1 Model I

We use the itemized-bid version of the Kelly mechanism [1]. Briefly, (i) every edge e has fixed capacity C_e and implements proportional pricing, meaning that if user u pays price p_{ue} to e, it gets a bandwidth allocation $x_{ue} = \frac{p_{ue}}{\sum_{u \in U} p_{ue}} C_e$ at e, and (ii) every user u purchases bandwidth separately from each edge that it wants to use. The total end-to-end bandwidth f_u that is usable by u can be obtained by a maxflow computation (with edge capacities given by x_{ue}).

The net utility Q_b of each *benign* user b is a bilinear combination of its link utility (which is defined by a scalar function U_b of its end-to-end bandwidth allocation f_b), and the total price $P_b = \sum_e p_{be}$ it has paid into the network:

$$Q_b(\dots) = U_b(f_b) - P_b$$

We define the social utility S as the sum of the link utilities of all the benign users in the system ($S = \sum_b U_b(f_b)$). Let S_{ben} and S_{opt} denote the social utilities in a Nash equilibrium of the game and in an optimal allocation (dictated by a powerful controller with perfect information trying to optimize the social utility) respectively. It has been shown [1] that if all the users of the network are benign and their link utilities are concave and non-decreasing

$$S_{ben} \geq \frac{3}{4} S_{opt}$$

We now introduce an additional Byzantine user (referred to henceforth as the *malicious* user) that can play an arbitrary combination of prices at the edges in the network. We continue to define the social utility S as the sum of the link utilities of the *benign* users of the network. Let S_{mal} denote the social utility in a Nash equilibrium in the presence of a malicious user and S_{ben} the social utility in a Nash equilibrium in the game consisting of only the benign users of the system (i.e., the game with the malicious player removed). As above, S_{opt} will denote the social utility in an optimal allocation. Then it can be shown that for any given strategy of the malicious user, under the same constraints as above,

$$S_{mal} \geq \frac{3}{4} S_{ben} - P_m$$

or, equivalently,

$$\frac{S_{mal} + P_m}{S_{ben}} \geq \frac{3}{4}$$

It can also be shown that

$$\frac{S_{mal} + P_m}{S_{opt}} \geq \frac{3}{4}$$

Both these bounds are tight (there are games which achieve the worst-case 0.75 values for both these ratios).

Observe that in the game without a malicious player, the social utility $S = \sum_b U_b = \sum_b Q_b + \sum_b P_b$ can be viewed as the sum of the net utilities of the benign users and the total payment into the network. The corresponding sum in the game with the malicious player is $\sum_b Q_b + (\sum_b P_b + P_m) = \sum_b U_b + P_m$ – the same as the numerator above. Thus, if we take this sum (the sum of the net utilities of the benign users and the total payment into the network) as the *definition* of the social utility, then the results above essentially state that in games with a malicious player, the worst-case efficiency loss is the same 25% that it is in games without a malicious player.

2 Model II

We are currently working on analyzing a (multi-stage) variant of the above model in which each user has a fixed total budget that it needs to spend over a finite time horizon. Essentially, every user buys a fixed number of "congestion credits" every month (say) from its ISP, and is free to spend these credits however it wants over this one-month period. We believe this is closer to the way congestion pricing would be implemented in an actual deployment – less variability from the end-users' perspective.

References

[1] R. Johari and J. N. Tsitsiklis, "Efficiency loss in a network resource allocation game," *Mathematics of Operations Research*, vol. 29, no. 3, pp. 407–435, 2004.