Mathematical Expectation or Expected value of a Random variable

The mathematical expression for computing the expected value of a discrete random variable X with probability mass function (p.m.f) $f(x)$ is given below:

$$E(X) = \sum_{x} x\, f(x) \quad \text{———} \quad (M1)$$

The mathematical expression for computing the expected value of a continuous random variable X with probability density function (p.d.f) $f(x)$ is, however, as follows:

$$E(X) = \int_{-\infty}^{\infty} x\, f(x) \quad \text{———} \quad (M2)$$

Properties:- ① The series in (M1) and (M2) Eq.s are absolutely convergent: i.e.

$$\int_{-\infty}^{\infty} |x f(x)|\, dx = \int_{-\infty}^{\infty} |x|\, f(x)\, dx < \infty$$

$$\sum_{x} |x f(x)| \leq \sum_{x} |x|\, f(x)\, dx < \infty.$$

Expected value of a function of a Random variable

Consider a random variable X with p.d.f / p.m.f $f(x)$ and distribution function $F(x)$. If $g(\cdot)$ is a function such that $g(X)$ is a random variable and $E[g(X)]$ exists, then

$$E[g(X)] = \int_{-\infty}^{\infty} g(x) f(x)\, dx \quad ; \text{ for cont}^n \text{ r.v.}$$

$$E[g(X)] = \sum_{x} g(x) f(x) \quad ; \text{ for discrete r.v.}$$

These results extend into higher dimensions. If X and Y have a joint p.d.f $f(x,y)$ and $Z = h(x,y)$ is a random variable for some function h and if $E(Z)$ exists, then

$$E(Z) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} h(x,y) f(x,y)\, dx\, dy.$$

for cont^n r.v.

$$E(Z) = \sum_{x} \sum_{y} h(x,y) f(x,y).$$

Note:- * The terms average, mean, expectation and first moment are all alternative names for the concept of expected value.

Mean $= \mu = E(X) = \mu_1$.

* The corresponding results for a discrete r.v can be obtained on replacing integration by summation(Σ) over the given range of the variable in the further properties.

Properties of Expectation

Property 1:- If X and Y are r.v. then $E(X+Y) = E(X)+E(Y)$, provided all the expectations exist.

Proof:- Let X and Y be continuous r.v.'s with joint p.d.f $f_{XY}(x,y)$ and marginal p.d.f's $f_X(x)$ and $f_Y(y)$ respectively. Then by definition

$$E(X) = \int_{-\infty}^{\infty} x f_X(x)\, dx$$

$$E(Y) = \int_{-\infty}^{\infty} y f_Y(y)\, dy$$

$$E(X+Y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (x+y) f_{XY}(x,y)\, dx\, dy$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x f_{XY}(x,y)\, dx\, dy + \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} y f_{XY}(x,y)\, dx\, dy$$

$$= \int_{-\infty}^{\infty} x \underbrace{\left[\int_{-\infty}^{\infty} f_{XY}(x,y)\, dy\right]}_{\text{marginal of } X} dx + \int_{-\infty}^{\infty} y \underbrace{\left[\int_{-\infty}^{\infty} f_{XY}(x,y)\, dx\right]}_{\text{marginal of } Y} dy$$

$$= \int_{-\infty}^{\infty} x f_X(x)\, dy + \int_{-\infty}^{\infty} y f_Y(y)\, dy$$

$$E(X+Y) = E(X) + E(Y).$$

That is; Expectation is a linear operator.

Generalisation:- The mathematical expectation of the sum of n random variables is equal to the sum of their expectations, provided all the expectations exist. Symbolically, if $X_1, X_2, — X_n$ are random variables then

$$E(X_1+X_2+ — +X_n) = E(X_1)+E(X_2)+ — + E(X_n).$$

or $$E\left(\sum_{i=1}^{n} X_i\right) = \sum_{i=1}^{n} E(X_i)$$; if all the expectations exist.

Property 2:- Multiplication theorem of Expectation:- If X and Y are independent random variables, then $E(XY) = E(X) \cdot E(Y)$.

Proof:- For then r.v. X and Y, we have

$$E(XY) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xy \, f_{XY}(x,y)\, dx\, dy$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xy \, f_X(x) \cdot f_Y(y)\, dx\, dy$$

since X and Y are independent

$$= \int_{-\infty}^{\infty} x f_X(x) dx \cdot \int_{-\infty}^{\infty} y f_Y(y)\, dy.$$

$$= E(X) \cdot E(Y).$$

Generalisation:- Symbolically, if $X_1, X_2, — X_n$ are r.Vs. (I.D) then

$$E(X_1, X_2, — X_n) = E(X_1) E(X_2) — \cdot E(X_n)$$

$$E\left(\prod_{i=1}^{n} X_i\right) = \prod_{i=1}^{n} E(X_i).$$

Property 3:- $E(aX+b) = a\,E(X) + b$ where a and b both are constants.

Property 4:- If $X \geq 0$ then $E(X) \geq 0$.

Property 5:- If X and Y are two r.v.'s such that $X \leq Y$, then $E(X) \leq E(Y)$, provided all the expectations exist.

Property 6:- $|E(X)| \leq E(|X|)$.

Property 7:- If X and Y are two random variables and g and h are the functions of 'x' and 'y' r.v.s separately (in respectively).

then

(i) $E(a\,g(X)) = a\,E[g(X)]$

(ii) $E[g(X)+b] = E[g(X)] + b$

(iii) $E[g(X)\cdot h(Y)] = E[g(X)]\cdot E[h(Y)]$, ($\because$ X and Y are I.D)

Property 8:- Expectation of a Linear combination of random variables:

Let $X_1, X_2 — X_n$ be any 'n' random variables and if $a_1, a_2 — a_n$ are any 'n' constants, then

$$E\left(\sum_{i=1}^{n} a_i X_i\right) = \sum_{i=1}^{n} a_i E(X_i),$$ provided all

the expectations exist.

Property 9:- $E(c) = \int_{-\infty}^{\infty} c\cdot f(x)\,dx = c\cdot\int_{-\infty}^{\infty} f(x)\,dx$

$= c$ (constant).

Variance

Let X be a random variable then the variance of X is $V(X) = E(X^2) - \{E(X)\}^2$

$= E[X - E(X)]^2$

$= \int_{-\infty}^{\infty} (x-\bar{x})^2 f(x)\, dx.$

where $\bar{x}$ is the mean.

Properties :-

- If X is a random variable, then $V(aX+b) = a^2 V(X)$. where a and b are constants.

