Unit IV **Multiple Integrals**
Overview: In this unit, we will learn to effectively evaluate double integrals and triple integrals. We will learn to apply double integrals to find the area. Further, we will use triple integrals to find volume.
Outcome: After completion of this unit, students would be able to: 1. employ appropriate mathematical techniques in evaluating Multiple Integrals. 2. apply various techniques of Multiple Integration in solving engineering problems.

Detailed Syllabus:
1.1 Double integrals (Cartesian)
1.2 Change of order of integration in double integrals
1.3 Change of variables (Cartesian to polar), Jacobian
1.4 Application of Double Integral to find area.
1.5 Triple Integral, Change of variable to spherical and cylindrical co-ordinates
1.6 Application of Triple Integral to find volume

Evaluation of Double Integration

Double integral over a region R may be evaluated by two successive integration. Double integral depend upon the nature of the curves bounding the region R. Let, R is bounded by the curves $x = x_1$, $x = x_2$ and $y = y_1, y = y_2$

i.e. $\boldsymbol{\int\int f(x,y)dx.dy = \int_{x_1}^{x_2}\int_{y_1}^{y_2} f(x,y)dx.dy}$

The order of integration is depend on the nature of limit in the region R.

Case-I

If R is the region bounded by the curves $x_1 = f_1(y), x_2 = f_2(y)$ and two straight lines $y_1 = b_1, y_2 = b_2$. Then,

$$\int\int_R \boldsymbol{f(x,y)\,dx.dy} = \int_{b_1}^{b_2}\left[\int_{f_1(y)}^{f_2(y)} \boldsymbol{f(x,y)dx}\right] \boldsymbol{dy}$$

In this integral we first integrate the function $f(x,y)$w.r.t. x keeping y constant from $f_1(y)$ to $f_2(y)$ and the resulting function of y from $b_1 to$ b_2.

Case-II

If R is the region bounded by the curves $y_1 = f_1(x), y_2 = f_2(x)$ and two straight lines $x_1 = b_1, x_2 = b_2$. Then,

$$\int \int_R \boldsymbol{f(x,y)\,dx.dy} = \int_{b_1}^{b_2} \left[\int_{f_1(x)}^{f_2(x)} \boldsymbol{f(x,y)dy} \right] \boldsymbol{dx}$$

In this integral we first integrate the function $f(x,y)$w.r.t. y keeping x constant from $f_1(x)\ to\ f_2(x)$ and the resulting function of x from $b_1 to\ b_2$.

Case-III

If R is the region bounded by the straight lines $x_1 = a_1, x_2 = a_2$ and $y_1 = b_1, y_2 = b_2$. Then,

$$\int \int_R \boldsymbol{f(x,y)\,dx.dy} = \int_{a_1}^{a_2} \left[\int_{b_1}^{b_2} \boldsymbol{f(x,y)dy} \right] \boldsymbol{dx} = \int_{b_1}^{b_2} \left[\int_{a_1}^{a_2} \boldsymbol{f(x,y)dx} \right] \boldsymbol{dy}$$

In this integration the order of integration is immaterial, provided the limits of the integration are changed accordingly.

Change of order of integration

As discussed in the case-I and case-II of evaluation of double integration there is a specified order to integrate double integral. Some time it is difficult to integrate them directly. In such case it may be convenient to integrate by changing the order of integration. The process of changing the double integral $\int \int_R V dxdy$ in to $\int \int_R V dydx$ or vice-versa is called the change of order of integration.

Working Rule:

When it is required to change the order of integration in an integral for which limit are given, we first find out the region R of integration from the limits. When region R of integration is known, we next assign limits of integration in the reverse order from geometric considerations. Now,

Suppose we want to change $\int_{x=a}^{b} \int_{y=f_{1(x)}}^{f_{2(x)}} \boldsymbol{f(x,y)dydx}$ to$\int_{y=a_1}^{b_1} \int_{x=\emptyset_{1(y)}}^{\emptyset_{2(y)}} \boldsymbol{f(x,y)dxdy}$.

First we see that the region bounded by the curves $x = a, x = b, y = f_1(x)$ and $f_2(x)$ we draw the rough sketch and find the region R which is called the region of integration and bounded by the given limits of integration.

