
BIM203 Logic Design

Gate Circuits and Boolean Equations

Overview

- **Gate Circuits and Boolean Equations**
 - **Binary Logic and Gates**
 - **Boolean Algebra**
 - **Standard Forms**

Binary Logic and Gates

- Binary variables take on one of two values.
- Logical operators operate on binary values and binary variables.
- Basic logical operators are the logic functions AND, OR and NOT.
- Logic gates implement logic functions.
- Boolean Algebra: a useful mathematical system for specifying and transforming logic functions.
- We study Boolean algebra as a foundation for designing and analyzing digital systems!

Binary Variables

- **Recall that the two binary values have different names:**
 - True/False
 - On/Off
 - Yes/No
 - 1/0
- **We use 1 and 0 to denote the two values.**
- **Variable identifier examples:**
 - A, B, y, z, or X_1 for now
 - RESET, START_IT, or ADD1 later

Logical Operations

- **The three basic logical operations are:**
 - **AND**
 - **OR**
 - **NOT**
- **AND is denoted by a dot (\cdot).**
- **OR is denoted by a plus ($+$).**
- **NOT is denoted by an overbar ($\bar{}$), a single quote mark ($'$) after, or (\sim) before the variable.**

Notation Examples

■ Examples:

- $Y = A \cdot B$ is read “Y is equal to A AND B.”
- $z = x + y$ is read “z is equal to x OR y.”
- $X = \bar{A}$ is read “X is equal to NOT A.”

■ Note: The statement:

$1 + 1 = 2$ (read “one plus one equals two”)

is not the same as

$1 + 1 = 1$ (read “1 or 1 equals 1”).

Operator Definitions

- Operations are defined on the values "0" and "1" for each operator:

AND

$$0 \cdot 0 = 0$$

$$0 \cdot 1 = 0$$

$$1 \cdot 0 = 0$$

$$1 \cdot 1 = 1$$

OR

$$0 + 0 = 0$$

$$0 + 1 = 1$$

$$1 + 0 = 1$$

$$1 + 1 = 1$$

NOT

$$\bar{0} = 1$$

$$\bar{1} = 0$$

Truth Tables

- *Truth table* – a tabular listing of the values of a function for all possible combinations of values on its arguments
- Example: Truth tables for the basic logic operations:

AND		
X	Y	$Z = X \cdot Y$
0	0	0
0	1	0
1	0	0
1	1	1

OR		
X	Y	$Z = X + Y$
0	0	0
0	1	1
1	0	1
1	1	1

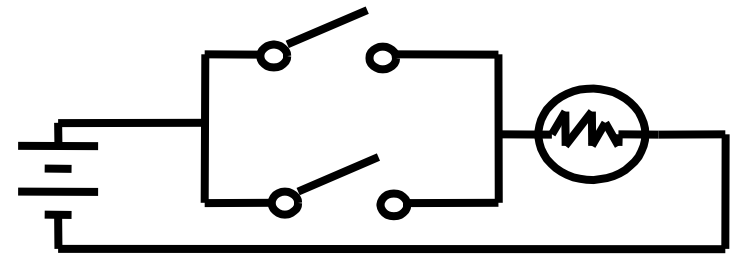
NOT	
X	$Z = \overline{X}$
0	1
1	0

Logic Function Implementation

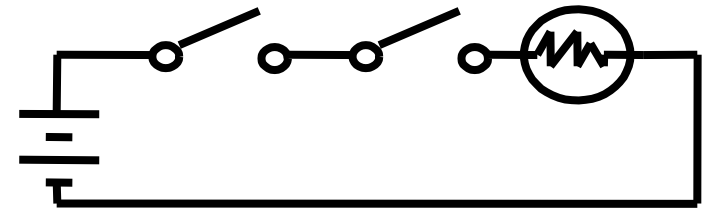
■ Using Switches

- For inputs:
 - logic 1 is switch closed
 - logic 0 is switch open
- For outputs:
 - logic 1 is light on
 - logic 0 is light off.
- NOT uses a switch such that:
 - logic 1 is switch open
 - logic 0 is switch closed

