Regularized Linear Regression

Solving the Problem of Overfitting Regularization

Regularized linear regression

$$J(\theta) = \frac{1}{2m} \left[\sum_{i=1}^{m} (h_{\theta}(x^{(i)}) - y^{(i)})^2 + \lambda \sum_{j=1}^{n} \theta_j^2 \right]$$

$$\min_{\theta} J(\theta)$$

Repeat {

$$\theta_{j} := \theta_{j} - \alpha \qquad \frac{1}{m} \sum_{i=1}^{m} (h_{\theta}(x^{(i)}) - y^{(i)}) x_{j}^{(i)}$$

$$(j = 0, 1, 2, 3, \dots, n)$$

Repeat {

$$\theta_0 := \theta_0 - \alpha \frac{1}{m} \sum_{i=1}^m (h_\theta(x^{(i)}) - y^{(i)}) x_0^{(i)}$$

$$\theta_{j} := \theta_{j} - \alpha \qquad \frac{1}{m} \sum_{i=1}^{m} (h_{\theta}(x^{(i)}) - y^{(i)}) x_{j}^{(i)}$$

$$(j = \mathbb{X}, 1, 2, 3, \dots, n)$$

}

O.



Repeat {

$$\rightarrow \theta_0 := \theta_0 - \alpha \frac{1}{m} \sum_{i=1}^m (h_\theta(x^{(i)}) - y^{(i)}) x_0^{(i)}$$

$$\Rightarrow \theta_j := \theta_j - \alpha \left[\frac{1}{m} \sum_{i=1}^m (h_\theta(x^{(i)}) - y^{(i)}) x_j^{(i)} + \frac{\lambda}{m} \Theta_j \right]$$

$$(j = \mathbf{X}, 1, 2, 3, \dots, n)$$

}

O.

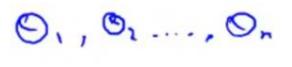
O,, O, ..., O,

Repeat {

$$\rightarrow \theta_0 := \theta_0 - \alpha \frac{1}{m} \sum_{i=1}^m (h_\theta(x^{(i)}) - y^{(i)}) x_0^{(i)}$$

$$\Rightarrow \theta_j := \theta_j - \alpha \underbrace{\left[\frac{1}{m} \sum_{i=1}^m (h_{\theta}(x^{(i)}) - y^{(i)}) x_j^{(i)} + \frac{\lambda}{m} \Theta_j\right]}_{(j = \mathbf{X}, \underline{1, 2, 3, \dots, n})}$$

0.



Repeat {

$$\theta_0 := \theta_0 - \alpha \frac{1}{m} \sum_{i=1}^m (h_\theta(x^{(i)}) - y^{(i)}) x_0^{(i)}$$

$$\theta_{j} := \theta_{j} - \alpha \left[\frac{1}{m} \sum_{i=1}^{m} (h_{\theta}(x^{(i)}) - y^{(i)}) x_{j}^{(i)} + \frac{\lambda}{m} \Theta_{j} \right]$$

$$(j = \mathbf{X}, 1, 2, 3, \dots, n)$$

$$\theta_j := \theta_j (1 - \alpha \frac{\lambda}{m}) - \alpha \frac{1}{m} \sum_{i=1}^m (h_\theta(x^{(i)}) - y^{(i)}) x_j^{(i)}$$

Exercise

• Suppose you are doing gradient descent on a training set of m>0 examples, using a fairly small learning rate $\alpha>0$ and some regularization parameter $\lambda>0$. Consider the update rule:

$$\theta_j := \theta_j (1 - \alpha \frac{\lambda}{m}) - \alpha \frac{1}{m} \sum_{i=1}^m (h_{\theta}(x^{(i)}) - y^{(i)}) x_j^{(i)}$$

- Which of the following statements about the term $\left(1-\alpha\frac{\lambda}{m}\right)$ is true?
 - $1 \alpha \frac{\lambda}{m} > 1$
 - $1 \alpha \frac{\lambda}{m} < 1$
 - $1 \alpha \frac{\lambda}{m} = 0$





Repeat {

$$\rightarrow \theta_0 := \theta_0 - \alpha \frac{1}{m} \sum_{i=1}^m \theta_i$$

$$\alpha \frac{1}{m} \sum_{i=1}^{m} (h_{\theta}(x^{(i)}) - y^{(i)}) x_0^{(i)}$$

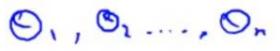
$$\theta_{j} := \theta_{j} - \alpha \left[\frac{1}{m} \sum_{i=1}^{m} (h_{\theta}(x^{(i)}) - y^{(i)}) x_{j}^{(i)} + \frac{\lambda}{m} \Theta_{j} \right]$$

$$(j = \mathbf{X}, 1, 2, 3, \dots, n)$$

$$\theta_j := \theta_j (1 - \alpha \frac{\lambda}{m}) - \alpha \frac{1}{m} \sum_{i=1}^m (h_\theta(x^{(i)}) - y^{(i)}) x_j^{(i)}$$

$$\left|-\frac{\lambda}{m}\right|$$

<u>O</u>.



