Solving Dual Model

- If the primal is a maximization problem, and the dual, of course, is a minimization problem

Example

Primal Problem: A plant makes two major products. Each of the products requires a certain quantity of raw materials and yields different profits. The pertinent information is summarized below.

	Product		
Resource	**1 (X_1)**	***2(X_2)***	**Resource Availability**
Raw material A	**1.5**	**2**	**11**
Raw material B	**1**	**2**	**8**
Profit	**10**	**24**	

- The objective function is to maximize profit,

$Z=10X_1 +24 X_2$

- Subject to the following constraints

$1.5 X_1 +2 X_2 \leq 11$ (A raw materials)

$1 X_1 + 2X_2 \leq 8$ (B raw materials)

$X_1, X_2 \geq 0$ (positivity)

Simplex Solution to Primal Problem

The simplex tableau for the problem can be set up and solved as

Initial Step

Basis	***Z***	**X_1**	**X_2**	***S_1***	S_2	**Solution**	***Intercept***
Z	**1**	**-10**	**-24**	0	0	0	
S_1	0	**1.5**	**2**	**1**	0	**11**	**5.5**
S_2	0	**1**	**2**	0	**1**	**8**	**4**

Second Step

Basis	***Z***	**X_1**	**X_2**	***S_1***	S_2	**Solution**	***Intercept***
Z	**1**	**2**	**0**	0	**12**	**96**	
S_1	0	**0.5**	**0**	**1**	**-1**	**3**	
X_2	0	**1/2**	**2/2**	0	**1/2**	**4**	

- Result is final, since there is no negative coefficient remains in the objective function row.
- The final solution is $X_2= 4$, $S_1= 3$ which gives the maximum objective function of $Z= 96$.

Since $S_1= 3$ means that 3 units of raw material A is not used and

$X_2= 4$ means that 4 unit of product 2 (X_2) is produced.

Simplex Solution to the Dual Problem

Dual Problem: Minimize

- The objective function is to minimize cost,

$Z=11\ Y_1 + 8\ Y_2$

- Subject to the following constraints

$1.5\ Y_1 + 1\ Y_2 \geq 10$ (cost of Product 1(X_1))

$2\ Y_1 + 2Y_2 \geq 24$ (cost of Product 2(X_2))

$Y_1, Y_2 \geq 0$ (positivity)

This is seen in the Simplex tableau

			Choice variable		***Surplus variable***		***Artifical* variable**		
Basic Variables	***Eq.* Number**	*Z*	Y_1	Y_2	Y_3	Y_4	Y_5	Y_6	**RHS**
Z	**0**	**1**	**-11**	**-8**	***0***	**0**	-100	-100	0
	1	0	**1.5**	**1**	***-1***	0	**1**	0	**10**
	2	*0*	*2*	*2*	*0*	***-1***	*0*	*1*	***24***

- The Row 0 has coefficients of –100 for Y_5, and Y_6. These will have to be replaced by zeros in the order for the initial basic solution to the problem to be read from RHS column.

- Add 100 times to Row 1 to Row 0, and add 100 times to Row 2 to Row 0, gives us.

			Choice variable		***Surplus variable***		***Artifical* variable**		
Basic Variables	***Eq.* Number**	*Z*	Y_1 ↓	Y_2	Y_3	Y_4	Y_5	Y_6	**RHS**
Z	**0**	**1**	**339**	**292**	***-100***	***-100***	0	0	3400
Y_1	**1**	0	**1.5/1.5**	**1/1.5**	***-1*/1.5**	0	**1/1.5**	0	**10/1.5**
Y_6	*2*	*0*	*2*	*2*	*0*	***-1***	*0*	*2*	***24***

Initial Step

			Choice variable		***Surplus variable***		***Artifical* variable**		
Basic Variables	***Eq.* Number**	*Z*	Y_1	Y_2	Y_3 ↓	Y_4	Y_5	Y_6	**RHS**
Z	**0**	**1**	**0**	**66**	***126***	**-100**	**-226**	0	1140
Y_1	**1**	0	**1**	**0.667**	***-0.667***	0	**0.667**	0	**6.667**
← **Y_3**	*2*	*0*	***0***	***0.667***	*1.334*	***-1***	***-1.334***	*2*	***10.667***

Second Step

			Choice variable		*Surplus variable*		*Artifical* variable		
Basic Variables	***Eq.* Number**	*Z*	Y_1	Y_2 ↓	Y_3	Y_4	Y_5	Y_6	**RHS**
Z	**0**	**1**	**189**	**192**	***0***	**-100**	**26**	0	2400
← Y_2	**1**	0	**-1.5**	**-1**	*1*	0	**-1**	0	**-10**
Y_3	*2*	*0*	*2*	*2*	*0*	***-1***	*0*	*2*	***24***

Third Step

			Choice variable		*Surplus variable*		*Artifical* variable		
Basic Variables	***Eq.* Number**	*Z*	Y_1	Y_2 ↓	Y_3	Y_4	Y_5	Y_6	**RHS**
Z	**0**	**1**	**-3**	**0**	***0***	**-4**	**-100**	-192	**96**
← Y_2	**1**	0	**1/2**	**0**	***-1***	1/2	**1**	-1	**2**
Y_3	*2*	*0*	*1*	*1*	*0*	***-1/2***	*0*	***1***	***12***

- **Result is final, since there is no positive coefficient remains in the objective function row.**
- **The final solution is Y_2= 2 , Y_3= 12 which gives the maximum objective function of Z= 96.**

Since Y_2= 2 means that cost of 1 unit of raw material B is 2 and
Y_1=0 means that cost of raw materials is zero, which means we have more raw material A than we need.