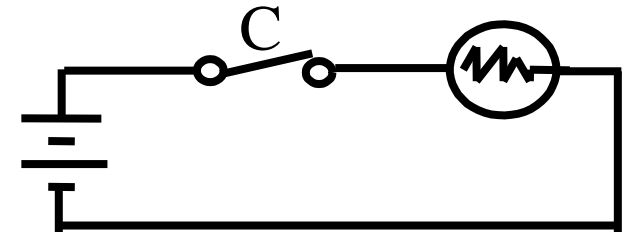
Switches in parallel => OR



Switches in series => AND

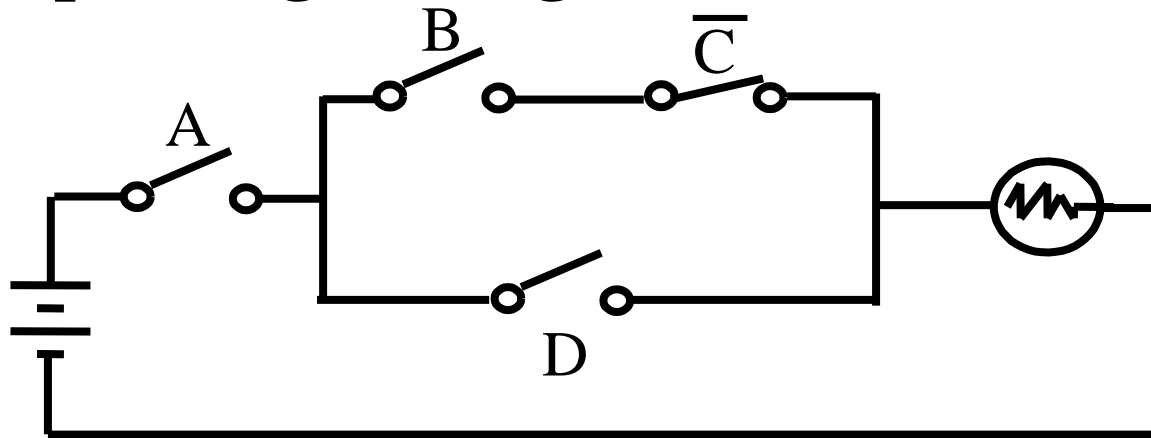


Normally-closed switch => NOT



Logic Function Implementation (Continued)

- **Example: Logic Using Switches**



- **Light is on ($L = 1$) for**
$$L(A, B, C, D) =$$

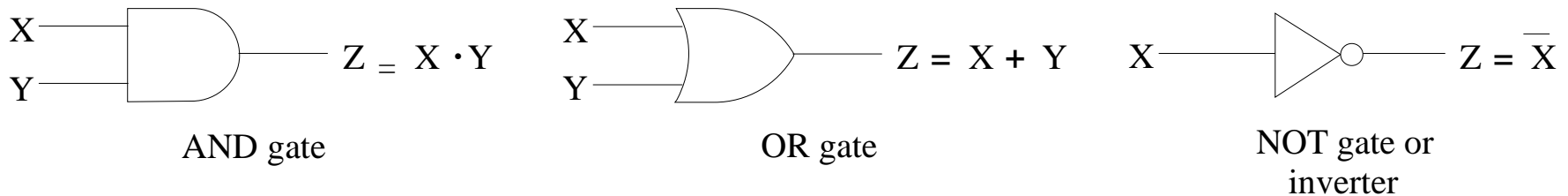
and off ($L = 0$), otherwise.
- **Useful model for relay circuits and for CMOS gate circuits, the foundation of current digital logic technology**

Logic Gates

- In the earliest computers, switches were opened and closed by magnetic fields produced by energizing coils in *relays*. The switches in turn opened and closed the current paths.
- Later, *vacuum tubes* that open and close current paths electronically replaced relays.
- Today, *transistors* are used as electronic switches that open and close current paths.
- **Optional: Chapter 6 – Part 1: The Design Space**

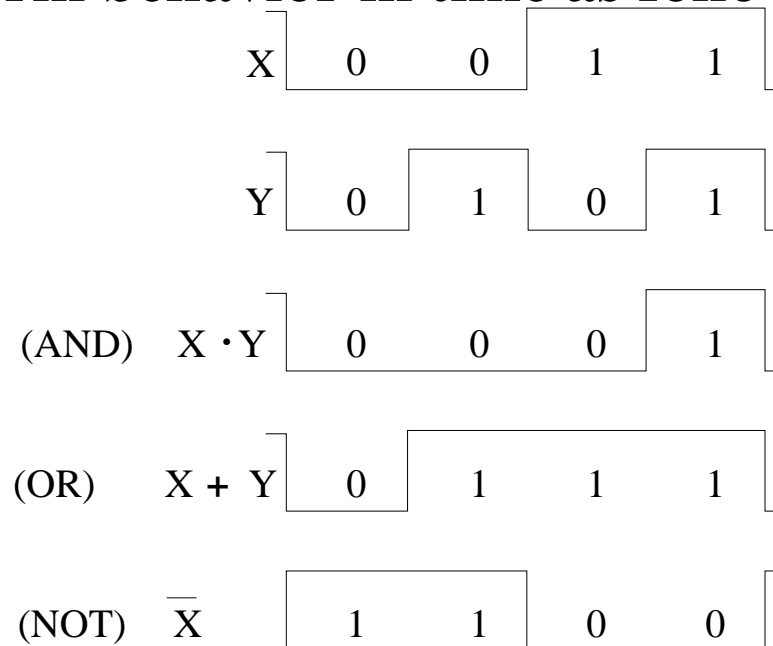
Logic Gate Symbols and Behavior

- Logic gates have special symbols:



