# The Simplex Algorithm

## Why Use a Special Algorithm

- Exhaustive search takes too long
  - Too many feasible solutions
- We want to ask many *"what if"* questions
  - So we run the model over and over
- We want to perform sensitivity analysis
  - What constraints are binding?
  - How much do the constraints cost us?
  - Which products are the most profitable?

# What is The Simplex Algorithm?

- Efficient procedure that searches a subset of the feasible solutions
- Hill climbing procedure that finds better and better solutions
- Procedure that calculates the cost of the constraints and value of the products ***at the margin***

# Terminology

- Feasible solution
  - Solution where are constraints are satisfied
  - Many are possible

- Optimal solution
  - Feasible solution with highest (or lowest) objective function value
  - Usually are one, but ties are possible and happen frequently

- Boundary equation
  - Constraint with inequality replaced by an equality
  - These define the ***feasible region***

# The Simplex Method

- This method on the assumption that the optimal solution will be an extreme point.
  - The constraint equations are reformulated as equalities by introducing slack variables.

**For Example:** From problem I

Cons(1) $7\ X_1 + 11\ X_2 \leq 77$

A slack variable $\mathbf{S_1}$ as the amount of **raw** gas that is not used for a particular production level ($X_1$, $X_2$). If this quantity is added to the left side of constraint (1), it becomes

Cons(1) $7\ X_1 + 11\ X_2 + \mathbf{S_1} = 77$

What $\mathbf{S_1}$ **tells us?**

- $\mathbf{S_1 > 0}$**,** have some slack for this constrain.
- $\mathbf{S_1 < 0}$**,** have exceeded this constrain.
- $\mathbf{S_1 = 0}$**,** exactly meet this constrain.

# The Simplex Method

A different slack variable is developed for each constraint equation, and form ***fully augmented version***.

$Z = 150\ X_1 + 175\ X_2$

$$
\begin{array}{llllll}
7\ X_1 + 11\ X_2 + \mathbf{S_1} & & & & = 77 & \text{(a)} \\
10\ X_1 + 8\ X_2 & + \mathbf{S_2} & & & = 80 & \text{(b)} \\
X_1 & & + \mathbf{S_3} & & = 9 & \text{(c)} \\
\quad X_2 & & & + \mathbf{S_4} & = 6 & \text{(d)}
\end{array}
$$

$X_1, X_2, S_1, S_2, S_3, S_4 \geq 0$, Now we formed a system of linear equations.

- This problem involves solving 4 equations with 6 unknowns.
- The difference between the number of unknown and number of equations equal to 2.
- So every feasible point has **2 variables** out of 6 equal to **zero**.

# The Simplex Method

For example, five corner points of the area ABCDE have the following zero values:

| **Extreme point** | **Zero Variables** |
|---|---|
| A | $X_1, X_2$ |
| B | $X_2$ , $S_2$ |
| C | $S_1, S_2$ |
| D | $S_1, S_4$ |
| E | $X_1, S_4$ |

- Thus, extreme points can be determined from the standard form by setting two of variables equal to zero.
  - For example, setting point E, $X_1 = S_4 = 0$, reduces the standard form to

$$
\begin{array}{llll}
11\,X_2 + \mathbf{S_1} & & & = 77 \\
8\,X_2 & + \mathbf{S_2} & & = 80 \\
 & & + \mathbf{S_3} & = 9 \\
X_2 & & & = 6
\end{array}
$$

System of equation can be solved for $X_2 = 6$, $S_1 = 11$, $S_2 = 32$, and $S_3 = 9$ And $X_1 = S_4 = 0$, These values define point E

# The Simplex Method

- To generalize, a basic solution for **m** linear equations with **n** unknowns is developed by setting **n-m** variables to zero.
- Solve the **m** equations for the **m** remaining unknowns.
- The zero variables are formally referred to as ***nonbasic variables***, whereas the remaining **m** variables are called ***basic variables***.
- If all the basic variables are nonnegative, the result is called a **basic feasible solution**. The optimum will be one of these.
- Now a direct approach to determining the optimal solution would be to calculate all the basic solutions, determine which were feasible, and among those, which had the highest value of Z.
- But this is not a **wise approach**
  - If m = 10 with n = 16, you have to 8008 [= 16! / (10! 6!)] 10 x 10 systems of equations to solve!
  - A significant portion of these may be infeasible.