Repeat {

$$\theta_0 := \theta_0 - \alpha \frac{1}{m} \sum_{i=1}^m (h_\theta(x^{(i)}) - y^{(i)}) x_0^{(i)}$$

$$\theta_j := \theta_j - \alpha$$

$$\frac{1}{m} \sum_{i=1}^{m} (h_{\theta}(x^{(i)}) - y^{(i)}) x_j^{(i)}$$



$$\theta_j := \theta_j (1 - \alpha \frac{\lambda}{m}) - \alpha \frac{1}{m} \sum_{i=1}^m (h_\theta(x^{(i)}) - y^{(i)}) x_j^{(i)}$$



$$X = \begin{bmatrix} (x^{(1)})^T \\ \vdots \\ (x^{(m)})^T \end{bmatrix}$$

$$\min_{\theta} J(\theta)$$

$$y = \begin{bmatrix} y^{(1)} \\ \vdots \\ y^{(m)} \end{bmatrix}$$

$$X = \begin{bmatrix} (x^{(1)})^T \\ \vdots \\ (x^{(m)})^T \end{bmatrix} \leftarrow \min_{\theta} J(\theta)$$

$$y = \begin{bmatrix} y^{(1)} \\ \vdots \\ y^{(m)} \end{bmatrix} \qquad \mathbb{R}^{n}.$$

$$y = \begin{bmatrix} y^{(1)} \\ \vdots \\ y^{(m)} \end{bmatrix} \qquad \mathbb{R}^{n}$$

$$X = \begin{bmatrix} (x^{(1)})^T \\ \vdots \\ (x^{(m)})^T \end{bmatrix} \leftarrow$$

$$\Rightarrow \min_{\theta} J(\theta)$$

$$\Rightarrow 0 = (X^T X)$$

$$y = \begin{bmatrix} y^{(1)} \\ \vdots \\ y^{(m)} \end{bmatrix} \quad \mathbb{R}^{n}$$

$$\frac{\partial}{\partial \phi_{5}} \quad \mathcal{J}(\phi) \stackrel{\text{Set}}{=} 0 \quad \text{and} \quad \mathcal{J}$$

$$\int_{-1}^{1} x^{T} dx$$

$$X = \begin{bmatrix} (x^{(1)})^T \\ \vdots \\ (x^{(m)})^T \end{bmatrix}$$

$$\Rightarrow \min_{\theta} J(\theta)$$

$$\Rightarrow 0 = (X^T \times + \lambda) \begin{bmatrix} y^{(1)} \\ \vdots \\ y^{(m)} \end{bmatrix}$$

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$$\Rightarrow \min_{\theta} J(\theta)$$

$$\Rightarrow 0 = (x^T \times + \lambda) \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

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Andrew Ng

Non-invertibility (optional/advanced).

If
$$\lambda > 0$$
,
$$\theta = \left(X^T X + \lambda \begin{bmatrix} 0 & & & \\ & 1 & & \\ & & \ddots & \\ & & \ddots & \\ & & & 1 \end{bmatrix} \right)^{-1} X^T y$$

Non-invertibility (optional/advanced).

Suppose
$$m \le n$$
, \leftarrow (#examples) (#features)

$$\theta = \underbrace{(X^T X)^{-1} X^T y}_{\text{Non-invertible / singular}}$$

If
$$\lambda > 0$$
,
$$\theta = \left(X^T X + \lambda \begin{bmatrix} 0 & 1 & 1 & 1 \\ & 1 & & \\ & & \ddots & 1 \end{bmatrix}\right)^{-1} X^T y$$

Summary

```
Repeat {  \theta_0 := \theta_0 - \alpha \, \, \frac{1}{m} \, \, \sum_{i=1}^m (h_\theta(x^{(i)}) - y^{(i)}) x_0^{(i)}   \theta_j := \theta_j - \alpha \, \left[ \left( \frac{1}{m} \, \, \sum_{i=1}^m (h_\theta(x^{(i)}) - y^{(i)}) x_j^{(i)} \right) + \frac{\lambda}{m} \, \theta_j \right]  j \in \{1, 2...n\} }
```

Summary

• With some manipulation our update rule can also be represented as:

$$\theta_j := \theta_j \big(1 - \alpha \frac{\lambda}{m}\big) - \alpha \frac{1}{m} \sum_{i=1}^m \big(h_\theta\big(x^{(i)}\big) - y^{(i)}\big) x_j^{(i)}$$

- Intuitively you can see it as reducing the value of θj by some amount on every update.
- Notice that the second term is now exactly the same as it was before.

Summary

• For normal equation, we have the following:

$$heta = \left(X^T X + \lambda \cdot L
ight)^{-1} X^T y$$
 where $L = \left[egin{array}{ccc} 0 & & & & \ & 1 & & & \ & & 1 & & \ & & \ddots & & \ & & & 1 \end{array}
ight]$

• Even though for m < n, X^TX is non-invertible, adding the term $\lambda \cdot L$, makes $X^TX + \lambda \cdot L$ invertible.