Unit III
Integral Calculus of functions of one variable

Overview:
This unit aims at providing adequate exposure to the theory and applications of Integral Calculus. It also aims to gradually develop in students an ability to apply these theoretical constructs to solve problems within Engineering domain. This unit covers Integration of single variable functions.

Outcome:
After completion of this unit, students would be able to:
employ the tool of beta, Gamma functions and evaluate integrals; apply the knowledge of integrals to find surface areas and volumes of revolutions.

Detailed Syllabus:

1.1 Evaluation of definite and improper integrals

1.2 Evaluation of definite and improper integrals

1.3 Beta, Gamma functions and their Properties

1.4 Applications of definite integrals to evaluate surface areas and volumes of revolutions.

1. *Prerequisite to the topic/unit* : Definite and Indefinite Integration
2. *Definitions*

IMPROPER INTEGRALS

An integral $\int_a^b f(x)dx$ is improper if one or both of the following conditions are satisfied:

1. *f(x)* becomes infinite at one or more points in the interval of integration $[a,b]$,
2. One or/& both of the limits of integration is infinite.

I. IMPROPER INTEGRALS OF FIRST KIND

Range of integration is infinite.

i.e. $\int_a^\infty f(x)dx$ or $\int_\infty^b f(x)dx$ or $\int_\infty^\infty f(x)dx$

Methods to evaluate improper integrals of first kind:

1. $\int_a^\infty f(x)dx = \lim_{p\to\infty}\int_a^p f(x)dx$. If the limit exists and is finite (say l_1) then the improper integral converges and has value l_1. Otherwise the integral diverges.
2. $\int_{-\infty}^b f(x)dx = \lim_{p\to-\infty}\int_p^b f(x)dx$. If the limit exists and is finite (say l_2) then the improper integral converges and has value l_2. Otherwise the integral diverges.
3. $\int_{-\infty}^\infty f(x)dx = \lim_{a\to-\infty}\int_a^c f(x)dx + \lim_{b\to\infty}\int_c^b f(x)dx$, where c is any finite constant including zero. If both the limits on the right hand side exists separately and are finite, say equal to l_3 and l_4 respectively, then the improper integral converges and has value l_3+l_4. If one or both of the limits do not exist or is infinite, then the improper integral diverges.

II. IMPROPER INTEGRALS OF SECOND KIND

Suppose $f(x)\to\infty \ as \ x\to a$, then the integral has a singularity at the lower point a. Then this singularity is cut-off by considering

$$\int_{a+\varepsilon}^b f(x)dx$$

where ε is a small positive number. Thus for a convergent improper integral of the second kind

$$\int_a^b f(x)dx = \lim_{\varepsilon\to 0}\int_{a+\varepsilon}^b f(x)dx \qquad (1)$$

which ignores the contribution of the singularity.

Similarly when $f(x)$ is discontinuous at the upper limit b then

$$\int_a^b f(x)dx = \lim_{\varepsilon\to 0}\int_a^{b-\varepsilon} f(x)dx \qquad (2)$$

Finally when $f(x)$ has a singularity at an intermediate point $c\,; a<c<b$ then

$$\int_a^b f(x)dx = \lim_{\varepsilon\to 0}\int_a^{c-\varepsilon} f(x)dx + \lim_{\varepsilon\to 0}\int_{c+\varepsilon}^b f(x)dx \qquad (3)$$

The RHS limit above is known as the Cauchy's principal value of the integral.

When the limit in the RHS of (1), (2), (3) fails to exist (or infinite) then the improper integral is said to diverge.

III. GAMMA FUNCTION

Definition: Gamma Function

The function of n $(n>0)$ defined by the integral $\int_0^\infty e^{-x}x^{n-1}dx$ is called Gamma function and is denoted by $\Gamma(n)$. Thus, $\Gamma(n)=\int_0^\infty e^{-x}x^{n-1}dx$

Properties of Gamma function:

1. $\Gamma(n+1)=n\,\Gamma(n)$

Alternatively the result means $\Gamma(n)=(n-1)\Gamma(n-1)$

2. If n is positive integer then $\Gamma(n+1)=n!$
3. $\Gamma(1)=1$
4. $\Gamma(0)=\infty$
5. $\Gamma\left(\frac{1}{2}\right)=\sqrt{\pi}$
6. $\Gamma(p)\,\Gamma(1-p)=\dfrac{\pi}{\sin p\pi},\ 0<p<1$