# Implementation of the Simplex Method

The simplex method avoids inefficiencies outlined in the previous section.

- It starts with a basic feasible solution.
  - It moves through a sequence of other basic feasible solutions and
  - Each step improves the value of the objective function.
  - Eventually, the optimal value is reached and the method is terminated.

**Examples:**

- Start at a basic feasible solution (an extreme corner point of the feasible space).
  - Starting point A; $X_1 = X_2 = 0$.
  - The original 6 equations with 4 unknowns become

$$\begin{aligned} S_1 \qquad\qquad\qquad &= 77 \\ S_2 \qquad\qquad &= 80 \\ S_3 \qquad &= 9 \\ S_4 &= 6 \end{aligned}$$

Thus, the starting values for the basic variables are given automatically as being equal to the right-hand sides of the constraints.

# Implementation of the Simplex Method

Summarize key information in a ***tableau***.

| Basic | Z | $X_1$ | $X_2$ | $S_1$ | $S_2$ | $S_3$ | $S_4$ | Solution | Intercept |
|---|---|---|---|---|---|---|---|---|---|
| Z | 1 | -150 | -175 | 0 | 0 | 0 | 0 | 0 | |
| $S_1$ | 0 | 7 | 11 | 1 | 0 | 0 | 0 | 77 | 11 |
| $S_2$ | 0 | 10 | 8 | 0 | 1 | 0 | 0 | 80 | 8 |
| $S_3$ | 0 | 1 | 0 | 0 | 0 | 1 | 0 | 9 | 9 |
| $S_4$ | 0 | 0 | 1 | 0 | 0 | 0 | 1 | 6 | $\infty$ |

- The objective function is $Z - 150 X_1 - 175 X_2 - 0 S_1 - 0 S_2 - 0 S_3 - 0 S_4 = 0$
- The next step involves moving to a new basic feasible solution that leads to an improvement of the objective function.
  - To accomplish this, increase a current nonbasic variable to basic (nonzero) (at this point, X1 or X2) so that Z increases. This variable is called the *entering variable.*
  - Extreme points must have 2 zero values. So, one of the current basic variables ($S_1$, $S_2$, $S_3$, or $S_4$) must also be set to zero (nonbasic). This variable is called the *leaving variable.*

# Implementation of the Simplex Method

Develop a mathematical approach for choosing the entering and leaving variables.

- The entering variable can be any variable in the objective function having a negative coefficient (because it will make Z bigger). For our case, $X_2$.
- Choose a leaving variable from among basic variables.
  - One way is to calculate the values at which the constraint lines intersect the axis or line corresponding to the leaving variable (in our case, the $X_1$ axis). We can calculate this value as the ratio of the right-hand side of the constraint (the "Solution" column of the tableau) to the corresponding coefficient of $X_1$.
    - For example, for the first constraints slack variable $S_1$, the result is, Intercept= 77/7 = 11,

| Intercept |
|---|
| (Solution/ $X_1$) |
| $S_1 = 11$ |
| $S_2 = 8$ |
| $S_3 = 9$ |
| $S_4 = \infty$ |

Because 8 is the smallest positive integer, it means that the second constraint line will be reached first as $X_1$ is increased.
Therefore, $S_2$ should be the entering variable.

# Implementation of the Simplex Method

At this point, we have moved to point B ($X_2 = S_2 = 0$), and the new basic solution becomes

$$\begin{aligned}
7X_1 + S_1 &= 77 \\
10X_1 &= 80 \\
X_1 \quad + S_3 &= 9 \\
+S_4 &= 6
\end{aligned}$$

The solution of this system of equations at point B: $X_1 = 8$, $S_1 = 21$, $S_3 = 1$, and $S_4 = 6$.

- The tableau can be used to make the same calculation by employing the Gauss-Jordan method.