(a) Graphic symbols

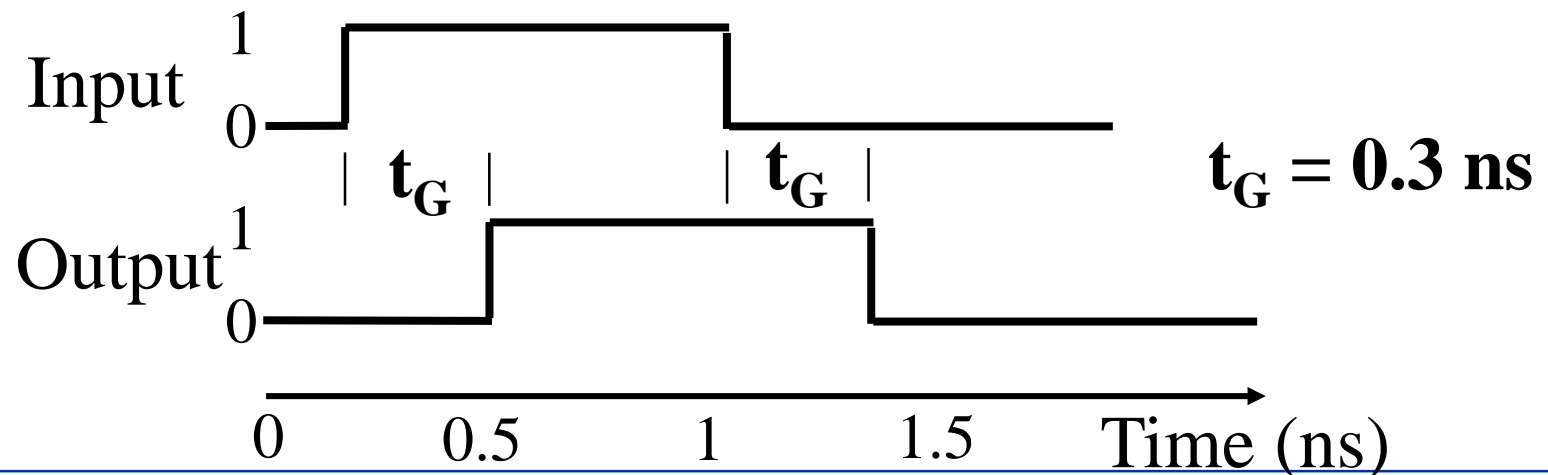
- And waveform behavior in time as follows:



(b) Timing diagram

Gate Delay

- In actual physical gates, if one or more input changes causes the output to change, the output change does not occur instantaneously.
- The delay between an input change(s) and the resulting output change is the *gate delay* denoted by t_G :



Logic Diagrams and Expressions

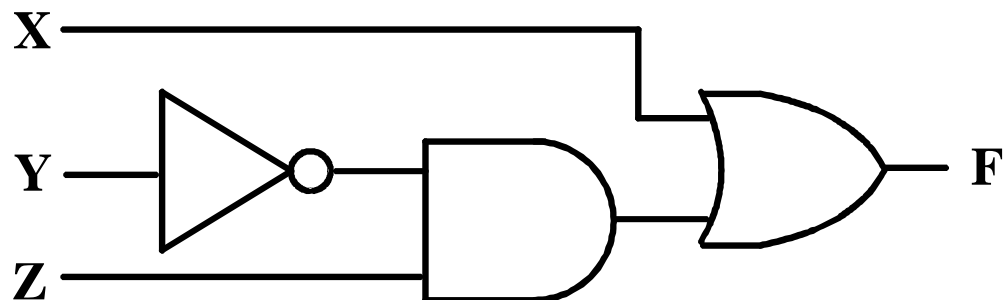
Truth Table

X Y Z	$F = X + \bar{Y} \cdot Z$
0 0 0	0
0 0 1	1
0 1 0	0
0 1 1	0
1 0 0	1
1 0 1	1
1 1 0	1
1 1 1	1

Equation

$$F = X + \bar{Y} Z$$

Logic Diagram



- Boolean equations, truth tables and logic diagrams describe the same function!
- Truth tables are unique; expressions and logic diagrams are not. This gives flexibility in implementing functions.

