Minimal degrees for faithful permutation representations of groups of order p^5 where p is an odd prime

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Abstract. In a 2018 paper, Behravesh and Delfani computed the minimal degree of a faithful permutation representation for groups of order p^5 where p is an odd prime. However, their results are incorrect and incomplete: some groups are not considered, some minimal degrees are incorrectly determined, and the reported list of degrees is incomplete. We determine the minimal degree of a faithful permutation representation for each group of order p^5 where p is an odd prime. We also record how to obtain such a representation.

1. Introduction

The minimal faithful permutation degree $\mu(G)$ of a finite group G is the least positive integer n such that G is isomorphic to a subgroup of the symmetric group S_n. Clearly, $\mu(G)$ is the minimal degree of a faithful representation of G by permutation matrices. A *quasi-permutation matrix* is a square matrix over the complex field $\mathbb{C}$ with non-negative integral trace. Wong [18] defined a *quasi-permutation representation* of G as one consisting of quasi-permutation matrices. Let $c(G)$ denote the minimal degree of a faithful quasi-permutation representation of G. Behravesh and Ghaffarzadeh [3, Theorem 3.2] proved that $c(G) = \mu(G)$ for a p-group G of odd order.

In [2], Behravesh and Delfani claimed to have computed µ(G) for all the groups of order p^5 where p is an odd prime. However, the results obtained in their article are inaccurate. The $\mu(G)$ values $3p^2$ and $2p^3$ are missing from [2, Theorem A]. Moreover, there are several errors in the paper. For example, $\mu(G_j) = p^3$ $(j = 1, 4, 5)$ in [2, Theorem 5.1], instead of p^2 as claimed. Further, they fail to report the minimal degrees for many groups: it appears that their results cover just 32 of the 70 non-abelian groups of order 5^5.

Here we compute the minimal degrees for all groups of p^5, so obtaining the following result.

Theorem 1.1. *The minimal degrees for faithful permutation representations of non-abelian groups of order p^5 where $p \geq 3$ is prime are: $p^2, p^2 + p, p^2 + 2p, 2p^2, 2p^2 + p, 3p^2, p^3, p^3 + p, p^3 + p^2, 2p^3, p^4$; no group of order 3^5 has minimal degree 9.*

We not only determine $\mu(G)$ for each non-abelian group G of order p^5, but we also record how to realize a minimal degree faithful permutation representation for G. Our results – and related code – are available publicly in Magma [4] via a GitHub repository [14], and can be used to construct explicitly this representation of G for a given prime p.

The groups of order p^5 are classified into 10 isoclinism families labelled Φ_i for $1 \leq i \leq 10$. The family Φ_1 consists of all abelian groups. We reference extensively the list of presentations and other data available in [10], and use its group identifiers.

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2. Main results

Throughout this section, $p \geq 5$ is a prime, unless stated otherwise; we deal with the groups of order 3^5 in Section 3. Let $d(G)$ and G' denote respectively the minimal number of generators and the commutator subgroup of a finite group G. Let $\mathrm{Core}_G(H)$ denote the normal core in G of a subgroup H. Let $\mathrm{Irr}(G)$, $\mathrm{lin}(G)$ and $\mathrm{nl}(G)$ be the set of irreducible complex characters, the set of linear characters and the set of nonlinear irreducible characters of G respectively. We denote the character degree set $\{\chi(1) \mid \chi \in \mathrm{Irr}(G)\}$ of G by $\mathrm{cd}(G)$. If $\chi \in \mathrm{Irr}(G)$ then $\Gamma(\chi)$ denotes the Galois group of $\mathbb{Q}(\chi)$ over $\mathbb{Q}$. Let ν be the smallest positive integer which is a non-quadratic residue modulo p and let ω be the smallest positive integer which is a primitive root modulo p.

To prove Theorem 1.1, we exploit extensively the equality of $\mu(G)$ and $c(G)$. For some groups we determine $\mu(G)$ directly; for the remainder we determine $c(G)$. In Section 2.1 we calculate $\mu(G)$ for some groups of order p^5. By refining these values, we then compute $c(G)$ for all outstanding groups in Section 2.2. We summarize our results in Tables 1, 2 and 3. We discuss the GitHub repository in Section 4.

2.1. Minimal faithful permutation representation degrees of groups of order p^5. We first mention a few known results used throughout the article.

It is well known that every permutation representation of G is a direct sum of transitive permutation representations, and every transitive representation is equivalent to a representation of the set of cosets of some subgroup of G. Hence we identify a faithful representation of G with the set of subgroups $\{H_1, \ldots, H_m\}$ determining its *transitive constituents*; the degree is $\sum_{i=1}^{m} |G : H_i|$, and $\cap_{i=1}^{m} \mathrm{Core}_G(H_i) = 1$. Now $\mu(G)$ is the minimum degree over all such sets of subgroups.

Lemma 2.1. [11, Theorem 3] *Let G be a p-group whose center is minimally generated by d elements, and let $\{H_1, \ldots, H_m\}$ determine a minimal degree faithful permutation representation of G. Then $m = d$.*

Lemma 2.2. [15, Lemma 14] *Let G be a non-abelian p-group of order p^n. If G is not a direct product of an abelian and a non-abelian subgroup, then p^2 divides $\mu(G)$.*

Lemma 2.3. [19, Corollary 2.2] *If H and K are non-trivial nilpotent groups, then $\mu(H \times K) = \mu(H) + \mu(K)$.*

We partition the groups into two types.

