**Butterworth Filters**

Butterworth filters are characterized by the property that the magnitude characteristic is maximally flat at the origin, and there are no 'ripples', either in the pass band or in the stop band. The sequence of topics covered is as follows:

- The Butterworth filter, its specifications
- The $n^{th}$ order Low-pass Butterworth filter with unity cut-off
- Design steps in obtaining the transfer function of the filter to meet the specifications
- Frequency transformation to obtain high-pass, band-pass and band-elimination filters
- Practical realization of Butterworth filters
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- Simulation results of few implementations (using the Multisim circuit simulation tool)
- Matlab code used in obtaining the results

1. **Design steps in obtaining the transfer function of the filter to meet the specifications**
2. **Practical realization of Butterworth filters**
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4. **Simulation results of few implementations (using the Multisim circuit simulation tool)**
5. **Matlab code used in obtaining the results**

**1. The Butterworth filter**

Butterworth filters are characterized by the property that the magnitude characteristic is maximally flat at the origin of the $s$ plane. The transfer function of an $n^{th}$ order Low-pass-Butterworth filter is given by:

$$H(s) = \frac{K_0}{(s-s_1)(s-s_2)\ldots(s-s_n)} \qquad (1)$$

where $K_0$ is a normalizing constant and

$$s_k = e^{(j\pi/2)[1+(2k-1)/n]} \qquad k = 1, 2, \ldots, n \qquad (2)$$

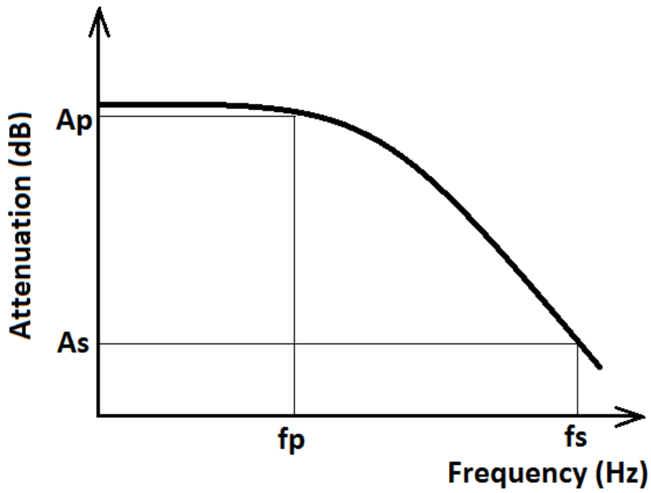
Some features of the Butterworth filters represented by equation (1) are:

- They are all-pole filters

- The poles lie on the left half of the s-plane
- They have the 3 dB cut-off at $\omega_C = 1$ rad/sec
- The complete filter is specified by the order *n*
- Through frequency transformation, it is possible to transform the above transfer function to another filter of specified cut-off or type
- The attenuation at a given frequency for a given order is given by

$$A_{dB} = 10\log_{10}\left(1+\left(\frac{\omega}{\omega_C}\right)^{2n}\right) \tag{3}$$

- The design steps for obtaining the transfer function of the Butterworth filter are as follows:
  1. Given the frequency specifications, obtain the equivalent specifications of the low-pass filter (need two frequencies with desired attenuation)
  2. Assume we need to have attenuation of As, Ap at frequencies fs and fp respectively. Then this corresponds to the specifications of low-pass filter




  3. The order n, of the desired filter is a smallest integer that satisfies the equation given below:

$$n \ge \frac{\log_{10}\left(\frac{10^{As/10}-1}{10^{Ap/10}-1}\right)}{2\log_{10}\left(\omega_S/\omega_P\right)} \tag{4}$$

  4. Using equations (1) and (2), we obtain the transfer function of the $n^{th}$ order Butterworth filter, which has cut-off 1 rad/sec

$$H(s)=\frac{K_0}{(s-s_1)(s-s_2)\ldots(s-s_n)}$$

5. Using the frequency transformations, we then obtain the transfer function of the filter with desired filter to meet the specifications.
   The continuous frequency transformation is given by:

   Low-pass to Low-pass $\quad s\rightarrow\frac{s}{\omega_C}$

   Low-pass to High-pass $\quad s\rightarrow\frac{\omega_C}{s}$

   Low-pass to Band-pass $\quad s\rightarrow\frac{\left(s^2+\omega_H\omega_L\right)}{s\left(\omega_H-\omega_L\right)}$

   Low-pass to Band-stop $\quad s\rightarrow\frac{s\left(\omega_H-\omega_L\right)}{\left(s^2+\omega_H\omega_L\right)}$

   where $\omega_H$ is the upper cut-off frequency

   and $\omega_L$ is the lower cut-off frequency

6. Once we have the transfer function of the desired Butterworth filter, we can obtain the following:

   i) The pole-zero plot

   ii) The frequency response

   iii) The cascade representation of first and second order filters

   iv) The practical realization of the filter

   v) The impulse and step response of the filter

   vi) Compare the frequency response of the designed filter with that of the desired filter

7. The realization of Band-pass filters
   Band pass filters have a frequency response as shown in figure below. The difference between the two cut-off frequencies $f_L$ (the lower cut-off ) and $f_H$ (the upper cut-off) is known as the bandwidth $Bw$.

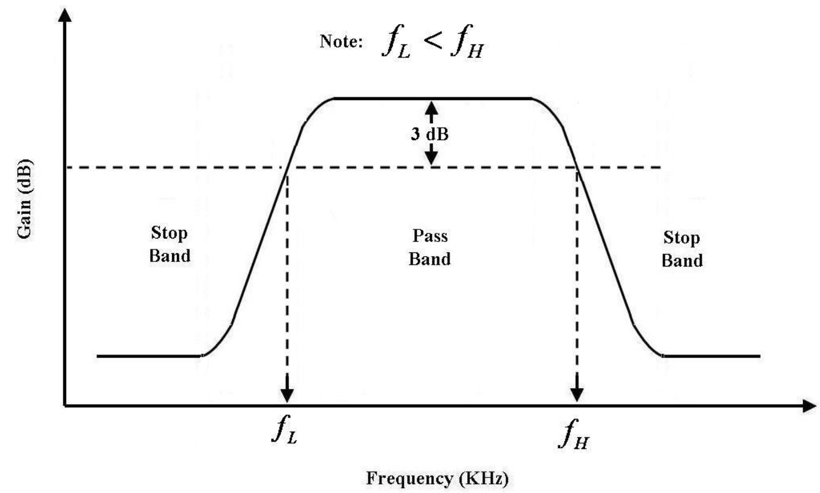
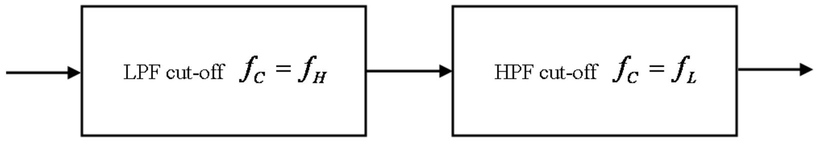



**Figure:** Frequency response of a band-pass filter

Wide-BPF can be realized by cascading a low-pass and a high-pass filter as shown in figure below, however, this method is not economical in terms of use of components, and also when the bandwidth is narrow.




**Band-pass filter**

8. The realization of Band-elimination filters

   Band elimination filters (or band reject filters) have a frequency response as shown in figure below.

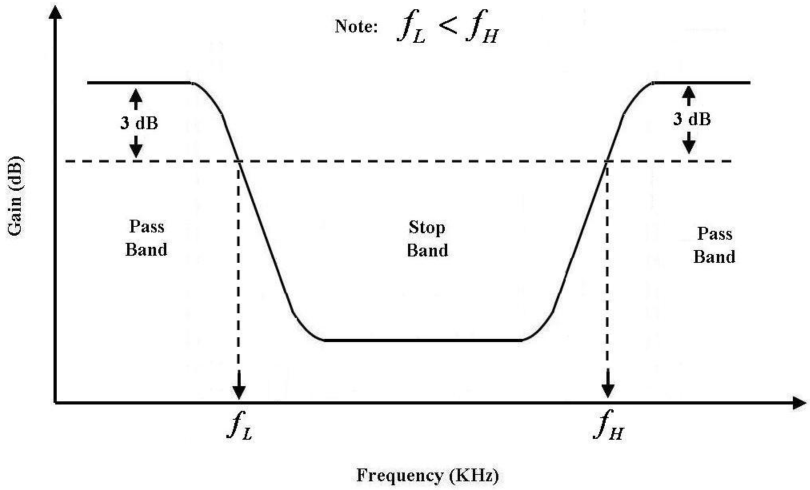



**Figure** : Frequency response of band elimination filter

It is possible to realize a BEF using a parallel connection of LPF and HPF as shown in figure below. But this method requires more components, and hence is not used.

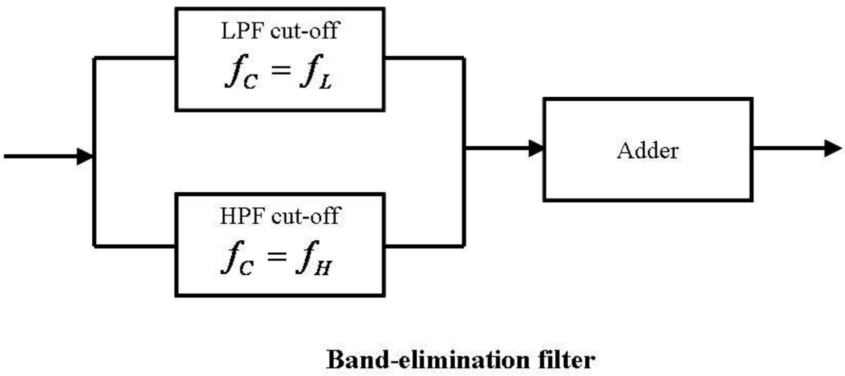
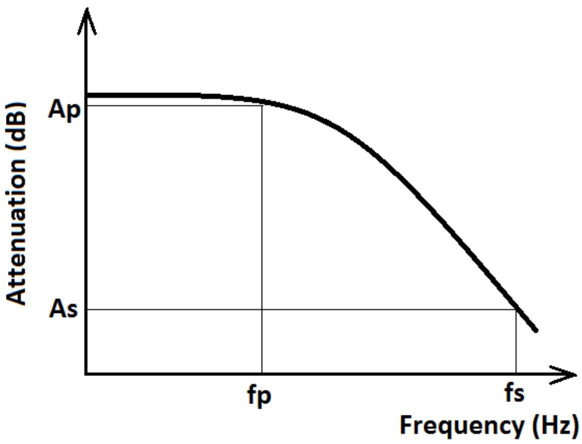



**Figure**: Realization of BEF using LPF and HPF

**Example:** Obtain the order of the Butterworth filter that has an attenduation of 1dB at 2 KHz and atleast 40 dB at 16 KHz.




**Solution**: For the given example

fp=2 KHz Ap= 1 dB

fs=16 KHz As=40 dB

$$n \geq \frac{\log_{10}\left(\frac{10^{40/10}-1}{10^{1/10}-1}\right)}{2\log_{10}(16/2)}$$

$$n \geq \frac{\log_{10}\left(\frac{9999}{0.2589}\right)}{2\log_{10}(8)}$$

$$n \geq 2.539$$

Since the order of the filter is an integer, we have the order of the filter as: n=3

**Example:** Obtain the order of the Butterworth filter which has atleast 66 dB attenuation at 2000 π rad/sec and 3 dB attenuation at 1000 π rad/sec .

**Solution:** For the given example

ωp=1000 π rad/sec Ap= 3 dB

ωs=2000 π rad/sec As=66 dB

$$n \geq \frac{\log_{10}\left(\frac{10^{66/10}-1}{10^{3/10}-1}\right)}{2\log_{10}(2000/1000)}$$

$$n \geq \frac{\log_{10}\left(4x10^{6}\right)}{2\log_{10}(2)}$$

$$n \geq 10.97$$

Since the order of the filter is an integer, we have the order of the filter as: n=11

## I. The First Order Butterworth Low-Pass-Filter with cut-off ($\omega_C$=1)

Butterworth filters are characterized by the property that the magnitude characteristic is maximally flat at the origin of the *s* plane. The transfer function of an $n^{th}$ order Low-pass-Butterworth filter is given by:

$$H(s) = \frac{K_0}{(s-s_1)(s-s_2)...(s-s_n)}$$

where $K_0$ is a normalizing constant and

$$s_k = e^{(j\pi/2)[1+(2k-1)/n]} \qquad k = 1, 2, ..., n$$

Hence, we have the transfer function of the first order Butterworth filter as:

$$H(s) = \frac{K_0}{(s-s_1)}$$

where $K_0$ is a normalizing constant and

$$s_1 = e^{j\pi} = (-1)$$

$$H(s) = \frac{1}{(s+1)}$$

We know that the following Laplace Transform relation:

$$\frac{1}{(s+a)} \leftrightarrow e^{-at}u(t)$$

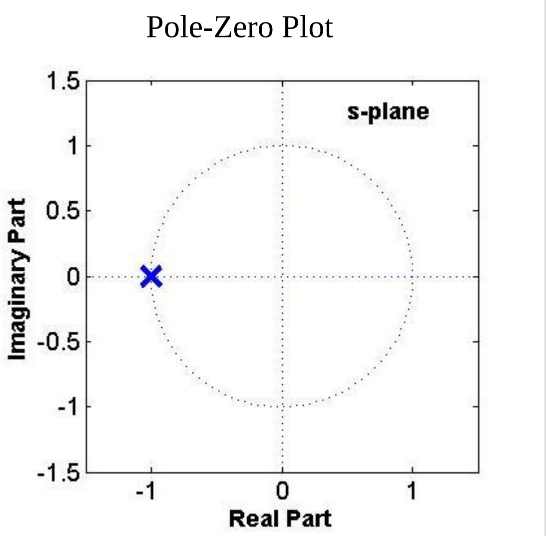
And hence the impulse response of the first order Butterworth filter is given by:

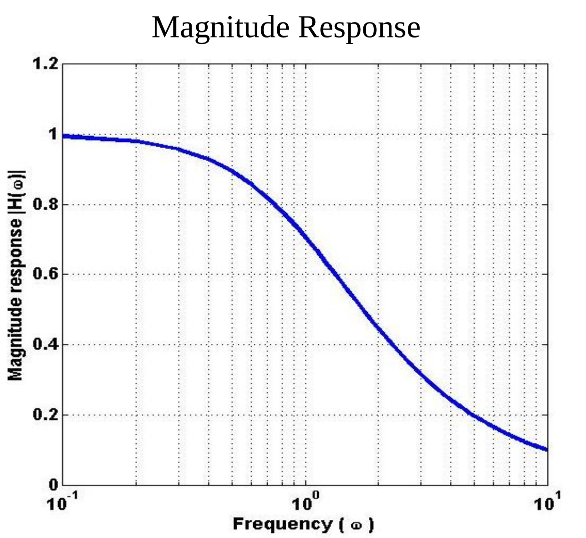
$$h(t) = e^{-t}u(t)$$

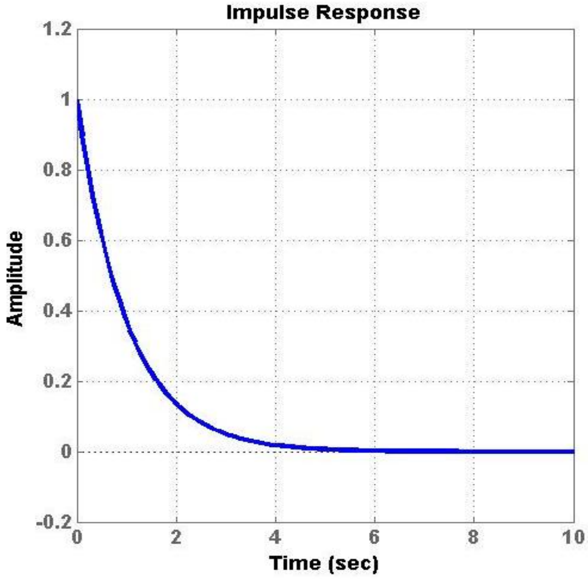
While the step response is given by:

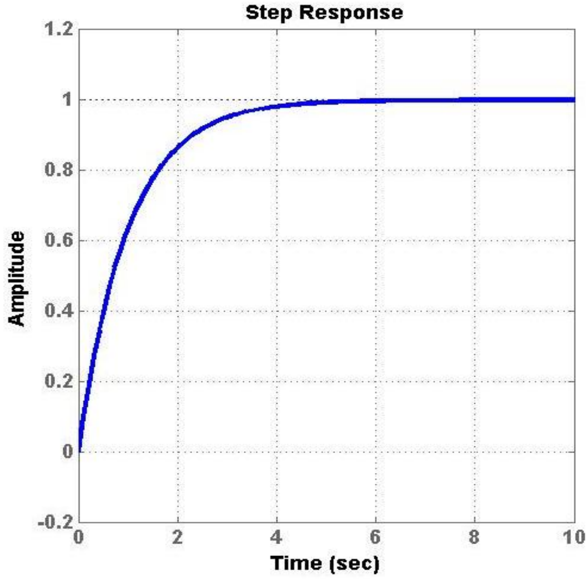
$$g(t) = \int_{-\infty}^{t} h(t)dt$$

$$= (1-e^{-t})u(t)$$






## The First Order Butterworth LPF with cut-off ($\omega_C$)

Applying the continuous frequency transformation,

Low-pass to Low-pass $\quad s \to \dfrac{s}{\omega_C}$

we have the first order Butterworth low-pass filter with cut-off $\omega_C$ is given by:

$$H(s) = \frac{1}{\left(\frac{s}{\omega_C} + 1\right)}$$

Using the Inverse Laplace Transform, we have impulse response of the first order Butterworth filter as:

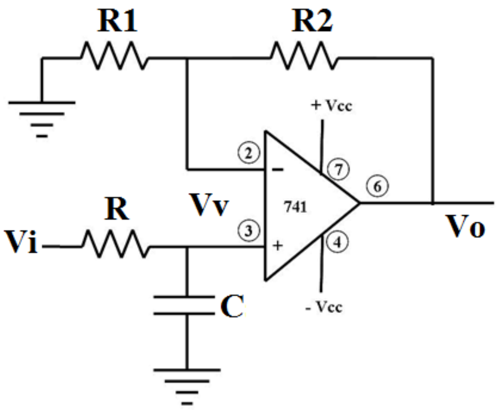
$$h(t) = \omega_C e^{-\omega_C t} u(t)$$

While the step response is given by:

$$\begin{aligned} g(t) &= \int_{-\infty}^{t} h(t)dt \\ &= \int_{0}^{t} \omega_C e^{-\omega_C t} dt \\ &= \left(1 - e^{-\omega_C t}\right) u(t) \end{aligned}$$

## Practical realization of First Order Butterworth Low-Pass Filter

Let us obtain the transfer function of the filter given below.




We shall attempt to obtain the transfer function of the above system using Laplace Transform. Applying the voltage division across resistors R1 and R2, we have

$$V_V(s) = \frac{R_1}{(R_1 + R_2)} V_O(s)$$

$$= \frac{V_O(s)}{A_{DC}} \qquad \text{where } A_{DC} = \frac{(R_1 + R_2)}{R_1}$$

Applying the voltage division across resistor and capacitor, we have

$$V_V(s) = \frac{(1/Cs)}{(R + (1/Cs))} V_i(s)$$

$$= \frac{1}{(1 + RCs)} V_i(s)$$

Hence,

$$V_i(s) = (1 + RCs) V_V(s) \qquad = (1 + RCs) \frac{V_O(s)}{A_{DC}}$$

$$H(s) = \frac{V_O(s)}{V_i(s)} = \frac{A_{DC}}{(RCs + 1)}$$

$$H(s) = \frac{A_{DC}}{\left(\frac{s}{(1/RC)} + 1\right)}$$

Comparing the above transfer function, with the transfer function of the First Order Butterworth Low-Pass filter, we conclude that the above circuit can be used to realize the First order LPF with cut-off

$$\omega_c = 1/RC \text{ rad/sec} \qquad \text{or} \quad f_c = \frac{1}{2\pi RC} Hz$$

The impulse response and the step response of the filter is given by:

$$h(t) = \frac{1}{RC} e^{-t/RC} u(t)$$

$$g(t) = \left(1 - e^{-t/RC}\right) u(t)$$

## II. The Second Order Butterworth LPF with cut-off ($\omega_C$=1)

Butterworth filters are characterized by the property that the magnitude characteristic is maximally flat at the origin of the $s$ plane. The transfer function of an $n^{th}$ order Low-pass-Butterworth filter is given by:

$$H(s) = \frac{K_0}{(s-s_1)(s-s_2)...(s-s_n)} \qquad (1)$$

where $K_0$ is a normalizing constant and

$$s_k = e^{(j\pi/2)[1+(2k-1)/n]} \qquad k = 1,2,...,n \qquad (2)$$

Hence, we have the transfer function of the second order Butterworth filter as:

$$H(s) = \frac{K_0}{(s-s_1)(s-s_2)}$$

where $K_0$ is a normalizing constant and

$$s_1 = e^{(j3\pi/4)} = (-0.707 + 0.707j)$$

$$s_2 = e^{(j5\pi/4)} = (-0.707-0.707j)$$

$$H(s) = \frac{1}{(s^2 + 1.414s + 1)}$$

$$= \frac{0.707j}{(s + 0.707 + 0.707j)} - \frac{0.707j}{(s + 0.707 - 0.707j)}$$

We know that the following Laplace Transform relation:

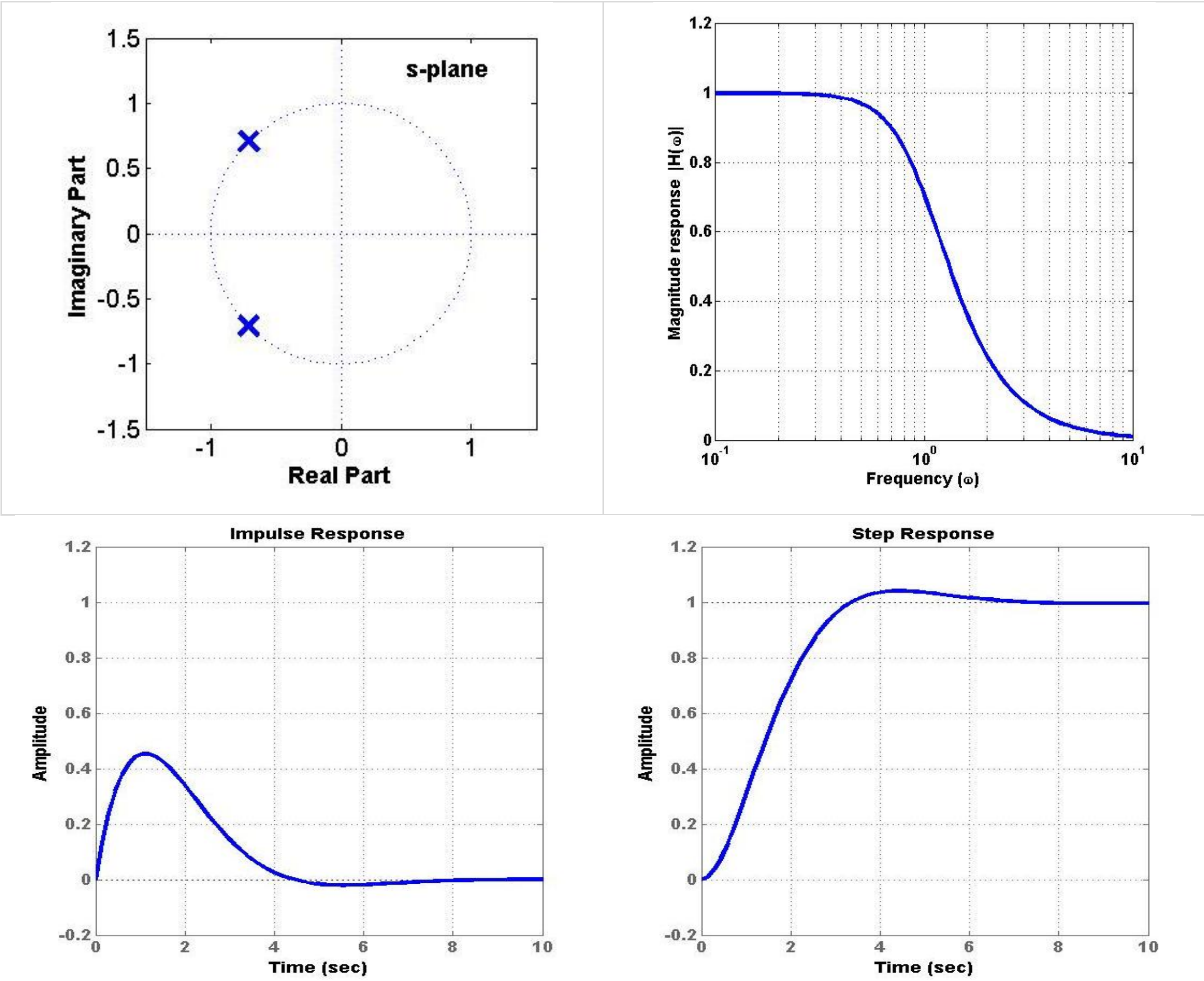
$$\frac{1}{(s+a)} \leftrightarrow e^{-at}u(t)$$

And hence the impulse response of the second order Butterworth filter is given by:

$$h(t) = 0.707\left(e^{-(0.707+0.707j)t} - e^{-(0.707-0.707j)t}\right)u(t)$$

$$= 1.414e^{-0.707t}\sin(0.707t)u(t)$$

| Pole-Zero Plot | Magnitude Response |
|---|---|




### The Second Order Butterworth LPF with cut-off ($\omega_C$)

Applying the continuous frequency transformation,

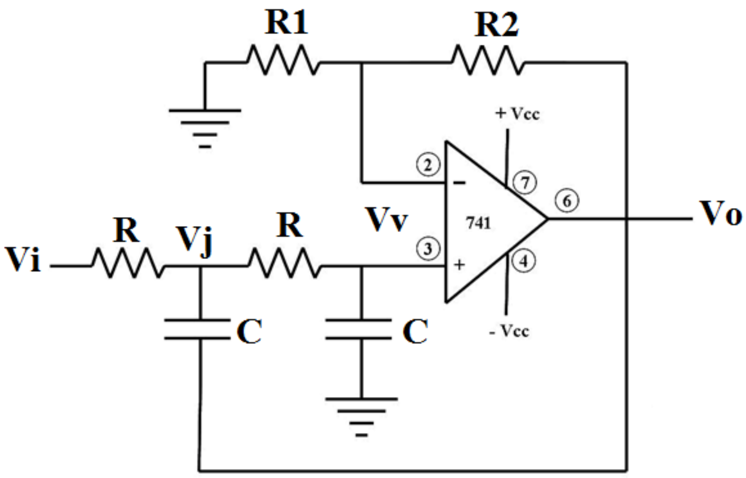
Low-pass to Low-pass $\quad s \rightarrow \dfrac{s}{\omega_C}$

we have the second order Butterworth low-pass filter with cut-off $\omega_C$ is given by:

$$H(s) = \frac{1}{\left(\left(\frac{s}{\omega_C}\right)^2 + 1.414\left(\frac{s}{\omega_C}\right) + 1\right)}$$

## Practical realization of Second Order Butterworth Low-Pass Filter

Let us obtain the transfer function of the filter given below.




We shall attempt to obtain the transfer function of the above system using Laplace Transform. Applying the voltage division across resistors R1 and R2, we have

$$V_V(s) = \frac{R_1}{(R_1 + R_2)} V_O(s)$$

$$= \frac{V_O(s)}{A_{DC}} \qquad \text{where } A_{DC} = \frac{(R_1 + R_2)}{R_1}$$

Applying the voltage division across resistor and capacitor, we have

$$V_V(s) = \frac{(1/Cs)}{(R + (1/Cs))} V_j(s)$$

$$= \frac{1}{(1 + RCs)} V_j(s)$$

Hence,

$$V_j(s) = (1 + RCs) V_V(s)$$

$$= (1 + RCs) \frac{V_O(s)}{A_{DC}}$$

Applying the Current division property, we have

$$\frac{\left(V_i(s)-V_j(s)\right)}{R}=\frac{\left(V_j(s)-V_O(s)\right)}{\left(1/Cs\right)}+\frac{V_j(s)}{\left(R+1/Cs\right)}$$

$$\Rightarrow \qquad \frac{V_i(s)}{R}=\left(\frac{1}{R}+\frac{1}{\left(1/Cs\right)}+\frac{1}{\left(R+1/Cs\right)}\right)V_j(s)-\frac{V_O(s)}{\left(1/Cs\right)}$$

$$\Rightarrow \qquad V_i(s)=\left(1+RCs+\frac{RCs}{\left(RCs+1\right)}\right)V_j(s)-RCsV_O(s)$$

$$\Rightarrow \qquad A_{DC}V_i(s)=\left(\left(\left(RCs+1\right)^2+RCs\right)-RCsA_{DC}\right)V_O(s)$$

$$\Rightarrow \qquad A_{DC}V_i(s)=\left(\left(RCs\right)^2+\left(3-A_{DC}\right)RCs+1\right)V_O(s)$$

Hence the transfer function is given by:

$$H(s)=\frac{V_O(s)}{V_i(s)}=\frac{A_{DC}}{\left(\left(RCs\right)^2+\left(3-A_{DC}\right)RCs+1\right)}$$

Comparing the above transfer function, with the transfer function of the Second Order Butterworth Low-Pass filter, we conclude that the above circuit can be used to realize the First order LPF with cut-off

$\omega_c = 1/RC$ rad/sec  or  $f_c = \dfrac{1}{2\pi RC}Hz$

However, the DC gain is **fixed** to

$$\left(3-A_{DC}\right)=1.414$$

and hence,

$$\begin{aligned} A_{DC} &= (3-1.414) \\ &= 1.586 \\ &= \frac{\left(R_1+R_2\right)}{R_1} \end{aligned}$$