Proof :- Let $Y = aX+b$. Then

$E(Y) = E(aX+b)$

$= a\,E(X)+b$

$Y - E(Y) = aX + \cancel{b} - a\,E(X) - \cancel{b}$

$= a[X - E(X)]$

squaring and taking expectation of both sides,

$E[Y-E(Y)]^2 = E[a^2[X-E(X)]^2]$

$= a^2 E(X - E(X))^2$

$E(Y) = a^2 V(X)$

$E(aX+b) = a^2 V(X).$

Discrete R.V :- conditional expectation of function

The conditional expectation or mean value of a continuous function $g(X,Y)$ given that $Y = y_j$ is defined by

$$E\left[g(X,Y)/_{Y=y_j}\right] = \sum_{i=1}^{\infty} g(x_i, y_j)\, P\left(X = x_i/_{Y=y_j}\right)$$

$$= \frac{\sum_{i=1}^{\infty} g(x_i, y_j) \cdot P(X = x_i \cap Y = y_j)}{P(Y = y_j)}$$

In other words: $E[g(X,Y)/Y = y_j)$ is the expectation of the function $g(X, y_j)$ of X w.r.to the conditional distribution of X when $y = y_j$.

conditional expectation of Random variable :- The conditional Expectation of a discrete r.v. X when given $Y = y_j$:-

$$E(X/Y = y_j) = \frac{\sum_{i=1}^{\infty} x_i \; P(X = x_i \cap Y = y_j)}{P(Y = y_j)}.$$

Similarly for 'Y.'

Continuous Random variable :- For a function $g(X,Y)$, given $Y = y_j$

$$E[g(X,Y)/Y = y] = \int_{-\infty}^{\infty} g(x,y) \cdot f_{X/Y}(x/y)\, dx.$$

$$= \frac{\int_{-\infty}^{\infty} g(x,y) \cdot f(x,y)\, dx}{f_Y(y)}$$

The conditional expectation of r.v. X when given Y=y.

$$E(X/Y=y) = \frac{\int_{-\infty}^{\infty} x\, f(x,y)\, dx}{f_Y(y)} = \int_{-\infty}^{\infty} x \cdot f_{X/Y}\, dx.$$

Similarly for 'Y' variable.

Conditional Variance

Discrete case : Discrete R.V

For X; $V(X/Y=y_j) = E\left[\{X - E[X/Y=y_j]\}^2 / Y=y_j\right]$

For Y; $V(Y/X=x_i) = E\left[\{Y - E[Y/X=x_i]\}^2 / X=x_i\right]$

Continuous case : Cont^y R.V.

For X; $V(X/Y=y) = E\left[\{X - E(X/Y=y)\}^2 / Y=y\right]$

For Y; $V(Y/X=x) = E\left[\{Y - E(Y/X=x)\}^2 / X=x\right]$

Examples:-

Ex 1:- Let X be a random variable with the following probability distribution:

x	-3	6	9
p(x)	$\frac{1}{6}$	$\frac{1}{2}$	$\frac{1}{3}$

Find E(X) and $E(X^2)$ and using the laws of expectation, evaluate $E(2X+1)^2$.

Sol^n:- $E(X) = \sum x.p(x) = (-3)\left(\frac{1}{6}\right) + (6)\left(\frac{1}{2}\right) + (9)\left(\frac{1}{3}\right)$

$= \frac{11}{2}$

$E(X^2) = \sum x^2.p(x) = (9)\left(\frac{1}{6}\right) + (36)\left(\frac{1}{2}\right) + (81)\left(\frac{1}{3}\right)$

$= \frac{93}{2}$

$E(2X+1)^2 = E(4X^2+1+4X) = 4E(X^2) + 4(E(X)) + 1$

$= 4 \times \frac{93}{2} + 4 \times \frac{11}{2} + 1 = \underline{209.}$

Ex 2:- A coin is tossed untill a head appears. What is the expectation of the number of tosses required?

Sol^n:- Let X denote the number of tosses required to get the first head. Then X can materialise in the following ways:

Event	x	Probability; p(x).
H	1	$\frac{1}{2}$
TH	2	$\frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$
TTH	3	$\frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} = \frac{1}{8}$
TTTH	4	$\frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} = \frac{1}{16}$
⋮	⋮	⋮

$$E(X) = \sum_{x=1}^{\infty} x \cdot p(x) = 1 \cdot \frac{1}{2} + 2 \cdot \frac{1}{4} + 3 \frac{1}{8} + 4 \cdot \frac{1}{16} + \text{———}$$

This is an arithmetic - geometric series with ratio of G.P being $r = \frac{1}{2}$.

Let $S = 1 \cdot \frac{1}{2} + 2 \cdot \frac{1}{4} \cdot 3 \cdot \frac{1}{8} + 4 \cdot \frac{1}{16} + \text{———}$

$\frac{S}{2} = 1 \cdot \frac{1}{4} + 2 \cdot \frac{1}{8} \cdot 3 \cdot \frac{1}{16} + \text{———}$

$S - \frac{S}{2} = 1 \cdot \frac{1}{2} + (2-1)\frac{1}{4} + (3-2)\frac{1}{8} + (4-3)\frac{1}{16} + \text{———}$

$\left(1 - \frac{1}{2}\right) S = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \text{———}$

This is geometric series with ratio '$\frac{1}{2}$'

(since the sum of infinite geometric progression with first term 'a' and ratio 'r' is

$\frac{a}{1-r}$; $r < 1$.)

$$\frac{1}{2} \cdot S = \frac{\frac{1}{2}}{1 - \frac{1}{2}} = 1.$$

$S = 2$, which implies $\underline{\underline{E = 2}}$.

Ex! 3 :- An urn contains 7 white and 3 red balls. Two balls are drawn together, at random from this urn. Compute the probability the neither of them is white. Find also the probability of getting one white and one red ball. Hence compute the expected number of white balls drawn.

Sol^n:- Let X be denote the number of white balls drawn. The probability distribution of X is obtained as follows:

x	0	1	2
$p(x)$	${}^3C_2 / {}^{10}C_2 = \frac{1}{15}$	$\frac{{}^7C_1 \times {}^3C_1}{{}^{10}C_2} = \frac{7}{15}$	$\frac{{}^7C_2}{{}^{10}C_2} = \frac{7}{15}$

Expected no. of white ball drawn is:

$$E(X) = \sum x \cdot p(x) = 0 \cdot \frac{1}{15} + 1 \cdot \frac{7}{15} + 2 \cdot \frac{7}{15}$$

$$= \frac{21}{15}$$

→ P(Neither of two balls are white) $= \frac{1}{15}$

→ P(one white and one red ball) $= \frac{7}{15}$.

Ex 4:- Two r.v.s' X and Y have the following joint probability density function:

$$f(x,y) = \begin{cases} 2 - x - y ; & 0 \le x \le 1,\ 0 \le y \le 1 \\ 0 ; & \text{otherwise.} \end{cases}$$

Find (i) marginal p.d.f of X & Y.

(ii) conditional density function

(iii) V(X) and V(Y).

Sol^n:- (i) $f_X(x) = \int_{-\infty}^{\infty} f(x,y)\, dy = \int_0^1 (2 - x - y)\, dy = \left[(2-x)y - \frac{y^2}{2}\right]_0^1$

$$= 2 - x - \frac{1}{2} = \begin{cases} \frac{3}{2} - x ; & 0 < x < 1 \\ 0 ; & \text{o.w.} \end{cases}$$

Similarly; $f_Y(y) = \begin{cases} \frac{3}{2} - y & ; 0<y<1 \\ 0 & ; \text{o.w.} \end{cases}$

(ii) $f_{X/Y}(x/y) = \dfrac{f_{XY}(x,y)}{f_Y(y)} = \dfrac{(2-x-y)}{\left(\frac{3}{2}-y\right)}$; $0<x<1$, $0<y<1$.