- Gauss-Jordan involved converting type pivot element to 1 and then eliminating the coefficients in the same column above and below the pivot element

# Implementation of the Simplex Method

For this example, the pivot row is $S_2$ (the entering variable) and the pivot element is 10 (the coefficient of the leaving variable, $X_1$). Dividing the row by 10 and replacing $S_2$ by $X_1$ gives

| Basic | Z | $X_1$ | $X_2$ | $S_1$ | $S_2$ | $S_3$ | $S_4$ | Solution | Intercept |
|---|---|---|---|---|---|---|---|---|---|
| Z | 1 | -150 | -175 | 0 | 0 | 0 | 0 | 0 | |
| $S_1$ | 0 | 7 | 11 | 1 | 0 | 0 | 0 | 77 | |
| $X_1$ | 0 | 1 | 0.8 | 0 | 0.1 | 0 | 0 | 8 | |
| $S_3$ | 0 | 1 | 0 | 0 | 0 | 1 | 0 | 9 | |
| $S_4$ | 0 | 0 | 1 | 0 | 0 | 0 | 1 | 6 | |

Next, the $X_1$ coefficients in the other rows can be eliminated, for example the pivot row is multiplied by –150 and subtracted from the first row to give

| Z | $X_1$ | $X_2$ | $S_1$ | $S_2$ | $S_3$ | $S_4$ | Solution |
|---|---|---|---|---|---|---|---|
| 1 | -150 | -175 | 0 | 0 | 0 | 0 | 0 |
| -0 | -(-150) | -(-120) | -0 | -(-15) | 0 | 0 | -(-1200) |
| 1 | 0 | -55 | 0 | 15 | 0 | 0 | 1200 |

# Implementation of the Simplex Method

Similar operations can be performed on the remaining rows to give the new ***Tableau***

| Basic | Z | $X_1$ | $X_2$ | $S_1$ | $S_2$ | $S_3$ | $S_4$ | Solution | Intercept |
|---|---|---|---|---|---|---|---|---|---|
| Z | 1 | 0 | -55 | 0 | 15 | 0 | 0 | 1200 | |
| $S_1$ | 0 | 0 | 5.4 | 1 | -0.7 | 0 | 0 | 21 | 3.889 |
| $X_1$ | 0 | 1 | 0.8 | 0 | 0.1 | 0 | 0 | 8 | 10 |
| $S_3$ | 0 | 0 | -0.8 | 0 | -0.1 | 1 | 0 | 1 | -1.25 |
| $S_4$ | 0 | 0 | 1 | 0 | 0 | 0 | 1 | 6 | 6 |

The objective function has increased to Z = 1200.

- Next, and in this case final step.
- Only one more variable, $X_2$, has a negative value in the objective function, and it is therefore chosen as the leaving variable.
  - According to the intercept values (now calculated as the solution column over the coefficients in the $X_2$ column), the first constraint has the smallest positive value, and therefore, $S_1$ is selected as the entering variable.

# Implementation of the Simplex Method

Thus, the simplex method moves us from points *B* to C in Figure. Finally, the Gauss-Jordan elimination can be implemented to solve the simultaneous equations.

| Basic | Z | $X_1$ | $X_2$ | $S_1$ | $S_2$ | $S_3$ | $S_4$ | Solution | Intercept |
|---|---|---|---|---|---|---|---|---|---|
| Z | 1 | 0 | -55 | 0 | 15 | 0 | 0 | 1200 | |
| $X_2$ | 0 | 0 | 5.4/5.4 | 1/5.4 | -0.7/5.4 | 0 | 0 | 21/5.4 | |
| $X_1$ | 0 | 1 | 0.8 | 0 | 0.1 | 0 | 0 | 8 | |
| $S_3$ | 0 | 0 | -0.8 | 0 | -0.1 | 1 | 0 | 1 | |
| $S_4$ | 0 | 0 | 1 | 0 | 0 | 0 | 1 | 6 | |

Next, the $X_2$ coefficients in the other rows can be eliminated, for example the pivot row is multiplied by –55 and subtracted from the first row to give

| Z | $X_1$ | $X_2$ | $S_1$ | $S_2$ | $S_3$ | $S_4$ | Solution |
|---|---|---|---|---|---|---|---|
| 1 | 0 | -55 | 0 | 15 | 0 | 0 | 1200 |
| -0 | -0 | -(-55x1) | -(-55x 0.1852) | -(-55x (-0.1296)) | 0 | 0 | -(-55x (3.889)) |
| 1 | 0 | 0 | 10.186 | 7.8704 | 0 | 0 | 1413.895 |