Boolean Algebra

- An algebraic structure defined on a set of at least two elements, B , together with three binary operators (denoted $+$, \cdot and $\overline{}$) that satisfies the following basic identities:
-

1. $X + 0 = X$

2. $X \cdot 1 = X$

3. $X + 1 = 1$

4. $X \cdot 0 = 0$

5. $X + X = X$

6. $X \cdot X = X$

7. $X + \overline{X} = 1$

8. $X \cdot \overline{X} = 0$

9. $\overline{\overline{X}} = X$

10. $X + Y = Y + X$

11. $XY = YX$

Commutative

12. $(X + Y) + Z = X + (Y + Z)$

13. $(XY)Z = X(YZ)$

Associative

14. $X(Y + Z) = XY + XZ$

15. $X + YZ = (X + Y)(X + Z)$

Distributive

16. $\overline{X + Y} = \overline{X} \cdot \overline{Y}$

17. $\overline{X \cdot Y} = \overline{X} + \overline{Y}$

DeMorgan's

Boolean Operator Precedence

- **The order of evaluation in a Boolean expression is:**
 1. Parentheses
 2. NOT
 3. AND
 4. OR
- **Consequence: Parentheses appear around OR expressions**
- **Example: $F = A(B + C)(C + \overline{D})$**

Useful Theorems

- $x \cdot y + \bar{x} \cdot y = y$ $(x + y)(\bar{x} + y) = y$ **Minimization**
- $x + x \cdot y = x$ $x \cdot (x + y) = x$ **Absorption**
- $x + \bar{x} \cdot y = x + y$ $x \cdot (\bar{x} + y) = x \cdot y$ **Simplification**
- $x \cdot y + \bar{x} \cdot z + y \cdot z = x \cdot y + \bar{x} \cdot z$ **Consensus**
 $(x + y) \cdot (\bar{x} + z) \cdot (y + z) = (x + y) \cdot (\bar{x} + z)$
- $\overline{x + y} = \bar{x} \cdot \bar{y}$ $\overline{x \cdot y} = \bar{x} + \bar{y}$ **DeMorgan's Laws**

Boolean Function Evaluation

$$F1 = xy\bar{z}$$

$$F2 = x + \bar{y}z$$

$$F3 = \bar{x}\bar{y}\bar{z} + \bar{x}y z + x\bar{y}$$

$$F4 = x\bar{y} + \bar{x}z$$

x	y	z	F1	F2	F3	F4
0	0	0	0	0		
0	0	1	0	1		
0	1	0	0	0		
0	1	1	0	0		
1	0	0	0	1		
1	0	1	0	1		
1	1	0	1	1		
1	1	1	0	1		

Expression Simplification

- An application of Boolean algebra
- Simplify to contain the smallest number of literals (complemented and uncomplemented variables):

$$\begin{aligned} & \mathbf{A B + \bar{A} C D + \bar{A} B D + \bar{A} C \bar{D} + A B C D} \\ &= \mathbf{A B + A B C D + \bar{A} C D + \bar{A} C \bar{D} + \bar{A} B D} \\ &= \mathbf{A B + A B (C D) + \bar{A} C (D + \bar{D}) + \bar{A} B D} \\ &= \mathbf{A B + \bar{A} C + \bar{A} B D = B (A + \bar{A} D) + \bar{A} C} \\ &= \mathbf{B (A + D) + \bar{A} C \quad 5 \text{ literals}} \end{aligned}$$

Complementing Functions

- **Use DeMorgan's Theorem to complement a function:**
 1. Interchange AND and OR operators
 2. Complement each constant value and literal
- **Example: Complement $F = \bar{x}y\bar{z} + x\bar{y}\bar{z}$**
$$\bar{F} = (x + \bar{y} + z)(\bar{x} + y + z)$$
- **Example: Complement $G = (\bar{a} + bc)\bar{d} + e$**
$$\bar{G} =$$

Overview – Canonical Forms

- **What are Canonical Forms?**
- **Minterms and Maxterms**
- **Index Representation of Minterms and Maxterms**
- **Sum-of-Minterm (SOM) Representations**
- **Product-of-Maxterm (POM) Representations**
- **Representation of Complements of Functions**
- **Conversions between Representations**

