A Wrap up for Normal Distribution

Statistics

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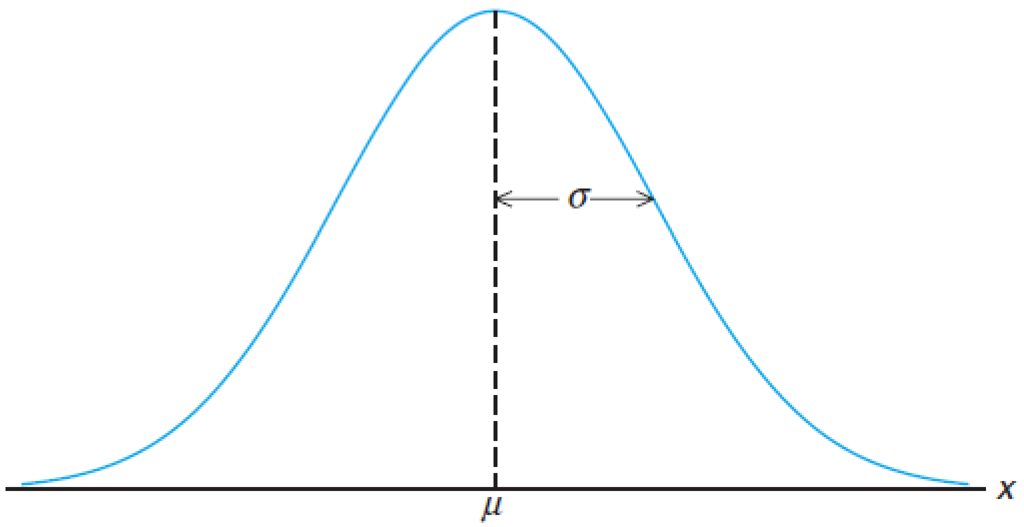
Normal Distribution

- The most important continuous probability distribution…
- It approximately describes many phenomena that occur in nature, industry, and research.
  - meteorological experiments,
  - rainfall studies
  - measurements of manufactured parts
- Sometimes referred as the Gaussian Distribution.

Normal Distribution

- The normal curve:



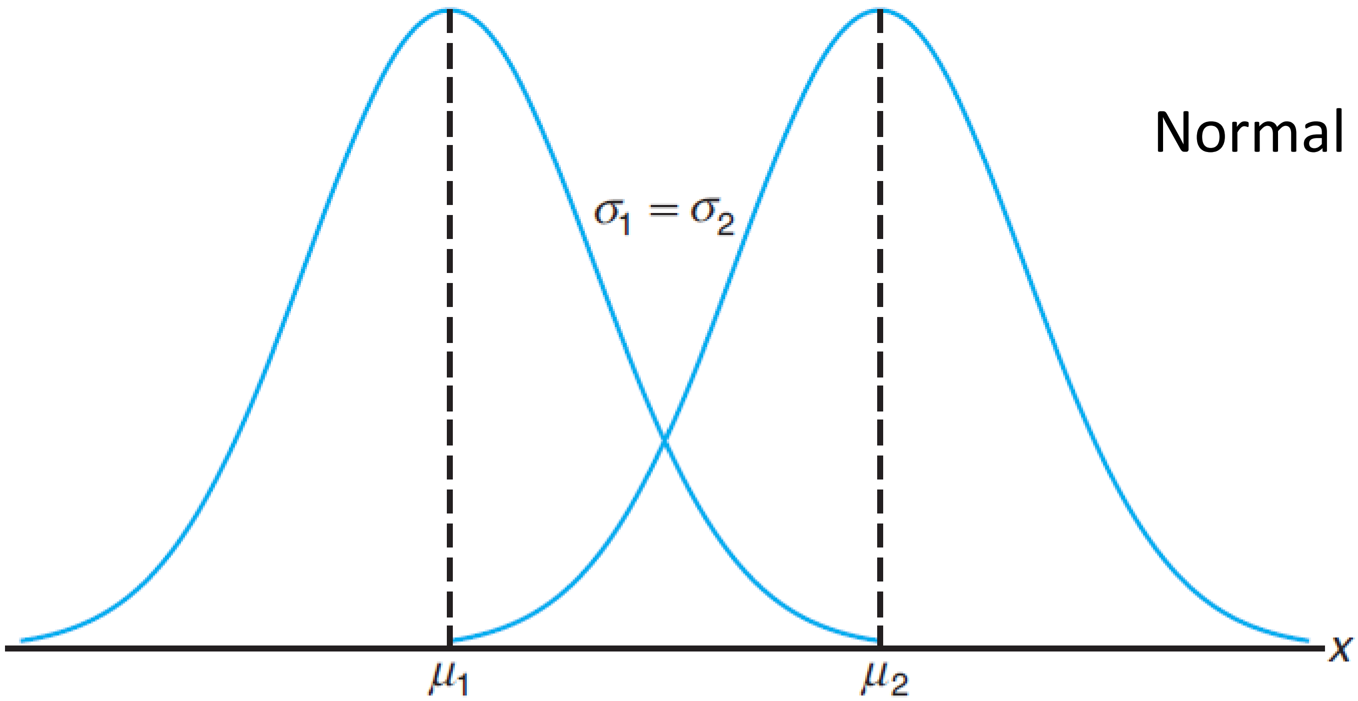
Normal Distribution

- A continuous RV X having the bell-shaped distribution is called a **normal random variable**.
- The mathematical equation for the probability distribution of the normal variable depends on the two parameters μ and σ, its mean and standard deviation, respectively.
- Hence, we denote the values of the density of X by $\boldsymbol{n(x;\, \mu,\, \sigma)}$.
- The density of the normal RV X, with mean μ and variance σ^2, is

$$f(x) = n(x;\mu,\sigma) = \frac{1}{\sqrt{2\pi\sigma}} e^{-\left(\frac{1}{2\sigma^2}\right)(x-\mu)^2} \qquad -\infty < x < \infty$$