Next we find the limit of x in term of y and limit of y in term of constant. For this, we select the region such as all given curve bounded by strip.

1. For the limit of x in term of y, we draw the strip parallel to X axis. Such as the strip cover the minimum and maximum limit of y.
2. Next we find the limits of y in term of constant.

Similarly,

We change order of integration$\int_{y=a}^{b} \int_{x=f_{1(y)}}^{f_{2(y)}} \boldsymbol{f(x,y)dxdy}$ to$\int_{x=a_1}^{b_1} \int_{y=\emptyset_{1(x)}}^{\emptyset_{2(x)}} \boldsymbol{f(x,y)dydx}$

Change of Co-ordinate system:

Double integration in polar co-ordinates

Let D be the domain in xy plane and let x, y be the rectangular Cartesian coordinates of any point P in D. let u, v be new variables in the domain D^* such that x, y and u, v are connected through the continuous functions

$$\boldsymbol{x = g(u,v), y = h(u,v)} \qquad \textbf{---(1)}$$

Then u, v are said to be curvilinear coordinates of point P^* in D^* which uniquely corresponds to P in D. Solving (1) for $u \; and \; v$, we get

$$\boldsymbol{u = g^*(x,y),\ v = h^*(x,y)} \qquad \textbf{---(2)}$$

Then a given double integral in the given (old) variables x,y can be transformed to a double integral in the new variables u, v as follows:

$$\int \int_D \boldsymbol{f(x,y)\, dx.dy} = \int \int_{D^*} \boldsymbol{F(u,v)\, |J| du.dv} \qquad \text{---(3)}$$

Here, $f(x,y) = f\{x_{(u,v)}, y_{(u,v)} = F(u,v)$ and J is the **Jacobian** (functional determinant) defined as

$$\boldsymbol{J = J\left(\frac{x,y}{u,v}\right) = \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}}$$

(3) is known as formula for transformation of coordinates in double integral.

For a double integral in Cartesian coordinates $\boldsymbol{x, y}$ the change of variables to polar coordinates $\boldsymbol{r}$, $\boldsymbol{\theta}$ can be done through the transformation $\boldsymbol{x = rcos\theta, y = rsin\theta}$.

Therefore,

$$\int \int_D \boldsymbol{f(x,y)\, dx.dy} = \int \int_{D^*} \boldsymbol{f(rcos\theta, rsin\theta)\, dx.dy} = \int_\theta \int_r \boldsymbol{F(r,\theta)\, r\, dr.d\theta}$$

where, $\boldsymbol{F(r,\theta)} = f\boldsymbol{(rcos\theta, rsin\theta) = f(x,y)}$and the Jacobian in thiscase is r.

<u>Applications of double integrals:</u>

1. If $f(x,y) = 1$, then the area A of the region R is given by $\iint_R dxdy$.

a) Cartesian coordinates : $A = \int_{x=a}^{x=b} \int_{y=f_2(x)}^{y=f_2(x)} dxdy$ or $A = \int_{y=c}^{y=d} \int_{x=f_2(y)}^{x=f_2(y)} dxdy$

b) Polar coordinates: $A = \int\limits_{\theta=\alpha_1}^{\theta=\alpha_2} \int\limits_{r=f_1(\theta)}^{r=f_2(\theta)} r dr d\theta$

Triple Integral:

The concept of double integral of a function $f(x,y)$ over a given region in xy-plane can be extended a step further to define triple integral.

Consider a function $f(x,y,z)$ defined over a finite region V of three dimensional space. Let the region be sub divided into n sub intervals $\delta V_1, \delta V_2, \ldots\ldots, \delta V_n$. Let $P(x_r, y_{r,} z_r)$ be a point in the rth sub interval. We now form the sum $\sum\limits_{r=1}^{n} f(x_r, \mathrm{y}_r, \mathrm{z}_r)$.

The limit of the above when it exists, as n tends to infinity and the volume of each sub region tends to zero is called triple integral of $f(x,y,z)$ over the region V and is denoted by $\iiint\limits_V f(x,y,z) dV$.