Types of integrals evaluated by Gamma function:

No.	Types of integrals evaluated by Gamma function	Substitution
1	$\int_0^\infty e^{-ax^n}dx$	$ax^n=t$
2	$\int_0^\infty x^m e^{-ax^n}dx$	$ax^n=t$
3	$\int_0^1 x^m(\log x)^n dx$	$\log x=-t$

VI. BETA FUNCTION

Definition: Beta Function

The function of m and n $(m,n>0)$ defined by the integral $\int_0^1 x^{m-1}(1-x)^{n-1}dx$ is called the Beta Function and is denoted by $\beta(m,n)$.

Thus, $\beta(m,n)=\int_0^1 x^{m-1}(1-x)^{n-1}dx$

The above definition of Beta function is equivalent to

1. $\beta(m,n)=2\int_0^{\pi/2}\sin^{2m-1}\theta\cos^{2n-1}\theta\,d\theta$
2. $\int_0^{\pi/2}\sin^p\theta\cos^q\theta\,d\theta=\frac{1}{2}\beta\left(\frac{p+1}{2},\frac{q+1}{2}\right)$

Properties of Beta function:

1. $\beta(m,n)=\beta(n,m)$
2. The relation between Beta and Gamma functions is $\beta(m,n)=\frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$

Types of integrals evaluated by Beta function:

1. $\int_0^a x^m(a-x)^n\,dx$ **put x = a t**

2. $\int_0^a x^m(a^n-x^n)^p\,dx$ **put** $x^n=a^nt$

3. $\int_0^1\left(1-\sqrt[n]{x}\right)^m dx$ **put** $x^{\frac{1}{n}}=t$

VII. AREA OF THE SURFACE OF A SOLID OF REVOLUTION

a) Area of the surface of a solid of revolution generated by revolving the arc AB of the curve $y = f(x)$ about the $x-axis$ is given by

$$S = \int_{x=a}^{x=b} 2\pi y \sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx$$

b) Area of the surface of a solid of revolution generated by revolving the arc AB of the curve $x = g(y)$ about the $y-axis$ is given by

$$S = \int_{y=c}^{y=d} 2\pi x \sqrt{1+\left(\frac{dx}{dy}\right)^2}\, dy$$

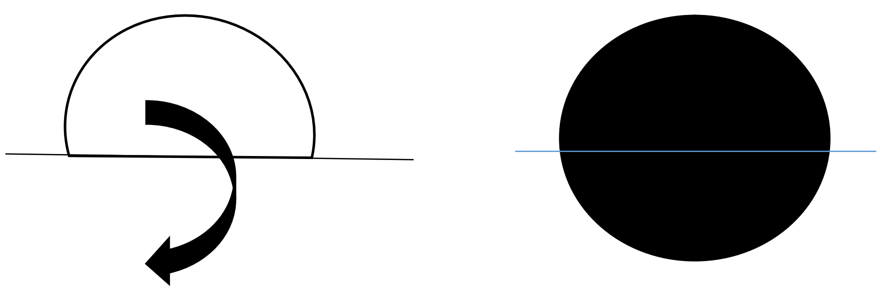
VIII. VOLUME OF A SOLID OF REVOLUTION

A solid of revolution is generated by revolving a plane area R about a line L in the plane.

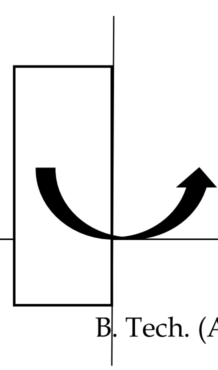
Line L is known as the axis of revolution. Line L does not intersect the plane area R but may touch the boundary of R.

