Network Problems

There is a multitude of situations that can be conveniently modeled and solved as networks (nodes connected by branches). As illustrations, consider the following situations:

1. The design of an offshore natural-gas pipeline network connecting well- heads in the Gulf of Mexico to an inshore delivery point. The objective of the model is to minimize the cost of constructing the pipeline.
2. The determination of the shortest route between two cities in an existing network of roads.
3. The determination of the maximum capacity (in tons per year) of a coal slurry pipeline network joining the coal mines in Wyoming with the power plants in Houston. (Slurry pipelines transport coal by pumping water through specially de- signed pipes.)
4. The determination of the minimum-cost flow schedule from oil fields to refineries through a pipeline network.
5. The determination of the time schedule (start and completion dates) for the activities of a construction project.

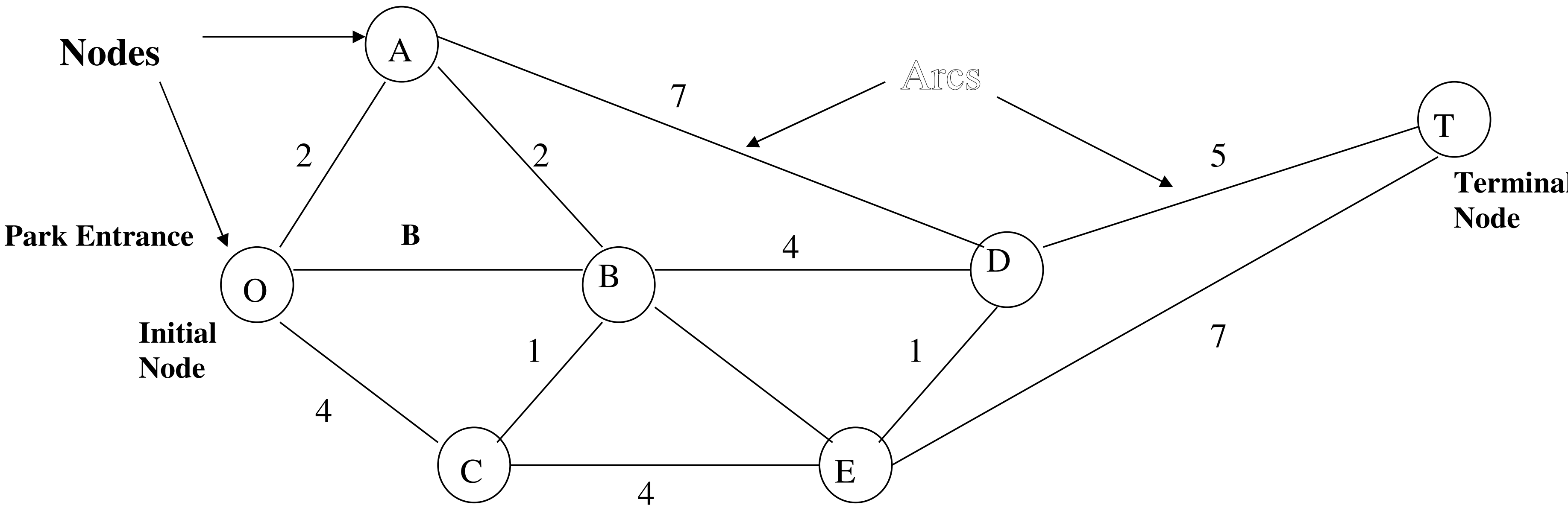
Network Problems

The solution of these situations is accomplished through a variety of network optimization algorithms.

Types of Network Algorithms:

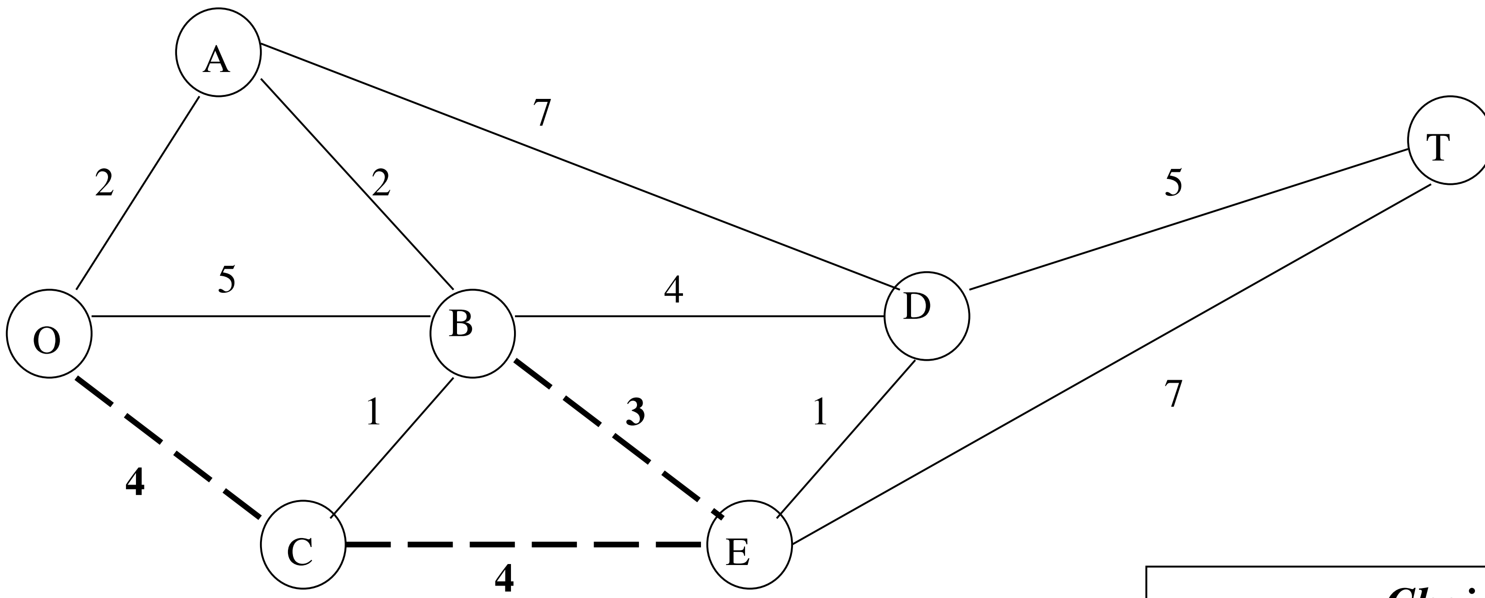
- Shortest Path Problems
- Minimum Spanning Tree Problems
- Maximum Flow Problems
- Critical Path (CPM) and PERT
- Maximum Cost Network Flow Problems

Graph



A Network consists of set of nodes.

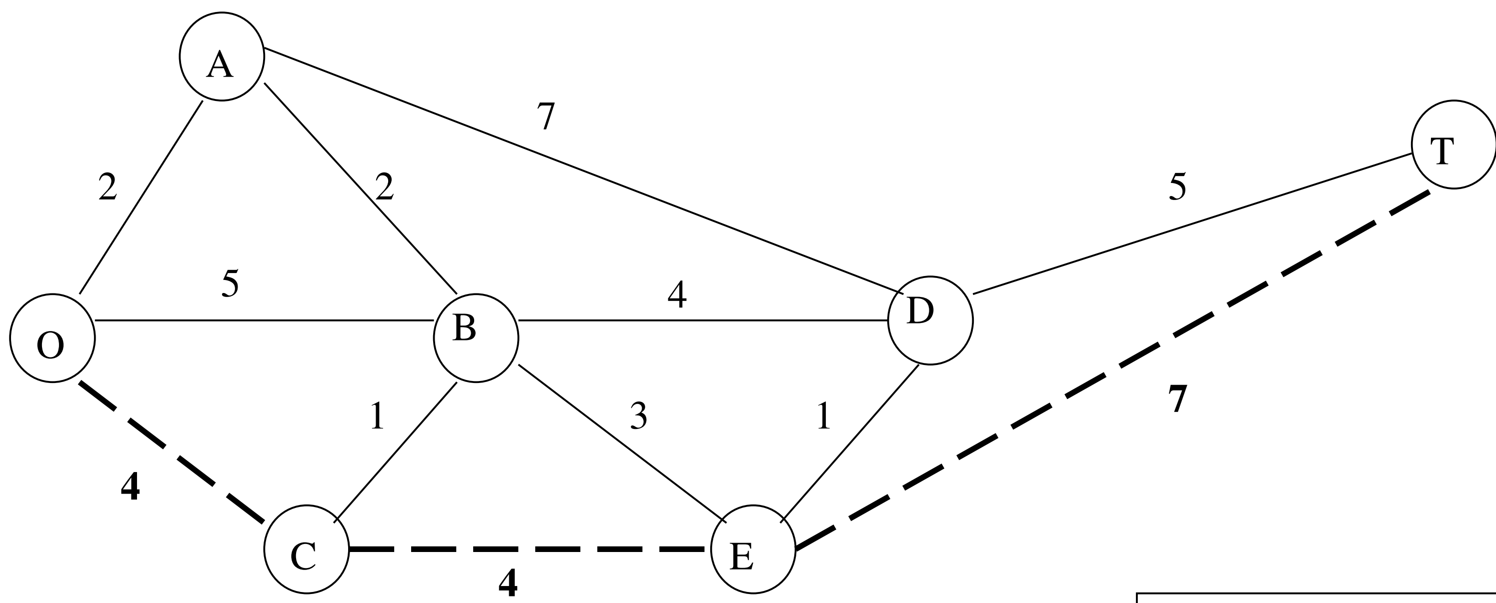
Chain



(O,C) - (C,E) - (B,E)

Chain
Sequence of arcs with exactly one node in common with the previous arc.

Path



(O,C) - (C,E) - (E,T)

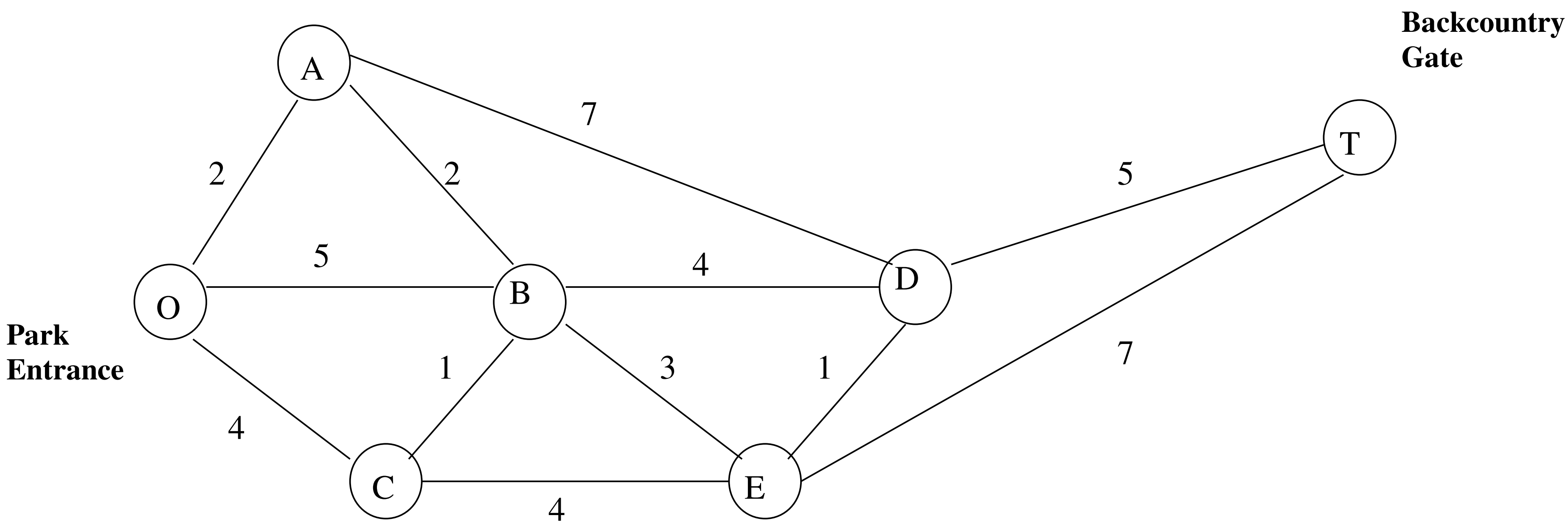
Arc
A chain in which the terminal node of each arc is identical to the initial node of the next arc.

Typical Networks

Flow	Arcs	Nodes
Sortyards	Roads	Pallet
Warehouses	Water Routes	Forklift
Factories	Cable Road	Skidder
Cutblocks	Rail Line	Job Order
Machine Centres	Conveyor Line	Log Boom

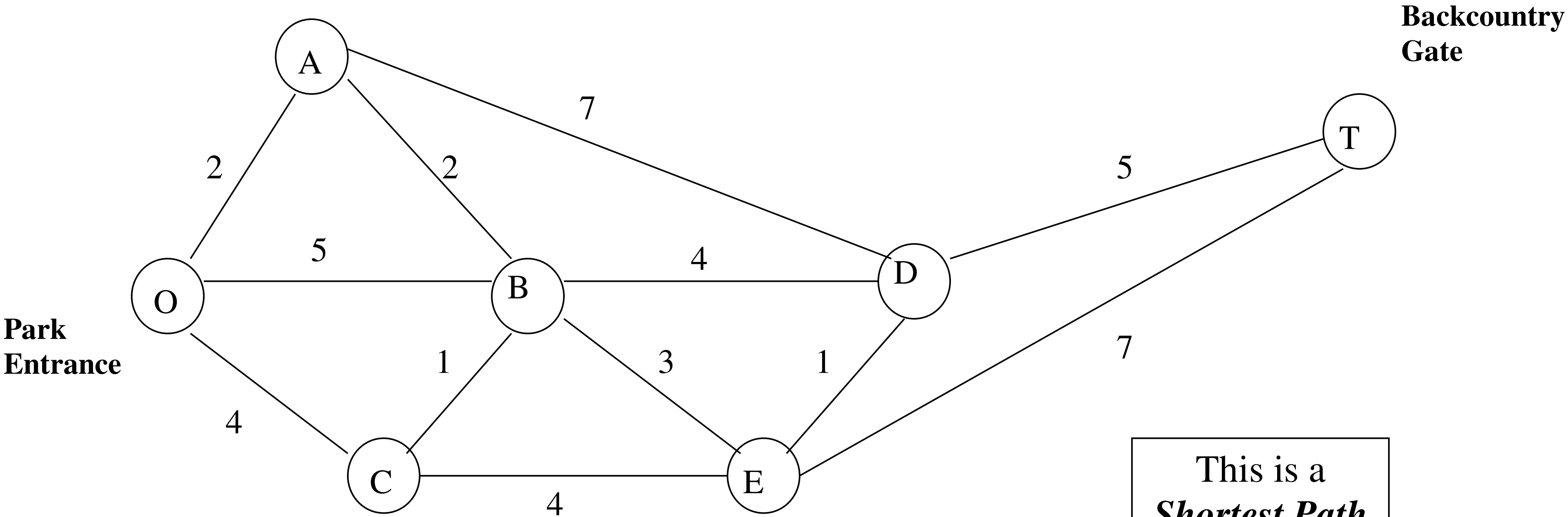
Seaside Park Example

- Wilderness Park with a narrow road system that can be used to develop a tram system to deliver hikers from Entrance **O** to Backcountry Gate **T**.