$f_{Y/X} = \dfrac{f_{XY}}{f_X} = \dfrac{(2-x-y)}{\left(\frac{3}{2}-x\right)}$; $0<x<1$, $0<y<1$.

(iii) $E(X) = \int_0^1 x \cdot f_X(x)\,dx = \int_0^1 x \cdot \left(\frac{3}{2}-x\right) dx$

$= \left[\frac{3}{2} \cdot \frac{x^2}{2} - \frac{x^3}{3}\right]_0^1$

$= \frac{3}{4} - \frac{1}{3} = \frac{9-4}{12} = \frac{5}{12}.$

$E(Y) = \int_0^1 y \cdot f_Y(x)\,dx = \int_0^1 y \cdot \left(\frac{3}{2}-y\right) dx.$

$= \frac{5}{12}.$

$E(X^2) = \int_0^1 x^2\left(\frac{3}{2}-x\right) dx = \left[\frac{3}{2} \cdot \frac{x^3}{3} - \frac{x^4}{4}\right]_0^1$

$= \frac{3}{6} - \frac{1}{4} = \frac{12-6}{24} = \frac{6}{24} = \frac{1}{4}$

Similarly, $E(Y^2) = \frac{1}{4}.$

$V(X) = E(X^2) - [E(X)]^2.$

$= \frac{1}{4} - \frac{25}{144} = \frac{36-25}{144} = \frac{11}{144}.$

Similarly; $V(Y) = \frac{11}{144}.$

$E\cancel{(X/Y=y)} = \int$

Ex:5 :- Let $f(x,y) = \begin{cases} 8xy & ; \ 0 < x < y < 1 \\ 0 & ; \ \text{o.w.} \end{cases}$

Find (a) $E(Y/X=x)$ (b) $E(XY/X=x)$.

(c) $V(Y/X=x)$.

Solⁿ :- $f_X(x) = \int_{-\infty}^{\infty} f(x,y)\,dy$

$= 8\int_{x}^{1} xy\,dy = 8x\left[\frac{y^2}{2}\right]_x^1 = 4x(1-x^2)\,; \ 0<x<1.$

$f_Y(y) = \int_{-\infty}^{\infty} f(x,y)\,dx.$

$= \int_0^y 8xy\,dx = 8y\left[\frac{x^2}{2}\right]_0^y = 4y^3, \quad 0<y<1.$

$f_{X/Y} = \frac{f_{XY}}{f_Y} = \frac{2x}{y^2},$

$f_{Y/X} = \frac{2y}{1-x^2}\,; \ 0<x<y<1.$

a) $E(Y/X=x) = \int_{-\infty}^{\infty} y\cdot f_{Y/X}\,dy = \int_x^1 y\cdot\frac{2y}{1-x^2}\,dy$

$= \frac{2}{1-x^2}\left(\frac{y^3}{3}\right)_x^1 = \frac{2}{1-x^2}\cdot\frac{1-x^3}{3} = \frac{2}{3}\frac{(1-x^3)}{(1-x^2)}.$

b.) $E(XY/X=x) = x\,E(Y/X=x) = \frac{2x}{3}\frac{(1-x^3)}{(1-x^2)}$

c.) $V(Y/X=x) = E(Y^2/X=x) - E(Y/X=x)$

$= \int_x^1 y^2 f_{Y/X}\,dy - \left[\frac{2}{3}\frac{(1-x^3)}{(1-x^2)}\right]^2$

$= \frac{1+x^2}{2} - \frac{4}{9}\frac{(1-x^3)^2}{(1-x^2)^2}.$

Moments

Moment generating function (about the origin) :-

The moment generating function (m.g.f) of a random variable X (about origin) having probability function $f(x)$ is given by :

$$M_X(t) = E(e^{tX}) = \begin{cases} \int e^{tx} f(x)\,dx, & \text{for cont}^n \text{ prob. dist}^n \\ \sum_x e^{tx} f(x), & \text{for discrete} \end{cases}$$

Alternatively, (on the based of absolutely convergent series)

$$M_X(t) = E(e^{tX}) = E\left(1 + tX + \frac{t^2 X^2}{2!} + \text{———}\right)$$

(By exponential function's expansion)

$$= 1 + t\,E(X) + \frac{t^2}{2!} E(X^2) + \text{———}$$

$$= 1 + t\mu_1' + \frac{t^2}{2!}\mu_2 + \text{—} + \frac{t^r}{r!}\mu_r + \text{—}$$

$$= \sum_{r=0}^{\infty} \frac{t^r}{r!}\mu_r.$$

where $\mu_r = E(X^r) = \begin{cases} \int_{-\infty}^{\infty} x^r f(x) & ;\text{ for cont}^n \\ \sum_x x^r f(x) & ;\text{ for discrete.} \end{cases}$

Remarks :- (1) μ_r is the r^{th} moment of X about origin.

Properties:-

- $M_{cX}(t) = M_X(ct)$, c being a constant.
- If $X_1, X_2 — X_n$ are independent random variables, then moment generating function of their sum $X_1+X_2+ — X_n$ is given by:

$$M_{X_1+X_2+ — +X_n}(t) = M_{X_1}(t) \cdot M_{X_2}(t) \cdot \ldots \cdot M_{X_n}(t).$$

- The first moment about the origin of a random variable is called the mean and is denoted by μ.

Central Moments

The n^{th} central moment of a random variable X is defined as;

$$E[(X-\mu_X)^n] = \begin{cases} \int (x-\mu_X)^n f_X(x)\,dx. & (\text{cont}^n.) \\ \sum_k (x_k-\mu_X)^k P_X(x_k). & (\text{Discrete}) \end{cases}$$

Remark:- • The second moment about the mean or second central moment is variance i.e. σ_X^2, (which is the lowest central moment of real interest because the first central moment is zero as $E[X-\mu_X] = E(X)-\mu_X = \mu_X-\mu_X = 0$.)

- The standard deviation can be defined as;

$$\sigma_X = \sqrt{E[(X-\mu_X)^2]}.$$

- Both the variance and standard deviation serve as a measure of the width of the pdf of a random variable.

~~The~~ . Skewness

The third central moment is known as the skewness.

Coefficient of Skewness :- The coefficient of skewness is

$$C_X = \frac{E[(X-\mu_X)^3]}{\sigma_X^3},$$

this is a dimensionless quantity that is +ve if the random variable has a pdf skewed to the right and -ve if skewed to the left.