(1) Type 1: a group that is a direct product of an abelian and a non-abelian subgroup; these are listed in Table 1.
(2) Type 2: all others; these are listed in Table 2.

The column labelled "Group G" identifies the group G of order p^5 by its identifier in [10]. In Table 1, the column labelled "Group H" lists a non-abelian direct factor H of G. The column labelled "Argument for lower bound (H)" gives (information about) a lower bound for $\mu(H)$; that labelled "$\mathcal{H}_H$" identifies a minimal degree faithful permutation representation of H; and that labelled "$\mu(H)$" lists the value of $\mu(H)$. In Table 2, the column labelled "Argument for lower bound" gives a lower bound for $\mu(G)$ and that labelled "$\mathcal{H}_G$" identifies a minimal degree faithful permutation representation of G. In both cases, the column labelled "$\mu(G)$" lists the value of $\mu(G)$.

Suppose $G = H \times K$ where H is non-abelian and K is abelian. If $\{H_1, H_2, \ldots, H_n\}$ and $\{K_1, K_2, \ldots, K_m\}$ are minimal degree faithful permutation representations of H and K respectively, then

$$\{H_1 \times K, \ldots, H_n \times K, H \times K_1, \ldots, H \times K_m\}$$

determines one for G (see [11, Proposition 2]).

Example 2.4. Consider the group of Type 1 listed in [10]:

$$\begin{aligned} \phi_2(311)a &= \phi_2(31) \times C_p = \langle \alpha, \alpha_1, \alpha_2 \mid [\alpha_1, \alpha] = \alpha^{p^2} = \alpha_2, \alpha_1^p = \alpha_2^p = 1 \rangle \times C_p \\ &= H \times \langle \beta \rangle, \text{ where } H = \phi_2(31) \text{ and } \langle \beta \rangle \cong C_p. \end{aligned}$$

Observe that $Z(H) = \langle \alpha^p \rangle \cong C_{p^2}$. Lemma 2.1 implies that we search for a subgroup H_1 of H with a trivial core. Since $H_1 = \langle \alpha_1 \rangle$ has this property, we deduce that $\mu(H) \leq |H : H_1| = p^3$. On the other hand, $H \geq S = \langle \alpha \rangle \cong C_{p^3}$, so $\mu(S) = p^3 \leq \mu(H)$. Therefore, $p^3 \leq \mu(H) \leq p^3$. Lemma 2.3 now implies that $\mu(G) = p^3 + p$. Since $\{\langle \alpha_1 \rangle\}$ is a minimal degree faithful permutation representation of H, the set $\{\langle \alpha_1, \beta \rangle, H\}$ determines one for $\phi_2(311)a$.

Example 2.5. Consider the group of Type 2 listed in [10]:

$$\phi_2(32)a_1 = \langle \alpha, \alpha_1, \alpha_2 \mid [\alpha_1, \alpha] = \alpha^{p^2} = \alpha_2, \alpha_1^{p^2} = \alpha_2^p = 1 \rangle.$$

Observe that $Z(G) = \langle \alpha^p, \alpha_1^p \rangle \cong C_{p^2} \times C_p$, so Lemma 2.1 implies that $|\mathcal{H}_G| = 2$. Let $H_1 = \langle \alpha \rangle$ and $H_2 = \langle \alpha_1 \rangle$. Since $\mathrm{Core}_G(H_1) \cap \mathrm{Core}_G(H_2) = 1$ we deduce that $\mu(G) \leq p^3 + p^2$. On the other hand, $G \geq \langle \alpha, \alpha_1^p \rangle \cong C_{p^3} \times C_p$ so $\mu(G) \geq p^3 + p$. Therefore, from Lemma 2.2, $\mu(G) = p^3 + p^2$.

