Chapter 9-10
Confidence Intervals and Hypothesis Testing

Confidence Intervals and Hypothesis Testing for the Variance

Statistics

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CI and HT for the Variance

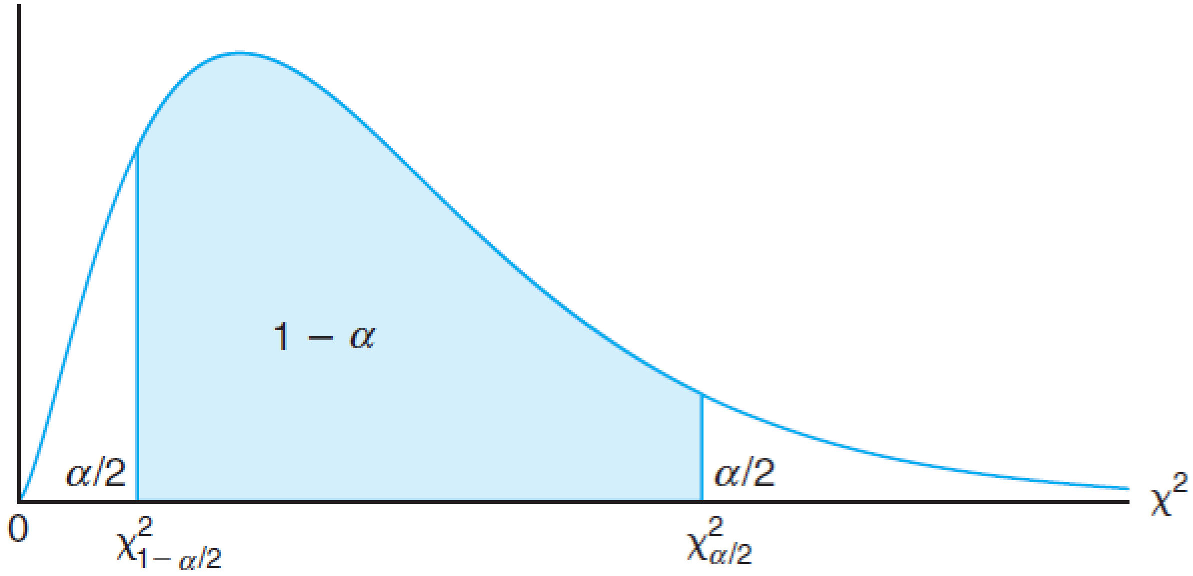
Sampling Distribution for s^2

Let $X_1, X_2, \ldots, X_n$ be a random sample from a normal distribution with unknown variance σ^2.

- We already know that a point estimator of σ^2 is S^2
  - we use S^2 to infer information about σ^2
  - Just like we use $\bar{X}$ to infer information about μ
- Well, for CI and HT we can use the following
  - **The statistic $\chi^2 = \frac{(n-1)S^2}{\sigma^2}$ has the χ^2 dist.with $\nu = n-1$ df.**