Chebychev's Inequality

Definition :- If X is a random variable with mean μ and variance σ^2, then for any positive number k, we have,

$$P\{|X-\mu| \geq k\sigma\} \leq \frac{1}{k^2}$$

or

$$P\{|X-\mu| < k\sigma\} \geq 1-\frac{1}{k^2}$$

Markov's Inequality

If x is a random variable with mean μ, and for a non-n for $k>0$,

$P\{|X| \geq k\} \leq \frac{E\{|X|\}}{k}$, is the markov's inequality.

Markov's Inequality

Let X be a non-negative random variable, for any real number $k > 0$,

$$P\{X \geq k\} \leq \frac{E(X)}{k},$$ is known as Markov's inequality.

Chernoff's bounds

Suppose X is a random variable whose moment generating function is $M_X(s)$. Then for any $k \in R$.

$$P\{X \geq k\} \leq \min_{s \geq 0} e^{-sk} M_X(s),$$

$$P\{X \leq k\} \leq \min_{s < 0} e^{-sk} M_X(s).$$

Distribution Functions with their mean, variance and other Properties

1 Binomial Distribution function / Binomial Random V.

Binomial distribution was discovered by James Bernou in the year 1700.

Let us visualize a concept or practical situation, where a trial or an experiment results in only two outcomes; say success and failure, the results of one trial doesn't influence the result of next trial, and the probability of the outcomes (success/failure) at each trial is the same from trial to trial.

Some of these situations are;

- Tossing a coin - Head or Tail.
- Birth of a baby - Girl or Boy
- Auditing a bill - Contains an error or not
- An item is defective or not.

Conditions for the applicability of Binomial distribution:-

1) There are only two possible outcomes
2.) The trial is performed repeatedly for a finited fixed number of trials.
3.) All the trials are independent.
4.) The probability of the each outcome ~~remo~~ remains constant.

a) Distribution function:-

$$F_X(x) = P(X \le x) = \sum_{r=0}^{n} {}^nC_r\, p^r q^{n-r},$$

where p is the probability of success and $q = 1-p$ is the probability of failure.

b.) Expected Value :-

$$E(X) = \sum_{x=0}^{n} x \cdot P(x) \quad ; \quad P(X=r) = p(r) = {}^nC_r\, p^r q^{n-r}$$

$r = 0, 1, 2 —— n.$

$$= \sum_{r=0}^{n} r \cdot {}^nC_r\, p^r q^{n-r}$$

$$= 0 + {}^nC_1\, p\, q^{n-1} + 2\cdot {}^nC_2\, p^2 q^{n-2} + —— + n \cdot {}^nC_n\, p^n q^0$$

$$= \frac{n!}{1!(n-1)!}\, p \cdot q^{n-1} + 2 \cdot \frac{n!}{2!(n-2)!}\, p^2 q^{n-2} ——$$

$$+ n \frac{n!}{n!(n-n)!}\, p^n q^0$$

$$= npq^{n-1} + n(n-1)\, p^2 q^{n-2} + —— + np^n$$

$$= np\left[q^{n-1} + (n-1)\, p q^{n-2} + \frac{(n-1)(n-2)}{2!}\, p^2 q^{n-3} + —— p^{n-1}\right]$$

$$= np\left[{}^{n-1}C_0\, p^0 q^{n-1} + {}^{n-1}C_1\, p^1 q^{n-2} + {}^{n-1}C_2\, p^2 q^{n-3}\right.$$

$$\left. + —— + {}^{n-1}C_{n-1}\, p^{n-1} q^0\right]$$

$$= np\,[p+q]^{n-1} \quad \because p+q = 1.$$

$$\boxed{E(X) = np = \mu}$$

c.) Variance :-

$$v(x) = \sum_{r=0}^{n} r^2 p(r) - \mu^2 \qquad \left(\because var = \sigma^2 = \sum_{i=0}^{n} p_i x_i^2 - (mean)^2 \right)$$

$$\sum_{r=0}^{n} r^2 p(r) = \sum_{r=0}^{n} [r + r(r-1)]\, {}^nC_r p^r q^{n-r} \qquad \left(\because r^2 = r + r(r-1) = \cancel{r} + r^2 - \cancel{r} = r^2 \right)$$

$$= \sum_{r=0}^{n} r\, {}^nC_r p^r q^{n-r} + \sum_{r=0}^{n} r(r-1)\, {}^nC_r p^r q^{n-r}$$

$$= np + \sum_{r=0}^{n} r(r-1)\, {}^nC_r p^r q^{n-r}$$

$$= np + 0 + 0 + \sum_{r=2}^{n} r(r-1)\, {}^nC_r p^r q^{n-r}$$

$$= np + \sum_{r=2}^{n} \cancel{r(r-1)} \cdot \frac{n(n-1)(n-2)\cdots(n-(r-1))\cancel{(n-r)!}}{\cancel{r(r-1)}(r-2)!\cancel{(n-r)!}} \cdot p^r q^{n-r}$$

$$= np + \sum_{r=2}^{n} \frac{n(n-1)(n-2)\cdots(n-(r-1))}{(r-2)!} p^r q^{n-r}$$

$$= np + n(n-1)p^2 \sum_{r=2}^{n} \frac{(n-2)\cdots(n-(r-1))}{(r-2)!} p^{r-2} q^{[(n-2)-(r-2)]}$$

$$= np + n(n-1)p^2 \sum_{r=2}^{n} \frac{(n-2)\cdots(n-(r-1))(n-r)!}{(r-2)![(n-2)-(r-2)]!} p^{r-2} \times q^{[(n-2)-(r-2)]}$$

$$= np + n(n-1)p^2 \sum_{r=2}^{n} {}^{n-2}C_{r-2}\, p^{r-2} q^{[(n-2)-(r-2)]}$$

$$= np + n(n-1)p^2 \left[{}^{n-2}C_0 \, p^0 q^{[(n-2)-0]} + {}^{n-2}C_1 p^1 q^{[(n-2)-1]} + \text{———} \; {}^{n-2}C_{n-2} \, p^{n-2} \, q^{[(n-2)-(n-2)]} \right]$$

$$= np + n(n-1)p^2 \left[\sum_{r=0}^{n-2} {}^{n-2}C_r \, p^r \, q^{[(n-2)-r]} \right]$$

$$= np + n(n-1)p^2 (p+q)^{n-2}$$ ($\because$ Binomial Expansion) ($p+q=1.$)

$$= np + n(n-1)p^2$$

variance $= V(x) = \sigma^2 = np + \cancel{n^2p^2} - np^2 - \cancel{(np)^2}$

$$= np(1-p)$$

$$= \underline{\underline{npq}}.$$

d) standard deviation $= \sqrt{\sigma^2} = \sqrt{npq}$.

e.) moment generating function :-

$$M_X(t) = E(e^{Xt}) = \sum_{x=0}^{n} e^{tx} p(x)$$

$$= \sum_{x=0}^{n} e^{tx} \, {}^nC_x \, p^x (q)^{n-x}$$

$$= \sum_{x=0}^{n} {}^nC_x (pe^t)^x q^{n-x}$$

$$= (pe^t + q)^n$$

$\because$ By using binomial expansion.

