Unit Test: Probability and Statistics
2024-2025 (Odd Semester): Department of Computer Science and Engineering

1. There are two bags A and B. A contains n white and 2 black balls and B contains 2 white and n black balls. One of the two bags is selected at random and two balls are drawn from it without replacement. If both the balls drawn are white and the probability that the bag A was used to draw the balls is $6/7$, find the value of n.

2. A coin is tossed $(m+n)$ times, $(m>n)$. Show that the probability of at least m consecutive heads is $\frac{n+2}{2^{m+1}}$.
3. A random variable X has the following probability function:

Value of $X = x_i$	0	1	2	3	4	5	6	7
$p(x_i)$	0	k	$2k$	$2k$	$3k$	k^2	$2k^2$	$7k^2+k$

   a) Find k,
   b) Evaluate $P(0<X<5)$,
   c) If $P(X \leq a) > \frac{1}{2}$, find the minimum value of a.

4. Suppose that X is a continuous random variable has probability density function given by:

$$f(x) = \frac{k}{1+x^2}, \quad -\infty < x < \infty.$$

   a) What is the value of k?
   b) Find the probability distribution of X.

5. Suppose that two-dimensional continuous random variable (X,Y) has joint probability density function given by:

$$f(x,y) = e^{-(x+y)}; 0 \leq x < \infty, 0 \leq y < \infty.$$

   Are X and Y are independent? Find
   a) $P(X>1)$
   b) $P(X<Y \mid X<2Y)$
   c) $P(1< X+Y<2)$.

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Example 4·5. *There are two bags A and B. A contains n white and 2 black balls and B contains 2 white and n black balls. One of the two bags is selected at random and two balls are drawn from it without replacement. If both the balls drawn are white and the probability that the bag A was used to draw the balls is $\frac{6}{7}$, find the value of n.*

Solution. Let E_1 denote the event that bag A is selected and E_2 denote the event that bag B is selected. Let E be the event that two balls drawn are white. We have

$$P(E_1) = P(E_2) = \frac{1}{2}$$

$$P(E \mid E_1) = \frac{{}^nC_2}{{}^{n+2}C_2} = \frac{n(n-1)}{(n+2)(n+1)}$$

and
$$P(E \mid E_2) = \frac{{}^2C_2}{{}^{n+2}C_2} = \frac{2}{(n+2)(n+1)}$$

Using Baye's Theorem, the probability that the two white balls drawn are from the bag A, is given by :

$$P(E_1 \mid E) = \frac{P(E_1)\,P(E \mid E_1)}{P(E_1)\,P(E \mid E_1) + P(E_2)\,P(E \mid E_2)} = \frac{6}{7} \qquad \text{(Given)}$$

$$\Rightarrow \quad \frac{\frac{1}{2}\cdot\frac{n(n-1)}{(n+2)(n+1)}}{\frac{1}{2}\cdot\frac{n(n-1)}{(n+2)(n+1)} + \frac{1}{2}\cdot\frac{2}{(n+2)(n+1)}} = \frac{6}{7} \quad \Rightarrow \quad \frac{n(n-1)}{n(n-1)+2} = \frac{6}{7}$$

$\therefore \quad 7n(n-1) = 6n(n-1) + 12 \quad \Rightarrow \quad n^2 - n - 12 = 0 \quad \Rightarrow \quad n = 4, -3.$

Since n cannot be negative, we get $n = 4$.

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Example 3·75. *A coin is tossed $(m + n)$ times, $(m > n)$. Show that the probability of at least m consecutive heads is $\frac{n+2}{2^{m+1}}$.*

Solution. Since $m > n$, only one sequence of m consecutive heads is possible. This sequence may start either with the first toss or second toss or third toss, and so on, the last one will be starting with $(n + 1)th$ toss.

Let E_i denote the event that the sequence of m consecutive heads starts with ith toss. Then the required probability is : $\quad P(E_1) + P(E_2) + \ldots + P(E_{n+1}).$ $\quad \ldots (*)$

$$P(E_1) = P \text{ [Consecutive heads in first } m \text{ tosses and head or tail in the rest]} = \left(\frac{1}{2}\right)^m$$

$$P(E_2) = P \text{ [Tail in the first toss, followed by } m \text{ consecutive heads and head or tail in the next]} = \frac{1}{2}\left(\frac{1}{2}\right)^m = \frac{1}{2^{m+1}}$$

In general,

$$P(E_r) = P \text{ [tail in the } (r-1)\text{th trial followed by } m \text{ consecutive heads and head or tail in the next]}$$

$$= \frac{1}{2}\left(\frac{1}{2}\right)^m = \frac{1}{2^{m+1}}, \forall\, r = 2, 3, \ldots, n+1.$$

Substituting in (*), required probability $= \frac{1}{2^m} + \frac{n}{2^{m+1}} = \frac{2+n}{2^{m+1}}$.

