Probability Theory

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Your current condition do not reflect your ultimate potential. —By Tony Robbins.

→ There are only few things that happen when we know, it will happen.

→ When a unique thing happens and we know the outcome, it is called a deterministic or predictable phenomenon.

For example :- Some established laws of Science

→ But most of the things that happen in our day to day life do not follow a set of rules, hence the results cannot be predicted, nor we ever know that it will happen, these are called unpredictable phenomenon.

For example :-

1. In business scenario; most of the managerial decisions are uncertain and we cannot foresee the future with surety, therefore, we have to depend on the assumption "the best possible" or "would be tomorrow".
2. The life of a battery or a bulb.
3. Chances of rain.
4. Tossing of a coin.
5. Throwing a dice.
6. Playing cards.
7.

In our daily life, we often use such language as "I'm not sure, whether I'm going to party or not." "We are not sure whether we will win or not." These all show the concept of probability as real life situation.

Development of Probability

Origin of the word probability is from the games of gambling, such as throwing of a dice or a coin or a game of cards.

Different Areas of Utilization of Probability Theory

> Insurance
> Weather prediction
> Finance Industries :- used to create mathematical models of the stock market to predict future trends.
> Sports
> Biological science
> Engineering
> Politics
> On-line shopping

Probability is a concept which numerically measures the chances of happening of an event.

Some definitions:-

Random experiment: It is an experiment in which the outcome of each trial is not unique but may be any one of the possible outcomes, when performed under identical conditions.

Trial:- Performing the random experiment.

Sample space:- A set of all possible outcomes of a random experiment.

Event:- Each subsets of a sample space is called an event or case

$$\underbrace{\boxed{\text{Trial}} \longrightarrow \boxed{\text{Outcome} = \text{Event}} \subseteq \boxed{\text{sample space}}}_{\text{Random experiment}}$$

Example:- Tossing of a coin three times:-

$$S = \{HHH, HHT, HTH, THH, HTT, THT, TTH, TTT\}$$

$\downarrow$

sample space

No. of the elements $= 2^8$

Events:-

(a) Single element Events:

$F_1 = \{HHH\}$, $F_3 = \{HTH\}$, $F_5 = \{THH\}$, $F_7 = \{TTH\}$

$F_2 = \{HHT\}$, $F_4 = \{HTT\}$, $F_6 = \{THT\}$, $F_8 = \{TTT\}$

No.s of elements in all F_i's = '1' = $n(F_i)$

4

(6) Mixed / more than one element :-

E_1 = only head / Three heads / (atmost three head & no tail)
= {HHH} ; $n(E_1)$ = '1'

E_2 = at least two head
= {HHH, HTH, THH, HHT} ; $n(E_2) = 4$

E_3 = at least one head
= {HHH, HTH, THH, TTH, HHT, HTT, THT} ; $n(E_3) = 7$

E_4 = atmost three head
= Sample Space ; $n(E_4) = n(S) = 8$

E_5 = No head = {TTT} ; $n(E_5)$ = '1'

E_6 = Exactly four heads
= { } = ϕ ; $n(E_6) = 0$

E_7 = getting all tails
= {TTT} ; $n(E_7) = 1$

E_8 = all heads or all tails
= {HHH, TTT} ; $n(E_8) = 2$

E_9 = ~~all~~ ~~the~~ same no. of head and tail
= { } = ϕ ; $n(E_9) = 0$.

Different types of Events :-

- **Null Event** :- An event having no sample point.
  Ex : E_6 & E_9. Also called impossible event.

- **Sure Event** :- An event which is sure to occur is said to be a sure events

  Ex : E_4

- **Simple / Elementary Event** :- An event having only one sample point.

  Ex : All F_i's and E_1, E_5, E_7

- **Compound Event** :- When an event is composed of a number of simple events

  Ex :- E_2, E_3, E_4, E_8.

- **Favourable Events** :- The number of outcomes of a random experiment which results in happening of an event is called the favourable cases to that event.

  Ex : E = exactly one head
  Favourable Events to the event 'E' : {HTT, THT, TTH}

- **Equally Likely Events** :- Events are said to be equally likely if we do not expect happening of one event in preference of others. In other words; if we can assign same probability to each outcomes.

Ex :→ {F_i}'s ; all these elements are equally likely.
→ Tossing of a coin → {H, T} ; happening of H or T having same probability.

- **Mutually Exclusive Events:-** Two or more events are said to be mutually exclusive events if the happening of any one of them excludes the happening of other events.

i.e. $A \cap B = \phi$ (let's take two events A & B).

Ex:- E_1 & E_7 are mutually exclusive
→ E_7 & E_8 are not mutually exclusive.

- **Exhaustive Events:-** Total no. of possible outcomes is called Exhaustive Events.

Ex: E_4.

Definition of Probability

Classical / Priori Probability:- (Mathematical)

If a random experiment or a trial results in 'n' exhaustive and equally likely events/outcomes, out of which 'm' are favourable to occurrence of any event 'E', then the probability of that Event E is given as:

$$P(E) = \frac{\text{no. of favourable cases}}{\text{total no. of cases}} = \frac{m}{n}$$

Axioms:-

→ $0 \leq P(E) \leq 1$

→ $P(E) = 0 \iff E$ is impossible event

→ $P(E) = 1 \iff E$ is sure event.

→ Probability of 'not occurrence of E i.e $\overline{E} / E^c$ $= 1 - P(E)$.

Limitations of Classical Definition :-

(i) If outcomes are not equally likely ;

(a) The probability that a candidate will pass or fail in a certain test is not 50% of pass / fail,

(b) The probability that a ceiling fan in a room will fall is 1/2, though the cases a & b as "(a) fall (b) not fall".

are mutually exclusive and exhaustive, but are not equally likely.

(ii) If the exhaustive number of outcomes of any random experiment is infinite or unknown.

Empirical Probability :- (Statistical)

If an experiment is performed repeatedly under essentially homogeneous and identical conditions, then the limiting value of the ratio of the number of times the event occurs to the number of trials, as the number of trials becomes indefinitely large, is called the probability of happening of the event, it being assumed that the limit is finite and unique.

Thus; $P(E) = \lim\limits_{n \to \infty} \frac{m}{n}$; When event A occurs m times in N repetitions of a random experiment and N is sufficiently large.

Here $\left(\frac{m}{n}\right)$ ratio is referred to as 'relative frequency' of the event in 'n' trials.

Relation between Classical and Empirical

Suppose, there are a large number of experiments to establish the chance of occurrence of an event. Such a case of empirical probability.

* Let us understand by the following example;

J. E. Kerrich conducted coin tossing experiment with 10 different sets of 1000 times tosses per set during his confinement in World War II.

The number of heads found by him were like; 502, 511, 497, 529, 476, 507, 504, 520, 504, 529.

The scenario gives the probability of getting a head in a toss of coin as

$$\therefore \frac{5079}{10,000} = 0.5079 \simeq \frac{1}{2}.$$

Thus, the empirical probability approaches the classical probability as the no. of trials becomes indefinitely large.

Some facts :-

1.) At least one of A or B $\rightarrow$ $A \cup B$

2.) Both A & B occur $\rightarrow$ $A \cap B$

3.) Neither A nor B occurs $\rightarrow$ $\bar{A} \cap \bar{B}$

4.) Event A occurs and B doesnot $\rightarrow$ $A \cap \bar{B}$

5.) Exactly one of the events A or B occurs $\rightarrow$ $A \Delta B$

6.) $P(\bar{A} \cap B) = P(B) - P(A \cap B)$

7.) $P(\overline{A \cup B}) = P(\bar{A} \cap \bar{B})$

8.) $P(\overline{A \cap B}) = P(\bar{A} \cup \bar{B})$.

Addition Laws of probability

If A and B are any two events, then

$$P(A \cup B) = P(A) + P(B) - P(A \cap B).$$

If A and B are mutually disjoint, then

$$P(A \cup B) = P(A) + P(B)$$

For three non-mutually exclusive events A, B and C,

$$P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(A \cap C) - P(B \cap C) + P(A \cap B \cap C).$$

Conditional Probability

The probability for the event A to occur while it is known that the event B has already occurred is called conditional probability and defined as;

$$P(A/B) = \frac{P(A\cap B)}{P(B)} \quad ; P(B) \neq 0.$$

It is also known as Multiplicative law of probability if we write the above expression as

$$P(A\cap B) = P(B) \cdot P(A/B). \quad ; P(B) > 0.$$

Independent Events

An event A is said to be independent of another event B, if the ~~cond~~ happening of A does not ~~get~~ get influenced by the happening of B.

⇒ Which implies to the formula of conditional probability as:

$$P(A/B) = P(A) \quad (\because \text{free of the happening of } B).$$

Multiplicative law for independent event became

$$P(A\cap B) = P(B) \cdot P(A).$$

Remark :- Two independent events (both of which are possible events), cannot be mutually disjoint.

Explanation :→ A & B are possible events so.

$P(A) \neq 0$ & $P(B) \neq 0$

i.e $P(A) \cdot P(B) \neq 0.$

$P(A\cap B) \neq 0.$

i.e. A & B are not mutually disjoint or exclusive.

In other words :. Two mutually disjoint events with possible probabilities are always dependent events.