Note that, the mean is the first moment:

$$\mu = \frac{d}{dt} M_X(t) \Big|_{at\ t=0} = n(pe^t+q)^{n-1} pe^t \Big|_{at\ t=0}$$

$$= n(pe^0+q)^{n-1} pe^0$$

$$= n(p+q)^{n-1} p = np \quad \because (p+q=1.)$$

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Poisson Distribution

The poisson distribution can be derived as limiting case of the binomial distribution under the following conditions:

i) n, the number of trials is indefinitely large i.e., $n \to \infty$.

ii) p, the constant probability of success is very small i.e., $p \to 0$.

iii) np is finite, say $np = \lambda$, then λ is a real +ve number as well as called the parameter of poisson distribution.

Derivation through Binomial distribution:

(a) The probability of 'r' successes in a series of indepent trials is given by

$$p(r) = {}^nC_r \, p^r q^{n-r}$$

$$= {}^nC_r \, p^r (1-p)^{n-r}$$

$$= \frac{n(n-1) \text{———} (n-(r-1)) \cdot \cancel{(n-r)!}}{r! \, \cancel{(n-r)!}} \cdot p^r \, \frac{(1-p)^n}{(1-p)^r}$$

put, $np = \lambda \to \cancel{p = \frac{\lambda}{n}}$ ~~and~~ $n = \frac{\lambda}{p}$

$$= \frac{\frac{\lambda}{p} \cdot \left(\frac{\lambda}{p} - 1\right) \text{———} \left(\frac{\lambda}{p} - (r-1)\right) \frac{\lambda}{p} \, p^r}{r!} \frac{\left(1 - \frac{\lambda}{n}\right)^n}{(1-p)^r}$$

$$= \frac{\lambda(\lambda - p) \text{———} (\lambda - p(r-1)) \, \cancel{p^r}}{\cancel{p^r} \, r!} \times \frac{\left(1 - \frac{\lambda}{n}\right)^n}{(1-p)^r}$$

Now, use $p = \frac{\lambda}{n}$.

$$p(r) = \frac{\lambda \cdot \lambda \cdot \text{---} \lambda \left(1-\frac{1}{n}\right)\cdot\left(1-\frac{2}{n}\right) \text{---} \left(1-\frac{r-1}{n}\right)}{r!} \times \frac{\left(1-\frac{\lambda}{n}\right)^n}{(1-p)^r}$$

using $p \to 0$, as $n \to \infty$.

$$= \frac{\lambda^r}{r!} \lim_{n\to\infty} \left(\left(1-\frac{1}{n}\right)\cdot\left(1-\frac{2}{n}\right) \text{---} \left(1-\frac{(r-1)}{n}\right)\right)$$

$$\times \lim_{n\to\infty} \left(1-\frac{\lambda}{n}\right)^n \times \lim_{p\to 0} \frac{1}{(1-p)^r}$$

using $\lim_{n\to\infty}\left(1+\frac{a}{n}\right)^n = e^a$.

and $\lim_{n\to\infty}\left(\frac{1}{1-n}\right)^a = 1$.

$\lim_{n\to\infty}\left(1+\frac{1}{n}\right) = 1$

$p(r) = \frac{\lambda^r}{r!} \cdot e^{-\lambda}$, This is the probability of r success and called as poisson distribution, where $r = 0, 1, 2, \text{---}$.

Conditions of poisson distribution :- The poisson distribution is used under the following conditions and examples :-

1. The variate (no. of occurrence) is a discrete variable.
2. No. of trials is large
3. For examples :-
   1. The no. of car accidents in a unit time at a busy place.

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(2) no. of printing mistake at each page of a good book.

(3) no. of telephone calls received at a particular switch board per minute.

(4) no. of deaths in a city in a year by a rare disease.

and many more.

(b) Expected value of Poisson distribution:-

$$E(X) = \sum_{x=0}^{\infty} x\, p(x) = \sum_{x=0}^{\infty} x \cdot \frac{e^{-\lambda}\lambda^{x}}{x!}$$

$$= \lambda e^{-\lambda} \sum_{x=0}^{\infty} \frac{\lambda^{(x-1)} \cdot \cancel{x}}{\cancel{x}(x-1)!}$$

$$= \lambda e^{-\lambda} \sum_{x=0}^{\infty} \frac{\lambda^{x-1}}{x-1}$$

$$= \lambda e^{-\lambda} \left[1 + \lambda + \frac{\lambda^2}{2!} + \text{——}\right] = \lambda \cancel{e^{-\lambda}} \cdot \cancel{e^{\lambda}} = \lambda.$$

(c) variance :-

$$V(X) = E(X^2) - [E(X)]^2.$$

$$E(X^2) = \sum_{x=0}^{\infty} x^2 \cdot p(x).$$

$$= \sum_{x=0}^{\infty} x^2 \cdot \frac{e^{-\lambda}\lambda^{x}}{x!}$$

$$= e^{-\lambda}\sum_{x=0}^{\infty} \frac{x^2 \lambda^x}{(x\cancel{8})!} = e^{-\lambda}\sum_{x=0}^{\infty} \frac{(x(x-1)+x)\lambda^x}{x!}$$

$\because x^2 = x(x-1) + x$

$$= e^{-\lambda}\left[\sum_{x=0}^{\infty} x(x-1)\frac{\lambda^x}{x!} + \sum_{x=0}^{\infty} \frac{x\lambda^x}{x!}\right]$$

$$= e^{-\lambda}\left[\sum_{x=2}^{\infty} \cancel{x}\cancel{(x-1)} \frac{\lambda^x}{\cancel{x}\cdot\cancel{(x-1)}(x-2)!} + \sum_{x=1}^{\infty} \cancel{x}\frac{\lambda^x}{\cancel{x}\cdot(x-1)!}\right]$$

$$= e^{-\lambda}\left[\sum_{x=2}^{\infty} \frac{\lambda^x}{(x-2)!} + \sum_{x=1}^{\infty} \frac{\lambda^x}{(x-1)!}\right]$$

$$= e^{-\lambda}\left[\lambda^2\sum_{x=2}^{\infty} \frac{\lambda^{x-2}}{(x-2)!} + \lambda\sum_{x=1}^{\infty} \frac{\lambda^{(x-1)}}{(x-1)!}\right]$$

$$= e^{-\lambda}\left[\lambda^2\left(1+\frac{\lambda}{1!}+\frac{\lambda^2}{2!}+\text{—}\right) + \lambda\left(1+\frac{\lambda}{1!}+\frac{\lambda^2}{2!}+\text{—}\right)\right]$$

$$= e^{-\lambda}(\lambda^2 e^{\lambda} + \lambda e^{\lambda})$$

$E(x^2) = \lambda^2+\lambda$

$$V(x) = \cancel{\lambda^2}+\lambda-\cancel{\lambda^2}$$
$$= \lambda$$

(b) standard deviation $= \sqrt{\lambda}$.