Table 1: $\mu(G)$ for groups of order p^5 of Type 1

Group G	Group H	Argument for lower bound (H)	$\mathcal{H}_H$	$\mu(H)$	$\mu(G)$
$\phi_2(311)a$	$\phi_2(31)$	$\mu(\langle \alpha \rangle) = p^3$	$\{\langle \alpha_1 \rangle\}$	p^3	$p^3 + p$
$\phi_2(221)a$	$\phi_2(22)$	$\mu(\langle \alpha, \alpha_1^p \rangle) = p^3$	$\{\langle \alpha \rangle, \langle \alpha_1 \rangle\}$	$2p^2$	$2p^2 + p$
$\phi_2(221)b$	$\phi_2(21)$	$\mu(\langle \alpha \rangle) = p^2$	$\{\langle \alpha_1 \rangle\}$	p^2	$2p^2$
$\phi_2(2111)a$	$\phi_2(21)$	$\mu(\langle \alpha \rangle) = p^2$	$\{\langle \alpha_1 \rangle\}$	p^2	$p^2 + 2p$
$\phi_2(2111)b$	$\phi_2(211)b$	$\mu(\langle \alpha_1, \gamma \rangle) = p^2 + p$	$\{\langle \alpha_1 \rangle\}$	p^3	$p^3 + p$
$\phi_2(2111)c$	$\phi_2(211)c$	$\mu(\langle \alpha, \alpha_2 \rangle) = p^2 + p$	$\{\langle \alpha \rangle, \langle \alpha_1, \alpha_2 \rangle\}$	$2p^2$	$2p^2 + p$
$\phi_2(2111)d$	$\phi_2(111)$	$\mu(H) = p^2$ from [12, Table 1]	$\{\langle \alpha_1 \rangle\}$	p^2	$2p^2$
$\phi_2(1^5)$	$\phi_2(111)$	$\mu(H) = p^2$ from [12, Table 1]	$\{\langle \alpha_1 \rangle\}$	p^2	$p^2 + 2p$
$\phi_3(2111)a$	$\phi_2(211)a$	$\mu(\langle \alpha \rangle) = p^2$	$\{\langle \alpha_1, \alpha_2 \rangle\}$	p^2	$p^2 + p$
$\phi_3(2111)b_r$, for $r = 1$ or ν	$\phi_2(211)b_r$	$\mu(\langle \alpha_2 \rangle) = p^2$	$\{\langle \alpha_2 \rangle\}$	p^3	$p^3 + p$
$\phi_3(1^5)$	$\phi_2(1^4)$	$\mu(\langle \alpha, \alpha_2 \rangle) = p^2$	$\{\langle \alpha_1, \alpha_2 \rangle\}$	p^2	$p^2 + p$

Table 2: $\mu(G)$ for groups of order p^5 of Type 2

Group G	Argument for lower bound	$\mathcal{H}_G$	$\mu(G)$
$\phi_2(41)$	$\mu(\langle \alpha \rangle) = p^4$	$\{\langle \alpha_1 \rangle\}$	p^4
$\phi_2(32)a_1$, $\phi_2(32)a_2$	$\mu(\langle \alpha, \alpha_1^p \rangle) = p^3 + p$	$\{\langle \alpha \rangle, \langle \alpha_1 \rangle\}$	$p^3 + p^2$
$\phi_2(311)b$	$\mu(\langle \alpha_1, \gamma \rangle) = p^3 + p$	$\{\langle \alpha_1 \rangle\}$	p^4
$\phi_2(311)c$	$\mu(\langle \alpha, \alpha_2 \rangle) = p^3 + p$	$\{\langle \alpha \rangle, \langle \alpha_1, \alpha_2 \rangle\}$	$p^3 + p^2$
$\phi_2(221)d$	$\mu(\langle \alpha, \alpha_1, \alpha_2 \rangle) = p^2 + 2p$	$\{\langle \alpha, \alpha_2 \rangle, \langle \alpha_1, \alpha_2 \rangle, \langle \alpha, \alpha_1^p \rangle\}$	$3p^2$
$\phi_3(311)a$	$\mu(\langle \alpha \rangle) = p^3$	$\{\langle \alpha_1, \alpha_2 \rangle\}$	p^3
$\phi_3(311)b_r$, for $r = 1$ or ν	$\mu(\langle \alpha_1, \alpha_2 \rangle) = p^3 + p$	$\{\langle \alpha_2 \rangle\}$	p^4
$\phi_3(221)a$	$\mu(\langle \alpha_1, \alpha_2 \rangle) = p^2 + p$	$\{\langle \alpha, \alpha_2 \rangle, \langle \alpha_1, \alpha_2 \rangle\}$	$2p^2$
$\phi_3(2111)c$	$\mu(\langle \alpha_2, \gamma \rangle) = p^2 + p$	$\{\langle \alpha_1, \alpha_2 \rangle\}$	p^3
$\phi_3(2111)d$	$\mu(\langle \alpha, \alpha_3 \rangle) = p^2 + p$	$\{\langle \alpha^p, \alpha_1, \alpha_2 \rangle, \langle \alpha_1, \alpha_2, \alpha_3 \rangle\}$	$2p^2$
$\phi_3(2111)e$	$\mu(\langle \alpha_1, \alpha_2 \rangle) = p^2 + p$	$\{\langle \alpha, \alpha_2, \alpha_3 \rangle, \langle \alpha_1, \alpha_2 \rangle\}$	$2p^2$
$\phi_4(221)a$	$\mu(\langle \alpha_1, \alpha_2 \rangle) = p^2 + p$	$\{\langle \alpha, \alpha_2 \rangle, \langle \alpha_1, \alpha_2 \rangle\}$	$2p^2$
$\phi_4(221)c$	$\mu(\langle \alpha_1, \alpha_2 \rangle) = p^2 + p$	$\{\langle \alpha, \alpha_1 \rangle, \langle \alpha, \alpha_2 \rangle\}$	$2p^2$
$\phi_4(221)d_r$, where $k = \omega^r$ for $r = 1, 2, \ldots, (p-1)/2$	$\mu(\langle \alpha_1, \alpha_2 \rangle) = 2p^2$	$\{\langle \alpha, \alpha_1 \rangle, \langle \alpha, \alpha_2 \rangle\}$	$2p^2$
$\phi_4(2111)a$	$\mu(\langle \alpha, \beta_1 \rangle) = p^2 + p$	$\{\langle \alpha, \alpha_2 \rangle, \langle \alpha_1, \alpha_2, \beta_1 \rangle\}$	$2p^2$
$\phi_4(2111)b$	$\mu(\langle \alpha_1, \alpha_2 \rangle) = p^2 + p$	$\{\langle \alpha, \alpha_1 \rangle, \langle \alpha, \alpha_2, \beta_2 \rangle\}$	$2p^2$
$\phi_4(2111)c$	$\mu(\langle \alpha_1, \alpha_2 \rangle) = p^2 + p$	$\{\langle \alpha, \beta_2 \rangle, \langle \alpha_1, \alpha_2 \rangle\}$	$2p^2$
$\phi_4(1^5)$	$\mu(\langle \alpha_1, \alpha_2 \rangle) = p^2 + p$	$\{\langle \alpha, \alpha_1, \beta_1 \rangle, \langle \alpha, \alpha_2, \beta_2 \rangle\}$	$2p^2$
$\phi_5(2111)$	$\mu(\langle \alpha_1, \alpha_3 \rangle) = p^2 + p$	$\{\langle \alpha_2, \alpha_3 \rangle\}$	p^3
$\phi_5(1^5)$	$\mu(\langle \alpha_1, \alpha_2, \alpha_3, \beta \rangle) = p^2 + p$	$\{\langle \alpha_2, \alpha_3 \rangle\}$	p^3
$\phi_6(221)a$	$\mu(\langle \alpha_1, \alpha_2^p \rangle) = p^2 + p$	$\{\langle \alpha_1, \beta, \beta_1 \rangle, \langle \alpha_2, \beta, \beta_2 \rangle\}$	$2p^2$