Change in Unit Profits *(c_j)*

- If the variable is not in the solution, it means that the constant ***(c_j)*** of this variable is not big enough.
    - The decrease in this constant will not affect the optimal solution.
    - When ***c_j-Z_j=0,*** it will not effect the value of objective function, but optimal solution will be changed.
    - When ***c_j>Z_j*** (Z_1 –c_1=2, so Z=12), optimal solution and its value will be effected.
- If the variable is in the solution,
    - The increase in constant will increase the value of objective function.
    - The decrease in the constant would change the optimal solution. Another variable can be in the solution.

  Let's find the range, which not affect the optimal solution.
  For Example, C_2=24+Δ

Basis	*Z*	X_1	X_2 ↓	S_1	S_2	**Solution**	***Intercept***
Z	**1**	**-10**	-24-Δ	0	0	0	
S_1	0	**1.5**	**2**	**1**	0	**11**	
← S_2	0	**1/2**	**2/2**	0	**1/2**	**8/2**	

Basis	*Z*	X_1	X_2	S_1	S_2	Solution	*Intercept*
Z	**1**	**2+1/2Δ**	**0**	0	**12+1/2Δ**	**96+4Δ**	
S_1	0	**0.5**	**0**	**1**	**-0.5**	**3**	
X_2	0	**1/2**	**2/2**	0	**1/2**	**4**	

Let's find the Δ, in order to X_1 and S_2 (X_4) be in the solution

$X_1 = 2+1/2\Delta = 0$, so $\Delta=-4$

and $S_4 = 12+1/2\Delta = 0$, so $\Delta=-24$

Thus when $C_2=20$, X_1 and

$C_2=0$, X4 will be in the solution.

Therefore $C_2>20$ will not affect the solution,

But when $C_2<20$ either X_1 or X_4 will be in the solution, and effect the optimal solution.

Change in Resource input (b_i)

By varying the b_i values affects the optimal output.

Let's look at previous example:

- When the slack variable is in the solution, corresponding resource have surplus.
  - Since S_1 is in the solution, increase in b_1 will not be effected optimal solution.
  - Since $S_1= 3$, therefore when the unit of less than 3 is deducted from b_1, it will not effect the solution.
  - But if 3 or more units deducted from b_1, optimal solution will change.
- When the slack variable is not in the solution, corresponding resource is used up.
  - If we get slack variable S_2 into solution and put the S_1 and X_2 out of solution
  - We can calculate the intercepts, these are equal to3/-1= -3 and 4/0.5= 8.
  - If $b_2-8 < b_2 < b_2+3$ $(0< b_2 < 11)$in this range optimal solution will be same.

EXAMPLE Problem I

Gucci Firm produces purses, shaving kit bags and rucksacks. Although each product is made of leather and synthetic material, leather is a limited raw material. During production process, two people are needed, a skilled sewer and a polisher. In the table below, raw material usage amounts for each unit of product and unit selling prices are shown. Find the optimum production amount of each product to maximize the total revenue.

Source	Raw Material Usage Amounts For Each Unit			Daily Usage
	Purse	Shaving Kit Bag	Rucksack	
Leather(unit^2)	2	1	3	42
Sewing(hour)	2	1	2	40
Polishing(hour)	1	0,5	1	45
Price	24	22	45	

Problem I

SOLUTION:

x_1 **:** daily purse production
x_2 **:** daily shaving kit bag production
x_3 **:** daily rucksack production

Max $z = 24x_1 + 22x_2 + 45x_3$

Constraints:

$$2\,x_1 + x_2 + 3\,x_3 \leq 42$$
$$2\,x_1 + x_2 + 2\,x_3 \leq 40$$
$$x_1 + 0{,}5x_2 + x_3 \leq 45$$
$$x_{1,}\, x_{2,}\, x_3 \geq 0$$

Problem I

Max $z = 24x_1 + 22x_2 + 45x_3 + 0s_1 + 0s_2 + 0s_3$

$2x_1 + x_2 + 3x_3 + s_1 = 42$

$2x_1 + x_2 + 2x_3 + s_2 = 40$

$x_1 + 0{,}5x_2 + x_3 + s_3 = 45$

Basic	z	x1	x2	x3	s1	s2	s3	Solution	Ratio
z	1	-24	-22	-45	0	0	0	0	0
s1	0	2	1	3	1	0	0	42	14
s2	0	2	1	2	0	1	0	40	20
s3	0	1	0,5	1	0	0	1	45	45

Entering Variable: x_3 is selected as the pivot column because it has the most negative coefficient in the z Row. s_1 row is selected as the pivot row since it has the minimum positive ratio. The new basic solution is maintained by Gauss-Jordan elimination method.

