Chapter 9-10
Hypothesis Testing and Confidence Intervals

HT and CI for the Mean

Statistics

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Point Estimation

Point Estimation of Process Parameters

- Parameters:
  - μ, σ, λ
- We never know them
- A **point estimator** is a statistic that produces a single numerical value as the estimate of the unknown parameter
  - $\bar{X}$ for μ
  - s^2 for σ^2
- Note that, the mean μ and variance σ^2 of a distribution are NOT necessarily the parameters of the distribution
  - Poisson $\mu = \lambda$ and $\sigma^2 = \lambda$
  - Binomial $\mu = np$ and $\sigma^2 = npq$

Point Estimation of Process Parameters

- We can show that following are good point estimators

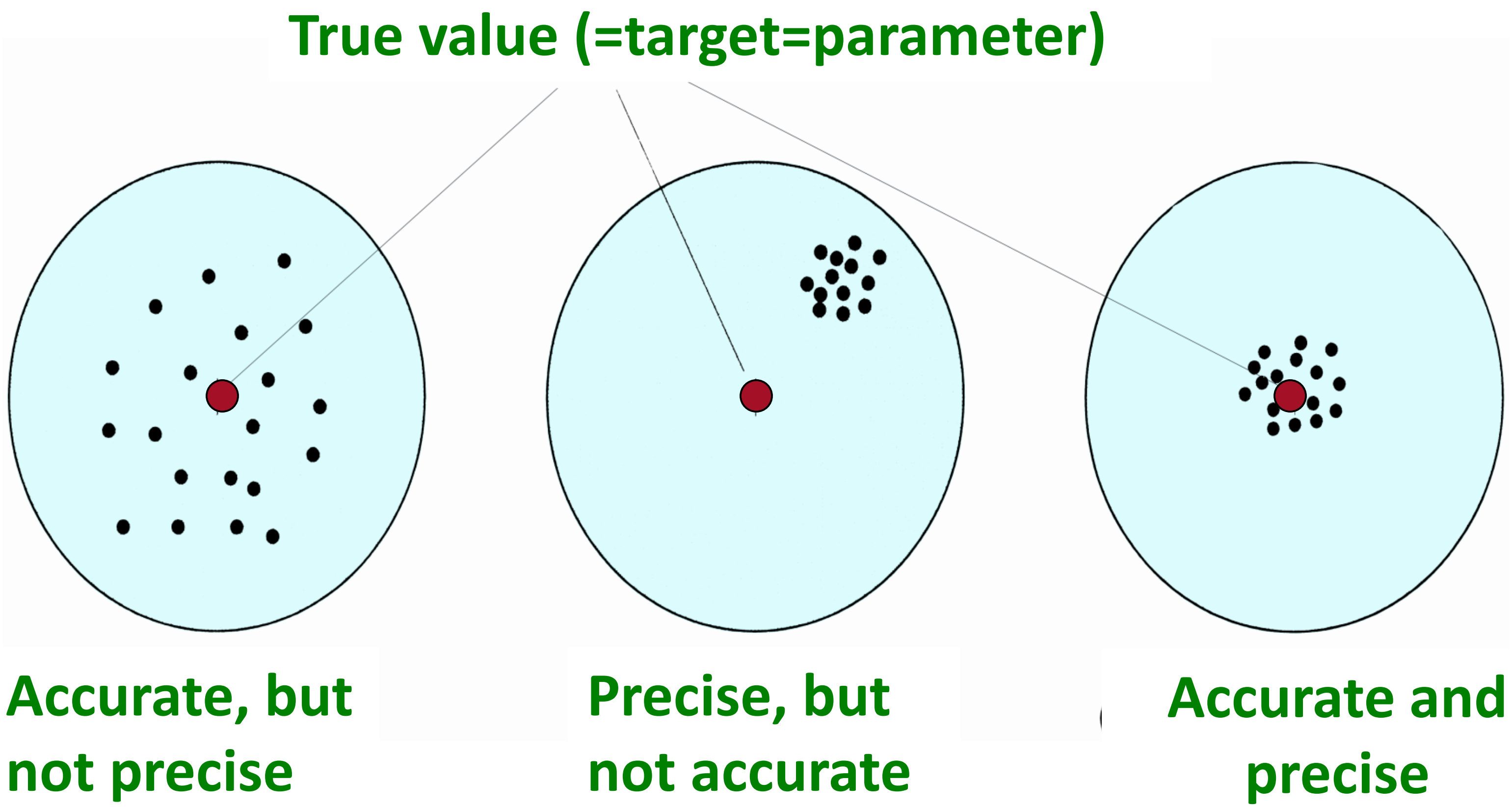
$$\hat{\lambda} = \frac{1}{n}\sum_{i=1}^{n} x_i = \bar{x}$$