Canonical Forms

- It is useful to specify Boolean functions in a form that:
 - Allows comparison for equality.
 - Has a correspondence to the truth tables
- Canonical Forms in common usage:
 - Sum of Minterms (SOM)
 - Product of Maxterms (POM)

Minterms

- **Minterms** are AND terms with every variable present in either true or complemented form.
- Given that each binary variable may appear normal (e.g., x) or complemented (e.g., \overline{x}), there are 2^n minterms for n variables.
- **Example:** Two variables (X and Y) produce $2 \times 2 = 4$ combinations:
 - XY (both normal)
 - $X\overline{Y}$ (X normal, Y complemented)
 - $\overline{X}Y$ (X complemented, Y normal)
 - $\overline{X}\overline{Y}$ (both complemented)
- Thus there are **four minterms** of two variables.

Maxterms

- Maxterms are OR terms with every variable in true or complemented form.
- Given that each binary variable may appear normal (e.g., x) or complemented (e.g., \bar{x}), there are 2^n maxterms for n variables.
- Example: Two variables (X and Y) produce $2 \times 2 = 4$ combinations:

$X + Y$ (both normal)

$X + \bar{Y}$ (x normal, y complemented)

$\bar{X} + Y$ (x complemented, y normal)

$\bar{X} + \bar{Y}$ (both complemented)

Maxterms and Minterms

- **Examples: Two variable minterms and maxterms.**

Index	Minterm	Maxterm
0	$\bar{x} \bar{y}$	$x + y$
1	$\bar{x} y$	$x + \bar{y}$
2	$x \bar{y}$	$\bar{x} + y$
3	$x y$	$\bar{x} + \bar{y}$

- **The index above is important for describing which variables in the terms are true and which are complemented.**

Standard Order

- Minterms and maxterms are designated with a subscript
- The subscript is a number, corresponding to a binary pattern
- The bits in the pattern represent the complemented or normal state of each variable listed in a standard order.
- All variables will be present in a minterm or maxterm and will be listed in the same order (usually alphabetically)
- Example: For variables a, b, c:
 - Maxterms: $(a + b + \bar{c})$, $(a + b + c)$
 - Terms: $(b + a + c)$, $a \bar{c} b$, and $(c + b + a)$ are NOT in standard order.
 - Minterms: $a \bar{b} c$, $a b c$, $\bar{a} \bar{b} c$
 - Terms: $(a + c)$, $\bar{b} c$, and $(\bar{a} + b)$ do not contain all variables

Purpose of the Index

- The index for the minterm or maxterm, expressed as a binary number, is used to determine whether the variable is shown in the true form or complemented form.
- For Minterms:
 - “1” means the variable is “Not Complemented” and
 - “0” means the variable is “Complemented”.
- For Maxterms:
 - “0” means the variable is “Not Complemented” and
 - “1” means the variable is “Complemented”.

Index Example in Three Variables

- **Example:** (for three variables)
- Assume the variables are called X, Y, and Z.
- The standard order is X, then Y, then Z.
- The Index 0 (base 10) = 000 (base 2) for three variables). All three variables are complemented for minterm 0 ($\bar{X}, \bar{Y}, \bar{Z}$) and no variables are complemented for Maxterm 0 (X,Y,Z).
 - Minterm 0, called m_0 is $\bar{X}\bar{Y}\bar{Z}$.
 - Maxterm 0, called M_0 is $(X + Y + Z)$.
 - Minterm 6 ?
 - Maxterm 6 ?

Index Examples – Four Variables

Index	Binary	Minterm	Maxterm
i	Pattern	m_i	M_i
0	0000	$\bar{a}\bar{b}\bar{c}\bar{d}$	$a + b + c + d$
1	0001	$\bar{a}\bar{b}\bar{c}d$?
3	0011	?	$a + b + \bar{c} + \bar{d}$
5	0101	$\bar{a}b\bar{c}d$	$a + \bar{b} + c + \bar{d}$
7	0111	?	$a + \bar{b} + \bar{c} + \bar{d}$
10	1010	$a\bar{b}c\bar{d}$	$\bar{a} + b + \bar{c} + d$
13	1101	$ab\bar{c}d$?
15	1111	$abcd$	$\bar{a} + \bar{b} + \bar{c} + \bar{d}$

Minterm and Maxterm Relationship

- **Review: DeMorgan's Theorem**

$$\overline{x \cdot y} = \bar{x} + \bar{y} \text{ and } \overline{x + y} = \bar{x} \cdot \bar{y}$$

- **Two-variable example:**

$$M_2 = \bar{x} + y \text{ and } m_2 = x \cdot \bar{y}$$

Thus M_2 is the complement of m_2 and vice-versa.