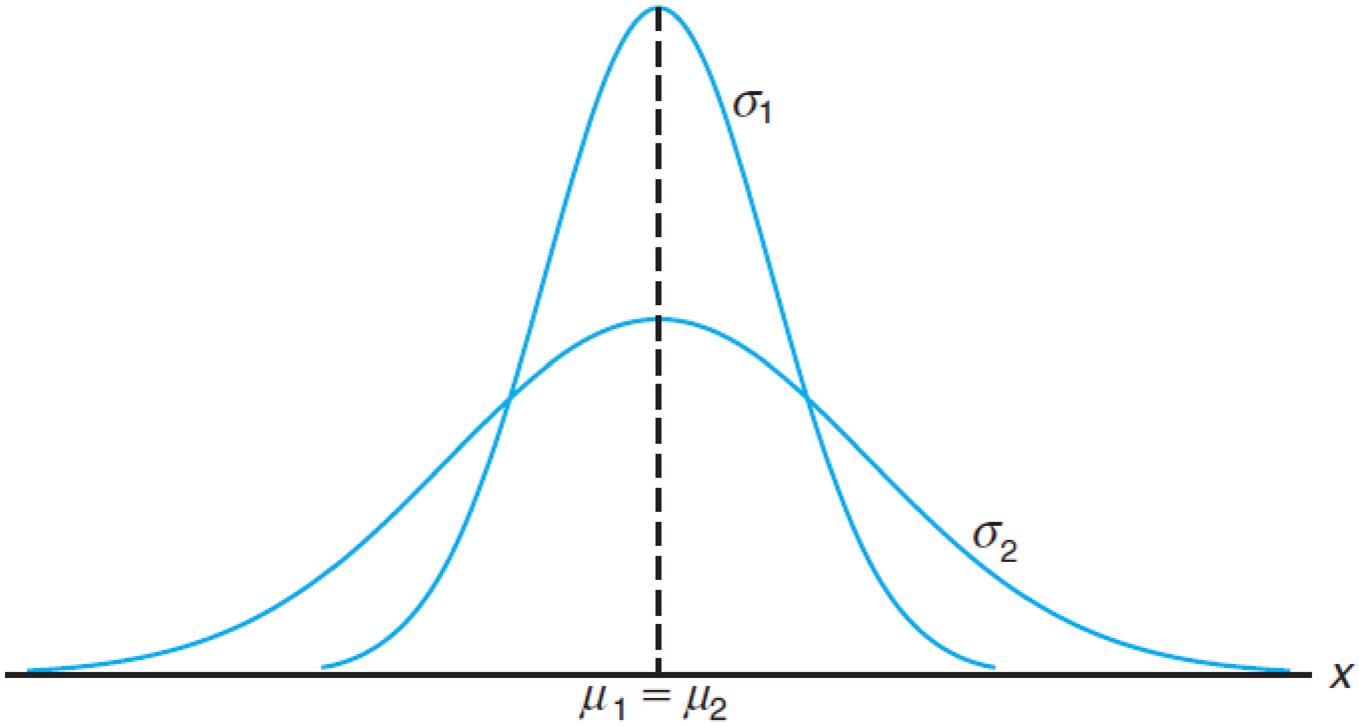
Normal Distribution

- Once μ and σ are specified, the normal curve is completely determined.
- For example, if $\mu = 50$ and $\sigma = 5$, then the ordinates $n(x; 50, 5)$ can be computed for various values of x and the curve drawn.



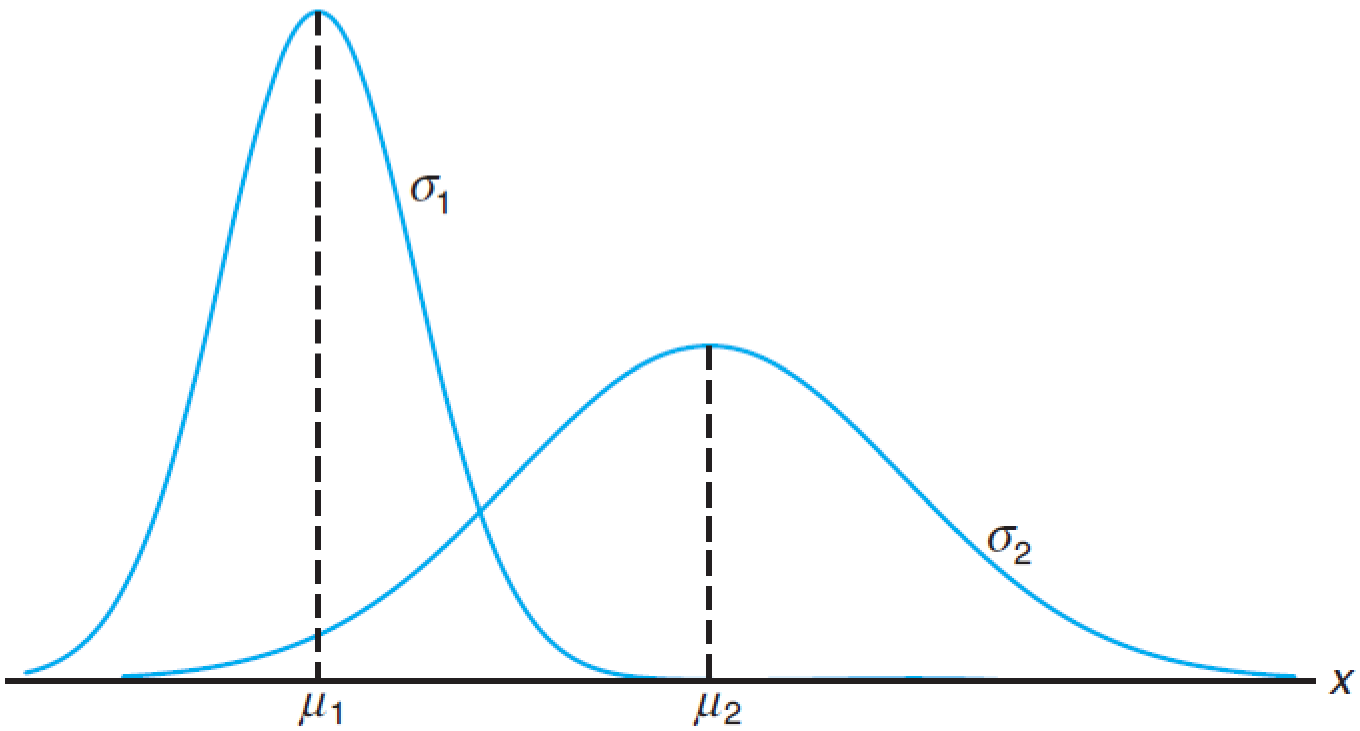
Normal curves with $\mu_1 < \mu_2$ and $\sigma_1 = \sigma_2$