Thus $\iiint\limits_V f(x,y,z) dV = \lim\limits_{\substack{n\to\infty \\ \delta V \to 0}} \sum\limits_{r=1}^{n} f(x_r, \mathrm{y}_r, \mathrm{z}_r) \delta V_r$

Evaluation of Triple Integral:

The triple integral can be evaluated by successive single integrals as follows,

$\int\limits_{x=a}^{x=b} \int\limits_{y=\Phi_1(x)}^{y=\Phi_2(x)} \int\limits_{z=f_1(x,y)}^{z=f_2(x,y)} f(x,y,z) dxdydz$ Where the integration with respect to z is performed first by treating x *and* y constant. Then the integration with respect to y is performed treating x constant and finally the integration with respect to x is performed.

Spherical coordinates:

$x = r\sin\theta\cos\phi,\ y = r\sin\theta\sin\phi,\ z = r\cos\theta$

$dxdydz = r^2 \sin\theta dr d\theta d\phi$

Cylindrical coordinates:

$x = r\cos\theta,\ y = r\sin\theta,\ z = z$

$dxdydz = r dr d\theta dz$

Application of triple integrals:

To express the volume of a solid as a triple integral, we note that the volume of a n elementary solid is $dxdydz$, and so the volume of the solid is given by

volume = $\iiint dxdydz$

If $f(x,y,z)=1$, then the volume V of the region T is given by $\iiint_T dxdydz$

Session 1

1. Evaluate $\int_0^3 \int_1^2 xy(x+y)dxdy$.   Ans: 24

2. Evaluate $\int_0^2 \int_0^{x^2} e^{y/x} dydx$.   Ans:e^2-1

3. Evaluate$\int_0^1 \int_0^1 \frac{dxdy}{\sqrt{(1-x^2)(1-y^2)}}$.   Ans:$\frac{\pi^2}{4}$

4. Evaluate $\int_0^1 \int_{x^2}^{2-x} ydxdy$.   Ans: $\frac{16}{15}$

5. Evaluate $\int_0^1 \int_{y^2}^{y} (1+xy^2)dx.dy$ Ans. $\frac{41}{210}$

6. Evaluate $\int_0^{\pi} \int_{2\sin\theta}^{4\sin\theta} r^3 drd\theta$.   Ans: $\frac{45\,\pi}{2}$

7. Evaluate $\int_0^{\pi} \int_0^{\cos\theta}$ r sin θ $drd\theta$. Ans: $\frac{1}{3}$

Session 2

1. Evaluate $\int\int_R (x^2+y^2)dxdy$, R being the region bounded by $y=x^2, x=2$ and$y=1$.

   Ans. $\frac{1286}{105}$

2. Evaluate $\iint_R e^{ax+by} dxdy$ where R is the area of triangle bounded by $x=0, y=0$ & $ax+by=1$.

   Ans: $\frac{1}{ab}$

3. Evaluate $\iint_R x^2 dxdy$ where R is the region in the first quadrant bounded by

   $y=\frac{16}{x}, x=8, y=0$ & $y=x$.   Ans:448

Session 3

4. Evaluate $\iint (x^2 - y^2) x \, dxdy$ over the positive quadrant of the circle $x^2 + y^2 = a^2$.

   Ans: $\frac{2a^5}{5}$.

5. Evaluate $\iint (x^2 + y^2) dxdy$ over the area of the triangle whose vertices are $(0,1), (1,1), (1,2)$.

   Ans: 7/6

6. Evaluate: $\iint xy(x+y) \, dxdy$ over the region bounded by the curves $y = x^2$ and $y = x$.

   Ans: 37/420

Session 4

1. Change the order of integration $\int_1^2 \int_3^4 f(x,y) dydx$ Ans: $\int_3^4 \int_1^2 f(x,y) dxdy$.

2. Change the order of integration and solve $\int_0^5 \int_{2-x}^{2+x} dydx$.

   Ans: 2

3. Change the order of integration $\int_{-1}^1 \int_0^{\sqrt{1-x^2}} f(x,y) dydx$ Ans: $\int_0^1 \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} f(x,y) dxdy$.