- **Since DeMorgan's Theorem holds for n variables, the above holds for terms of n variables**
- **giving:**

$$M_i = \overline{m_i} \text{ and } m_i = \overline{M_i}$$

Thus M_i is the complement of m_i .

Function Tables for Both

- **Minterms of 2 variables**

x y	m₀	m₁	m₂	m₃
0 0	1	0	0	0
0 1	0	1	0	0
1 0	0	0	1	0
1 1	0	0	0	1

Maxterms of 2 variables

x y	M₀	M₁	M₂	M₃
0 0	0	1	1	1
0 1	1	0	1	1
1 0	1	1	0	1
1 1	1	1	1	0

- **Each column in the maxterm function table is the complement of the column in the minterm function table since M_i is the complement of m_i .**

Observations

- In the function tables:
 - Each minterm has one and only one 1 present in the 2^n terms (a minimum of 1s). All other entries are 0.
 - Each maxterm has one and only one 0 present in the 2^n terms. All other entries are 1 (a maximum of 1s).
 - We can implement any function by "ORing" the minterms corresponding to "1" entries in the function table. These are called the minterms of the function.
 - We can implement any function by "ANDing" the maxterms corresponding to "0" entries in the function table. These are called the maxterms of the function.
 - This gives us two canonical forms:
 - Sum of Minterms (SOM)
 - Product of Maxterms (POM)
- for stating any Boolean function.

Minterm Function Example

- **Example:** Find $F_1 = m_1 + m_4 + m_7$

- $F_1 = \bar{x} \bar{y} z + x \bar{y} \bar{z} + x y z$

x y z	index	$m_1 + m_4 + m_7 = F_1$				
0 0 0	0	0	+	0	+	0 = 0
0 0 1	1	1	+	0	+	0 = 1
0 1 0	2	0	+	0	+	0 = 0
0 1 1	3	0	+	0	+	0 = 0
1 0 0	4	0	+	1	+	0 = 1
1 0 1	5	0	+	0	+	0 = 0
1 1 0	6	0	+	0	+	0 = 0
1 1 1	7	0	+	0	+	1 = 1

Minterm Function Example

- $F(A, B, C, D, E) = m_2 + m_9 + m_{17} + m_{23}$
- $F(A, B, C, D, E) =$

Maxterm Function Example

- Example: Implement F1 in maxterms:

$$F_1 = M_0 \cdot M_2 \cdot M_3 \cdot M_5 \cdot M_6$$

$$F_1 = (x + y + z) \cdot (x + \bar{y} + z) \cdot (x + \bar{y} + \bar{z}) \cdot (\bar{x} + y + \bar{z}) \cdot (\bar{x} + \bar{y} + z)$$

x y z	i	$M_0 \cdot M_2 \cdot M_3 \cdot M_5 \cdot M_6 = F1$
0 0 0	0	$0 \cdot 1 \cdot 1 \cdot 1 \cdot 1 = 0$
0 0 1	1	$1 \cdot 1 \cdot 1 \cdot 1 \cdot 1 = 1$
0 1 0	2	$1 \cdot 0 \cdot 1 \cdot 1 \cdot 1 = 0$
0 1 1	3	$1 \cdot 1 \cdot 0 \cdot 1 \cdot 1 = 0$
1 0 0	4	$1 \cdot 1 \cdot 1 \cdot 1 \cdot 1 = 1$
1 0 1	5	$1 \cdot 1 \cdot 1 \cdot 0 \cdot 1 = 0$
1 1 0	6	$1 \cdot 1 \cdot 1 \cdot 1 \cdot 0 = 0$
1 1 1	7	$1 \cdot 1 \cdot 1 \cdot 1 \cdot 1 = 1$

Maxterm Function Example

- $F(A, B, C, D) = M_3 \cdot M_8 \cdot M_{11} \cdot M_{14}$
- $F(A, B, C, D) =$

Canonical Sum of Minterms

- **Any Boolean function can be expressed as a Sum of Minterms.**
 - For the function table, the minterms used are the terms corresponding to the 1's
 - For expressions, expand all terms first to explicitly list all minterms. Do this by “ANDing” any term missing a variable v with a term $(v + \bar{v})$.
- **Example: Implement $f = x + \bar{x} \bar{y}$ as a sum of minterms.**

First expand terms: $f = x(y + \bar{y}) + \bar{x} \bar{y}$

Then distribute terms: $f = xy + x\bar{y} + \bar{x} \bar{y}$

Express as sum of minterms: $f = m_3 + m_2 + m_0$

Another SOM Example

- **Example: $F = A + \bar{B} C$**
- **There are three variables, A, B, and C which we take to be the standard order.**
- **Expanding the terms with missing variables:**
- **Collect terms (removing all but one of duplicate terms):**
- **Express as SOM:**

Shorthand SOM Form

- From the previous example, we started with:

$$F = A + \bar{B} C$$

- We ended up with:

$$F = m_1 + m_4 + m_5 + m_6 + m_7$$

- This can be denoted in the formal shorthand:

$$F(A, B, C) = \Sigma_m(1, 4, 5, 6, 7)$$

- Note that we explicitly show the standard variables in order and drop the “m” designators.