Normal Distribution



Normal curves with
$\mu_1 = \mu_2$ *and* $\sigma_1 < \sigma_2$

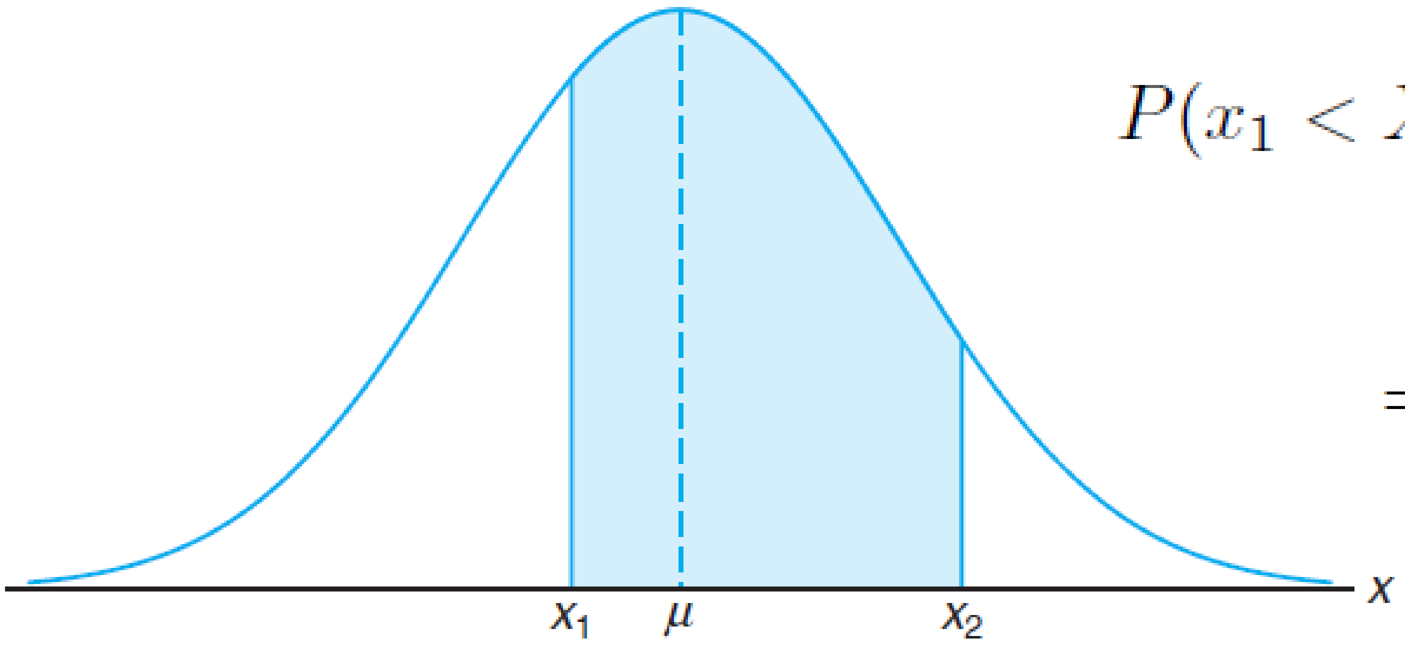
Normal Distribution



Normal curves with
$\mu_1 < \mu_2$ *and* $\sigma_1 < \sigma_2$

Normal Distribution

- The mean and variance of $n(x; \mu, \sigma)$ are μ and $\sigma 2$, respectively.
- Hence, the standard deviation is σ.



$$P(x_1 < X < x_2) = \int_{x_1}^{x_2} n(x; \mu, \sigma)\, dx$$

$$= \frac{1}{\sqrt{2\pi}\sigma} \int_{x_1}^{x_2} e^{-\frac{1}{2\sigma^2}(x-\mu)^2} dx$$

Figure 6.6: $P(x_1 < X < x_2) =$ area of the shaded region.

Normal Distribution

- Hard to find the integral!!!
- How to deal with it?
  - Standardize
  - Use tables

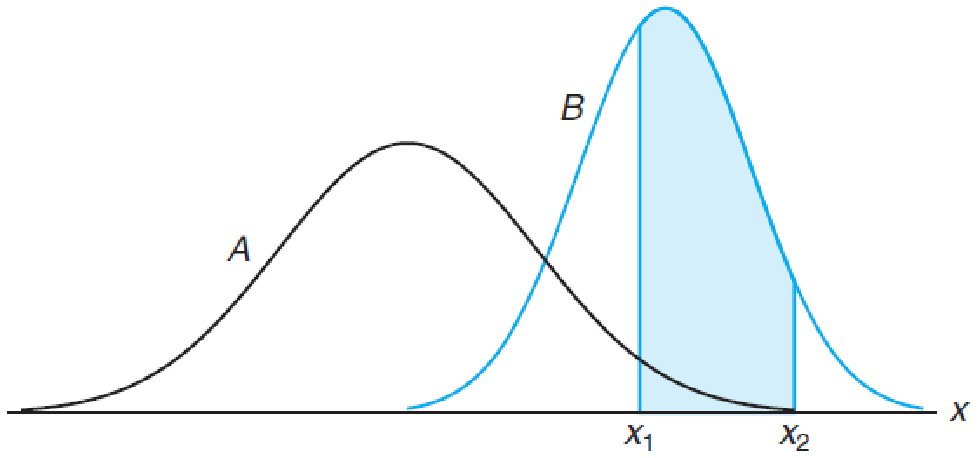


Figure 6.7: $P(x_1 < X < x_2)$ for different normal curves.

Normal Distribution

- The distribution of a normal random variable with mean 0 and variance 1 is called a **standard normal distribution**.
- We can convert all random variables into standard normal RV by:

$$Z = \frac{X - \mu}{\sigma}$$

- Whenever $X = x$, the corresponding value of Z=$z = (x - \mu)/\sigma$.
- Moreover we have:

$$P(x_1 < X < x_2) = P\left(\frac{x_1 - \mu}{\sigma} < \frac{X - \mu}{\sigma} < \frac{x_2 - \mu}{\sigma}\right) = P(z_1 < Z < z_2)$$

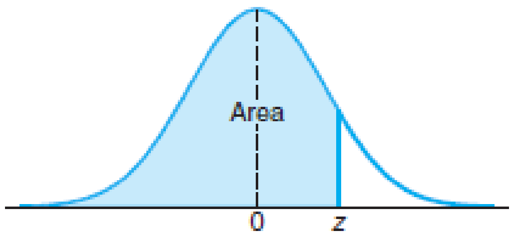


Table A.3 Areas under the Normal Curve

z	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
−3.4	0.0003	0.0003	0.0003	0.0003	0.0003	0.0003	0.0003	0.0003	0.0003	0.0002
−3.3	0.0005	0.0005	0.0005	0.0004	0.0004	0.0004	0.0004	0.0004	0.0004	0.0003
−3.2	0.0007	0.0007	0.0006	0.0006	0.0006	0.0006	0.0006	0.0005	0.0005	0.0005
−3.1	0.0010	0.0009	0.0009	0.0009	0.0008	0.0008	0.0008	0.0008	0.0007	0.0007
−3.0	0.0013	0.0013	0.0013	0.0012	0.0012	0.0011	0.0011	0.0011	0.0010	0.0010
−2.9	0.0019	0.0018	0.0018	0.0017	0.0016	0.0016	0.0015	0.0015	0.0014	0.0014
−2.8	0.0026	0.0025	0.0024	0.0023	0.0023	0.0022	0.0021	0.0021	0.0020	0.0019
−2.7	0.0035	0.0034	0.0033	0.0032	0.0031	0.0030	0.0029	0.0028	0.0027	0.0026
−2.6	0.0047	0.0045	0.0044	0.0043	0.0041	0.0040	0.0039	0.0038	0.0037	0.0036
−2.5	0.0062	0.0060	0.0059	0.0057	0.0055	0.0054	0.0052	0.0051	0.0049	0.0048
−2.4	0.0082	0.0080	0.0078	0.0075	0.0073	0.0071	0.0069	0.0068	0.0066	0.0064
−2.3	0.0107	0.0104	0.0102	0.0099	0.0096	0.0094	0.0091	0.0089	0.0087	0.0084
−2.2	0.0139	0.0136	0.0132	0.0129	0.0125	0.0122	0.0119	0.0116	0.0113	0.0110
−2.1	0.0179	0.0174	0.0170	0.0166	0.0162	0.0158	0.0154	0.0150	0.0146	0.0143
−2.0	0.0228	0.0222	0.0217	0.0212	0.0207	0.0202	0.0197	0.0192	0.0188	0.0183

