Test 2: Probability and Statistics

Tuesday: 8:30 – 9:30 am
E5 Sec. 2.

Max. Marks: 10

(1) A problem in mathematics is given to three students A, B, C, whose chances of solving are $\frac{2}{3}$, $\frac{1}{2}$ and $\frac{1}{3}$ respectively. What is the probability that the problem will be solved?

(2) There are $(N+1)$ identical urns marked $0, 1, 2, —, N$ each of which contains N white and red balls. The k^{th} urn contains k red and $N-k$ white balls, ($k = 0, 1, 2, — N$). An urn is chosen at random and 'n' random drawings of a ball are made from it, the ball drawn being replaced after each draw. If the balls drawn are all red, show that the probability that the next drawing will also yield a red ball is approximately $\frac{(n+1)}{(n+2)}$, when N is large.

(3) The diameter, say X, of an electric cable, is assumed to be continuous random variable with probability density function: $f(x) = 6x(1-x)$, $0 \leq x \leq 1$;

(a) check that the above is a p.d.f.,

(b) Obtain an expression for the distribution function of X.

(c). Determine the number k such that $P(X<k) = P(X>k)$.

(4) If, $p(x) = \begin{cases} \frac{x}{15}; & x = 1, 2, 3, 4, 5 \\ 0; & \text{otherwise,} \end{cases}$

Find

(i) $P(X = 1 \text{ or } 2)$, (ii) $P\left\{\frac{1}{2} < X < \frac{5}{2} \middle/ X > 1\right\}$.

(5) A gun is aimed at a certain point (origin of the coordinate system). Because of the random factors, the actual hit point can be any point (X, Y) in a circle of radius 'R' about the origin. Assume that the joint density of X and Y is constant in the circle given by:

$$f_{XY}(x, y) = \begin{cases} k, & x^2 + y^2 \leq R^2, \\ 0, & \text{otherwise;} \end{cases}$$

(i) compute k,

(ii) Show that

$$f_X(x) = \begin{cases} \frac{2}{\pi R}\left\{1 - \left(\frac{x}{R}\right)^2\right\}^{1/2}, & -R \leq x \leq R \\ 0, & \text{otherwise.} \end{cases}$$

Example 8 A problem in mathematics is given to three students A, B and C whose chances of solving are $\frac{2}{3}$, $\frac{1}{2}$ and $\frac{1}{3}$ respectively. What is the probability that the problem will be solved?

Solution: Probability of A solving the problem $P(A) = \frac{2}{3} \Rightarrow P(\bar{A}) = \frac{1}{3}$

Probability of B solving the problem $P(B) = \frac{1}{2} \Rightarrow P(\bar{B}) = \frac{1}{2}$

Also probability of C solving the problem $P(C) = \frac{1}{3} \Rightarrow P(\bar{C}) = \frac{2}{3}$

Now probability that A, B and C do not solve the problem is $\frac{1}{3} \times \frac{1}{2} \times \frac{2}{3} = \frac{1}{9}$

$\therefore$ The probability that the problem is solved $= 1 -$ (Probability that problem is not solved) $= 1 - \frac{1}{9} = \frac{8}{9}$

Example 4·19. *There are (N + 1) identical urns marked 0, 1, 2,..., N each of which contains N white and red balls. The kth urn contains k red and N – k white balls, (k = 0, 1, 2, ..., N). An urn is chosen at random and n random drawings of a ball are made from it, the ball drawn being replaced after each draw. If the balls drawn are all red, show that the probability that the next drawing will also yield a red ball is approximately (n + 1) /(n + 2) when N is large.*

Solution. Let E_k denote the event that the kth urn is chosen, $(k = 0, 1, 2, \ldots, N)$. Then the event, $E_0, E_1, E_2, \ldots, E_n$ are pairwise mutually exclusive, one of which certainly occurs.

Then $\quad P(E_1) = P(E_2) = \ldots P(E_n) = \dfrac{1}{N+1}$ (From symmetry) $\quad \ldots (i)$

Let A denote the event of getting n red balls successively in n draws (with replacement from an urn chosen at random).

