Unit II
Partial Differentiation

Overview:

This unit aims at providing exposure to the theory and applications of Differential calculus of multivariable functions; It also aims to gradually develop in students an ability to apply these theoretical constructs to solve problems within Engineering domain.

Outcome:

After completion of this unit, students would be able to:

operate and analyse functions of several variables and relate the results to real life problems.

Detailed Syllabus:

2.1 Limit and continuity
2.2 partial derivatives
2.3 Taylor's theorem of function of two variables
2.4 Maxima, minima and saddle points
2.5 Method of Lagrange multipliers

Unit Details:

2.1 Limits and Continuity of Functions of Two Variables

Limits :

Definition: We write $\lim_{(x,y)\to(a,b)} f(x,y) = L$ and we read the limit of $f(x,y)$ as (x,y) approaches (a,b) is L, if we can make $f(x,y)$ as close as we want to L, simply by taking (x,y) close enough to (a,b) but not equal to it.

Properties of Limits of Functions of Several Variables

We list these properties for functions of two variables. Similar properties hold for functions of more variables. Let us assume that L, M, and k are real numbers and that $\lim_{(x,y)\to(a,b)} f(x,y) = L$ and $\lim_{(x,y)\to(a,b)} g(x,y) = M$

then the following holds

1) $\lim_{(x,y)\to(a,b)} x = a$

$\lim_{(x,y)\to(a,b)} y = b$

$\lim\limits_{(x,y)\to(a,b)} c = c$, if c is constant.

2) Sum and difference rules

$\lim\limits_{(x,y)\to(a,b)} f(x,y) \pm g(x,y) = L \pm M$

3) Constant multiple rule

$\lim\limits_{(x,y)\to(a,b)} [k\, f(x,y)] = k\, L$

4) Product rule

$\lim\limits_{(x,y)\to(a,b)} f(x,y)\, g(x,y) = L\, M$

5) Quotient rule

$\lim\limits_{(x,y)\to(a,b)} \dfrac{f(x,y)}{g(x,y)} = \dfrac{L}{M}$ Provided $M \neq 0$

6) Power rule

If r and s are integers with no common factors, and $s \neq 0$

Then $\lim\limits_{(x,y)\to(a,b)} [f(x,y)]^{\frac{r}{s}} = L^{\frac{r}{s}}$

Problems:

1. Evaluate the following

1) $\lim\limits_{(x,y)\to(1,3)} \dfrac{x^2+3x}{x-y}$ Ans: -2

2) $\lim\limits_{(x,y)\to(0,0)} \dfrac{xy-y-2x+2}{x-1}$ Ans: -2

3) $\lim\limits_{(x,y)\to(2,0)} \dfrac{\sqrt{2x-y}-2}{2x-y-4}$ Ans: $\dfrac{1}{4}$

4) $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^2-xy}{\sqrt{x}-\sqrt{y}}$ Ans: 0

5) $\lim\limits_{(x,y)\to(0,0)} \dfrac{e^y \sin x}{x}$ Ans: 1

2. Determine the existence of following limits

1) $\lim\limits_{(x,y)\to(0,0)} \dfrac{x\ y}{x^2+y^2}$ Ans: Does not exist

2) $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^2}{x^4+y^2}$ Ans: Does not exist

3) $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^2 y}{x^4+y^4}$ Ans: Does not exist

Continuity:

Definition: A function $f(x,y)$ is said to be continuous at a point (a ,b) if the following is true

i) f(a,b) should be well defined.

ii) $\lim_{(x,y)\to(a,b)} f(x,y)$ exist

iii) $\lim_{(x,y)\to(a,b)} f(x,y) = f(a,b)$

Definition: If a function f is not continuous at a point $(a\ ,b)$,
Then it is called as **discontinuous at** $(a\ ,b)$.

Problems:

1. Test the continuity for the following functions at origin

i) $$f(x,y) = \frac{2x^2y^2}{x^4+y^4} \quad if\ (x,y) \neq (0,0)$$
$$= 0 \qquad (x,y) = (0,0)$$
Ans: Discontinuous

ii) $$f(x,y) = \frac{x}{\sqrt{x^2+y^2}} \quad if\ (x,y) \neq (0,0)$$
$$= 2 \qquad (x,y) = (0,0)$$
Ans: Discontinuous

2.2 Partial Derivatives (Basic)

- **Partial Derivative of first order:**

  Let $z = f(x,y)$ be a function of two independent variables x and y, then partial derivative of z with respect to x is denoted by $\frac{\partial z}{\partial x}$ or $\frac{\partial f}{\partial x}$ or f_x and is defined as

  $$\frac{\partial z}{\partial x} = \lim_{\delta x\to 0} \frac{f(x+\delta\ ,y)-f(x,y)}{\delta x}$$ Provided limit on RHS exists.