Properties of Independent Events:-

1) If A & B are independent then
   (a) A & $\overline{B}$
   (b) $\overline{A}$ & B
   (c) $\overline{A}$ & $\overline{B}$
   (d) A & $\overline{A}$

2.) **Pairwise Independent Events:-** The events $A_1, A_2, \text{---} A_n$ are said to be pairwise independent iff

$$P(A_i \cap A_j) = P(A_i) \cdot P(A_j), \quad i \neq j = 1, 2, \text{---} n.$$

3.) **Mutually Independent events:-** The n events $A_1, A_2, A_3 \text{---} A_n$ in a sample spaces are said to be mutually independent if

$$P(A_{i_1} \cap A_{i_2} \cap A_{i_3} \text{---} \cap A_{i_k}) = P(A_{i_1}) \cdot P(A_{i_2}) \text{---} P(A_{i_k})$$

where $k = 2, 3, \text{---} n$.

4.) Let A_1, A_2, A_3 are mutually independent events.

$$P(A_1 \cap A_2) = P(A_1) \cdot P(A_2)$$
$$P(A_1 \cap A_3) = P(A_1) \cdot P(A_3)$$
$$P(A_2 \cap A_3) = P(A_2) \cdot P(A_3)$$

$$P(A_1 \cap A_2 \cap A_3) = P(A_1) \cdot P(A_2) \cdot P(A_3)$$

Joint Probability

The probability of two events, occurring together, is called joint probability.

$$P(A \cap B) = P(A) \cdot P(B).$$

The law of total probability:-

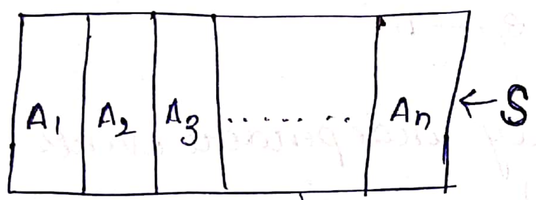
The general multiplication rule leads to an alternative rule called the "Rule of total probability."

~~An event~~

→ Let set S is the union of $A_1, A_2, \text{——} A_n$

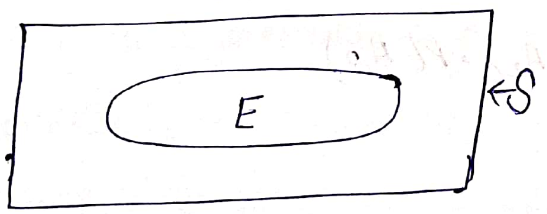
i.e $S = \bigcup_{i=1}^{n} A_i$

then we can say that $\{A_i\}_{i=1}^{n}$ be a partition of S.



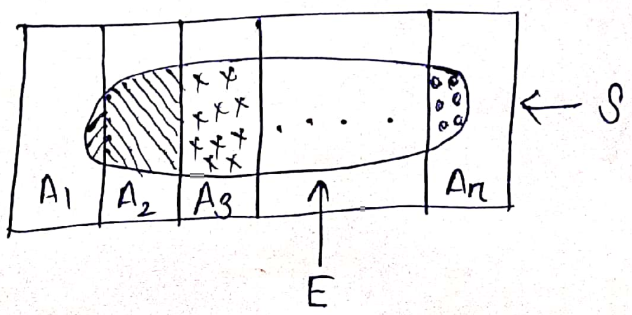
→ $S = \bigcup_{i=1}^{n} A_i$

→ Let E be any event of S i.e $E \subseteq S$



then $E = E \cap S = E \cap (A_1 \cup A_2 \text{——} \cup A_n)$

$= (E \cap A_1) \cup \text{——} \cup (E \cap A_n)$



→ Let A_i's are mutually disjoint and form the partition of E, then probability of E can be written as

$$P(E) = P(E \cap S) = P(E \cap A_1) + P(E \cap A_2) + \text{——} P(E \cap A_n)$$

using the formula of conditional probability for $P(E \cap A_i)$

$$P(E) = P(A_1) \cdot P(E/A_1) + \text{——} P(A_n) \cdot P(E/A_n) \text{——} (i)$$

The relation (i) is called total probability.

Baye's Theorem.

→ Depends on new information of the event.

→ Application of above facts
⇒ conditional probability
⇒ total probability

Prior Probability → New Information → Application of Baye's th'm → posterior Probability

Th'm :- Baye's Theorem :-

If $A_1, A_2, \text{——} A_n$ are mutually disjoint events with $P(A_i) \neq 0$ $(i = 1, 2, \text{—} n)$, then for any event $E \subseteq S = \bigcup_{i=1}^{n} A_i$ such that $P(E) > 0$, we have

$$P(A_i/E) = \frac{P(E/A_i) \cdot P(A_i)}{P(E)} = \frac{P(E/A_i) \cdot P(A_i)}{\sum_{i=1}^{n} P(E/A_i) \cdot P(A_i)} ; \; i = 1, 2, \text{—} n.$$

Some Solved Examples

use of intersection and union :-

Ex1. An urn contains 4 white, 5 red and 6 black balls. Three balls are drawn at random. Find the probability that balls are white ~~and~~ red and black.

Soln:- Let the even E = Balls are white, red and black.

$W = 4, \quad R = 5, \quad B = 6$

use the combination method

$$P(E) = \frac{{}^4C_1 \times {}^5C_1 \times {}^6C_1}{{}^{15}C_3} = \frac{\frac{4!}{1!\,3!} \times \frac{5!}{1!\,4!} \times \frac{6!}{1!\,5!}}{\frac{15!}{3!\,12!}}$$

$$= \frac{4\times5\times6}{15\times14\times13} \times 3! = \frac{24}{91}$$

Ex2:- An urn contains 5 white, 6 red, 4 black balls. Two balls are drawn at random. Find the probability that both are red. Also find the probability of one white & one black ball.

Soln:- $W = 5, R = 6, B = 4$

Let E : Both are red

$P(E) = P(R)$ and $P(R)$

$= P(R) \times P(R)$

$= \frac{6}{15} \times \frac{5}{14} = \frac{1}{7}$ ($\because$ witho replacement)

F : one white and one black

$P(F) = [P(W) \text{ and } P(B)] / (\text{or}) \ [P(B) \text{ and } P(W)]$

$= P(W) \times P(B) + P(B) \times P(W)$

$= \frac{5}{15} \times \frac{4}{14} + \frac{4}{15} \times \frac{5}{14} = \frac{4}{21}$

Note :- Questions in which replacement is not allowed can be attempted in a better manner using combination.

Alternative method :-

$P(E) = \frac{^6C_2}{^{15}C_2} = \frac{6 \times 5}{15 \times 14} = \frac{1}{7}$

$P(F) = \frac{^5C_1 \times {^4C_1}}{^{15}C_2} = \frac{5 \times 4 \times 2}{15 \times 14} = \frac{4}{21}.$

Ex8 :- Four cards are drawn without replacement from a well shuffled pack of 52 cards. Find the probability that :

(i) All cards are spades

(ii) There are two spades and two hearts

(iii) All cards are black

Also compute the probability if four cards are drawn with replacement.

Sol^n :- E_1 = All cards are spades

E_2 = There are two spades and two hearts

E_3 = All cards are black

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Without replacement :-

i) $P(E_1) = \dfrac{^{13}C_4}{^{52}C_4} = \dfrac{\frac{13.12.11.10}{4!}}{\frac{52.51.50.49}{4!}} = \dfrac{11}{4165}$

ii) $P(E_2) = \dfrac{^{13}C_2 \times {^{13}C_2}}{^{52}C_4} = \dfrac{\frac{13.12}{2!} \times \frac{13.12}{2!}}{\frac{52.51.50.49}{4!}} = \dfrac{468}{20825}$

iii) $P(E_3) = \dfrac{^{26}C_4}{^{52}C_4} = \dfrac{\frac{26.25 \times 24.23}{4!}}{\frac{52.51,50.49}{4!}} = \dfrac{46}{833}.$

With replacement →

i.) $P(E_1) = \dfrac{13}{52} \times \dfrac{13}{52} \times \dfrac{13}{52} \times \dfrac{13}{52} = \dfrac{1}{256}$

(ii) $P(E_2) = P(S) \cdot P(S) \cdot P(H) \cdot P(H) + P(H)\, P(H)\, P(S)\, P(S)$
$+ P(S) \cdot P(H)\, P(S)\, P(H) + P(S) P(H)\, P(H)\, P(S)$
$+ P(H)\, P(S)\, P(H)\, P(S) + P(H) \cdot P(S)\, P(S)\, P(H)$

$= 6\left[\dfrac{13}{52} \times \dfrac{13}{52} \times \dfrac{13}{52} \times \dfrac{13}{52}\right] = \dfrac{3}{128}$

(iii) $P(E_3) = \dfrac{26}{52} \times \dfrac{26}{52} \times \dfrac{26}{52} \times \dfrac{26}{52} = \dfrac{1}{16}.$

Ex 4:→ What is the probability that a leap year selected at random will have 53 Mondays?

Sol^n:- A leap year has 366 days = 52 full weeks + 2 extra day

$S = \{(M,T), (T,W), (W,TH), (TH,F), (F,S), (S,Sun) (Sun,M)\}$

E = The leap year selected will have 53 mondays

$= \{(M,T) (Sun,M)\}$

$P(E) = \frac{2}{7}$

Ex 5:- The 'n' persons are seated on 'n' chairs at a round table. Find the probability that two specified persons are sitting next to each other.