## III. The Third Order Butterworth LPF with cut-off ($\omega_C$=1)

Butterworth filters are characterized by the property that the magnitude characteristic is maximally flat at the origin of the *s* plane. The transfer function of an $n^{th}$ order Low-pass-Butterworth filter is given by:

$$H(s) = \frac{K_0}{(s-s_1)(s-s_2)\ldots(s-s_n)} \qquad (1)$$

where $K_0$ is a normalizing constant and

$$s_k = e^{(j\pi/2)[1+(2k-1)/n]} \qquad k = 1, 2, \ldots, n \qquad (2)$$

Hence, we have the transfer function of the third order Butterworth filter as:

$$H(s) = \frac{K_0}{(s-s_1)(s-s_2)(s-s_3)}$$

where $K_0$ is a normalizing constant and

$$s_k = e^{(j\pi/2)[1+(2k-1)/3]} \qquad k = 1, 2, 3$$

$$s_1 = e^{(j2\pi/3)} = (-0.5+0.866j)$$

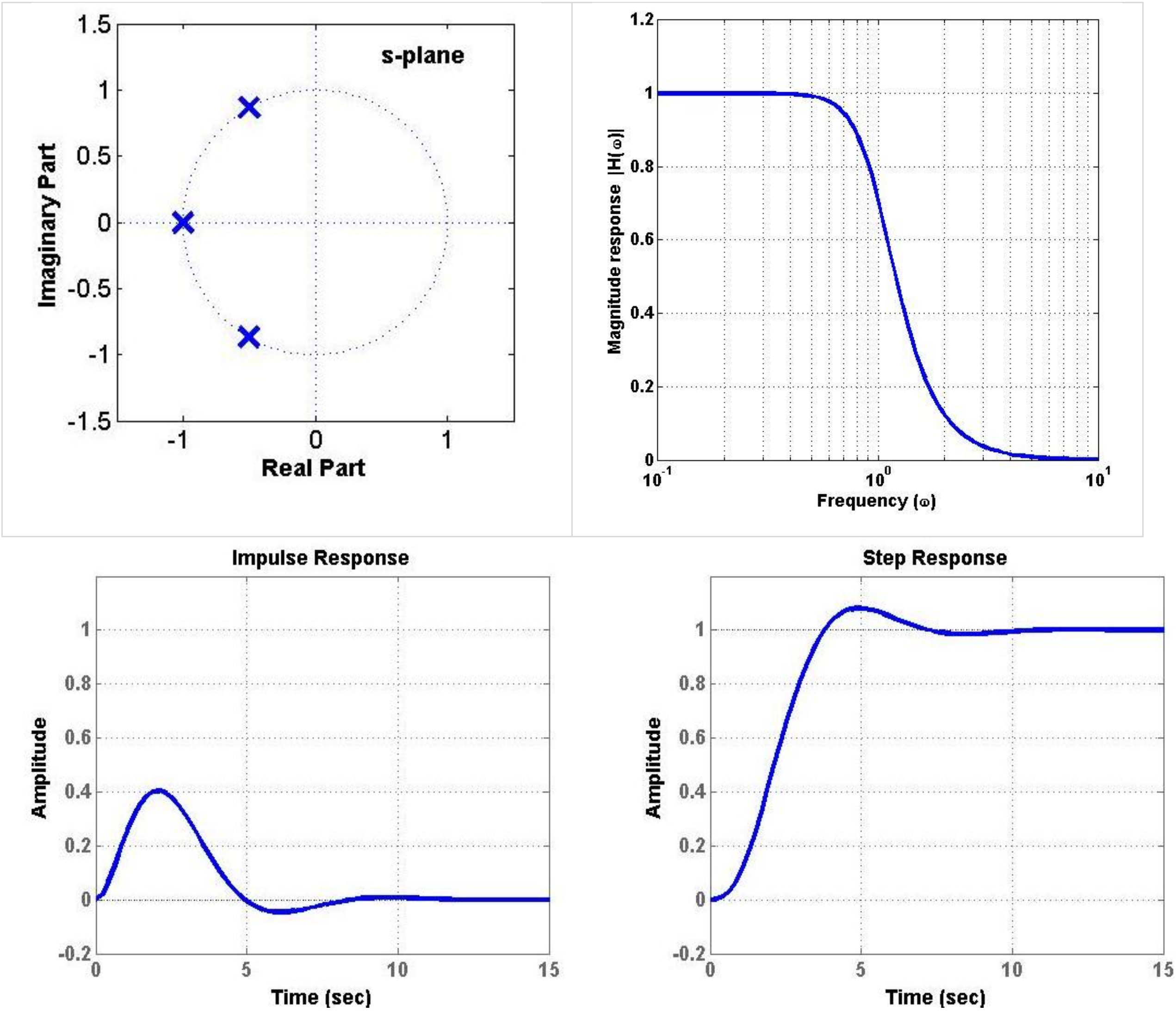
$$s_2 = e^{(j\pi)} = (-1)$$

$$s_3 = e^{(j4\pi/3)} = (-0.5-0.866j)$$

$$H(s) = \frac{1}{(s+1)(s^2+s+1)}$$

$$= \frac{1}{(s^3+2s^2+2s+1)}$$

| Pole-Zero Plot | Magnitude Response |
|---|---|




## Practical realization of Third Order Butterworth Low-Pass Filter

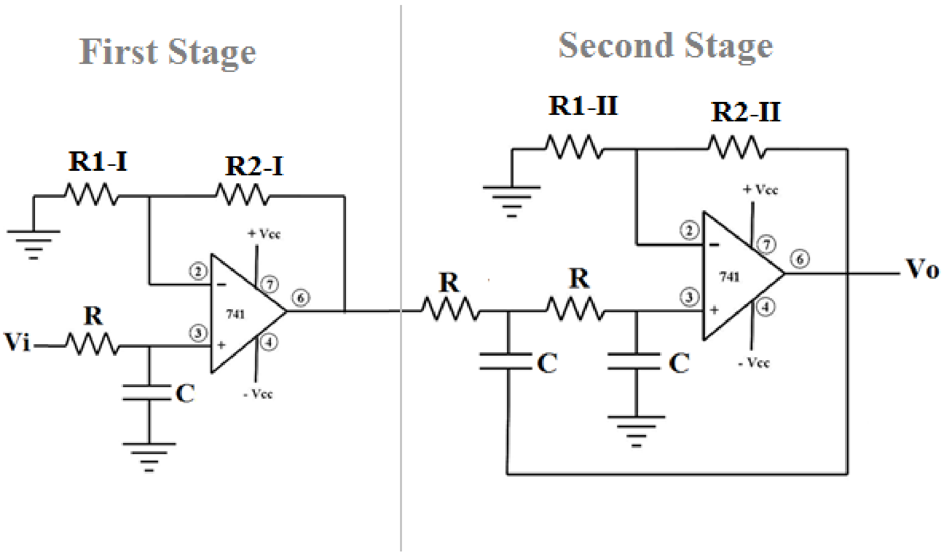
The transfer function of the third order Low-Pass Butterworth filter with $\omega_C = 1$ is given by:

$$H(s) = \frac{1}{(s+1)\left(s^2+s+1\right)}$$

Applying the low-pass to low-pass frequency transformation, we have the first order Butterworth low-pass filter with cut-off $\omega_C$ is given by:

$$H(s) = \frac{1}{\left(\frac{s}{\omega_C}+1\right)} \; \frac{1}{\left(\left(\frac{s}{\omega_C}\right)^2+\left(\frac{s}{\omega_C}\right)+1\right)}$$

The third order Butterworth LPF is realized as a cascade of the first-order and the second-order filters as shown below:




The filter cut-off is given by:

$$\omega_c = 1/RC \text{ rad/sec} \qquad \text{or} \qquad f_c = \frac{1}{2\pi RC} Hz$$

The DC gain of the First-Stage, may be set to a desired value through

$$A_{DC-I} = \frac{(R_{1-I} + R_{2-I})}{R_{1-I}}$$

The DC gain of the Second-Stage, is obtained from the filter transfer function, and is given by

$$(3 - A_{DC-II}) = 1$$

and hence,

$$A_{DC-II} = 2$$
$$= \frac{(R_{1-II} + R_{2-II})}{R_{1-II}}$$

## IV. The Fourth Order Butterworth LPF with cut-off ($\omega_C$=1)

Butterworth filters are characterized by the property that the magnitude characteristic is maximally flat at the origin of the $s$ plane. The transfer function of an $n^{th}$ order Low-pass-Butterworth filter is given by:

$$H(s)=\frac{K_0}{(s-s_1)(s-s_2)\ldots(s-s_n)} \tag{1}$$

where $K_0$ is a normalizing constant and

$$s_k = e^{(j\pi/2)[1+(2k-1)/n]} \qquad k=1,2,\ldots,n \tag{2}$$

Hence, we have the transfer function of the fourth order Butterworth filter as:

$$H(s)=\frac{K_0}{(s-s_1)(s-s_2)(s-s_3)(s-s_4)}$$

where $K_0$ is a normalizing constant and

$$s_k = e^{(j\pi/2)[1+(2k-1)/4]} \qquad k=1,2,\ldots,4$$

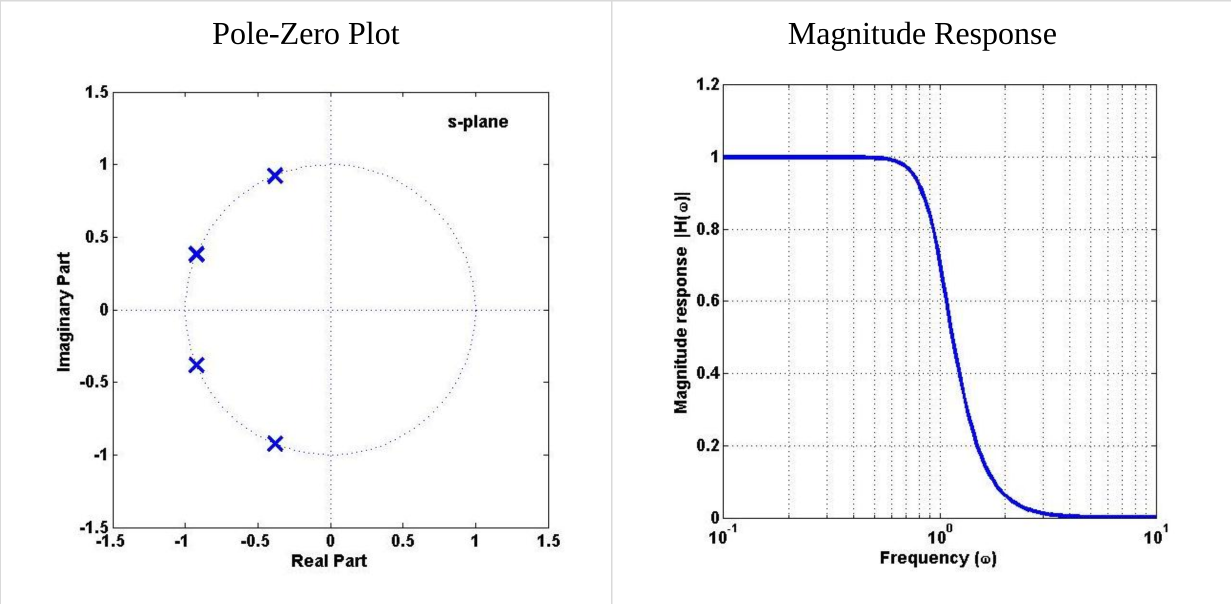
$$s_1 = e^{(j5\pi/8)} = (\text{-}0.3827 + 0.9239\text{j})$$

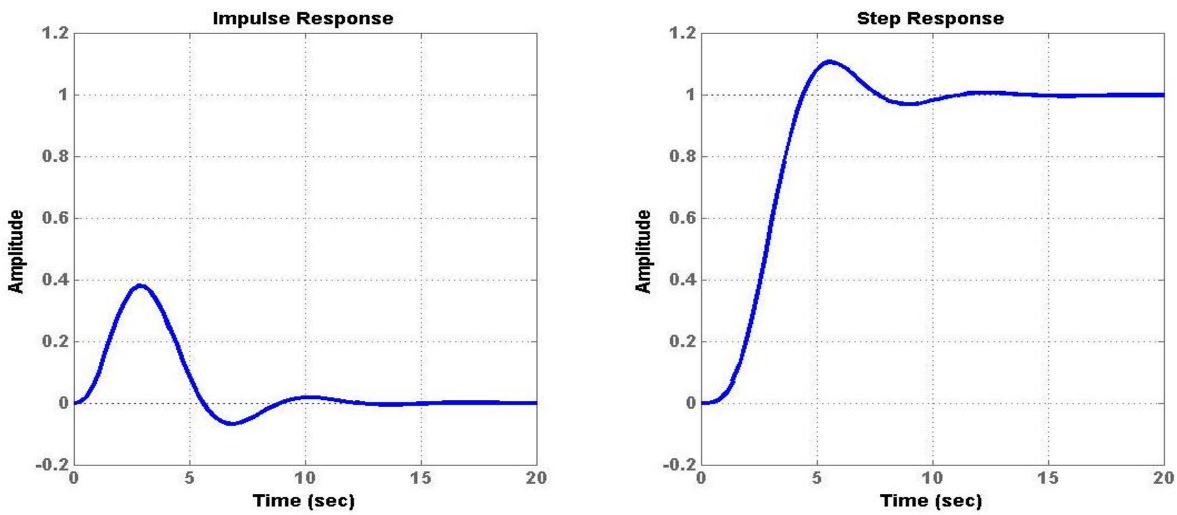
$$s_2 = e^{(j7\pi/8)} = (\text{-}0.9239 + 0.3827\text{j})$$

$$s_3 = e^{(j9\pi/8)} = (\text{-}0.9239 - 0.3827\text{j})$$

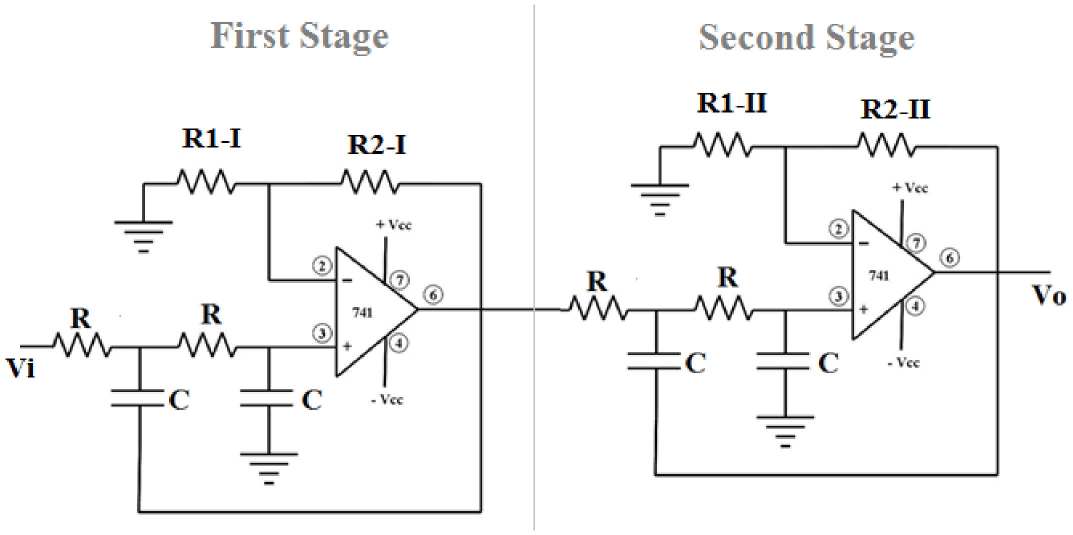
$$s_4 = e^{(j11\pi/8)} = (\text{-}0.3827 - 0.9239\text{j})$$

$$H(s)=\frac{1}{(s^2+0.765s+1)(s^2+1.848s+1)}$$
$$=\frac{1}{(s^4+2.61s^3+3.41s^2+2.61s+1)}$$




## Practical Realization of the Fourth order filter



## V. The Fifth Order Butterworth LPF with cut-off ($\omega_C$=1)

Butterworth filters are characterized by the property that the magnitude characteristic is maximally flat at the origin of the $s$ plane. The transfer function of an $n^{th}$ order Low-pass-Butterworth filter is given by:

$$H(s)=\frac{K_0}{(s-s_1)(s-s_2)\ldots(s-s_n)} \tag{1}$$

where $K_0$ is a normalizing constant and

$$s_k = e^{(j\pi/2)[1+(2k-1)/n]} \qquad k = 1,2,\ldots,n \tag{2}$$

Hence, we have the transfer function of the fifth order Butterworth filter as:

$$H(s)=\frac{K_0}{(s-s_1)(s-s_2)(s-s_3)(s-s_4)(s-s_5)}$$

where $K_0$ is a normalizing constant and

$$s_k = e^{(j\pi/2)[1+(2k-1)/5]} \qquad k = 1,2,\ldots,5$$

$$s_1 = e^{(j3\pi/5)} = (\text{-0.3090 + 0.9511j})$$

$$s_2 = e^{(j4\pi/5)} = (\text{-0.8090 + 0.5878j})$$

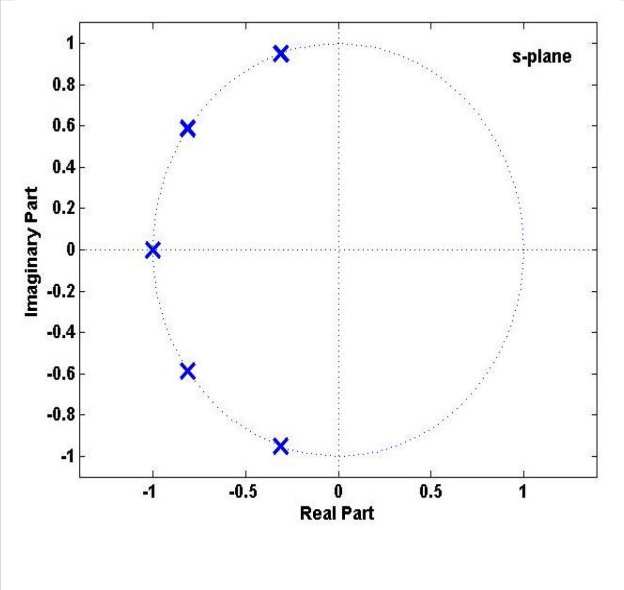
$$s_3 = e^{(j\pi)} = (-1)$$

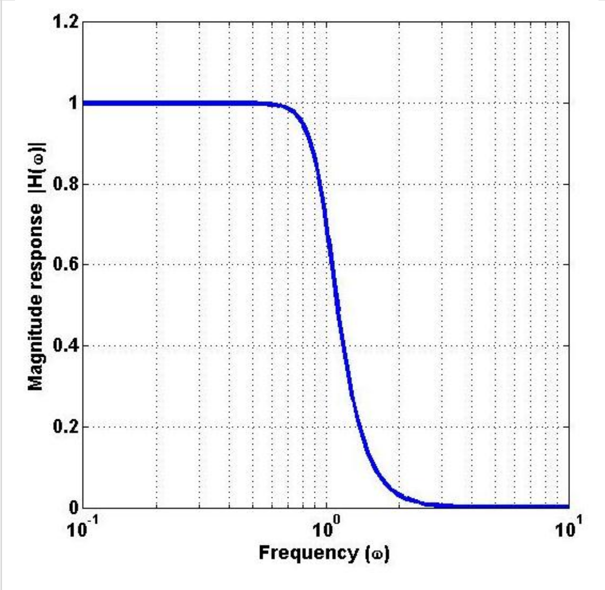
$$s_4 = e^{(j6\pi/5)} = (\text{-0.8090 - 0.5878j})$$

