SVKM's NarseeMonjee Institute of Management Studies
Mukesh Patel School of Technology Management & Engineering

Unit V
Vector Calculus
(session 37 to session 45)

Overview:

Outcomes:
After completion of the course, students would be able to:

Detailed Syllabus:
5.1 Gradient
5.2 Directional Derivative
5.3 Divergence, Curl, Scalar Potential
5.4 Harmonic function
5.5 Line Integrals
5.6 Greens Theorem
5.7 Surface integral and Stokes Theorem
5.8 Surface integral and Gauss Divergence Theorem

Gradient, Directional Derivative, total derivative

The gradient of a scalar field $f(x, y, z)$ is given by

$$grad f = \nabla f = \left(\hat{\imath}\frac{\partial}{\partial x} + \hat{\jmath}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right) f(x, y, z)$$

The directional derivative of f in the direction of a unit vector $\hat{u}$ is $(\nabla f).\hat{u}$

Curl and Divergence

The **divergence** of a vector field F is given by $div F = \nabla . F = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$
A vector field F is **solenoidal** if $\nabla \cdot F = 0$ everywhere.
The **curl** of a vector field F is given by

$$curl\ F = \nabla \times F = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ F_1 & F_2 & F_3 \end{vmatrix}$$

A vector field F is **irrotational** if $\nabla \times F = 0$ everywhere.

Harmonic Functions

Any function of x, y which has continuous partial derivatives of the first and second order and satisfies Laplace's equation $\nabla^2\emptyset = \frac{\partial^2\emptyset}{\partial x^2} + \frac{\partial^2\emptyset}{\partial y^2} = 0$ is called a Harmonic Function.

Line Integral

A line integral of a vector function F over a curve C is defined by $\int_C \overline{F}.d\overline{r}$

Where $\overline{F} = f_1(x,y,z)i + f_2(x,y,z)j + f_3(x,y,z)k$ and $d\overline{r} = dx\,i + dy\,j + dz\,k$

Note:1. If $\overline{F}$ represents the variable force acting on a particle along arc AB, then total work done = $\int_A^B \overline{F}.d\overline{r}$

2. When the path of integration is closed curve then notation of integration is $\oint$

3. When $r(t)$ is the parametric representation of the curve C the line integral is given by $\int_C F(r).dr = \int_a^b F(r(t)).r'(t)\,dt$

Surface Integral

A surface $s = f(u,v)$ is called smooth if $f(u,v)$ possess continuous first order partial derivative.

Surface integral of a vector function $\overline{F}$ over the surface S is defined as the integral of the components of $\overline{F}$ along the normal to the surface.

It is given by $\int_S \overline{F}.\hat{n}ds$ where $\hat{n} = \frac{grad\ f}{|grad\ f|}$ and $ds = \frac{dx\,dy}{(\hat{n}.\hat{k})}$

Note: $\int_S \overline{F}.\hat{n}ds = \iint_S \overline{F}.\hat{n}\frac{dxdy}{(\hat{n}.\hat{k})}$

Green's Theorem

If $\phi(x,y), \psi(x,y)$, $\frac{\partial\phi}{\partial y}$ and $\frac{\partial\psi}{\partial x}$ be continuous functions over a region R bounded by simple closed curve C in xy - plane, then

$$\oint_C (\phi\,dx + \psi\,dy) = \iint_R \left(\frac{\partial\psi}{\partial x} - \frac{\partial\phi}{\partial y}\right)dx\,dy$$

Stoke's Theorem

The circulation of $\overline{F} = Mi + Nj + Pk$ around the boundary C of the oriented surface S in the direction counter clockwise with respect to the surface's unit normal vector n equals the integral of $\nabla \times F \bullet \hat{n}$ over S.

$$\oint_C \overline{F}.d\overline{r} = \int_S \nabla \times F \bullet \hat{n}\, ds$$

Note: If C is a curve in the xy plane, oriented counter clockwise and R is a region in the xy plane bounded by C, then $ds = dxdy$ and $\left(\nabla \times F\right) \bullet \hat{n} = \left(\nabla \times F\right) \bullet k = \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}$.

Gauss Divergence Theorem

The flux of a vector field $\overline{F} = Mi + Nj + Pk$ across a closed oriented surface S in the direction of the surface's outward unit normal vector n equals the integral of $\nabla \bullet F$ over the region D enclosed by the surface: $\int_S \overline{F}.\hat{n}ds = \int_D \nabla \bullet \overline{F}\, dV$.