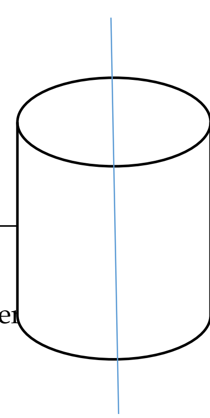
Examples:

1. Sphere is a solid of revolution generated by revolving the semicircle region R about its diameter L.



2. Right circular cylinder is a solid of revolution obtained by revolving a rectangle R about its edge L.





1. If the area bounded by the curve $y = f(x)$, the line $y = p$ and the lines $x = a, x = b$ is revolved about the line $y = p$ (a line parallel to the X-axis), then the volume of the solid of revolution is given by

$$V = \pi \int_a^b (y-p)^2 \, dx$$

2. If the area bounded by the curve $x = g(y)$, the line $x = q$ and the lines $y = c, y = d$ is revolved about the line $x = q$ (a line parallel to the Y-axis), then the volume of the solid of revolution is given by

$$V = \pi \int_c^d (x-q)^2 \, dy$$

<u>Classwork problems</u>

Session 1

a. **Problems on Evaluation of improper integrals(first kind)**

Evaluate the following integrals:

1) $\int_{-\infty}^{1} e^x dx$ Ans: e

2) $\int_{1}^{\infty} \frac{1}{x^4} dx$ Ans: $\frac{1}{3}$

3) $\int_{-\infty}^{-1} \frac{dx}{x^4}$ Ans. $\frac{1}{3}$

4) $\int_{-\infty}^{\infty} \frac{dx}{1+x^2}$ Ans. π

5) $\int_{1}^{\infty} \frac{1}{x^2} dx$ Ans: 1

6) $\int_{0}^{\infty} \frac{1}{(1+x)\sqrt{x}} dx$ Ans: π

Session 2

1.2 Problems on Evaluation of improper integrals (second kind)

Evaluate the following integrals:

1) $\int_{-a}^{a} \frac{dx}{\sqrt{a^2 - x^2}}$ Ans: π

2) $\int_{-1}^{1} \frac{dx}{x^{2/3}}$ Ans: 6

3) $\int_{0}^{1} \frac{x}{\sqrt{1-x^2}}\, dx$ Ans: 1

Session 3

1.3 Problems on Beta, Gamma functions and their Properties

Class Room Problems

1. Evaluate $\int_0^\infty e^{-h^2x^2}\, dx$ Ans. $\frac{\sqrt{\pi}}{2h}$.
2. Evaluate $\int_0^\infty e^{-x^4}\, dx$ Ans. $\frac{1}{4}\Gamma\left(\frac{1}{4}\right)$.
3. Evaluate $\int_0^\infty e^{-x^3}\, dx$ Ans. $\frac{1}{3}\Gamma\left(\frac{1}{3}\right)$.
4. Evaluate $\int_0^\infty \sqrt{x}\, e^{-x^3}\, dx$ Ans. $\frac{\sqrt{\pi}}{3}$.
5. Evaluate $\int_0^\infty x^2 e^{-h^2x^2}\, dx$ Ans. $\frac{\sqrt{\pi}}{4h^3}$.
6. Evaluate $\int_0^\infty x^2 e^{-x^4}\, dx$ Ans. $\frac{1}{4}\Gamma\left(\frac{3}{4}\right)$.
7. Show that $\int_0^\infty x e^{-x^8}\, dx \cdot \int_0^\infty x^2 e^{-x^4}\, dx = \frac{\pi}{16\sqrt{2}}$.
8. Evaluate $\int_0^\infty x^{1/4} e^{-\sqrt{x}}\, dx.$ Ans. $\frac{3}{2}\sqrt{\pi}$

Session 4

9. Evaluate $\int_0^1 \left(x\log x\right)^3 dx$ Ans. $\frac{-3}{128}$.

10. Evaluate $\int_0^1 x^m \left(\log\frac{1}{x}\right)^n dx$ Ans. $\frac{\Gamma(n+1)}{\left(m+1\right)^{n+1}}$.

11. Evaluate $\int_0^1 \frac{dx}{\sqrt{x\cdot\log(1/x)}}$ Ans. $\sqrt{2\pi}$

12. Evaluate $\int_0^1 \left(\log\frac{1}{x}\right)^{p-1} dx$ Ans. Γp

13. Evaluate $\int_0^1 \frac{dx}{\sqrt{-\log x}}$ Ans. $\sqrt{\pi}$

Session 5

Beta Function:

