Tutorial No.1

Based on Limit and continuity

Evaluate the following

1) $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^3 - y^3}{x - y}$

Ans:0

2) $\lim\limits_{(x,y)\to(2,0)} \dfrac{x \sin y}{x^2 + 1}$

Ans: 0

3) $\lim\limits_{(x,y)\to(0,0)} \dfrac{x - y + 2\sqrt{x} - 2\sqrt{y}}{\sqrt{x} - \sqrt{y}}$

Ans: 2

Determine the existence of following limits

4) $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^3 y^3}{x^6 + y^6}$

Ans: Does not exist

5) $\lim\limits_{(x,y)\to(0,0)} \dfrac{y^2 - x^2}{x^2 + y^2}$

Ans: Does not exist

6) Is the following function continuous

i) $f(x,y) = \dfrac{2xy}{x^2 + y^2} \quad if\ (x,y) \neq (0,0)$

$\quad = 0 \qquad (x,y) = (0,0)$

Ans: Discontinuous

ii) $f(x,y) = \dfrac{y}{\sqrt{x^2 + y^2}} \quad if\ (x,y) \neq (0,0)$

$\quad = 0 \qquad (x,y) = (0,0)$

Ans: Discontinuous

iii)v) $f(x,y)=\dfrac{x^3-y^3}{x-y} \quad if \ (x,y)\neq(0,0)$

$\qquad =0 \qquad\qquad (x,y)=(0,0)$

Ans: continuous

iv) $f(x,y)=\dfrac{xy}{x^2+y^2} \quad if \ (x,y)\neq(0,0)$

$\qquad =0 \qquad\qquad (x,y)=(0,0)$

Ans: Discontinuous

Tutorial 2

Based on Partial Differentiation

1. If $z=e^{3x^2+4y}$, find $\dfrac{\partial z}{\partial x}$ and $\dfrac{\partial z}{\partial y}$.
2. If $z=y\sin(xy)$, find $\dfrac{\partial z}{\partial x}$ and $\dfrac{\partial z}{\partial y}$.
3. If $f(x,y)=x^2+3xy+y-1$, find $\dfrac{\partial z}{\partial x}$ and $\dfrac{\partial z}{\partial y}$ at the point $(4,-5)$.
4. If $f(x,y,z)=x\sin(y+3z)$, find $\dfrac{\partial f}{\partial x},\dfrac{\partial f}{\partial y} and \dfrac{\partial f}{\partial z}$.
5. If resistors r_1, r_2 and r_3 ohms are connected in parallel to make an r ohm resistor, the value of r can be found from the equation $\dfrac{1}{r}=\dfrac{1}{r_1}+\dfrac{1}{r_2}+\dfrac{1}{r_3}$. Find the value of rate of change of r in the direction of r_2 when $r_1=30, r_2=45$ and $r_3=90$ ohms.
6. In a study of frost penetration it was found that the temperature T at time t (measured in days) at a depth x (measured in meters) can be modeled by the function $T(x,t)=T_0+T_1e^{-\lambda x}\sin(\omega t-\lambda x)$ where $\omega=\dfrac{2\pi}{365}$ and λ is a positive constant.

   a)Find $\dfrac{\partial T}{\partial x}$ .What is its physical significance?

   b)Find $\dfrac{\partial T}{\partial t}$. What is its physical significance?
7. The temperature at a point (x,y) on a flat metal plate is given by $T(x,y)=\dfrac{60}{1+x^2+y^2}$ where T is measured in ${}^0C$ and x,y in meters. Find the rate of change of temperature with respect to distance at the point (2,1) in the x-direction and the y-direction.
8. The wind-chill index is modeled by the function $W=13.12+0.6215T-11.37\upsilon^{0.16}+0.3965T\upsilon^{0.16}$ where T is temperature in ${}^0C$ and υ is the

wind speed (km/h). When $T = -15\,^{\circ}C$ and $\upsilon = 30km/h$ by how much would you expect the apparent temperature W to drop if the actual temperature decreases by $1\,^{\circ}C$ what if the wind speed increases by 1 km/h?