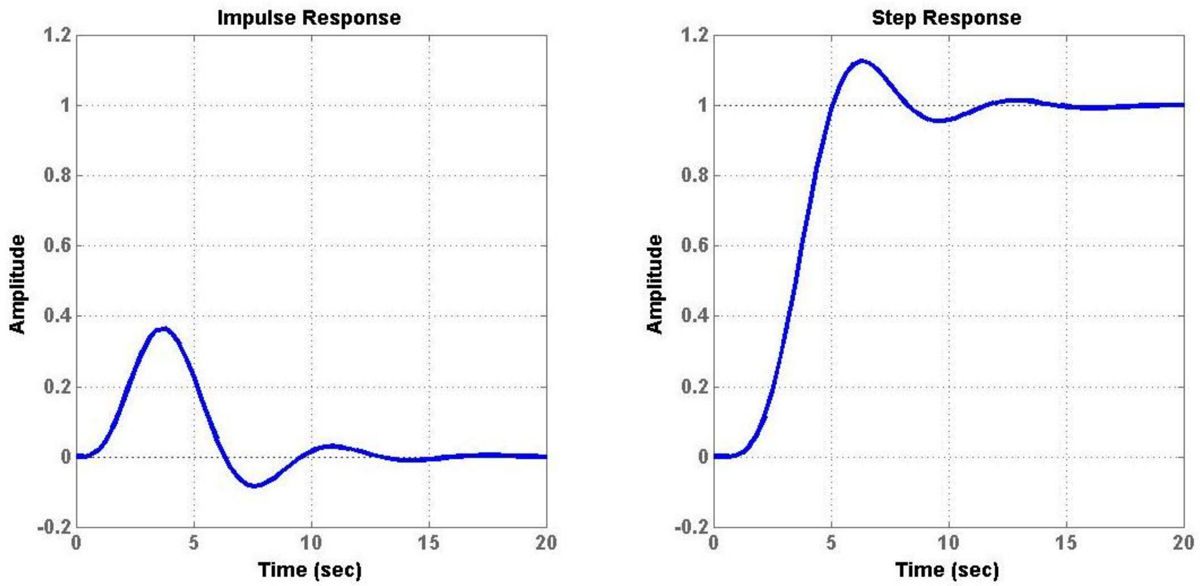
$$s_5 = e^{(j7\pi/5)} = (\text{-0.3090 - 0.9511j})$$

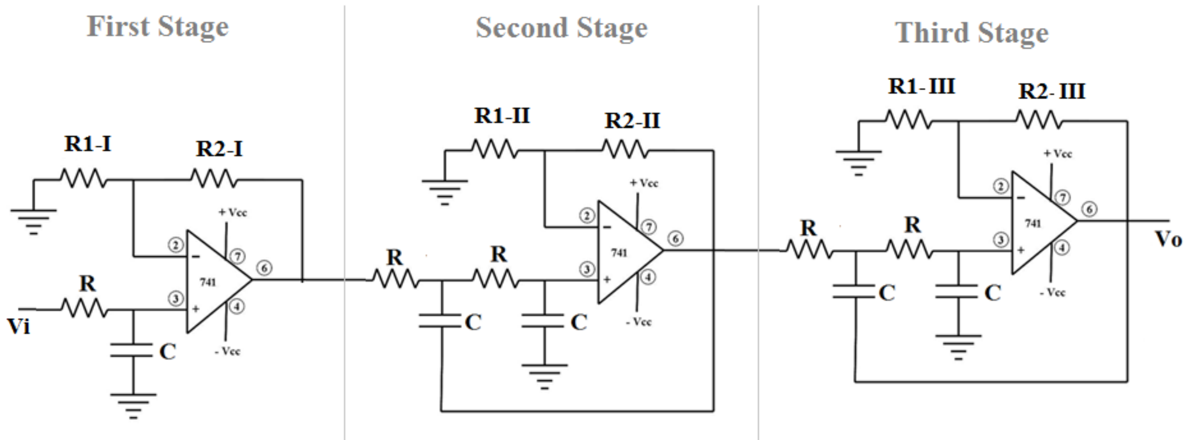
$$\begin{aligned} H(s) &= \frac{1}{(s+1)(s^2+0.618s+1)(s^2+1.618s+1)} \\ &= \frac{1}{(s^5+3.263s^4+5.236s^3+5.236s^2+3.263s+1)} \end{aligned}$$

| Pole-Zero Plot | Magnitude Response |
|---|---|


s-plane
Imaginary Part
Real Part
1
0.8
0.6
0.4
0.2
0
-0.2
-0.4
-0.6
-0.8
-1
-0.5
0.5


Magnitude response |H(ω)|
Frequency (ω)
1.2
1
0.8
0.6
0.4
0.2
0


Impulse Response
Amplitude
Time (sec)
Step Response
Amplitude
Time (sec)


First Stage
Second Stage
Third Stage
R1-I
R2-I
R1-II
R2-II
R1-III
R2- III
R
C
Vi
Vo
741
+ Vcc
- Vcc

## The Tenth Order Butterworth LPF with cut-off ($\omega_C$=1)

Butterworth filters are characterized by the property that the magnitude characteristic is maximally flat at the origin of the $s$ plane. The transfer function of an $n^{th}$ order Low-pass-Butterworth filter is given by:

$$H(s) = \frac{K_0}{(s-s_1)(s-s_2)\ldots(s-s_n)} \qquad (1)$$

where $K_0$ is a normalizing constant and

$$s_k = e^{(j\pi/2)[1+(2k-1)/n]} \qquad k = 1,2,\ldots,n \qquad (2)$$

Hence, we have the transfer function of the tenth order Butterworth filter as:

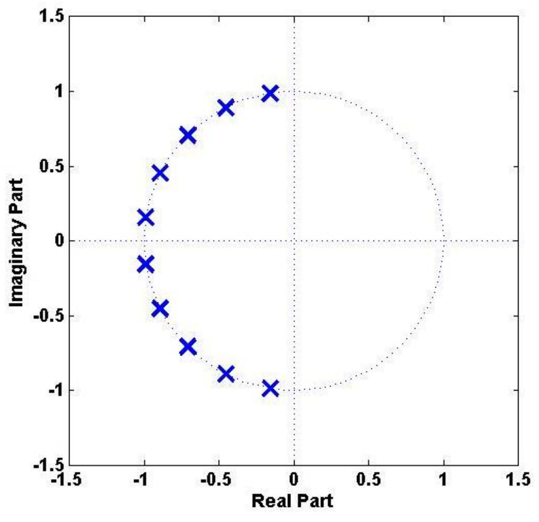
$$H(s) = \frac{K_0}{(s-s_1)(s-s_2)\ldots(s-s_{10})}$$

where $K_0$ is a normalizing constant and

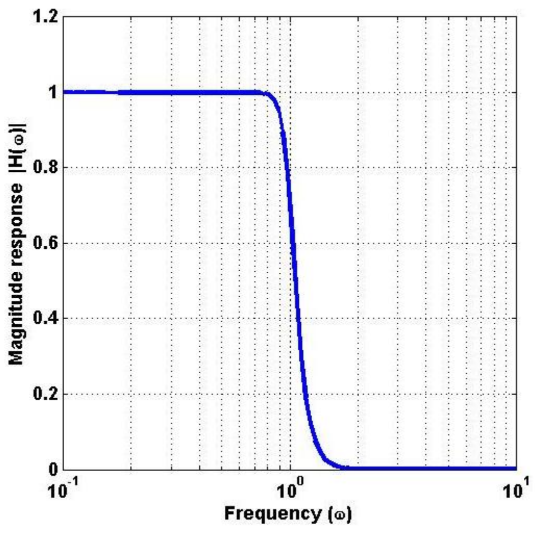
$$s_k = e^{(j\pi/2)[1+(2k-1)/10]} \qquad k = 1,2,\ldots,10$$

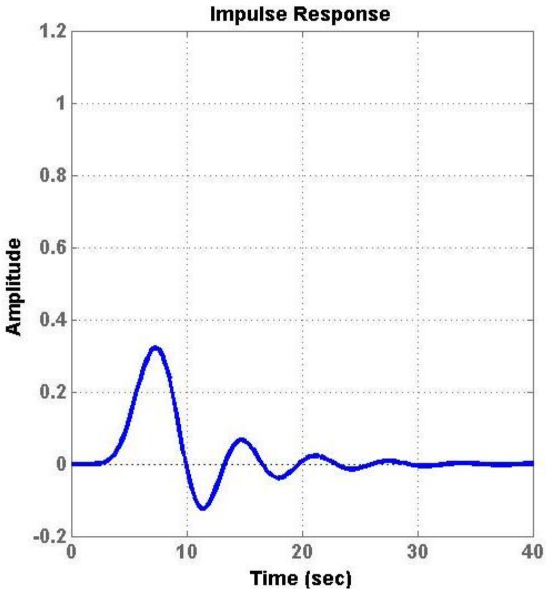
$$H(s) = \frac{1}{\left(s^{10} + 6.4s^9 + 20.4s^8 + 42.8s^7 + 64.9s^6 + 74.2s^5 + 64.9s^4 + 42.8s^3 + 20.4s^2 + 6.4s + 1\right)}$$

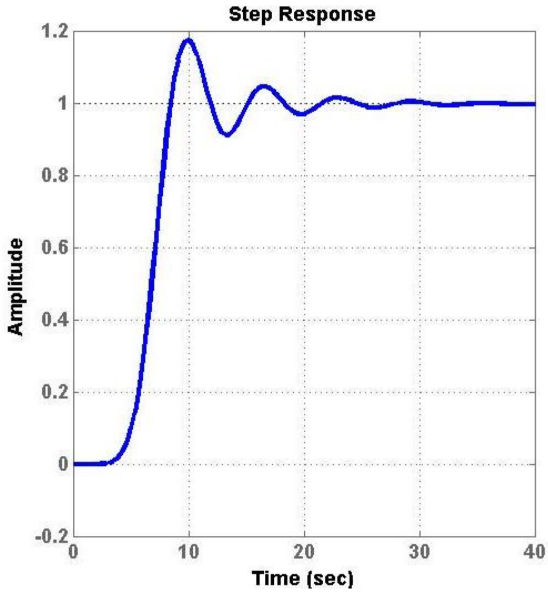
Pole-Zero Plot




Magnitude Response




Impulse Response
Amplitude
Time (sec)
1.2
1
0.8
0.6
0.4
0.2
0
-0.2
0
10
20
30
40


Step Response
Amplitude
Time (sec)
1.2
1
0.8
0.6
0.4
0.2
0
-0.2
0
10
20
30
40

## I. Active Low Pass Filter

**1. Experiment**:

Design a second order active low pass Butterworth filter with a cut-off of ………kHz. Obtain the roll-off factor and cutoff frequency of the filter designed. Compare the designed cut-off frequency with the desired cut-off frequency. Give reasons for difference if any.

**Theory:**

The $n$ poles of an $n^{th}$ order Butterworth filter lie on the left half of the s-plane, on a circle of radius unity (the poles have an angular separation of $\pi/n$ radians). The second order Butterworth filter has two poles $p1$ and $p2$ located on the left half of the s-plane as shown in the figure 1. Hence the transfer function of the filter with cut-off frequency $\omega_c = 1$ radians/second is given by

$$H(s) = \frac{1}{(s - p1)(s - p2)} \qquad (1)$$

Where, $p1 = (-0.707 + j0.707)$ and $p2 = (-0.707 - j0.707)$

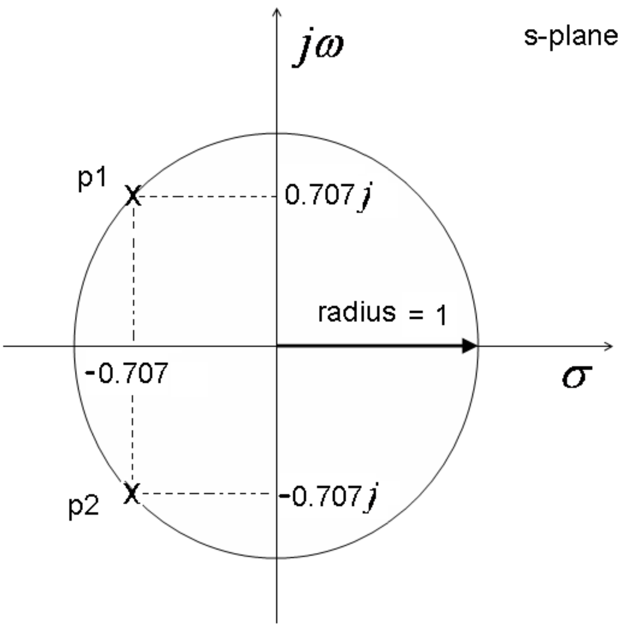



**Figure 1**: The two poles of the second order Butterworth low pass filter

Which is equivalent to

$$H(s) = \frac{1}{\left(s^2 + 1.414s + 1\right)} \tag{2}$$

To obtain the transfer function of a filter with cut-off $\omega_c$ radians/second, we replace $s$ by $s/\omega_c$, and we have,

$$H(s) = \frac{1}{\left((s/\omega_c)^2 + 1.414(s/\omega_c) + 1\right)} \tag{3}$$

The practical realization of the equation (3) is given by figure 2, has the transfer function

$$H_f(s) = \frac{(1/RC)}{\left(s^2 + ((3 - A_F)/RC)s + (1/RC)^2\right)} \tag{4}$$

given below:

Applying the voltage division across resistors R1 and R2, we have

Comparing equations (3) and (4), we have the cut-off frequency as

$$\omega_c = 1/RC \text{ radians/second}$$

or

$$f_c = \frac{1}{2\pi RC} Hz \tag{5}$$

with feedback gain $A_F$

$$A_F = \left(1 + \frac{R_2}{R_1}\right) = 1.586 \tag{6}$$

**Procedure:**

**Step 1:** Obtain the internal impedance $R_S$ of the signal source

**Step 2:** Given cut-off frequency $f_c$ Hz, assume suitable **R**, (greater than $R_S$),

Calculate **C** using equation (5)

**Step 3:** Assume $\mathbf{R_1}$ and compute $\mathbf{R_2}$ using equation (6)

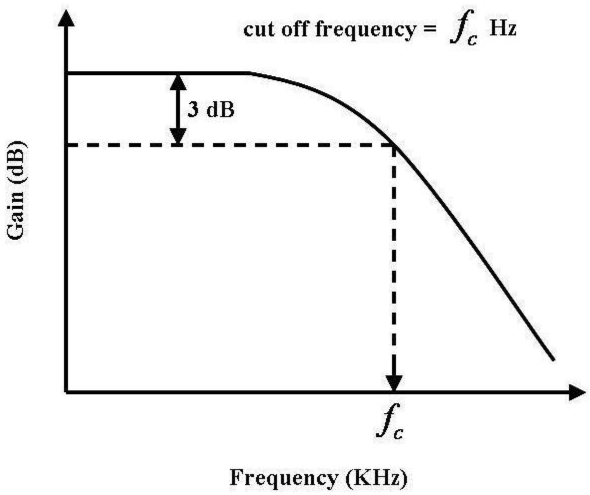
**Step 4:** Rig up the circuit of figure 2, the active second order LPF

