Mathematical Programming (MP)

Programming is used in the sense of planning

Features of Mathematical Programing Models:

- Involve optimization
  - *maximization, ie profit maximization*
  - *minimization, ie cost minimization*

- Involve mathematical relationships
  - *equations, equalities, inequalities*
  - *logical dependencies*

Mathematical Programming

Types of MP Models

- **Linear programming models**
  - ***all equations are linear***

- **Non-linear programming models**
  - ***• equations can be non-linear***

- **Integer programming models**
  - ***can force solution in integers***
  - ***Knapsack algorithm can be formulated as an IP***

Steps in Model Building

- **Determine type of model to use**
  - ***Many "classical" models in use***
  - ***Recognize key assumptions of the model***
- **Formulate the model**
- **Parameterize the model**
  - ***Numerical values assigned to the parameters***
- **Validate the model**
  - ***Model should accurately represent the system***

Linear Programming

Linear programming (LP) applies to optimization models in which the objective and constraint functions are strictly linear.

- **Standard tool used in many business applications**
- **The technique is used in a wide range of applications**
- **It also boasts efficient computational algorithms for problems with thousands of constraints and variables.**
- **Simple and easy to solve with computer using Excel or commercial software**
- **Flexible formulation allows modeling many situations**

- Well understood by a variety of people.
- LP forms the backbone of the solution algorithms for other models, including integer, stochastic, and nonlinear programming

When To Use Linear Programming

Five conditions:

1. You have limited or scarce resources to allocate
2. You have an objective function to maximize or minimize
3. There are linear constraints to the solution
4. Homogeneity in resources (i.e. resources can be classed in homogenous group)
5. Fractions of resources are possible

Linear Programming (LP) model has three basic components

1. Decision variables that we seek to determine
2. Objective (goal) that we aim to optimize
3. Constraints that we need to satisfy

Characteristics of an LP

- **A single linear equation to be maximized or minimized (the objective function)**
- **A set of bounds on the variables**
  - ***– Upper Bounds***
  - ***– Lower Bounds***
- **A set of constraints that can be**
  - ***– Equalities***
  - ***– Inequalities (<=, >=)***
- **A set of scarce resources associated with the constraints (the Right Hand Sides, RHS)se**

Example Problem

Seray Furniture Ltd.

- Discontinued several products and have excess plant capacity

- Has three plants:
  - *ANKARA 4 Hours Excess Capacity*
  - *ISTANBUL 12 Hours Excess Capacity*
  - *İZMİT 18 Hours Excess Capacity*

Example Problem

- **Will make two new products:**
  - ***End Table 30 M TL profit per unit***
  - ***Side Table 50 M TL profit per unit***

- **Each unit of production uses capacity as follows:**

	End Table	Side Table
ANKARA	0.1	0.0
ISTANBUL	0.0	0.2
İZMİT	0.2	0.5

Objective Function

What is the objective function?

Questions

- What do you want to maximize or minimize?

Answer:

X1 = Number of End table
X2 = Number of Side table

Maximize:

$30X_1 + 50X_2$

Constraints

Maximize: Objective Function

$30X_1 + 50X_2$

Where:
X_1 = Number of End table produced
X_2 = Number of Side produced.

What are these constraints? Constrains

Subject to:

$0.1X_1$	$<= 12$	(Ankara capacity)
$0.2X_2$	$<= 4$	(Istanbul capacity)
$0.2X_1 + 0.5X_2$	$<= 18$	(İzmit capacity)

Solve this by trial and error

Mathematical Formulation

Maximize:

$30X_1 + 50X_2$

Subject to:

$0.1X_1$	<= 12	(Ankara capacity)
$0.2X_2$	<= 4	(Istanbul capacity)
$0.2X_1 + 0.5X_2$	<= 18	(İzmit capacity)

Because of the inequalities there are many feasible solutions.
We should find the best one.

Solving the LP

Maximize:

$30X_1 + 50X_2$

Try X_1 =10 and X_2 = 5

Subject to:

$0.1X_1$	<= 12	(Ankara capacity)
$0.2X_2$	<= 4	(Istanbul capacity)
$0.2X_1 + 0.5X_2$	<= 18	(İzmit capacity)

Solve all equations, check for feasibility and increase X_1 and X_2

Example Problem 2: The Reddy Mikks Company

Reddy Mikks produces both interior and exterior paints from two raw materials, M1 and M2. The following table provides the basic data of the problem:

	Tons of raw material per ton of		*Maximum daily availability (tons)*
	Exterior paint	*interior paint*	
Raw material, M1	6	4	24
Raw material, M2	1	2	6
Profit per ton ($1000)	5	4	.