Sol^n:- Exhaustive no. of outcomes for 'n' persons to be sitting on 'n' chairs $= (n-1)!$

→ Two specified persons A & B who sit together

→ Consider sitting of these persons as one persons

→ Remainning persons are $n-1$

→ '$n-1$' person can be seated at a round table in the ways $= (n-2)!$

→ A & B can interchange their seats, then no. of ways $= 2! \times (n-2)!$

Required probability $= \frac{(n-2)! \times 2!}{(n-1)!} = \frac{2}{n-1}$ ← $\frac{\text{favourable ways}}{\text{total ways}}$

Ex 6:-

Examples on Addition and Multiplication Laws

Ex 6:- In a certain college, the students enage in various sports in the following proportions

Foot ball(F) : 60% of all students

Basket ball (B): 50% of all students

Both F & B : 30% of all students

If a student is selected at random, what is the probability that he will

i) Play foot ball or basket ball

ii) Play neither sports?

Sol^n:- $P(F) = \frac{6}{10}$, $P(B) = \frac{5}{10}$, $P(F \cap B) = \frac{3}{10}$

(i.) $P(F \cup B) = P(F) + P(B) - P(F \cap B)$

$= \frac{6}{10} + \frac{5}{10} - \frac{3}{10} = \frac{8}{10}$

(ii) $P(\bar{F} \cap \bar{B}) = P(\overline{F \cup B})$

$= 1 - P(F \cup B)$

$= 1 - \frac{8}{10} = \frac{2}{10} = \frac{1}{5}$

Ex 7:- Three newspapers A, B, C are published in a city and a survey on readers reveals the following information:

25% read A, 30% read B, 20% read C, 10% read both A & B, 5% read both A and C, 8% read both B and C, 3% read all three newspapers. For a person chosen at random, find the probability that he reads none of the newspapers.

Sol^n:- $P(A) = \frac{25}{100}$, $P(B) = \frac{30}{100}$, $P(C) = \frac{20}{100}$

$P(A\cap B) = \frac{10}{100}$, $P(A\cap C) = \frac{5}{100}$, $P(B\cap C) = \frac{8}{100}$

$P(A\cap B\cap C) = \frac{3}{100}$

$$P(A\cup B\cup C) = P(A) + P(B) + P(C) - P(A\cap B) - P(B\cap C) - P(A\cap C) + P(A\cap B\cap C)$$

$$= \frac{25}{100} + \frac{30}{100} + \frac{20}{100} - \frac{10}{100} - \frac{5}{100} - \frac{8}{100} + \frac{3}{100}$$

$$= \frac{11}{20}$$

$$P(\overline{A\cup B\cup C}) = 1 - P(A\cup B\cup C)$$

$= 1 - \frac{11}{20} = \frac{9}{20}$ The required probability.

Ex8:- The odds that person X speaks the truth are 3:2 and the odds that person Y speaks the truth are 5:3. In what percentage of cases are they likely to contradict each other on an identical point.

Sol^n:- Let us define the events:

A = X speaks the truth; $\bar{A}$ = tell a lie

B = Y speaks the truth; $\bar{B}$ = tell a lie

Given:- $P(A) = \frac{3}{3+2} = \frac{3}{5}$, $P(\bar{A}) = \frac{2}{5}$

$P(B) = \frac{5}{5+3} = \frac{5}{8}$, $P(\bar{B}) = \frac{3}{8}$

The event E = X & Y contradict

Contradictions happen as:

(i) X speaks truth & Y tells a lie

$= A\cap \bar{B}$

(ii) Y speaks truth and X tells a lie

$= \bar{A} \cap B$

By ~~addition~~ (multiplicative) Laws, the required probability is

$$P(E) = P(A \cap \bar{B}) \text{ or } P(\bar{A} \cap B)$$

$$= P(A \cap \bar{B}) + P(\bar{A} \cap B)$$

$$= P(A) \cdot P(\bar{B}) + P(\bar{A}) \cdot P(B) \quad (\because A \& \bar{B} \text{ and } \bar{A} \& B \text{ are independent})$$

$$= \frac{3}{5} \times \frac{3}{8} + \frac{2}{5} \times \frac{5}{8} = \underline{\underline{\frac{19}{40}}}.$$

Examples on Conditional probability

Ex 9:- Data on the readership of a certain magazine show that the proportion of male readers under 35 is 0.40 and over 35 is 0.20. If the proportion of readers under 35 is 0.70, find the proportion of subscribers that are females over 35 years. Also calculate the probability that a randomly selected male subscriber is under 35 years of age.

Sol^n:- Defining the events:-

A : Reader of the magazine is a male

B : Reader of the magazine is ~~a female~~ over 35 years of age.

Given:- $P(A \cap B) = .40$, $P(A \cap \bar{B}) = 0.20$

$P(\bar{B}) = 0.70$, $P(B) = 1 - P(\bar{B}) = 0.30$.

Find:- (i) $P(\bar{A} \cap B) = ?$ (ii) $P(\bar{B}/A) = ?$

(i) The proportion of subscribers that are females over 35 year is

$$P(\bar{A} \cap B) = P(B) - P(A \cap B)$$

$$= 0.30 - 0.20 = \underline{\underline{0.10}}$$

(ii) The probability that a randomly selected male subscriber is the under 25 years is:

$$P(\overline{B}/A) = \frac{P(A\cap\overline{B})}{P(A)} = \frac{P(A\cap\overline{B})}{P(A\cap B)+P(A\cap\overline{B})}$$

$$= \frac{0.40}{(0.20+0.40)} = \frac{2}{3}$$

Ex 10. A bag contain 17 counters marked with the number 1 to 17. A counter is drawn and replaced; a second drawing is then made. What is probability that:

(i) the first number drawn is even and the second odd?

(ii) the first number is odd and the second even?

How will your results in (i) and (ii) be effected if the first counter drawn is not replaced?

Sol^n:- Define the events:

A: getting even numbered counter drawn at first

B: getting odd " " at second draw

(1) Replacement → Independency of A & B

$P(A) = \frac{8}{17}$, $P(B) = \frac{9}{17}$

(i) Event E: first draw is even and second is odd

E: A and B

$$P(E) = P(A\cap B)$$

$$= P(A)\cdot P(B)$$

$$= \frac{8\cdot 9}{17\times 17} = \frac{72}{289}$$

(ii) F: first draw is odd and second is even

: B and A

$P(F) = P(B) \cdot P(A)$

$= \frac{72}{289}$.

② Without replacement: A & B are not independent

(i) $P(A) = \frac{8}{17}$, $P(B/A) = \frac{9}{16}$ ∵ second draw is dependent on first draw.

$P(E) = P(A \cap B)$

$= P(A) \cdot P(B/A)$

$= \frac{8}{17} \cdot \frac{9}{16} = \frac{9}{34}$

(ii) $P(A/B) = \frac{8}{16}$ $P(B) = \frac{9}{17}$ (∵ Here A is at second draw so Prob. of A is dependent on B)

$P(F) = P(B \cap A)$

$= P(B) \cdot P(A/B)$

$= \frac{9}{17} \times \frac{8}{16} = \frac{9}{34}$

Ex.11:- A consignment of 15 record players contains 4 defectives. The record players are selected at random, one by one, and examined. Those examined are not put back. What is the probability that the 9th one examined is the last defective?

Soln:- Defining the events

A: getting exactly 3 defectives in examination of 8 record players

B: 9th piece examined is a defective one

Find :- E = 9^{th} piece examined is last defective
= 9^{th} piece examined is a defective
and
before 9^{th} piece examined, there are exactly 3 defective

E = B and A

$P(E) = P(A \cap B) = ?$

→ Without replacement :-
Defectives = 4
Non-defec. = 11
total = 15 , total examined = 8

$$P(A) = \frac{{}^4C_3 \times {}^{11}C_5}{{}^{15}C_8} = \frac{\frac{4!}{3!} \times \frac{11!}{5!\,6!}}{\frac{15!}{8!\,7!}} = \frac{4!\,11!\,8!\,7!}{3!\,5!\,6!\,15!}$$

$$= \frac{\cancel{4}\cdot\cancel{7}\cdot 8\cdot 7\cdot\cancel{6}}{15\cdot\cancel{14}\cdot 13\cdot\cancel{12}} = \frac{8\cdot 7}{15\cdot 13}$$

$P(B/A)$ = probability of B when A has already happened
Remaining defective = 4 − 3 = 1
Remaining = 15 − 8 = 7

$$P(B/A) = \frac{1}{7}$$

$$P(E) = P(A\cap B) = \frac{8\cdot\cancel{7}}{15\cdot 13} \times \frac{1}{\cancel{7}} = \frac{8}{195}$$
$= P(A)\cdot P(B/A)$

Examples on Baye's theorem

Ex. 12: → The contents of urns I, II, & III are as follows:

1 white, 2 black and 3 red balls,
2 white, 1 black and 1 red balls, and
4 white, 5 black, and 3 red balls

One urn is chosen at random and two balls drawn from it. They happen to be white and red. What is the probability that they come from urns I, II or III?

Sol^n:- → Let I, II, & III urns are chosen and denoted by the events E_1, E_2 and E_3.