**Step 5:** Record the input and output **peak-to-peak** voltage for various input frequencies, and complete the table below

| **Frequency (kHz)** | $V_i$ | $V_0$ | $Gain = 20\log\left(V_0/V_i\right) dB$ |
|---|---|---|---|
| **100 Hz** | | | |

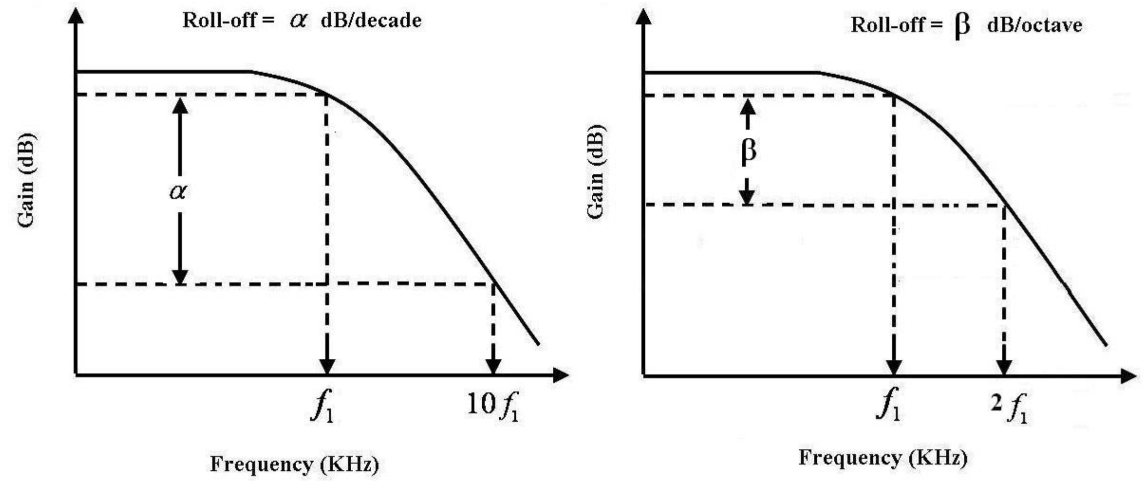
| | | | |
|---|---|---|---|
| **200 Hz** | | | |
| **….** | | | |
| **1 KHz** | | | |
| **2 KHz** | | | |
| **….** | | | |
| **10 KHz** | | | |
| **20 KHz** | | | |
| **….** | | | |
| **100 K Hz** | | | |
| **200 K Hz** | | | |
| **….** | | | |
| **1 M Hz** | | | |

**Step 6:** Plot the frequency response of the designed filter (Plot of Frequency Vs. Gain on a semi-log sheet), and hence obtain the cut-off frequency




**Step 7**: Compute the Roll-off factor of the designed filter

(The ideal value of roll-off factor is **- 40dB/decade** or **-12dB/octave**)

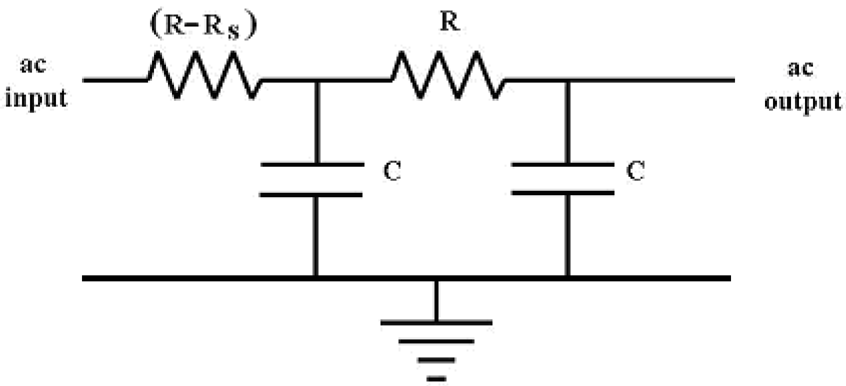



**Observations:**

- **i)** The designed filter has a cut-off frequency …………Hz
- **ii)** The designed filter has a roll-off factor ……….. dB/decade

**The passive LPF**

Repeat the frequency response readings for the circuit below. How does it compare with earlier one?




Sketch the frequency response of the passive and active second order LPF on the same graph sheet. What is the observation?

**Some interesting experiments:**

**i)** Give an audio signal as input to the filter and the filter output to a speaker, and hear the output when cut-off is 1kHz. Repeat for filter cut-off of 5kHz and 10kHz individually. Comment on the result.

**ii)** Give a square wave equal to the designed cut-off frequency and record the output. Give reason if the output waveform is a sine wave.

# III Active Band Pass Filter

## 1 Experiment :

Design a second order active wide-band-pass filter with a lower cut-off frequency 3KHz, and the upper cut-off-frequency of 30 KHz, having voltage gain of 25. Sketch the frequency response and compare the designed parameters with the desired parameters.

Band pass filters have a frequency response as shown in figure 1. The difference between the two cut-off frequencies $f_L$ (the lower cut-off ) and $f_H$ (the upper cut-off) is known as the bandwidth $Bw$.

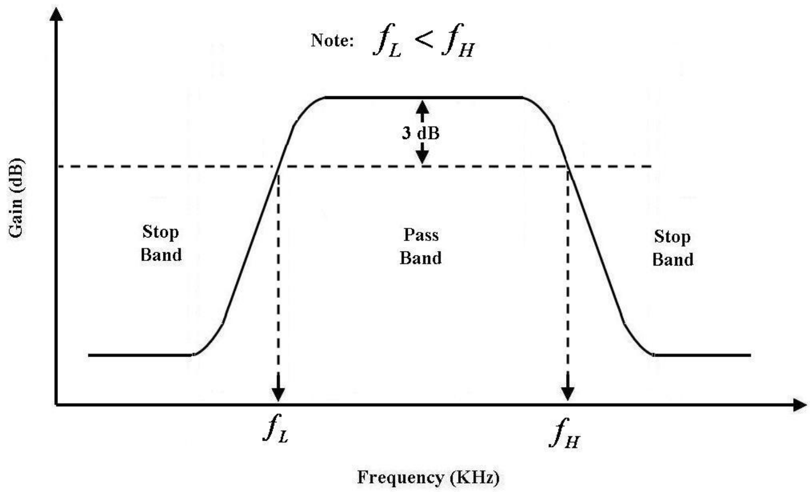



**Figure 1**: Frequency response of a band-pass filter

Wide-BPF can be realized by cascading a low-pass and a high-pass filter as shown in figure 2. This method is not economical in terms of use of components, and also when the bandwidth is narrow.

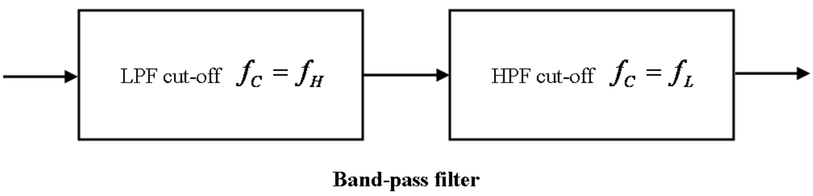



**Figure 2**: Realization of a band-pass filter

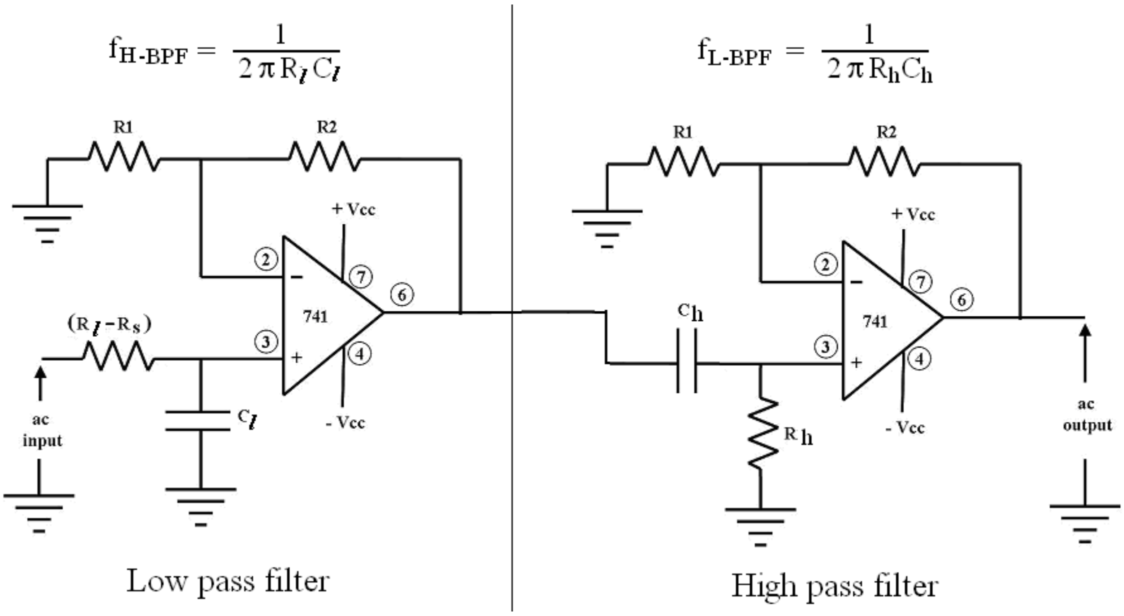



**Figure 3**: Realization of a wide-band-pass filter

**Procedure:**

**Step 1:** Given the upper cut-off frequency $f_H$ Hz, compute the values of the resistor and the capacitor of the LPF

**Step 2:** Given the lower cut-off frequency $f_L$ Hz, compute the values of the resistor and the capacitor of the HPF

**Step 3:** Given the gain of the BPF, assume $\mathbf{R_1}$ and compute $\mathbf{R_2}$ using equation

$$Gain = \left(1 + \frac{R_2}{R_1}\right)$$

**Step 4:** Record the input and output **peak-to-peak** voltage for various input frequencies, and complete the table below.

| Frequency (kHz) | $V_i$ | $V_0$ | $Gain = 20\log\left(V_0/V_i\right) dB$ |
|---|---|---|---|
| **100 Hz** | | | |
| **200 Hz** | | | |
| **….** | | | |
| **1 M Hz** | | | |

**Step 5:** Plot the frequency response of the designed filter (Plot of Frequency Vs. Gain on a semi-log sheet), and hence obtain the lower and upper cut-off frequencies

**Experiment 2: Active narrow-band pass filters/ Resonant filter**

Design a second order active Bandpass filter with a centre frequency 5KHz, quality factor 10, and voltage gain 25. Sketch the frequency response and compare the designed parameters with the desired parameters.

Band pass filters have a frequency response as shown in figure 1. The difference between the two cut-off frequencies $f_L$ (the lower cut-off ) and $f_H$ (the upper cut-off) is known as the bandwidth $Bw$.

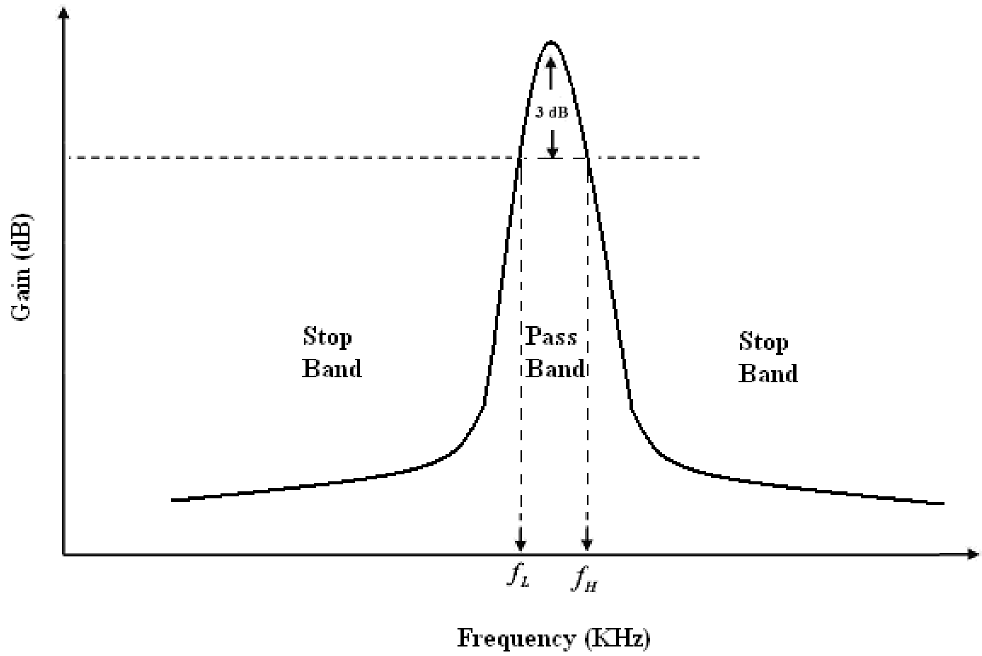



**Figure 1**: Frequency response of a narrow band-pass filter

When the bandwidth $Bw$ is small compared to either $f_L$ or $f_H$, the circuit is known as a resonant circuit with frequency response shown in figure 1. This can be implemented using figure 2. Resonant filters are characterized by the center or resonant frequency $f_0$, and a high quality factor $Q$, where,

$$Q = f_0/Bw \qquad \text{and} \qquad f_0 = \sqrt{f_L f_H}$$

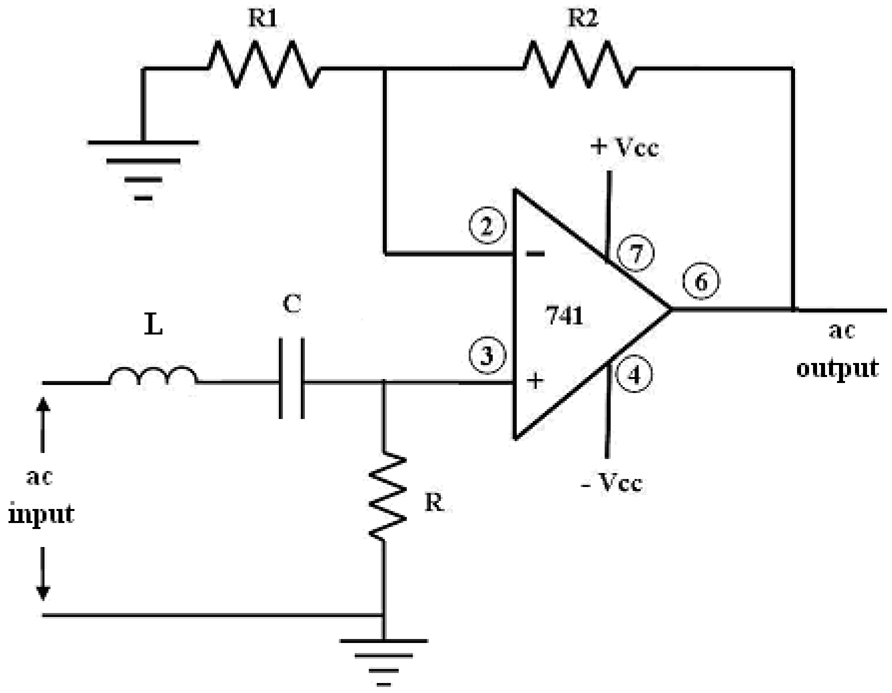



**Figure 2: Active resonant/narrow band pass filter**

**Procedure**

**Step 1:** Centre frequency $f_0 = \dfrac{1}{2\pi\sqrt{LC}}$ Hz

Choose L, compute C

**Step 2:** Given voltage gain $A_0$, choose $R_1$ and compute $R_2$

$$A_O = \left(1 + \frac{R_2}{R_1}\right)$$

**Step 3:** Given quality factor $Q$, compute resistor R, using

$$R = \frac{1}{Q}\sqrt{\frac{L}{C}}$$

**Step 4:** Connect the circuit of figure 2

**Step 5:** Record the input and output **peak-to-peak** voltage for various input frequencies, and complete the table below.

| **Frequency (kHz)** | $V_i$ | $V_0$ | $Gain = 20\log\left(V_0/V_i\right) dB$ |
|---|---|---|---|
| **100 Hz** | | | |
| **200 Hz** | | | |

| | | | |
|---|---|---|---|
| **....** | | | |
| **1 M Hz** | | | |

**Step 6:** Plot the frequency response of the designed filter (Plot of Frequency Vs. Gain on a semi-log sheet).

**Results:** From the frequency response obtain

i) The lower cut-off frequency $f_L$ and the higher cut-off frequency $f_H$ .

ii) The bandwidth $Bw = f_H - f_L$

iii) The quality factor, $Q = f_0/Bw$

iv) The voltage gain $A_0 = (V_0/V_i)$, in the pass band of the filter

If the designed parameters are not equal to the desired ones, give reasons.