A market survey indicates that the daily demand for interior paint cannot exceed that of exterior paint by more than 1 ton. Also the maximum daily demand of interior paint is 2 tons.

Reddy Mikks wants to determine the optimum (best) product mix of interior and exterior paints that maximizes the total daily profit.

Example Problem 2: Objective Function

The **variables** of the model are defined as

X1 = Tons produced daily of exterior paint
X2 = Tons produced daily of interior paint

- The **objective** function, the company wants to increase its profit as much as possible.
- Letting *z* represent the total daily profit.

The objective of the company is expressed as

Maximize z = 5X1 + 4X2

Example Problem 2: Constraints

- Next, we construct the constraints that restrict raw materials usage and demand.
- The raw materials restrictions are expressed verbally as

(Usage of a raw material by both paints) $\leq$ (Maximum raw material availability)

From the data of the problem,
Usage of raw material M1 per day = 6 X1 + 4 X2 tons
Usage of raw material M2 per day = 1 X1 + 2 x2 tons

- Because the daily availabilities of raw materials M1 and M2 are limited to 24 and 6 tons, respectively.
- The associated restrictions are given as

$$6\,X1 + 4\,X2 \leq 24 \quad \text{(Raw material Ml)}$$
$$X1 + 2\,X2 \leq 6 \quad \text{(Raw material M2)}$$

Example Problem 2: Constraints (Continue)

- The first demand restriction says that the difference between the daily production of interior and exterior paints, X2-X1, does not exceed 1 ton.

$$X2 - X1 \leq 1$$

- The second demand restriction stipulates that the maximum daily demand of interior paint is limited to 2 tons,

$$X2 \leq 2$$

- An implicit restriction is that variables X1 and X2 cannot assume negative values.

$$X1 \geq 0,$$
$$X2 \geq 0,$$

Example Problem 2: Mathematical Formulation

The complete Reddy Mikks model is

$$\text{Maximize } z = 5 X1 + 4 X2$$

Subject to

$$6X1 + 4X2 \le 24$$

$$X1 + 2 X2 \le 6$$

$$-X1 + X2 \le 1$$

$$X2 \le 2$$

$$X1, X2 \ge 0$$

Example Problem 2: Solving the LP

Determine the *best feasible* solution among the following (feasible and infeasible) solutions of the Reddy Mikks model:

1. Try X_1 =3 Tons and X_2 = 1 Ton per day

For $\mathbf{X_1 = 3}$ Tons **and** $\mathbf{X_2 = 1}$ Ton per day,
In the first constraint,
$6 X1 + 4 X2 = 6 \times 3 + 4 \times 1 = 22 < 24$

And it does not violate any of the constraints,

The value of the objective function associated with the solution is
$z = 5 \times 3 + 4 \times 1 = 19$ (thousand dollars).

2. Try X_1 =1 Tons and X_2 = 4 Ton per day

In the first constraint,

6 X1 + 4 X2 = 6 x 1 +4 x 4 = 22 (<24)

X1+ 2 X2= 1x1 + 2x4 =9 (≤ 6) so infeasible

3. Try X_1 =3 Tons and X_2 = 1.5 Ton per day

In the first constraint,

6 X1 + 4 X2 = 6 x 3 +4 x 1.5 = 24 (=24)

X1+ 2 X2= 1x3 + 2x1.5 =6 (= 6)

-X1 + X2= -3 +1.5=-1.5 (≤1)

X2= 1.5 (≤2)

z = 5 x 3 + 4 x 1.5 = 21 (thousand dollars).

So It gives the best feasible solution.

Example Problem 3: The Reddy Mikks Company

For the Reddy Mikks model, construct each of the following constraints and express them with a constant right-hand side:

X1 = Tons of exterior paint.

X2 = Tons of interior paint.

(a) The daily demand for interior paint exceeds that of exterior paint by *at least* 1 ton.

X2-X1≥1

(b) The daily usage of raw material M2 is *at most* 6 tons and *at least* 3 tons.

X1+2X2 ≥3

X1+2X2 ≤6

(c) The proportion of interior paint to the total production of both interior and exterior paints must not exceed 0.5.