→ Let A be an event that the two balls taken from the selected urn are white and red.

$P(E_1) = P(E_2) = P(E_3) = 1/3$; urns are three

$$P(A/E_1) = \frac{P(E_1 \cap A)}{P(E_1)}$$

$P(A|E_1) = P(A)$; consider only I urn

$$= \frac{{}^1C_1 \times {}^3C_1}{{}^6C_2} = \frac{\frac{1 \times 3!}{1!\,2!}}{\frac{6!}{2!\,4!}} = \frac{\frac{3\cancel{2!}}{\cancel{2!}}}{\frac{6.5.\cancel{4!}}{2.\cancel{4!}}} = \frac{\cancel{3}}{\cancel{3}.5} = \frac{1}{5}$$

$P(A|E_2) = P(A)$; consider only II urn

$$= \frac{{}^2C_1 \times {}^1C_1}{{}^4C_2} = \frac{\frac{2!}{1!} \times 1}{\frac{4!}{2!\,2!}} = \frac{\cancel{2!} \times \cancel{2!} \times \cancel{2!}}{\cancel{4}.3.\cancel{2!}} = \frac{1}{3}$$

$P(A|E_3) = P(A)$; consider only III urn

$$= \frac{{}^4C_1 \times {}^3C_1}{{}^{12}C_2} = \frac{\frac{4!}{\cancel{3!}} \times \frac{\cancel{3!}}{2!}}{12!/2!\,.\,10!} = \frac{\frac{4.3!}{2}}{12!} \times 2! \times 10!$$

$$= \frac{4 \cdot 8!}{12! \times 2} \times 2 \times 10! = \frac{\cancel{4} \cdot \cancel{8} \cdot 2 \cdot \cancel{2}}{\cancel{12} \cdot 11 \times \cancel{2}} = \frac{2}{11}$$

(i) The probability of urn I, when we have drawn two balls; 1 is white & 1 is red.

$$P(E_2/A) = \frac{P(E_2) \cdot P(A/E_2)}{\sum_{i=1}^{3} P(E_i) \cdot P(A/E_i)}$$

$$= \frac{\frac{1}{3} \times \frac{1}{3}}{\frac{1}{3} \times \frac{1}{5} + \frac{1}{3} \times \frac{1}{3} + \frac{1}{3} \times \frac{1}{11}} = \frac{55}{118}$$

Similarly,

(ii) $$P(E_3/A) = \frac{\frac{1}{3} \times \frac{2}{11}}{\frac{1}{3} \times \frac{1}{5} + \frac{1}{3} \times \frac{1}{3} + \frac{1}{3} \times \frac{2}{11}} = \frac{30}{118}$$

(iii) $$P(E_1/A) = \frac{\frac{1}{3} \times \frac{1}{5}}{\frac{1}{3} \times \frac{1}{5} + \frac{1}{3} \times \frac{1}{3} + \frac{1}{3} \times \frac{2}{11}} = \frac{33}{118}.$$

Ex.13 :→ A letter is known to have come either from TATANAGAR or from CALCUTTA. On the envelope just two consecutive letters TA are visible. What is the probability that the letter came from CALCUTTA?

Solⁿ :- Let E_1 & E_2 are the events that define the letter came from TATANAGAR and CALCUTTA respectively.

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Let A denote the event that two consecutive visible letters on the envelop are TA.

$P(E_1) = P(E_2) = 1/2$ (treat them just like two boxes)

$P(A/E_1) = \frac{2}{8}$, $P(A/E_2) = \frac{1}{7}$; $P(E_2/A) = ?$

using Baye's theorem:-

$$P(E_2/A) = \frac{P(E_2)\,P(A/E_2)}{P(E_1)\cdot P(A/E_1) + P(E_2)\cdot P(A/E_2)}$$

$$= \frac{\frac{1}{2}\cdot\frac{1}{7}}{\frac{1}{2}\cdot\frac{2}{8}+\frac{1}{2}\cdot\frac{1}{7}} = \frac{4}{11}.$$

Ex.14. A speaks truth 4 out of 5. A die is tossed. He report that there is six. What is the chance that actually there was six?

Sol^n:- Define the following events;

E_1 : A speaks truth, E_2 = A tells a lies,

E : A reports a six.

Given:- $P(E_1) = \frac{4}{5}$, $P(E_2) = 1/5$, $P(E/E_1) = \frac{1}{6}$

$P(E/E_2) = 5/6$

Find:- $P(E_1/E) = ?$

$$P(E_1/E) = \frac{P(E_1)\cdot P(E/E_1)}{P(E_1)\cdot P(E/E_1) + P(E_2)\,P(E/E_2)} = \frac{\frac{4}{5}\cdot\frac{1}{6}}{\frac{4}{5}\cdot\frac{1}{6}+\frac{1}{5}\cdot\frac{5}{6}}$$

$$= \frac{\frac{4}{5.6}}{\frac{9}{5.6}} = \frac{4}{9}.$$

Random variable and distribution functions

The decision making gets complicated when there are no. of alternative available and the outcomes of each alternative is not very easy to determine. The quantity and quality of data available for decision making becomes important to understand the quality of decision made. Due to dynamic situation of the Business, a decision maker has to depend on uncertain outcomes which is termed as "random variable".

For Example :- In many experiment, we are not interested in knowing which of the outcomes has occurred, but in the numbers associated with them.

Ex :- ① When n coins are tossed, one may be interested in knowing the number of heads obtained.

② When a pair of dice are tossed, one may seek information about the sum of points.

Conclusion :- We associate a real number with each outcome of an experiment.

Random Variable:- A random variable is a variable whose values are determined by the outcomes of a random experiment i.e. to each outcomes of the experiment E_i (sample point) of sample space 'S', there corresp-onds a unique real number X_i known as the value of the random variable.

Ex1:→ Tossing of two coins :

$S = \{HH, HT, TH, TT\}$

Let us define the random variable X = no. of heads

→ In any experiment possible outcomes : HH, HT, TH, TT

$X(HH) = 2$

$X(HT) = 1$, $X(HT) = 1$

$X(TT) = 0$

$X(\omega)$: 0 1 2

$\omega \in \{HH, HT, TH, TT\} = S$

ω = sample point

value of this RV is determined by these outcomes (each) of RE

TT	HT	TH	HH
	'1'	'1'	'2'

$\boxed{X(\omega)}$: random variable 'X' takes a unique value '0' when outcome is 'TT' (sample point) ω = 'TT'

Definition:- Random variable is a function $X(\omega)$ whose domain is sample space 'S' and range is real line i.e $(-\infty, \infty)$, such that for every real number 'a', the event $[\omega : X(\omega) \leq a] \in B$, where B is ~~a field subset~~ a field of subsets of 'S'.

Note:- Making the probability statements about a random variable $X(\omega)$ as:-

$$P(X=0) = \frac{1}{4}, \quad P(X=1) = \frac{2}{4}, \quad P(X=2) = \frac{1}{4}$$

$$P(X(\omega) \leq 1) = P(X=0) + P(X=1) = \frac{1}{4} + \frac{2}{4} = \frac{3}{4}.$$

$$P[HT, TH, TT] = \frac{3}{4}.$$

That is; $P(X \leq a)$ is simply the probability of the set of outcome 'ω' for which $X(\omega) \leq a$ or $P(X(\omega) \leq a)$

$= P[\omega : X(\omega) \leq a]$.

Types of Random Variables:-

1. Discrete random variables:- (DRV)
2. Continuous random variable (CRV)

(1) Discrete random variable:- A random variable is said to be discrete if it takes only a countable number of real value. In other words; a real valued function defined on a discrete sample space is called a DRV.

Ex:- In example '1'; X is a DRV, as it contains finite number of real values $\{0, 1, 2\}$.

→ For instance, a random variable representing the number of automobiles sold at a particular dealership on one day would be discrete.

(2) Continuous Random variable:- A random variable is said to be continuous RV if it can take all possible values (integral as well as fraction) between a certain limits.

In other words:- A random variable is said to be continuous when its different values cannot be put in 1-1 correspondence with a set of positive integers.

Ex:- Weight of a person in kilograms (or pounds) would be continuous.

Distribution Function

Let X be a random ~~experiment~~ variable. The function F defined for all real x by

$$F(x) = P(X \leq x) = P[\omega : X(\omega) \leq x] \quad x \in R,$$

is called the distribution function (DF; d.f) of the random variable X.

Note:-> A distribution function is also called the cumulative distribution function.

→ $F(x) \leq F(y)$ if $x \leq y$

→ $0 \leq F(x) \leq 1$.

→ If a random variable X assume values $x_1, \ldots, x_n$ with probability $p_1 \ldots p_n$ resp.

~~then~~ such that: (i) $0 \leq p_i \leq 1$ for $i = 1, 2, \ldots n$

(ii) $p_1 + p_2 + \ldots + p_n = 1$.

then then X possesses the following probability distribution.

X :	x_1	x_2	x_3	....	x_n
$p(x)$:	p_1	p_2	p_3	....	p_n

The probability distribution for a R.V. describes how the probabilities are distributed over the values of R.V.

① **For a discrete RV** :- the probability distribution is defined by a probability mass function.