Continuation of Table 2			
Group G	Argument for lower bound	$\mathcal{H}_G$	$\mu(G)$
$\phi_6(221)b_r$, where $k=\omega^r$ for $r=1,2,\ldots,(p-1)/2$	$\mu(\langle\alpha_1,\alpha_2^p\rangle)=p^2+p$	$\{\langle\alpha_1,\beta,\beta_1\rangle,\langle\alpha_2,\beta,\beta_2\rangle\}$	$2p^2$
$\phi_6(2111)a$	$\mu(\langle\alpha_1,\beta_2\rangle)=p^2+p$	$\{\langle\alpha_1,\beta\rangle,\langle\alpha_2,\beta,\beta_2\rangle\}$	$2p^2$
$\phi_6(1^5)$	$\mu(\langle\alpha_1,\beta,\beta_1,\beta_2\rangle)=p^2+p$	$\{\langle\alpha_1,\beta,\beta_1\rangle,\langle\alpha_2,\beta,\beta_2\rangle\}$	$2p^2$
$\phi_7(2111)a$ $\phi_7(2111)c$	$\mu(\langle\alpha,\beta\rangle)=p^2+p$	$\{\langle\alpha_1,\alpha_2\rangle\}$	p^3
$\phi_7(2111)b_r$, for $r=1$ or ν	$\mu(\langle\alpha_1,\alpha_2\rangle)=p^2+p$	$\{\langle\alpha,\beta\rangle\}$	p^3
$\phi_7(1^5)$	$\mu(\langle\alpha,\alpha_2,\alpha_3,\beta\rangle)=p^2+p$	$\{\langle\alpha_1,\alpha_2\rangle\}$	p^3
$\phi_8(32)$	$\mu(\langle\alpha_1\rangle)=p^3$	$\{\langle\alpha_2\rangle\}$	p^3
$\phi_9(2111)a$	$\mu(\langle\alpha\rangle)=p^2$	$\{\langle\alpha_1,\alpha_2,\alpha_3\rangle\}$	p^2
$\phi_9(2111)b_r$, where $k=\omega^r$ for $r+1=1,2,\ldots,\gcd(p-1,3)$	$\mu(\langle\alpha_1,\alpha_2\rangle)=p^2+p$	$\{\langle\alpha_2,\alpha_3\rangle\}$	p^3
$\phi_9(1^5)$	$\mu(\langle\alpha,\alpha_3,\alpha_4\rangle)=p^2$	$\{\langle\alpha_1,\alpha_2,\alpha_3\rangle\}$	p^2
$\phi_{10}(2111)a_r$, where $k=\omega^r$ for $r+1=1,2,\ldots,\gcd(p-1,4)$	$\mu(\langle\alpha_1,\alpha_2,\alpha_3,\alpha_4\rangle)=p^2+p$	$\{\langle\alpha_2,\alpha_3\rangle\}$	p^3
$\phi_{10}(2111)b_r$, where $k=\omega^r$ for $r+1=1,2,\ldots,\gcd(p-1,3)$	$\mu(\langle\alpha_1,\alpha_3\rangle)=p^2+p$	$\{\langle\alpha_2,\alpha_3\rangle\}$	p^3
$\phi_{10}(1^5)$	$\mu(\langle\alpha_1,\alpha_2,\alpha_3,\alpha_4\rangle)=p^2+p$	$\{\langle\alpha_2,\alpha_3\rangle\}$	p^3

2.2. Minimal faithful quasi-permutation representation degrees of groups of order p^5. In Section 2.1, we computed $\mu(G)$ for some groups G of order p^5. Now, we compute $c(G)$ for the remaining groups of order p^5. We first determine the values that $c(G)$ can take for each of the relevant isoclinism families.

