Unit I

Differential Calculus of functions of one variable

Overview:

This unit covers the mean value theorems (Rolle's theorem, Lagrange's mean value theorem and Cauchy's mean value theorem), Convergence of Sequences and series, Taylor's and Maclaurin's Series Expansion, Indeterminate forms and finding limits using L-Hospital's rule. (09 hours)

Outcome:

After completion of this unit, students would be able to:

1. Interpret mean value theorems and implement the concepts comprehensively; deploy power series for advanced mathematical analysis.
2. Operate and analyze functions of single and relate the results to real life problems.

https://teacher.desmos.com/activitybuilder/custom/56be50c3801a4d4d0eb42d6a

https://www.geogebra.org/m/fnNwDNmS

https://www.geogebra.org/m/qTneMSa4

Detailed Syllabus:

1.1 Mean value theorems
1.2 Convergence of Sequences and series
1.3 Taylor's and Maclaurin's Series Expansion
1.4 Indeterminate forms and L' Hospital's rule

1.1 Mean value theorems

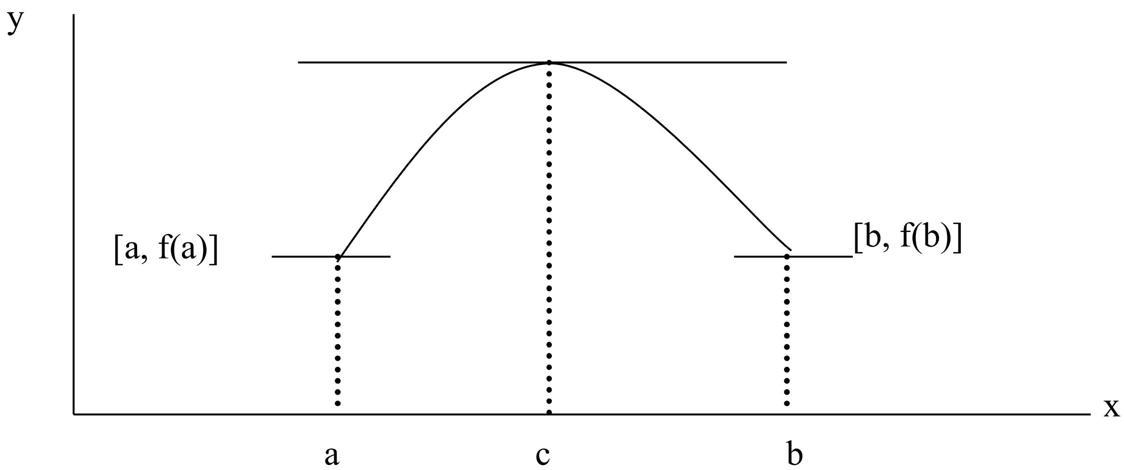
Rolle's Theorem (French Mathematician Michelle Rolle 1652-1679) :

Statement: If

- $f(x)$ is continuous in the closed interval $[a,\ b]$,
- $f(x)$ is differential in the open interval (a, b),
- $f(a) = f(b)$

Then there is at least one value c in the open interval (a, b) such that $f'(c) = 0$.

Geometrical Meaning of Rolle's Theorem: Consider a curve $f(x)$ that satisfies the conditions of the Rolle's Theorem as shown in figure:



As we see the curve $f(x)$ is continuous in the closed interval $[a,b]$, the curve is smooth i.e. there can be a unique tangent to the curve at any point in the open interval (a,b) and also $f(a) = f(b)$. Hence by Rolle's Theorem there exist at least one point c belonging to (a,b) such that $f'(c) = 0$. In other words there exists at least one point at which the tangent drawn to the curve will have its slope zero or lies parallel to x-axis.

Problems:

1. Verify Rolle's Theorem for $f(x) = x^2$ in $[-1,1]$.
2. Verify Rolle's Theorem for $f(x) = (x-a)^m (x-b)^n$ in [a, b] where a <b and a, b>0.