Normal Distribution

Table A.3 (continued) Areas under the Normal Curve

z	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
0.0	0.5000	0.5040	0.5080	0.5120	0.5160	0.5199	0.5239	0.5279	0.5319	0.5359
0.1	0.5398	0.5438	0.5478	0.5517	0.5557	0.5596	0.5636	0.5675	0.5714	0.5753
0.2	0.5793	0.5832	0.5871	0.5910	0.5948	0.5987	0.6026	0.6064	0.6103	0.6141
0.3	0.6179	0.6217	0.6255	0.6293	0.6331	0.6368	0.6406	0.6443	0.6480	0.6517
0.4	0.6554	0.6591	0.6628	0.6664	0.6700	0.6736	0.6772	0.6808	0.6844	0.6879
0.5	0.6915	0.6950	0.6985	0.7019	0.7054	0.7088	0.7123	0.7157	0.7190	0.7224
0.6	0.7257	0.7291	0.7324	0.7357	0.7389	0.7422	0.7454	0.7486	0.7517	0.7549
0.7	0.7580	0.7611	0.7642	0.7673	0.7704	0.7734	0.7764	0.7794	0.7823	0.7852
0.8	0.7881	0.7910	0.7939	0.7967	0.7995	0.8023	0.8051	0.8078	0.8106	0.8133
0.9	0.8159	0.8186	0.8212	0.8238	0.8264	0.8289	0.8315	0.8340	0.8365	0.8389
1.0	0.8413	0.8438	0.8461	0.8485	0.8508	0.8531	0.8554	0.8577	0.8599	0.8621
1.1	0.8643	0.8665	0.8686	0.8708	0.8729	0.8749	0.8770	0.8790	0.8810	0.8830
1.2	0.8849	0.8869	0.8888	0.8907	0.8925	0.8944	0.8962	0.8980	0.8997	0.9015
1.3	0.9032	0.9049	0.9066	0.9082	0.9099	0.9115	0.9131	0.9147	0.9162	0.9177
1.4	0.9192	0.9207	0.9222	0.9236	0.9251	0.9265	0.9279	0.9292	0.9306	0.9319

Normal Distribution

- For example
- $P(Z<0.56)$
- $P(Z<0.56) = 0.7123$
- $P(Z<1.42)$
- $P(Z<0.56) = 0.9222$
- $P(Z>0.72) =$
- $P(Z>0.72) = 1 – P(Z<0.72)$
- $= 1-0.7642 = 0.2358$

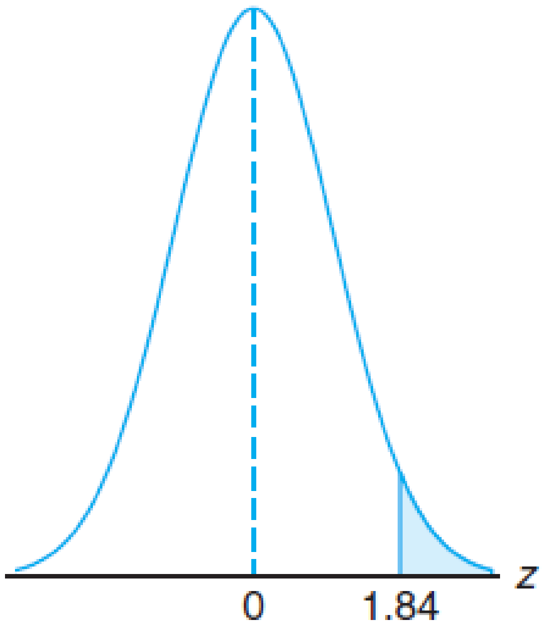
- $P(Z<k) =0.5987$ ➔ Find k?
- From the table, k=0.25

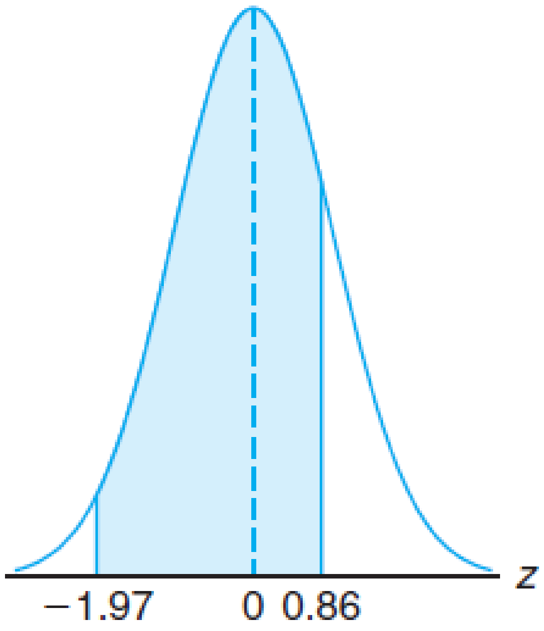
- $P(Z>k) = 0.3085$ ➔ Find k?

- $P(Z<k) = 1-0.3085 = 0.6915$
- From the table, k=0.5

Normal Distribution

- **Example 6.2:** Given a standard normal distribution, find the area under the curve that lies
- (a) to the right of $z = 1.84$ and
- (b) between $z = -1.97$ and $z = 0.86$.





Normal Distribution

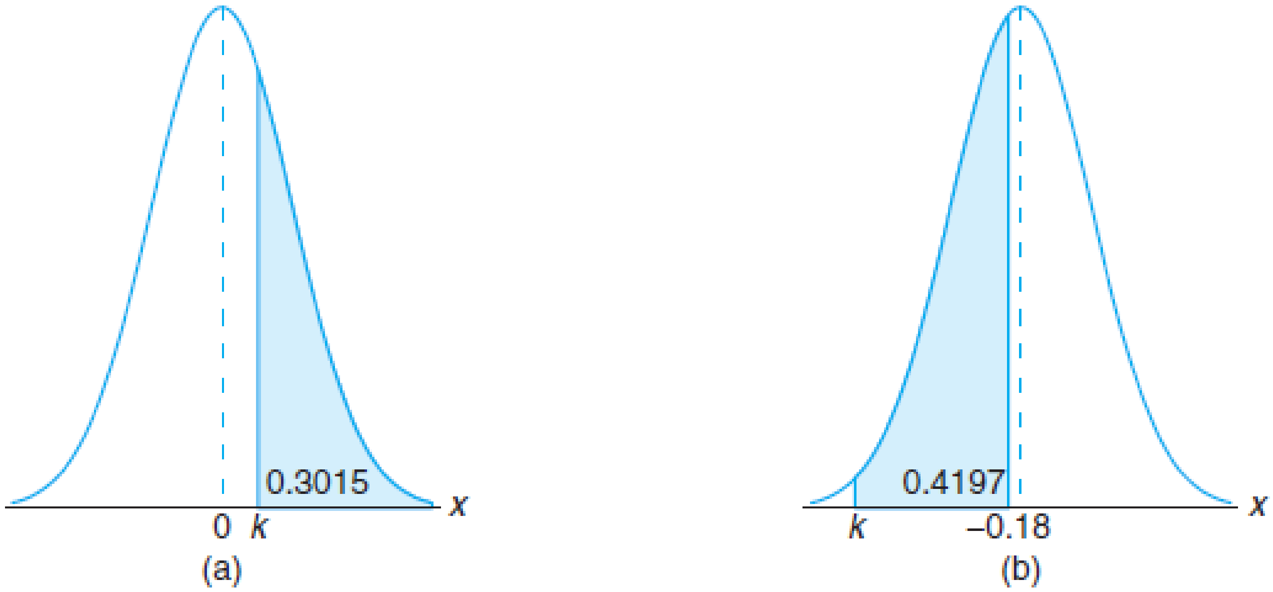
- (a) The area in Figure to the right of $z = 1.84$ is equal to 1 minus the area in Table to the left of $z = 1.84$, namely, $1 - 0.9671 = 0.0329$
- (b) The area in Figure between $z = -1.97$ and $z = 0.86$ is equal to
  - The area to the left of $z = 0.86$ minus the area to the left of $z = -1.97$.
  - From Table we find the desired area to be $0.8051 - 0.0244 = 0.7807$

