Chapter 9-10

Hypothesis Testing and Confidence Intervals

CI for the mean with known variance

Statistics

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Confidence Interval for the Mean

CI for the mean – σ known

Confidence interval for μ

- $\bar{X} \sim N(\mu, \frac{\sigma}{\sqrt{n}})$,
  - If X is normal, then this follows
  - If X is not normal, then this still follows from Central Limit Theorem.

- Recall
  - $P(-z_{\alpha/2} < Z < z_{\alpha/2}) = 1 - \alpha$
  - $Z = \frac{\bar{X}-\mu}{\sigma/\sqrt{n}}$

CI for the mean – σ known

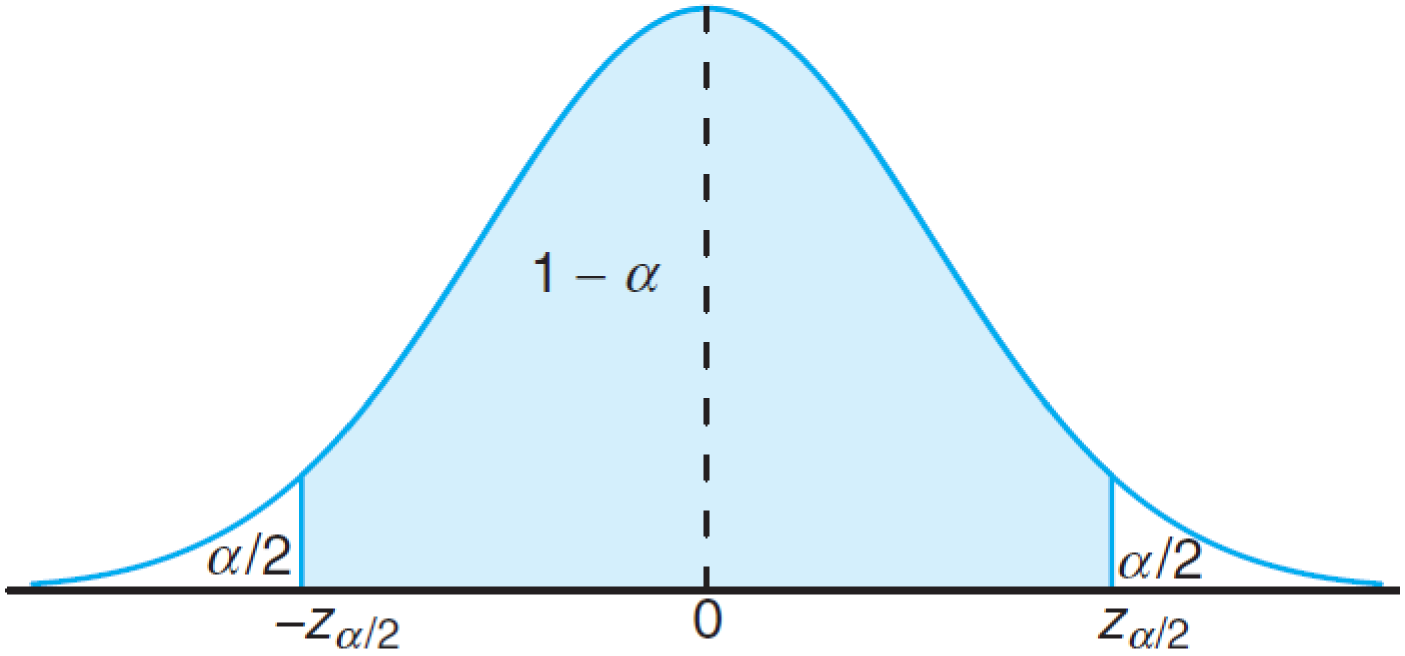


Figure 9.2: $P(-z_{\alpha/2} < Z < z_{\alpha/2}) = 1 - \alpha$.