Definition:- (Probability mass function)-

If X is a discrete random variable with distinct values x_1 — x_n then the function $p(x)$ defined as:

$$p(x) = \begin{cases} P(X=x_i) = p_i \; ; & \text{if } x = x_i \\ 0 \; ; & \text{if } x \neq x_i \; ; \; i = 1,2,\text{—} \end{cases}$$

is called the probability mass function of R.V. 'X'.

Note :- ⓐ $p_i \geq 0 \; \forall i$

ⓑ $\sum_{i=1}^{\infty} p_i = 1.$

→ Also known as Discrete probability distribution.

Ex 2 :- Recalling ex 1; the probability distribution of random variable X (no. of heads) is given by the following table:

X :	0	1	2
$P(X) = p_i$:	$\frac{1}{4}$	$\frac{2}{4}$	$\frac{1}{4}$

ⓐ $p_i \geq 0 \; \forall i$ ⓑ $\frac{1}{4} + \frac{2}{4} + \frac{1}{4} = \sum_{i=1}^{3} p_i$

$= 1.$

Ex8:- Let X be the random variable which represents the sum of numbers of points on throwing two unbiased dice. If the point shown in each die is equal to one, the minimum ~~sum~~ is equal to 2. If both dice show six, then the sum of no's at the maximum is 12.

$\Rightarrow$:

	1	2	3	4	5	6
1	(1,1)	(1,2)	(1,3)	(1,4)	(1,5)	(1,6)
2	(2,1)	(2,2)	(2,3)	(2,4)	(2,5)	(2,6)
3	(3,1)	(3,2)	(3,3)	(3,4)	(3,5)	(3,6)
4	(4,1)	(4,2)	(4,3)	(4,4)	(4,5)	(4,6)
5	(5,1)	(5,2)	(5,3)	(5,4)	(5,5)	(5,6)
6	(6,1)	(6,2)	(6,3)	(6,4)	(6,5)	(6,6)

→ The sum 'X' must be an integer between 2 and 12.

The probability distribution in this case is given by the following table:

	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9	x_{10}	x_{11}
$X = x_i$	2	3	4	5	6	7	8	9	10	11	12
$P(X = x_i) = p(x_i) = p_i$	$\frac{1}{36}$	$\frac{2}{36}$	$\frac{3}{36}$	$\frac{4}{36}$	$\frac{5}{36}$	$\frac{6}{36}$	$\frac{5}{36}$	$\frac{4}{36}$	$\frac{3}{36}$	$\frac{2}{36}$	$\frac{1}{36}$
	p_1	p_2	p_3	p_4	p_5	p_6	p_7	p_8	p_9	p_{10}	p_{11}

i) $p_i \geq 0 \quad \forall i$

ii) $\sum_{i=1}^{11} p_i = p_1 + p_2 + \text{---} p_{11} = 1.$

(a) $P(X \geq 10) = p_9 + p_{10} + p_{11} = \frac{1}{6}$

(b) $P(X \leq 6) = p_1 + p_2 + p_3 + p_4 + p_5 + \cancel{p_6} = \frac{5}{12}.$

(c) $P(7 \leq X < 12) = p_7 + p_8 + p_9 + p_{10} = \frac{7}{18}.$

(2) Probability distribution for a continuous Random variable:-

→ In continuous case; the counterpart of the probability mass function is the probability density function.

"The probability density function provides the value of the function at any particular value of x."

→ It doesn't directly give the probability of the ~~function~~ random variable taking on a specific value.

Note:- (i) PDF denoted by $f(x)$ and some properties;

(ii) $f(x) \geq 0 \ \forall x$ (vi) $f(x) = \dfrac{dF(x)}{dx}$.

(iii) $$\int_{-\infty}^{\infty} f(x)\, dx = 1.$$

(iv.) The probability $P(E)$ given by

$F(x) = P(E) = \int_E f(x)\, dx$, is well defined for any event E.

(v.) In case of discrete RV; the probability at a point i.e. $P(x=c)$ is not zero for some fixed 'c'. but in case of CRV; the probability ~~of~~ at a point $P(x=c) = 0$.

Which implies:-

$$P(\alpha < X < \beta) = P(\alpha \leq X \leq \beta) = P(\alpha < X \leq \beta) = P(\alpha \leq X < \beta).$$

Here; It doesn't matter whether we include end points or not.

Definition: Probability density function / density function (p.d.f)

The function $f_X(x)$, so defined is known as p.d.f and defined as

$$f_X(x) = \lim_{\delta x \to 0} \frac{P(x \leq X \leq x + \delta x)}{\delta x} \text{ for interval } (x, x+\delta x)$$

Ex 4:→ A continuous random variable X has a p.d.f $f(x) = 3x^2 \quad 0 \le x \le 1$. Find a and b such that

(i) $P(X \le a) = P(X > a)$ and

(ii) $P(X > b) = 0.05$.

Soln:- As total probability is always '1'.

So, $P(X \le a) + P(X > a) = 1$

Since both terms are equal given in (i) part

$$P(X \le a) = \frac{1}{2} \Rightarrow \int_0^a f(x)\,dx = \frac{1}{2} \quad \because x \ge 0$$

$$\Rightarrow \int_0^a 3x^2\,dx = \frac{1}{2} \Rightarrow 3\left|\frac{x^3}{3}\right|_0^a = \frac{1}{2}$$

$$\Rightarrow a^3 = \frac{1}{2} \Rightarrow a = \left(\frac{1}{2}\right)^{1/3}$$

(ii) $P(X > b) = 0.05 \quad \because x \le 1.$

$$\Rightarrow \int_b^1 3x^2\,dx = 0.05$$

$$\Rightarrow 3\left|\frac{x^3}{3}\right|_b^1 = 0.05 \Rightarrow 1 - b^3 = 0.05$$

$$\Rightarrow b^3 = 1 - \frac{1}{20} \Rightarrow b = \left(\frac{19}{20}\right)^{1/3}$$

Ex 5:- Let X be a continuous random variable with p.d.f

$$f(x) = \begin{cases} ax & ; 0 \le x \le 1 \\ a & ; 1 \le x \le 2 \\ -ax + 3a & ; 2 \le x \le 3 \\ 0 & ; \text{elsewhere,} \end{cases}$$

(a) Determine the constant 'a'. (ii) compute $P(X \le 1.5)$

Sol^n:- Total probability is unity.

i.e. $\int_{-\infty}^{\infty} f(x)\,dx = 1.$

$$\Rightarrow \int_{-\infty}^{0} f(x)\,dx + \int_{0}^{1} f(x)\,dx + \int_{1}^{2} f(x)\,dx + \int_{2}^{3} f(x)\,dx + \int_{3}^{\infty} f(x)\,dx = 1$$

$$\Rightarrow \int_{0}^{1} ax\,dx + \int_{1}^{2} a\,.dx + \int_{2}^{3} (-ax+3a)\,dx = 1$$

$$\Rightarrow \left|\frac{ax^2}{2}\right|_{0}^{1} + \left|ax\right|_{1}^{2} + \left|-\frac{ax^2}{2} + 3ax\right|_{2}^{3} = 1.$$

$$\Rightarrow \frac{a}{2} + (2a - a) + \left(-a\frac{9}{2} + 9a\right) - \left(-a\cancel{\frac{4}{2}} + 6a\right) = 1$$

$$\Rightarrow a\left[\frac{1}{2} + 1 - \frac{9}{2} + 9 + 2 - 6\right] = 1$$

$$\Rightarrow a\left[-\cancel{\frac{8}{2}}^{4} + 6\right] = 1.$$

(i) $\Rightarrow \quad a = 1/2$

(ii) $P(X \leq 1.5) = \int_{-\infty}^{1.5} f(x)\,dx = \int_{-\infty}^{0} f(x)\,dx + \int_{0}^{1} f(x)\,dx + \int_{1}^{1.5} f(x)\,dx$

$$= a\int_{0}^{1} x\,dx + \int_{1}^{1.5} a\,dx$$

$$= a\left|\frac{x^2}{2}\right|_{0}^{1} + a\left|x\right|_{1}^{1.5}$$

$$= \frac{a}{2} + a(1.5-1)$$

$$= \frac{a}{2} + \frac{a}{2} = a \quad \text{from (i)}$$

$$= 1/2$$

Continuous Distribution Function

If x is a continuous random variable with p.d.f $f(x)$, then the function

$$F(x) = P(X \le x) = \int_{-\infty}^{x} f(t)\,dt \;; \quad -\infty < x < \infty$$

is called the distribution function (d.f.) or sometimes the cumulative distribution function (c.d.f) of the random variable X.

Properties :- ① $0 \le F(x) \le 1$, $-\infty < x < \infty$

② $F(x)$ is non-decreasing function.

③ $F(x)$ is a continuous function

④ The discontinuities of $F(x)$ are at the most countable.

⑤ $P(a \le X \le b) = F(b) - F(a)$

Ex 6 :- Verify, the following is a distribution function :

$$F(x) = \begin{cases} 0 & ; \; x < -a \\ \frac{1}{2}\left(\frac{x}{a}+1\right) & ; \; -a \le x \le a \\ 1 & ; \; x > a. \end{cases}$$

Sol^n :- Check for above properties.