**Experiment 3: Resonant filter without inductor**

Design a second order active Bandpass filter with a centre frequency 5 KHz, quality factor 10, and voltage gain 25. Sketch the frequency response and compare the designed parameters with the desired parameters.

Usually filters without inductors are preferred. In such case, the active resonant filter can be realized using figure 3.

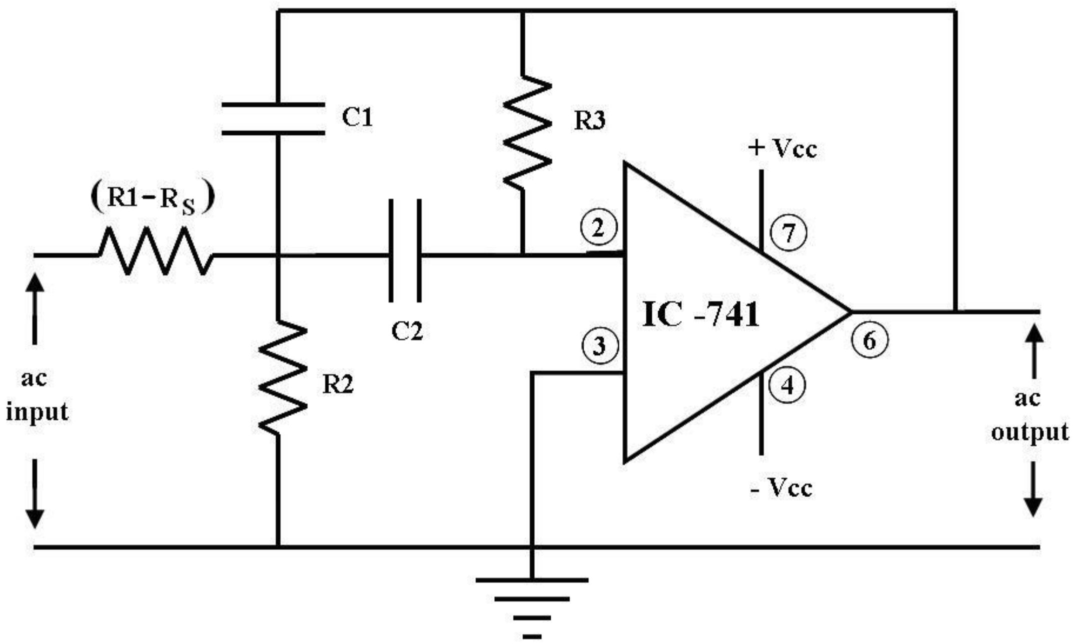



**Figure 3**: Active second order band-pass filter without inductor

Given the following parameters:

voltage gain $A_0 = (V_0/V_i)$,

the center frequency $\omega_0$, with $f_0 = \sqrt{f_L f_H}$,

and the quality factor, $Q = f_0/Bw$, where $Bw = f_H - f_L$

**Procedure:**

**Step 1:** Compute $R_S$, the internal resistance of the signal source.

**Step 2:** Choose $R_1$, higher than $R_S$

**Step 3:** Obtain $C_1$ from the following equation

$$-R_1C_1 = Q/(\omega_0 A_0)$$

**Step 4:** Assume $C_1 = C_2$

**Step 5:** Computer $R_3$ from,

$$R_3 \frac{C_1C_2}{C_1+C_2} = \frac{Q}{\omega_0}$$

**Step 6:** Compute $R_2$ from

$$(R_1 \parallel R_2) R_3C_1C_2 = \frac{1}{\omega_0^2}$$

**Step 7:** Using the designed components, connect the active BPF of figure 3.

**Step 8:** Record the input and output **peak-to-peak** voltage for various input frequencies, and complete the table below.

| **Frequency (kHz)** | $V_i$ | $V_0$ | $Gain = 20\log(V_0/V_i)\,dB$ |
|---|---|---|---|
| **100 Hz** | | | |
| **200 Hz** | | | |
| **….** | | | |
| **1 M Hz** | | | |

**Step 9:** Plot the frequency response of the designed filter (Plot of Frequency Vs. Gain on a semi-log sheet).

**Results:** From the frequency response obtain

v) The lower cut-off frequency $f_L$ and the higher cut-off frequency $f_H$ .

vi) The bandwidth $Bw = f_H - f_L$

vii) The quality factor, $Q = f_0/Bw$

viii) The voltage gain $A_0 = (V_0/V_i)$, in the pass band of the filter

If the designed parameters are not equal to the desired ones, give reasons.

## Active band elimination filters

**Experiment:**

Design a second order active Notch filter with a centre frequency ……..Hz. Sketch the frequency response, obtain the rejection frequency, and the band of frequencies being rejected.

Band elimination filters (or band reject filters) have a frequency response as shown in figure 1. The difference between the two cut-off frequencies $f_L$ (the lower cut-off ) and $f_H$ (the upper cut-off) is known as the bandwidth $Bw$.

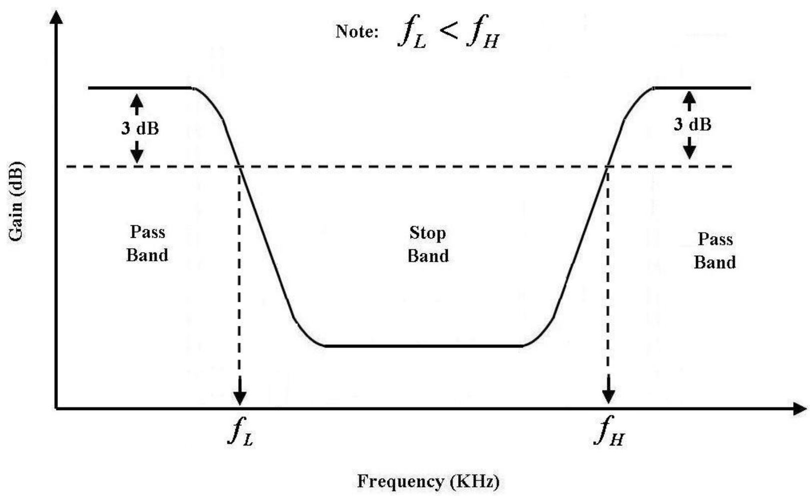



**Figure 1**: Frequency response of band elimination filter

It is possible to realize a BEF using a parallel connection of LPF and HPF as shown in figure 2. But this method requires more components, and hence is not used.

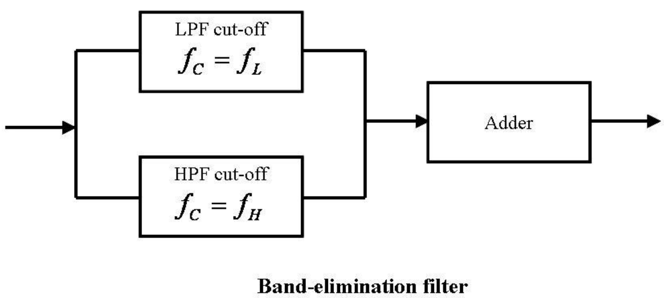



**Figure 2**: Realization of BEF using LPF and HPF

References:

1. Jacob Millman and Christos C Halkias, ‘Integrated Electronics: Analog and Digital Circuits and Systems’, Tata McGraw-Hill Publishing Company Ltd, New Delhi, Edition 1991
2. Lawrence R Rabiner and Bernard Gold, ‘Theory and Application of Digital Signal Processing’, Prentice Hall International, Seventh Indian Reprint, 1993
3. https://en.wikipedia.org/wiki/Stephen_Butterworth
4.

**Stephen Butterworth** (1885–1958) was a British physicist who invented the Butterworth filter,[1] a class of electrical circuits that are used to separate different frequencies of electrical signals.

Stephen Butterworth was born on 11 August 1885 in Rochdale, Lancashire, England (a town located about 10 miles north of the city of Manchester). He was the son of Alexander Butterworth, a postman, and Elizabeth (maiden name unknown).[2] He was the second of four children.[3] In 1904, he entered the University of Manchester, from which he received, in 1907, both a Bachelor of Science degree in physics (first class) and a teacher's certificate (first class). In 1908 he received a Master of Science degree in physics.[4] For the next 11 years he was a physics lecturer at the Manchester Municipal College of Technology. He subsequently worked for several years at the National Physical Laboratory, where he did theoretical and experimental work for the determination of standards of electrical inductance. In 1921 he joined the Admiralty's Research Laboratory. Unfortunately, the classified nature of his work prohibited the publication of much of his research there. Nevertheless, it is known that he worked in a wide range of fields. For example, he determined the electromagnetic field around submarine cables carrying alternating current,[5] and he investigated underwater explosions and the stability of torpedoes. In 1939, he was a "Principal Scientific Officer" at the Admiralty Research Laboratory in the Admiralty's Scientific Research and Experiment Department.[6] During World War II, he investigated both magnetic mines and the degaussing of ships (as a means of protecting them from magnetic mines).

He was a first-rate applied mathematician. He often solved problems that others had regarded as insoluble. For his successes, he employed judicious approximations, penetrating physical insight, ingenious experiments, and skilful use of models. He was a quiet and unassuming man. Nevertheless, his knowledge and advice were widely sought and readily offered. He was respected by his colleagues and revered by his subordinates.[7]

In 1942 he was awarded the Order of the British Empire.[8] In 1945 he retired from the Admiralty Research Laboratory. He died on 28 October 1958 at his home inCowes on the Isle of Wight, England at the age of 73.[9][10][11]

Low-pass to High-pass $\quad s \rightarrow \frac{\omega_C}{s}$

## The First Order Butterworth HPF with cut-off ($\omega_C$=1)

We know that the first order LPF is given by

$$H(s) = \frac{1}{(s+1)}$$

We apply the continuous frequency transformation to obtain the corresponding HPF

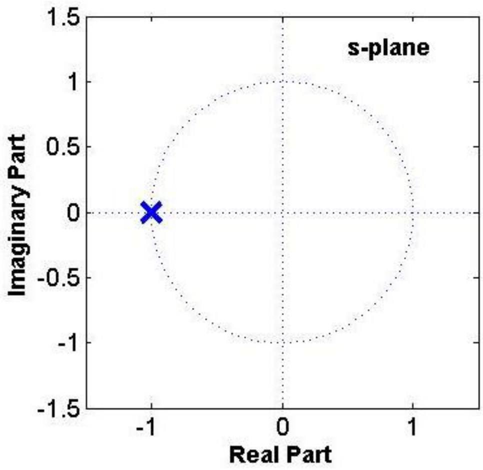
Low-pass to High-pass $\quad s \rightarrow \frac{\omega_C}{s}$

Hence the first order HPF is given by (for $\omega_C$ =1):

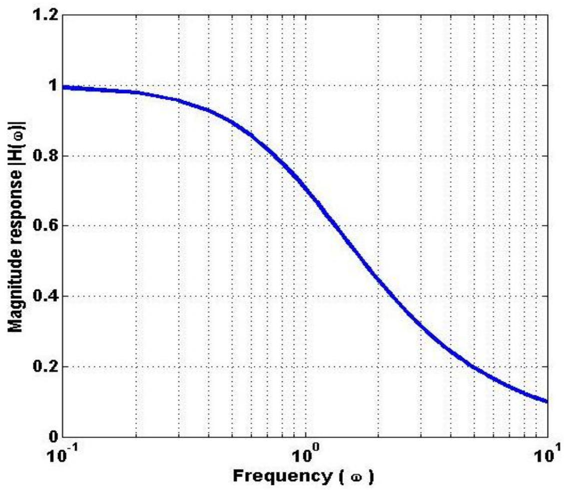
$$H(s) = \frac{1}{\left(\frac{1}{s}+1\right)}$$