Table A.3 Areas under the Normal Curve

z	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
−1.9	0.0287	0.0281	0.0274	0.0268	0.0262	0.0256	0.0250	0.0244	0.0239	0.0233
−1.8	0.0359	0.0351	0.0344	0.0336	0.0329	0.0322	0.0314	0.0307	0.0301	0.0294
−1.7	0.0446	0.0436	0.0427	0.0418	0.0409	0.0401	0.0392	0.0384	0.0375	0.0367
−1.6	0.0548	0.0537	0.0526	0.0516	0.0505	0.0495	0.0485	0.0475	0.0465	0.0455
−1.5	0.0668	0.0655	0.0643	0.0630	0.0618	0.0606	0.0594	0.0582	0.0571	0.0559

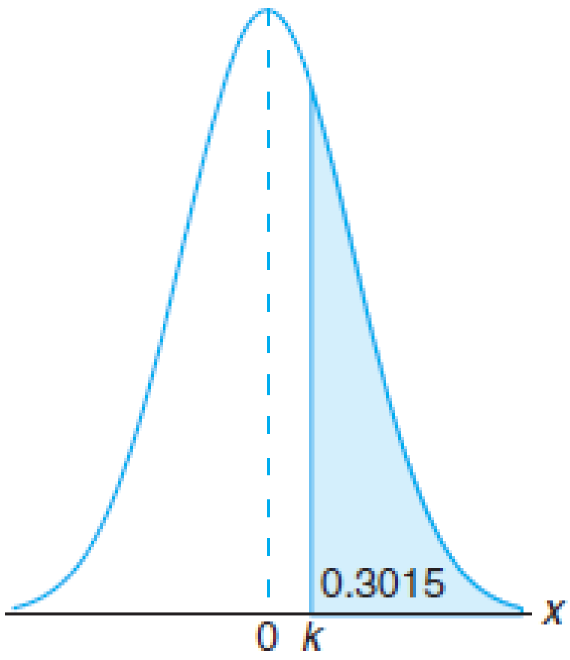
Normal Distribution

- **Example 6.3:** Given a standard normal distribution, find the value of k such that
- (a) $P(Z > k) = 0.3015$ and
- (b) $P(k < Z < -0.18) = 0.4197$.



Normal Distribution

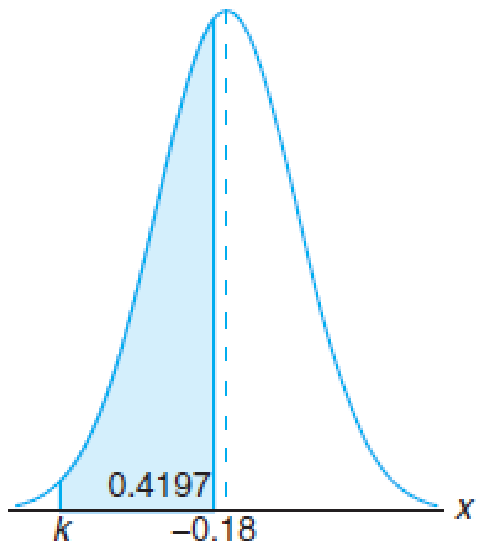
- (a) In Figure (a), we see that the k value leaving an area of 0.3015 to the right must then leave an area of 0.6985 to the left. From Table A.3 it follows that $k = 0.52$.



(a)

Normal Distribution

- (b) From Table A.3 we note that the total area to the left of −0.18 is equal to 0.4286.
- In Figure (b), we see that the area between k and −0.18 is 0.4197,
- so the area to the left of k must be 0.4286 − 0.4197 = 0.0089.
- Hence, from Table A.3, we have $k = -2.37$.



(b)

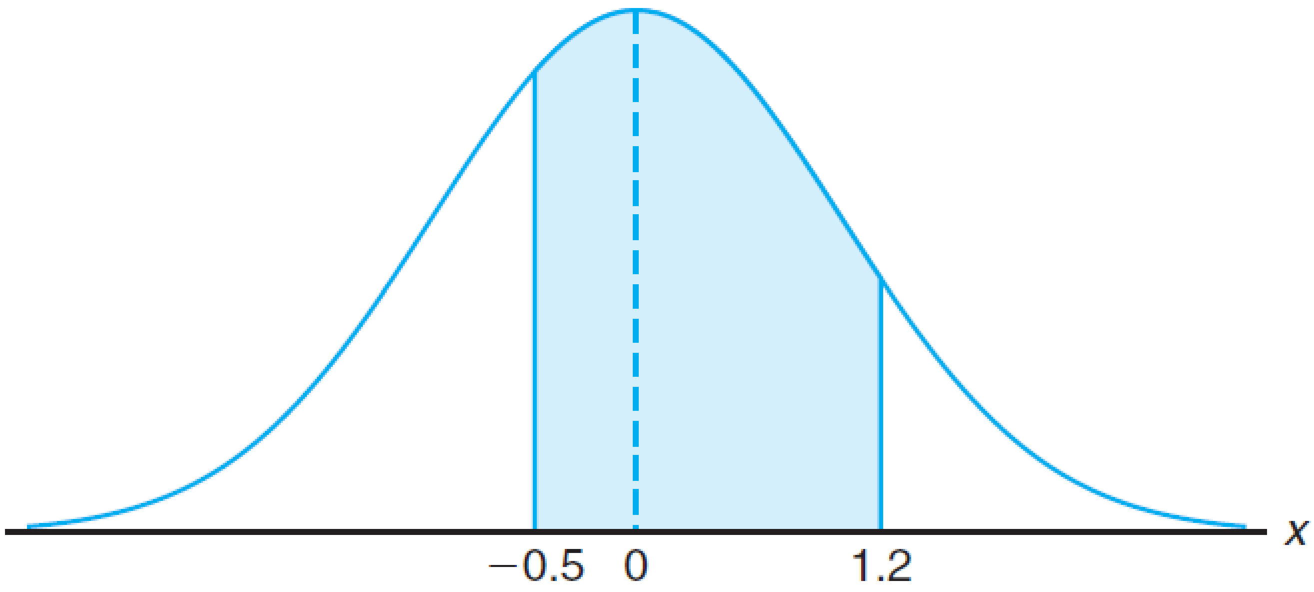
Normal Distribution

- **Example 6.4:** Given a random variable X having a normal distribution with $\mu = 50$ and $\sigma = 10$, find the probability that X assumes a value between 45 and 62.
- Solution: In order to use Table A.3, we need to convert these values to standard normal:
- The z values corresponding to $x1 = 45$ and $x2 = 62$ are