Sampling Distribution for s^2

$$P\left(\chi^2_{1-\alpha/2} < \chi^2 < \chi^2_{\alpha/2}\right) = 1 - \alpha$$



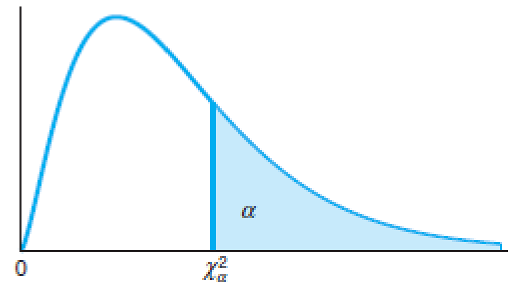


Table A.5 Critical Values of the Chi-Squared Distribution

	α									
v	**0.995**	**0.99**	**0.98**	**0.975**	**0.95**	**0.90**	**0.80**	**0.75**	**0.70**	**0.50**
1	0.0^4393	0.0^3157	0.0^3628	0.0^3982	0.00393	0.0158	0.0642	0.102	0.148	0.455
2	0.0100	0.0201	0.0404	0.0506	0.103	0.211	0.446	0.575	0.713	1.386
3	0.0717	0.115	0.185	0.216	0.352	0.584	1.005	1.213	1.424	2.366
4	0.207	0.297	0.429	0.484	0.711	1.064	1.649	1.923	2.195	3.357
5	0.412	0.554	0.752	0.831	1.145	1.610	2.343	2.675	3.000	4.351
6	0.676	0.872	1.134	1.237	1.635	2.204	3.070	3.455	3.828	5.348
7	0.989	1.239	1.564	1.690	2.167	2.833	3.822	4.255	4.671	6.346
8	1.344	1.647	2.032	2.180	2.733	3.490	4.594	5.071	5.527	7.344
9	1.735	2.088	2.532	2.700	3.325	4.168	5.380	5.899	6.393	8.343
10	2.156	2.558	3.059	3.247	3.940	4.865	6.179	6.737	7.267	9.342
11	2.603	3.053	3.609	3.816	4.575	5.578	6.989	7.584	8.148	10.341
12	3.074	3.571	4.178	4.404	5.226	6.304	7.807	8.438	9.034	11.340
13	3.565	4.107	4.765	5.009	5.892	7.041	8.634	9.299	9.926	12.340
14	4.075	4.660	5.368	5.629	6.571	7.790	9.467	10.165	10.821	13.339
15	4.601	5.229	5.985	6.262	7.261	8.547	10.307	11.037	11.721	14.339

HT for σ^2

- **Step1:** The hypothesis are:
  - $H_0 : \sigma = \sigma_0$
  - $H_1 : \sigma \neq \sigma_0$
- **Step2:** The test statistic $\chi^2 = \dfrac{(n-1)s^2}{\sigma_0^2}$ with v=n-1 d.o.f :
- **Step 3**: R- $[\chi^2_{1-\alpha\backslash 2}, \chi^2_{\alpha\backslash 2}]$,
- **Step 4**: Calculate χ^2_{obs} using the formula in step2
- **Step5:** if χ^2_{obs} is in the critical region, we reject the null hypothesis.

HT for σ^2

- **Example**: Recall the example where we discussed about the heights of the students.
- We have the following heights :

171	175	156	151	179
175	170	164	167	162

- The average height turns out to be $\overline{\mathrm{X}} = 167$
- The sample standard deviation is $s = 8.87$

HT for σ^2

Example:

- Test the hypothesis for the heights of students here for $\alpha = 0.05$:

  $H_0: \sigma = 6$

  $H_1: \sigma \neq 6$

Solution:

- using Table A.5 with *v* = 9 dof, we find
  - $\chi 2_{0.025}$ = **19.023** and $\chi 2_{\ 0.975}$ = 2.700

- Observed value is $\chi^2 = \frac{\left((n-1)s^2\right)}{\sigma_0^2} = 9 \times \frac{8.87^2}{6^2} = 19.67$ **>19.023**: hence reject H_0.

CI for σ^2

Substitute χ^2 and after some algebra, we can construct a CI for σ^2:

$$P\left(\chi^2_{1-\alpha/2} < \frac{(n-1)S^2}{\sigma^2} < \chi^2_{\alpha/2}\right) = 1 - \alpha$$

$$P\left(\frac{(n-1)S^2}{\chi^2_{\alpha/2}} < \sigma^2 < \frac{(n-1)S^2}{\chi^2_{1-\alpha/2}}\right) = 1 - \alpha$$

CI for σ^2

If s^2 is the variance of a random sample of size n from a normal population, a $100(1-\alpha)\%$ confidence interval for σ^2 is

$$\frac{(n-1)s^2}{\chi^2_{\alpha/2}} < \sigma^2 < \frac{(n-1)s^2}{\chi^2_{1-\alpha/2}},$$

where $\chi^2_{\alpha/2}$ and $\chi^2_{1-\alpha/2}$ are χ^2-values with $v = n-1$ degrees of freedom, leaving areas of $\alpha/2$ and $1-\alpha/2$, respectively, to the right.