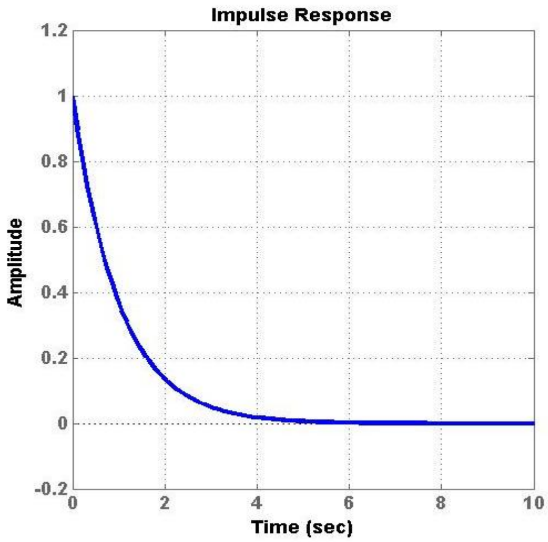
$$= \frac{s}{(s+1)}$$

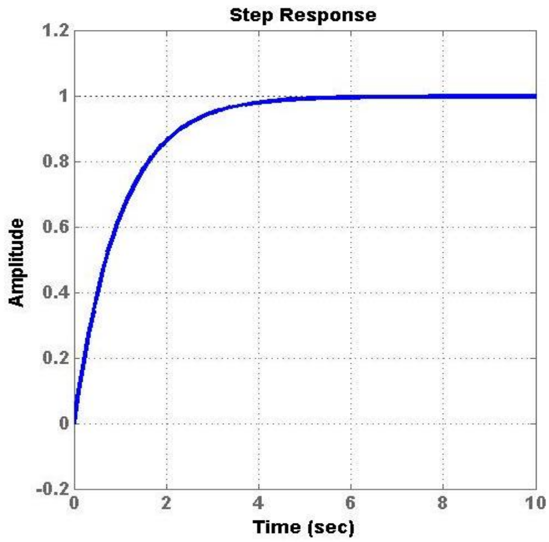
Pole-Zero Plot

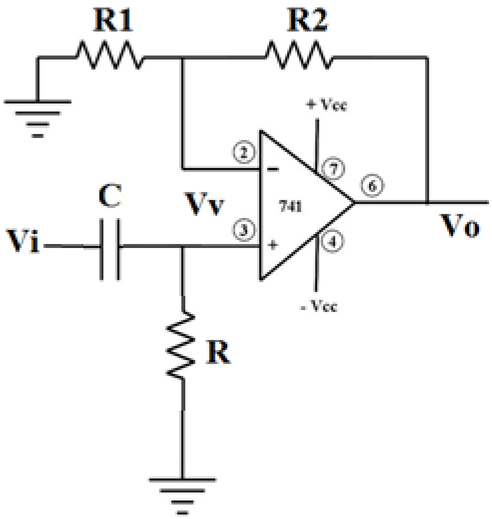



Magnitude Response



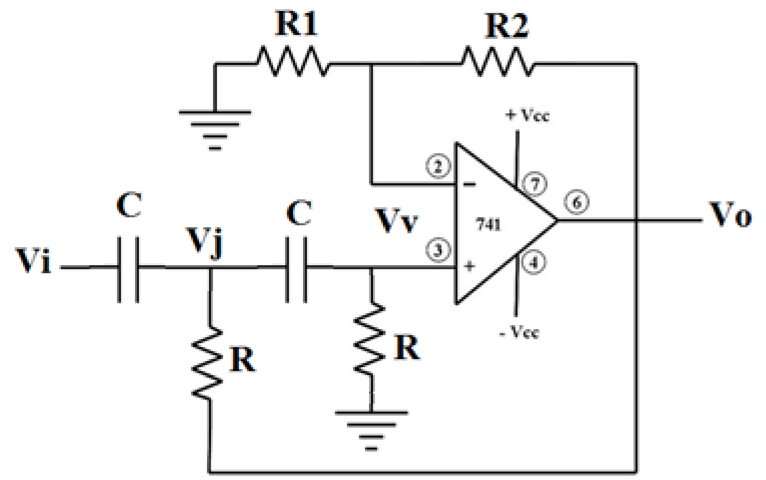


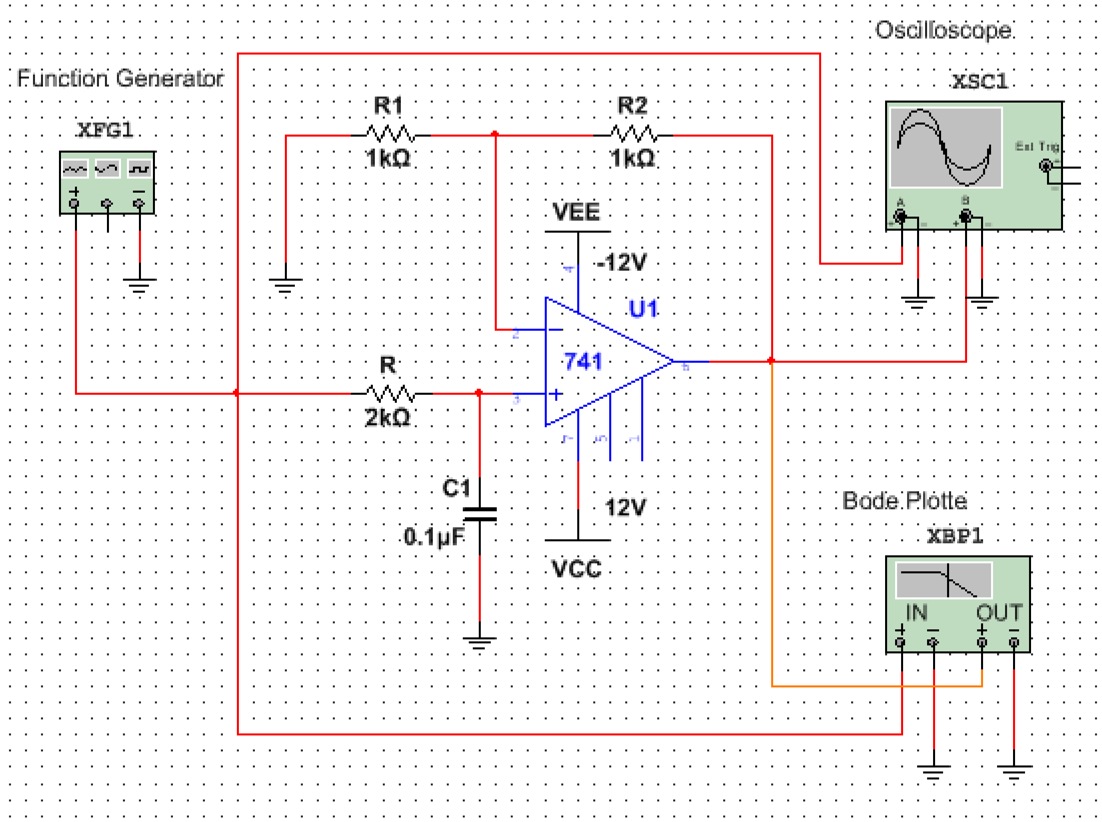




$$H(s) = \frac{s^2}{\left(s^2 + 1.414s + 1\right)}$$

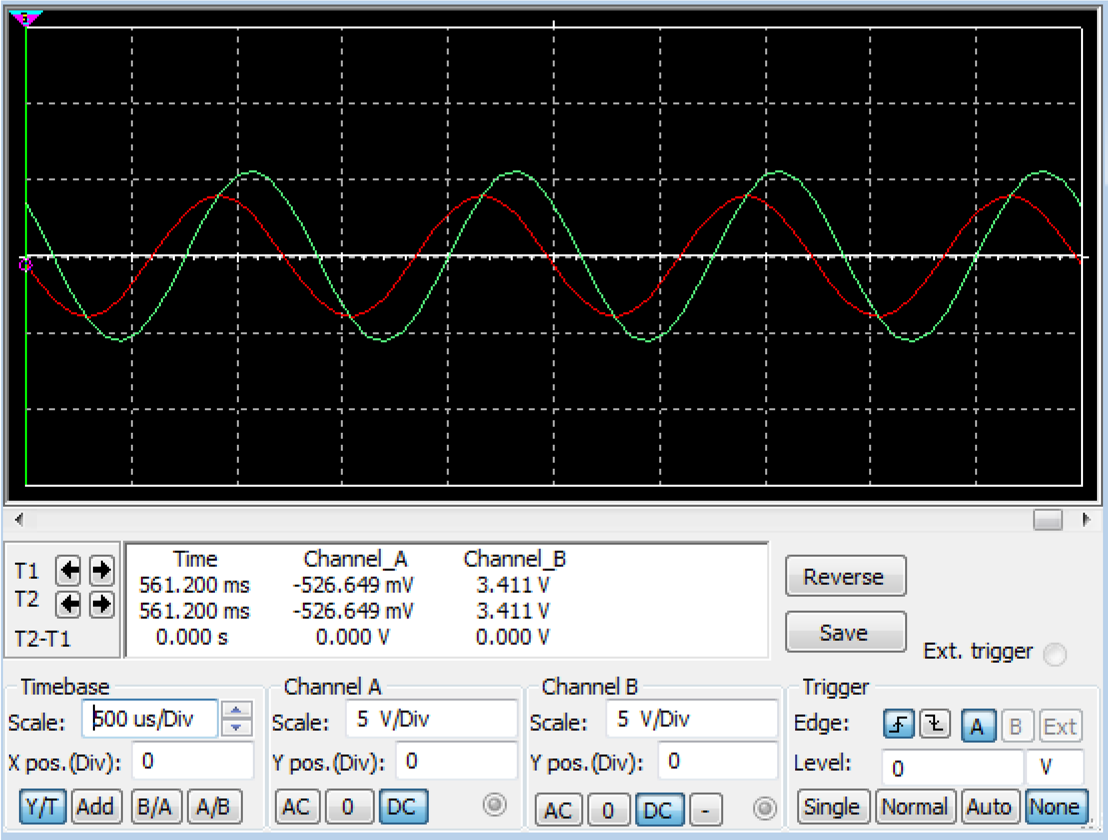
$$H(s) = \frac{1}{\left(s^2 + 1.414s + 1\right)}$$



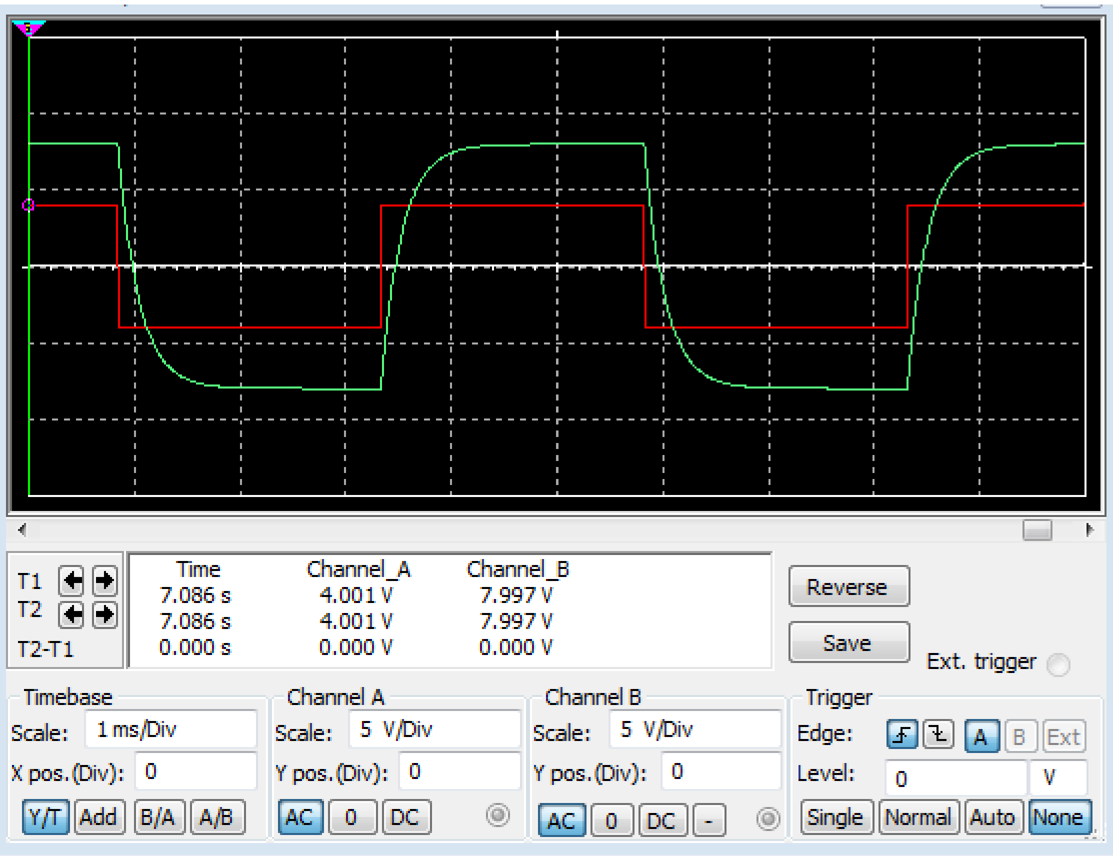

First order Low pass filter



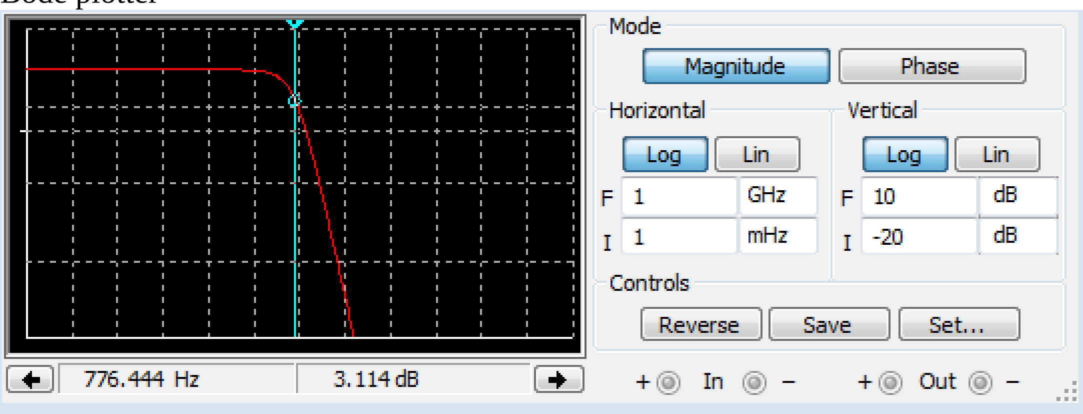
Oscilloscope waveforms  I/P frequency 800Hz- 2vp



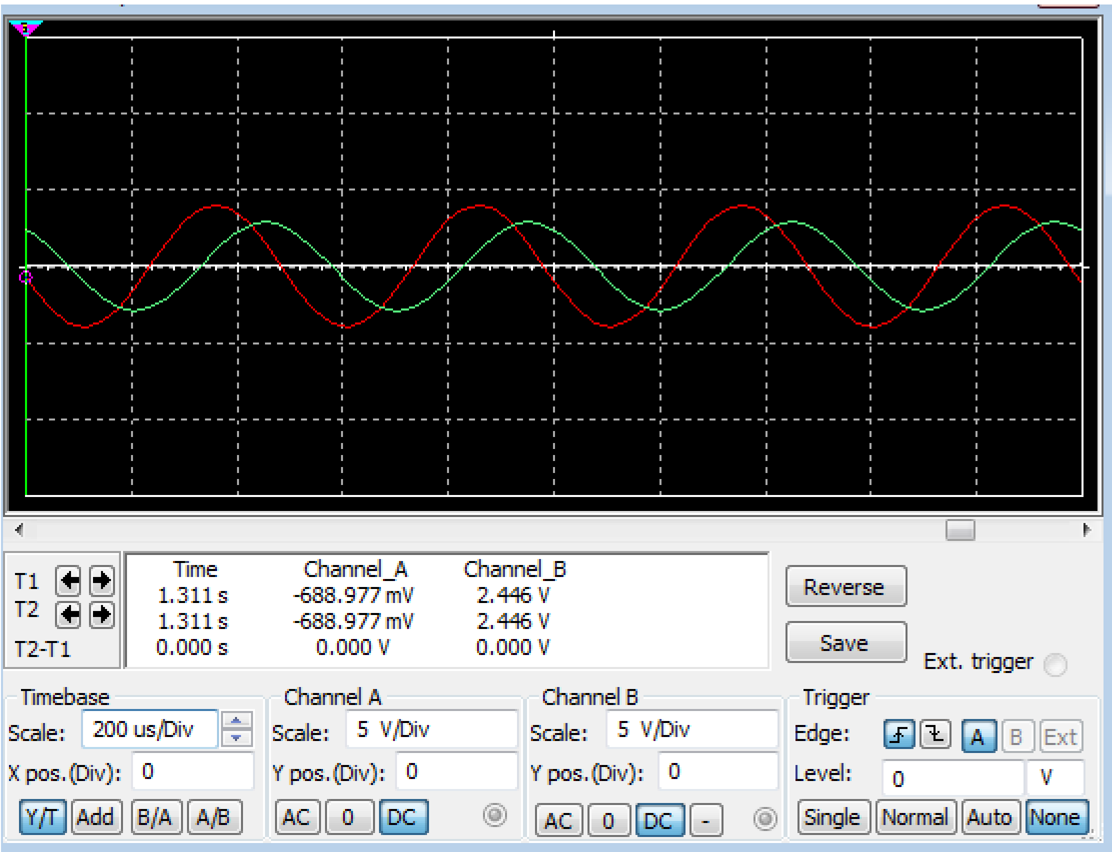

Oscilloscope waveforms  I/P frequency 200Hz- 2vp



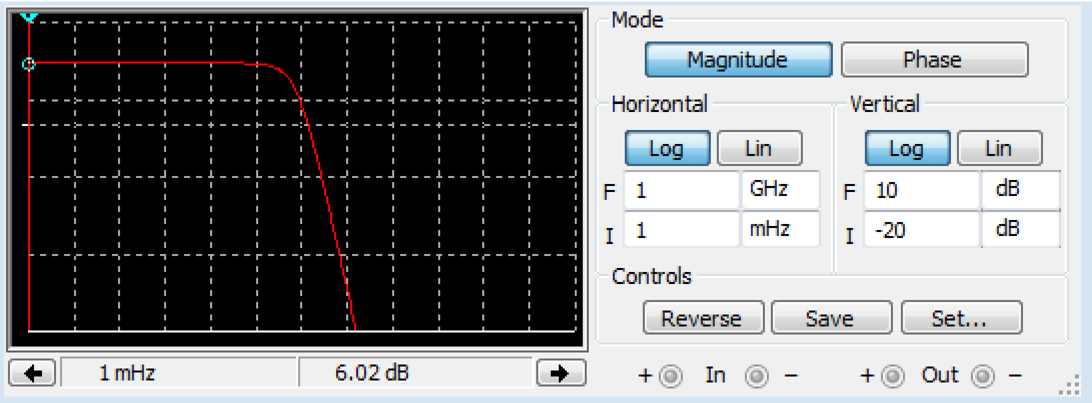

Bode plotter



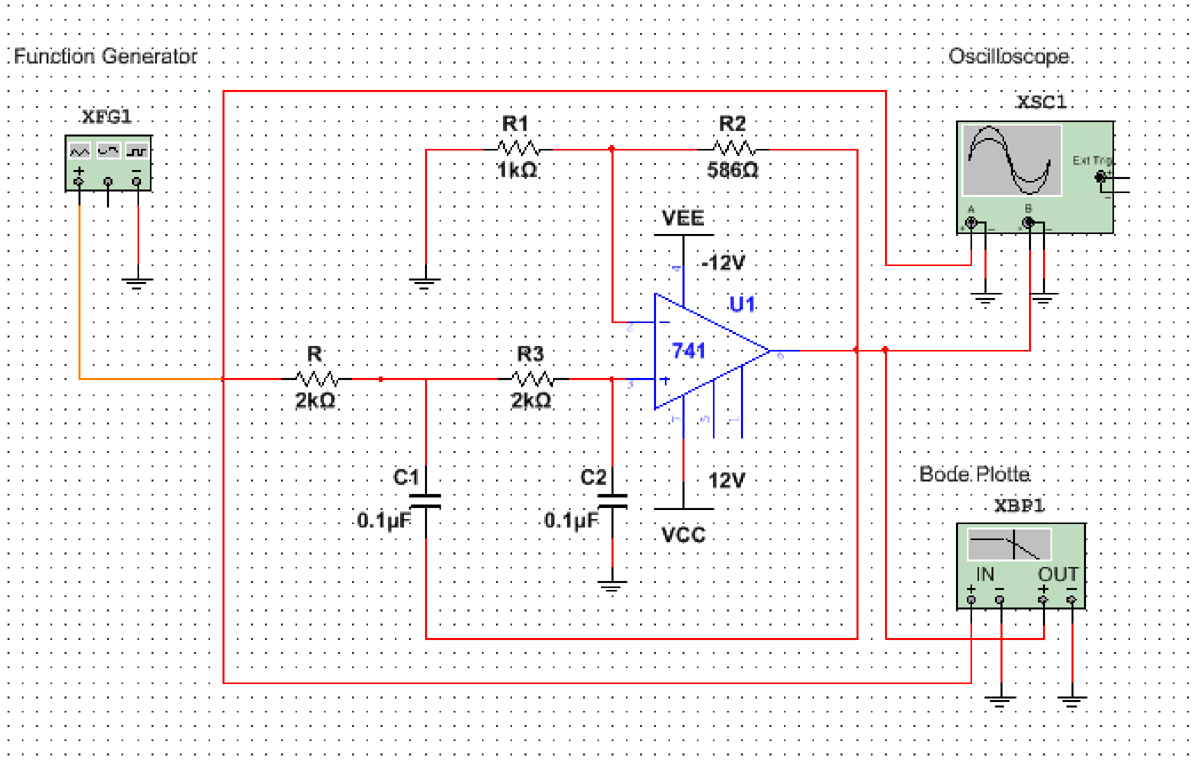
Oscilloscope waveforms I/P frequency 2000Hz- 2vp