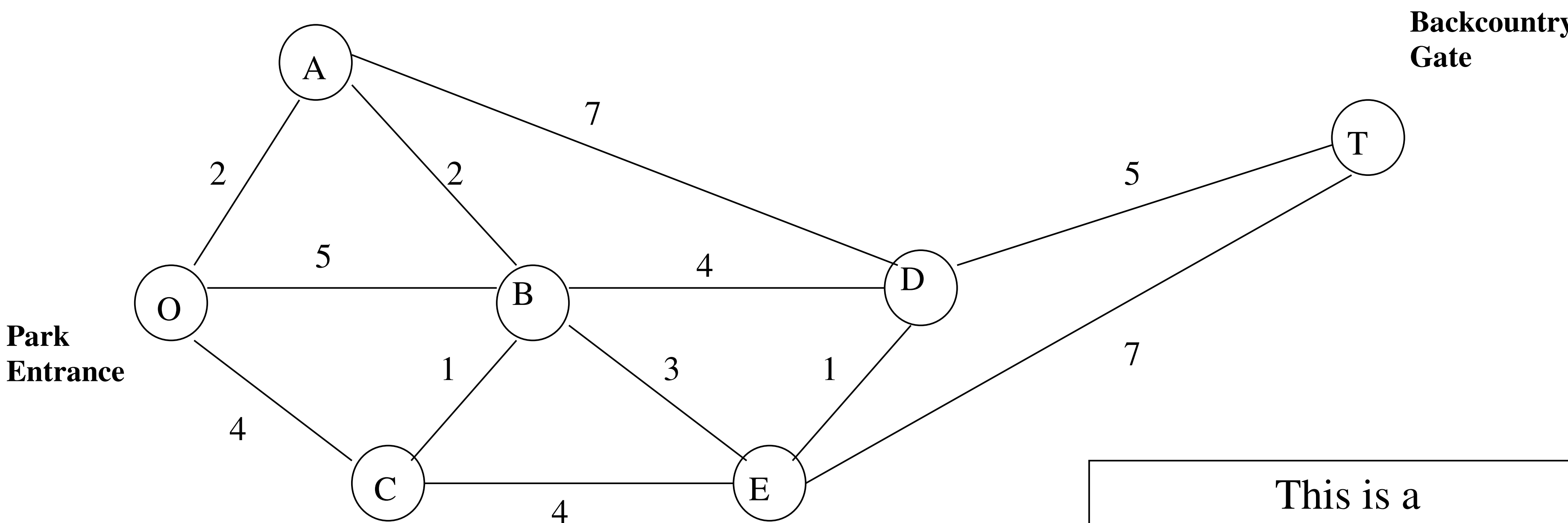
3 Problems

1. Shortest total distance for the trams



This is a ***Shortest Path*** Problem

2. Connect all stations with telephone line with minimal miles of line.



This is a
Minimum Spanning Tree
Problem

3 Problems

3. Maximum Flow on Each Arc

- Limits on number of trips on each arc due to ecology
- At peak season different routes must be taken to reduce to satisfy restrictions

Problem: How to route various trips to maximize number of trips without violating the limits on an individual road.

This is a ***Maximum Flow*** Problem

Shortest Path Problem

- Shortest Route Problem

The shortest-route problem determines the shortest route from a origin to a destination through a connecting network

Solution Mechanism:

1. ***Exhaustive Search***: Search all of the possible arcs in the network and pick the best one.

2. ***Network Algorithm*****:**
   a) Dijkstra's Algorithm
   b) Transshipment Problem

Table of Nodes & Arcs

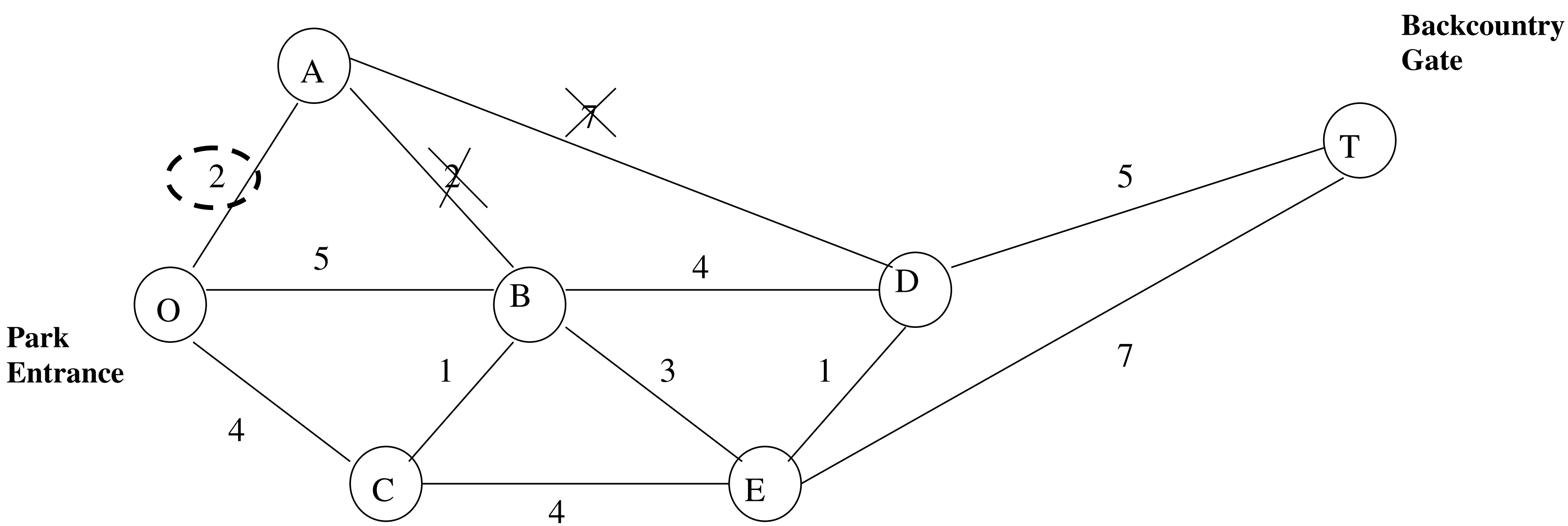
O	A	B	C	D	E	T
OA–2	AB–2	BC–1	CB–1	DE–1	ED–1	
OC–4	AD–7	BA–2	CE–4	DB–4	EB–3	
OB–5		BE–3		DT–5	EC–4	
		BD–4		DA–7	ET–7	

Nearest Node

Step 1. N=1

Find nearest node to origin

Then eliminate all other ways of getting to that node.



Shortest Path Problem

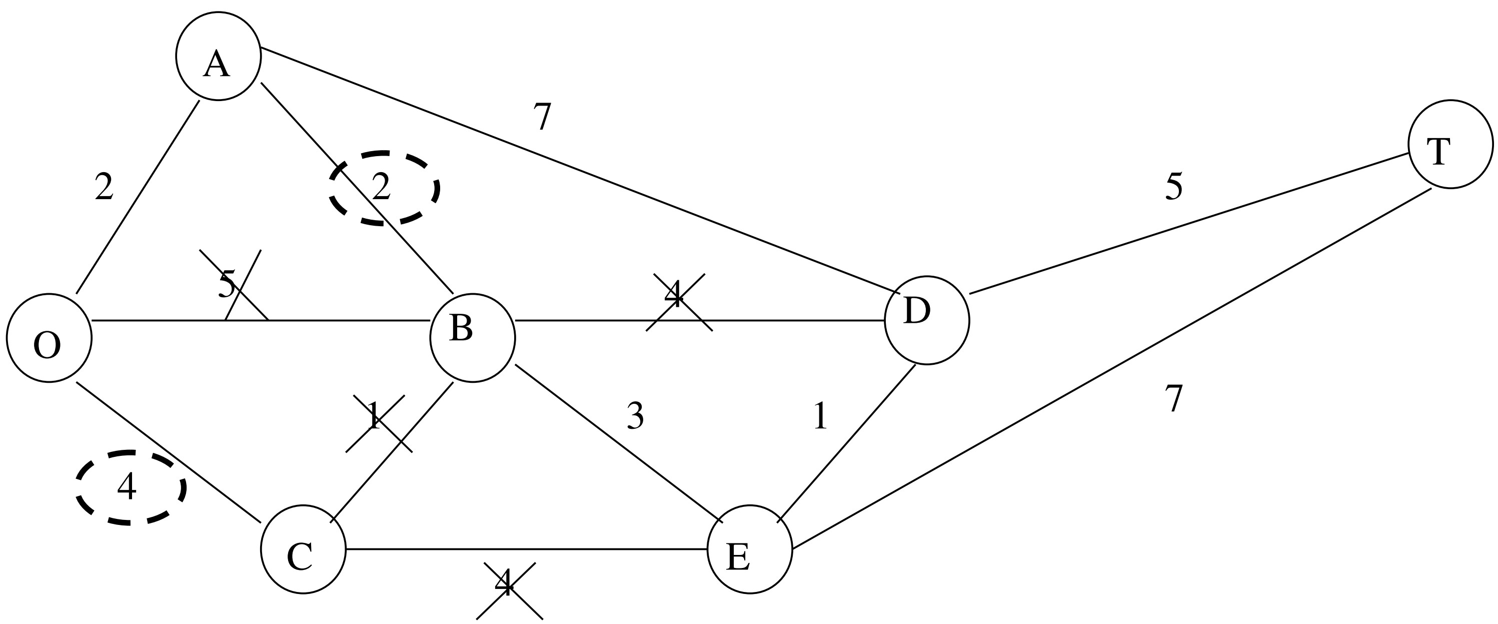
2

O	A	B	C	D	E	T
OA–2	AB–2	BC–1	CB–1	DE–1	ED–1	
OC–4	AD–7	~~BA–2~~	CE–4	DB–4	EB–3	
OB–5		BE–3		DT–5	EC–4	
		BD–4		~~DA–7~~	ET–7	

Second Nearest Node

Step 2. N=2

1. Find candidates for 2nd nearest node to origin (C & B)
2. Find shortest route to C & B (tied at 4)
3. Eliminate all other branches to C & B



Second Nearest Node

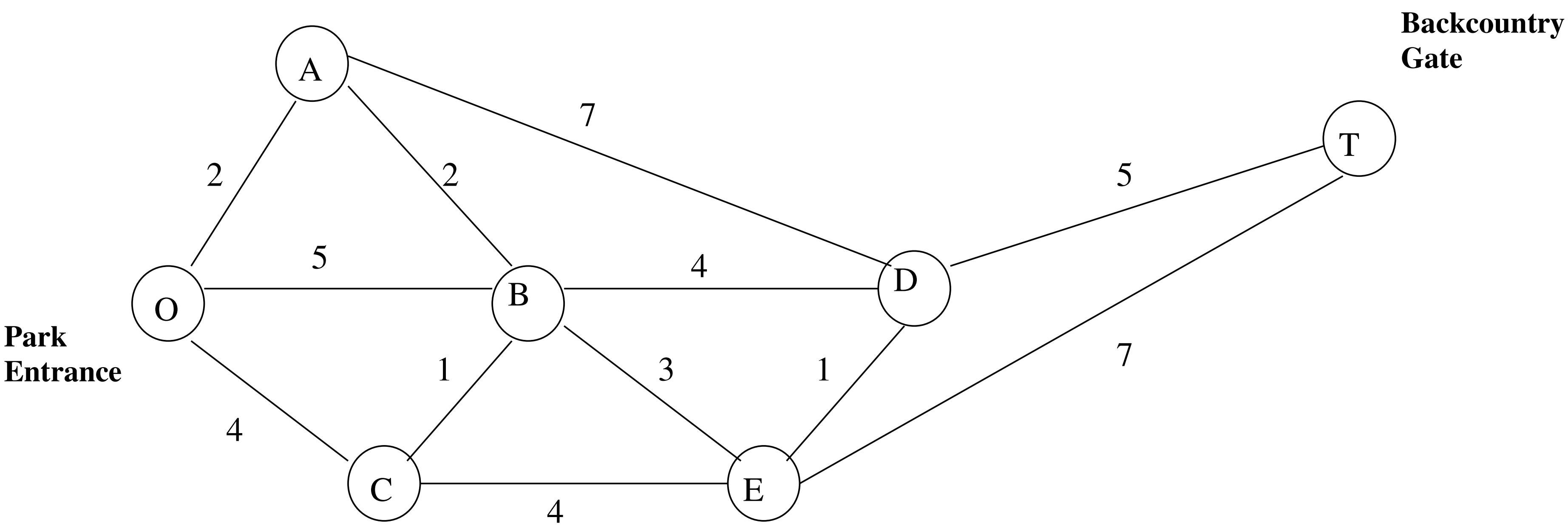
	2	4	4			
O	**A**	**B**	**C**	**D**	**E**	**T**
OA–2	AB–2	~~BC–1~~	~~CB–1~~	DE–1	ED–1	
OC–4	AD–7	~~BA–2~~	CE–4	~~DB–4~~	~~EB–3~~	
~~OB–5~~		BE–3		DT–5	~~EC–4~~	
		BD–4		~~DA–7~~	ET–7	

Fourth Nearest Node

Step 3. N=4 (2nd and 3rd were ties)

1. Find candidates for 4th nearest node to origin

Use table on next slide:



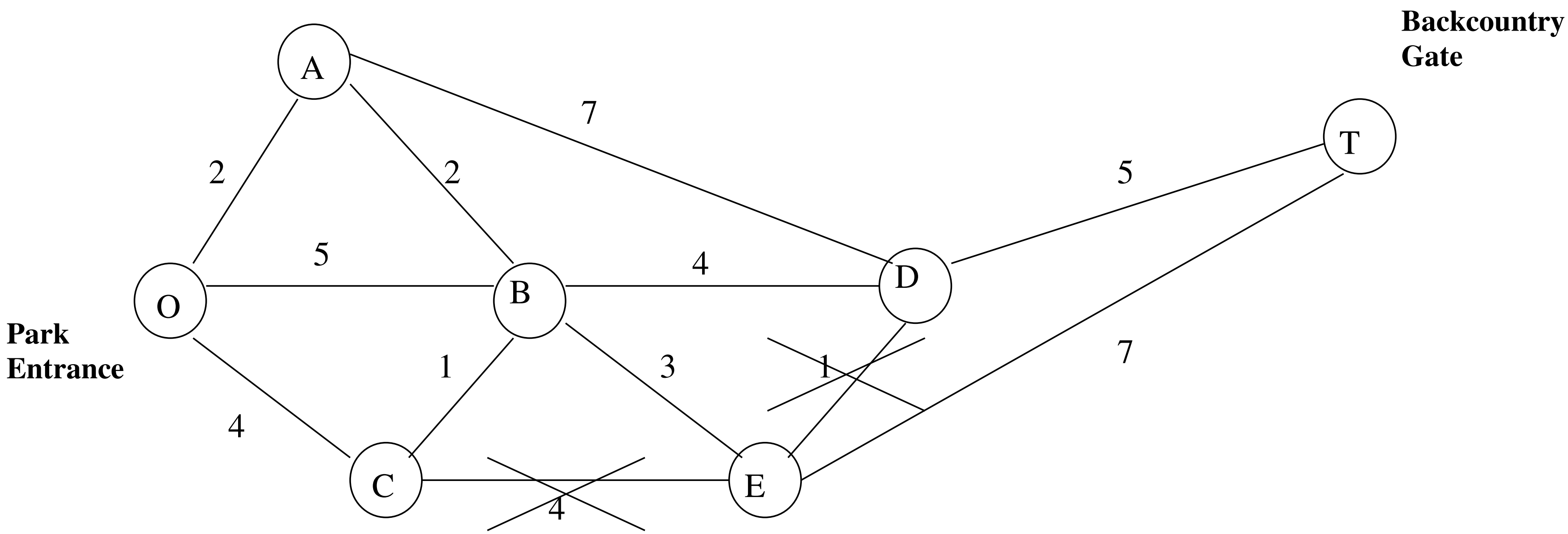
Finding 4th Nearest Node

	2	**4**	**4**		**7**	
O	**A**	**B**	**C**	**D**	**E**	**T**
OA–2	AB–2	~~BC–1~~	~~CB–1~~	DE–1	ED–1	
OC–4	AD–7 **9**	~~BA–2~~	CE–4 **8**	~~DB–4~~	~~EB–3~~	
~~OB–5~~		BE–3 **7**		DT–5	~~EC–4~~	
		BD–4 **8**		~~DA–7~~	ET–7	

Cross Out other Nodes

Step 1. N=4

1. Find candidates for 4 th nearest node to origin (E)
2. Cross out all other branches leading to E



Finding 4th Nearest Node

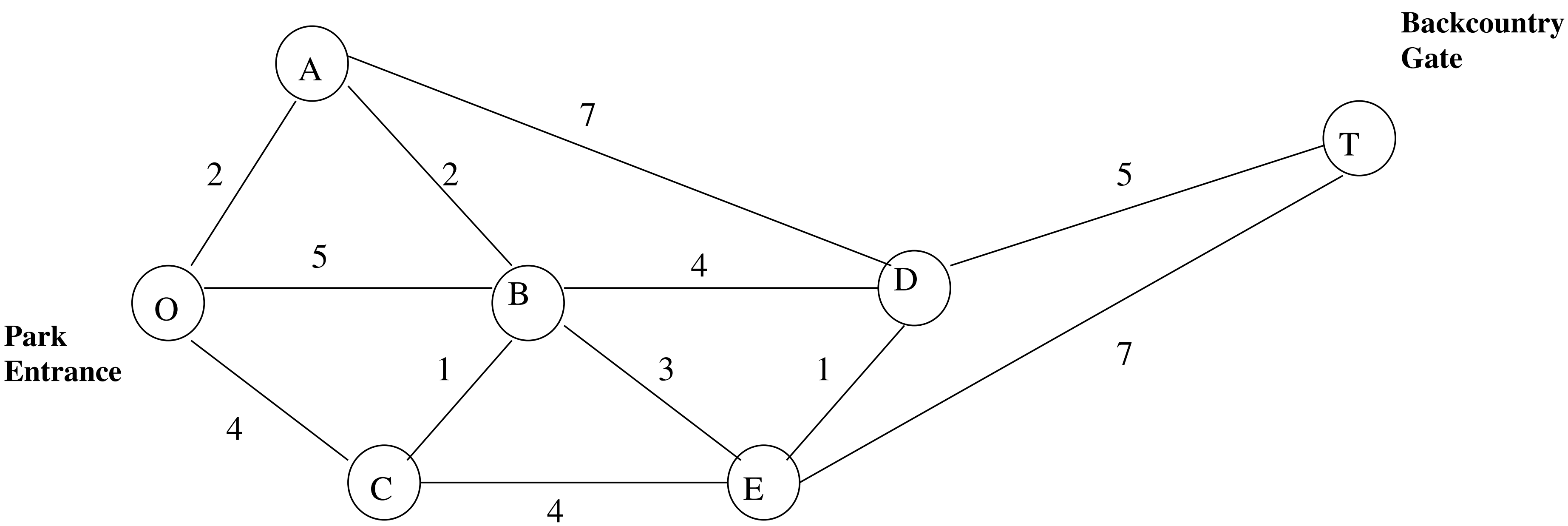
	2	4	4		7	
O	**A**	**B**	**C**	**D**	**E**	**T**
OA–2	AB–2	~~BC–1~~	~~CB–1~~	~~DE–1~~	ED–1	
OC–4	AD–7	~~BA–2~~	~~CE–4~~	~~DB–4~~	~~EB–3~~	
~~OB–5~~		BE–3		DT–5	~~EC–4~~	
		BD–4		~~DA–7~~	ET–7	