X2/(X1+X2) ≤0.5

or 0.5 X1-0.5 X2≥ 0

Example Problem 4: The Reddy Mikks Company

For the feasible solution X1 = 2, X2 = 2 of the Reddy Mikks model, determine

(a) The unused amount of raw material Ml.

(b) The unused amount of raw material M2.

Solution:

Let S1 and S2 be the unused capacity of M1 and M2, then,

Since $6X1 + 4X2 \leq 24$ (Raw material Ml)

$X1 + 2X2 \leq 6$ (Raw material M2)

Then

$$6X1 + 4X2 + S1 = 24$$

$$S1 = 24-(6x2+4x2) = 4 \text{ tons}$$

$$X1 + 2X2 + S2 = 6$$

$$S2 = 6-(1x2+2x2) = 0 \text{ ton.}$$

Example Problem 5: Minimization Problem

A furniture company manufactures a single product. The estimated demand for the product for the next three months are 1000, 800, and 1200 respectively. The company has a regular time capacity of 800 per month and an overtime capacity of 200 per month. The cost of regular time production is $20 per unit and cost of over time production is $25 per unit. The company can carry inventory to the next month and the holding cost is $3 per unit month. The demand has to be met every month. Solve the problem.

Example Problem 5: Minimization Problem

Solution 1:

Decision variables: Let

Xi: quantity produced during regular time in month i.

Yi: quantity produced during overtime time in month i.

Ii: quantity carried at the end of month I to next month.

Objective function: **Minimize 20 ΣXi+ 20 ΣYi + 3 ΣIi**

Constraints:

$X1+ Y1=1000+ I1$ (1st month requirement)

$I1+ X2+ Y2=800+I2$ (2nd month requirement)

$I2+ X3+ Y3=1200$ (3rd month requirement)

$Xi \leq 800$ (i=1,2,3)

$Yi \leq 200$ (i=1,2,3)

$Xi, Yi, Ii \geq 0$

8 variables and 9 constraints.

Example Problem 5: Minimization Problem

Solution 2:

The variable I can be eliminated by rewriting the constrains as:

$X1+ Y1 \geq 1000$ (1st month requirement)

$X1+ Y1 + X2+ Y2 \geq 1800$ (2nd month requirement)

$X1+ Y1 + X2+ Y2+X3+ Y3 \geq 3000$ (3rd month requirement)

$Xi \leq 800$ (i=1,2,3)

$Yi \leq 200$ (i=1,2,3)

$Xi, Yi, \geq 0$

Objective function becomes:

Minimize 20 ΣXi+ 20 ΣYi +

3 (X1+ Y1 -1000 **+** X1+ Y1 + X2+ Y2-1800

+ X1+ Y1 + X2+ Y2+X3+ Y3-3000)

6 variables and 9 constraints.

So we have fewer variables and have inequalities.

Compare to these two solutions which one better.

Example Problem 5: Minimization Problem

Solution 3: Xijk : quantity produced in month i to meet the demand of month j using production type k. (k=1means for regular time, k=2 overtime)

$$X111 + X112 = 1000 \quad \text{(1st month requirement)}$$
$$X121 + X122 + X221 + X222 = 800 \quad \text{(2nd month requirement)}$$
$$X131 + X132 + X231 + X232 + X331 + X332 = 1200 \quad \text{(3rd month requirement)}$$
$$X111+X121+X131 \le 800$$
$$X221+X231 \le 800$$
$$X331 \le 800$$
$$X112+X122+X132 \le 200$$
$$X222+X232 \le 200$$
$$X332 \le 200$$
$$X111,X121,X131,X221,X231,X331,X112,X122,X132,X222,X232,X332 \ge 0$$

Objective function becomes:

Minimize 20 (X111**+** X121**+** X131+ X221 + X231+X331)
+25 (X112**+** X122**+** X132+ X222 + X232+X332)
+3 (X121+ X122+ X231**+** X232)+**6 (**X131+ X132)

12 variables and 9 constraints. So this is not efficient compared to second formulation

Graphical LP Solution

The graphical procedure includes two steps:

1. Determination of the solution space that defines *all* feasible solutions of the model.
2. Determination of the optimum solution from among all the feasible points in the solution space.