1. Prove that $\int_0^4 \sqrt{x}\left(4-x\right)^{3/2} dx = 4\pi$

2. Evaluate $\int_0^9 x^{3/2}\left(9-x\right)^{1/2} dx$ Ans. $\frac{729\pi}{16}$

3. Evaluate $\int_0^a \frac{x^m}{\sqrt{a-x}} dx$ Ans. $a^{m+(1/2)}\beta\left(m+1,\frac{1}{2}\right)$

4. Evaluate $\int_0^2 x^3\sqrt{2-x}\, dx$ Ans. $\frac{512}{315}\sqrt{2}$

5. Evaluate $\int_0^1 x^6\left(1-x^2\right)^{1/2} dx$ Ans. $\frac{5\pi}{256}$

6. Evaluate $\int_0^2 y^4(8-y^3)^{-1/3} dy$ Ans. $\frac{16}{3}\beta\left(\frac{5}{3},\frac{2}{3}\right)$

7. Evaluate $\int_0^{2a} x^2\sqrt{2ax-x^2}\, dx$ Ans. $\frac{5}{8}a^4\pi$

Session 6

8. Evaluate $\int_0^1 \left(1-\sqrt[3]{x}\right)^{11} dx$ Ans. $\frac{1}{26}$
9. Evaluate $\int_0^1 \frac{dx}{\sqrt{1-x^8}}$ Ans. $\frac{1}{8}\beta\left(\frac{1}{8},\frac{1}{2}\right)$
10. Evaluate $\int_0^1 \sqrt{1-x^6}\,dx$ Ans- $\frac{1}{6}\beta\left(\frac{1}{6},\frac{3}{2}\right)$
11. Evaluate $\int_0^{\pi/4} \sin^7 2\theta\, d\theta$ Ans. $\frac{8}{35}$
12. Evaluate $\int_0^1 \frac{x^7}{\sqrt{1-x^4}}\, dx$ Ans. $\frac{1}{3}$
13. Evaluate $\int_0^a (a^2-x^2)^{5/2} dx$ Ans. $\frac{5a^6\pi}{32}$
14. Evaluate $\int_0^{\pi} \sin^2\theta\left(1+\cos\theta\right)^4 d\theta$ Ans. $\frac{21\pi}{16}$
15. Evaluate $\int_0^{\pi/6} \cos^6 3\theta\ \sin^2 6\theta\ d\theta$ Ans. $\frac{7\pi}{384}$

Session 7

1.4 Problems on Applications of definite integrals to evaluate surface areas and volumes of revolutions.

Area of surface of solid of revolution

1) Find the area of the surface of the solid of revolution generated by revolving the parabola $y^2 = 4ax,\ 0 \le x \le 3a$ about the $x-axis$.

   Ans: $\frac{56\pi a^2}{3}$

2) Determine the surface area of the paraboloid generated by revolving the curve $y = x^2$ included between $x = 0\ and\ x = 6/5$ about $y-axis$.

   Ans: $\frac{1036\pi}{375}$

3) Determine the surface area of the solid obtained by rotating $y=\sqrt{9-x^2}$, $-2 \le x \le 2$ about the $x-axis$.

   Ans: 24π

4) Determine the surface area of sphere of radius a Ans: $4\pi a^2$

Session 8

Volume of surface of solid of revolution

1) Find the volume of solid of revolution generated by revolving the area bounded by parabola $x^2 = y,\ x = 3$ and *X-axis* about the *X-axis* .

   Answer: $\frac{243\pi}{5}$

2) Determine the volume of solid generated by revolving the plane area bounded by $y^2 = 4x$ and $x = 4$ about the line $x = 4$.

   Ans: $\frac{1024\pi}{15}$.

3) The area between the curve $y = 1/x$, the y-axis and the lines $y = 1$ and $y = 2$ is rotated about the y-axis. Find the volume of the solid of revolution formed.

   Ans: $\frac{\pi}{2}$.

4) Find the volume of solid of revolution generated by revolving the area bounded by parabola $x^2 = y,\ x = 3$ and X-axis about the $x = 3$.

   Ans: $\frac{27\pi}{2}$