3. Is Rolle's Theorem applicable to the following functions?

(i) $f(x) = |x|$ in $-1 \le x \le 1$ (ii) $f(x) = x$ in $1 \le x \le 2$

(iii) $f(x) = |\sin x|$ in $\left[\frac{-\pi}{4}, \frac{\pi}{4}\right]$

Ans. (i) No (ii) No (iii) No

4. Verify Rolle's Theorem for the function $f(x) = \begin{cases} x^2 + 2 & -1 \le x \le 0 \\ x + 2 & 0 \le x \le 1 \end{cases}$

Ans: Not Applicable.

5. Verify Rolle's Theorem for the function $x(x+3)e^{-x/2}$ *in* $[-3,0]$.

Ans. Theorem is verified and $c = -2$.

6. Verify Rolle's Theorem for the function $\frac{x^2-4x}{x+2}$ *in* $[0,4]$.

Ans. Theorem is verified and $c=-2+2\sqrt{3}$.

7. Apply Rolle's Theorem to $f(x)=\sin x\sqrt{\cos 2x}$ *in* $\left[0,\pi/4\right]$ *and find c.*

Ans: $c=\pi/6$.

8. Find 'c' of Rolle's Theorem for $\log\left[\frac{x^2+ab}{(a+b)x}\right]$ *in* $[a,b]$, $a>0, \mathrm{b}>0$.

Ans: $c=\sqrt{ab}$.

9. Verify Rolle's Theorem for $(x-a)^m(x-b)^n$ in $[a,b]$, where m, n are positive integers.

Ans. Theorem is verified and $c=\frac{mb+na}{m+n}, a<c<b$.

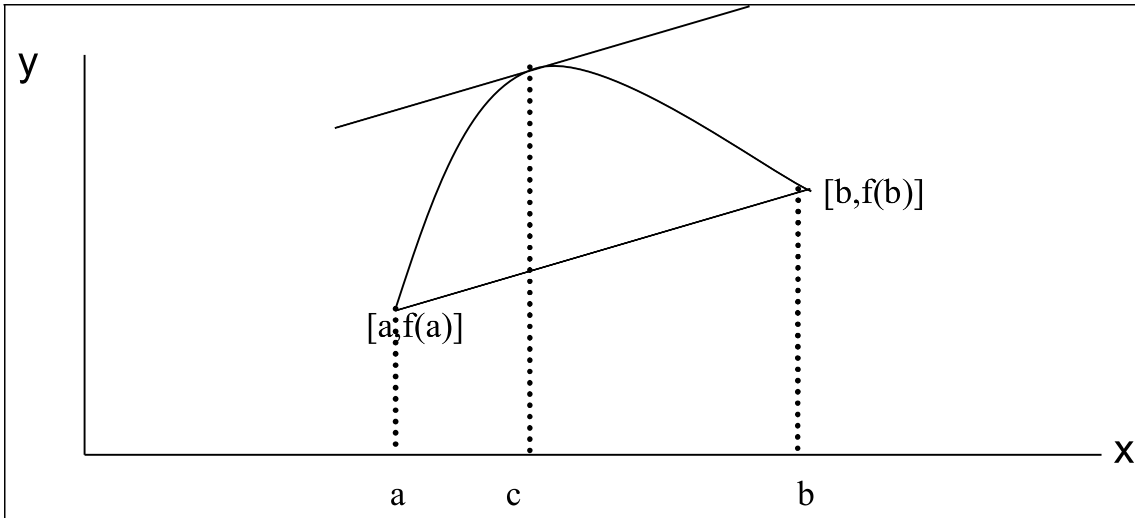
Lagrange's Mean Value Theorem (Italian-French Mathematician J. L. Lagrange 1736-1813):

Statement: If

- $f(x)$ is continuous in the closed interval $[a,\ b]$,
- $f(x)$ is differential in the open interval (a,b),

Then there is at least one value c in the open interval (a,b) such that $\frac{f(b)-f(a)}{b-a}=f'(c)$.