Now, we have to show that;

$$\int_{-\infty}^{\infty} f(x)\,dx = 1.$$

$$\Rightarrow \int_{-\infty}^{\infty} f(x)\,dx = \int_{-a}^{a} f(x)\,dx = \int_{-a}^{a} \frac{1}{2a}\,dx$$

$$= \frac{1}{2a}[x]_{-a}^{a} = \frac{2a}{2a} = 1.$$

as we know :

$$F(x) = \int_{-\infty}^{x} f(x)\,dt, \; -\infty < x < \infty$$

$$f(x) = \frac{d}{dx}F(x) = \begin{cases} \frac{1}{2a} & ; \; -a \le x \le a \\ 0 & ; \; \text{o.w.} \end{cases}$$

Ex 7:-

Mixed Random Variable

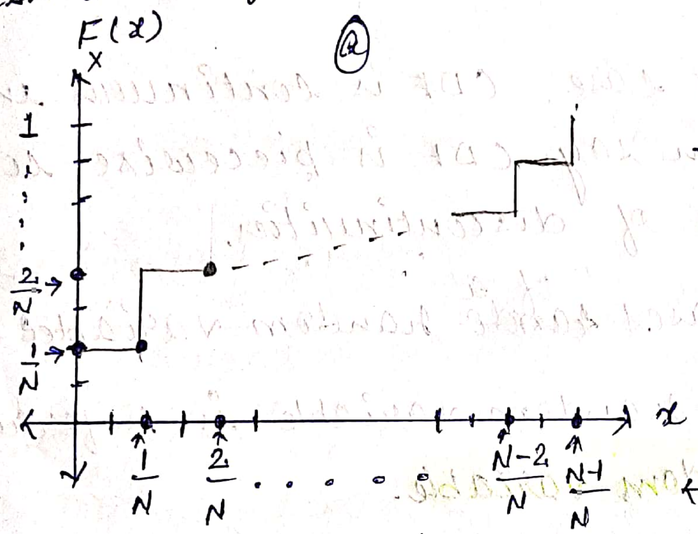
Recalling the cumulative distribution function; x (CDF)

$$F_X(x) = P(X \le x) = \begin{cases} P(\omega : X(\omega) \le x) = \sum_{i=1} p_i \\ \int_{-\infty}^{x} f(x)\,dx \end{cases} \quad \text{——} (1)$$

→ For an integer 'k', such that $0 < k < N$, the CDF is

$$F_X(x) = \frac{k}{N}, \quad \text{as} \quad \frac{k-1}{N} \le x < \frac{k}{N} \quad \text{——} (2)$$

The graphical representation of this cumulative distribution function is



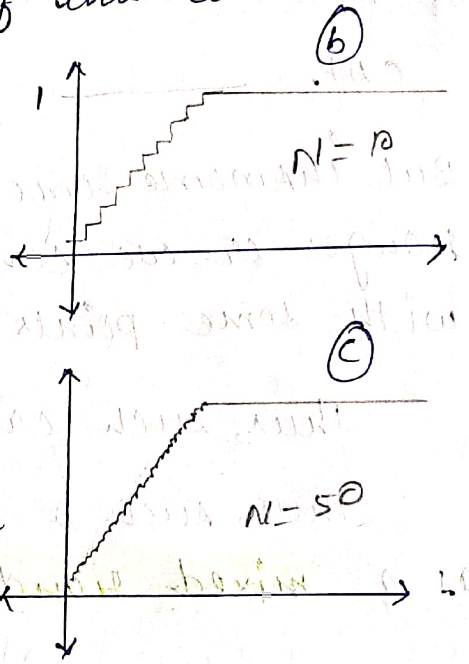


Fig:- General CDF of random variable X, which is staircase type function

→ Another CDF is given as:

$$F_X(x) = \begin{cases} 0 & x \le 0 \\ x & 0 < x \le 1 \\ 1 & x > 1. \end{cases} \quad \text{——} (3)$$

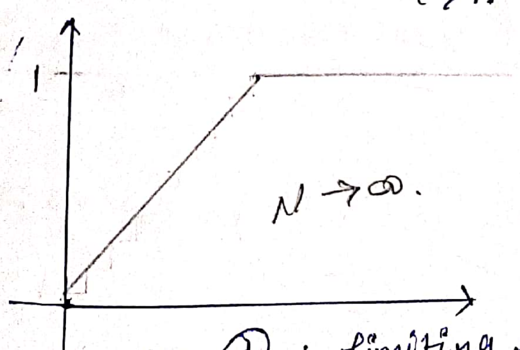


Fig (d) : Limiting case of eq. (2) as $N \to \infty$, as well as for Eq (3)

Points:-

(a) In the limiting case; r.v. X is a continuous r.v. and takes on value in the range [0,1) with equal probability.

Note:- In further study, we will see that this will be referred to as a uniform random variable.

(b) From above fig; when r.v is discrete r.v.; the CDF is discontinuous with a staircase type function.

If r.v is continuous, the CDF will be continuous CDF.

(c) But ~~in some~~ some case, CDF is continuous in some ranges or we can say CDF is piecewise continuous with some points of discontinuities.

~~Thus, such cases of a random variables~~

Thus, such a random variable is referred to as a **mixed random variable**.

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Some specific Random Variables

Random Variables

Discrete RV

- Binomial R.V.
- Poisson R.V.
- Uniform R.V.

Continuous R.V

- Normal / Gaussian R.V.
- Exponential R.V.
- Uniform R.V
- Rayleigh R.V.

Discrete Random Variables with their probability mass functions

a) Binomial Random Variables :-

A random variable X is said to (follow binomial distribution) binomial random variable if the probability mass function is defined by

$$P(X=x) = p(x) = \begin{cases} {}^{n}C_{x}\, p^{x} q^{n-x}, & x = 0,1,2,\ldots n \\ & q = 1-p \\ 0, & \text{otherwise.} \end{cases}$$

Here, two independent constants n&p are the parameters

b.) Poisson Random variable :- A random variable X is said to be poisson r.v., if the probability mass function is defined as;

$$P(X=x) = p(x, \lambda) = \begin{cases} \dfrac{e^{-\lambda}\lambda^{x}}{x!} & ; x = 0, 1, 2, \ldots ; \lambda > 0 \\ 0 & ; \text{otherwise,} \end{cases}$$

here λ is a parameter.

c) <u>Discrete Uniform Random variable :-</u>

A random variable X is said to be unifor r.v. if probability mass function of X is given by

$$P(X=x) = \begin{cases} \frac{1}{n} & , \text{for } x = 1, 2, \ldots n \\ 0 & , \text{otherwise} \end{cases}$$

where n is a parameter. It is also known as discrete rectangular.

Continuous R.V's with their probability density functions

a.) <u>Gaussian / Normal R.V :-</u> A random variable X is said to be normal random variable if the probabili[ty] density function can be defined by,

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}, \quad \begin{array}{l} -\infty < x < \infty \\ -\infty < \mu < \infty \\ \sigma > 0. \end{array}$$

where, μ and σ are the parameters ; mean and standard deviation respectively.

b.) Exponential R.V :- A random variable X is said to be exponential random variable if the probability density function is defined as

$$f(x) = \begin{cases} \lambda e^{-\lambda x}, & x \geq 0,\ \lambda > 0 \\ 0, & \text{otherwise.} \end{cases}$$

here λ is a parameter.

c.) Uniform / Rectangular R.V :- A r.v X is said to be continuous uniform r.v. if the probability density function is given as

$$f(x; a, b) = \begin{cases} \frac{1}{b-a}, & \text{if } a < x < b \\ 0, & \text{o.w.} \end{cases}$$

here a and b are the parameters.

d.) Rayleigh Random variable :- A r.v. X is said to be Rayleigh r.v., if the p.d.f of x is defined with scale parameter σ as follows,

$$f(x; \sigma) = \frac{x}{\sigma^2} e^{\frac{-x^2}{2\sigma^2}}, \quad x \geq 0.$$

Two-dimensional Random V.

- So, far we have defined only one random variable on a sample space.
- It is also possible to define more than one r.v. on the same space.

For example:- We may be interested in recording height as well as weight of every person in a group.

Joint probability mass function

[A1]
Definition:- If (X, Y) is a two-dim. discrete random variables, then the joint discrete function of X, Y, also called joint probability mass function of X, Y, denoted by $p_{X,Y}$ $P[X=x_i, Y=y_j]$ is defined as:

$$P[X=x_i \cap Y=y_j] = p_{XY}(x_i, y_j) = \begin{cases} P(X=x_i, Y=y_j), & \text{for a value } (x_i, y_j) \text{ of } (X,Y) \\ 0 & \text{otherwise.} \end{cases}$$

It can be represented in the form of table as follows:

X \ Y	y_1	y_2	y_3	—	y_j	—	y_m	Total
x_1	p_{11}	p_{12}	p_{13}	—	p_{1j}	—	p_{1m}	$p_{1\cdot}$
x_2	p_{21}	p_{22}	p_{23}	—	p_{2j}	—	p_{2m}	$p_{2\cdot}$
|	|	|	|		|		|	|
x_i	p_{i1}	p_{i2}	p_{i3}	—	p_{ij}	—	p_{im}	$p_{i\cdot}$
|	|	|	|		|			|
x_n	p_{n1}	p_{n2}	p_{n3}	—	p_{nj}	—	p_{nm}	$p_{n\cdot}$
Total	$p_{\cdot 1}$	$p_{\cdot 2}$	$p_{\cdot 3}$	—	$p_{\cdot j}$	—	$p_{\cdot m}$	1

Remark:- → $\sum_{i}\sum_{j} p_{XY}(x_i, y_j) = 1$

i.e The summation is taken over all possible values of (X,Y).