$$\hat{p} = \frac{1}{n}\sum_{i=1}^{n} x_i = \bar{x}$$

Point Estimation of Process Parameters

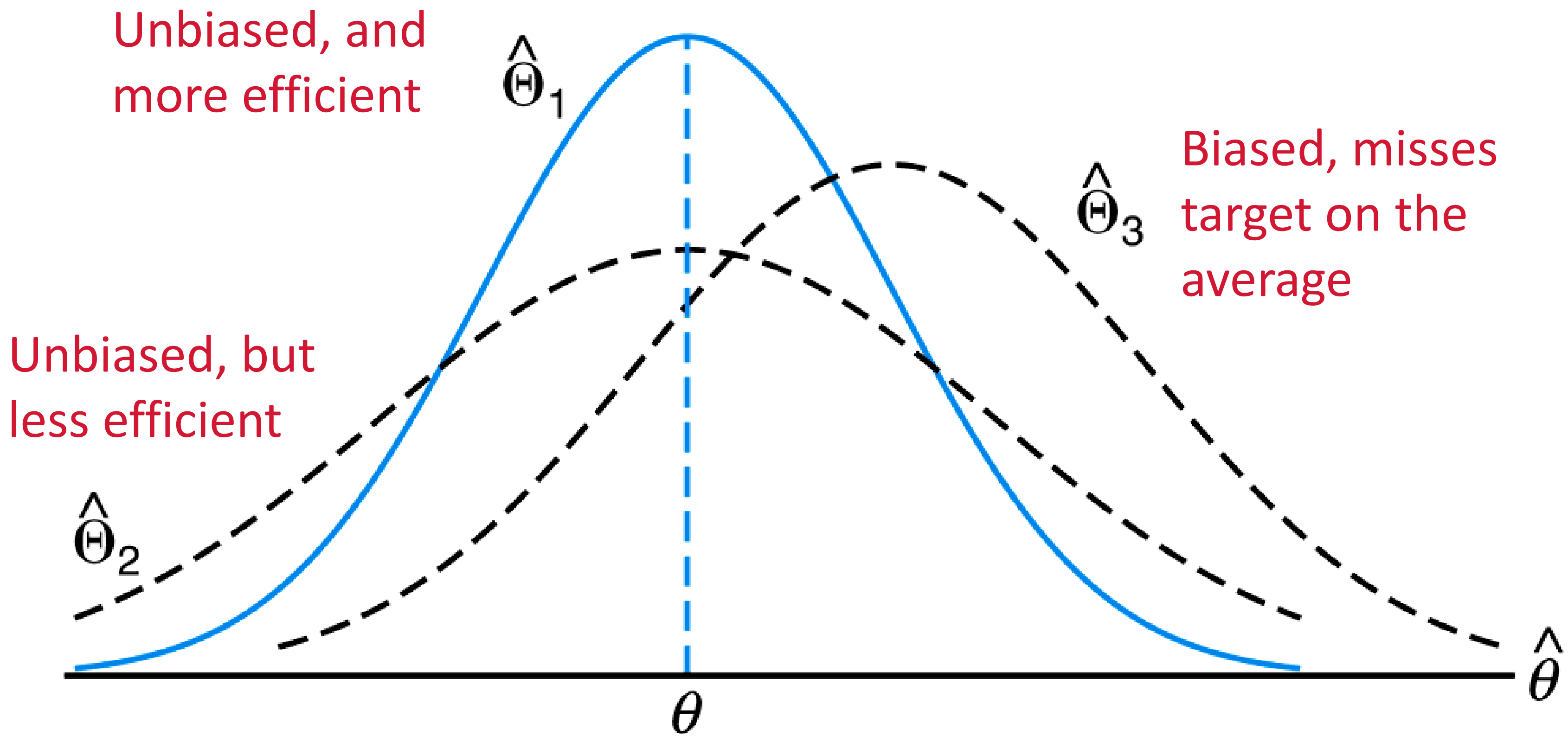
- What do we mean by good?
- Unbiased:
  - E [Estimator] = Parameter to be estimated
- Minimum Variance:
  - $\sigma_{\bar{X}}$ should be minimum
- For example
  - $E[\bar{X}] \ = \mu$
  - $E[s^2] = \sigma^2$