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Example 5·2. *A random variable X has the following probability distribution :*

x :	0	1	2	3	4	5	6	7
$p(x)$:	0	k	$2k$	$2k$	$3k$	k^2	$2k^2$	$7k^2 + k$

(i) Find k, (ii) Evaluate $P(X < 6)$, $P(X \geq 6)$, and $P(0 < X < 5)$, (iii) If $P(X \leq c) > \frac{1}{2}$, find the minimum value of c, and (iv) Determine the distribution function of X. [Madurai Univ. B.Sc., Oct. 1988]

Solution. Since $\sum_{x=0}^{7} p(x) = 1$, we have

$\Rightarrow \quad k + 2k + 2k + 3k + k^2 + 2k^2 + 7k^2 + k = 1$

$\Rightarrow \quad 10k^2 + 9k - 1 = 0$

$\Rightarrow \quad (10k - 1)(k + 1) = 0 \Rightarrow k = 1/10$

[$\because k = -1$, is rejected, since probability canot be negative.]

(ii) $P(X < 6) = P(X = 0) + P(X = 1) + \ldots + P(X = 5)$

$$= \frac{1}{10} + \frac{2}{10} + \frac{2}{10} + \frac{3}{10} + \frac{1}{100} = \frac{81}{100}$$

$$P(X \geq 6) = 1 - P(X < 6) = \frac{19}{100}$$

$P(0 < X < 5) = P(X = 1) + P(X = 2) + P(X = 3) + P(X = 4) = 8k = 4/5$

(iii) $P(X \leq c) > \frac{1}{2}$. By trial, we get $c = 4$.

(iv)

X	$F_X(x) = P(X \leq x)$
0	0
1	$k = 1/10$
2	$3k = 3/10$
3	$5k = 5/10$
4	$8k = 4/5$
5	$8k + k^2 = 81/100$
6	$8k + 3k^2 = 83/100$
7	$9k + 10k^2 = 1$

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Problems

1.If the probability density function $f(x) = \frac{k}{1+x^2}$ $-\infty < x < \infty$. Find the value of 'k' and probability distribution function of $f(\mathbf{x})$.

Sol: Since total area under the probability curve is 1 i. e, $\int_a^b f(x)dx = 1$.

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$$\int_{-\infty}^{\infty} \frac{k}{1+x^2} dx = 1.$$

$$2k(\tan^{-1} x)\Big|_{0}^{\infty} = 1$$

$$2k(\tan^{-1} \infty - \tan^{-1} 0) = 1$$

$$\therefore k = \frac{1}{\pi}$$

Cumulative distribution function of f(x) is given by

$$\int_{-\infty}^{x} f(x)dx = \int_{-\infty}^{x} \frac{k}{1+x^2} dx = \frac{1}{\pi}(\tan^{-1} x)\Big|_{-\infty}^{x} = \frac{1}{\pi}[\frac{\pi}{2} + (\tan^{-1} x)].$$

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Example 5·44. *Given :* $f(x, y) = e^{-(x+y)}\, I_{(0,\infty)}(x) \cdot I_{(0,\infty)}(y)$*. Are X and Y independent ? Find (i)* $P(X > 1)$, *(ii)* $P(X < Y \mid X < 2Y)$, *(iii)* $P(1 < X + Y < 2)$.