  Similarly partial derivative of z with respect to y is denoted by $\frac{\partial z}{\partial y}$ or $\frac{\partial f}{\partial y}$ or f_y and is defined as

  $$\frac{\partial z}{\partial y} = \lim_{\delta y\to 0} \frac{f(x,y+\delta y)-f(x,y)}{\delta y}$$ Provided limit on RHS exists.

- **Standard Rules:**

  If u and v are functions of x and y possessing partial derivatives of the first order then we can use standard rules of differentiation of sum, difference, product and quotient of u and v as follows:

  1. If $z = u \pm v$ then $\frac{\partial z}{\partial x} = \frac{\partial u}{\partial x} \pm \frac{\partial v}{\partial x}$, $\frac{\partial z}{\partial y} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}$

2. If $z = uv$, then $\frac{\partial z}{\partial x} = u\frac{\partial v}{\partial x} + v\frac{\partial u}{\partial x}$, $\frac{\partial z}{\partial y} = u\frac{\partial v}{\partial y} + v\frac{\partial u}{\partial y}$
3. If $z = \frac{u}{v}$, then $\frac{\partial z}{\partial x} = \frac{v\frac{\partial u}{\partial x} - u\frac{\partial v}{\partial x}}{v^2}$, $\frac{\partial z}{\partial y} = \frac{v\frac{\partial u}{\partial y} - u\frac{\partial v}{\partial y}}{v^2}$

Problems:

1) If $z = \log\left(x^2 + y^2\right)$, find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$.
2) If $z = \tan^{-1}\left(\frac{y}{x}\right)$, find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$.
3) If $u = 4x^3y^2 - zy^4 + z^3 + 4y - x^{16}$, find $\frac{\partial u}{\partial x}$, $\frac{\partial u}{\partial y}$ and $\frac{\partial u}{\partial z}$.
4) Find all second order derivatives of $z = x^4y - y^2x^2 + xy^3$.
5) Find all second order derivatives of $z = \log\left(x^2 + y^2\right)$.
6) If $u = x^3 - 3xy^2$, Prove that $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$.
7) If $u = \cos\left(\sqrt{x} + \sqrt{y}\right)$, Prove that

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + \frac{1}{2}\left(\sqrt{x} + \sqrt{y}\right)\sin\left(\sqrt{x} + \sqrt{y}\right) = 0$$

8) If $u = \log\left(x^2 + y^2 + z^2\right)$, Prove that $x\frac{\partial^2 u}{\partial y \partial z} = y\frac{\partial^2 u}{\partial z \partial x} = z\frac{\partial^2 u}{\partial x \partial y}$.
9) If $u = e^{xyz}$, Prove that $\frac{\partial^3 u}{\partial x \partial y \partial z} = \left(1 + 3xyz + x^2y^2z^2\right) e^{xyz}$.
10) If $z = x^2 \tan^{-1}\left(\frac{y}{x}\right) - y^2 \tan^{-1}\left(\frac{x}{y}\right)$, Prove that $\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial^2 z}{\partial y \partial x} = \frac{x^2 - y^2}{x^2 + y^2}$.
11) If $u = e^r$, $r^2 = x^2 + y^2$, prove that $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = e^r\left(1 + \frac{2}{r}\right)$.

10) If $u = r^m$, $r^2 = x^2 + y^2 + z^2$, prove that $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = m(m+1)r^{m-2}$.

2.3 Taylor's theorem of function of two variables

Taylor's Series of Two Variables

If $f(x,y)$ and all its partial derivatives up to the nth order are finite and continuous for all points (x,y), where $a \le x \le a+h,\ a \le y \le b+k$

Then

$$f(x,y) = f(a,b) + \left[(x-a)f_x(a,b) + (y-b)f_y(a,b)\right] + \frac{1}{2!}\left[(x-a)^2 f_{xx}(a,b) + 2(x-a)(y-b)f_{xy}(a,b) + (y-b)^2 f_{yy}(a,b)\right] + ...$$

Maclaurin's series:

$$f(x,y) = f(0,0) + \left[xf_x(0,0) + yf_y(0,0)\right] + \frac{1}{2!}\left[x^2 f_{xx}(0,0) + 2xyf_{xy}(0,0) + y^2 f_{yy}(0,0)\right] + ...$$

Problems:

1. Expand $e^x \sin y$ in powers of x and y as far as terms of second degree.
2. Find the expansion for $\cos x \cos y$ in powers of x, y upto second order terms.
3. Find the first six terms of the expansion of the function $e^x \log(1+y)$ in a Taylor's series in the neighborhood of the point $(0,0)$.
4. Expand $e^x \cos y$ near the point $\left(1, \frac{\pi}{4}\right)$ by Taylor's Theorem.
5. Obtain Taylor's expansion of $\tan^{-1}\frac{y}{x}$ about $(1,1)$ up to and including the second degree terms.