We summarize some preliminary results. Let $\chi,\ \psi\in \mathrm{Irr}_{\mathbb{C}}(G)=\mathrm{Irr}(G)$. We say that χ and ψ are Galois conjugate over $\mathbb{Q}$ if there exists $\sigma\in\Gamma(\chi)$ such that $\chi^\sigma=\psi$, where $\mathbb{Q}(\chi)$ denotes the field obtained by adjoining the values $\chi(g)$, for all $g\in G$, to $\mathbb{Q}$. Galois conjugacy defines an equivalence relation on $\mathrm{Irr}(G)$. Moreover, if $\mathcal{C}$ denotes the equivalence class of χ with respect to Galois conjugacy over $\mathbb{Q}$, then $|\mathcal{C}|=|\mathbb{Q}(\chi):\mathbb{Q}|$ (see [9, Lemma 9.17]). Under the above setup, we have following definition.

Definition 2.6. *Let G be a finite group.*

(i) *For $\psi\in\mathrm{Irr}(G)$, define $d(\psi)=|\Gamma(\psi)|\psi(1)$.*
(ii) *For each complex character χ, define*

$$m(\chi)=\begin{cases}0 & \text{if } \chi(g)\geq 0 \text{ for all } g\in G,\\ -\min\left\{\sum_{\sigma\in\Gamma(\chi)}\chi^\sigma(g): g\in G\right\} & \text{otherwise.}\end{cases}$$

Lemma 2.7. [3, Lemma 2.2] *Let G be a finite group. Let $X\subset\mathrm{Irr}(G)$ be such that $\cap_{\chi\in X}\ker(\chi)=1$ and $\cap_{\chi\in Y}\ker(\chi)\neq 1$ for every proper subset Y of X. Let $\xi_X=\sum_{\chi\in X}\left[\sum_{\sigma\in\Gamma(\chi)}\chi^\sigma\right]$ and let $m(\xi_X)$ be the absolute value of the minimum value that ξ_X takes over G. Then*

$$c(G)=\min\{\xi_X(1)+m(\xi_X)\mid X\subset\mathrm{Irr}(G) \text{ satisfying the above property}\}.$$

We identify $X_G\subset\mathrm{Irr}(G)$ with a minimal degree faithful quasi-permutation representation of G if

$$\bigcap_{\chi\in X_G}\ker(\chi)=1 \text{ and } \bigcap_{\chi\in Y}\ker(\chi)\neq 1 \text{ for every } Y\subset X_G \tag{1}$$

and $c(G)=\xi_{X_G}(1)+m(\xi_{X_G})$.

Lemma 2.8. [7, Theorem 2.3] *Let G be a p-group whose center is minimally generated by d elements and let X_G be a minimal degree faithful quasi-permutation representation of G as defined in Equation (1).*

(i) $|X_G| = d$.
(ii) $m(\xi_{X_G}) = \frac{1}{p-1}\sum_{\chi\in X_G}\left[\sum_{\sigma\in\Gamma(\chi)}\chi^{\sigma}(1)\right]$.

Lemma 2.9. [3, Theorem 3.2] *If G is a finite p-group, then $c(G) = \mu(G)$.*

Lemma 2.10. [15, Remark 20] *Let G be a non-abelian group of order p^5. Assume $d(Z(G)) = m$ where $1 \le m \le 3$, $\exp(G) = p^b$ where $1 \le b \le 4$, and $\max \operatorname{cd}(G) = p^e$ where $1 \le i \le 2$. Then*

$$\sum_{k=1}^{b} a_k p^k \le c(G) \le \sum_{l=1}^{b+e} c_l p^l, \text{ where } a_k, c_l \in \mathbb{Z}_{\ge 0} \text{ with } a_b \ne 0, c_{b+e} \ne 0 \text{ and } \sum_{k=1}^{b} a_k = \sum_{l=1}^{b+e} c_l = m.$$

Now, we prove the main result.

Theorem 2.11. *Let G be a non-abelian group of order p^5.*
(i) *If $G \in \Phi_2$, then $c(G) = p^2+2p, 2p^2, 2p^2+p, 3p^2, p^3+p, p^3+p^2$, or p^4.*
(ii) *If $G \in \Phi_3$, then $c(G) = p^2+p, 2p^2, p^3, p^3+p, p^3+p^2$, or p^4.*
(iii) *If $G \in \Phi_i$ where $i \in \{4, 6\}$, then $c(G) = 2p^2, p^3+p^2$, or $2p^3$.*
(iv) *If $G \in \Phi_i$ where $i \in \{5, 7, 8, 10\}$, then $c(G) = p^3$.*
(v) *If $G \in \Phi_9$, then $c(G) = p^2$, or p^3.*