(c) <u>Moment generating function :-</u>

$$M_X(t) = E(e^{tX}) = \sum_{x=0}^{\infty} e^{xt}\, p(x).$$

$$= \sum_{x=0}^{\infty} e^{xt}\frac{e^{-\lambda}\lambda^x}{x!}$$

$$= e^{-\lambda}\sum_{x=0}^{\infty} \frac{(e^t\lambda)^x}{x!}$$

$$= e^{-\lambda}e^{\lambda e^t}$$

Remark :- verify for mean as first moment.

$$\frac{d}{dt}M_x(t)\Big|_{\text{at } t=0} = e^{-\lambda}e^{\lambda e^t}\lambda e^t\Big|_{\text{at } t=0} = e^{-\lambda}e^{\lambda e^0}\lambda e^0 = \lambda.$$

(2) Variance = 2nd moment − (first moment)^2.

Discrete Uniform Distribution

In this distribution, the different values of the random variable become equally likely.

(a) **Distribution** :- Recalling the mass function $p(x)$

$$p(x) = P(X = x) = \begin{cases} \frac{1}{n}, & x = 1, 2, \ldots, n \\ 0, & \text{o.w.} \end{cases}$$

$$P(X \leq x) = \begin{cases} \sum_{i=1}^{x} p(x_i) \Rightarrow x = 1, \ldots, N \\ 0, & \text{o.w.} \end{cases}$$

(b) **Mean** (μ).

$$E(X) = \frac{1}{n} \sum_{i=1}^{n} i = \frac{1}{n}[1 + 2 + 3 + \cdots + n]$$

$$= \frac{1}{\cancel{n}} \frac{\cancel{n}(n+1)}{2} = \frac{n+1}{2}.$$

(c) **Variance** :-

$$V(X) = E(X^2) - (E(X))^2$$

$$= \sum_{i=1}^{n} \frac{i^2}{n} - \frac{(n+1)^2}{4} = \frac{1}{n}[1^2 + 2^2 + \cdots + n^2] - \frac{(n+1)^2}{4}$$

$$= \frac{1}{\cancel{n}} \left(\frac{\cancel{n}(n+1)(2n+1)}{6} \right) - \frac{(n+1)^2}{4}$$

$$= \frac{(n+1)}{2} \left[\frac{2n+1}{3} - \frac{(n+1)}{2} \right] = \frac{(n+1)}{2} \left[\frac{4n+2-3n-3}{6} \right]$$

$$= \frac{(n+1)}{2} \left[\frac{n-1}{6} \right] = \frac{(n-1)(n+1)}{12}.$$

(d) standard deviation : $\sqrt{\frac{(n-1)(n+1)}{12}}$

(e) **Moment generating function :-**

$$M_X(t) = E(e^{Xt}) = \sum_{x=1}^{n} e^{xt} p(x).$$

$$= \frac{1}{n} \sum_{x=1}^{n} e^{xt}$$

$$= \frac{1}{n} \cdot [e^t + e^{2t} + \text{———} \; e^{nt}]$$

$\therefore$ G.P series

$$r = \frac{e^{2t}}{e^t} = e^t$$

$$\text{Sum} = \begin{cases} a\left(\frac{1-r^n}{1-r}\right) & ; \; |r|<1 \\ a\left(\frac{r^n-1}{r-1}\right) & ; \; |r|>1 \\ an & ; \; r=1. \end{cases}$$

$$= \frac{1}{n} e^t \left[\frac{1-e^{nt}}{1-e^t}\right]$$

Continuous Uniform Distribution

(a) The distribution function;

$$F(x) = \int_a^x f(x)\,dx = \int_a^x \frac{1}{b-a}\,dx.$$

$$= \begin{cases} \dfrac{(x-a)}{b-a}, & a < x < b \\ 0 & x \leq a \\ 1 & x \geq b \end{cases}.$$

(b) Expected value:

$$E(X) = \int_a^b x.f(x)\,dx = \int_a^b x.\frac{1}{b-a} = \frac{1}{(b-a)}\left(\frac{x^2}{2}\right)_a^b$$

$$= \frac{b^2-a^2}{2(b-a)} = \frac{(b+a)}{2}.$$

(c) Variance :-

$$V(x) = E(X^2) - (E(X))^2$$

$$E(X^2) = \int_a^b x^2 f(x)\,dx = \frac{x^3}{3}\Big|_a^b \times \frac{1}{(b-a)}$$

$$= \frac{1}{(b-a)} \cdot \frac{b^3-a^3}{3}$$

$$= \frac{b^2+a^2+ab}{3}$$

$$V(x) = \frac{b^2}{3} + \frac{a^2}{3} + \frac{ab}{3} - \left(\frac{b^2}{4} + \frac{a^2}{4} + \frac{2ab}{4}\right)$$

$$= \frac{(b-a)^2}{12}$$

(4) Moment generating function:-

$$M_x(t) = \int_a^b e^{tx} f(x)\,dx = \int_a^b \frac{e^{tx}}{b-a} = \frac{e^{bt} - e^{at}}{t(b-a)}, \quad t \neq 0$$

Remark:- Verify the first moment equal to mean.

Normal Distribution

The normal distribution is a limiting case of the binomial distribution under the following conditions;

(i) n, the no. of trials is indefinitely large i.e. $n \to \infty$

(ii) neither p nor q is very small.

Applicable in the condition (Ex.)

1. Length and diameter of certain products like pipe, discs etc.
2. Height and weight of a baby at birth.
3. Weekly sales of an item. etc.

(a.) <u>Distribution function:-</u>

$$F(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}\,\sigma} \exp\left[\frac{-1}{2\sigma^2}(x-\mu)^2\right] dx.$$

where μ & σ are the parameters.

(b) <u>Mean:-</u>

$$E(x) = \frac{1}{\sqrt{2\pi}\,\sigma} \int_{-\infty}^{\infty} x \cdot e^{-\frac{1}{2\sigma^2}(x-\mu)^2} dx$$

Let $x-\mu = z \rightarrow dx = dz$

$x = z+\mu$

$$= \frac{1}{\sqrt{2\pi}\,\sigma} \int_{-\infty}^{\infty} (z+\mu)\, e^{-\frac{1}{2\sigma^2}z^2} dz$$

$$= \frac{1}{\sqrt{2\pi}\,\sigma} \int_{-\infty}^{\infty} z\, e^{-\frac{1}{2\sigma^2}z^2} dz + \frac{1}{\sqrt{2\pi}\,\sigma} \int_{-\infty}^{\infty} \mu\, e^{-\frac{z^2}{2\sigma^2}} dz.$$

$$= \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} z\, e^{-\frac{z^2}{2\sigma^2}} dz + \mu \underbrace{\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{z^2}{2\sigma^2}} dz}_{=}$$

$$= \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} z\, e^{-\frac{z^2}{2\sigma^2}} dz + \mu \underbrace{\int_{-\infty}^{\infty} (\text{p.d.f of standard normal dist}^n)\, dz}_{=1.}$$