Canonical Product of Maxterms

- Any Boolean Function can be expressed as a Product of Maxterms (POM).
 - For the function table, the maxterms used are the terms corresponding to the 0's.
 - For an expression, expand all terms first to explicitly list all maxterms. Do this by first applying the second distributive law, “ORing” terms missing variable v with a term equal to $V \cdot \bar{V}$ and then applying the distributive law again.

- Example: Convert to product of maxterms:

$$f(x, y, z) = x + \bar{x} \bar{y}$$

Apply the distributive law:

$$x + \bar{x} \bar{y} = (x + \bar{x})(x + \bar{y}) = 1 \cdot (x + \bar{y}) = x + \bar{y}$$

Add missing variable z :

$$x + \bar{y} + z \cdot \bar{z} = (x + \bar{y} + z)(x + \bar{y} + \bar{z})$$

Express as POM: $f = M_2 \cdot M_3$

Another POM Example

- Convert to Product of Maxterms:

$$f(A, B, C) = A \bar{C} + B C + \bar{A} \bar{B}$$

- Use $x + y z = (x+y) \cdot (x+z)$ with $x = (A \bar{C} + B C)$, $y = \bar{A}$, and $z = \bar{B}$ to get:

$$f = (A \bar{C} + B C + \bar{A})(A \bar{C} + B C + \bar{B})$$

- Then use $x + \bar{x} y = x + y$ to get:

$$f = (\bar{C} + B C + \bar{A})(A \bar{C} + C + \bar{B})$$

and a second time to get:

$$f = (\bar{C} + B + \bar{A})(A + C + \bar{B})$$

- Rearrange to standard order,

$$f = (\bar{A} + B + \bar{C})(A + \bar{B} + C) \text{ to give } f = M_5 \cdot M_2$$

Function Complements

- The complement of a function expressed as a sum of minterms is constructed by selecting the minterms missing in the sum-of-minterms canonical forms.
- Alternatively, the complement of a function expressed by a Sum of Minterms form is simply the Product of Maxterms with the same indices.
- Example: Given $F(x, y, z) = \Sigma_m(1, 3, 5, 7)$
 $\bar{F}(x, y, z) = \Sigma_m(0, 2, 4, 6)$
 $\bar{F}(x, y, z) = \Pi_M(1, 3, 5, 7)$

Conversion Between Forms

- To convert between sum-of-minterms and product-of-maxterms form (or vice-versa) we follow these steps:
 - Find the function complement by swapping terms in the list with terms not in the list.
 - Change from products to sums, or vice versa.
- Example: Given F as before: $F(x, y, z) = \Sigma_m(1, 3, 5, 7)$
- Form the Complement: $\bar{F}(x, y, z) = \Sigma_m(0, 2, 4, 6)$
- Then use the other form with the same indices – this forms the complement again, giving the other form of the original function: $F(x, y, z) = \Pi_M(0, 2, 4, 6)$

Standard Forms

- **Standard Sum-of-Products (SOP) form:**
equations are written as an OR of AND terms
- **Standard Product-of-Sums (POS) form:**
equations are written as an AND of OR terms
- **Examples:**
 - SOP: $A B C + \bar{A} \bar{B} C + B$
 - POS: $(A + B) \cdot (A + \bar{B} + \bar{C}) \cdot C$
- These “mixed” forms are **neither SOP nor POS**
 - $(A B + C) (A + C)$
 - $A B \bar{C} + A C (A + B)$

Standard Sum-of-Products (SOP)