Then $\quad P(A \mid E_k) = \dfrac{k}{N} \cdot \dfrac{k}{N} \cdot \ldots \dfrac{k}{N} = \left(\dfrac{k}{N}\right)^n \quad \ldots (ii)$

and
$$P(A) = \sum_{k=0}^{N} P(E_k)\, P(A \mid E_k) = \frac{1}{N+1} \sum_{k=0}^{N} \left(\frac{k}{N}\right)^n$$

Now, using Baye's theorem, we get

$$P(E_k \mid A) = \frac{P(E_k)\, P(A \mid E_k)}{\sum\limits_{k=0}^{N} P(E_k)\, P(A \mid E_k)} = \frac{\dfrac{1}{N+1} \cdot \left(\dfrac{k}{N}\right)^n}{\dfrac{1}{N+1} \sum\limits_{k=0}^{N} \left(\dfrac{k}{N}\right)^n}$$

Let C be the future event that $(n + 1)$th draw also yields a red ball. Then on the same lines, it can be shown that

$$P(A \cap C) = \frac{1}{N+1} \sum_{k=0}^{N} \left(\frac{k}{N}\right)^{n+1}$$

Hence, the required probability is,

$$P(C \mid A) = \frac{P(A \cap C)}{P(A)} = \frac{\dfrac{1}{N+1} \sum\limits_{k=0}^{N} \left(\dfrac{k}{N}\right)^{n+1}}{\dfrac{1}{N+1} \sum\limits_{k=0}^{N} \left(\dfrac{k}{N}\right)^{n}}$$

If N is very large, then

$$P(A) \simeq \frac{1}{N} \sum_{n=0}^{N} \left(\frac{k}{N}\right)^{n} \simeq \int_0^1 x^n \, dx = \frac{1}{n+1}$$

and

$$P(A \cap C) \simeq \frac{1}{N} \sum_{k=0}^{N} \left(\frac{k}{N}\right)^{n+1} \simeq \int_0^1 x^{n+1} \, dx = \frac{1}{n+2}$$

$\therefore$

$$P(C \mid A) = \frac{n+1}{n+2}.$$

Example 5·24. *The diameter, say X, of an electric cable, is assumed to be a continuous random variable with p.d.f. :* $f(x) = 6x(1-x),\ 0 \le x \le 1$

(i) Check that the above is a p.d.f.,

(ii) Obtain an expression for the c.d.f. of X.,

(iii) Compute $P\left(X \le \frac{1}{2} \,\middle|\, \frac{1}{3} \le X \le \frac{2}{3}\right)$, *and*

(iv) Determine the number k such that $P(X < k) = P(X > k)$.

Solution.

(*i*) Since $\displaystyle\int_0^1 f(x)\,dx = \int_0^1 6x(1-x)\,dx = 6\left|\frac{x^2}{2} - \frac{x^3}{3}\right|_0^1 = 1$, $f(x)$ is a *p.d.f.*

$$(ii)\quad F(x) = \begin{cases} 0, & \text{if } x \le 0 \\ \displaystyle\int_0^x 6t(1-t)\,dt = (3x^2 - 2x^3), & 0 < x \le 1 \\ 1, & \text{if } x > 1 \end{cases}$$

$$(iii)\quad P\left(X \le \tfrac{1}{2} \,\middle|\, \tfrac{1}{3} \le X \le \tfrac{2}{3}\right) = \frac{P\left(\frac{1}{3} \le X \le \frac{1}{2}\right)}{P\left(\frac{1}{3} \le X \le \frac{2}{3}\right)} = \frac{\displaystyle\int_{1/3}^{1/2} 6x(1-x)\,dx}{\displaystyle\int_{1/3}^{2/3} 6x(1-x)\,dx} = \frac{11/54}{13/27} = \frac{11}{26}.$$

(*iv*) We have $P(X < k) = P(X > k)$

$$\Rightarrow \quad \int_0^k 6x(1-x)\,dx = \int_k^1 6x(1-x)\,dx \quad \text{or} \quad 3k^2 - 2k^3 = 3(1-k^2) - 2(1-k^3)$$

$$\Rightarrow \quad 4k^3 - 6k^2 + 1 = 0 \quad \Rightarrow \quad k = \frac{1}{2},\ \frac{1 \pm \sqrt{3}}{2}.$$

The only admissible value of k in the given range is $\frac{1}{2}$. Hence the value of k is $\frac{1}{2}$.