Finding Fifth Nearest Node

Step 4. N=5

1. Find candidates for 5 th nearest node to origin

Use table on next slide:



Finding 5th Nearest Node

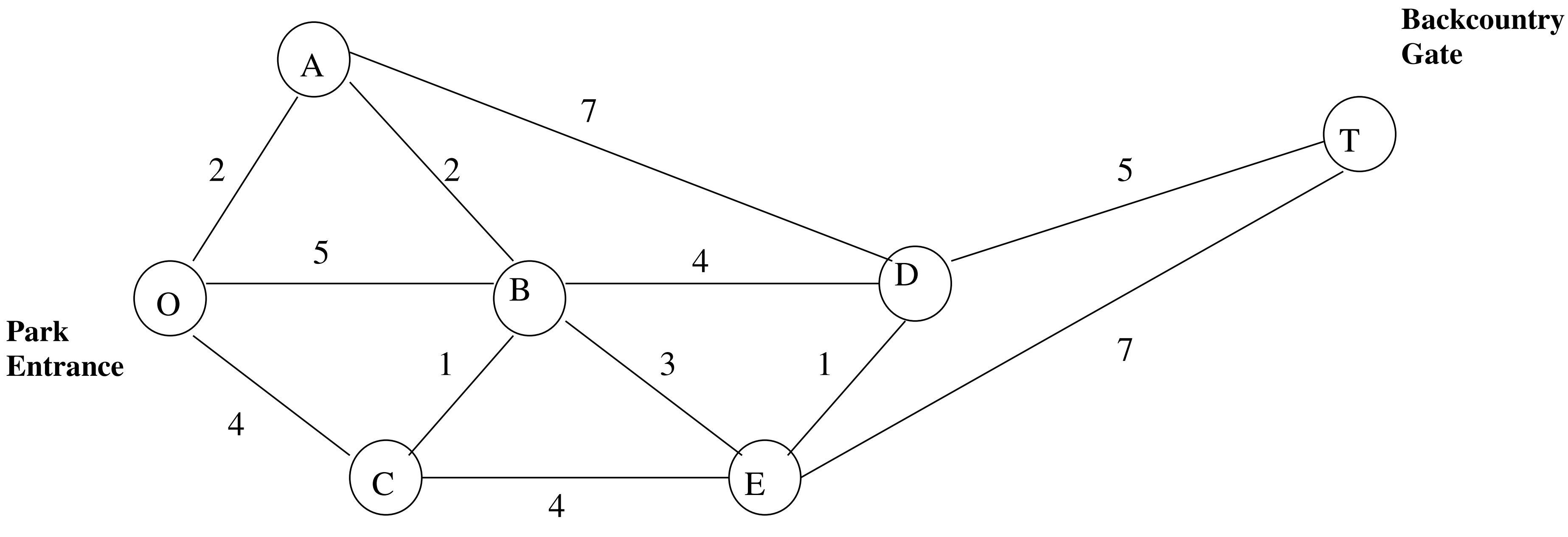
	2	4	4	8	7	
O	**A**	**B**	**C**	**D**	**E**	**T**
OA–2	AB–2	~~BC–1~~	~~CB–1~~	~~DE–1~~	ED–1	
OC–4	~~AD–7~~	~~BA–2~~	~~CE–4~~	~~DB–4~~	~~EB–3~~	
~~OB–5~~		BE–3		DT–5	~~EC–4~~	
		BD–4		~~DA–7~~	ET–7	

Finding Last Node

Step 5. N=6

1. Compare last nodes to T

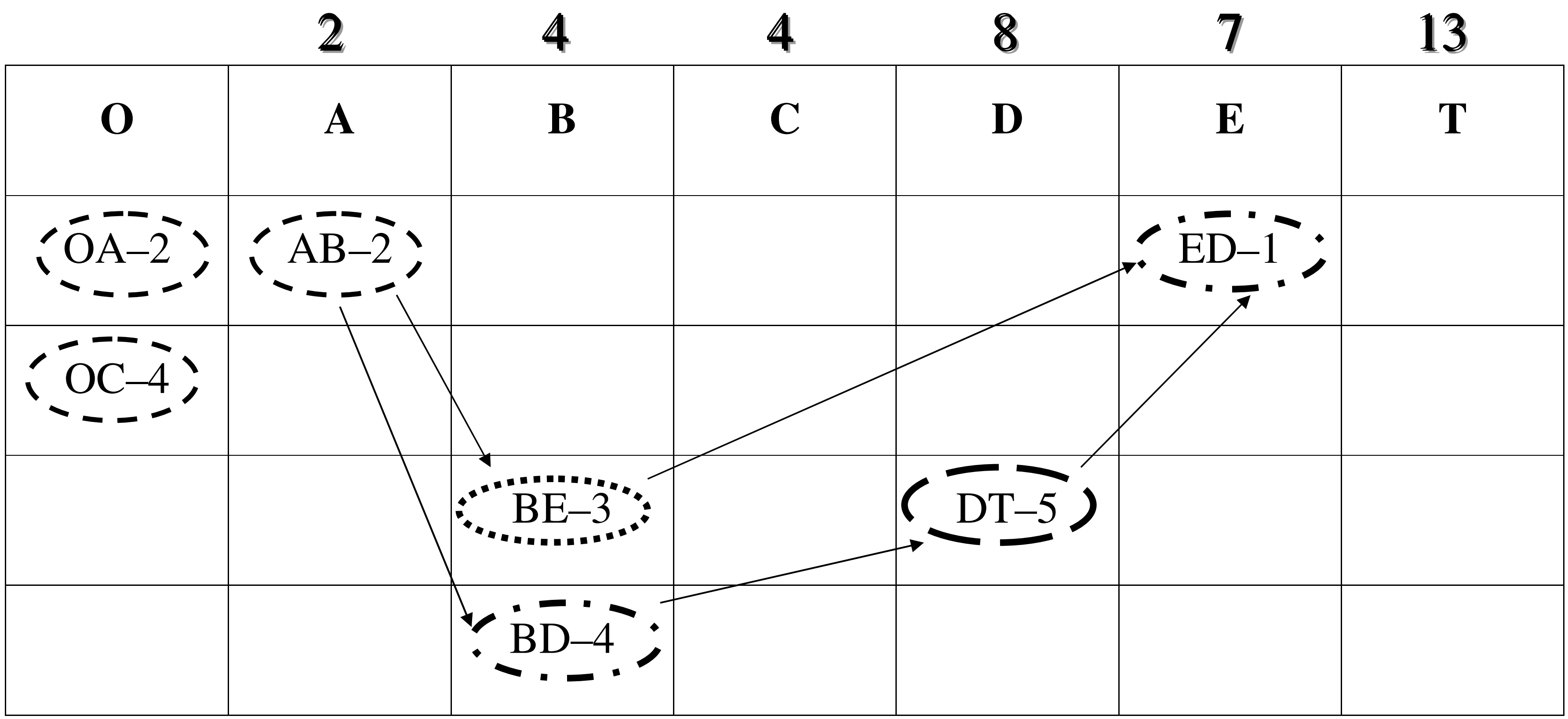
Use table on next slide:



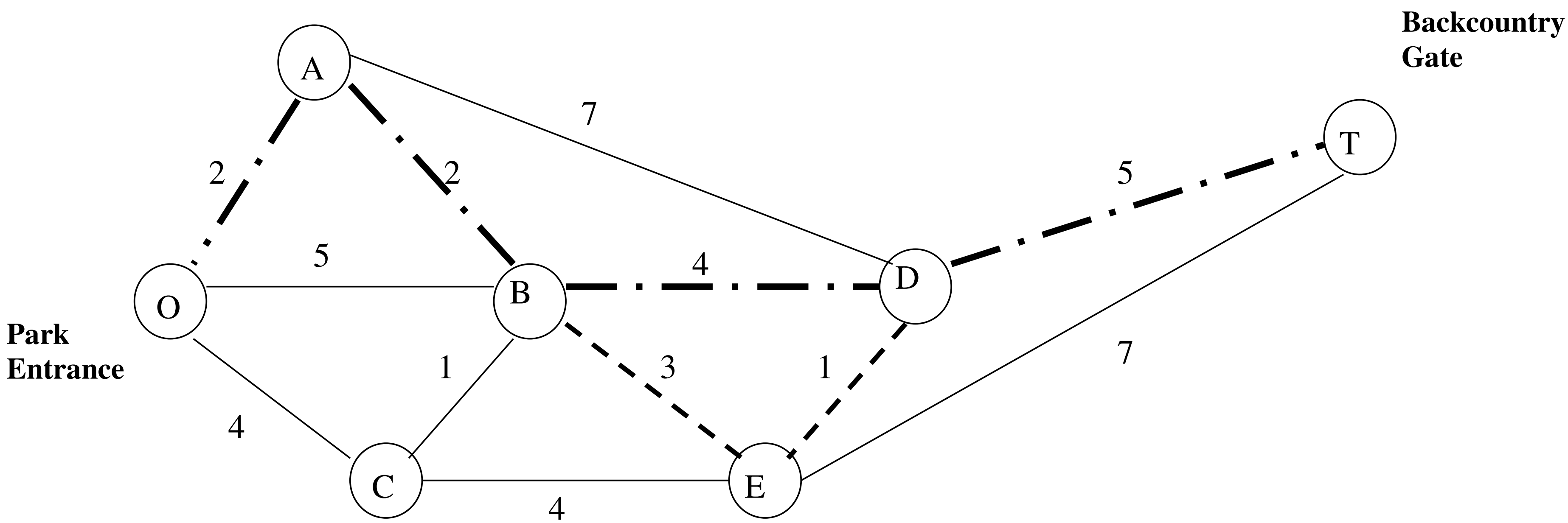
Finding Last Node

	2	4	4	8	7	13
O	**A**	**B**	**C**	**D**	**E**	**T**
OA–2	AB–2	~~BC–1~~	~~CB–1~~	~~DE–1~~	ED–1	
OC–4	~~AD–7~~	~~BA–2~~	~~CE–4~~	~~DB–4~~	~~EB–3~~	
~~OB–5~~		BE–3		DT–5	~~EC–4~~	
		BD–4		~~DA–7~~	~~ET–7~~	

Tracing Through the Solution



Solution



Shortest Path Problem

Dijkstra's algorithm is designed to determine the shortest routes between the source node and every other node in the network.

Dijkstra's algorithm. The computations of the algorithm advance from a node **i** to an immediately succeeding node ***j*** using a special labeling procedure. Let U_i be the shortest distance from source node **1** to node **i**, and define d_{ij} (≥ 0) as the length of arc (i, j).

Then the label for node ***j*** is defined as

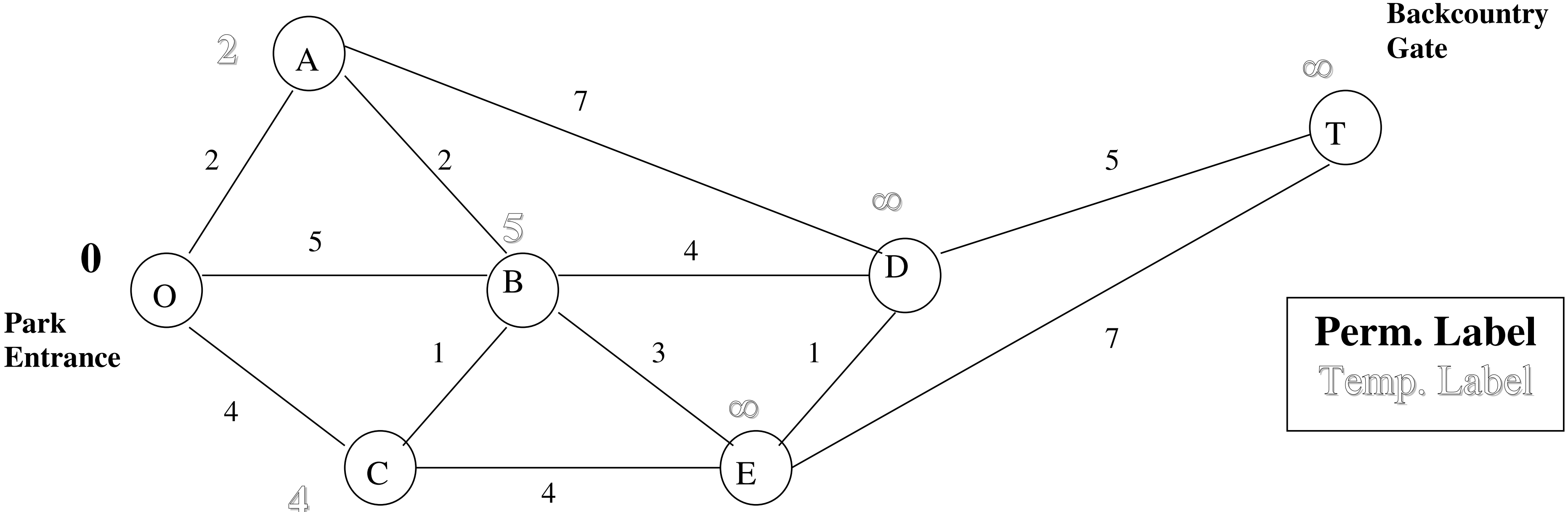
$$\left[u_i, i\right] = \left[u_i + d_{ij}, i\right], \quad d_{ij} \geq 0$$

- Node labels in Dijkstra 's algorithm are of two types: *temporary* and *permanent*.
- A temporary label can be replaced with another label ifa shorter route to the same node can be found. At the point when it becomes evident that no better route can be found, the status of the temporary label is changed to permanent.
- The steps of the algorithm are summarized as follows.