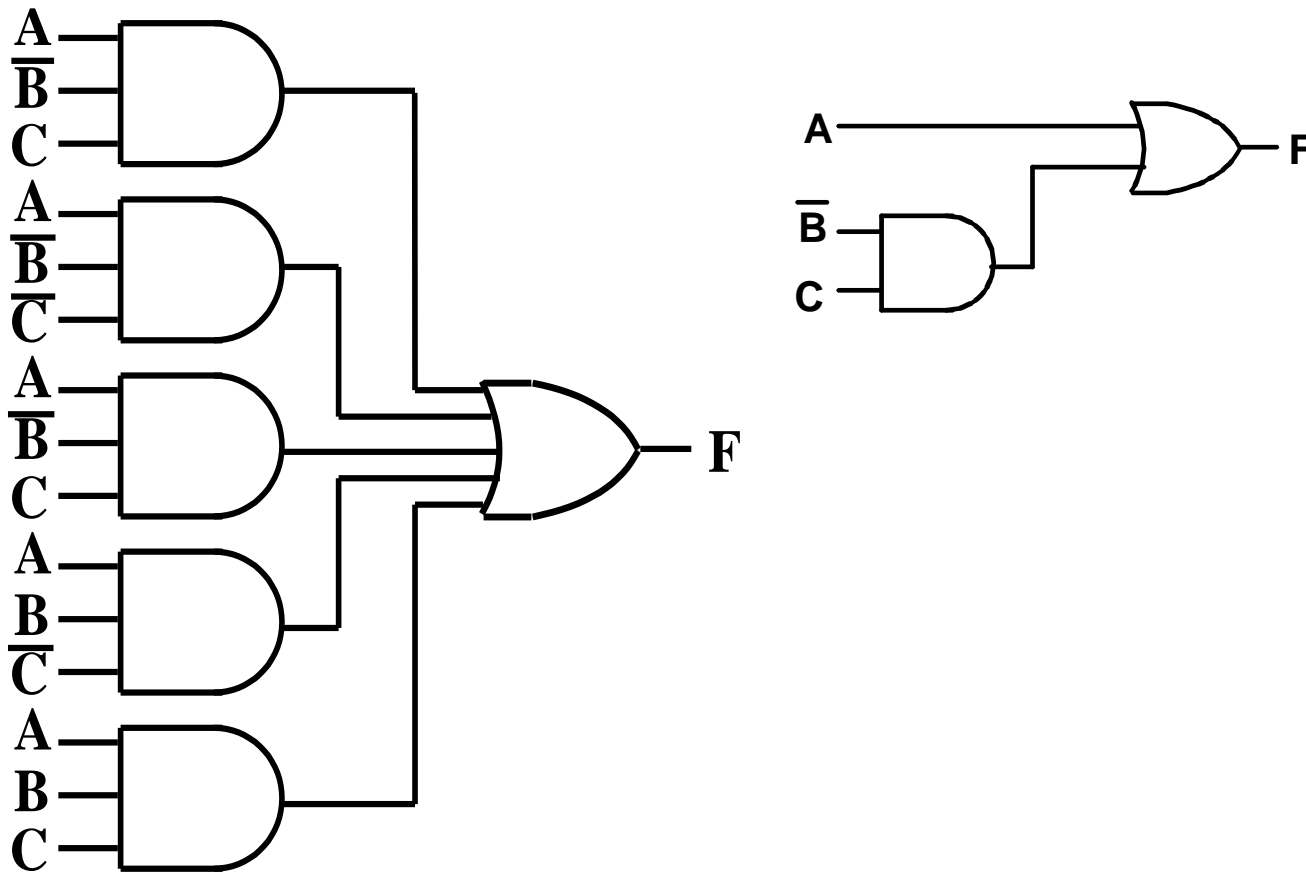
- A sum of minterms form for n variables can be written down directly from a truth table.
 - Implementation of this form is a two-level network of gates such that:
 - The first level consists of n -input AND gates, and
 - The second level is a single OR gate (with fewer than 2^n inputs).
- This form often can be simplified so that the corresponding circuit is simpler.

Standard Sum-of-Products (SOP)

- A Simplification Example:
- $F(A, B, C) = \Sigma m(1, 4, 5, 6, 7)$
- Writing the minterm expression:
$$F = \overline{A} \overline{B} C + A \overline{B} \overline{C} + A \overline{B} C + A B \overline{C} + A B C$$
- Simplifying:
$$F =$$
- Simplified F contains 3 literals compared to 15 in minterm F

AND/OR Two-level Implementation of SOP Expression

- The two implementations for F are shown below – it is quite apparent which is simpler!



SOP and POS Observations

- **The previous examples show that:**
 - **Canonical Forms (Sum-of-minterms, Product-of-Maxterms), or other standard forms (SOP, POS) differ in complexity**
 - **Boolean algebra can be used to manipulate equations into simpler forms.**
 - **Simpler equations lead to simpler two-level implementations**
- **Questions:**
 - **How can we attain a “simplest” expression?**
 - **Is there only one minimum cost circuit?**
 - **The next part will deal with these issues.**

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