Geometry

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Objectives

- Introduce the elements of geometry
  - Scalars
  - Vectors
  - Points
- Develop mathematical operations among them in a coordinate-free manner
- Define basic primitives
  - Line segments
  - Polygons

Basic Elements

- Geometry is the study of the relationships among objects in an n-dimensional space
  - In computer graphics, we are interested in objects that exist in three dimensions
- Want a minimum set of primitives from which we can build more sophisticated objects
- We will need three basic elements
  - Scalars
  - Vectors
  - Points

Coordinate-Free Geometry

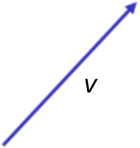
- When we learned simple geometry, most of us started with a Cartesian approach
  - Points were at locations in space **p**=(x,y,z)
  - We derived results by algebraic manipulations involving these coordinates
- This approach was nonphysical
  - Physically, points exist regardless of the location of an arbitrary coordinate system
  - Most geometric results are independent of the coordinate system
  - Example Euclidean geometry: two triangles are identical if two corresponding sides and the angle between them are identical

Scalars

- Need three basic elements in geometry
  Scalars, Vectors, Points
- Scalars can be defined as members of sets which can be combined by two operations (addition and multiplication) obeying some fundamental axioms (associativity, commutivity, inverses)
- Examples include the real and complex number systems under the ordinary rules with which we are familiar
- Scalars alone have no geometric properties

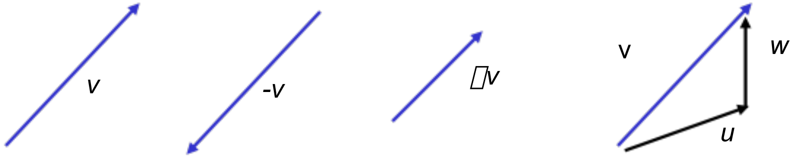
Vectors

- Physical definition: a vector is a quantity with two attributes
  - Direction
  - Magnitude
- Examples include
  - Force
  - Velocity
  - Directed line segments
    - Most important example for graphics
    - Can map to other types



Vector Operations

- Every vector has an inverse
  - Same magnitude but points in opposite direction
- Every vector can be multiplied by a scalar
- There is a zero vector
  - Zero magnitude, undefined orientation
- The sum of any two vectors is a vector
  - Use head-to-tail axiom



Linear Vector Spaces

- Mathematical system for manipulating vectors
- Operations

  Scalar-vector multiplication u=□v

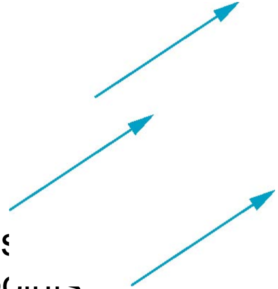
  Vector-vector addition: $w=u+v$
- Expressions such as

  $v=u+2w-3r$

Make sense in a vector space

Vectors Lack Position

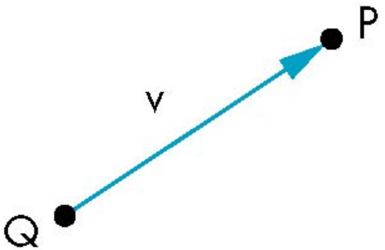
- These vectors are identical
  - Same length and magnitude



- Vectors s ent for geometry
  - Need points

Points

- Location in space
- Operations allowed between points and vectors
  - Point-point subtraction yields a vector
  - Equivalent to point-vector addition



v=P-Q

P=v+Q

Affine Spaces

- Point + a vector space
- Operations
  - Vector-vector addition
  - Scalar-vector multiplication
  - Point-vector addition
  - Scalar-scalar operations
- For any point define
  - $1 \cdot P = P$
  - $0 \cdot P = \mathbf{0}$ (zero vector)

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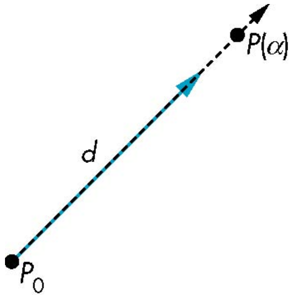
Lines

- Consider all points of the form

  $P(a)=P0 + a\,\mathbf{d}$

  Set of all points that pass through P0 in the direction of the vector **d**



Parametric Form

- This form is known as the parametric form of the line
  - More robust and general than other forms
  - Extends to curves and surfaces
- Two-dimensional forms
  - Explicit: $y = mx + h$
  - Implicit: $ax + by + c = 0$
  - Parametric:

$$x(a) = ax_0 + (1-a)x_1$$

$$y(a) = ay_0 + (1-a)y_1$$

Rays and Line Segments

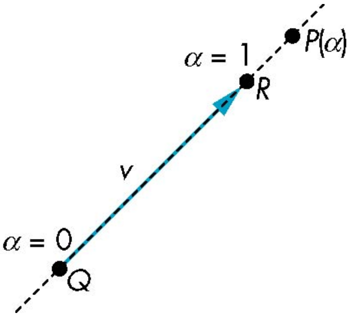
- If a >= 0, then P(a) is the *ray* leaving P0 in the direction **d**