Dijkstra's Algorithm

Step 1. N=1

1. Label Starting Node as Permanent Node 0
2. Label Each Node i connected to 0 with temporary label equal to the length from node 1 to node i, all others label ∞**.**
3. Choose smallest temp label and make it permanent



Dijkstra's Algorithm

Step 1. N=1

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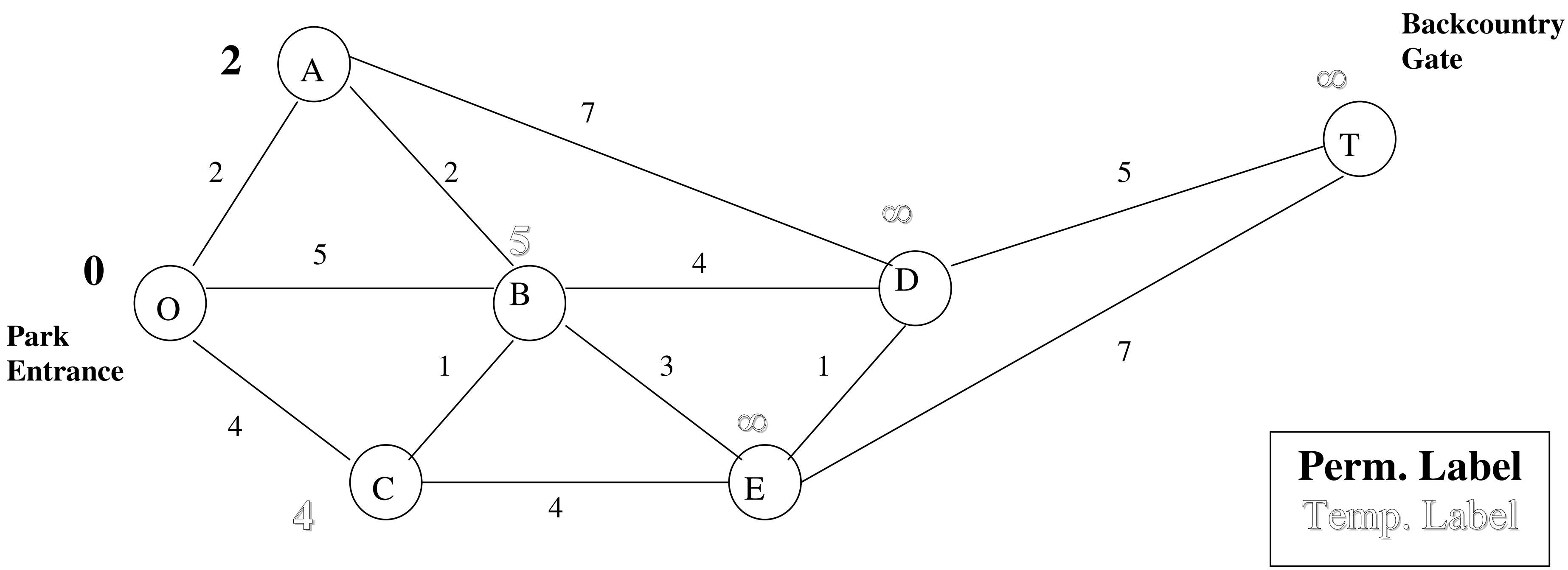
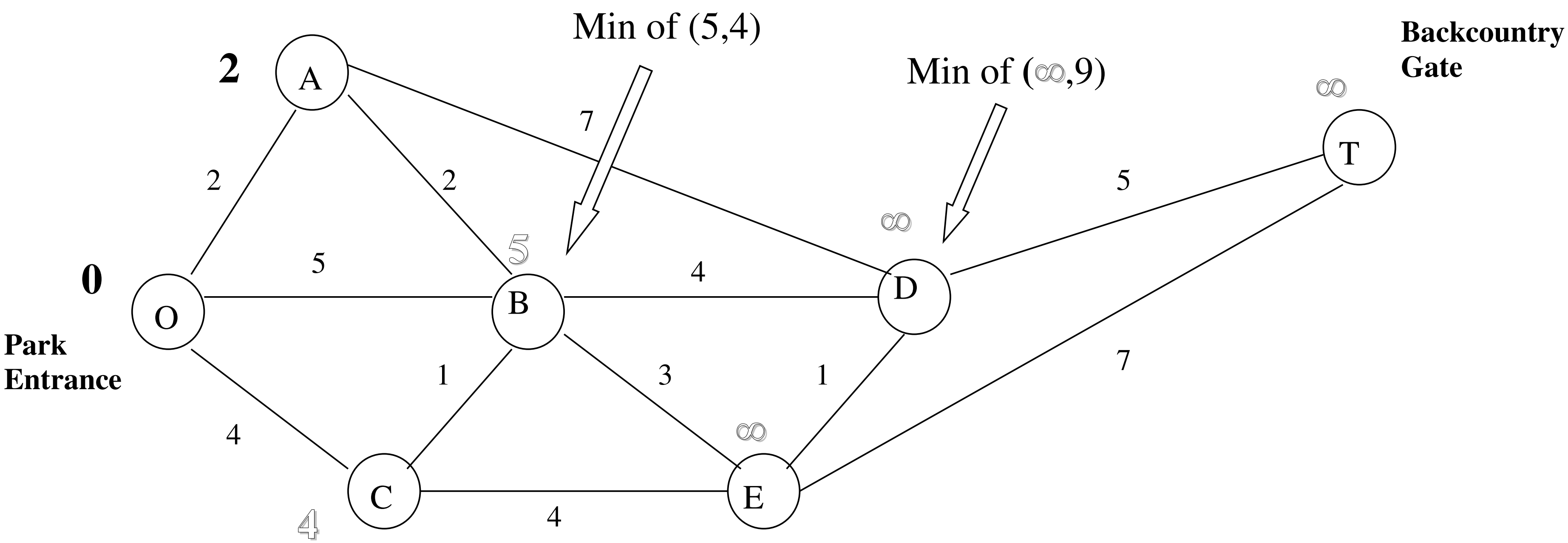


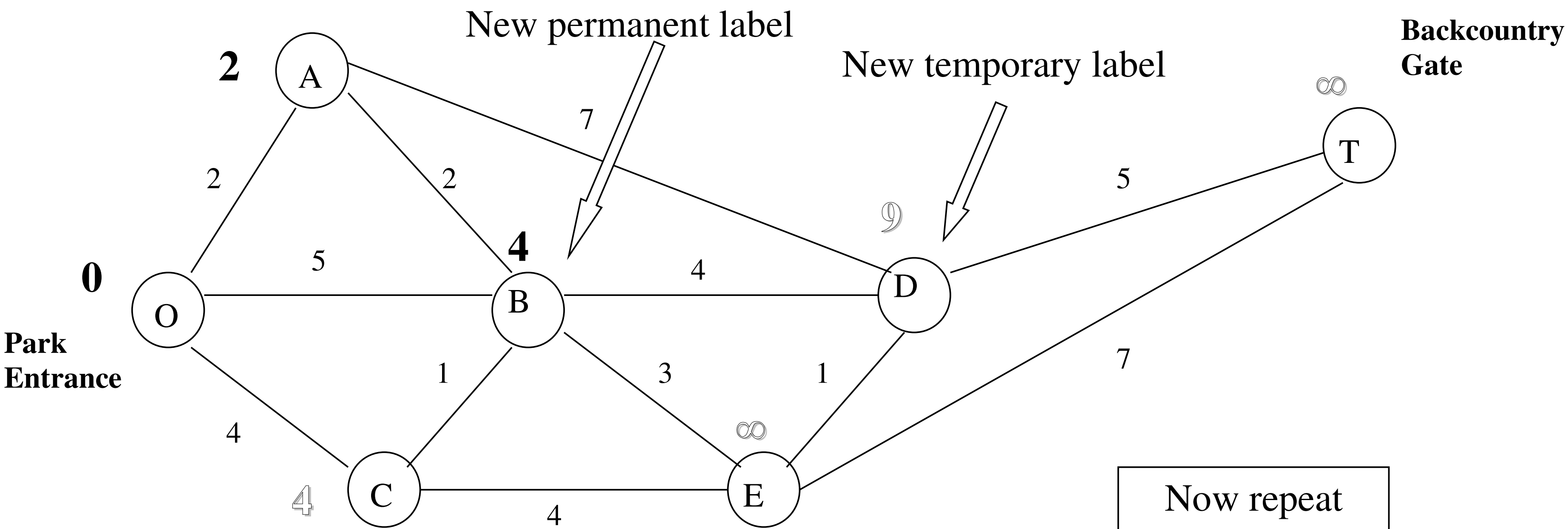
Dijkstra's Algorithm

Step 2. N>1 (step 2 is repeated)

1. Node i is the k th closest node to node 1
2. For each node j with temporary label, replace node j's temporary label with minimum of :
   - Node j's current temporary label
   - Node i's temporary label plus length of arc (i, j)