Marginal Probability mass function of X & Y:→

Let (X,Y) be a discrete two-dimensional r.v. which takes up countable number of values (x_i, y_i). Then the probability distribution of X is determined as follows.

$$P_X(x_i) = P(X=x_i)$$
$$= P(X=x_i, Y=y_1) + P(X=x_i, Y=y_2) + \text{——} \; P(X=x_i \cap Y=y_m)$$
$$= P(X=x_i \cap Y=y_1) + P(X=x_i \cap y_2) + \text{——} \; P(X=x_i \cap y_m).$$
$$= p_{i1} + p_{i2} + \text{——} \; p_{im}$$
$$= \sum_{j=1}^{m} p_{ij} = \sum_{j=1}^{m} p(x_i, y_j) = p_{i\cdot}$$

$$P_X(x_i) = P_{XY}(x, \infty) \quad \text{———} \quad ★$$

The equation '★' is known as marginal probability mass function of X.

Similarly for Y.

$$P_Y(y_j) = P_{XY}(\infty, y) = \sum_{i=1}^{m} p(x_i, y_j) = p_{\cdot j}$$

is known as marginal probability mass function of Y.

Conditional probability mass function

Definition:- Let (X, Y) be a discrete two-dim. r.v. Then the conditional ~~[discrete random]~~ probability mass function of X, given $Y = y$, denoted by $p_{X/Y}(x/y)$ and defined as:

$$p_{X/Y}(x/y) = \frac{P(X = x, Y = y)}{P(Y = y)}, \quad P(Y = y) \neq 0.$$

Similarly, for Y,

$$p_{Y/X}(y/x) = \frac{P(X = x, Y = y)}{P(X = x)}, \quad P(X = x) \neq 0.$$

Discrete distribution functions for all above

Joint distribution function:- The distribution function of the (two-dimensional random variable or) joint p.m.f of (X, Y) is a real valued function F defined for all real X and Y by the relation:

$$F_{XY}(x, y) = P(X \leq x, Y \leq y).$$

Properties:

1) $F(-\infty, y) = F(x, -\infty) = 0 = F(-\infty, -\infty)$.

2.) $F(\infty, \infty) = 1$

3.) $0 \le F_{XY}(x, y) \le 1$.

marginal distribution function:- In the context of joint distribution, we can obtain the individual distributions of X and Y are marginal distribution functions of X & Y respectively. (Just like as obtained in p.m.f case).

i.e.
$$F_X(x) = P(X \le x, Y < \infty) = P(X \le x).$$
$$= F_{XY}(x, \infty)$$
$$= \sum_y P(X \le x, Y = y)$$

similarly for Y,

$$F_Y(y) = P(X < \infty, Y \le y) = P(Y \le y).$$
$$= F_{XY}(\infty, y)$$
$$= \sum_x P(X = x, Y \le y).$$

conditional distribution Function:- The distribution function of Y when X has already assumed the particular value x;

→ $$F_{Y|X}(y|x) = P(Y \le y / X = x).$$

→ for X when Y has already assumed the particular value y;

$$F_{X/Y}(x|y) = P(X \le x / Y = y).$$

Continuous Two-dim. R.V.

Joint density function :- The joint p.d.f is given by it's differentiation form ~~in~~ as

$$f_{XY}(x,y) = \frac{\partial^2 F_{XY}(x,y)}{\partial x\, \partial y}$$

$$= \lim_{\substack{\delta x \to 0 \\ \delta y \to 0}} \frac{P(x \le X \le x+\delta x,\ y \le Y \le y+\delta y)}{\delta x, \delta y}.$$

Marginal density function :-

For X: $f_X(x) = \dfrac{dF_X(x)}{dx} = \displaystyle\int_{-\infty}^{\infty} f_{XY}(x,y)\,dy$

For Y: $f_Y(y) = \dfrac{dF_Y(x)}{dy} = \displaystyle\int_{-\infty}^{\infty} f_{XY}(x,y)\,dx$

Joint distribution function :-

$$F_{XY}(x,y) = P(X \le x, Y \le y)$$
$$= \int_{-\infty}^{y}\int_{-\infty}^{x} f_{X,Y}(u,v)\,du\,dv.$$

Marginal distribution function :-

$$F_X(x) = \int_{-\infty}^{x}\left(\int_{-\infty}^{\infty} f_{XY}(x,y)\,dy\right)dx.$$

$$F_Y(y) = \int_{-\infty}^{y}\left(\int_{-\infty}^{\infty} f_{XY}(x,y)\,dy\right)dx.$$

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Conditional probability density function and conditional distribution

Conditional p.d.f :- If X and Y have a joint p.d.f $f_{X,Y}$. then the conditional p.d.f is given as: (of X, given that Y = y)

$$f_{X/Y}(x/y) = \begin{cases} \dfrac{f_{XY}(x,y)}{f_Y(y)} & \text{if } f_Y(y) > 0 \\ 0 & \text{o.w.} \end{cases}$$

Similarly for Y.

Conditional distribution / conditional cumulative D.F

:- The conditional (cumulative) probability distribut-ion function of X, given Y = y, as:

$$F_{X/Y}(x/y) = P(X \leq x \mid Y = y)$$

$$= \int_{-\infty}^{x} f_{X/Y}(s/y)\, ds.$$

Similarly for Y, given X = x, as

$$F_{Y/X}(y/x) = P(Y \leq y \mid X = x).$$

$$= \int_{-\infty}^{y} f_{Y/X}(s/x)\, ds.$$

Independent Random Variables :- Two r.v.'s X & Y with joint p.d.f (p.m.f) $f_{XY}(x,y)$ and marginal p.d.f's (p.m.f's) $f_X(x)$ and $f_Y(y)$ respectively are said to be stochastically independent if and only if

$$f_{XY}(x,y) = f_X(x) \cdot f_Y(y).$$

Note :- In terms of distribution functions: The joint distribution function is the product of their marginal dist^n functions;

$$F_{XY}(x,y) = F_X(x) \cdot F_Y(y).$$

Examples :-

Ex:7 The joint probability density function of a two-dimensional random variable (X, Y) is given by;

$$f(x,y) = \begin{cases} 2 & ; 0 < x < 1, \quad 0 < y < x; \\ 0 & ; \text{o.w.} \end{cases}$$

(i) Find the marginal density function of X & Y.

(ii) Find the condition density function of Y given $X = x$, and conditional density function of X given $Y = y$.

(iii) Check for independence of X and Y.

Solution :-

i) The marginal p.d.f of X is

$$f_X(x) = \begin{cases} \int_{-\infty}^{\infty} f_{XY}(x,y)\, dy & ; \quad 0 < x < 1, \quad 0 < y < x. \\ 0 & ; \text{o.w.} \end{cases}$$

$$= \begin{cases} \int_0^x 2\, dy = 2x & , \quad 0 < x < 1 \\ 0, & \text{otherwise.} \end{cases}$$

of Y is

$$f_Y(y) = \begin{cases} \int_{-\infty}^{\infty} f_{XY}(x,y)\, dx & ; \quad 0 < y < x < 1 = 0 < y < 1 \\ & \quad y < x < 1 \\ = \int_y^1 2\, dx = 2(1-y). & \\ 0 & , \quad \text{o.w.} \end{cases}$$

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(ii) The conditional density function of Y given X $(0<x<1)$ is :

$$f_{Y/X}(y/x) = \frac{f_{XY}(x,y)}{f_X(x)} = \frac{\text{Joint of } X \& Y}{\text{marginal of } X}$$

$$= \frac{\cancel{2}}{\cancel{2}x} = \frac{1}{x} \; ; \quad 0<y<x.$$

of X given Y $(0<y<1)$ is :

$$f_{X/Y}(x/y) = \frac{f_{XY}(x,y)}{f_Y(y)}$$

$$= \frac{\cancel{2}}{\cancel{2}(1-y)} = \frac{1}{(1-y)} \; , \quad y<x<1.$$

(iii) $f_X(x) \cdot f_Y(y) = 2(2x)(1-y)$

$$\neq 2$$

$$\neq f_{X,Y}(x,y)$$

Joint p.d.f $\neq$ marginal p.d.f of X

$*$

marginal p.d.f of Y.

X and Y are dependent.

Ex :- 8 A random variable (X, Y) can yield one of the following pairs of values with probabilities noted against them :

For each observations	probability
(1,1) ; (2,1) ; (3,3) ; (4,3)	$\frac{1}{20}$
(3,1) ; (4,1) ; (1,2) ; (2,2) ; (3,2) ; (4,2) ; (1,3) ; (2,3)	$\frac{1}{10}$

Find the probability that $Y=2$ given that $X=4$. Also, find the probability that $Y=2$. Examine if the two events $X=4$ and $Y=2$ are independent.