Figure 5. Difference between accuracy and precision



Consider each shooting (black dot) as a possible value of the statistic from a random sample. The larger red dot is the target (value of the parameter to be estimated)

Figure 4. Sampling distributions of different estimators of θ



Distributions of three different estimators of θ.

Hypothesis Testing

HT for the mean – σ known

- I conjecture the following:
  - The average height of students in this university is 170cm
- What can you say about this?
- How to test this conjecture?

HT for the mean – σ known

- **Example:**
- A random sample of 100 recorded deaths in Turkey during the past year showed an average life span of 71.8 years.
- We want to test whether the mean life span today is 70 or not, i.e.,
    - ***Test whether the real underlying $\boldsymbol{\mu = 70}$ or not***.
- Assuming a population st. dev of 8.9 years (i.e., we know that $\sigma = 8.9$), what can you say about this conjecture?

HT – General Concepts

- There is an evidence, 71.8, which shows that the real mean is not 70.

- We will use hypothesis testing to test the conjecture.
  - *A **statistical hypothesis** is an assertion or conjecture concerning one or more populations*

- *How to do this?*

HT for the mean – σ known

Null Hypothesis Vs **Alternative Hypothesis**

- $H_0: \mu = 70$
- $H_1: \mu \neq 70$
- ***Two decisions:***
  - ***reject H0*** *in favor of H1 because of sufficient evidence in the data* or
  - ***fail to reject*** **H0** *because of insufficient evidence in the data.*
- *How to decide? Any guess? Recall* $\bar{X} = \mathbf{71.8}$
  - *Close enough?*
  - *What does **close** mean?*

HT for the mean – σ known

A clever way is the following:

- Assume that the null hypothesis is true.
- Under this hypothesis, we know the distribution of $\bar{X}$!
  - Plot it!
- Then put your observed test statistic, $\bar{x}_{obs}$ on the plot
- Check whether it is an extreme observation or not