Problem I

Basic	z	x1	x2	x3	s1	s2	s3	Solution	Ratio
z	1	6	-7	0	15	0	0	630	0
x3	0	2/3	1/3	1	1/3	0	0	14	42
s2	0	2/3	1/3	0	- 2/3	1	0	12	36
s3	0	1/3	1/6	0	- 1/3	0	1	31	186

We still have a negative coefficient in the z Row. This means that we did not reach the optimum solution yet. The new pivot column and pivot row is selected with the same logical approach.

Problem I

Basic	z	x1	x2	x3	s1	s2	s3	Solution
z	1	22	0	0	1	21	0	882
x3	0	0	0	1	1	-1	0	2
x2	0	2	1	0	-2	3	0	36
s3	0	0	0	0	0	-0,5	1	25

Since there is no negative coefficient in the z row we can say that the solution is optimum. In the optimum solution

Max $z = 882$, $x_2 = 36$, $x_3 = 2$, and $s_3 = 25$

$$2\,x_1 + x_2 + 3\,x_3 + s_1 = 42$$
$$2\,x_1 + x_2 + 2\,x_3 + s_2 = 40$$
$$x_1 + 0{,}5x_2 + x_3 + s_3 = 45$$

From the last equation we find $x_1 = 0$. Also $s_1 = 0$ and $s_2 = 0$.

The maximum total revenue is 882. This can be maintained by producing 36 shaving kit bags, 2 rucksacks and no purses daily.

Problem I

SENSITIVITY ANALYSIS

BV(Basic Variables) = { x_3, x_2, s_3 }
NBV(Non Basic Variables)= { x_1,s_1,s_2}.

$$c_j' = C(BV)*B^{-1}*a_j - c_j$$

c_j' : coefficient of x_j in the optimal table's z row.
a_j : column in the constraints for the variable x_j.
C(BV) : 1*m vector [$C(VB_1)$, $C(VB_2)$, $C(VB_3)$,….$C(VB_m)$]
B : m*m matrix whose j^{th} column is the column for BV_j.

Problem I

Changing the right hand side of a constraint:

If we change the amount of sewing hour's b2 to b2+Δ, the right hand side of the constraints in the optimal table will become:

$$B^{-1} \text{ x } \begin{bmatrix} 42 \\ 40+\Delta \\ 45 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 \\ -2 & 3 & 0 \\ 0 & -0.5 & 1 \end{bmatrix} \begin{bmatrix} 42 \\ 40+\Delta \\ 45 \end{bmatrix} = \begin{bmatrix} 2 - \Delta \\ 36+3\Delta \\ 25-0.5\Delta \end{bmatrix}$$

The right hand side of each constraint in the optimal table should remain nonnegative.

$2 - \Delta \geq 0$ $(\Delta \leq 2)$
$36 + 3\Delta \geq 0$ $(\Delta \geq -12)$
$25 - 0.5\Delta \geq 0$ $(\Delta \leq 50)$

If $-12 \leq \Delta \leq 2$ the current basis remains feasible and therefore optimal.

Problem I

Changing the objective function coefficient of a basic variable:

Changing the coefficient of x_3 would affect the optimal solution of the problem. If c_3 is changed to 45+Δ from 45, C(BV) will be changed to [45+Δ 22 0]. B^{-1} can be found by Gauss-Jordan method:

$$B^{-1} = \begin{bmatrix} 1 & -1 & 0 \\ -2 & 3 & 0 \\ 0 & -0.5 & 1 \end{bmatrix}$$

If $c_3 = 45+\Delta$

$C(VB)^* B^{-1} = [\ 1+\Delta \quad 21-\Delta \quad 0]$

Coefficients of x_3, x_2, s_3 in the z row must still be 0 in the optimum solution.

Problem I

NBV in the new z row is

$$c_1' = C(BV)*B^{-1}*a_1 - c_1 = [\,1+\Delta \quad 21-\Delta \quad 0\,] * \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix} - (24) = 20$$

$s_1 = 1 + \Delta$
$s_2' = 21 - \Delta$

Z row of the optimal table is now:
$z + 20 * x_1 + (1 + \Delta)\, s_1 + (21 - \Delta)\, s_2$

BV will remain optimal since s_1 and $s_2 \geq 0 \rightarrow -1 \leq \Delta \leq 21$

Problem I

Changing the objective function coefficient of a nonbasic variable:

NB decision variable is x_1. The coefficient of x_1 (c_1) is 24. We should find the values of Δ that will make the current set of basic variables remain optimal while c_1 is 24+Δ instead of 24.

$$c_1' = [45\ 22\ 0] \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix} - (24+\Delta) = 110 - \Delta$$

For $c_1' \geq 0$ the solution remains optimal. $110 - \Delta \geq 0 \quad \Delta \leq 110$. This is, if $c_1 \leq$ 24+110 = 134 then BV remains optimal.