If we use two points to define v, then

P(a) = Q + a (R-Q)=Q+av

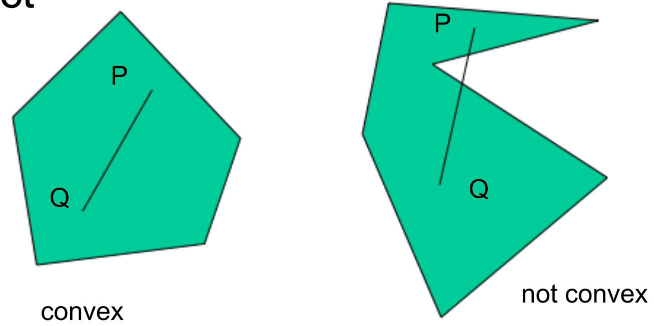
=aR + (1-a)Q

For 0<=a<=1 we get all the points on the *line segment* joining R and Q



Convexity

- An object is *convex* iff for any two points in the object all points on the line segment between these points are also in the object



Affine Sums

- Consider the "sum"

  $P=a_1P_1+a_2P_2+\ldots..+a_nP_n$

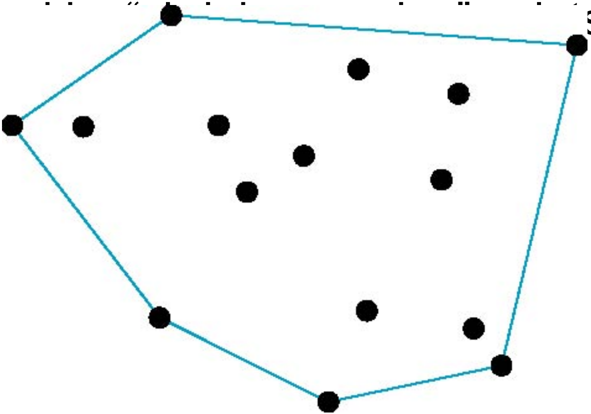
  Can show by induction that this sum makes sense iff

  $a_1+a_2+\ldots..a_n=1$

  in which case we have the *affine sum* of the points $P_1,P_2,\ldots..P_n$
- If, in addition, $a_i>=0$, we have the *convex hull* of $P_1,P_2,\ldots..P_n$

Convex Hull

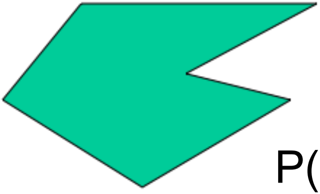
- Smallest convex object containing P1,P2,…..Pn
- Form



Curves and Surfaces

- Curves are one parameter entities of the form P(a) where the function is nonlinear
- Surfaces are formed from two-parameter functions P(a, b)
  - Linear functions give planes and polygons

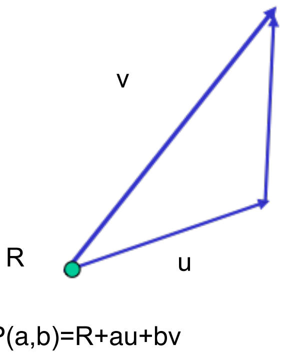
P(a)

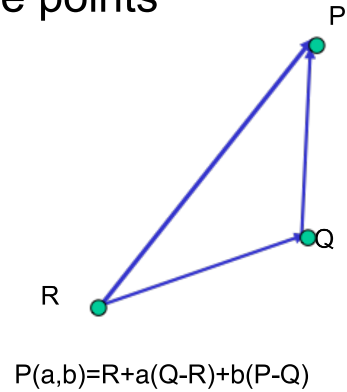


P(a, b)

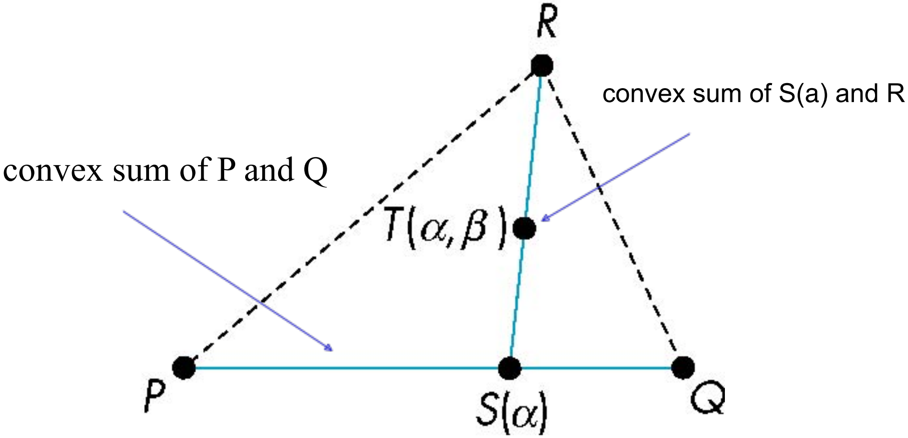
Planes

- A plane can be defined by a point and two vectors or by three points





Triangles



for 0<=a,b<=1, we get all points in triangle

Normals

- Every plane has a vector n normal (perpendicular, orthogonal) to it
- From point-two vector form $P(a,b)=R+au+bv$, we know we can use the cross product to find $n = u \times v$ and the equivalent form

$$(P(a)-P) \cdot n=0$$