Solution. We are given :

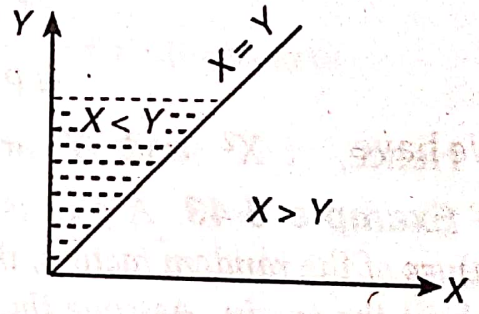
$$f(x, y) = e^{-(x+y)}\,;\, 0 \le x < \infty,\, 0 \le y < \infty$$

$$= (e^{-x})(e^{-y}) = f_X(x) \cdot f_Y(y)\,;\, 0 \le x < \infty\,,\, 0 \le y < \infty \qquad \dots (1)$$

$\Rightarrow$ X and Y are independent and $f_X(x) = e^{-x}\,;\, x \ge 0$ and $f_Y(y) = e^{-y}\,;\, y \ge 0$ ... (2)

(i)
$$P(X > 1) = \int_1^\infty f_X(x)\, dx$$

$$= \int_1^\infty e^{-x}\, dx = \left| \frac{e^{-x}}{-1} \right|_1^\infty = \frac{1}{e}$$



(ii)
$$P(X < Y \mid X < 2Y) = \frac{P(X < Y \cap X < 2Y)}{P(X < 2Y)}$$

$$= \frac{P(X < Y)}{P(X < 2Y)} \qquad \dots (3)$$

$$P(X < Y) = \int_0^\infty \left\{ \int_0^y f(x, y)\, dx \right\} dy = \int_0^\infty \left\{ e^{-y} \left| \frac{e^{-x}}{-1} \right|_0^y \right\} dy$$

$$= -\int_0^\infty e^{-y}(e^{-y} - 1)\, dy = -\left| \frac{e^{-2y}}{-2} + e^{-y} \right|_0^\infty = 1 - \frac{1}{2} = \frac{1}{2}$$

$$P(X < 2Y) = \int_0^\infty \left\{ \int_0^{2y} f(x, y)\, dx \right\} dy = -\int_0^\infty e^{-y}(e^{-2y} - 1)\, dy$$

$$= -\left| \frac{e^{-3y}}{-3} + e^{-y} \right|_0^\infty = 1 - \frac{1}{3} = \frac{2}{3}.$$

Substituting in (3), $\quad P(X < Y \mid X < 2Y) = \dfrac{1/2}{2/3} = \dfrac{3}{4}.$

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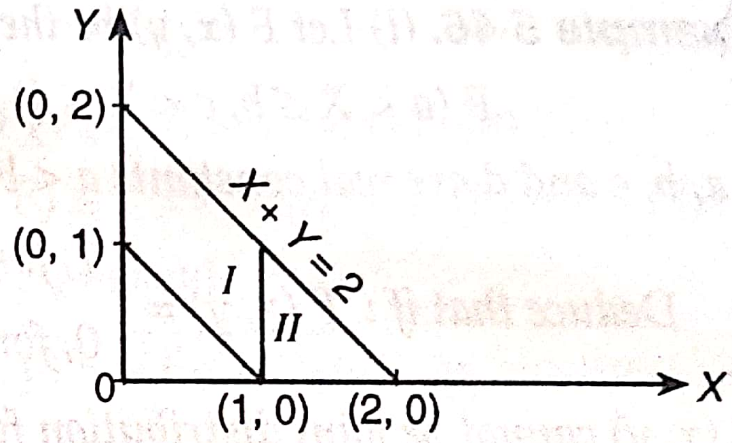
(iii) $P(1 < X + Y < 2)$



$$= \iint_{I} f(x, y)\, dx\, dy + \iint_{II} f(x, y)\, dx\, dy$$

$$= \int_0^1 \left(\int_{1-x}^{2-x} f(x,y)\, dy \right) dx + \int_1^2 \left(\int_0^{2-x} f(x, y)\, dy \right) dx$$

$$= \int_0^1 \left(e^{-x} \int_{1-x}^{2-x} e^{-y}\, dy \right) dx + \int_1^2 \left(e^{-x} \int_0^{2-x} e^{-y}\, dy \right) dx$$

$$= \int_0^1 \frac{e^{-x}}{-1} (e^{x-2} - e^{x-1})\, dx + \int_1^2 \frac{e^{-x}}{-1} (e^{x-2} - 1)\, dx$$

$$= -(e^{-2} - e^{-1}) \int_0^1 1.dx - \int_1^2 (e^{-2} - e^{-x})\, dx$$

$$= -(e^{-2} - e^{-1}) \left[\left| x \right|_0^1 \right] - \left[\left| e^{-2} . x + e^{-x} \right|_1^2 \right] = \frac{2}{e} - \frac{3}{e^2}.$$