CI for σ^2

- We found CI for σ^2.
- How about σ?
  - Take square root!

$$\sqrt{\frac{(n-1)s^2}{\chi^2_{\alpha/2}}} < \sigma < \sqrt{\frac{(n-1)s^2}{\chi^2_{1-\alpha/2}}}.$$

CI for σ^2

Example: Find a 95% CI for the σ^2 of heights of the students.

Solution: We know that $s = 8.87$, n=10.

- To obtain a 95% CI, we choose α = 0.05
- using Table A.5 with v = 9 dof, we find
  - $\chi2_{0.025}$ = 19.023
  - $\chi2_{\ 0.975}$ = 2.700

$$P\left(\frac{(n-1)S^2}{\chi^2_{\alpha/2}} < \sigma^2 < \frac{(n-1)S^2}{\chi^2_{1-\alpha/2}}\right) = 1 - \alpha$$

- $\frac{(9)(8.87^2)}{19.023} < \sigma^2 < \frac{(9)(8.87^2)}{2.700}$ = 37.22 < σ^2 < 262.26 => 6.1 < σ < 16.19

Ratio of two variances: ${\sigma_1^2}/{\sigma_2^2}$

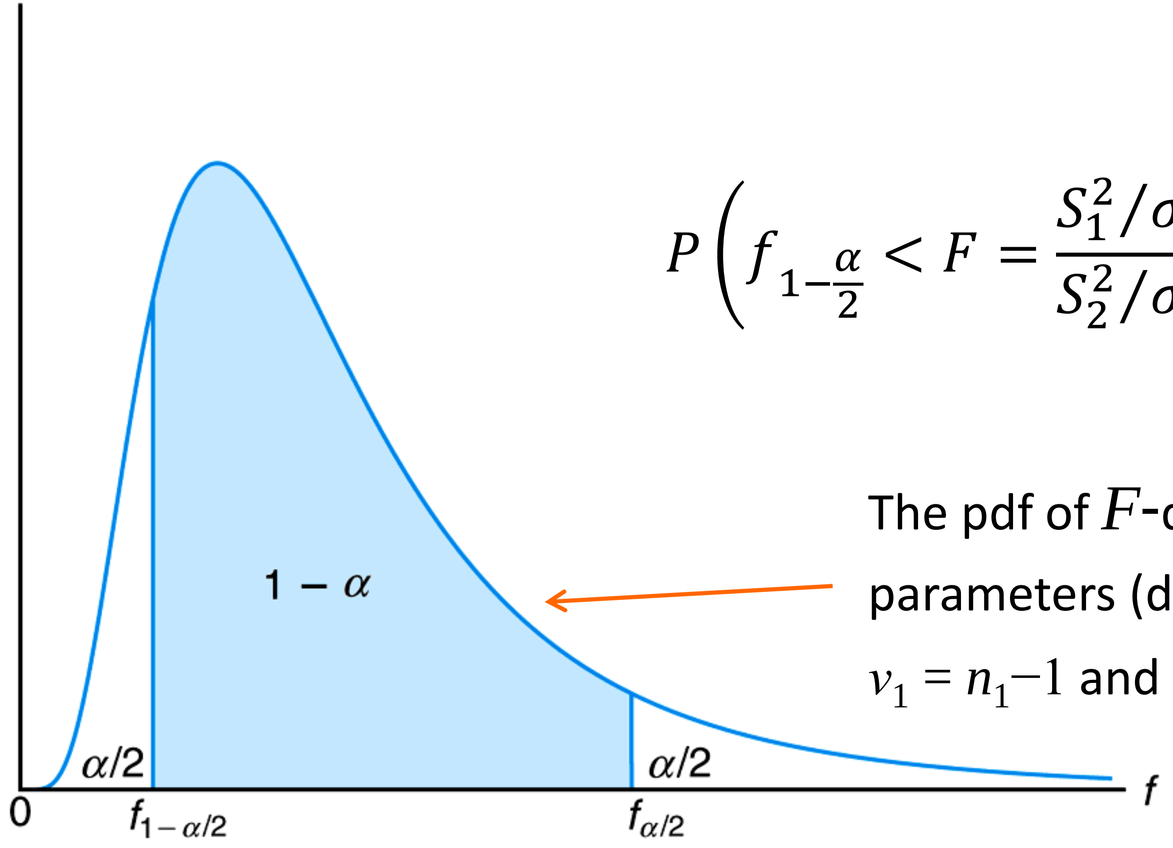
Sampling Distribution for S_1^2/S_2^2

- S_1^2/S_2^2 is a point estimator for σ_1^2/σ_2^2.
- New information:
  - The ratio of two independent chi-squared RVs has an *F*-dist.
  - $\chi_1^2 = \frac{(n_1-1)S_1^2}{\sigma_1^2}$ and $\chi_2^2 = \frac{(n_2-1)S_2^2}{\sigma_2^2}$ are chi-squared RVs with $v_1 = n_1 - 1$ and $v_2 = n_2 - 1$ dof, if samples are normal
- Hence:

$$F = \frac{S_1^2/\ \sigma_1^2}{S_2^2/\ \sigma_2^2} = \frac{\sigma_2^2 S_1^2}{\sigma_1^2 S_2^2}$$

has an F-distribution with v1 = n1−1 and v2 = n2 −1 d.o.f.

Sampling Distribution for S_1^2/S_2^2