# Implementation of the Simplex Method

The result is the final ***Tableau.***

| Basic | Z | $X_1$ | $X_2$ | $S_1$ | $S_2$ | $S_3$ | $S_4$ | Solution | Intercept |
|---|---|---|---|---|---|---|---|---|---|
| Z | 1 | 0 | 0 | 10.186 | 7.8704 | 0 | 0 | 1413.895 | |
| $X_2$ | 0 | 0 | 1 | 0.1852 | -0.1296 | 0 | 0 | 3.889 | |
| $X_1$ | 0 | 1 | 0 | -0.1481 | 0.2037 | 0 | 0 | 4.889 | |
| $S_3$ | 0 | 0 | 0 | 0.1481 | -0.2037 | 1 | 0 | 4.111 | |
| $S_4$ | 0 | 0 | 0 | -0.1852 | 0.1296 | 0 | 1 | 2.111 | |

- Result is final, since there is no negative coefficient remains in the objective function row.
- The final solution is $X_1$= 4.889, $X_2$= 3.889 which gives the maximum objective function of Z= 1413.895

# Example Problem I

**Problem Statement:**

A furniture company manufactures desks, tables, and chairs. The manufacture of each type of furniture requires lumber, finishing and carpentry. The amount of each resource needed to make each type of furniture is given in following Table,

| **Resource** | **Product** | | | **Available sources** |
|---|---|---|---|---|
| | **Desk** | **Table** | **Chair** | |
| Lumber | 8 board ft | 6 board ft | 1 board ft | 48 board ft |
| Finishing hours | 4 hours | 2 hours | 1.5 hours | 20 hours |
| Carpentry hours | 2 hours | 1.5 hours | 0.5 hours | 8 hours |
| Profit | **$60** | **$30** | **$20** | |

# Solving the LPP I with Simplex Method

Develop a linear programming formulation to maximize the profits for this operation.

**Solution**.
X1: number of desks produced
X2: number of tables produced
X3: number of chairs produced

- Total profit max $Z = 60\,X1 + 30\,X2 + 20\,X3$
- Subject to

$$
\begin{array}{lll}
8\,X1 + 6\,X2 + X3 & \le 48 & \\
4\,X1 + 2\,X2 + 1.5\,X3 & \le 20 & \\
2\,X1 + 1.5\,X2 + 0.5\,X3 & \le 8 & \\
X1,\ X2,\ X3 & \ge 0 & \text{Positivity constraint}
\end{array}
$$

# Simplex Solution to the LPP I

The simplex tableau for the problem can be set up and solved as

- Start at a basic feasible solution (an extreme corner point of the feasible space).

Summarize key information in a ***tableau***

| | Basic Variable | | | | | | | Solution |
|---|---|---|---|---|---|---|---|---|
| Row 0 | Z | -60 X1 | -30X2 | -20 X3 | | | | = 0 |
| Row 1 | S1 | 8 X1 | +6 X2 | + X3 | + S1 | | | = 48 |
| Row 2 | S2 | 4 X1 | +2 X2 | +1,5X3 | | + S2 | | = 20 |
| Row 3 | S3 | 2 X1 | +1.5 X2 | +0.5X3 | | | + S3 | = 8 |

Determining the Entering Value

- We choose the entering variable in a max problem to be the non-basic variable with the most negative coefficient in Row 0.
- In this problem each one unit increase of X1 increases Z by 60. We would like to make X1 as large as possible.

# Simplex Solution to the LPP I

S1 >=0 For X1 <= 48/8 = 6
S2 >=0 For X1 <= 20/4 = 5
S3 >=0 For X1 <= 8/2 = 4

- This means to keep all the basic variables nonnegative, the largest that we can make X1 is 4. If X1>4 then S3 will be negative and we will no longer have a basic feasible solution.

- To make X1 a basic variable in Row 3, we use elementary row operations to make X1 have a coefficient of 1 in Row 3, and coefficient of 0 in all other rows. Row 3 is called pivot row. We will now make iterations (IT) to make row 3's X1 coefficient 1 and other's 0.