4. Change the order of integration $\int_0^3 \int_{y^2/9}^{\sqrt{10-y^2}} f(x,y) \, dxdy$.

Session 5

1. Change to polar coordinates and evaluate $\int_0^4 \int_{\sqrt{4x-x^2}}^{\sqrt{16-x^2}} \frac{dydx}{\sqrt{16-x^2-y^2}}$. Ans : 4

2. Change to polar coordinates and evaluate $\int_0^a \int_y^a \frac{x^2}{\sqrt{(x^2+y^2)}} dxdy$. Ans: $\frac{a^3}{3} \log(1+\sqrt{2})$

3. Change to polar coordinates and evaluate $\int_0^a \int_0^{\sqrt{a^2-x^2}} y\sqrt{(x^2+y^2)} dydx$. Ans: $\frac{a^4}{4}$

Session 6

1. Find the area of the circle $x^2 + y^2 = a^2$.

2. Using double integration find the area between the curves $y^2 = 4x$ and $2x - 3y + 4 = 0$.

3. Find by double integration, the area bounded by the parabolas $y^2 = 4 - x$and $y^2 = 4 - 4x$.
Ans: 8 sq. units.

Session 7

1. Evaluate $\int_0^{\log 2}\int_0^{x}\int_0^{x+y} e^{x+y+z}\,dxdydz$. Ans: $\frac{5}{8}$

2. Evaluate $\int_0^{1}\int_0^{\sqrt{1-x^2}}\int_0^{\sqrt{1-x^2-y^2}} \frac{dxdydz}{\sqrt{1-x^2-y^2-z^2}}$. Ans: $\frac{\pi^2}{8}$

3. Evaluate $\int_0^{\log 2}\int_0^{x}\int_0^{x+\log y} e^{x+y+z}\,dxdydz$. Ans: $\frac{8}{3}\log 2 - \frac{19}{9}$

Session 8

1. Evaluate $\iiint \frac{dxdydz}{x^2+y^2+z^2}$ over the volume of the sphere $x^2 + y^2 + z^2 = a^2$. Ans: $4\pi a$

2. Using spherical coordinates, Evaluate $\iiint_V \frac{dxdydz}{\left(x^2+y^2+z^2\right)^{3/2}}$ where V is the volume bounded by spheres $x^2 + y^2 + z^2 = 16$ and $x^2 + y^2 + z^2 = 25$. Ans: $4\pi \log\left(\frac{5}{4}\right)$

3. Evaluate $\iiint_V \sqrt{x^2+y^2}\,dxdydz$ where V is the volume bounded by $x^2 + y^2 = z^2$, $z > 0$ and $z = 1$.

Ans: $\frac{\pi}{2}$

4. Evaluate $\iiint z^2\,dxdydz$ over the volume bounded by the cylinder $x^2 + y^2 = a^2$, paraboloid $x^2 + y^2 = z$ and the plane $z = 0$. Ans: $\frac{a^8\pi}{12}$

Session 9

1. Find the volume bounded by the coordinate planes and the plane $x+y+z=1$.

Ans : $\frac{1}{6}$

2. Find the volume of rectangular parallelepiped bounded by the set of inequalities $1 \le x \le 3, 3 \le y \le 6, 2 \le z \le 5$. Ans: 18

3. Find the volume of a sphere of radius a. Ans : $\frac{4}{3}\pi a^3$

4. Find the volume bounded by $y^2 = x$, $x^2 = y$ and the planes $z=0$ to $z=a$. Ans: $\frac{a}{3}$

Session 10

1. Find by triple integration the volume bounded by $y^2 = x,\ x^2 = y$ and the planes $z = 0\ and\ x + y + z = 2.$ Ans: $\frac{11}{30}$.
2. Find the volume bounded by the cylinder $x^2 + y^2 = 4$ and the planes $y + z = 3$ and $z = 0$
   Ans: 12π
3. Find the volume bounded by $x^2 + y^2 = az$ and the cylinder $x^2 + y^2 = a^2$.

   Ans : $\pi a^3/2$

References:

1. *Advanced Engineering Mathematics*, 10^{th} Edition, Erwin Kreyszig, Wiley India, 2017
2. *Advanced Engineering Mathematics*, 20^{th} Edition, H. K. Dass, S. Chand & Company Ltd, 2012