Geometrical Meaning of Lagrange's Mean Value Theorem: Consider a curve $f(x)$ that satisfies the conditions of the LMVT as shown in figure:



From the figure, we observe that the curve $f(x)$ is continuous in the closed interval $[a,b]$; the curve is smooth i.e. there can be a unique tangent to the curve at any point in the open interval (a,b). Hence by LMVT there exist at least one point c belonging to (a,b) such that $f'(c)=\dfrac{f(b)-f(a)}{b-a}$. In other words there exists at least one point at which the tangent drawn to the curve lies parallel to the chord joining the points $[a,f(a)]$ and $[b,f(b)]$.

Problems:

1. Verify LMVT for $f(x)=(x-1)(x-2)(x-3)$ in $[0,4]$

2. Verify LMVT for $f(x)=\log_e x$ in $[1,e]$

3. Examine the validity of the conditions and the conclusion of Lagrange's Mean Value theorem for the functions:

i) e^x *on* $[0,1]$ ii) $x^{1/3}$ *on* $[-1,1]$ iii) $\sqrt{x^2-4}$ *on* $[2,3]$ iv) $\tan^{-1}x$ *on* $[0,1]$

Ans: i) $c=\log(e-1)$, ii) Not applicable, iii) $c=\sqrt{5}$, iv) $c=\sqrt{(16-\pi^2)}\big/\pi$.

4. Find 'c' of the Lagrange Mean Value theorem, if $f(x)=x(x-1)(x-2),\ a=0, b=\dfrac{1}{2}$.

Ans. $c=\dfrac{6-\sqrt{21}}{6}$

5. Verify Lagrange's Mean Value Theorem for $f(x)=lx^2+mx+n \ \ in \left[a,b\right]$.

Ans. Theorem is verified and $c=\dfrac{b+a}{2}$

6. Verify Lagrange's Mean Value Theorem for $f(x)=\log_e x \ \ in \left[1,e\right]$.

Ans. Theorem is verified and $c=1.718$

7. Show that the chord joining the points $x=2, x=3$ on the curve $y=x^3$ is parallel to the tangent to the curve at $x=\sqrt{19/3}$.

Cauchy's Mean Value Theorem (French Mathematician A. L. Cauchy 1789-1857):

Statement: If

- $f(x)$ and $g(x)$ are continuous in the closed interval $[a,\ b]$,
- $f(x)$ and $g(x)$ are differential in the open interval (a,b),
- $g'(x)\neq 0$ for any x in (a,b)

Then there is at least one value c in the open interval (a,b) such that $\frac{f(b)-f(a)}{g(b)-g(a)}=\frac{f'(c)}{g'(c)}$.

1. Verify the Cauchy's MVT for $f(x)=x^2$ and $g(x)=x^4$ in $[a,b]$

2. Verify the Cauchy's MVT for $f(x)=\log x$ and $g(x)=\dfrac{1}{x}$ in $[1,e]$

3. Verify Cauchy's mean value theorem for the following functions:

$$f(x)=x^2,\ \ g(x)=x^3 \ \ in\ 1\le x\le 2\,.$$

Ans. Theorem is verified and $c=\dfrac{14}{9}$

4. Verify Cauchy's mean value theorem for $f(x)=e^x$ and $g(x)=e^{-x}$ and show that 'c' of Cauchy's mean value theorem is the average of a and b.

Ans. Theorem is verified and $c=\dfrac{a+b}{2}$

5. Verify Cauchy's mean value theorem by considering the functions $\sin x, \cos x$ for the Interval (a,b).

Ans. Theorem is verified and $c = \frac{a+b}{2}$

6. Considering the functions $\frac{1}{x^2}$ and $\frac{1}{x}$, prove that 'c' of Cauchy's mean value theorem is $\frac{2ab}{a+b}$.

7. Prove that $\frac{\sin b - \sin a}{e^b - e^a} = \frac{\cos c}{e^c}$. Also deduce that $e^c \sin x = \left(e^x - 1\right)\cos c$.