# Simplex Solution to the LPP I

IT 1 : We multiply Row3 by ½. The result will be

X1 + 0.75 X2 + 0.25 X3 + 0.5 S3 = 4    Row3'

IT 2 : To create a zero coefficient for X1 in Row0, we replace Row0 with 60*(Row3')+Row 0

Z + 15 X2 – 5 X3 + 30 S3 = 240    Row0'

IT 3 : To create a zero coefficient for X1 in Row1, we replace Row1 with –8(Row3')+Row1

- X3 + S1 – 4 S3 = 16    Row1'

IT 4 : To create a zero coefficient for X1 in Row 2, we replace Row 2 with –4(Row3')+Row2

- X2 + 0.5X3 + S2 – 2 S3 = 4    Row2'

# Simplex Solution to the LPP I

After these iterations our new table is shown below:

| | Basic Variable | | | | | | | Solution |
|---|---|---|---|---|---|---|---|---|
| Row 0' | Z | 0 | +15 X2 | -5 X3 | | | + 30 S3 | =240 |
| Row 1' | S1 | 0 | 0 | -X3 | + S1 | | - 4 S3 | =16 |
| Row 2' | S2 | 0 | -X2 | +0.5X3 | | + S2 | - 2 S3 | =4 |
| Row 3' | X1 | X1 | +0.75X2 | +0.25X3 | | | + 0.5S3 | =4 |

- The new objective function will be :

  $Z = 240 - 15\ X2 + 5\ X3 - 30\ S3$
- We still look for having a larger z value. If we increase X2 and S3, these variables will decrease z function. So we choose X3 (the most negative coefficient in Row0'); because if we increase X3 one unit the objective function will increase by 5.

# Simplex Solution to the LPP I

In Row1' there will be no ratio, because X3 has negative coefficient in Row1'.
Row 2' : 4/0.5 = 8
Row 3' : 4/0.25 = 16

The smallest ratio occurs in Row2' so we should make iterations to X3 in Row2'. We will make coefficient of X3 1 in Row2' and 0 to others.

IT 1 : We multiply Row2' by 2 to make 1 coefficient

$- 2X2 + X3 + 2S2 – 4S3 = 8$ Row2''

IT 2 : To create a zero coefficient of 0 for X3 in Row0', we replace Row0' with 5*(Row2'')+Row0'

$Z + 5X2 + 10S2 + 10S3 = 280$ Row0''

IT 3 : To create a zero coefficient of 0 for X3 in Row1', we replace Row1' with Row2''+Row1'

$- 2X2 + S1 + 2S2 – 8S3 = 24$ Row1''

IT 4 : To create a zero coefficient of 0 for X3 in Row3', we replace Row3' with – 1/4*(Row2'')+Row3'

$X1 + 1.25X2 – 0.5S2 + 1.5S3 = 2$ Row3''

# Simplex Solution to the LPP I

After these iterations our new table is shown below :

| | Basic Variable | | | | | | | Solution |
|---|---|---|---|---|---|---|---|---|
| Row 0" | Z | | +5 X2 | | | + 10 S2 | +10 S3 | =280 |
| Row 1" | S1 | | -2 X2 | | +S1 | + 2 S2 | -8 S3 | =24 |
| Row 2" | X3 | | -2 X2 | + X3 | | + 2 S2 | -4 S3 | =8 |
| Row 3" | X1 | X1 | + 1.25 X2 | | | -0.5 S2 | +1.5 S3 | =2 |

The new objective function will be :

$$Z = 280 - 5x2 - 10s2 - 10s3$$

We see that increasing X2 by 1 (while S2 and S3 = 0) will decrease Z by 5. Increasing S2 and S3 will decrease Z by 10. All coefficients in Row0’’ are nonnegative and we can’t choose any other variable which will make Z larger. We reach the optimal solution for furniture company.