(By taking $\frac{z}{\sigma}$ = new variable)

$$= \frac{1}{\sigma\sqrt{2\pi}} \underbrace{\int_{-\infty}^{\infty} z\, e^{-\frac{z^2}{2\sigma^2}}}_{(= \text{odd})} dz + \mu$$

$= \mu + 0$

[$\because$ Since $\int_{-a}^{a} f(x)\,dx = \begin{cases} 2\int_0^a f(x), & \text{if } f(x) = \text{even} \\ 0, & \text{if } f(x) = \text{odd.} \end{cases}$]

$= \underline{\mu}.$

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Variance:-

$$\cancel{V(x^2) = \frac{1}{\sigma\sqrt{2\pi}}\int_{-\infty}^{\infty} x^2 e^{-\frac{1}{2\sigma^2}(x-\mu)^2}\,dx - (E(x))^2}$$

$$V(x) = E(x^2) - [E(x)]^2$$

$$= E(x-\mu)^2$$

$$= \frac{1}{\sigma\sqrt{2\pi}}\int_{-\infty}^{\infty}(x-\mu)^2 e^{-\frac{1}{2\sigma^2}(x-\mu)^2}\,dx.$$

Let $z = \frac{x-\mu}{\sigma}$ $\quad dz = \frac{dx}{\sigma}$

$$= \frac{1}{\cancel{\sigma}\sqrt{2\pi}}\int_{-\infty}^{\infty}\sigma^2 z^2 \cdot e^{-\frac{1}{2}z^2}\cancel{\sigma}\,dz$$

$$= \frac{\sigma^2}{\sqrt{2\pi}}\int_{-\infty}^{\infty} z^2 e^{-\frac{1}{2}z^2}\,dz = \frac{2\sigma^2}{\sqrt{2\pi}}\int_0^{\infty} z^2 e^{-\frac{1}{2}z^2}\,dz$$

(due to integration property, $f(x)$ = even)

Let $\frac{z^2}{2} = y$

$z\,dz = dy \Rightarrow dz = \frac{dy}{\sqrt{2y}}$

$$= \frac{2\sigma^2}{\sqrt{2\pi}}\int_{0}^{\infty} 2y\, e^{-y}\,\frac{dy}{\sqrt{2}\sqrt{y}}$$

$$= \frac{\cancel{4}\sigma^2}{\cancel{2}\sqrt{\pi}}\int_{0}^{\infty}\sqrt{y}\, e^{-y}\,dy.$$

$$= \frac{2\sigma^2}{\sqrt{\pi}}\int_0^{\infty} y^{\frac{1}{2}=\frac{3}{2}-1}\, e^{-y}\,dy$$

∵ since the formula of Gamma func.

$$\Gamma(z) = \int_0^\infty e^{-x} x^{z-1} dx.$$

$$= \frac{2\sigma^2}{\sqrt{\pi}} \Gamma\left(\frac{3}{2}\right)$$

∵ $\Gamma n = (n-1)!$
$= (n-1)\Gamma(n-1)$

$$= \frac{2\sigma^2}{\sqrt{\pi}} \Gamma\left(\frac{1}{2}+1\right)$$

$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$

$$= \frac{\cancel{2}\sigma^2}{\sqrt{\pi}} \left(\frac{1}{\cancel{2}}\right) \Gamma\left(\frac{1}{2}\right)$$

$$= \frac{\sigma^2}{\cancel{\sqrt{\pi}}} \times \cancel{\sqrt{\pi}}$$

$$= \underline{\underline{\sigma^2}}.$$

(d) Stand deviation $= \underline{\underline{\sigma}}$.

(e) Moment generating function:-

$$M_x(t) = \int_{-\infty}^{\infty} e^{tx} f(x)\, dx$$

$$= \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{tx} e^{\frac{-(x-\mu)^2}{2\sigma^2}} dx$$

Let $\frac{x-\mu}{\sigma} = z \Rightarrow x - \mu = z\sigma$

$\Rightarrow x = (z\sigma + \mu)$

$dx = \sigma dz$

$$= \frac{1}{\cancel{\sigma}\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{t(\mu+z\sigma)} e^{-\frac{z^2}{2}} \cancel{\sigma} dz$$

$$= \frac{e^{\mu t}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{z(\sigma t)} e^{-z^2/2} dz$$

Taking $t\sigma z - \frac{z^2}{2} = -\frac{1}{2}\left[z^2 - 2t\sigma z + (\sigma t)^2 - (\sigma t)^2\right]$

$$= -\frac{1}{2}\left[(z - t\sigma)^2 - (\sigma t)^2\right]$$

$$= -\frac{1}{2}\left[z - t\sigma\right]^2 + \frac{(\sigma t)^2}{2}$$

$$= \frac{e^{\mu t}}{\sqrt{2\pi}} \times e^{\frac{(\sigma t)^2}{2}} \int_{-\infty}^{\infty} e^{-\frac{1}{2}(z-\sigma t)^2}\, dz$$

$$= \frac{e^{\left(\mu t + \frac{\sigma^2 t^2}{2}\right)}}{\sqrt{2\pi}} \underbrace{\int_{-\infty}^{\infty} e^{-\frac{1}{2}(z-\sigma t)^2}\, dz}_{\text{p.d.f of standard normal dist}^n = 1.}$$

$$= e^{\left(\mu t + \frac{\sigma^2 t^2}{2}\right)}$$

Standard Normal Distribution

In this case, we use a substitution as

$Z = \frac{x-\mu}{\sigma}$, is called standard normal variable.

Hence, we have p.d.f of 'x'

$$f(x) = \frac{1}{\sqrt{2\pi}\,\sigma} Exp\left[-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2\right]; -\infty < x < \infty.$$

became new p.d.f for 'z' variable.

$$g(z) = \frac{1}{\sqrt{2\pi}} Exp\left[-\frac{1}{2} z^2\right].$$

In this case mean = 0, σ = '1'.

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Exponential Distribution

\# The p.d.f is

$$f(x) = \begin{cases} \lambda e^{-\lambda x}, & x \geq 0 \\ 0 & \text{o.w.} \end{cases}$$

(a) Distribution :-

$$F(x) = \int_0^{x} f(s)\, ds$$

$$= \lambda \cdot \int_0^{x} e^{-\lambda s}\, ds$$

$$= \cancel{\lambda}\left(\frac{e^{-\lambda s}}{-\cancel{\lambda}}\right)_0^{x}$$

$$= -\left(e^{-\lambda x} - 1\right)$$

$$= \begin{cases} 1 - e^{-\lambda x}, & x \geq 0 \\ 0 & \text{o.w.} \end{cases}$$