2.4 Maxima, Minima and saddle points (second derivative test)

- **Maxima and Minima of $z = f(x,y)$:**
  i. A function $f(x,y)$ is said to be maximum at point (a, b) if $f(a,b) > f(a+h, b+k)$ for small values of h and k, positive or negative.
  ii. A function $f(x,y)$ is said to be minimum at point (a, b) if $f(a,b) < f(a+h, b+k)$ for small values of h and k, positive or negative.

Working rule to find maxima and minima:

Let $f(x, y)$ be a function of two variables.

Step1: Find $\frac{\partial f}{\partial x}$ & $\frac{\partial f}{\partial y}$

Step2: Solve $\frac{\partial f}{\partial x} = 0$ & $\frac{\partial f}{\partial y} = 0$ and find values of x & y.

Step 3: Find $\frac{\partial^2 f}{\partial x^2} = r$, $\frac{\partial^2 f}{\partial x \partial y} = s$, $\frac{\partial^2 f}{\partial y^2} = t$ at above points.

Case(i): **If** $rt - s^2 > 0$ **&** $r < 0$ **then f has maximum at that point.**

Case(ii): **If** $rt - s^2 > 0$ **&** $r > 0$ **then f has minimum at that point.**

Case(iii): **If** $rt - s^2 < 0$ **then f has neither maximum nor minimum (saddle point).**

Case(iv): **If** $rt - s^2 = 0$ **then failure case.**

Problems:

1. Find extreme values of $x^2 + y^2 + 6x + 12$. {Minima at (-3, 0)}

2. Find extreme values of $x^3 + xy^2 + 21x - 12x^2 - 2y^2$.
{Answer: Max value 10 at (1, 0) and min value -98 at (7, 0).}

3. A rectangular box open at the top has volume of 108 cubic cm. Find the dimensions of the box requiring least material. {Answer: (6, 6, 3)}

4. Divide 120 into three parts so that the sum of their products taken two at a time shall be maximum. {Answer: (40, 40, 40)}

2.5 Method of Lagrange Multipliers

- **Lagrange Multipliers Method: (with one constraint)**
  Let $f(x, y, z)$ be a function of three variables *x, y, z* and the variables be connected by the relation $\emptyset(x, y, z) = 0$……………. (1)
  Then to find values of *x, y, z* for which $f(x, y, z)$ is maximum and minimum construct an auxiliary equation

$$F(x, y, z) = f(x, y, z) + \lambda\phi(x, y, z)$$

Differentiating partially w. r. t. *x, y, z* and equating to zero we get

$$\frac{\partial F}{\partial x} = \frac{\partial f}{\partial x} + \lambda \frac{\partial \phi}{\partial x} = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots (2)$$

$$\frac{\partial F}{\partial y} = \frac{\partial f}{\partial y} + \lambda \frac{\partial \phi}{\partial y} = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots (3)$$

$$\frac{\partial F}{\partial z} = \frac{\partial f}{\partial z} + \lambda \frac{\partial \phi}{\partial z} = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots (4)$$

Solving equations (1), (2), (3) and (4) we can find the values of *x, y, z* and λ for which $f(x, y, z)$ has maximum and minimum values. This method of obtaining maximum and minimum values of $f(x, y, z)$ is called as Lagrange's method of undetermined multipliers and the equations (2), (3) and (4) are called Lagrange's equations. The term λ is called Lagrange multiplier.

Problems:

1) Use Lagrange Multipliers Method to determine extreme values of $x^2 + y^2 + 2x - 2y + 1$ subject to the constraint $x^2 + y^2 = 2$. {Ans: max 7 at (1,-1), min -1 at (-1,1)}

2) Use Lagrange Multipliers Method to determine maximum and minimum value of $x^2 + y^2$ subject to the constraint $xy = 1$. {Ans: min 2 at (1,1), (-1,-1)}

3) Use Lagrange Multipliers Method to determine minimum distance from origin to the plane $3x + 2y + z = 12$. {Answer: Minimum distance is $\sqrt{72/7}$ at $x = 18/7$, $y = 12/7$, $z = 6/7$}