Z= 280, X1=2 , X3=8. Slack Variables : S1 =24

# Example Problem II

**Problem Statement:**

Suppose that a gas processing plant receives a fixed amount of raw gas each week. The raw gas is processed into three grades of heating gas, regular, premium and supreme quality. These grades of gas are in high demand (that is, they are guaranteed to sell) and yield different profits to the company. However, their production involves both time and on-site storage constraints. For example, only one of the grades can be produced at a time, and the facility is open for only 80 hrs/week. Further, there is limited on-site storage for each of the products. All these factors are listed below (note that a metric ton is equal to 1000 kg):

| | Product | | | |
|---|---|---|---|---|
| **Resource** | **Regular** | **Premium** | **Supreme** | **Resource Availability** |
| Raw gas | 7 $m^3$ /Ton | 11 $m^3$ /Ton | 15 $m^3$ /Ton | 154 $m^3$ /Week |
| Production time | 10 hr /Ton | 8 hr /Ton | 12 hr /Ton | 80 hr /Week |
| Storage | 9 Ton | 6 Ton | 5 Ton | |
| Profit | 150 /Ton | 175 /Ton | 250 /Ton | |

# Solving the LPP II with Simplex Method

Develop a linear programming formulation to maximize the profits for this operation.

**Solution**. The engineer operating this plant must decide how much of each gas to produce to maximize profits. If the amounts of regular, premium and supreme produced weekly are designated as $X_1$ $X_2$, and $X_3$ respectively, the total weekly profit can be calculated as

- Total profit $Z = 150 X_1 + 175 X_2 + 250 X_3$
- Subject to

| | |
|---|---|
| $7 X_1 + 11 X_2 + 15 X_3 \leq 154$ m$^3$ /Week | Material constraint |
| $10 X_1 + 8 X_2 + 12 X_3 \leq 80$ hr /tonne | Time constraint |
| $X_1 \leq 9$ | "Regular" storage constraint |
| $X_2 \leq 6$ | "Premium" storage constraint |
| $X_3 \leq 5$ | "Supreme" storage constraint |
| $X_1, X_2, X_3 \geq 0$ | Positivity constraint |

# Simplex Solution to the LPP II

The simplex tableau for the problem can be set up and solved as

- Start at a basic feasible solution (an extreme corner point of the feasible space).
  - Starting point A; $X_1 = X_2 = X_3 = 0$.
  - The original 5 equations with 5 unknowns become

$$\begin{aligned} S_1 \quad\quad\quad\quad\quad &= 154 \\ S_2 \quad\quad\quad\quad &= 80 \\ S_3 \quad\quad\quad &= 9 \\ S_4 \quad\quad &= 6 \\ S_5 \quad &= 5 \end{aligned}$$

Summarize key information in a ***tableau***.

| Basis | P | $X_1$ | $X_2$ | $X_3$ | $S_1$ | $S_2$ | $S_3$ | $S_4$ | $S_5$ | ***Solution*** | **Intercept** |
|---|---|---|---|---|---|---|---|---|---|---|---|
| **P** | **1** | **-150** | **-175** | **-250** | 0 | 0 | 0 | 0 | 0 | 0 | |
| $S_1$ | 0 | **7** | **11** | **15** | **1** | 0 | 0 | 0 | 0 | **154** | **10.2667** |
| $S_2$ | 0 | **10** | **8** | **12** | 0 | **1** | 0 | 0 | 0 | **80** | **6.66667** |
| $S_3$ | 0 | **1** | 0 | 0 | 0 | 0 | **1** | 0 | 0 | **9** | ∞ |
| $S_4$ | 0 | 0 | **1** | 0 | 0 | 0 | 0 | **1** | 0 | **6** | ∞ |
| $S_5$ | 0 | 0 | 0 | **1** | 0 | 0 | 0 | 0 | **1** | **5** | **5** |

The objective function is $Z - 150 X_1 - 175 X_2 - 250 X_3 – 0 S_1 – 0 S_2 – 0 S_3 – 0 S_4 – 0 S_5 = 0$

# Simplex Solution to the LPP II

- The next step involves moving to a new basic feasible solution that leads to an improvement of the objective function.
  - The ***entering variable*** can be any variable in the objective function having a negative coefficient (because it will make Z bigger). For our case, $X_3$.
  - Choose a ***leaving variable*** from among basic variables with the smallest positive integer value of intercept, for our case, **$S_5$**.