Note: $\int_D dV = \iiint_D dxdydz$

5.1 Problems on Gradient

1. Find $\nabla \phi$ if $\phi = x^2 + y^2 + z^2$ at (1,-1,1). Ans: $2i - 2j + 2k$
2. Find the unit vector normal to the surface $x^2 + xy + z^2 = 4$ at the point (1,-1,2).
   Ans: $\frac{i + j + 4k}{\sqrt{18}}$
3. Find ∇f and $\left|\nabla f\right|$ if i) $f = 2xz^4 - x^2 y$ at (2,-2,-1), ii) $f = 2xz^2 - 3xy - 4x$ at (1,-1,2)
   Ans: i) $10i - 4j - 16k, 2\sqrt{93}$ ii) $7i - 3j + 8k, 2\sqrt{29}$

5.2 Problems on Directional Derivatives

1. Find directional derivative of $f = xy + yz + zx$ at (1, 2, 3) in the direction of $3i + 4j + 5k$. Ans: $46/5\sqrt{2}$
2. In what direction from (3, 1,-2) is the directional derivative of $\varphi = x^2 y^2 z^4$ maximum? And what is the magnitude of maximum change? $\left(Ans: 96\vec{i} + 288\vec{j} - 288\vec{k},\ 96\sqrt{19}\right)$
3. Find the rate of change of $\phi = xyz$ in the direction normal to the surface $x^2 y + y^2 x + yz^2 = 3$ at the point $(1,1,1)$. $\left(Ans: \frac{9}{\sqrt{29}}\right)$
4. Captain Astro is drifting in space near the sunny side of Mercury and notices that the hull of her ship is beginning to melt. The temperature in her vicinity is given by

$T = e^{-x} + e^{-2y} + e^{3z}$, where x, y and z are measured in meters. If she is at $(1,1,1)$, in what direction should she proceed in order to cool fastest? $\left(Ans : e^{-1}\vec{i} + 2e^{-2}\vec{j} - 3e^{3}\vec{k} \right)$

5.3 Problems on curl, divergence and scalar potential

1. If $\overline{f} = \left(3x^2\right)\hat{i} + \left(5xy\right)\hat{j} + \left(xyz^3\right)\hat{k}$, find $div\,\overline{f}$ & $curl\,\overline{f}$ at $(1,2,3)$.

Ans: $div\,\overline{f}$ =65 & $curl\,\overline{f}$ $=27\hat{i}-54\hat{j}+10\hat{k}$

2. If $\overline{F} = \left(x+3y^2\right)\hat{i} + \left(2y+2z^2\right)\hat{j} + \left(x^2+az\right)\hat{k}$ is solenoidal. Find the value of a. Ans: $a = -3$

3. Prove that $\overline{F} = \left(x+2y+az\right)\hat{i} + \left(bx-3y-z\right)\hat{j} + \left(4x+cy+2z\right)\hat{k}$ is solenoidal and determine a, b and c such that $\vec{F}$ is irrotational. Ans: $a = 4, b = 2, c = -1$

4. If $\overline{F} = 2xz\hat{i} + y^2\hat{j} + yz\hat{k}$ and $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$ find $div\left(\vec{F}\times\vec{r}\right)$ and $curl\left(\vec{F}\times\vec{r}\right)$

$\left(Ans : xz + 2xy, \left(6xz - 3xy\right)\vec{i} + \left(y^2 - 2yz\right)\vec{j} + \left(yz - 2z^2\right)\vec{k} \right)$

5. Show that $\overline{F} = \left(x^2 - yz\right)\hat{i} + \left(y^2 - zx\right)\hat{j} + \left(z^2 - xy\right)\hat{k}$ is irrotational and hence find its scalar potential. Ans: $\phi = \frac{x^3}{3} + \frac{y^3}{3} + \frac{z^3}{3} - xyz$

6. Prove that $\overline{F} = \left(z^2 + 2x + 3y\right)\hat{i} + \left(3x + 2y + z\right)\hat{j} + \left(y + 2zx\right)\hat{k}$ is irrotational and find scalar potential ϕ such that $\phi(1,1,0) = 4$. Ans: $\phi = x^2 + y^2 + xz^2 + 3xy + yz - 1$