HT for the mean – σ known

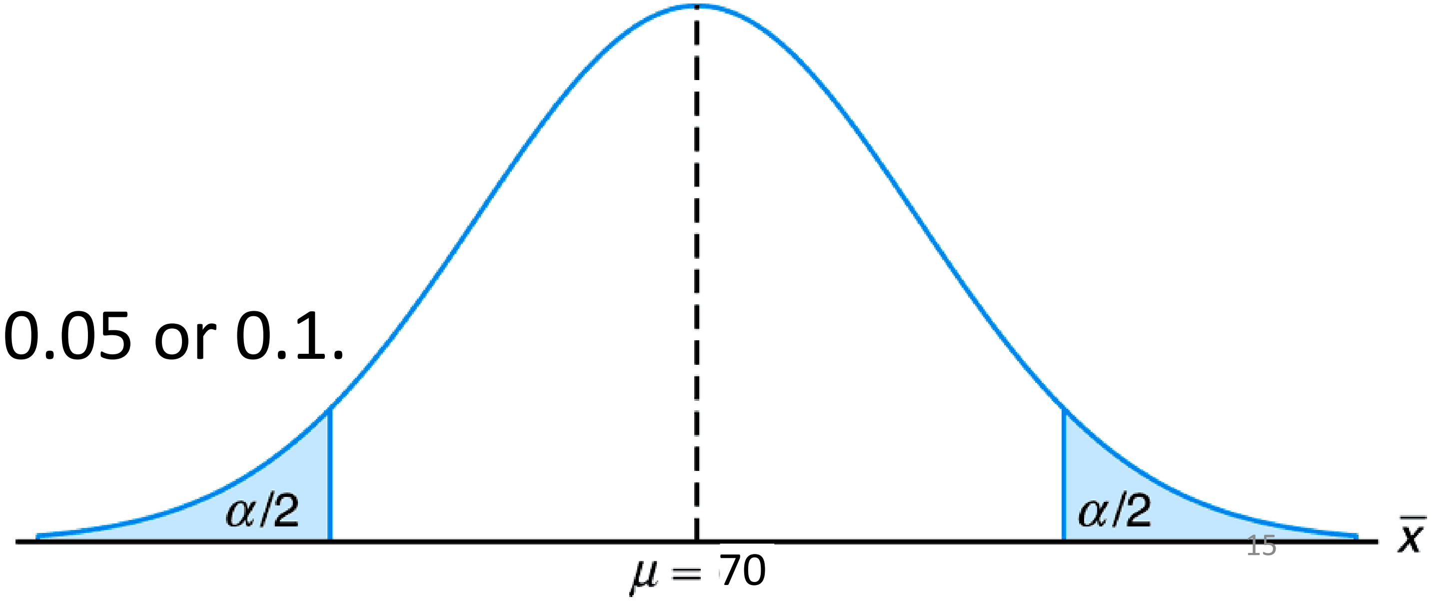
- Note that we start with the assumption that H_0 is correct.
- If $\bar{x}_{obs}$ is an extreme observation then
  - ***H_0 can be suspected and can be concluded not to be correct.***
- ***Hence we say that we reject H0***
- *Try it on your notes now…*

- *One problem…*
  - *How do we define a point to be extreme?*

HT for the mean – σ known

- We will be given the definition of extreme points through the significance level α.
- For example, if the significance level is given as $\alpha = 0.10$, then
  - The left most 5% and the right most %5 points are extreme.
- If our $\bar{x}_{obs}$ is there, then we can reject H0.