$$P\left(f_{1-\frac{\alpha}{2}} < F = \frac{S_1^2/\sigma_1^2}{S_2^2/\sigma_2^2} < f_{\frac{\alpha}{2}}\right) = 1 - \alpha$$

The pdf of F-distribution with parameters (degrees of freedom) $v_1 = n_1 - 1$ and $v_2 = n_2 - 1$

A useful property of F distribution is: $f_{1-a/2}(n_1, n_2) = \dfrac{1}{f_{a/2}(n_2, n_1)}$

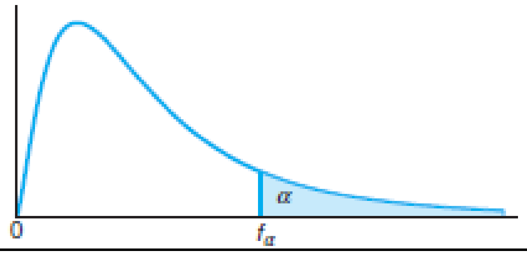


Table A.6 Critical Values of the F-Distribution

$f_{0.05}(v_1, v_2)$

v_2	v_1 = 1	2	3	4	5	6	7	8	9
1	161.45	199.50	215.71	224.58	230.16	233.99	236.77	238.88	240.54
2	18.51	19.00	19.16	19.25	19.30	19.33	19.35	19.37	19.38
3	10.13	9.55	9.28	9.12	9.01	8.94	8.89	8.85	8.81
4	7.71	6.94	6.59	6.39	6.26	6.16	6.09	6.04	6.00
5	6.61	5.79	5.41	5.19	5.05	4.95	4.88	4.82	4.77
6	5.99	5.14	4.76	4.53	4.39	4.28	4.21	4.15	4.10
7	5.59	4.74	4.35	4.12	3.97	3.87	3.79	3.73	3.68
8	5.32	4.46	4.07	3.84	3.69	3.58	3.50	3.44	3.39
9	5.12	4.26	3.86	3.63	3.48	3.37	3.29	3.23	3.18
10	4.96	4.10	3.71	3.48	3.33	3.22	3.14	3.07	3.02
11	4.84	3.98	3.59	3.36	3.20	3.09	3.01	2.95	2.90
12	4.75	3.89	3.49	3.26	3.11	3.00	2.91	2.85	2.80
13	4.67	3.81	3.41	3.18	3.03	2.92	2.83	2.77	2.71
14	4.60	3.74	3.34	3.11	2.96	2.85	2.76	2.70	2.65
15	4.54	3.68	3.29	3.06	2.90	2.79	2.71	2.64	2.59

HT for σ_1^2/σ_2^2

- **Step1:** The hypothesis are:
  - $H_0: \sigma_1^2 = \sigma_2^2$ (or $\sigma_1 = \sigma_2$)
  - $H_1: \sigma_1^2 \neq \sigma_2^2$ ($\sigma_1 \neq \sigma_2$)