5.4 Problems on Harmonic Function

1. Prove that $u = x^2 - y^2, v = \frac{-y}{x^2 + y^2}$ both u and v satisfy Laplace's equation.
2. Verify whether the following functions are harmonic or not
   - (i) xy **Ans.** Harmonic
   - (ii) $e^x \sin y$ **Ans.** Harmonic
   - (iii) $x^2 + y^2$ **Ans.** Not Harmonic
   - (iv) $e^{-2xy} \sin\left(x^2 - y^2\right)$ **Ans.** Harmonic

5.5 Problems on Line Integral

1. Evaluate $\int \left(2xy\hat{i} - x^2\hat{j}\right).d\overline{r}$ along

i) the straight line from $(0,0)$ to $(2,1)$

ii) parabola $x^2 = 4y$ from $(0,0)$ to $(2,1)$

2.Evaluate $\int_C \overline{F}.d\overline{r}$ where $\overline{F} = x^2\,\hat{i} + xy\,\hat{j}$ and C is the boundary of the square in the plane $z = 0$ and bounded by the lines $x = 0, y = 0, x = a$ and $y = a$. Ans. $\frac{a^3}{2}$

3.Find the work done when a force $\overline{F} = \left(x^2 - y^2 + x\right)\hat{i} - \left(2xy + y\right)\hat{j}$ moves a particle from origin to $(1,1)$ along a parabola $y^2 = x$. Ans. $\frac{2}{3}$

5.6 Problems on Green's Theorem

1. Using Green's theorem, evaluate $\int_C \left(x^2 y\,dx + x^2 dy\right)$ where C is the boundary described counter clockwise of the triangle with vertices $(0,0),(1,0),(1,1)$.

Ans. $\frac{5}{12}$

2. Using Green's theorem, evaluate $\int_C \left(x^2 - 2xy\right)dx + \left(x^2 y + 3\right)dy$ around the boundary C of the region $y^2 = 8x$ and $x = 2$.

5.7 Problems on Stoke's Theorem

1. Using Stoke's theorem, calculate the circulation of the field $F = x^2\hat{i} + 2x\hat{j} + z^2\hat{k}$ around the curve C :the ellipse $4x^2 + y^2 = 4$ in the xy plane counter clockwise when viewed from above.

Ans: 4π.

2. Using Stoke's theorem, calculate the circulation of the field $F = y\hat{i} + xz\hat{j} + x^2\hat{k}$ around the curve C :the boundary of triangle cut from the plane $x + y + z = 1$ by the first octant counter clockwise when viewed from above. Ans: $\frac{-5}{6}$.

5.8 Problems on Gauss Divergence's Theorem

1. Use Gauss Divergence Theorem to find the outwards flux of $F = (y - x)\hat{i} + (z - y)\hat{j} + (y - x)\hat{k}$ across the boundary of the region D: the cube bounded by the planes $x = \pm 1, y = \pm 1$ *and* $z = \pm 1$.

Ans: -16

2.Use Gauss Divergence Theorem to find the outwards flux of $F = y\hat{i} + xy\hat{j} - z\hat{k}$ across the boundary of the region D: the region inside the solid cylinder $x^2 + y^2 \le 4$ between

the plane $z = 0$ and the paraboloid $z = x^2 + y^2$. Ans: -8π . 3. Use Gauss Divergence Theorem to find the outwards flux of $F = x^2\hat{i} - 2xy\hat{j} + 3xz\hat{k}$ across the boundary of the region D: the region cut from the first octant by the sphere $x^2 + y^2 + z^2 = 4$. Ans: 3π .
Text Books: 1. B.V. Ramana (2017), "Higher Engineering Mathematics", *McGraw Hill Education*, *1st Edition.* Ch 15: Vector Differential Calculus Ch 16: Vector Integral Calculus *2.* B.S. Grewal (2017), Higher Engineering Mathematics, *Khanna Publishers, 44th Edition*. Ch : Ch 8: Vector Calculus
References: **1.** *Advanced Engineering Mathematics*, 10th Edition, Erwin Kreyszig, Wiley India, 2017 **2.** *Advanced Engineering Mathematics*, 20th Edition, H. K. Dass, S. Chand & Company Ltd, 2012