1.2 Convergence of Sequences and series:

Sequence

1) Determine the general term of each of the following sequence. Check whether the following sequences are convergent.

i. $\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \ldots.$ **Ans.** $\frac{1}{2^n}$

ii. $\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \ldots.$ **Ans.** $\frac{n}{n+1}$

iii. $1, -1, 1, -1, \ldots.$ **Ans.** $(-1)^{n-1}$

2) Test the convergence of following sequences:

1. $a_n = \frac{3n-1}{1+2n}$ (cgt)
2. $a_n = 1 + \frac{2}{n}$ (cgt)
3. $a_n = \frac{n^2 - 2n}{3n^2 + n}$ (cgt)

3) Test the convergence of following series:

Geometric Test:

The infinite series $1 + r + r^2 + \ldots\ldots + r^{n-1} + \ldots\ldots$ is

i) Convergent if $|r| < 1$ and its sum is $\frac{1}{1-r}$

ii) Divergent if $|r| \geq 1$.

Using geometric test:

1. $1 + \frac{2}{3} + \frac{4}{9} + \frac{8}{27} + \frac{16}{81} + \cdots$ (cgt)

2. $1-\frac{1}{2}+\frac{1}{2^2}-\frac{1}{2^3}+\frac{1}{2^4}........$ (cgt)

3. $\sum_{n=0}^{\infty}\frac{3^{2n}}{2^{3n}}$. (dgt)

D'Alembert's Ratio Test:

Let $\sum_{n=1}^{\infty}u_n$ be a nonnegative series. Assume that $u_n \neq 0$ for all n and that $\lim_{n\to\infty}\frac{u_{n+1}}{u_n}=l$

a. If $0 \le l < 1$, then $\sum_{n=1}^{\infty}u_n$ converges.

b. If $l > 1$, then $\sum_{n=1}^{\infty}u_n$ diverges.

If $l = 1$ then we cannot draw any conclusion from this test alone.

Using ratio test:

1. $\sum_{n=1}^{\infty}\frac{n}{n!}$ (cgt)
2. $\sum_{n=1}^{\infty}\frac{2^n}{n^2}$. (dgt)
3. $\frac{1}{1+2}+\frac{2}{1+2^2}+\frac{3}{1+2^3}+.......$ (cgt)

Cauchy's Root Test:

Let $\sum_{n=1}^{\infty}u_n$ be a nonnegative series and assume that $\lim_{n\to\infty}\sqrt[n]{u_n}=l$

a. If $0 \le l < 1$, then $\sum_{n=1}^{\infty}u_n$ converges.

b. If $l > 1$, then $\sum_{n=1}^{\infty}u_n$ diverges.

If $l = 1$ then we cannot draw any conclusion from this test alone.

Using Root test:

1. $\sum_{n=1}^{\infty}\left(\frac{n}{2n+5}\right)^n$ (cgt)
2. $\sum_{n=1}^{\infty}\left(\frac{\log n}{1000}\right)^n$ (dgt)

1.3 Taylor's and Maclaurin's series expansion:

Taylor's Series:-

$$f(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2!}f''(a) + \frac{(x-a)^3}{3!}f'''(a) + \cdots.$$

Using Taylor's series,

1. Expand $\cos x$ in powers of $\left(x-\frac{\pi}{2}\right)$.

$$\text{Ans. } \cos x = -\left(x-\frac{\pi}{2}\right) + \frac{1}{3!}\left(x-\frac{\pi}{2}\right)^3 - \frac{1}{5!}\left(x-\frac{\pi}{2}\right)^5 + \ldots$$

2. Expand $\log x$ in powers of $(x-2)$.

$$\text{Ans. } \log x = \log 2 + \frac{(x-2)}{2} - \frac{(x-2)^2}{8} + \frac{(x-2)^3}{24} - \ldots$$