| Basis | P | $X_1$ | $X_2$ | $X_3$ | $S_1$ | $S_2$ | $S_3$ | $S_4$ | $S_5$ | *Solution* | Intercept |
|---|---|---|---|---|---|---|---|---|---|---|---|
| P | 1 | -150 | -175 | 0 | 0 | 0 | 0 | 0 | 250 | 1250 | |
| $S_1$ | 0 | 7 | 11 | 0 | 1 | 0 | 0 | 0 | -15 | 79 | 7.18182 |
| $S_2$ | 0 | 10 | 8 | 0 | 0 | 1 | 0 | 0 | -12 | 20 | 2.5 |
| $S_3$ | 0 | 1 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 9 | ∞ |
| $S_4$ | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 1 | 0 | 6 | 6 |
| $X_3$ | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 1 | 5 | ∞ |

# Simplex Solution to the LPP II

- The next step involves moving to a new basic feasible solution that leads to an improvement of the objective function.
  - The ***entering variable*** can be $\mathbf{X_2}$.
  - Choose a ***leaving variable*** can be, $\mathbf{S_2}$.

| **Basis** | **P** | **$X_1$** | **$X_2$** | **$X_3$** | **$S_1$** | **$S_2$** | **$S_3$** | **$S_4$** | **$S_5$** | ***Solution*** | **Intercept** |
|---|---|---|---|---|---|---|---|---|---|---|---|
| **P** | **1** | **68.75** | **0** | **0** | 0 | **21.88** | 0 | 0 | **-12.5** | **1687** | |
| **$S_1$** | 0 | **-6.75** | **0** | **0** | **1** | **-1.375** | 0 | 0 | **1.5** | **51.5** | **34.3334** |
| **$X_2$** | 0 | **1.25** | **1** | **0** | 0 | **0.125** | 0 | 0 | **-1.5** | **2.5** | **-1.6667** |
| **$S_3$** | 0 | **1** | 0 | 0 | 0 | 0 | **1** | 0 | 0 | **9** | ∞ |
| **$S_4$** | 0 | **-1.25** | **0** | 0 | 0 | **-0.125** | 0 | **1** | **1.5** | **3.5** | **2.3334** |
| **$X_3$** | 0 | 0 | 0 | **1** | 0 | 0 | 0 | 0 | **1** | **5** | **5** |

# Simplex Solution to the LPP II

- The next step involves improvement of the objective function.
  - The ***entering variable*** can be $\mathbf{S_5}$.
  - Choose a ***leaving variable***, that can be, $\mathbf{S_4}$.

| **Basis** | **P** | $X_1$ | $X_2$ | $X_3$ | $S_1$ | $S_2$ | $S_3$ | $S_4$ | $S_5$ | ***Solution*** |
|---|---|---|---|---|---|---|---|---|---|---|
| **P** | **1** | **58.33** | **0** | **0** | 0 | **20.83** | 0 | **8.33** | **0** | **1716.7** |
| $S_1$ | 0 | **-5.5** | **0** | **0** | **1** | **-1.25** | 0 | **-1** | **0** | **48** |
| $X_2$ | 0 | **0** | **1** | **0** | 0 | **0** | 0 | **1** | **0** | **6** |
| $S_3$ | 0 | **1** | 0 | 0 | 0 | 0 | **1** | 0 | 0 | **9** |
| $S_5$ | 0 | **-0.833** | **0** | 0 | 0 | **-0.083** | 0 | **0.67** | **1** | **2.3334** |
| $X_3$ | 0 | **0.833** | 0 | **1** | 0 | **0.083** | 0 | **-0.67** | **0** | **2.6667** |

# Example Problem III

**Problem Statement:**

A chemical plant makes three major products on a weekly basis. Each of these products requires a certain quantity of raw chemical, different production times, and yields different profits. The pertinent information is summarized below:

| | **Product 1** | **Product 2** | **Product 3** | **Resource Availability** |
|---|---|---|---|---|
| Raw chemical | 5 kg/kg | 4 kg/kg; | 10 kg/kg | 3000 kg |
| Production time | 0.05 hr/kg | 0.1 hr/kg | 0.2 hr/kg | 55 hr/week |
| New Profit | \$30/kg | \$30/kg | \$35/kg | |

Note that there is sufficient warehouse space at the plant to store a total of 450 kg/week.