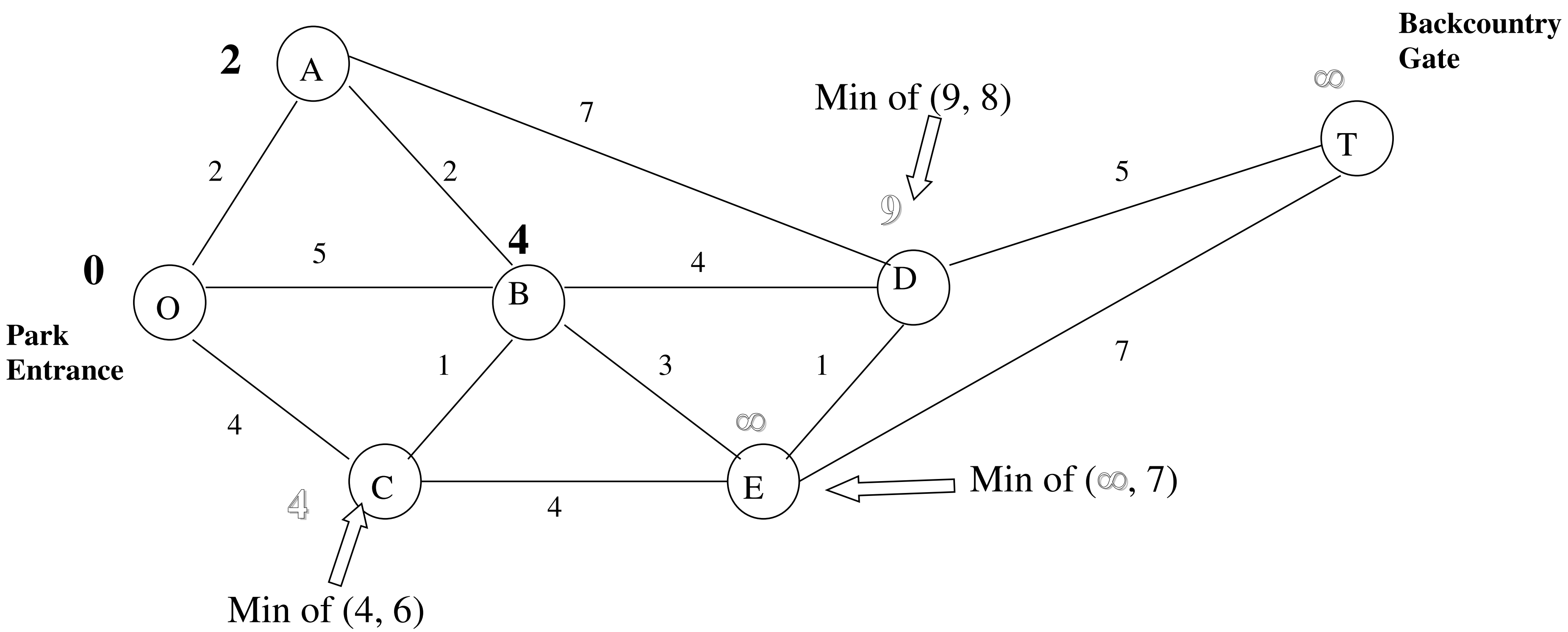


Dijkstra's Algorithm

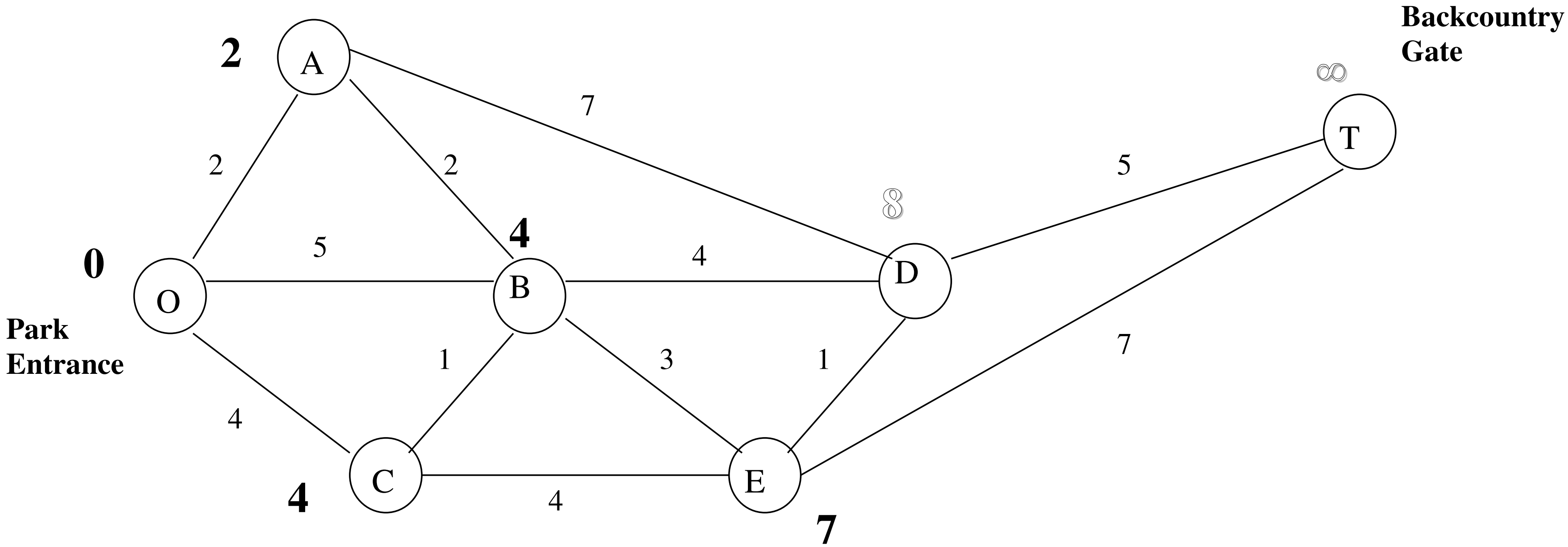


Now repeat
Step 2

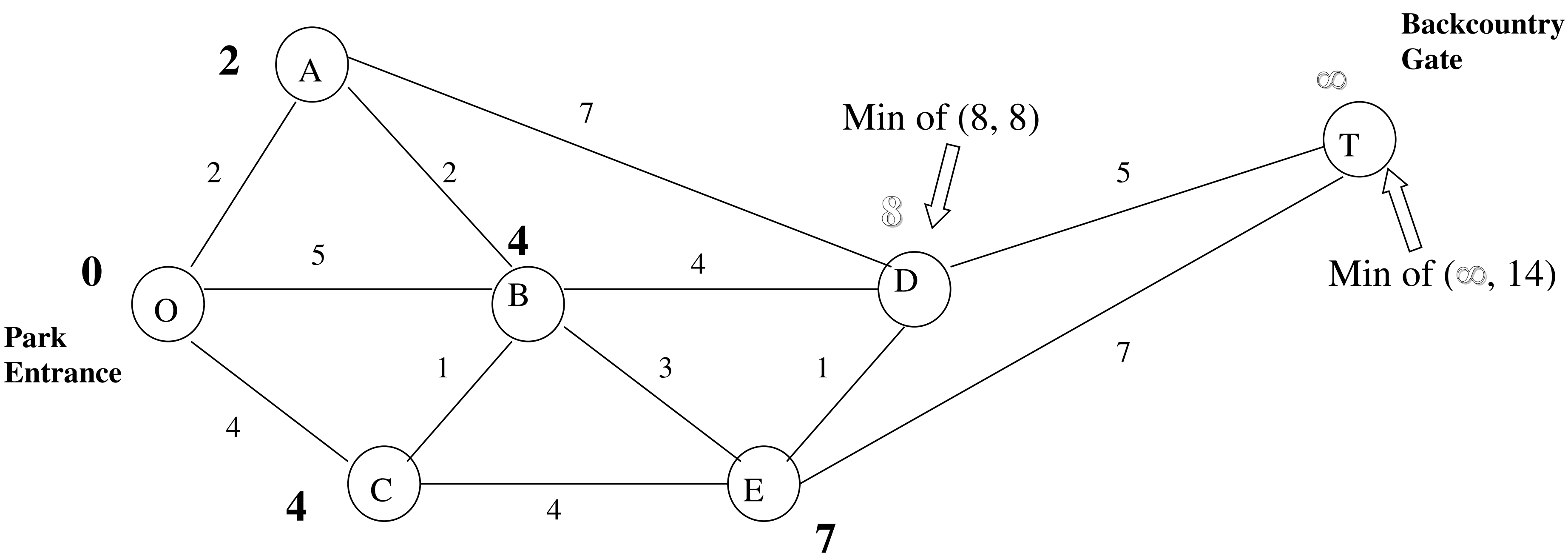
Dijkstra's Algorithm



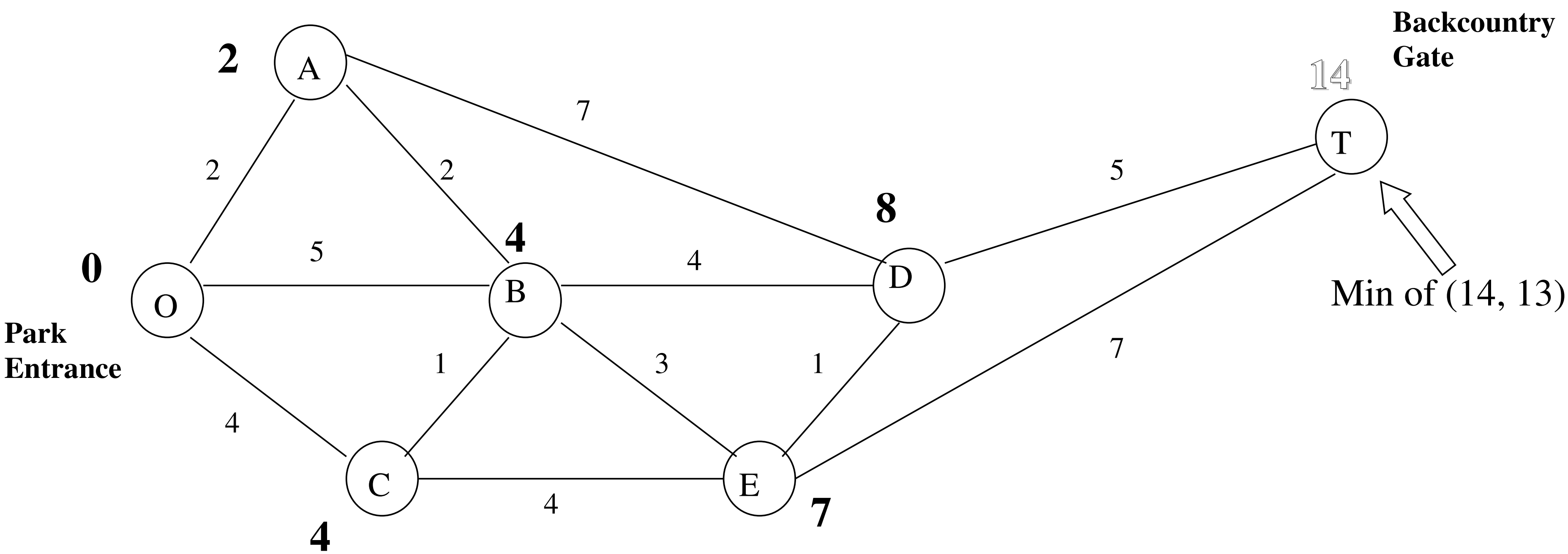
Dijkstra's Algorithm



Dijkstra’s Algorithm

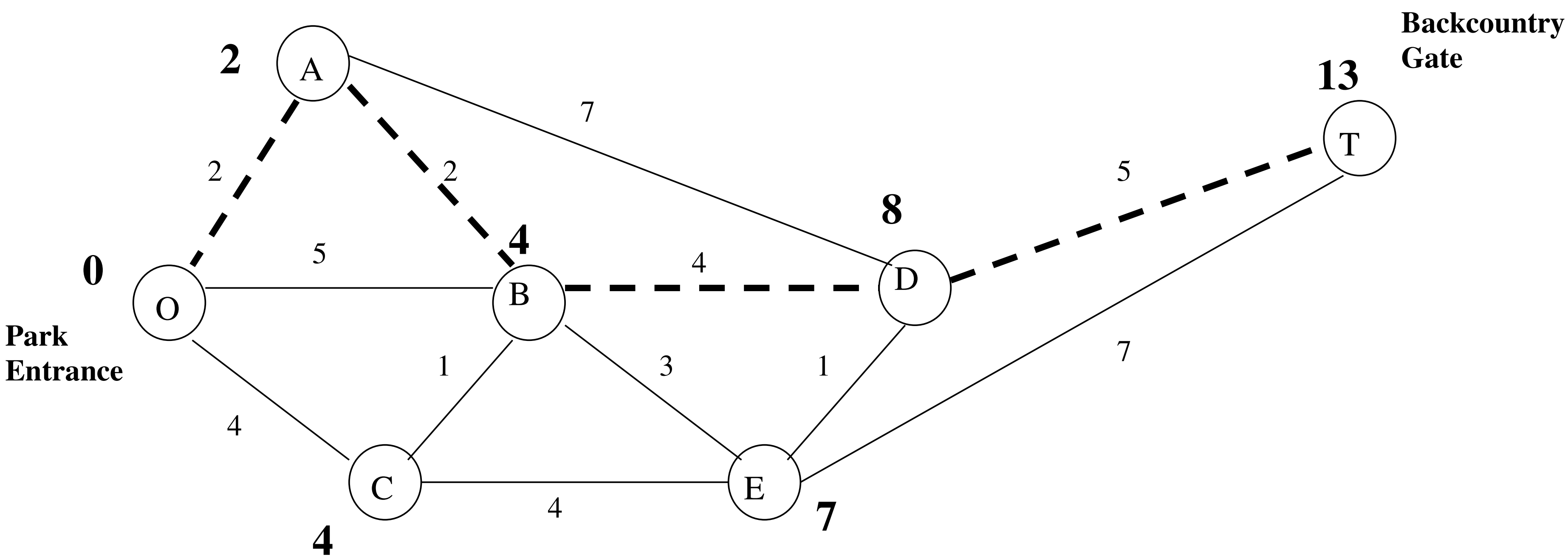


Dijkstra's Algorithm



Dijkstra's Algorithm

Tracing Back thru: Start at last node (j) and go to node (i) which is equal to node j's label minus arc (i,j)



Dijkstra's Algorithm

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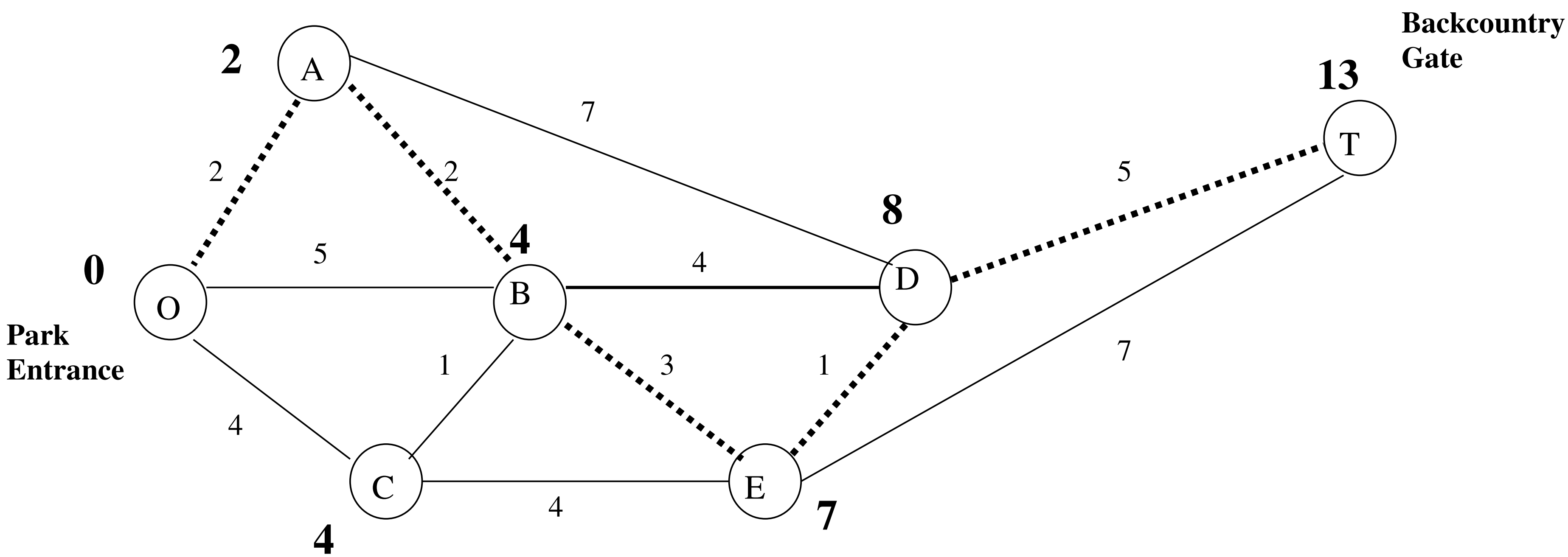


Examples of the Shortest-Route Applications

1. Equipment replacement

Rent Car is developing a replacement plan for its car fleet for a 5-year (1996 to 2002) planning horizon. At the start of each year, a decision is made as to whether a car should be kept in operation or replaced. A car must be in service at least 2 year but must be replaced after 4 years. The following table provides the replacement cost as a function of the year a car is acquired and the number of years in operation.

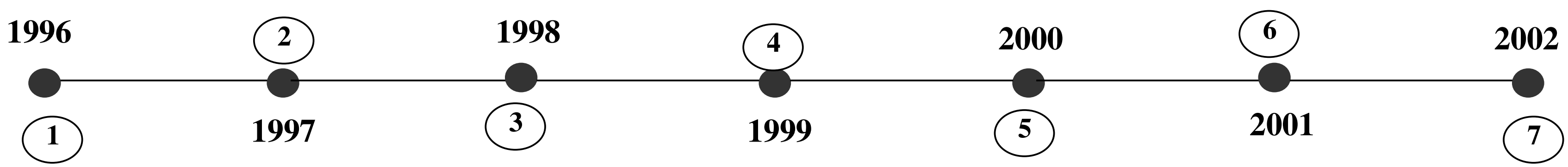
Replacement cost ($) for given years in operation

Year acquired	2	3	4
1996	3800	4100	6800
1997	4000	4800	7000
1998	4200	5100	7200
1999	4800	5700	---
2000	5300	---	---

1. Examples of the Shortest-Route Applications (**Equipment replacement**)

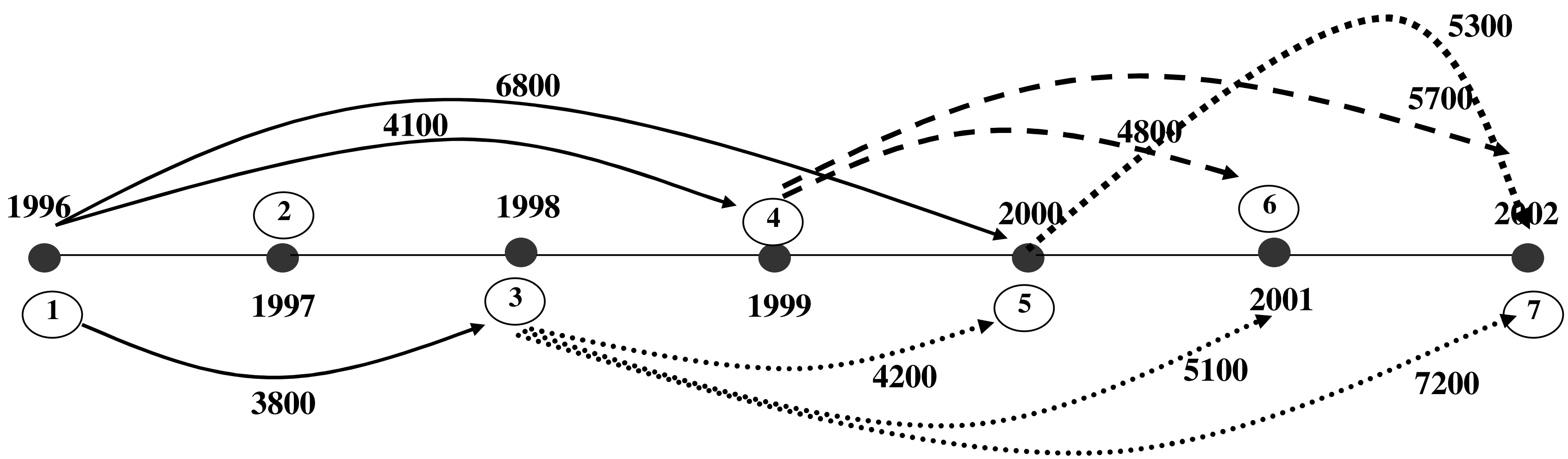
Definition of Problem:

- The problem can be formulated as a network in which nodes 1 to 7 represent years 1996 to 2002.

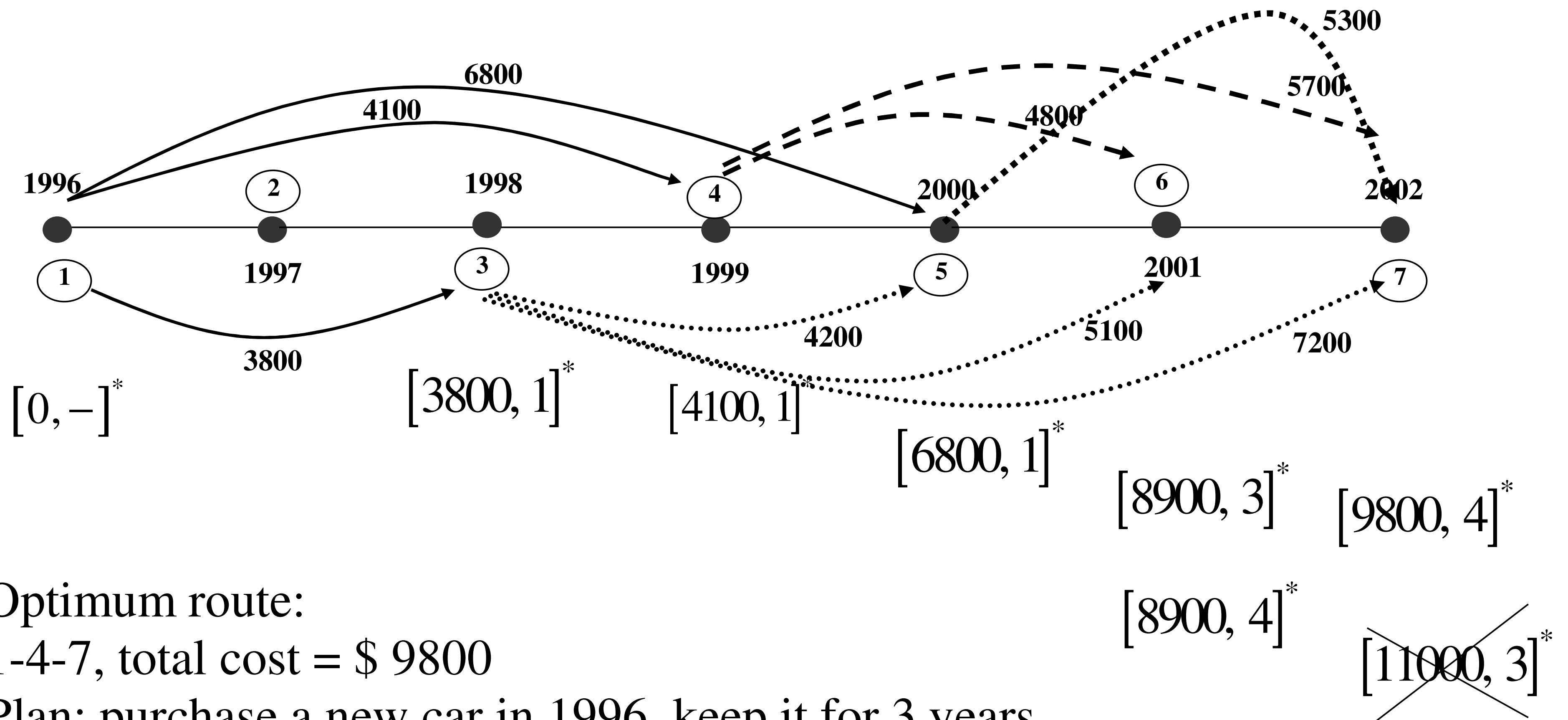


- Arcs from node 1 (year 1996) can reach only nodes 3, 4, and 5 because a car must be in operation between 2 and 4 years.
- The arcs from the other nodes can be interpreted similarly.
- The length of each arc equals the replacement cost.
- The solution of the problem is equivalent to finding the shortest route between nodes 1 and 7.