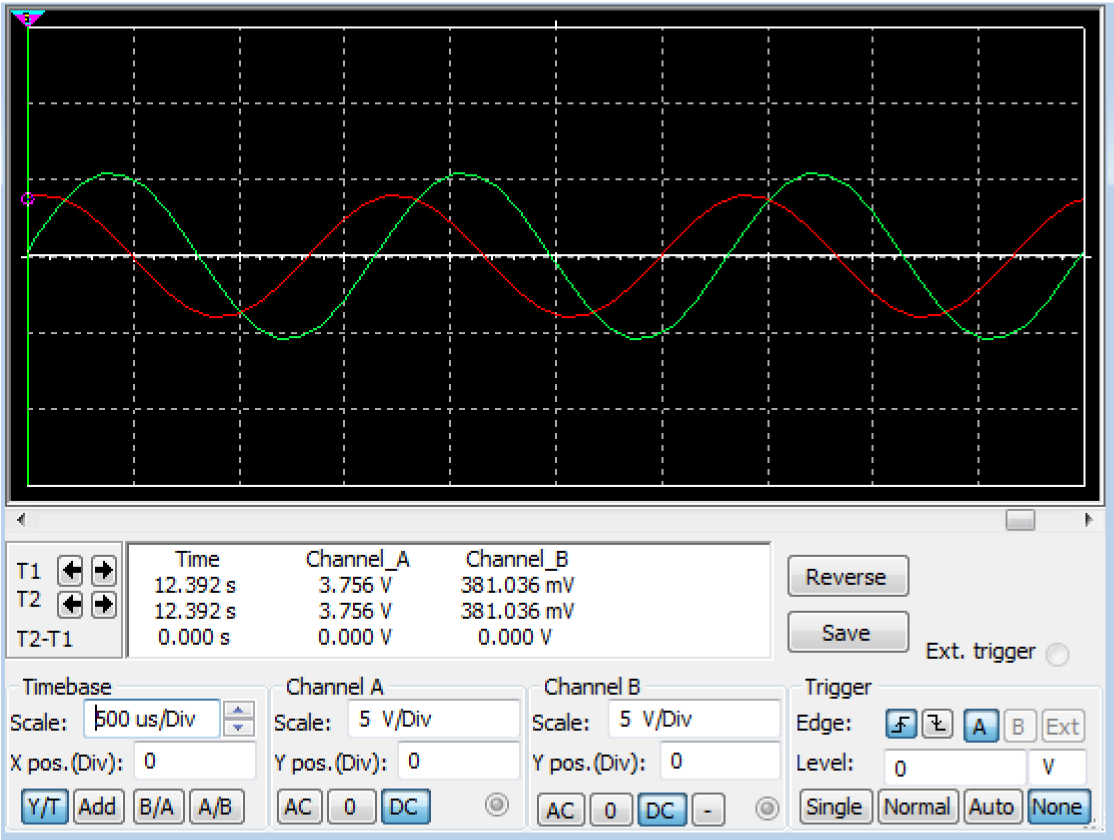
Bode plotter



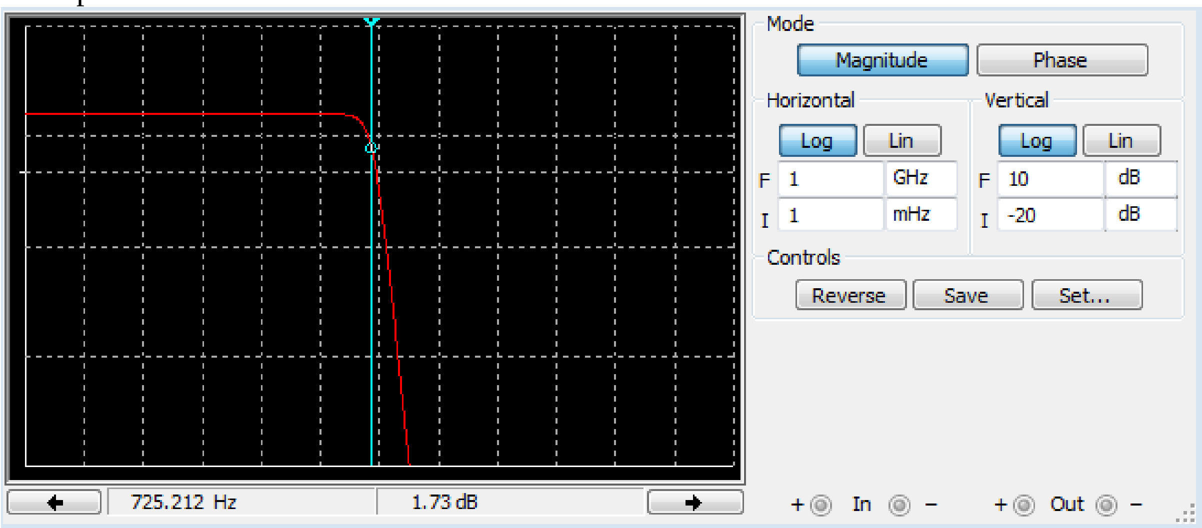

Second order LPF



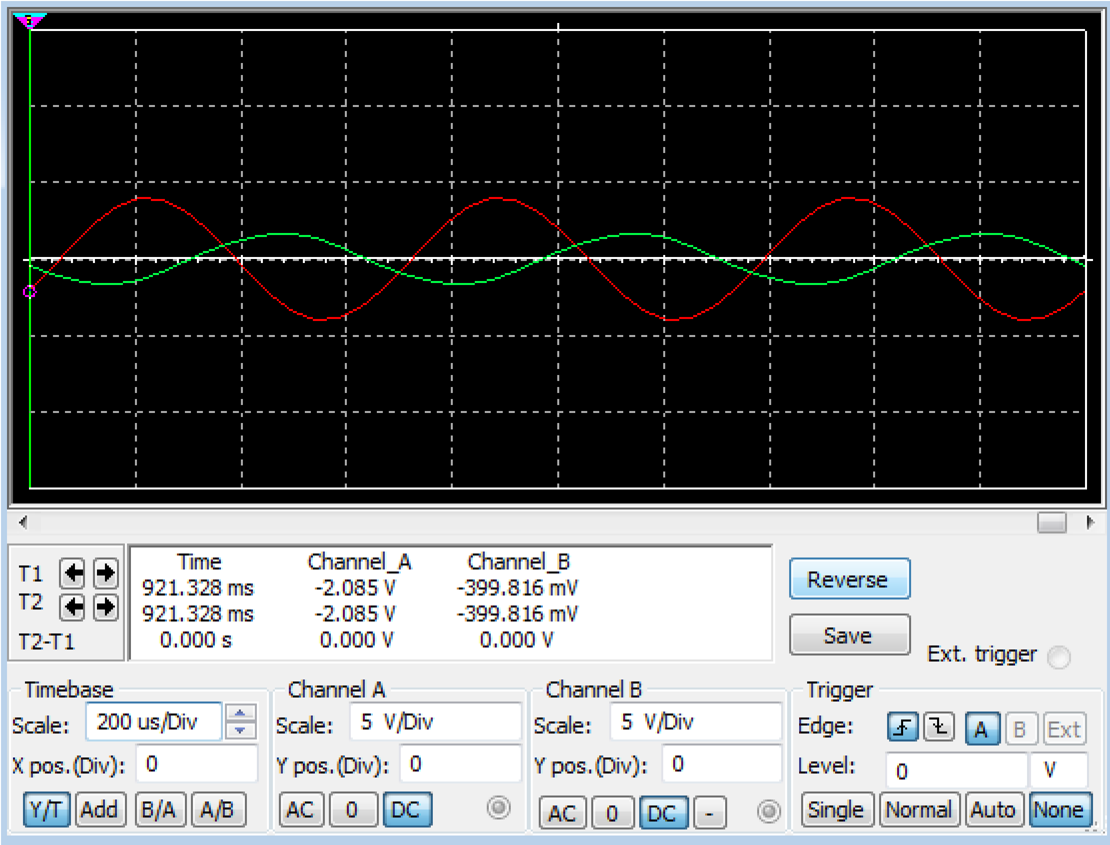

Oscilloscope waveforms  I/P frequency 600Hz- 2vp



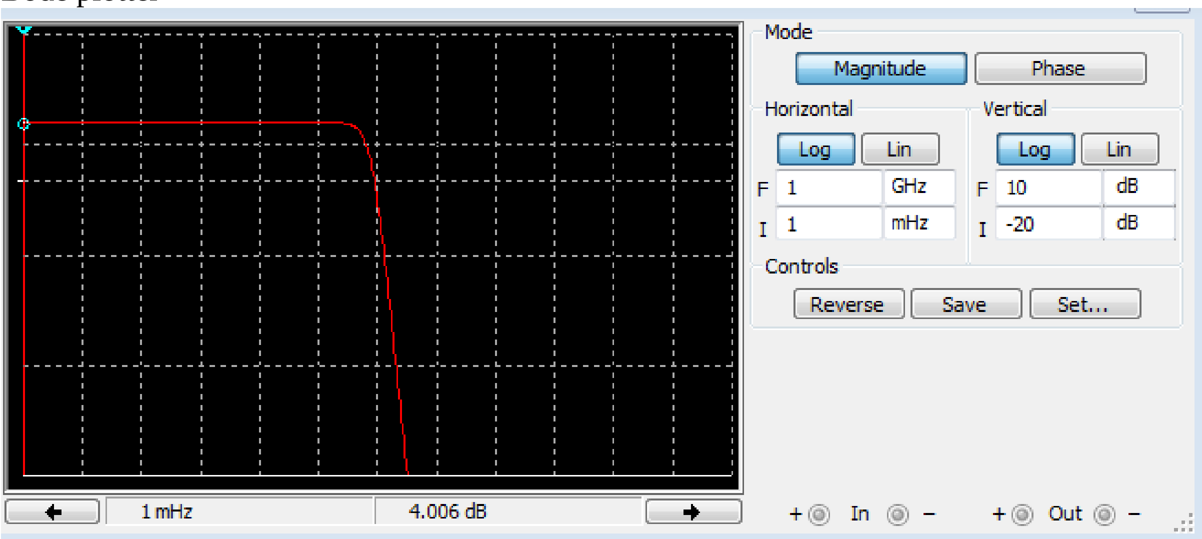

Bode plotter



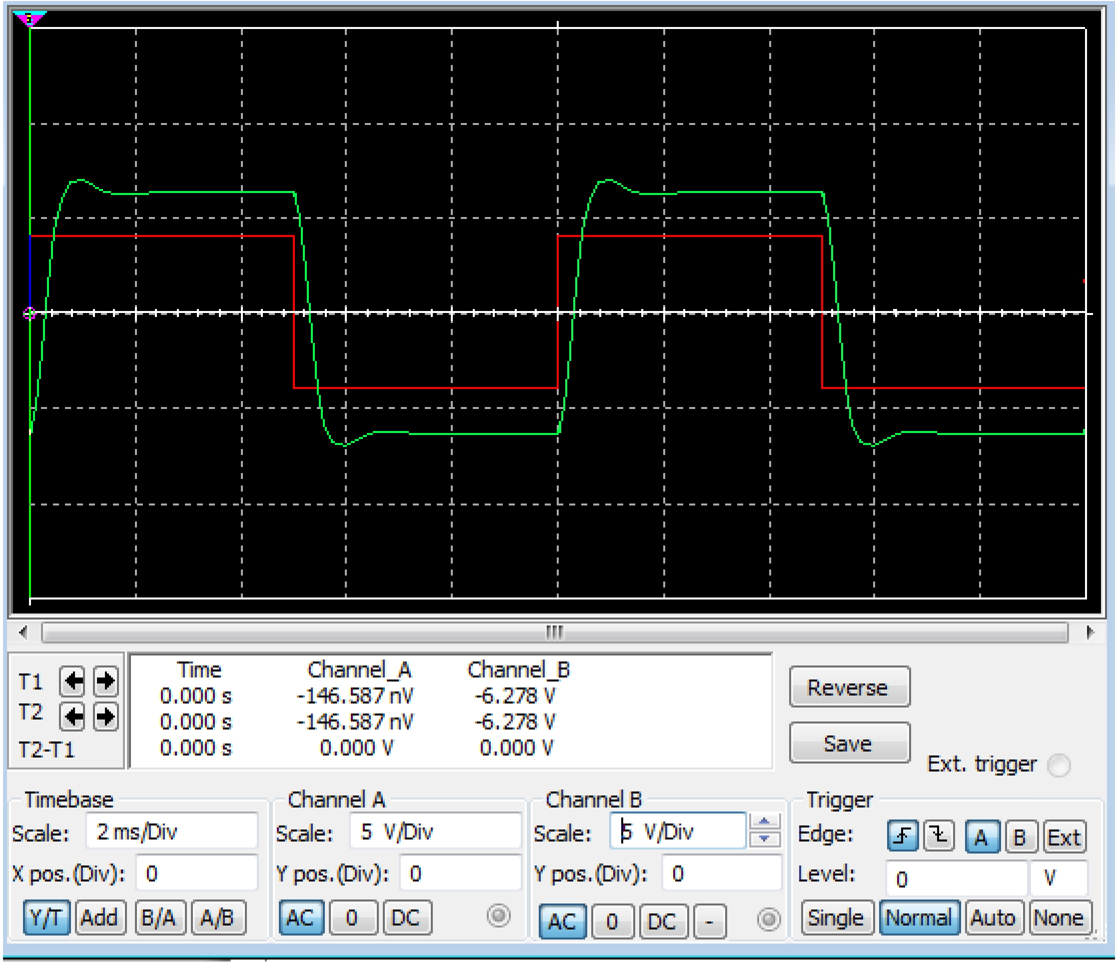

Oscilloscope waveforms I/P frequency 1500Hz- 2vp




Bode plotter




Oscilloscope waveforms I/P frequency 100Hz- 2vp



T1
T2
T2-T1
Time
0.000 s
0.000 s
0.000 s
Channel_A
-146.587 nV
-146.587 nV
0.000 V
Channel_B
-6.278 V
-6.278 V
0.000 V
Reverse
Save
Ext. trigger
Timebase
Scale: 2 ms/Div
X pos.(Div): 0
Y/T Add B/A A/B
Channel A
Scale: 5 V/Div
Y pos.(Div): 0
AC 0 DC
Channel B
Scale: 5 V/Div
Y pos.(Div): 0
AC 0 DC -
Trigger
Edge: A B Ext
Level: 0 V
Single Normal Auto None