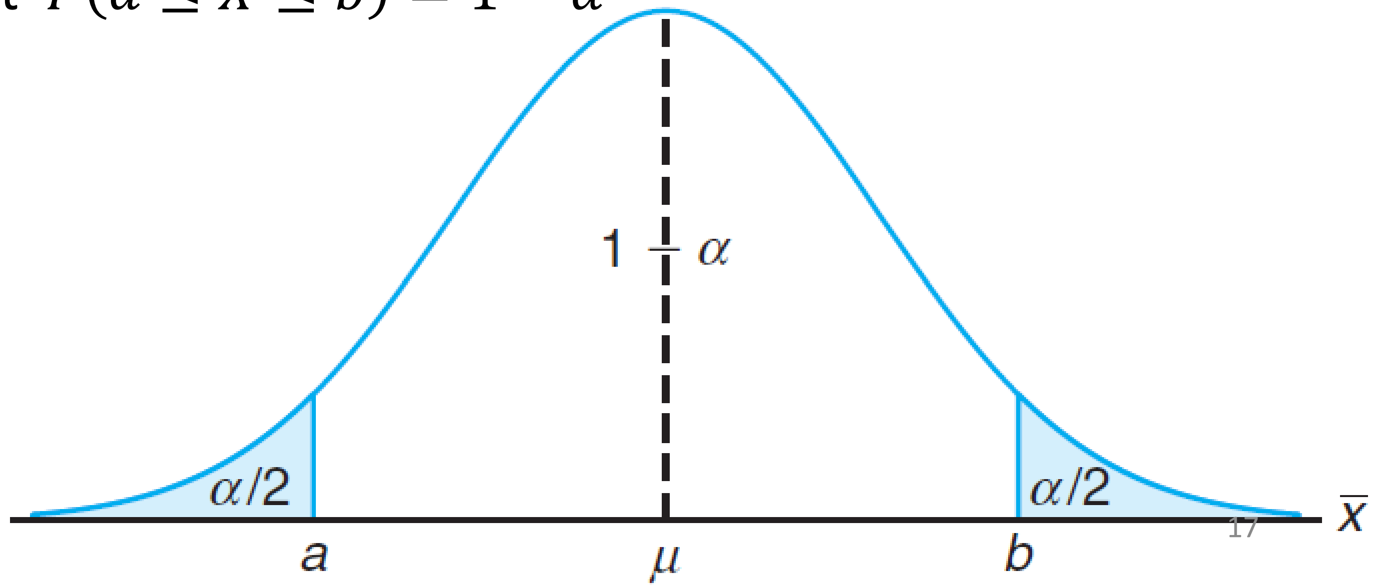
In general we pick α as 0.01, 0.05 or 0.1.



- Solve the same example with
  - $\alpha = 0.10$
  - $\alpha = 0.05$
  - $\alpha = 0.01$

HT for the mean – σ known

- **Critial Region / Acceptance Region:**
  - If the value of our test statistic is in the **Critical Region**,
    - then we reject H_0
  - We use the significance level α to find the critical region.
  - Pick two values a and b such that $P(a \leq \bar{X} \leq b) = 1 - \alpha$
  - Hence the critical region is
    - $\bar{x} \geq b \; or \; \bar{x} \leq a$
  - The remaining region is(?)
    - **acceptance region: [a,b]**



HT for the mean – σ known

- How to find **a** and **b** in this example?

- We know that
  - $\bar{X} \sim N(\mu, \frac{\sigma}{\sqrt{n}})$.
  - $P(-z_{\alpha/2} < Z < z_{\alpha/2})$=$1-\alpha$

- Determine $z_{\alpha/2}$.
  - For this example let $\alpha = 0.05$
  - Then, using the table A.3, we have $z_{\alpha/2} = z_{0.025} = 1.96$

- Finally we can use $Z = \frac{\bar{X}-\mu}{\sigma/\sqrt{n}}$, to find **a** and **b**
  - $\boldsymbol{a} = \boldsymbol{\mu} - \boldsymbol{z_{\alpha/2}} \times \boldsymbol{\sigma}/\sqrt{\boldsymbol{n}}$ = $68 - 1.96 \times 8.9/10$ = 66.25
  - $\mathbf{b} = \boldsymbol{\mu} + \boldsymbol{z_{\alpha/2}} \times \boldsymbol{\sigma}/\sqrt{\boldsymbol{n}}$ = $68 + 1.96 \times 8.9/10$ = 69.75

HT for the mean – σ known

Conclusion:

- The rejection region is $\left[-\infty, \boldsymbol{\mu} - \boldsymbol{z}_{\frac{\alpha}{2}} \times \frac{\boldsymbol{\sigma}}{\sqrt{\boldsymbol{n}}}\right], \left[\boldsymbol{\mu} + \boldsymbol{z}_{\frac{\alpha}{2}} \times \frac{\boldsymbol{\sigma}}{\sqrt{\boldsymbol{n}}}, \infty\right]$
  - $[-\infty, 68.25], [71.75, +\infty]$

- Our <u>**observed**</u> $\bar{X}$ is 71.8, which is greater than 71.75.