Example 5·2. *If,* $p(x) = \begin{cases} \dfrac{x}{15}; x = 1, 2, 3, 4, 5 \\ 0, \text{ elsewhere} \end{cases}$

Find (i) $P\{X = 1 \text{ or } 2\}$, *and (ii)* $P\left\{\frac{1}{2} < X < \frac{5}{2} \,\middle|\, X > 1\right\}$.

Solution. (i) $P(X = 1 \text{ or } 2) = P(X = 1) + P(X = 2) = \dfrac{1}{15} + \dfrac{2}{15} = \dfrac{1}{5}$

$$\text{(ii) } P\left(\frac{1}{2} < X < \frac{5}{2} \,\middle|\, x > 1\right) = \frac{P\left\{\left(\frac{1}{2} < X < \frac{5}{2}\right) \cap (X > 1)\right\}}{P(X > 1)} = \frac{P\{(X = 1 \text{ or } 2) \cap (X > 1)\}}{P(X > 1)}$$

$$= \frac{P(X = 2)}{1 - P(X = 1)} = \frac{2/15}{1 - (1/15)} = \frac{1}{7}.$$

Example 5·43. *A gun is aimed at a certain point (origin of the coordinate system). Because of the random factors, the actual hit point can be any point (X, Y) in a circle of radius R about the origin. Assume that the joint density of X and Y is constant in this circle given by :*

$$f_{XY}(x, y) = \begin{cases} k, \text{ for } x^2 + y^2 \le R^2 \\ 0, \text{ otherwise} \end{cases}$$

(i) Compute k,

(ii) show that $$f_X(x) = \begin{cases} \dfrac{2}{\pi R}\left\{1 - \left(\dfrac{x}{R}\right)^2\right\}^{1/2}, \text{ for } -R \le x \le R \\ 0 \qquad\qquad\qquad\quad, \text{ otherwise} \end{cases}$$

Solution. (*i*) The constant k is computed from the consideration that the total probability is 1, *i.e.,*

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x, y)\, dxdy = 1 \Rightarrow \iint_{x^2+y^2 \le R^2} k\, dx\, dy = 1 \Rightarrow 4\iint_{I} k\, dx\, dy = 1$$

where region I is the first quadrant of the circle $x^2 + y^2 = R^2$.

$$\Rightarrow \quad 4k \int_0^R \left(\int_0^{\sqrt{R^2 - x^2}} 1 \,.\, dy \right) dx = 1 \quad \Rightarrow \quad 4k \int_0^R \sqrt{R^2 - x^2}\, dx = 1$$

$$\Rightarrow 4k \left| x \sqrt{R^2 - x^2} + \frac{R^2}{2} \sin^{-1}\left(\frac{x}{R}\right) \right|_0^R = 1 \quad \Rightarrow \quad 4k\left(\frac{R^2}{2} \cdot \frac{\pi}{2}\right) = 1 \quad \Rightarrow \quad k = \frac{1}{\pi R^2}.$$

$$\therefore \qquad f_{XY}(x, y) = \begin{cases} 1/(\pi R^2)\,; & x^2 + y^2 \le R^2 \\ 0 & , \textit{otherwise} \end{cases}$$

$$(ii) \quad f_X(x) = \int_{-\infty}^{\infty} f(x, y)\, dy = \frac{1}{\pi R^2} \int_{-\sqrt{R^2 - x^2}}^{\sqrt{R^2 - x^2}} 1 .\, dy$$

$$\left[\because\ x^2 + y^2 \le R^2 \quad \Rightarrow \quad -(R^2 - x^2)^{1/2} \le y \le (R^2 - x^2)^{1/2}\right]$$

$$= \frac{2}{\pi R^2} \int_0^{\sqrt{R^2 - x^2}} 1 .\, dy = \frac{2}{\pi R^2} (R^2 - x^2)^{1/2}$$

$$= \frac{2}{\pi R} \left\{1 - \left(\frac{x}{R}\right)^2\right\}^{1/2}, \; -R \le x \le R$$