CI for the mean – σ known

- Hence we have
- $P(-z_{\alpha/2} < \frac{\bar{X}-\mu}{\sigma/\sqrt{n}} < z_{\alpha/2}) = 1 - \alpha$
- After some algebra, can be re-written as:

$$P\left(\bar{X} - z_{\alpha/2}\frac{\sigma}{\sqrt{n}} < \mu < \bar{X} + z_{\alpha/2}\frac{\sigma}{\sqrt{n}}\right) = 1 - \alpha.$$

- Hence we can write the confidence interval as follows:

CI for the mean – σ known

If $\bar{x}$ is the mean of a random sample of size n from a population with known variance σ^2, a $100(1-\alpha)\%$ confidence interval for μ is given by

$$\bar{x} - z_{\alpha/2}\frac{\sigma}{\sqrt{n}} < \mu < \bar{x} + z_{\alpha/2}\frac{\sigma}{\sqrt{n}},$$

where $z_{\alpha/2}$ is the z-value leaving an area of $\alpha/2$ to the right.

Note that the value of $\bar{x}$ is in fact the $\bar{x}_{obs}$.

CI for the mean – σ known

Example:

- A random sample of 100 recorded deaths in Turkey during the past year showed an average life span of 71.8 years.
- Assuming a population standard deviation of 8.9 years, please construct a 95% CI for μ.

Solution: From Table A.3: $z_{\alpha\backslash 2} = z_{0.025} = 1.96$

- $71.8 - 1.96\frac{8.9}{\sqrt{100}} < \mu < 71.8 + 1.96\frac{8.9}{\sqrt{100}}$
- => $70.06 < \mu < 73.54$
- We are %95 confident that our real μ lies in this interval.