(b) Mean :-

$$E(X) = \int_{-\infty}^{\infty} x f(x)\, dx.$$

$$= \int_{-\infty}^{\infty} x \lambda e^{-\lambda x}\, dx.$$

$$= \lambda \int_0^{\infty} x \cdot e^{-\lambda x}\, dx$$

$$= \lambda \left[\left[x \cdot \frac{e^{-\lambda x}}{-\lambda}\right]_0^{\infty} - \int_0^{\infty} \frac{e^{-\lambda x}}{-\lambda}\, dx\right]$$

$$= \lambda\left[0 - 0 - \left[\frac{e^{-\lambda x}}{(-\lambda)^2}\right]_0^\infty\right]$$

$$= \lambda\left(-\left(0 - 1 \cdot \frac{1}{\lambda^2}\right)\right)$$

$$= \cancel{\lambda} \cdot \frac{1}{\lambda^{\cancel{2}}} = \frac{1}{\lambda}.$$

(c) Variance :-

$$E(X^2) = \int_0^\infty x^2 f(x)\,dx$$

$$= \int_0^\infty x^2 \lambda e^{-\lambda x}\,dx$$

$$= \lambda \int_0^\infty x^2 e^{-\lambda x}\,dx$$

$$= \lambda\left[\left(x^2 \frac{e^{-\lambda x}}{-\lambda}\right)_0^\infty + \frac{2}{\lambda}\underbrace{\int_0^\infty x\, e^{-\lambda x}\,dx}_{=\frac{1}{\lambda^2}}\right]$$

$$= \cancel{\lambda}\left[\frac{2}{\cancel{\lambda}} \times \frac{1}{\lambda^2}\right]$$

$$= \frac{2}{\lambda^2}$$

$$V(X) = E(X^2) - (E(X))^2$$

$$= \frac{2}{\lambda^2} - \frac{1}{\lambda^2} = \frac{1}{\lambda^2}.$$

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Moment generating function :-

$$M_x(t) = \int_0^{\infty} e^{tx} f(x)\, dx$$

$$= \lambda \int_0^{\infty} e^{tx} e^{-\lambda x}\, dx$$

$$= \lambda \cdot \int_0^{\infty} e^{(t-\lambda)x}\, dx$$

$$= \lambda \left[\frac{e^{(t-\lambda)x}}{(t-\lambda)} \right]_0^{\infty}$$

$$= \frac{\lambda}{t-\lambda} \left[e^{-(\lambda-t)x} \right]_0^{\infty}$$

$$= \frac{\lambda}{t-\lambda} [0 - 1] \; ; \; \lambda > t.$$

$$= \frac{\lambda}{\lambda - t} \; ; \; \lambda > t.$$

Rayleigh Distribution

The probability density function is

$$f(x,\sigma) = \frac{x}{\sigma^2} e^{\frac{-x^2}{2\sigma^2}}, \quad x \geq 0. \quad \text{--- (1)}$$

where 'σ' is the parameter.

(a) **Distribution :-**

$$F(x,\sigma) = P(X \leq x) \quad \text{--- (2)}$$

$$= \int_{-\infty}^{x} f(x,\sigma)\, dx$$

$$= \int_{0}^{x} \frac{x}{\sigma^2} e^{\frac{-x^2}{2\sigma^2}} dx. \quad \text{--- (3)}$$

Let $\frac{x^2}{2\sigma^2} = z$

$$\frac{\cancel{2}x}{\cancel{2}\sigma^2} dx = dz$$

$$= \int_{0}^{x^2/2\sigma^2} e^{-z} dz \quad \text{--- (4)}$$

$$= \left[\frac{-e^{-z}}{(-1)}\right]_0^{\frac{x^2}{2\sigma^2}} = -e^{-\frac{x^2}{2\sigma^2}} + 1$$

$$= 1 - e^{-x^2/2\sigma^2}$$

$$F(x;\sigma) = 1 - e^{-x^2/2\sigma^2}, \quad x \geq 0.$$

(b) **Mean :-**

$$E(X) = \int_0^{\infty} x \cdot f(x)\, dx = \int_0^{\infty} x \cdot \frac{x}{\sigma^2} e^{-\frac{x^2}{2\sigma^2}} dx \quad \text{--- (5)}$$

$$= \int_0^\infty \frac{2 x^2}{2 \sigma^2} e^{-x^2/2\sigma^2} \, dx.$$

Let $\frac{x^2}{2\sigma^2} = z \Rightarrow \frac{\cancel{2}x \, dx}{\cancel{2}\sigma^2} = dz \Rightarrow \frac{dx}{\sigma^2} = \frac{dz}{\sqrt{2\sigma^2 z}}$

$x = \sqrt{2\sigma^2 z}$

$$= \int_0^\infty 2z \, e^{-z} \frac{\sigma^2 dz}{\sqrt{2}\,\sigma\sqrt{z}}$$

$$= \frac{\sqrt{2}\,\sigma^2}{\sigma} \int_0^\infty z^{\frac{3}{2}-1} e^{-z} dz$$

using the Gamma function as

$$\Gamma(\omega) = \int_0^\infty x^{\omega-1} e^{x} \, dx, \quad \text{Real}(\omega) > 0$$

and $\Gamma(n) = (n-1)\,\Gamma(n-1)$

$$= \frac{\sqrt{2}\,\sigma^2}{\sigma} \Gamma\left(\frac{3}{2}\right)$$

$$= \sigma^2 \frac{\sqrt{2}}{\sigma} \times \frac{1}{2} \Gamma\frac{1}{2} \qquad \because \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$$

$$= \frac{\sigma^2 \sqrt{\pi}}{\sqrt{2}\,\sigma} = \sqrt{\frac{\pi}{2}} \times \frac{\sigma^{\cancel{2}}}{\cancel{\sigma}}$$

$$= \sigma\sqrt{\frac{\pi}{2}}$$

(c) Variance :- $E(X^2) - [E(X)]^2 = E(X-\mu)^2$

$$E(X^2) = \int_0^\infty x^2 f(x)\, dx$$

$$= \int_0^\infty x^2 \frac{x}{\sigma^2} e^{-x^2/2\sigma^2} dx$$

$$= \Big[x^2 \underbrace{\int \frac{x}{\sigma^2} e^{-x^2/2\sigma^2} dx}_{A}\Big]_0^\infty - \int_0^\infty \frac{d[x^2]}{dx}\Big(\underbrace{\int \frac{x}{\sigma^2} e^{-x^2/2\sigma^2}}_{A}\Big) dx$$

(By integration rule)

Now using eq. (2) we get the integration of (A)

=

$$\int \frac{x}{\sigma^2} e^{-x^2/2\sigma^2} dx = -e^{-x^2/2\sigma^2} \text{ ——— (6)}$$

$$= \Big[-x^2 e^{-x^2/2\sigma^2}\Big]_0^\infty + \int_0^\infty 2x\, e^{-x^2/2\sigma^2} dx.$$

$$= 0 + 2\int_0^\infty x\, e^{-x^2/2\sigma^2} dx$$

$$= 2\sigma^2 \int_0^\infty \frac{x}{\sigma^2} e^{-\frac{x^2}{2\sigma^2}} dx$$

again using (6), we obtained

$$= 2\sigma^2 \Big[-e^{-x^2/2\sigma^2}\Big]_0^\infty = 2\sigma^2$$

$$V(X) = 2\sigma^2 - \sigma^2 \frac{\pi}{2} = \frac{\sigma^2[4-\pi]}{2}$$