1. Examples of the Shortest-Route Applications **(Equipment replacement)**



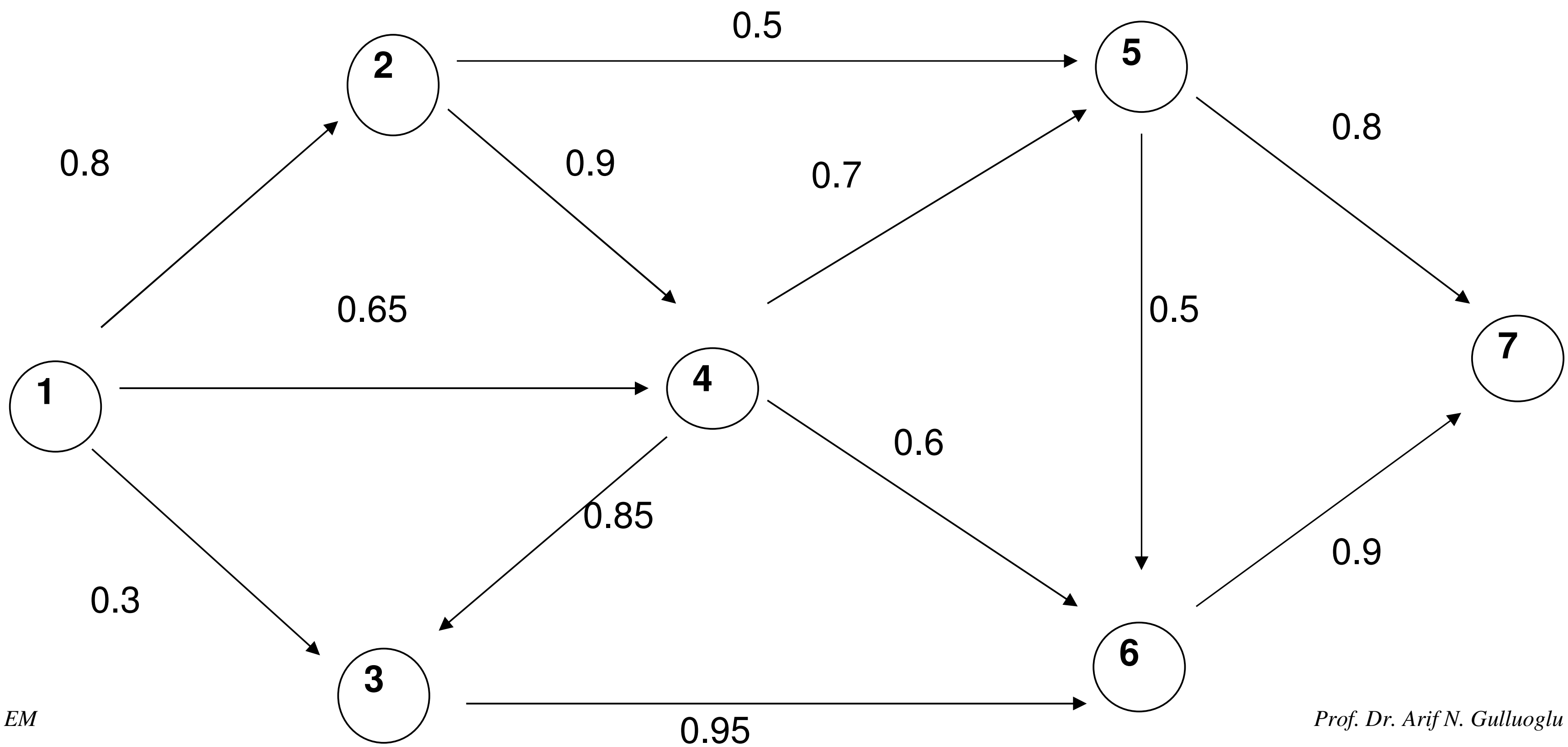
1. Examples of the Shortest-Route Applications **(Equipment replacement)**



Optimum route:
1-4-7, total cost = $ 9800
Plan: purchase a new car in 1996, keep it for 3 years.
Purchase another new car in 1999 and keep it until 2002.

2. Examples of the Shortest-Route Applications (**Most reliable route**)

Figure provides the communication network between two station, 1 and 7. The probability that a link in the network will operate without failure is shown on each arc. Messages are sent from stations 1 to stations 7 and the objective is to determine the route that will maximize the probability of a successful transmission. Formulate the situation as a shortest-route model.



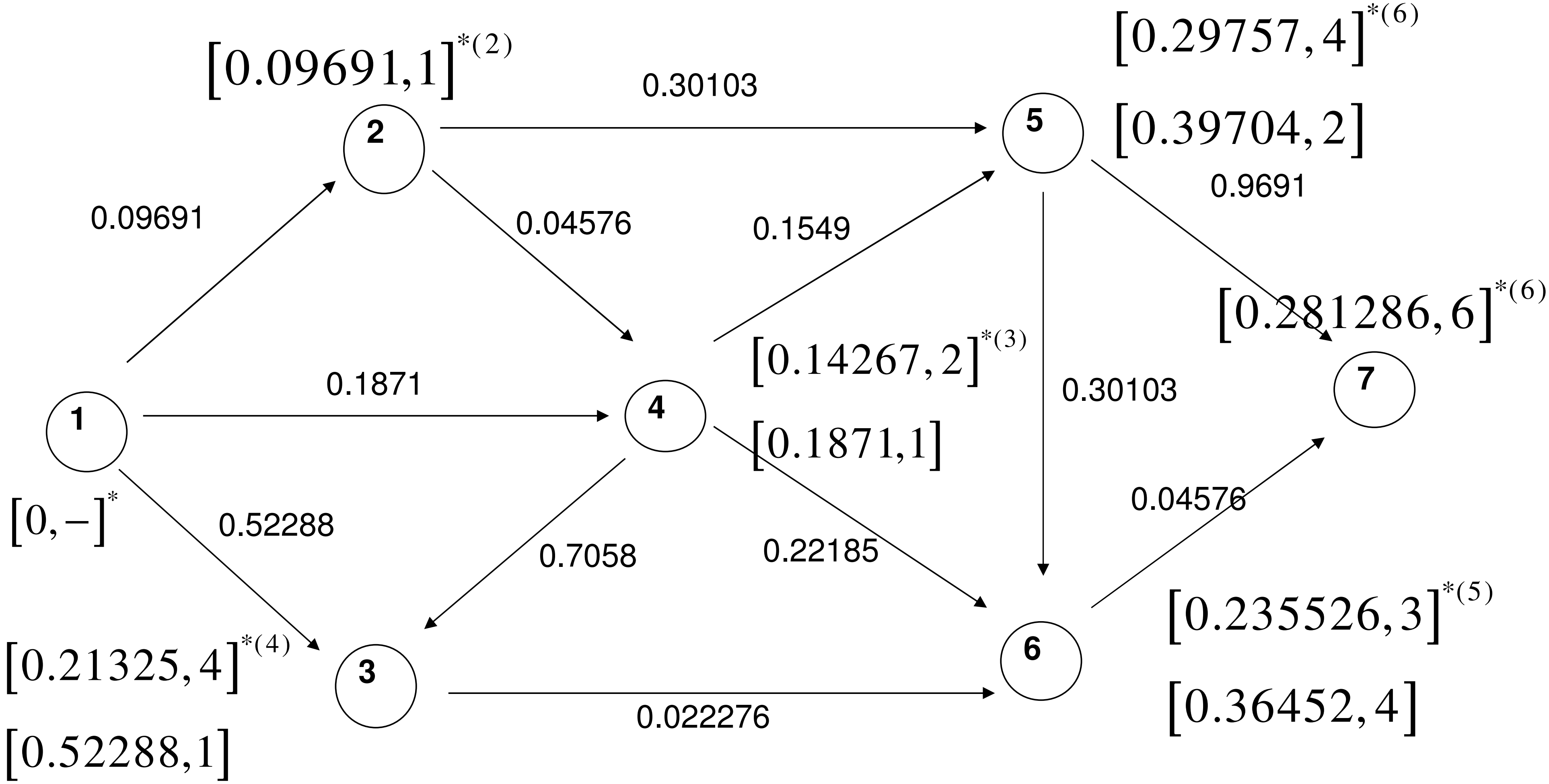
2. Examples of the Shortest-Route Applications (**Most reliable route**)

Problem can be formulated as a shortest-route model by using a logarithmic transformation that will convert the product probability into the sum of the logarithms of probabilities- that is, if $\mathbf{p_{1k}=p_1 \text{ x } p_2 \text{ x} \ldots \text{x } p_k}$ is the probability of not being stopped, then

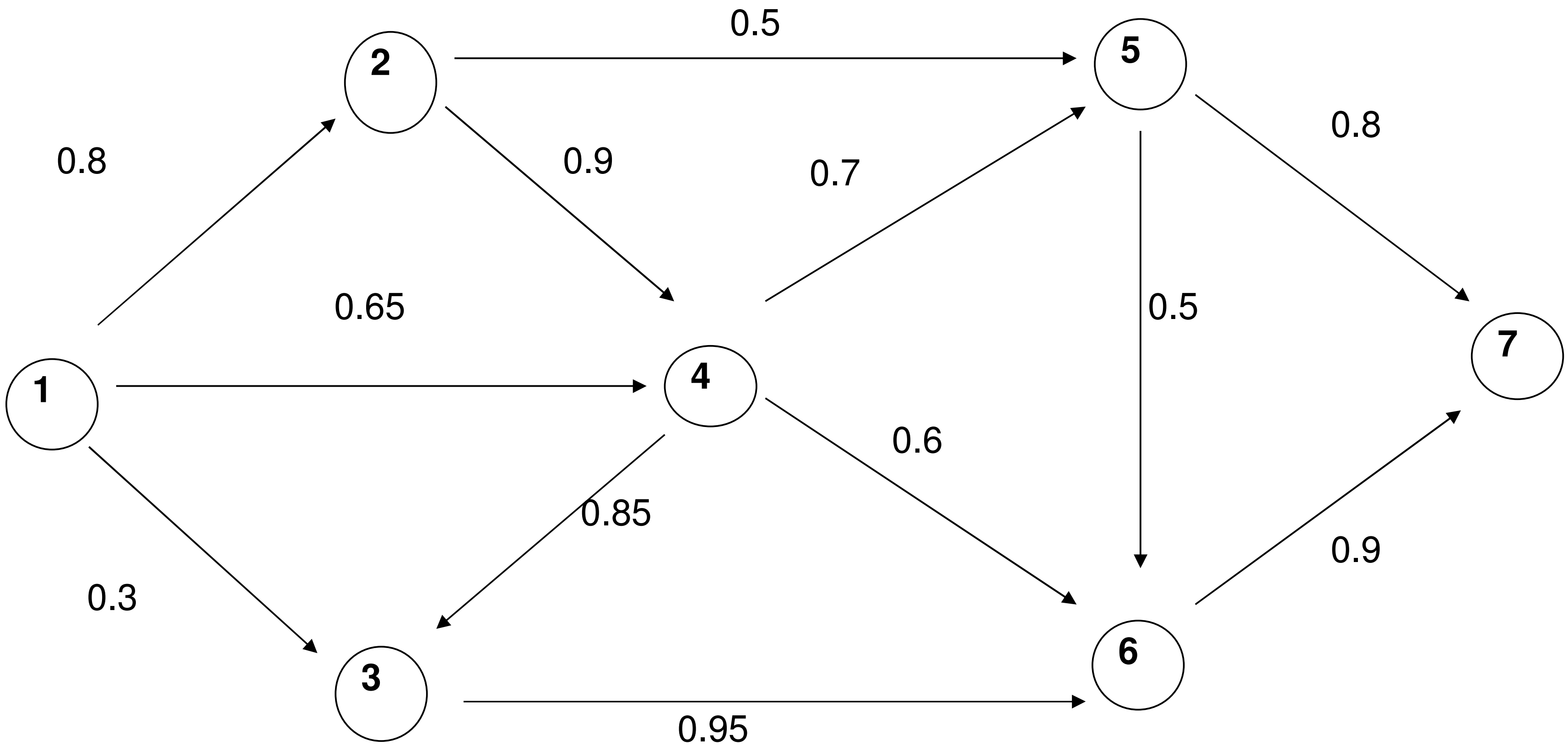
$$p_{1k}=p_1 \text{ x } p_2 \text{ x} . . . \text{x } p_k$$

Mathematically, the maximization of p_{1k} is equivalent to the maximization of log $\mathbf{p_{1k}}$. Because log $\mathbf{p_{1k} \leq 0}$**,** the maximization of **log**$\mathbf{p_{1k}}$ is in turn, equivalent to the minimization of **-log** $\mathbf{p_{1k}}$. Using this transformation, the individual probabilities pj in the figure are replaced with **- log** $\mathbf{p_j}$, for all j in the network, thus yielding the shortest-route network in the following Figure

2. Examples of the Shortest-Route Applications (**Most reliable route**)



2. Examples of the Shortest-Route Applications (**Most reliable route**)



Solution: 1-2-4-3-6-5,

Route value = 0.281286

Probability =10 $^{-0.281286}$= 0.52326

Minimal Spanning Tree Problem

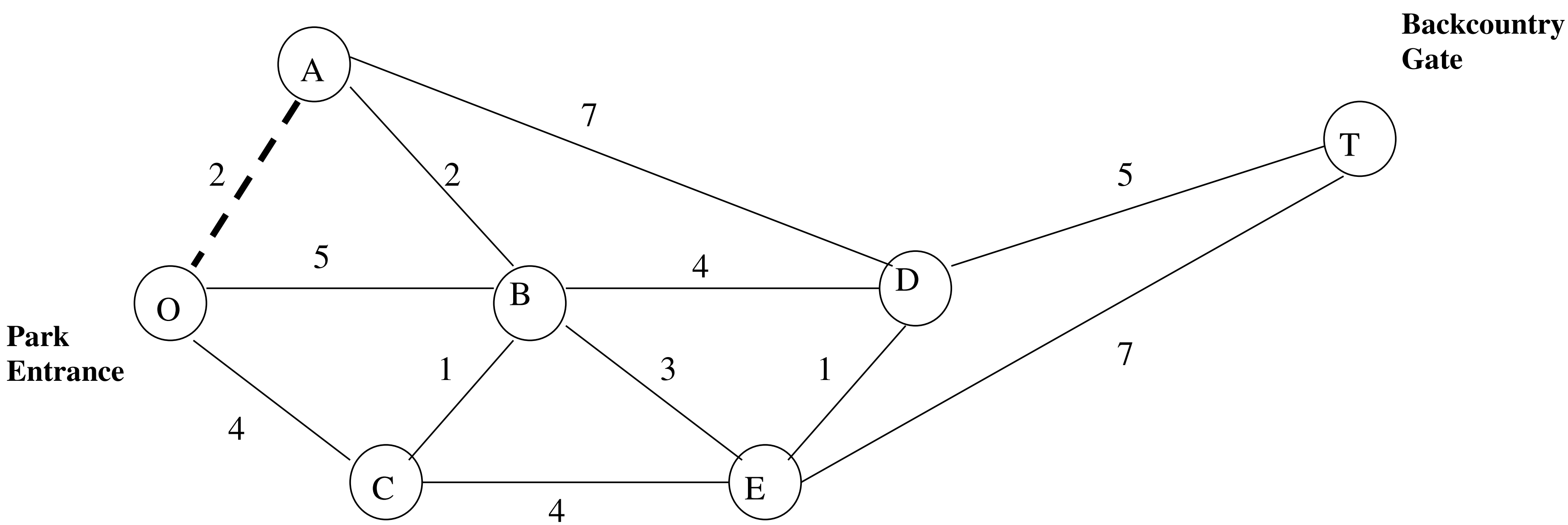
Summary of Solution Procedure

1. Select any node arbitrarily and connect to the ***nearest*** node

2. Identify the unconnected node that closest to a connected node and then connect these two nodes. Repeat until all nodes have been connected.