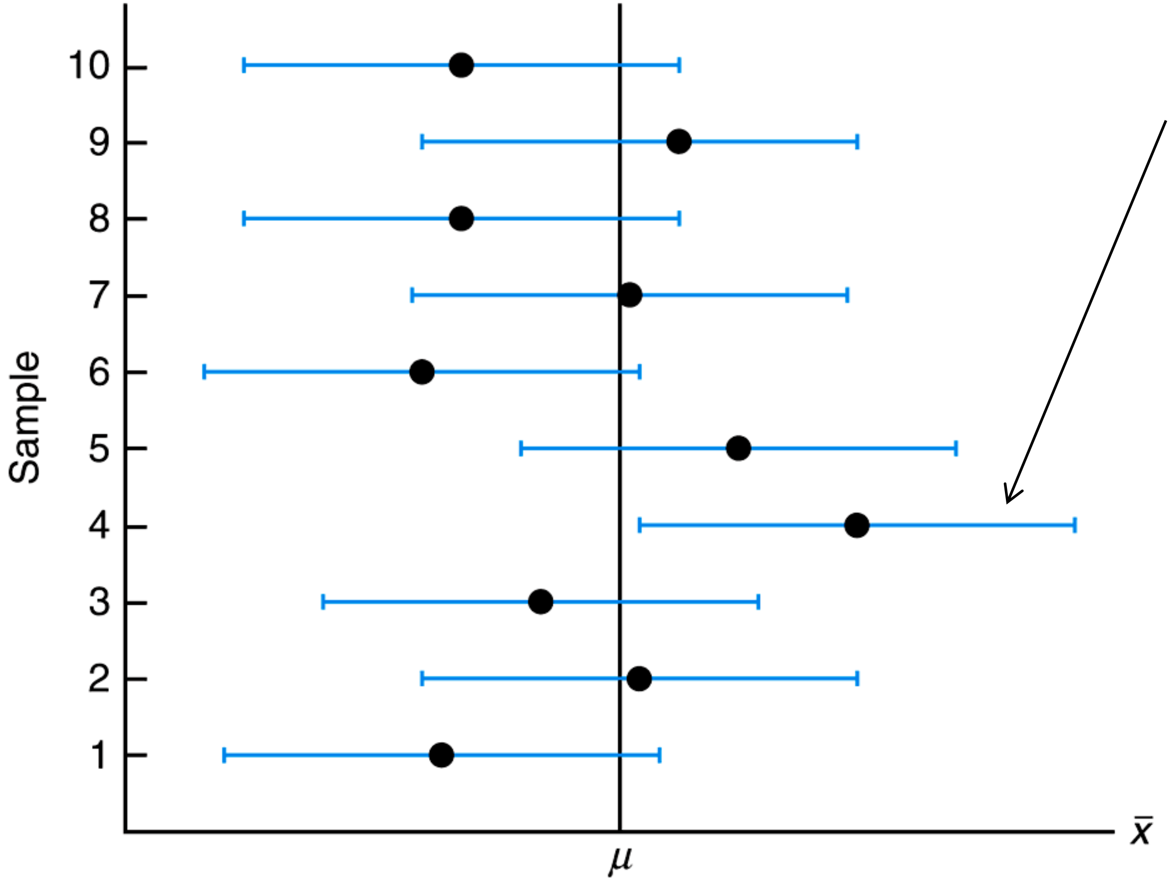
CI for the mean – σ known

Example (Ctd): Now construct a 90% confidence interval.

- First find $z_{\alpha\backslash 2} = z_{0.05} = 1.645$
- Then construct the confidence interval:
- $71.8 - 1.645 \frac{8.9}{\sqrt{100}} < \mu < 71.8 + 1.645 \frac{8.9}{\sqrt{100}}$
- => $70.34 < \mu < 73.26$
- compare with
- => $70.06 < \mu < 73.54$
- *If you want more confidence, then the interval becomes larger.*

Figure 7. Interval Estimates of μ for Different Samples

The upper and lower limits of a confidence interval are random variables, hence **each different sample** will yield a **different confidence interval** for μ.



Some of these intervals may even miss the true unknown) value of μ.

HT = CI !!!

- Return to our Hypothesis test question:
- Test whether
  - $H_0: \mu = 70$
  - $H_1: \mu \neq 70$

- In fact, we can draw the same conclusion by looking at the 95% CI:
  - $70.06 < \mu < 73.54$ ➔ 70 is not included in this CI.

- Exercise: Do the same thing for $\alpha = 0.10$

Error in Estimating μ by $\bar{x}$

Theorem says: With $(1-\alpha)\,100\%$ confidence the error ***e*** in estimating the population mean μ by using the sample mean can be at most equal to $z_{\alpha/2}\sigma/\sqrt{n}$

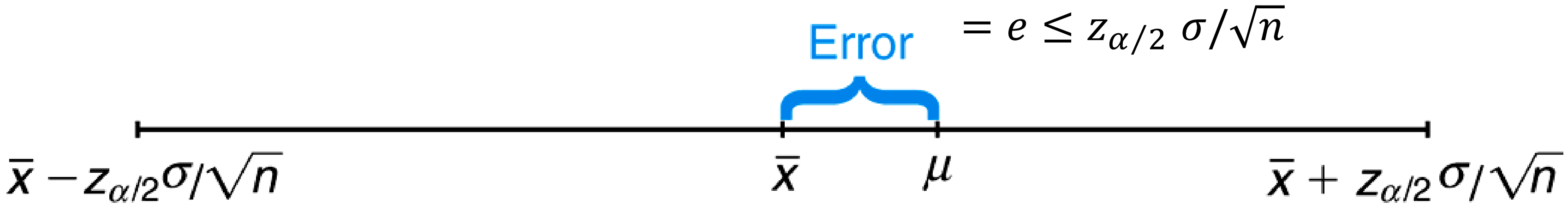


Figure 1. Confidence interval and the size of the error in estimating μ.

Estimating the Needed Sample Size for a Given Error Magnitude and Confidence Level

We will now rephrase the previous theorem as follows:

THEOREM. We can be $(1-\alpha)\ 100\%$ confident that the error in estimating the population mean μ by using the sample mean will not exceed a specified amount e when the **sample size** is taken as:

$$n = \left(\frac{z_{a/2}\, s}{e} \right)^2 .$$