3. Expand $x^5 - x^4 + x^3 - x^2 + x - 1$ in powers of $(x-1)$.

Ans: $3(x-1) + 6(x-1)^2 + 7(x-1)^3 + 4(x-1)^4 + (x-1)^5$

4. Expand $x^5 + 2x^4 - x^2 + x + 1$ in powers of $(x+1)$.

Ans: $(x+1)^2 + 2(x+1)^3 - 3(x+1)^4 + (x+1)^5$

Maclaurin's Series:-

$$f(x) = f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \ldots$$

Using Maclaurin's Series
Prove that

a) $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots.$

b) $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots.$

c) $\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots..$

d) $\log(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \ldots..$

1.4 Indeterminate form and L'Hospital's rule:

Note : Rule for finding the limits

(i) $\lim_{x\to a}[f(x)+g(x)] = \lim_{x\to a} f(x) + \lim_{x\to a} g(x)$

(ii) $\lim_{x\to a} f(x).g(x) = \lim_{x\to a} f(x) . \lim_{x\to a} g(x)$

(iii) $\lim_{x\to a}\dfrac{f(x)}{g(x)} = \dfrac{\lim_{x\to 0} f(x)}{\lim_{x\to 0} g(x)}$

Formulae : (i) $\lim_{x\to 0}\dfrac{\sin x}{x} = 1$ (ii) $\lim_{x\to 0}\dfrac{\sin^{-1} x}{x} = 1$

(iii) $\lim_{x\to 0}\dfrac{\tan x}{x} = 1$ (iv) $\lim_{x\to 0}(1+x)^{1/x} = e$

(v) $\lim_{x\to \infty}\left(1+\dfrac{1}{x}\right)^x = e$

L' Hospital Rule : If $f(x)$ and $g(x)$ are two functions which can be expanded by Taylor's series in the neighborhood of $x=a$ and if $f(a)=g(a)=0$, then $\lim_{x\to a}\dfrac{f(x)}{g(x)} = \lim_{x\to a}\dfrac{f'(x)}{g'(x)}$ provided the latter limit exists.

Problems on Indeterminate forms

Indeterminate type of the form $\dfrac{0}{0}$

1. Evaluate $\lim_{x\to 0}\dfrac{\log(1-x^2)}{\log\cos x}$ {**Ans:** 2 }
2. $\lim_{x\to 0}\dfrac{3^x-2^x}{x}$ {Ans: $\log\dfrac{3}{2}$ }

Indeterminate type of the form $\dfrac{\infty}{\infty}$

3. Evaluate $\lim_{x\to 1}\dfrac{\log(1-x)}{\cot \pi x}$ {**Ans:** 0 }
4. Evaluate $\lim_{x\to 0}\dfrac{\log_e \tan 2x}{\log_e \tan x}$. {**Ans:** 1}

Indeterminate type of the form $0\times\infty$

5. Evaluate $\lim_{x\to 0}\tan x\log x$ {**Ans:** 0 }

6. Evaluate $\lim_{x\to 0}\sin x\log x$. {**Ans:** 0 }

Indeterminate type of the form $\infty-\infty$

7. Evaluate $\lim_{x\to \pi/2}(\sec x-\tan x)$ {**Ans:** 0 }

8. Evaluate $\lim_{x\to 0}\left(\cot x-\frac{1}{x}\right)$ {**Ans:** 0 }

Indeterminate type of the form 1^{∞}

9. Evaluate $\lim_{x\to \pi/2}(\operatorname{cosec} x)^{\tan^2 x}$ {**Ans:** $\sqrt{e}$ }

10. Evaluate $\lim_{x\to 0}(\cos x)^{1/x^2}$ {**Ans:** $\frac{1}{\sqrt{e}}$ }

Indeterminate type of the form 0^0

11. Evaluate $\lim_{x\to 1}\left(1-x^2\right)^{1/\log(1-x)}$ {**Ans:** e }

12. Evaluate $\lim_{x\to 1}(x-1)^{(x-1)}$ {**Ans:** e }

Indeterminate type of the form ∞^0

13. Evaluate $\lim_{x\to 0}\left(\frac{1}{x}\right)^{x}$ {**Ans:** 1}