 7k + 9k$

$=25k \quad =0.694.$

iii)$P(x > 3)=P(4) + P(5) + P(6) = 7k + 9k + 11k = 27k = 0.75$

Sol^n 4 :- (a) Since f is a probability density function

$$\int_{-\infty}^{\infty} f(x)\, dx = 1$$

$$\int_{-\infty}^{\infty} c(4x-2x^2)dx = 1$$

$$c\int_{0}^{2} (4x-2x^2)\, dx = 1$$

$$c\left[\frac{4x^2}{2} - \frac{2x^3}{3}\right]_0^2 = 1$$

$$c\left(2\cdot 4 - \frac{2\cdot 8}{3}\right) = 1$$

$$c\left(8\left(1-\frac{2}{3}\right)\right) = 1$$

$$c = \frac{3}{8}$$

(b) $P[X > 1] = \int_{1}^{\infty} f(x)\, dx$

$$= c\int_{1}^{2} (4x-2x^2)\, dx$$

$$= \frac{3}{8}\int_{1}^{2} (4x-2x^2)\, dx$$

$$= \frac{1}{2}$$

Example 5·40. *The joint p.d.f. of two random variables X and Y is given by :*

$$f(x, y) = \frac{9(1+x+y)}{2(1+x)^4(1+y)^4} ; 0 \leq x < \infty, 0 \leq y < \infty$$

Find the marginal distributions of X and Y, and the conditional distribution of Y for X = x.

Solution. Marginal *p.d.f.* of X is given by :

$$f_X(x) = \int_0^\infty f(x, y)\, dy = \frac{9}{2(1+x)^4} \int_0^\infty \frac{(1+y)+x}{(1+y)^4} dy$$

$$= \frac{9}{2(1+x)^4} \int_0^\infty \{(1+y)^{-3} + x(1+y)^{-4}\} dy$$

$$= \frac{9}{2(1+x)^4} \left(\left| \frac{-1}{2(1+y)^2} \right|_0^\infty + x \left| \frac{-1}{3(1+y)^3} \right|_0^\infty \right)$$

$$= \frac{9}{2(1+x)^4} \cdot \left(\frac{1}{2} + \frac{x}{3} \right) = \frac{3}{4} \cdot \frac{3+2x}{(1+x)^4} ; \; 0 < x < \infty$$

Since $f(x, y)$ is symmetric in x and y, the marginal *p.d.f.* of Y is given by :

$$f_Y(y) = \int_0^\infty f(x, y)\, dx = \frac{3}{4} \cdot \frac{3+2y}{(1+y)^4} ; \; 0 < y < \infty$$

The conditional distribution of Y for X = x is given by :

$$f_{XY}(Y = y \mid X = x) = \frac{f_{XY}(x, y)}{f_X(x)} = \frac{9(1+x+y)}{2(1+x)^4(1+y)^4} \times \frac{4(1+x)^4}{3(3+2x)} = \frac{6(1+x+y)}{(1+y)^4(3+2x)} ; 0 < y < \infty$$