Maximize $Z = 30\ X_1 + 30\ X_2 + 35\ X_3$ Maximize Profits

Subject to

$5\ X_1 + 4\ X_2 + 10\ X_3 \leq 3000$ Material constraint

$0.05\ X_1 + 0.1\ X_2 + 0.2\ X_3 \leq 55$ Time constraint

$X_1 + X_2 + X_3 \leq 450$ storage constraint

$X_1, X_2, X_3 \geq 0$ Positivity constraint

# Simplex Solution to the LPP III

The simplex tableau for the problem can be set up and solved as

| Basis | P | $X_1$ | $X_2$ | $X_3$ | $S_1$ | $S_2$ | $S_3$ | *Solution* | Intercept |
|---|---|---|---|---|---|---|---|---|---|
| **P** | **1** | **-30** | **-30** | ***-35*** | 0 | 0 | 0 | 0 | |
| $S_1$ | 0 | **5** | **4** | ***10*** | **1** | 0 | 0 | **3000** | **300** |
| $S_2$ | *0* | ***0.05*** | ***0.1*** | ***0.2*** | *0* | *1* | *0* | ***55*** | ***275*** |
| $S_3$ | 0 | **1** | **1** | *1* | 0 | 0 | **1** | **450** | **450** |

- The next step involves moving to a new basic feasible solution that leads to an improvement of the objective function.
  - The ***entering variable*** can be $X_3$.
  - Choose a ***leaving variable*** can be, $S_2$.

| Basis | P | $X_1$ | $X_2$ | $X_3$ | $S_1$ | $S_2$ | $S_3$ | *Solution* | Intercept |
|---|---|---|---|---|---|---|---|---|---|
| **P** | **1** | ***-21.25*** | **-12.5** | **0** | 0 | **175** | 0 | **9625** | |
| $S_1$ | *0* | ***2.5*** | ***-1*** | ***0*** | *1* | ***-50*** | *0* | ***250*** | ***100*** |
| $X_3$ | 0 | ***0.25*** | **0.5** | **1** | 0 | **5** | 0 | **275** | **1100** |
| $S_3$ | 0 | ***0.75*** | **0.5** | **0** | 0 | **-5** | **1** | **175** | **233.3334** |

# Simplex Solution to the LPP III

- The next step involves moving to a new basic feasible solution that leads to an improvement of the objective function.
  - The ***entering variable*** can be $\mathbf{X_1}$.
  - Choose a ***leaving variable*** can be, $\mathbf{S_1}$.

| Basis | P | $X_1$ | $X_2$ | $X_3$ | $S_1$ | $S_2$ | $S_3$ | *Solution* | Intercept |
|---|---|---|---|---|---|---|---|---|---|
| P | 1 | 0 | -21 | 0 | 8.5 | -250 | 0 | 11750 | |
| $X_1$ | 0 | 1 | -0.4 | 0 | 0.4 | -20 | 0 | 100 | -250 |
| $X_3$ | 0 | 0 | 0.6 | 1 | -0.1 | 10 | 0 | 250 | 416.6667 |
| $S_3$ | 0 | 0 | 0.8 | 0 | -0.3 | 10 | 1 | 100 | 125 |

- The next step involves improvement of the objective function.
  - The ***entering variable*** can be $\mathbf{X_2}$.
  - Choose a ***leaving variable*** can be, $\mathbf{S_3}$.

| Basis | P | $X_1$ | $X_2$ | $X_3$ | $S_1$ | $S_2$ | $S_3$ | *Solution* |
|---|---|---|---|---|---|---|---|---|
| P | 1 | 0 | 0 | 0 | 0.625 | 12.5 | 26.25 | 14375 |
| $X_1$ | 0 | 1 | 0 | 0 | 0.25 | -15 | 0.5 | 150 |
| $X_3$ | 0 | 0 | 0 | 1 | 0.125 | 2.5 | -0.75 | 175 |
| $X_2$ | 0 | 0 | 1 | 0 | -0.375 | 12.5 | 1.25 | 125 |