Sol^n:- $P(Y=2) = P$(all the pairs that include Y as 2)

$$= P\{(1,2)\cup(2,2)\cup(3,2)\cup(4,2)\}$$

$$= \frac{1}{10}+\frac{1}{10}+\frac{1}{10}+\frac{1}{10} = \frac{4}{10} = \frac{2}{5}.$$

$P(X=4) = P$(all the pairs that include X as 4)

$$= P[(4,1)\cup(4,2)\cup(4,3)]$$

$$= \frac{1}{10}+\frac{1}{10}+\frac{1}{20} = \frac{5}{20} = \frac{1}{4}.$$

$$P(Y=2/X=4) = \frac{P(X=4\cap Y=2)}{P(X=4)} = \frac{P\{(4,2)\}}{P(X=4)}$$

$$= \frac{\frac{1}{10}}{\frac{1}{4}} = \frac{2}{5}.$$

Checking:

$P(X=4, Y=2) = P(X=4)\cdot P(Y=2)$

$\Rightarrow P(X=4, Y=2) = P\{(4,2)\}$

$$= \frac{1}{10} \quad \text{—— (1)}$$

$$\Rightarrow P(X=4)\cdot P(Y=2) = \frac{1}{4}\cdot\frac{2}{5} = \frac{1}{10} \quad \text{—— (2)}$$

Eq (1) and Eq. (2) are equal, Hence the events $X=4$ & $Y=2$ are indep

Ex:9 For the adjoining bivariate probability distribution of X and Y, find:

i) $P(X\le 1, Y=2)$

ii) $P(X\le 1)$,

iii) $P(Y\le 3)$, &

iv) $P(X<3, Y\le 4)$

X \ Y	1	2	3	4	5	6
0	0	0	1/32	2/32	2/32	3/32
1	1/16	1/16	1/8	1/8	1/8	1/8
2	1/32	1/32	1/64	1/64	0	2/64

Sol^n:- The marginal distributions are given as :

$\downarrow X$ \ $Y \rightarrow$	1	2	3	4	5	6	$p_X(x)$
0	0	0	$\frac{1}{32}$	$\frac{2}{32}$	$\frac{2}{32}$	$\frac{3}{32}$	$\frac{8}{32}$
1	$\frac{1}{16}$	$\frac{1}{16}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{10}{16}$
2	$\frac{1}{32}$	$\frac{1}{32}$	$\frac{1}{64}$	$\frac{1}{64}$	0	$\frac{2}{64}$	$\frac{8}{64}$
$p_Y(y)$	$\frac{3}{32}$	$\frac{3}{32}$	$\frac{11}{64}$	$\frac{13}{64}$	$\frac{6}{32}$	$\frac{16}{64}$	$\sum p(x)=1$ $\sum p(y)=1$

(i) $P(X \leq 1, Y = 2) = P(X=0, Y=2) + P(X=1, Y=2)$

$= 0 + \frac{1}{16} = \frac{1}{16}.$

(ii) $P(X \leq 1) = P(X=0) + P(X=1)$

$= \frac{8}{32} + \frac{10}{16} = \frac{8+20}{32} = \frac{28}{32} = \frac{7}{8}$

iii) $P(Y \leq 3) = P(Y=1) + P(Y=2) + P(Y=3).$

$= \frac{3}{32} + \frac{3}{32} + \frac{11}{64} = \frac{23}{64}$

(iv) $P(X < 3, Y \leq 4) = P(X=0, Y \leq 4) + P(X=1, Y \leq 4) + P(X=2, Y \leq 4).$

$= \left[0 + 0 + \frac{1}{32} + \frac{2}{32}\right] + \left[\frac{1}{16} + \frac{1}{16} + \frac{1}{8} + \frac{1}{8}\right] + \left[\frac{1}{32} + \frac{1}{32} + \frac{1}{64} + \frac{1}{64}\right]$

$= \frac{3}{32} + \frac{6}{16} + \frac{6}{64}$

$= \frac{6+24+6}{64} \Rightarrow \frac{\cancel{36}^{\,9}}{\cancel{64}_{\,16}} = \frac{9}{16}$

Ex : 10. If the joint distribution function of X and Y is given by:

$$F_{XY}(x,y) = \begin{cases} 1-e^{-x}-e^{-y}+e^{-(x+y)} & ; x>0, y>0. \\ 0 & ; \text{elsewhere.} \end{cases}$$

(a.) Find the marginal densities of X and Y.

(b) Are X and Y independent?

(c) Find $P(X \le 1 \cap Y \le 1)$ and $P(X+Y \le 1)$.

Sol^n:- The joint p.d.f of the r.v. (X, Y) is given by:

$$f_{XY}(x,y) = \frac{\partial^2 F_{XY}(x,y)}{\partial x \partial y}$$

$$= \frac{\partial}{\partial x}\left[e^{-y} - e^{-(x+y)}\right]$$

$$= \begin{cases} e^{-(x+y)} & ; x \ge 0, y \ge 0. \\ 0 & ; \text{O.W.} \end{cases}$$

(a) The marginal p.d.f of X and Y are;

$$f_X(x) = \int_{-\infty}^{\infty} f_{XY}(x,y)\, dy$$

$$= \int_0^{\infty} e^{-(x+y)}\, dy = e^{-x}\int_0^{\infty} e^{-y}\, dy = -e^{-x}\left[e^{-y}\right]_0^{\infty}$$

$$= e^{-x} \quad x \ge 0.$$

$$f_Y(x) = \int_{-\infty}^{\infty} f_{XY}(x,y)\, dx.$$

$$= \int_0^{\infty} e^{-(x+y)}\, dx = e^{-y}\left[-e^{-x}\right]_0^{\infty} = e^{-y} \quad , y \ge 0.$$

(b.) $f_X(x) \cdot f_Y(y) = e^{-x} \cdot e^{-y} = e^{-(x+y)} = f_{XY}(x,y).$

X & Y are independent.

$$(c.)\ P(X \le 1 \cap Y \le 1) = \int_{-\infty}^{x}\int_{-\infty}^{y} f_{XY}\, dx\, dy$$

$$= \int_0^1 \int_0^1 f_{XY}\, dx\, dy$$

$$= \left(\int_0^1 e^{-x}\, dx\right) \cdot \int_0^1 e^{-y}\, dy$$

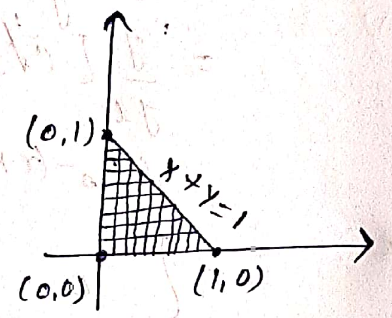
$$= (1-e^{-1})\,(1-e^{-1})$$

$$= (1-e^{-1})^2$$

$$P(X+Y \le 1) = \iint_{X+Y\le 1} f_{XY}\, dx\, dy$$

∴ for X : limit $0 \rightarrow 1$

then for Y :- $0 \rightarrow$ curve



$$= \int_0^1 \left\{ \int_0^{1-x} f_{XY}\, dy \right\} dx$$

$$= \int_0^1 \left\{ \int_0^{1-x} e^{-x} e^{-y}\, dy \right\} dx$$

$$= \int_0^1 e^{-x} \left[-e^{-y}\right]_0^{1-x} dx \quad = \int_0^1 -e^{-x}\left[e^{-(1-x)} - 1\right] dx$$

$$= -\int_0^1 \left[e^{-1} - e^{-x}\right] dx \quad = -\left[e^{-1}x + e^{-x}\right]_0^1$$

$$= -\left[e^{-1} + e^{-1} - e^{0}\right]$$

$$= 1 - 2e^{-1}$$

Note :- Alternatively, we can interchange the limits for x & y.

Ex 11:- The joint ~~distribution~~ density of X and Y is given by:

$$f(x,y) = 4xy\, e^{-(x^2+y^2)} \; ; \quad x \geq 0,\ y \geq 0.$$

Find the conditional density of X given Y=y.

Sol^n:- Marginal density of Y is given by:

$$f_Y(y) = \int_{-\infty}^{\infty} f_{XY}(x,y)\, dx$$

$$= \int_0^{\infty} 4xy\, e^{-(x^2+y^2)}\, dx$$

$$= 4y\, e^{-y^2} \int_0^{\infty} x \cdot e^{-x^2}\, dx$$

$$= \frac{4y \cdot e^{-y^2}}{2} \int_0^{\infty} \frac{-2x}{(-1)}\, e^{-x^2}\, dx$$

$\because$ Let $e^{-x^2} = z \Rightarrow e^{-x^2}(-2x)\,dx = dz$) → with the limits of z: $x=0 \to z=1$, $x=\infty \to z=0$.

$$= \frac{4y}{-2} \cdot e^{-y^2} \int_{\text{limit } z=1}^{0} 1\, dz.$$

$$= -\frac{4}{2}\, y \cdot e^{-y^2} (-1) \left[z \right]_{\text{limits } z=0}^{z=1} = -\frac{2y e^{-y^2}}{\cancel{2}}$$

$$= -2y\, e^{-y^2} \cdot (-1)\, [1-0]$$

$$= 2y\, e^{-y^2} \;;\ y \geq 0.$$

The conditional density function of X given Y=y:

$$f_{X/Y}(x/y) = \frac{f_{XY}(x,y)}{f_Y(y)} = \frac{\text{joint p.d.f}}{\text{marginal p.d.f of } Y.}$$

$$= \frac{4xy\, e^{-x^2} e^{-y^2}}{2y\, e^{-y^2}} = 2x\, e^{-x^2} \;; \quad x \geq 0.$$