Proof. **Case (i):** Let $G \in \Phi_2$. From [16, Table 1], $c(G) = p^4, p^3+p^2, p^3+p, 3p^2, 2p^2+p, 2p^2$, or p^2+2p.
Case (ii): Let $G \in \Phi_3$. Then $|Z(G)| = p^2$, $\operatorname{cd}(G) = \{1, p\}$ and $\exp(G) \le p^3$. If $Z(G)$ is cyclic, then from Lemma 2.2, 2.9 and 2.10, $c(G) = p^3$, or p^4. Now, suppose $Z(G) \cong C_p \times C_p$. Suppose $G = H \times K$, where H is a non-abelian subgroup and $K \cong C_p$. Then $|H| = p^4$, $H \in \Phi_3$, $Z(H) \cong C_p$ and $\exp(H) \le p^2$. Then from Lemma 2.2, 2.9 and 2.10, we get $c(H) = p^2$, or p^3. This implies that $c(G) = p^2+p$, or p^3+p. Now, suppose G is not a direct product of an abelian and a non-abelian subgroup. The relevant groups are $\phi_3(221)a$, $\phi_3(221)b_r$ ($r = 1$ or ν), $\phi_3(2111)d$, and $\phi_3(2111)e$.

$$\text{If } G = \phi_3(221)a, \text{ then } \operatorname{Core}_G\langle\alpha,\alpha_2\rangle \cap \operatorname{Core}_G\langle\alpha,\alpha_2\rangle = 1.$$
$$\text{If } G = \phi_3(221)b_r\ (r = 1,\nu), \text{ then } \operatorname{Core}_G\langle\alpha\rangle \cap \operatorname{Core}_G\langle\alpha,\alpha_2\rangle = 1.$$
$$\text{If } G = \phi_3(2111)d, \text{ then } \operatorname{Core}_G\langle\alpha^p,\alpha_1,\alpha_2\rangle \cap \operatorname{Core}_G\langle\alpha_1,\alpha_2,\alpha_3\rangle = 1.$$
$$\text{If } G = \phi_3(2111)e, \text{ then } \operatorname{Core}_G\langle\alpha,\alpha_2,\alpha_3\rangle \cap \operatorname{Core}_G\langle\alpha_1,\alpha_2\rangle = 1.$$

Thus, if $G \in \{\phi_3(221)a, \phi_3(221)b_r\ (r = 1,\nu), \phi_3(2111)d, \phi_3(2111)e\}$, then $\mu(G) \le p^3+p^2$. From Lemma 2.9, $c(G) \le p^3+p^2$. Since $\exp(G) = p^2$ (see [10]), from Lemma 2.1 and 2.2, $c(G) = 2p^2$, or p^3+p^2.
Case (iii): Let $G \in \Phi_i$ where $i \in \{4, 6\}$ Then $Z(G) \cong C_p \times C_p$, $\operatorname{cd}(G) = \{1, p\}$ and $\exp(G) \le p^2$. Further, G is not a direct product of an abelian and a non-abelian subgroup. By Lemma 2.2, 2.9 and 2.10, $c(G) = 2p^2$, p^3+p^2, or $2p^3$.
Case (iv): For $G \in \Phi_i$ where $i \in \{5, 7, 8, 10\}$, we have $Z(G) \cong C_p$. If $G \in \Phi_5$, then $\operatorname{cd}(G) = \{1, p^2\} = \{1, |G/Z(G)|^{1/2}\}$. From [16, Corollary 4], $c(G) = |G/Z(G)|^{1/2}c(Z(G)) = p^3$. If $G \in \Phi_i$ where $i \in \{7, 8, 10\}$, then $\operatorname{cd}(G) = \{1, p, p^2\}$. Let χ be a faithful irreducible character of G. From [15, Lemma 21], $\chi(1) = p^2 = |G/Z(G)|^{1/2}$. Since $Z(G) \cong C_p$, from [1, Theorem 4.6], $c(G) = \chi(1)|Z(G)| = p^3$.
Case (v): Let $G \in \Phi_9$. Then $Z(G) \cong C_p$, $\operatorname{cd}(G) = \{1, p\}$ and G is a maximal class p-group. By [17, Proposition 29], $c(G) = p^2$, or p^3. □

Tables 1 and 2 list $\mu(G)$ for some groups G of order p^5. In Table 3, we list $c(G)$ for the remaining groups. The column labelled "Group G" identifies a group G of order p^5. Suppose

$$X_G = \{\eta_{iG/G'}\}_{i=1}^{s} \cup \{\psi_j\}_{j=1}^{t} \text{ where } s+t = d(Z(G)), \text{ and } \{\eta_{iG/G'}\}_{i=1}^{s} \subset \operatorname{lin}(G/G'),$$

and $\psi_j = (\lambda_{R_j})\uparrow_{R_j}^{G}$ where $R_j \le G$ and $\lambda_{R_j} \in \operatorname{lin}(R_j)$ for $1 \le j \le t$, determines $c(G)$. The column labelled "X_G" lists the chosen $R_j \le G$, $\lambda_{R_j} \in \operatorname{lin}(R_j)$, and $\eta_{iG/G'} \in \operatorname{lin}(G)$. If $\langle a\rangle$ is a subgroup of $R_j \le G$, then $\lambda_{\langle a\rangle}$ and $1_{\langle a\rangle}$ denote a faithful and trivial character of $\langle a\rangle$ respectively. (For clarity in the tables, we use R and S to denote subgroups of G rather than R_1 and R_2.) We use $\eta_{G/G'}$ when η is a linear character of G; otherwise, we use λ_R for some $R \le G$ where $\lambda_R \in \operatorname{lin}(R)$ and $(\lambda_R)\uparrow_R^G \in \operatorname{nl}(G)$. Finally, the values of $c(G)$ are listed. For all groups G in Table 3, $\operatorname{cd}(G) = \{1, p\}$.