- **Step2:** The test statistic $F = \dfrac{S_1^2/\sigma_1^2}{S_2^2/\sigma_2^2} = \dfrac{\sigma_2^2 S_1^2}{\sigma_1^2 S_2^2}$ with $v_1=n_1-1$ and $v_2=n_2-1$ d.o.f

- **Step 3**: R- $[f_{1-\alpha\backslash 2}(v_1, v_2), f_{\alpha\backslash 2}(v_1, v_2)]$,

- **Step 4**: Calculate f_{obs} using the formula in step2

- **Step5:** if f_{obs} is in the critical region, we reject the null hypothesis.

HT for σ_1^2/σ_2^2

Example:

- Assume we take 8 students from ÖzÜ and 10 Students from Yildiz and measure their heights.
- It turns out that $s_1 = 4$ cm and $s_2 = 5$ cm.
- Using a significance level of $\alpha = 0.10$, can we say that their variances are equal?

HT for σ_1^2/σ_2^2

Solution:

$H_0: \sigma_1 = \sigma_2$

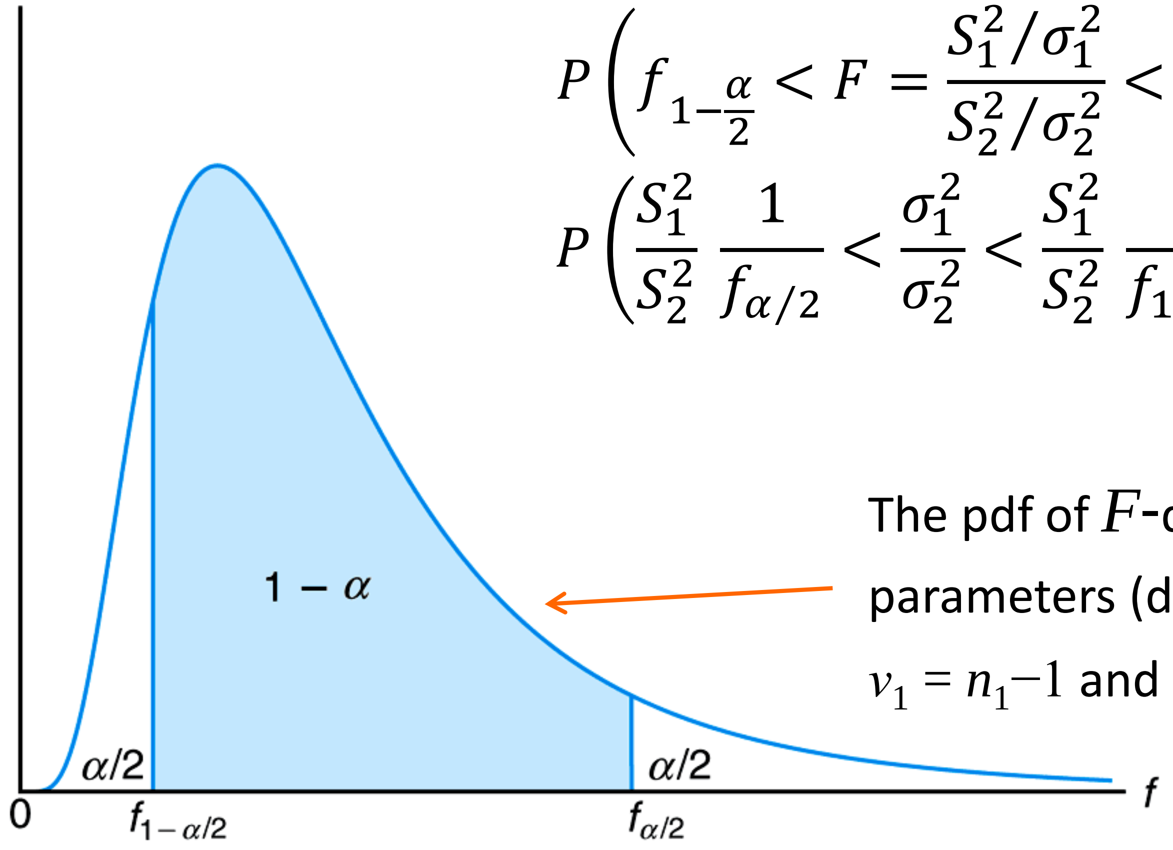
$H_1: \sigma_1 \neq \sigma_2$

- $f_{\alpha\backslash 2=0.05}(v_1 = 7, v_2 = 9)$=3.29
- $f_{1-\alpha\backslash 2=0.95}(v_1 = 7, v_2 = 9) = \frac{1}{f_{\alpha=0.05}(9,7)} = \frac{1}{3.68} = 0.27$
- Observed f is
  - $f = \frac{S_1^2\sigma_2^2}{S_2^2\sigma_1^2} = \frac{S_1^2}{S_2^2} = \frac{4^2}{5^2} = 0.64$
- Since 0.27 < 0.64 < 3.29 we do not reject H_0

CI for σ_1^2/σ_2^2

$$P\left(f_{1-\frac{\alpha}{2}} < F = \frac{S_1^2/\sigma_1^2}{S_2^2/\sigma_2^2} < f_{\frac{\alpha}{2}}\right) = 1-\alpha$$

$$P\left(\frac{S_1^2}{S_2^2}\frac{1}{f_{\alpha/2}} < \frac{\sigma_1^2}{\sigma_2^2} < \frac{S_1^2}{S_2^2}\frac{1}{f_{1-\alpha/2}}\right)$$



The pdf of F-distribution with parameters (degrees of freedom) $v_1 = n_1 - 1$ and $v_2 = n_2 - 1$

CI for σ_1^2/σ_2^2