$$z_1 = \frac{45 - 50}{10} = -0.5 \text{ and } z_2 = \frac{62 - 50}{10} = 1.2$$

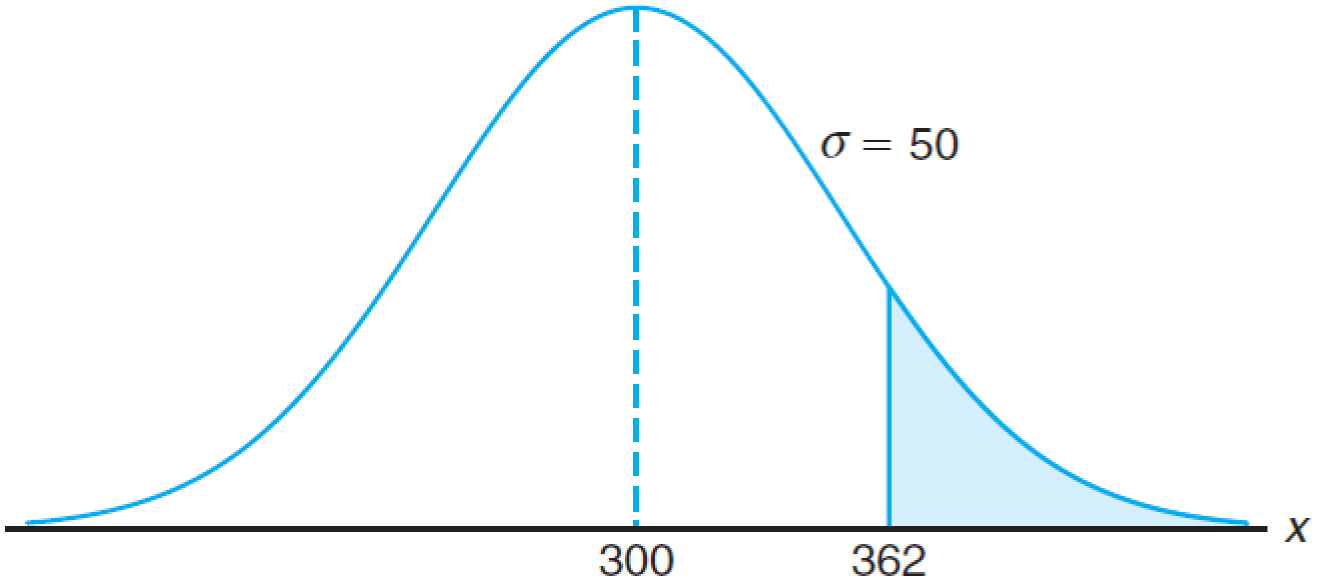
Normal Distribution

- Therefore
- $P(45 < X < 62) = P(-0.5 < Z < 1.2)$.
- $= P(Z < 1.2) - P(Z < -0.5)$
- $= 0.8849 - 0.3085 = 0.5764$.



Normal Distribution

- **Example 6.5:** Given that X has a normal distribution with $\mu = 300$ and $\sigma = 50$, find the probability that X assumes a value greater than 362.
- **Solution:** $P(X>362) = 1 – P(X<362)$



Normal Distribution

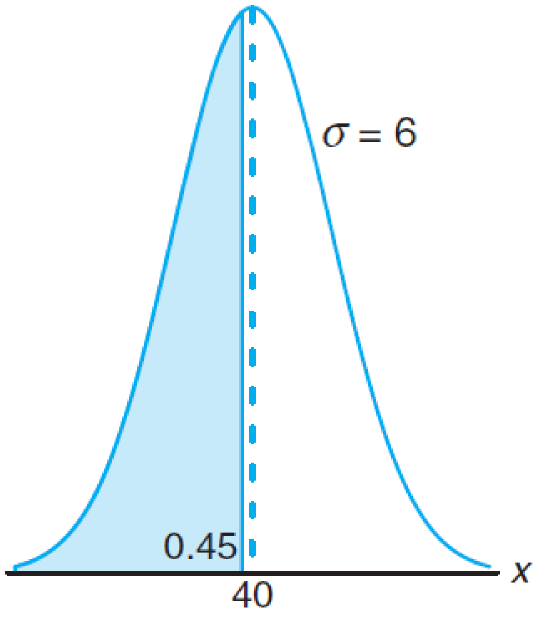
- Find the corresponding standard normal:

$$z = \frac{362 - 300}{50} = 1.24$$

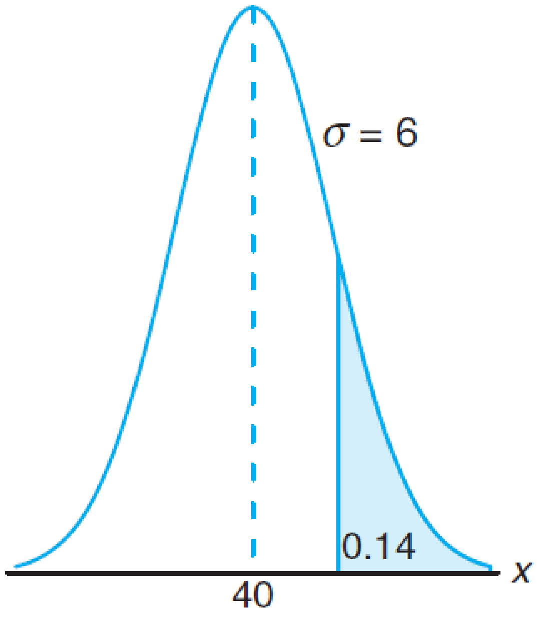
- *Hence we have*
- $P(X > 362) = 1 - P(X < 362)$
- $= 1 - P(Z < 1.24) = 1 - 0.8925 = 0.1075.$

Normal Distribution

- **Example 6.6:** For $\mu = 40$ and $\sigma = 6$, find the value of x that has
- (a) 45% of the area to the left and
- (b) 14% of the area to the right.



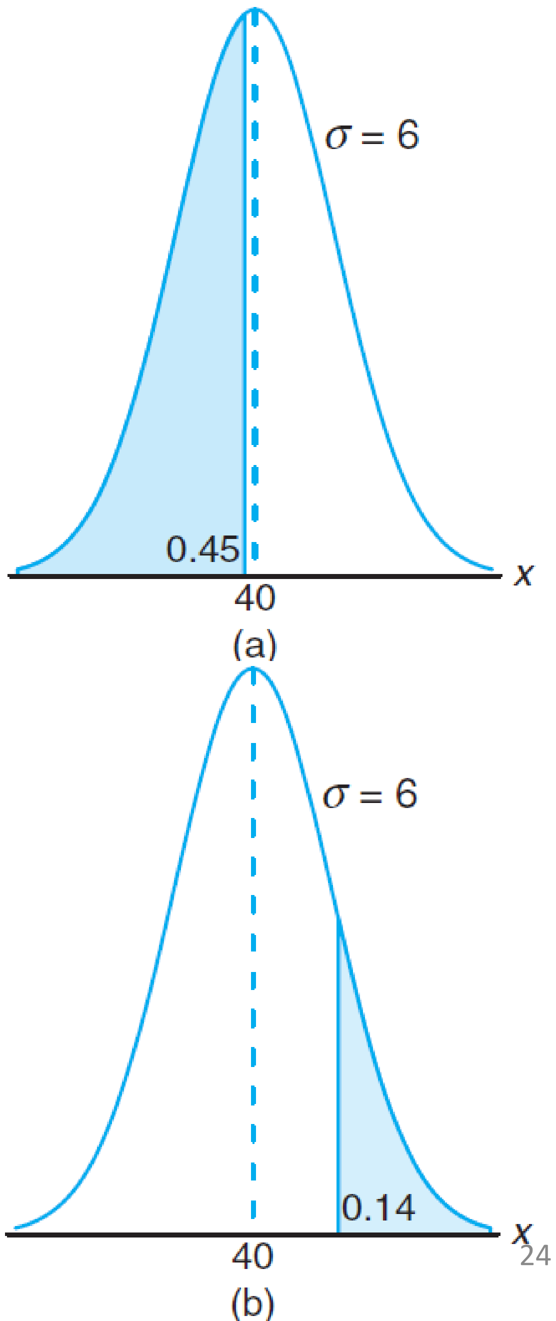
(a)