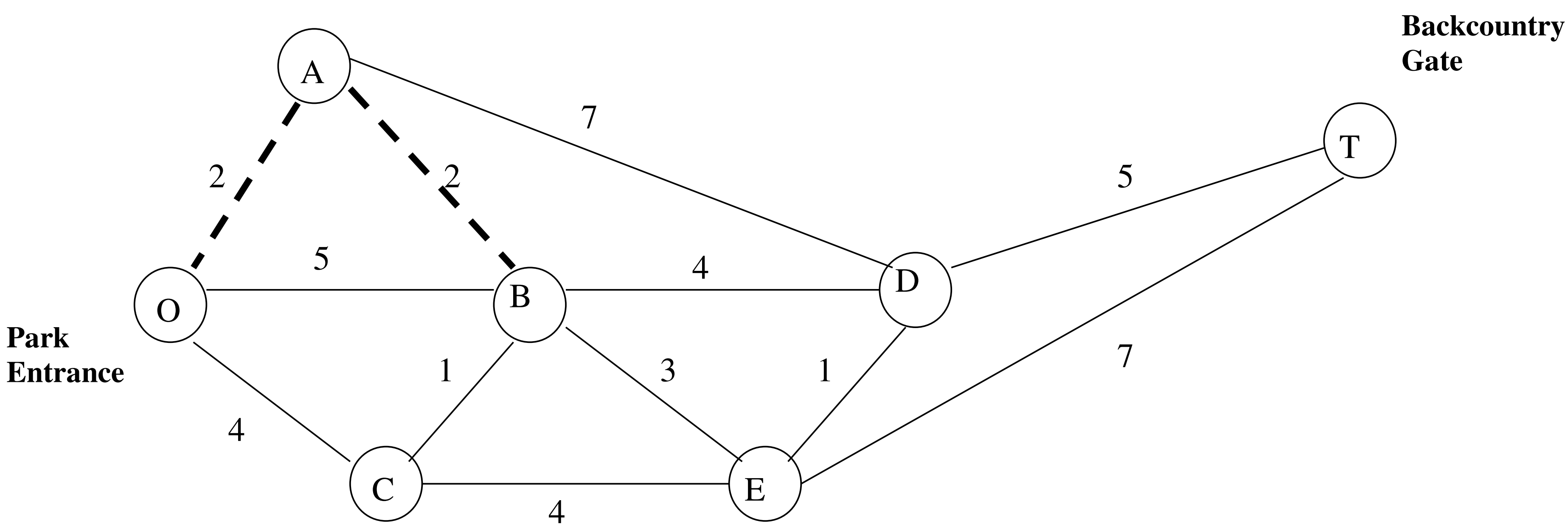
Nearest Node

Step 1. Arbitrarily select *any* node to start (O)
Nearest node to O is A



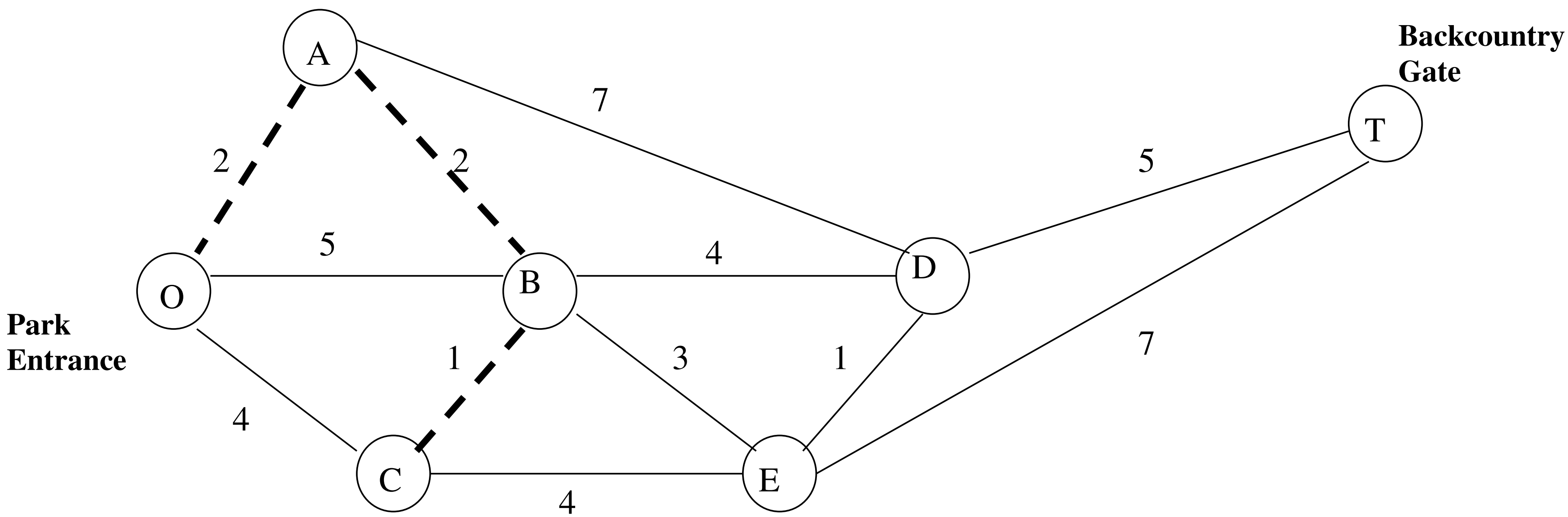
Nearest Node

Step 2. Unconnected node nearest to O or A is B
Connect B to A



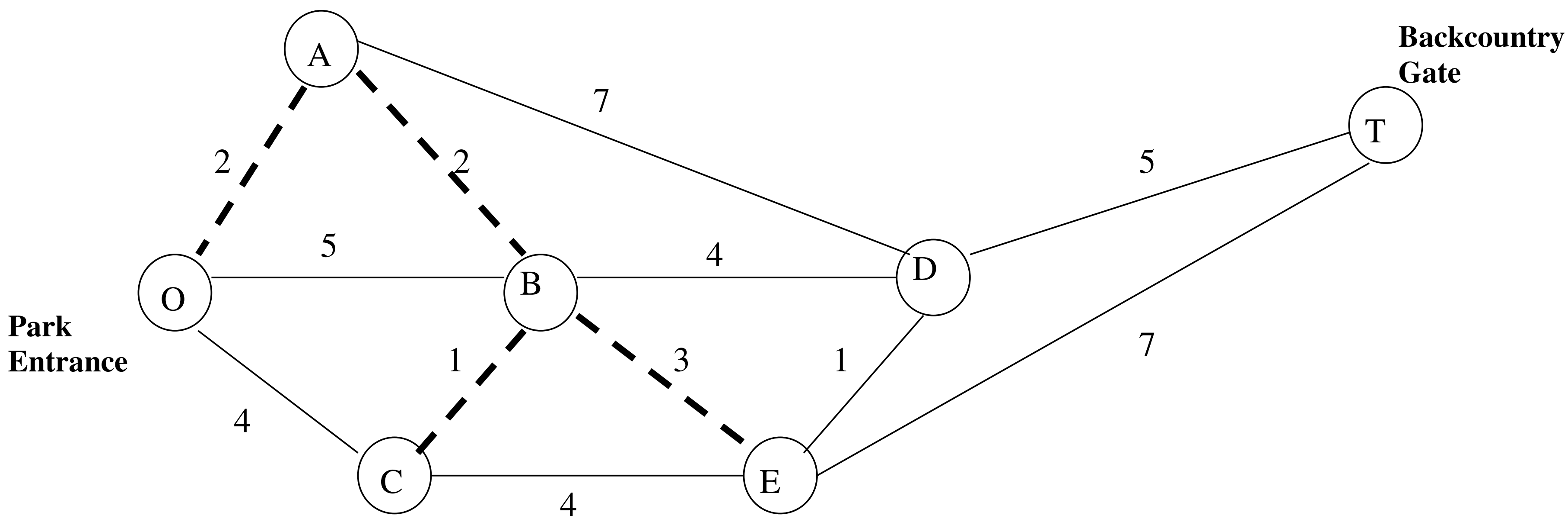
Nearest Node

Step 3. Unconnected node nearest to O, A or B is C
Connect C to B



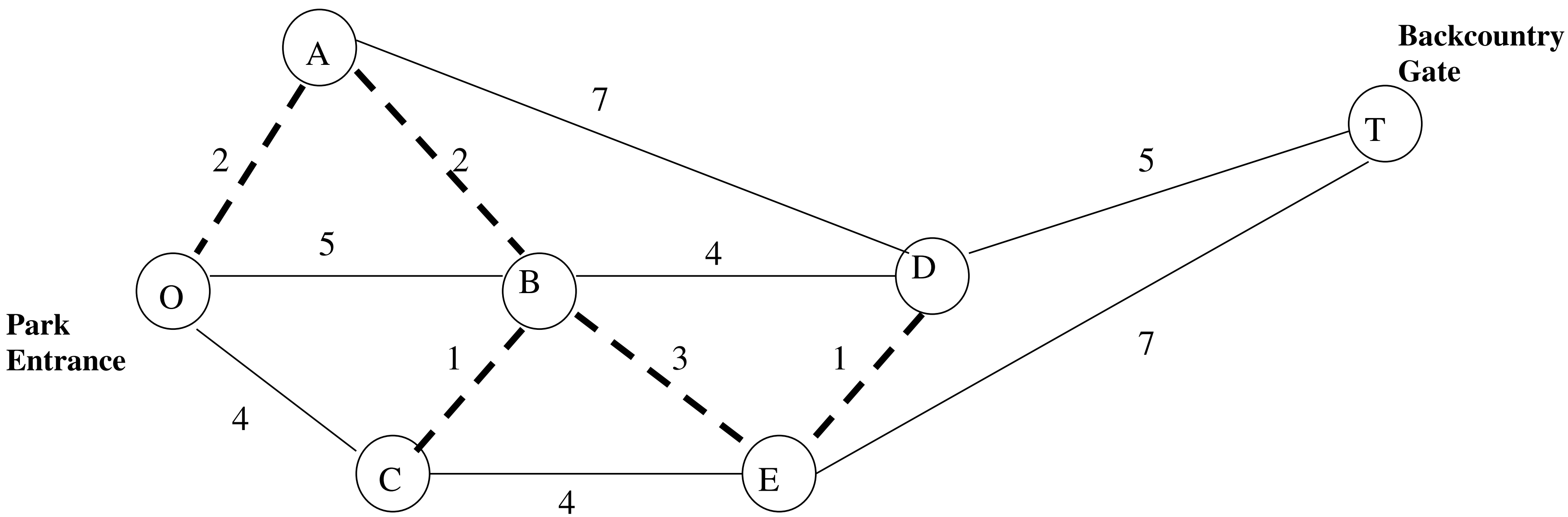
Nearest Node

Step 4. Unconnected node nearest to O, A, B or C is E
Connect E to B



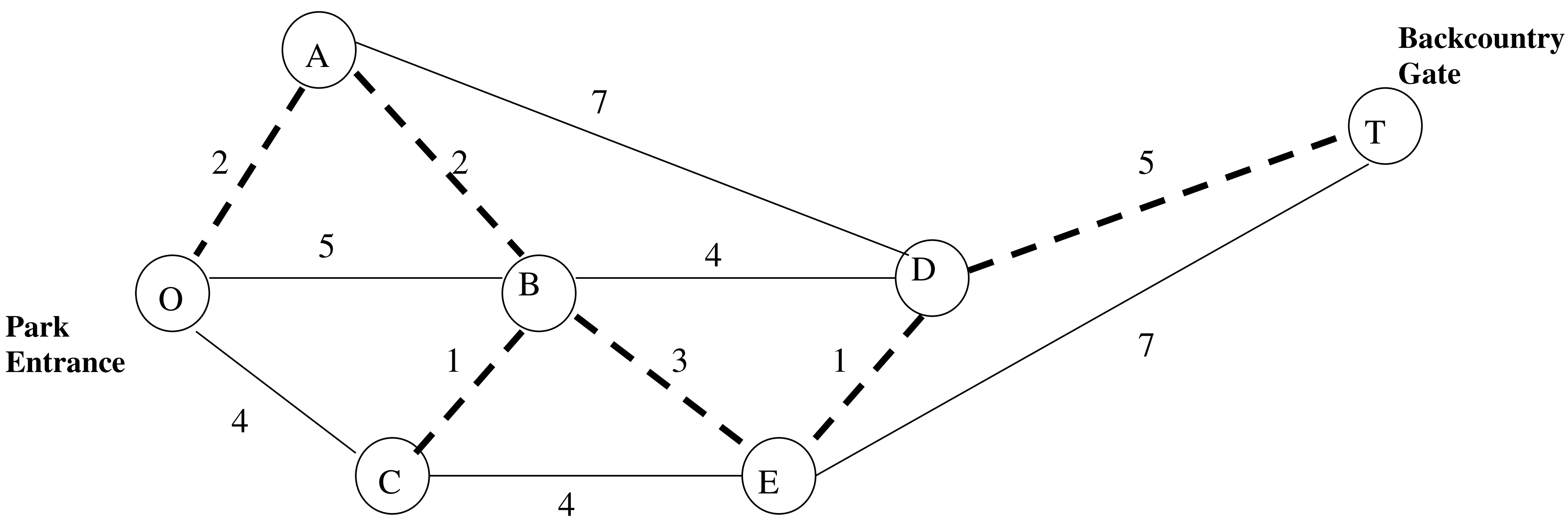
Nearest Node

Step 5. Unconnected node nearest to O, A, B, C or E is D
Connect D to E



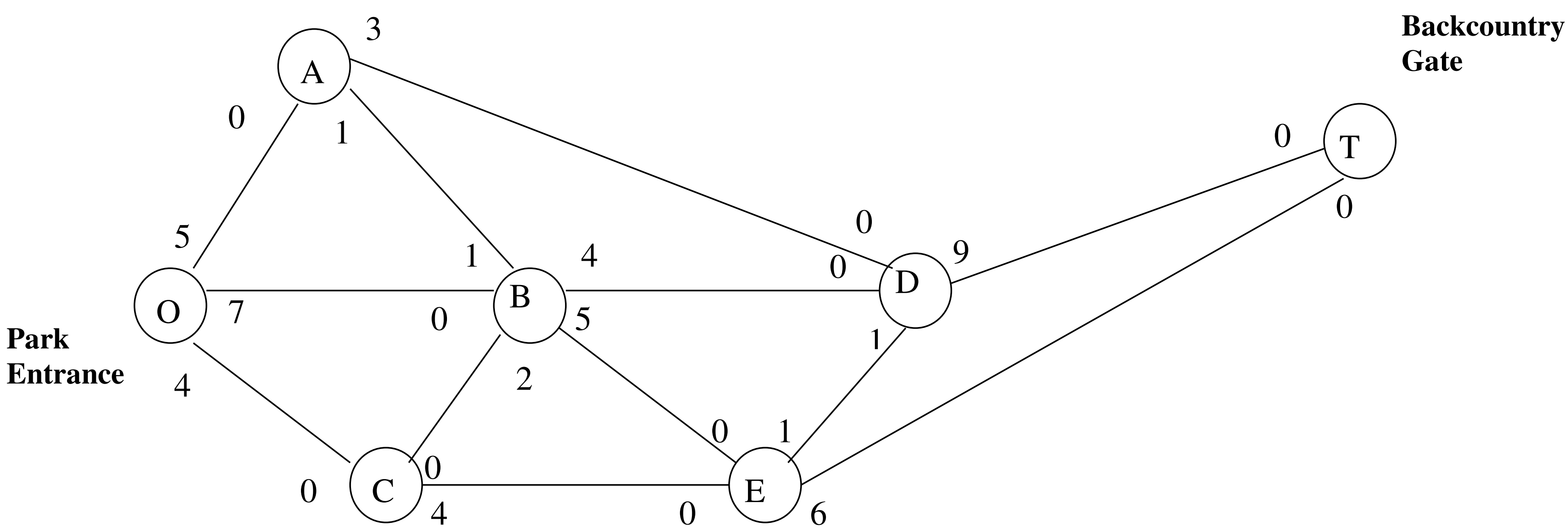
Nearest Node

Step 6. Unconnected node nearest to O, A, B, C, E or D is T
Connect D to T – All nodes connected
Minimal Span is 14 miles



Maximal Flow

- Flow from i to j can not exceed flow capacity c_{ij}
- Maximize total flow.

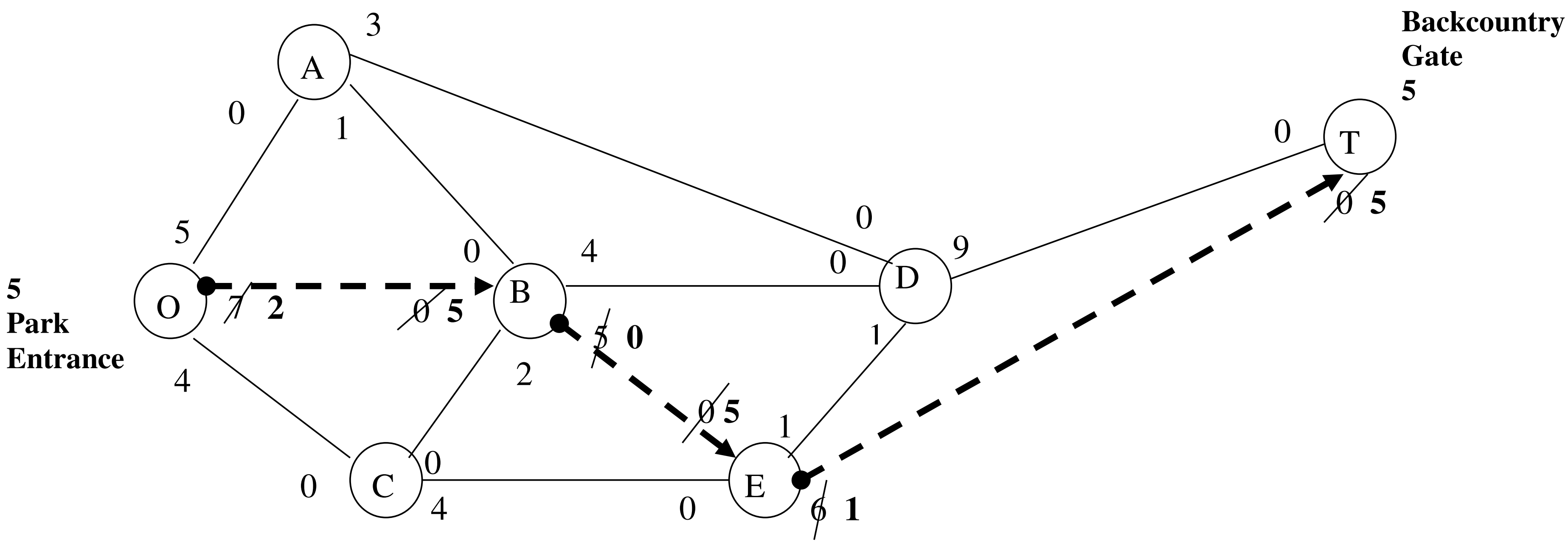


Summary of Solution Procedure

1. Find a path from source to sink with strictly positive flow capacity
2. Search this path for the branch with the smallest remaining flow capacity (c*)
   – increase the flow in the path by c*
3. Decrease by c* the remaining flow capacity, increase by c* the remaining flow capacity in opposite direction
4. Return to step 1, until no positive flow exists Network Problems