EXAMPLE 2.12. Consider the group of Type 2

$$G := \phi_2(221)c = \langle \alpha, \alpha_1, \alpha_2, \gamma \mid [\alpha_1, \alpha] = \gamma^p = \alpha_2, \alpha^{p^2} = \alpha_1^p = \alpha_2^p = 1 \rangle.$$

The column labelled "X_G" lists $R = \langle \alpha^p, \alpha_1, \gamma \rangle$; and $\eta_{G/G'} = \eta_{\langle \alpha G' \rangle} \cdot 1_{\langle \alpha_1 G' \rangle} \cdot 1_{\langle \gamma G' \rangle} \in \mathrm{lin}(G)$ where $\eta_{\langle \alpha G' \rangle}$ is a faithful linear character of $\langle \alpha G' \rangle$; and $\lambda_R = 1_{\langle \alpha^p \rangle} \cdot 1_{\langle \alpha_1 \rangle} \cdot \lambda_{\langle \gamma \rangle} \in \mathrm{lin}(R)$ where $\lambda_{\langle \gamma \rangle}$ is a faithful linear character of $\langle \gamma \rangle$. Hence $\{\eta_{G/G'}, \lambda_R\uparrow_R^G\}$ is a minimal degree faithful quasi-permutation representation of G, and so $c(G) = p^3 + p^2$.

By [13, Remark 4.34], for each group G in Table 3, by identifying kernels of appropriate characters, we also obtain a minimal degree faithful permutation representation of G.

Note: The groups G_8, G_9 and G_{46} listed in [8, Section 6.5] are isomorphic to $\Phi_6(221)c_r$ for $r = 1, \nu$ and $\Phi_4(221)_e$ respectively. Since both determining and reporting the required data for the former is easier, in Table 3 we record X_G using Girnat's presentations for these three groups, not those of James. For completeness, we list these groups.

$$G_8 = \langle g_1, g_2, g_3, g_4, g_5 \mid [g_2, g_1] = g_3, [g_3, g_1] = g_4 g_5, [g_3, g_2] = g_5, g_1^p = g_4, g_2^p = g_5,$$
$$\text{all other powers and commutators are trivial}\rangle.$$
$$G_9 = \langle g_1, g_2, g_3, g_4, g_5 \mid [g_2, g_1] = g_3, [g_3, g_1] = g_4 g_5^{\nu}, [g_3, g_2] = g_5, g_1^p = g_4, g_2^p = g_5,$$
$$\text{all other powers and commutators are trivial}\rangle.$$
$$G_{46} = \langle g_1, g_2, g_3, g_4, g_5 \mid [g_2, g_1] = g_4 g_5, [g_3, g_1] = g_5, g_2^p = g_4, g_3^p = g_5,$$
$$\text{all other powers and commutators are trivial}\rangle.$$

Table 3: $c(G)$ for groups of Type 2

Character degree set $= \{1, p\}$		
Group G	X_G	$c(G)$
$\phi_2(221)c$	$R = \langle \alpha^p, \alpha_1, \gamma \rangle$ $\eta_{G/G'} = \eta_{\langle \alpha G' \rangle} \cdot 1_{\langle \alpha_1 G' \rangle} \cdot 1_{\langle \gamma G' \rangle}$ $\lambda_R = 1_{\langle \alpha^p \rangle} \cdot 1_{\langle \alpha_1 \rangle} \cdot \lambda_{\langle \gamma \rangle}$	$p^3 + p^2$
$\phi_3(221)b_r$, for $r = 1$ or ν	$R = \langle \alpha^p, \alpha_1, \alpha_2 \rangle$ $\eta_{G/G'} = \eta_{\langle \alpha G' \rangle} \cdot 1_{\langle \alpha_1 G' \rangle}$ $\lambda_R = 1_{\langle \alpha^p \rangle} \cdot \lambda_{\langle \alpha_1 \rangle} \cdot 1_{\langle \alpha_2 \rangle}$	$p^3 + p^2$
$\phi_4(221)b$	$R = \langle \alpha^p, \alpha_1, \alpha_2 \rangle$ $\lambda_R = \lambda_{\langle \alpha^p \rangle} \cdot 1_{\langle \alpha_1 \rangle} \cdot 1_{\langle \alpha_2 \rangle}$ $\chi_R = 1_{\langle \alpha^p \rangle} \cdot 1_{\langle \alpha_1 \rangle} \cdot \chi_{\langle \alpha_2 \rangle}$	$p^3 + p^2$
G_{46}	$R = \langle g_1, g_2, g_5 \rangle$ $S = \langle g_1, g_3, g_4 \rangle$ $\lambda_R = 1_{\langle g_1 R' \rangle} \cdot \lambda_{\langle g_2 R' \rangle}$ $\chi_S = 1_{\langle g_1 S' \rangle} \cdot 1_{\langle g_3 S' \rangle} \cdot \chi_{\langle g_4 S' \rangle}$	$p^3 + p^2$
$\phi_4(221)f_0$, $\phi_4(221)f_r$ where $4k = \omega^{2r+1} - 1$ for $r = 1, 2, \ldots, (p-1)/2$	$R = \langle \alpha_1, \alpha_2 \rangle$ $\lambda_R = \lambda_{\langle \alpha_1 \rangle} \cdot 1_{\langle \alpha_2 \rangle}$ $\chi_R = 1_{\langle \alpha_1 \rangle} \cdot \chi_{\langle \alpha_2 \rangle}$	$2p^3$
$\phi_4(2111)c$	$R = \langle \alpha_1, \alpha_2, \beta_2 \rangle$ $\lambda_R = 1_{\langle \alpha_1 \rangle} \cdot \lambda_{\langle \alpha_2 \rangle} \cdot 1_{\langle \beta_2 \rangle}$ $\chi_R = 1_{\langle \alpha_1 \rangle} \cdot 1_{\langle \alpha_2 \rangle} \cdot \chi_{\langle \beta_2 \rangle}$	$p^3 + p^2$
G_8, G_9	$R = \langle g_1, g_3, g_4, g_5 \rangle$ $S = \langle g_2, g_3, g_4 \rangle$ $\lambda_R = \lambda_{\langle g_1 R' \rangle} \cdot 1_{\langle g_3 R' \rangle}$ $\chi_S = 1_{\langle g_2 S' \rangle} \cdot 1_{\langle g_3 S' \rangle} \cdot \chi_{\langle g_4 S' \rangle}$	$p^3 + p^2$
$\phi_6(221)d_0$, $\phi_6(221)d_r$ where $4k = \omega^{2r+1} - 1$ for $r = 1, 2, \ldots, (p-1)/2$	$R = \langle \alpha_1, \beta, \beta_1 \rangle$ $S = \langle \alpha_2, \beta, \beta_2 \rangle$ $\lambda_R = \lambda_{\langle \alpha_1 R' \rangle} \cdot 1_{\langle \beta R' \rangle}$ $\chi_S = \chi_{\langle \alpha_2 S' \rangle} \cdot 1_{\langle \beta S' \rangle}$	$2p^3$