9. Find all second order derivatives of $z = x^2y^2 - xy^4 + x^3y - xy$.
10. Find all second order derivatives of $z = e^{3x^2+4y}$.
11. If $\log\left(x^2 + y^2\right) = u$, Prove that $\dfrac{\partial^2 u}{\partial x \partial y} = \dfrac{\partial^2 u}{\partial y \partial x}$.
12. If $z = \dfrac{x^2 + y^2}{(x+y)}$, prove that $\left(\dfrac{\partial z}{\partial x} - \dfrac{\partial z}{\partial y}\right)^2 = 4\left(1 - \dfrac{\partial z}{\partial x} - \dfrac{\partial z}{\partial y}\right)$.
13. If $u = 3(ax+by+cz)^2 - (x^2+y^2+z^2)$ and $a^2+b^2+c^2 = 1$ then prove that $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} + \dfrac{\partial^2 u}{\partial z^2} = 0$.
14. If $z = \tan(y+ax) + (y-ax)^{\frac{3}{2}}$ then prove that $\dfrac{\partial^2 z}{\partial x^2} = a^2 \dfrac{\partial^2 z}{\partial y^2}$.
15. If $u = \log\left(\tan x + \tan y + \tan z\right)$, prove that $\sin 2x \dfrac{\partial u}{\partial x} + \sin 2y \dfrac{\partial u}{\partial y} + \sin 2z \dfrac{\partial u}{\partial z} = 2$.
16. If $u = \log\left(x^3 + y^3 - x^2y - xy^2\right)$ then prove that $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = 3$.
17. If $u = Ae^{-gx}\sin(nt - gx)$ where *A, g, n, a* are constants satisfies the equation

    $\dfrac{\partial u}{\partial t} = a^2 \dfrac{\partial^2 u}{\partial x^2}$ then, show that $ag = \sqrt{\dfrac{n}{2}}$.
18. If $\dfrac{1}{u^2} = x^2 + y^2 + z^2$ then prove that $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} + \dfrac{\partial^2 u}{\partial z^2} = 2u$.
19. If $u = e^{r^2}$, $r^2 = x^2 + y^2 + z^2$ then prove that $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} + \dfrac{\partial^2 u}{\partial z^2} = 4r^2e^{r^2} + 6e^{r^2}$
20. If $u = \sin r$, $r^2 = x^2 + y^2$, prove that $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = -\sin r + \dfrac{1}{r}\cos r$.

<u>Tutorial No.3</u>

Based on Taylor's Series, Maxima, minima and saddle points & Method of Lagrange multipliers

1. Expand $f(x,y) = x^3 + y^3 + xy^2$ in powers of $(x-1)$ *and* $(y-2)$ using Taylor's series.
2. Obtain the expansion of e^{xy} in powers of $(x-1)$ *and* $(y-1)$.
3. Find Taylor's expansion of x^y about (1,1).
4. Expand $f(x,y) = e^{x+y}$ in powers of x *and* y.
5. Find the maximum and minimum values of $x^3 + 3xy^2 - 3x^2 - 3y^2 + 4$.

   [Ans: Max=4 at (0, 0), Min=0 at (2, 0), saddle points are (1, 1) & (1, -1)]
6. Find all stationary values of $x^3 + 3xy^2 - 15x^2 - 15y^2 + 72x$.

   [Ans: Min=108 at (6, 0), Max=108 at (4, 0) and saddle points are (5, 1) & (5, -1)]
7. Show that the minimum value of $u = xy + a^3\left(\frac{1}{x} + \frac{1}{y}\right)$ is $3a^2$.
8. A rectangular box with open top has volume V. Find the dimensions of the box requiring least material. [Ans: $x=y=2z$]
9. Find the maximum and minimum values of the function $f(x,y) = 3x + 4y$ on the circle $x^2 + y^2 = 1$ using the method of Lagrange's Multiplier.

   [Ans: Min= -5 and Max=5]
10. Find the maximum and minimum distances of the point (3, 4, 12) from the sphere $x^2 + y^2 + z^2 = 1$. [Ans: Min=12 and Max=14]
11. Find a point on the plane $x + 2y + 3z = 13$ nearest to the point (1, 1, 1) using the method of Lagrange multipliers. [Ans: 3/2, 2, 5/2]
12. If the temperature T at any point (x, y, z) on the surface of the sphere $x^2 + y^2 + z^2 = 1$ is $T = 400xyz^2$, find the highest temperature.