If $S_1{}^2$ and $S_2{}^2$ are the variances of independent random samples of sizes n_1 and n_2 from normal populations, then a $1-\alpha$ confidence interval for $\sigma_1{}^2$ / $\sigma_2{}^2$ is

$$\frac{S_1^2}{S_2^2}\frac{1}{f_{\alpha/2}(v_1,v_2)} < \frac{\sigma_1^2}{\sigma_2^2} < \frac{S_1^2}{S_2^2} f_{\alpha/2}(v_2,v_1)$$

Where $f_{\alpha/2}$ is obtained from the F-distribution.

Note : We used the theorem that states $f_{1-a/2}(n_1,n_2)=\dfrac{1}{f_{a/2}(n_2,n_1)}$

CI for σ_1^2/σ_2^2

- **Example:** A confidence interval for the difference in the mean salt content measured in milligrams per liter, at two stations on İstanbul Boğazı.
- Station 1: 15 samples, 3.84 avg weight, st dev of 3.07
- Station 2: 12 samples, 1.49, avg weight std dev of 0.8
- We thinkthat the normal population variances are UNequal.
- Justify this assumption by constructing 98% confidence intervals for σ_1^2/σ_2^2 and for σ_1/σ_2

CI for σ_1^2/σ_2^2

- **Solution:**
- we have $n_1 = 15$, $n_2 = 12$, $s_1 = 3.07$, and $s_2 = 0.80$
- For a 98% confidence interval, $\alpha = 0.02$.
- Interpolating in Table A.6, we find $f_{0.01}(14, 11) \approx 4.30$ and $f_{0.01}(11, 14) \approx 3.87$.
- Therefore, the 98% confidence interval for σ_1^2/σ_2^2 *is*

$$\left(\frac{3.07^2}{0.80^2}\right)\left(\frac{1}{4.30}\right) < \frac{\sigma_1^2}{\sigma_2^2} < \left(\frac{3.07^2}{0.80^2}\right)(3.87) \qquad 3.425 < \frac{\sigma_1^2}{\sigma_2^2} < 56.991$$

CI for σ_1^2/σ_2^2

- Taking the square root of both sides we have:

$$1.851 < \frac{\sigma_1}{\sigma_2} < 7.549.$$

- Since the interval does not include 1, we were correct in assuming that the variances (or the st. dev.s) are not equal.