Unit III
Integral Calculus of functions of one variable

Tutorial No.1

Based on Evaluation of definite and improper integrals :

Answer the following:

1) $\int_{-\infty}^{\infty} \frac{1}{1+x^2}\,dx$ Ans: π

2) $\int_{2}^{\infty} \frac{1}{x\sqrt{x^2-1}}\,dx$ Ans: $\frac{\pi}{6}$

3) $\int_{1}^{\infty} \frac{1}{x^2+3}\,dx$ Ans: $\frac{\pi\sqrt{3}}{9}$

4) $\int_{-1}^{1} \sqrt{\frac{1+x}{1-x}}\,dx$ Ans: π

5) $\int_{1}^{\infty} \frac{dx}{x^2}$ Ans. 1

6) $\int_{4}^{\infty} \frac{dx}{\sqrt{x^3}}$ Ans. 1

Tutorial No.2

Based on Beta & Gamma functions and their Properties

Evaluate the following integrals:

1. $\int_{0}^{\infty} e^{-x^2}\,dx$ Ans. $\frac{\sqrt{\pi}}{2}$

2. $\int_{0}^{\infty} e^{-x^2/4}\,dx$ Ans. $\sqrt{\pi}$

3. $\int_{0}^{\infty} e^{-x^5}\,dx$ Ans. $\frac{1}{5}\left\lfloor\frac{1}{5}\right.$

4. $\int_0^\infty x^4\ e^{-x^4}\,dx$ Ans. $\frac{1}{16}\sqrt{\frac{1}{4}}$

5. $\int_0^\infty x^{1/4}\ e^{-\sqrt{x}}\,dx$ Ans. $\frac{3}{2}\sqrt{\pi}$

6. $\int_0^\infty \frac{e^{-x^3}}{\sqrt{x}}\,dx \int_0^\infty y^4 e^{-y^6}\,dy$ Ans. $\frac{\pi}{9}$

7. $\int_0^\infty x^2\ e^{-x^4}\,dx \int_0^\infty e^{-x^4}\,dx$ Ans. $\frac{\pi}{8\sqrt{2}}$

8. $\int_0^1 (\log x)^4\,dx$ Ans. 24

9. $\int_0^1 (x\log x)^4\,dx$ Ans. $\frac{24}{3125}$

10. $\int_0^1 \sqrt{\log\left(\frac{1}{x}\right)}\ dx$ Ans. $\frac{\sqrt{\pi}}{2}$

11. $\int_0^2 x^3\sqrt{2-x}\,dx$ Ans. $\frac{512}{315}\sqrt{2}$

12. $\int_0^2 x\ \sqrt[3]{8-x^3}\,dx$ Ans. $\frac{8}{3}\beta\left(2/3, 4/3\right)$

13. $\int_0^2 \frac{x^2}{\sqrt{2-x}}\,dx$ Ans. $2^{3/2}\beta\left(3, 1/2\right)$

14. $\int_0^3 \frac{x^{3/2}}{\sqrt{3-x}}\,dx \int_0^1 \frac{dx}{\sqrt{1-x^{1/4}}}$ Ans. $\frac{432}{35}\pi$

15. $\int_0^1 x^2\left(1-x^2\right)^4\ dx$ Ans. $\frac{1}{2}\beta\left(3/2, 5\right)$

16. $\int_0^{2a} x\sqrt{2ax-x^2}\ dx$ Ans. $\frac{\pi}{2}a^3$

17. $\int_0^1 \frac{dx}{\sqrt{1-x^4}}$ Ans. $\frac{1}{4}\beta\left(\frac{1}{4},\frac{1}{2}\right)$

18. Show that $\int_0^1 \sqrt{1-x^4}\ dx = \frac{\sqrt{\pi}}{6}\frac{\Gamma\left(1/4\right)}{\Gamma\left(3/4\right)}$

19. $\int_0^{\pi} (1-\cos\theta)^3\, dx$ Ans. $\frac{5\pi}{2}$

20. $\int_0^3 \frac{x^{3/2}}{(3-x)^{1/2}}\, dx$ Ans. $\frac{27\pi}{8}$

21. $\int_0^a \frac{x^4}{\sqrt{a^2-x^2}}\, dx$ Ans. $\frac{3\pi}{16}a^4$

22. $\int_0^1 \frac{x^7}{\sqrt{1-x^2}}\, dx$ Ans. $\frac{16}{35}$

23. $\int_0^1 \frac{x^9}{\sqrt{1-x^4}}\, dx$ Ans. $\frac{3\pi}{32}$

24. $\int_0^{\pi} (1+\cos\theta)^3\, dx$ Ans. $\frac{5\pi}{2}$

25. $\int_0^{\pi/6} \cos^3 3\theta \sin^2 6\theta\, dx$ Ans. $\frac{32}{315}$

26. $\int_0^{\pi} \sin^2\theta\,(1+\cos\theta)^3\, dx$ Ans. $\frac{7\pi}{8}$

27. $\int_0^1 x^4\sqrt{1-x^2}\ dx$ Ans. $\frac{\pi}{32}$

28. $\int_{-\pi}^{\pi} \sin^4 x \cos^2 x\, dx$ Ans. $\frac{\pi}{8}$

29. Prove that $\int_0^{\pi/2} \sqrt{\tan\theta}\, d\theta \int_0^{\pi/2} \sqrt{\cot\theta}\, d\theta = \frac{\pi^2}{2}$

30. Show that $\left(\int_0^{\pi/2} \sqrt{\sin x}\, dx\right)\left(\int_0^{\pi/2} \frac{1}{\sqrt{\sin x}}\, dx\right) = \pi$

Tutorial No.3

Applications of definite integrals to evaluate surface areas and volumes of revolutions

I) Answer the following:

1) Find the area of the surface that is generated by revolving the portion of the curve $y = x^2$ between $x = 1$ and $x = 2$ about the y-axis. Ans: $\frac{\pi}{6}(17^{3/2} - 5^{3/2})$

2) Show that the area of the surface of revolution generated by revolving about the curve $y = x^3$ included between the ordinates $x = 0$ *and* $x = 1$ about the $x - axis$ is $\frac{\pi}{27}(10^{3/2} - 1)$.

3) The given curve is rotated about the $y - axis$. Find the area of the resulting surface.

i) $y = \sqrt[3]{x}$, $1 \le y \le 2$ $\quad Ans: \pi(145\sqrt{145} - 10\sqrt{10})27$

ii) $x = \sqrt{a^2 - y^2}$, $0 \le y \le a/2$ $\quad Ans: \pi a^2$

II) Answer the following:

1) Find the volume of the solid generated by revolving the region bounded by $y = \sqrt{x}, y = 0$ and $x = 9$ about the x-axis and about the line x=9. Ans: $\frac{81\pi}{2}$, 171π.

2) Find the volume of the solid generated when the region enclosed by $y = \sqrt{x}$, $y = 2$ and $x = 0$ revolved about the y-axis. Ans: $\frac{32\pi}{5}$.

3) Find the volume of the solid generated by revolving the region bounded by $y = x,\ y = 0,\ x = 4$ about x-axis. Ans: $21\frac{1}{3}\pi$

4) Find the volume of the solid of revolution generated when the area of the curve $x^2 + y^2 = 16$ between $x = -1$ *and* $x = 1$ is rotated about the x-axis. Ans: $31\frac{1}{3}\pi$