(b)

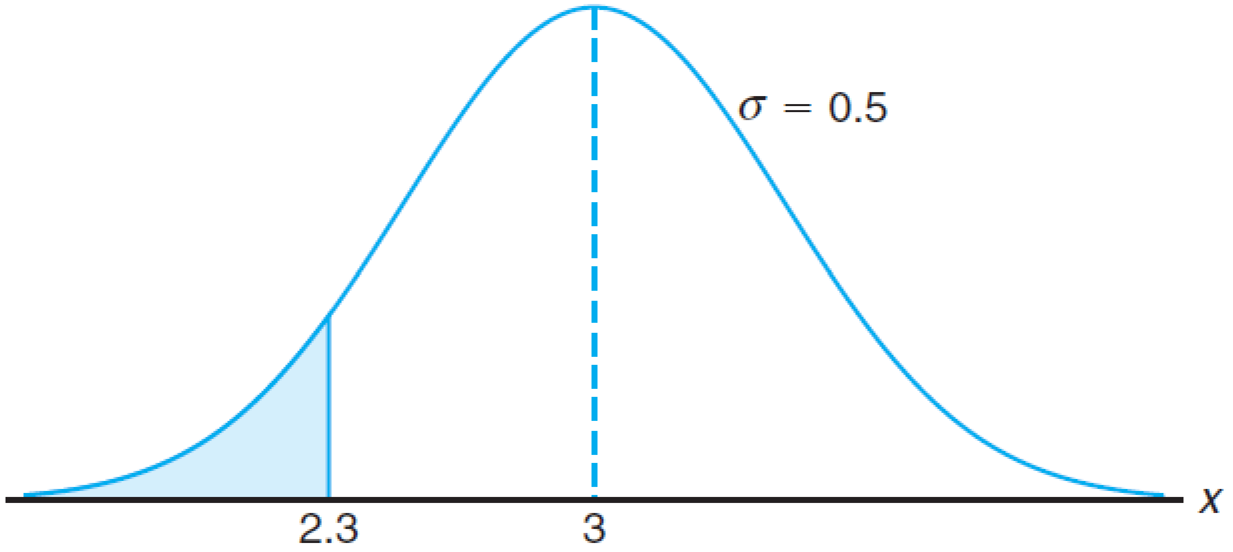
Normal Distribution

- (a) From Table A.3
- $P(Z < -0.13) = 0.45$. Hence
- $x = (6)(-0.13) + 40 = 39.22$
- (b) We need 0.86 to the left.
- From Table A.3
- $P(Z < 1.08) = 0.86$. Hence
- $x = (6)(1.08) + 40 = 46.48$



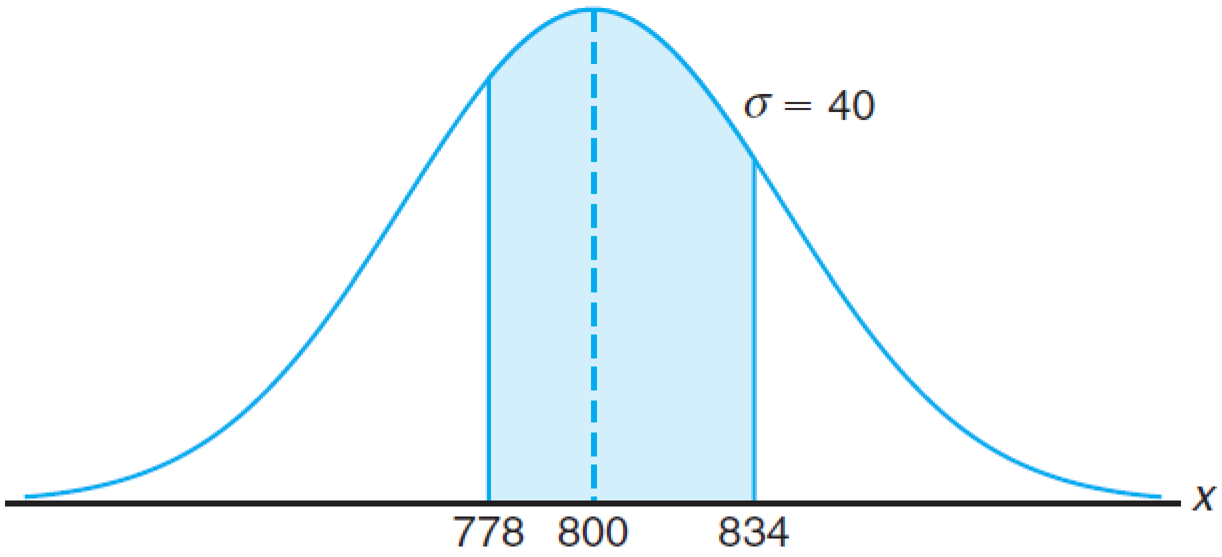
Normal Distribution

- **Example 6.7:** A certain type of storage battery lasts, on average, 3.0 years with a standard deviation of 0.5 year. Assuming that battery life is normally distributed, find the probability that a given battery will last less than 2.3 years.
- $z = \frac{2.3 - 3}{0.5} = -1.4$
- $P(X < 2.3) = P(Z < -1.4) = 0.0808$



Normal Distribution

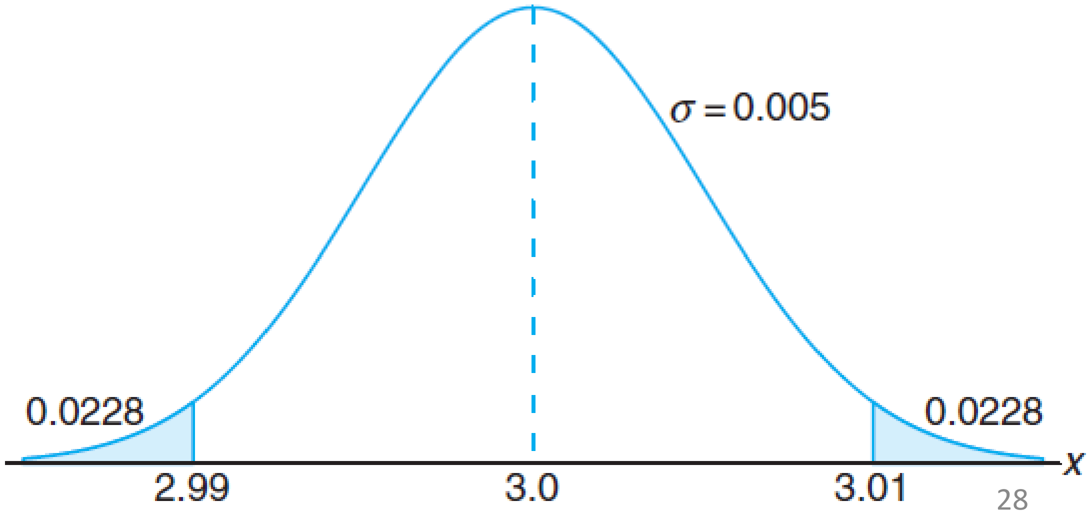
- **Example 6.8:** An electrical firm manufactures light bulbs that have a life, before burn-out, that is normally distributed with mean equal to 800 hours and a standard deviation of 40 hours. Find the probability that a bulb burns between 778 and 834 hours.
- $z_1 = \frac{778 - 800}{40} = -0.55$
- $z_2 = \frac{834 - 800}{40} = 0.85$
- $P(778 < X < 834) =$
- $P(-0.55 < Z < 0.85) =$
- $P(Z < 0.85) - P(Z < -0.55)$
- $= 0.8023 - 0.2912 = 0.5111$



Normal Distribution

- **Example 6.9:** In an industrial process, the diameter of a ball bearing is an important measurement.
- The buyer sets specifications for the diameter to be 3.0 ± 0.01 cm.
- The implication is that no part falling outside these specifications will be accepted.
- It is known that in the process the diameter of a ball bearing has a normal distribution with
  - mean $\mu = 3.0$ and standard deviation $\sigma = 0.005$.
- On average, how many manufactured ball bearings will be scrapped?