- Hence we reject the hypothesis that the mean of the underlying distribution is 70 at a significance level of $\alpha = 0.05$.

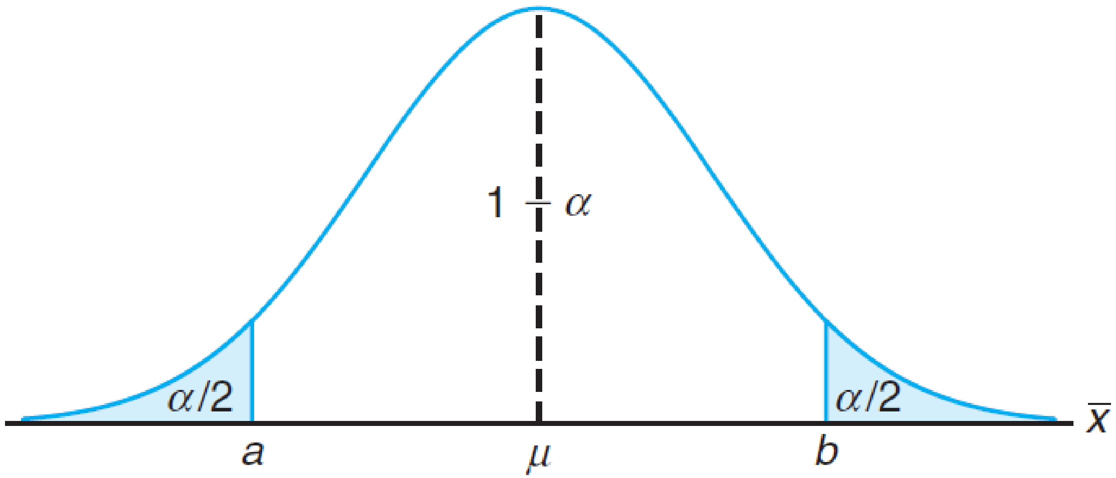
- In other words, our observation does not support the conjecture / hypothesis which says the real mean is 70.

HT for the mean – σ known – Standardizing…

- Another way is to choose the test statistic as $Z = \frac{\bar{X} - \mu}{\sigma/\sqrt{n}}$
- We found that $z_{\alpha/2} = z_{0.025} = 1.96$
- Now calculate **observed Z value is under null hypothesis is:**:
  - $z_{obs} = \frac{71.8 - 70}{8.9/10}$ = 2.022

  which is greater that 1.96.

Hence we reject the hypothesis.



HT Testing – Summary

1. *State the null and alternative hypotheses.*
2. *Choose an appropriate test statistic*
3. *Establish the critical region using the significance level α*
4. *Calculate test statistic's observed value under H_0*
5. *Reject H_0 if the computed test statistic is in the critical region. Otherwise, do not reject.*

Hypothesis Testing – General Concepts

- The alternative hypothesis H_1 usually represents
  - the question to be answered or
  - the theory to be tested.
- The null hypothesis H_0
  - is the opposite of H_1 and
  - is often the logical complement of H_1.
- At the end of a hypothesis testing procedure we will arrive at one of the following two conclusions:
  - *Reject* H_0 in favor of H_1, or
  - *Fail to reject* H_0 (insufficient evidence in data)…➔ not accepting H0

Hypothesis Testing – General Concepts

- Note that there is no such conclusion as to “Accept H0.”
  - only state that “we fail to reject H0.
- Consider the following example:
  - H_0 *: The defendant is innocent*
  - H_1 *: The defendant is guilty*
- There is the suspicion of guilt,
  - but ***any suspect is considered innocent*** until there is sufficient evidence
- Failure to reject H_0 may not always imply innocence,
  - but shows that the evidence was insufficient to convict.