Graphical LP Solution of theReddy Mikks Problem

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Determination of the Feasible Solution Space:

- The horizontal axis *X1* and the vertical axis *X2* represent the exterior- and interior-paint variables, respectively.
  - First, we account for the nonnegativity constraints $X1 \geq 0$ and $X2 \geq 0$.
  - To account for the remaining four constraints, first replace each inequality with equations and then graph the resulting straight line by locating two distinct points on it.

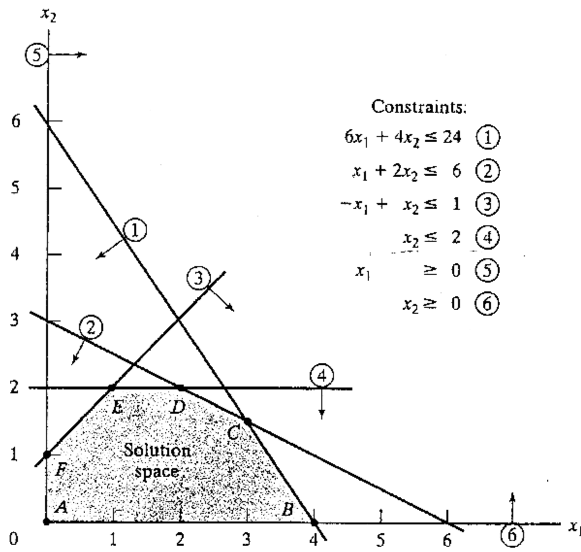
For example, after replacing $6X1 + 4X2 \leq 24$ with

$$6X1 + 4X2 = 24,$$

two distinct points that, the line passes through (0,6) and (4,0), as shown by line (1)

Graphical LP Solution of theReddy Mikks Problem

- Next, consider the effect of the inequality. To determine the correct side, choose any *reference point (*select (0, 0) as the reference) in the first quadrant. if it satisfies the inequality, then the side in which it lies is the feasible half-space. Else, the other side is.
- Application of the reference point procedure to all the constraints of the , model produces the feasible space shown in Figure



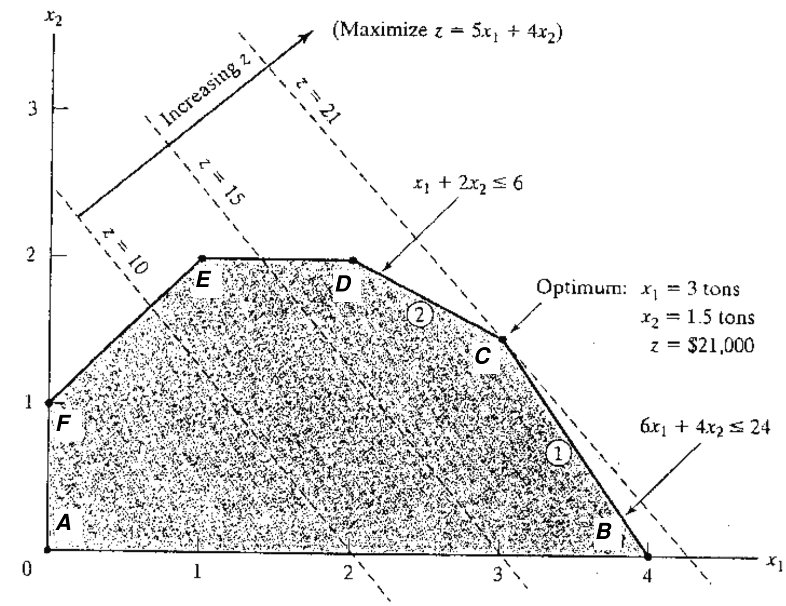
Determination of the optimum solution:

- The feasible space in Figure is delineated by the line segments joining the corner points *A, B,* C, *D, E,* and *F.* Any point within or on the boundary is feasible.
- The determination of the optimum solution requires identifying the direction in which the profit function **z = 5X1 + 4X2** increases (recall that we are *maximizing z).*
- We can do so by assigning *arbitrary* increasing values to *z.* For example, using *z* = 10 and *z* = 15 would be equivalent to graphing the two lines

  5X1 + *4X2* = 10 and

  5xı + *4X2* = 15.

Thus, the direction of increase in *z* is as shown Figure. The optimum solution occurs at C, which is the point in the solution space beyond which any further increase in *z* will put us outside the boundaries of *ABCDEF.*



Mathematical Programming

$$\text{Solving the LP with} \left\{\begin{array}{c}\text{Excel}\\ \text{The Simplex Algoritm}\end{array}\right\}$$