Normal Distribution

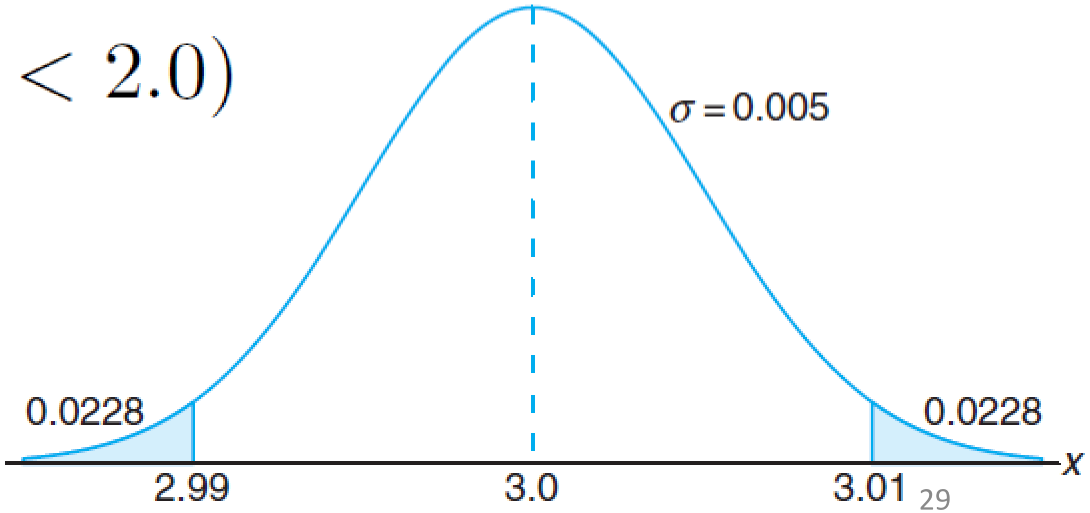


Normal Distribution

- Find the corresponding z values:

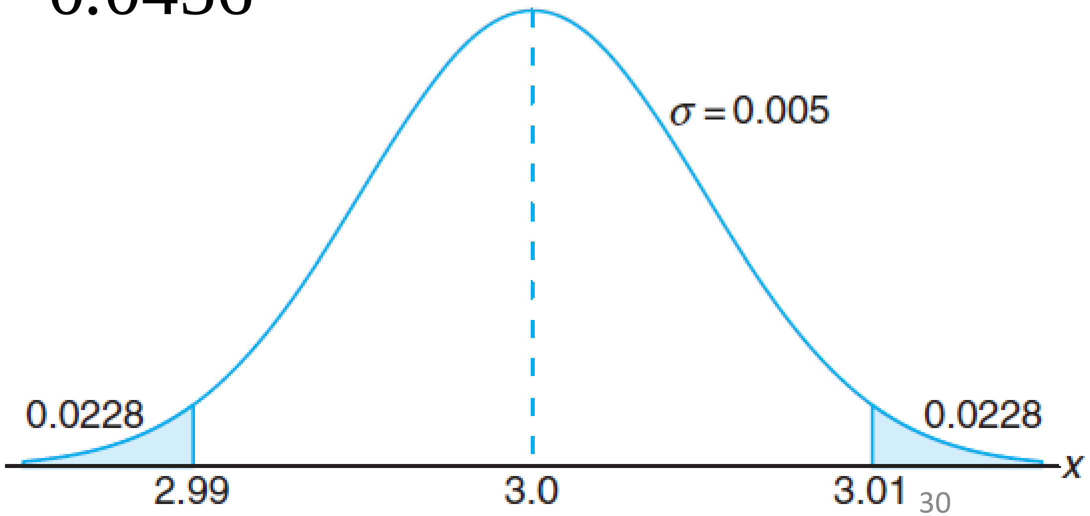
$$z_1 = \frac{2.99 - 3.0}{0.005} = -2.0 \text{ and } z_2 = \frac{3.01 - 3.0}{0.005} = +2.0.$$

$$P(2.99 < X < 3.01) = P(-2.0 < Z < 2.0)$$



Normal Distribution

- From Table A.3, $P(Z < -2.0) = 0.0228$
- Due to symmetry, we have:
- $P(Z > 2.0) = P(Z < -2.0) = 0.0228$
- Hence:
- $P(Z < -2.0) + P(Z > 2.0) = 2(0.0228) = 0.0456$
- As a result, on average, 4.56% of manufactured ball bearings will be scrapped



Normal Distribution

- **Example 6.10:** Gauges are used to reject all components for which a certain dimension is not within the specification $1.50 \pm d$.
- It is known that this measurement is normally distributed with mean 1.50 and standard deviation 0.2.
- Determine the value d such that the specifications "cover" **95%** of the measurements.

Normal Distribution

- The problem will be solved in two steps:
- Step 1: Find the corresponding z values.
- Step 2: Find the real normal values.
- **Step 1:**
- Due to symmetry, we can write the following
- $P(z_1 < Z < z_2)$
- Look at the board…

Normal Distribution

- $P(Z < z_2)$ = 0.975 => From Table A.3 we have z_2 = 1.96
- Hence from $P(Z > z_1)$ = 0.975 z_1 = -1.96

Table A.3 (continued) Areas under the Normal Curve

z	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
1.5	0.9332	0.9345	0.9357	0.9370	0.9382	0.9394	0.9406	0.9418	0.9429	0.9441
1.6	0.9452	0.9463	0.9474	0.9484	0.9495	0.9505	0.9515	0.9525	0.9535	0.9545
1.7	0.9554	0.9564	0.9573	0.9582	0.9591	0.9599	0.9608	0.9616	0.9625	0.9633
1.8	0.9641	0.9649	0.9656	0.9664	0.9671	0.9678	0.9686	0.9693	0.9699	0.9706
1.9	0.9713	0.9719	0.9726	0.9732	0.9738	0.9744	0.9750	0.9756	0.9761	0.9767

Normal Distribution

- Hence we can write
- $P(-1.96 < Z < 1.96) = 0.95$
- *We use* $Z = \frac{X-\mu}{\sigma}$ *to find the corresponding z values:*
- Hence we have $X = \mu + \sigma Z$
- $X_1 = \mu + \sigma\,(-1.96) = 1.5 - (0.2)(1.96) = 1.108$
- $X_2 = \mu + \sigma\,1.96 = 1.5 + (0.2)(1.96) = 1.892$
- Therefore: *1.5 + d = 1.892 => d = 0.392*

Normal Distribution

- We will use the following notation
- $z_{0.05}$= The z value such that it leaves an area of 0.05 to the right.
  - $z_{0.10}$= 1.282
  - $z_{0.05}$= 1.645
  - $z_{0.01}$= 2.326