Continuation of Table 3		
Group G	X_G	$c(G)$
$\phi_6(2111)b_r$, for $r = 1$ or ν	$R = \langle \alpha_1, \beta, \beta_1, \beta_2 \rangle$ $S = \langle \alpha_2, \beta, \beta_2 \rangle$ $\lambda_R = 1_{\langle \alpha_1 R' \rangle} \cdot 1_{\langle \beta R' \rangle} \cdot \lambda_{\langle \beta_2 R' \rangle}$ $\chi_S = \chi_{\langle \alpha_2 S' \rangle} \cdot 1_{\langle \beta S' \rangle}$	$p^3 + p^2$

3. $\mu(G)$ for groups of order 3^5

We used the standard function `MinimalDegreePermutationRepresentation` in GAP [6] to compute $\mu(G)$ for each of the 67 groups G of order 3^5. We present the results in Table 4.

Table 4: Value of $\mu(G)$ for the groups of order 3^5

$\mu(G)$	Group identifier
12	51
15	62, 63, 67
18	3, 4, 7, 13, 17, 18, 35, 36, 37, 38, 39, 41, 43, 47, 61
21	31, 32, 33
27	2, 16, 22, 25, 26, 27, 28, 29, 30, 55, 56, 57, 58, 59, 60, 65, 66
30	49, 52, 53, 54, 64
33	48
36	5, 6, 10, 11, 12, 14, 15, 21, 34, 40, 42, 46
54	8, 9, 44, 45
81	19, 20, 24, 50
84	23
243	1

4. Access to results

The data recorded in Tables 1, 2 and 3 is publicly available in Magma via a GitHub repository [14]; we also include code which identifies errors of [2]. The recording of data is similar to that of [13]; we summarize it here for completeness.

For each group G of order p^5 we recorded its presentation from [10]. For those G listed in Tables 1 and 2, we recorded either $\mathcal{H}_H$ or $\mathcal{H}_G$; so we can readily construct an explicit faithful permutation representation for G of degree $\mu(G)$.

For those G listed in Table 3 we recorded details which allow us to reconstruct the required characters. We illustrate this by reference to Example 2.12. Recall $\lambda_R{\uparrow}_R^G \in X_G$ for $G = \phi_2(221)c$. We did not record the chosen character λ_R, but instead listed generating sets for both R and $\ker(\lambda_R)$. Using these, we can first compute a faithful linear character of $R/\ker(\lambda_R)$, and then lift this to R to recover λ_R; finally we compute $\lambda_R{\uparrow}_R^G \in X_G$.

Using this data and related code, we verified readily that the description listed for one of $\mathcal{H}_H$, $\mathcal{H}_G$, X_H or X_G in Tables 1, 2 and 3 determines a faithful permutation representation for each group G of order p^5 for $5 \leq p \leq 97$. We checked that the elements of X_H or X_G are irreducible for $p \leq 13$. For this range we also verified the claimed value of $\mu(G)$ using the following:

- An improved version of the GAP standard function `MinimalFaithfulPermutationDegree` developed by Alexander Hulpke.
- An implementation in Magma of the algorithm of Elias *et al.* [5] developed by Neil Saunders.

The latter is included in the repository [14].

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