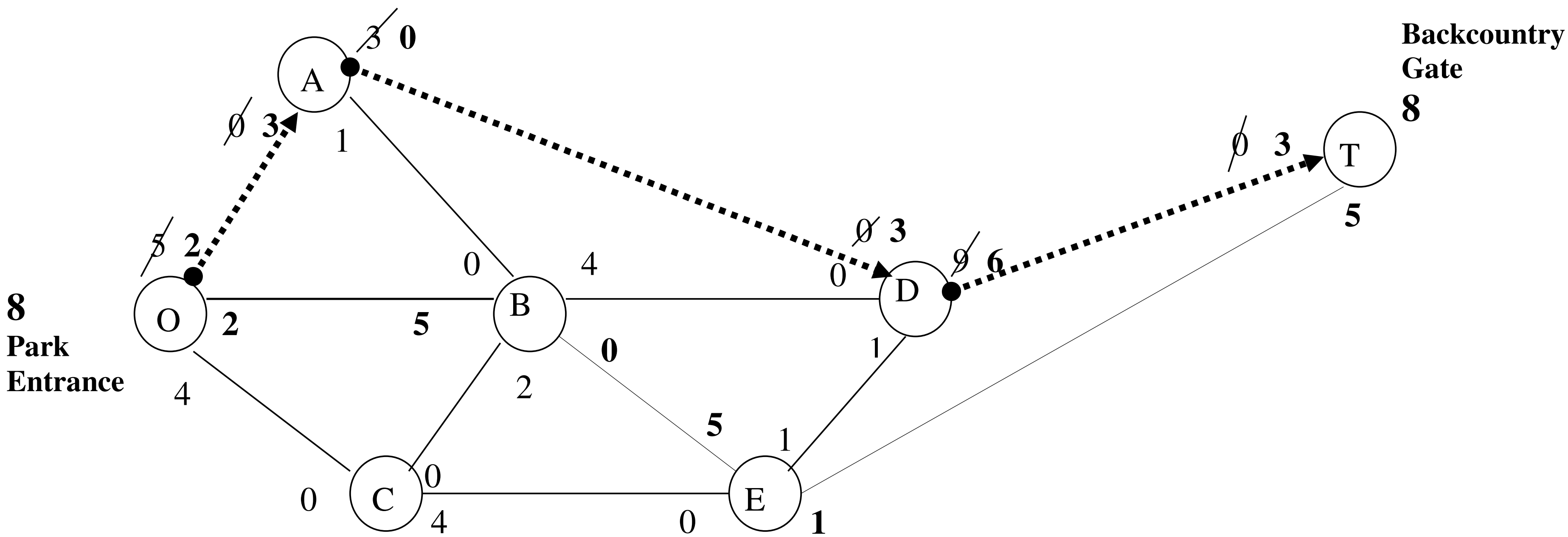
Iteration 1

- ***Iteration 1.*** Assign flow of 5 to O - B - E -T



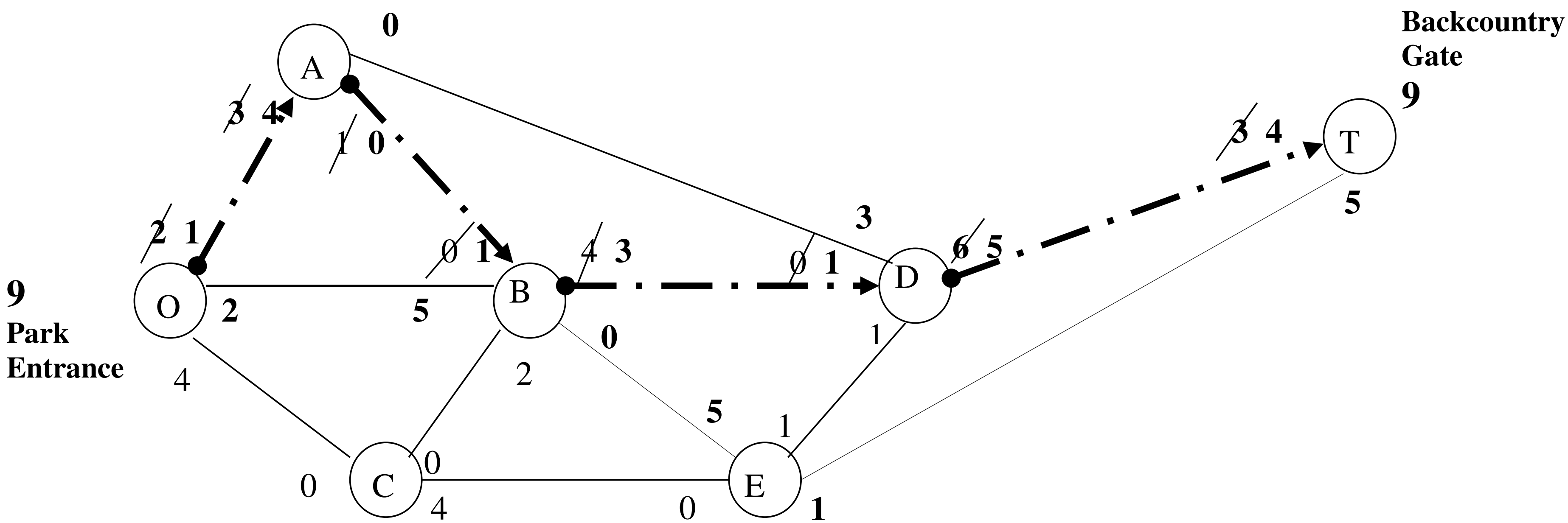
Iteration 2

Iteration 2. Assign flow of 3 to O - A - D - T



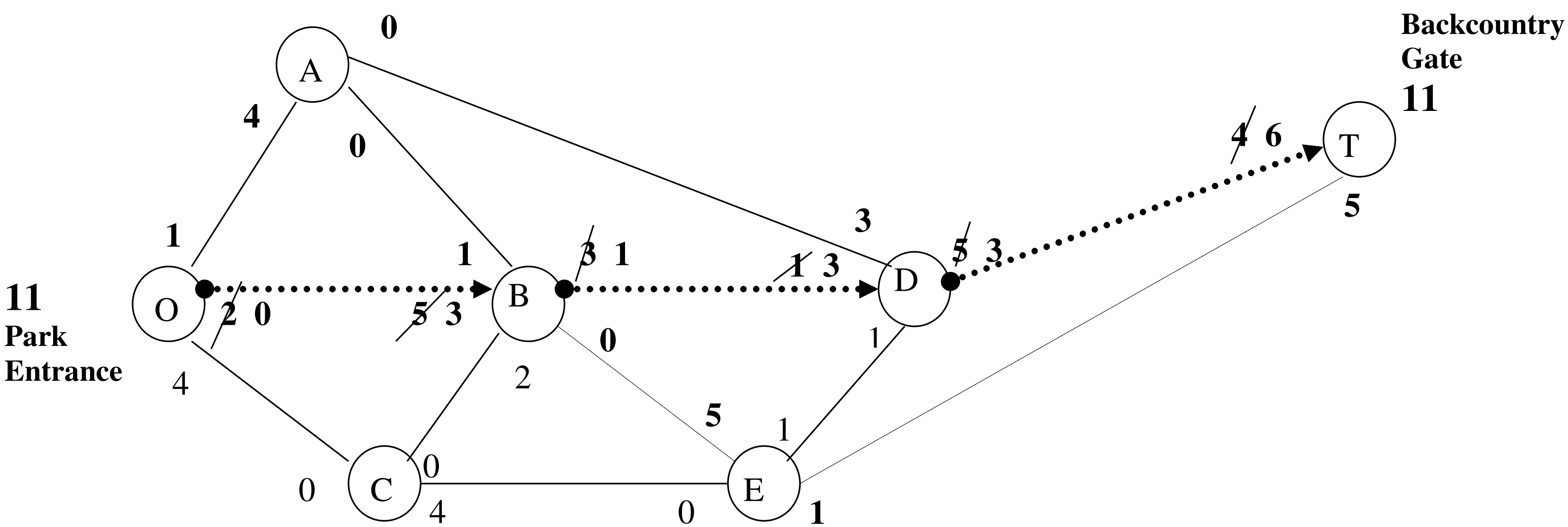
Iteration 3

- ***Iteration 3.*** Assign flow of 1 to O - A - B - D - T



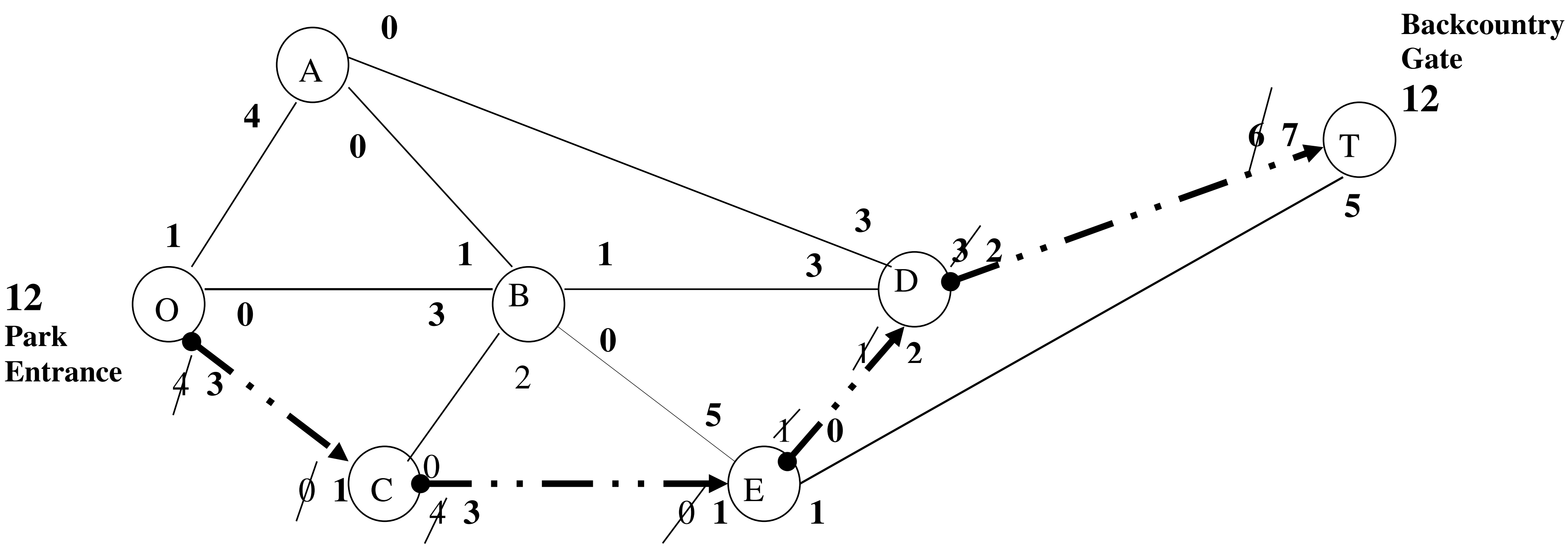
Iteration 4

- ***Iteration 4.*** Assign flow of 2 to O – B – D –T

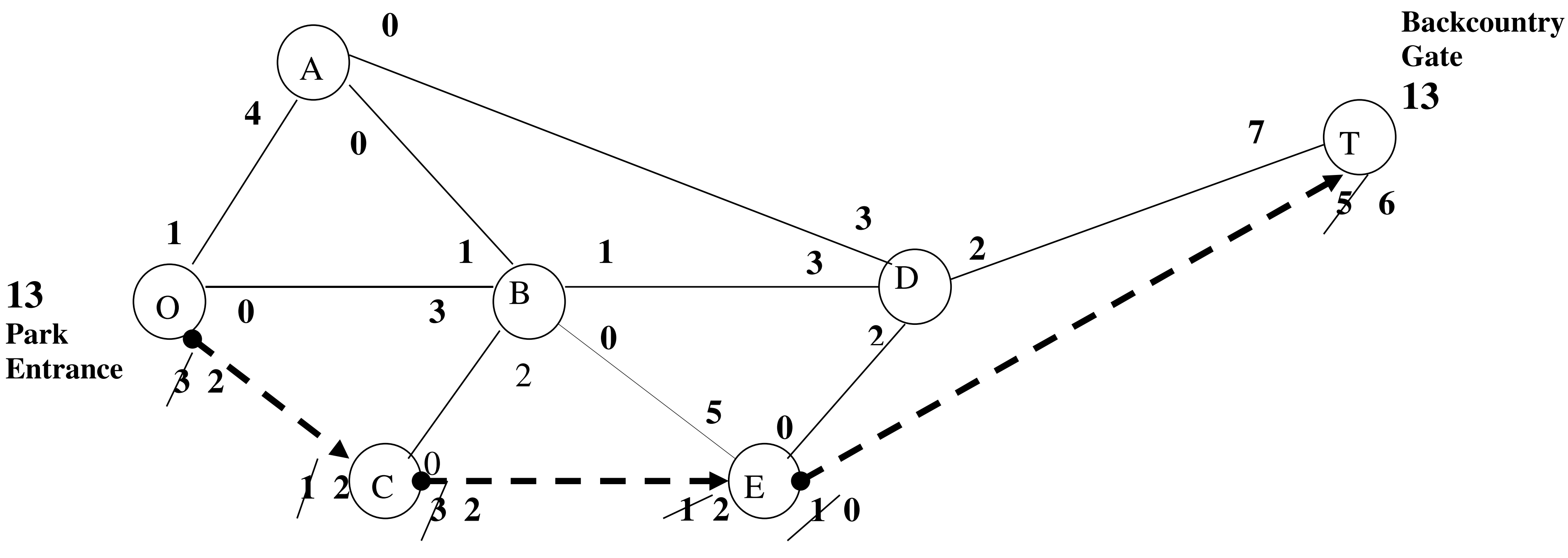


Iteration 5

- ***Iteration 5.*** Assign flow of 1 to O – C – E – D – T

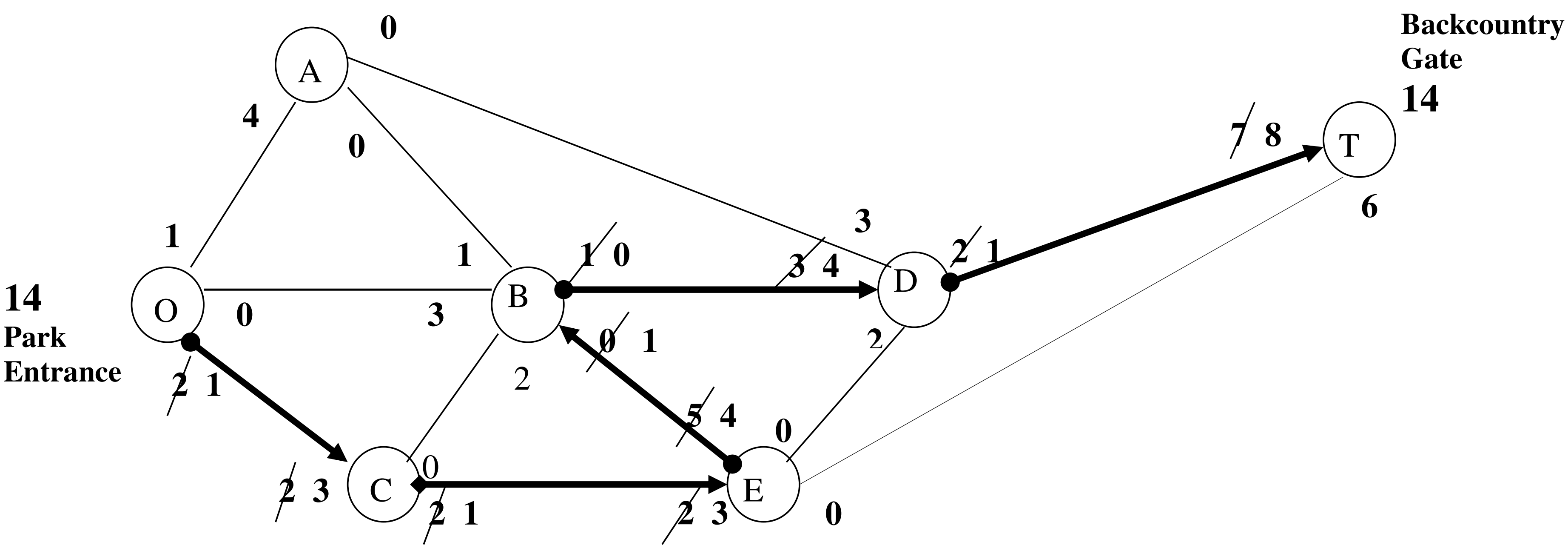


Iteration 6. Assign flow of 1 to O - C - E - T



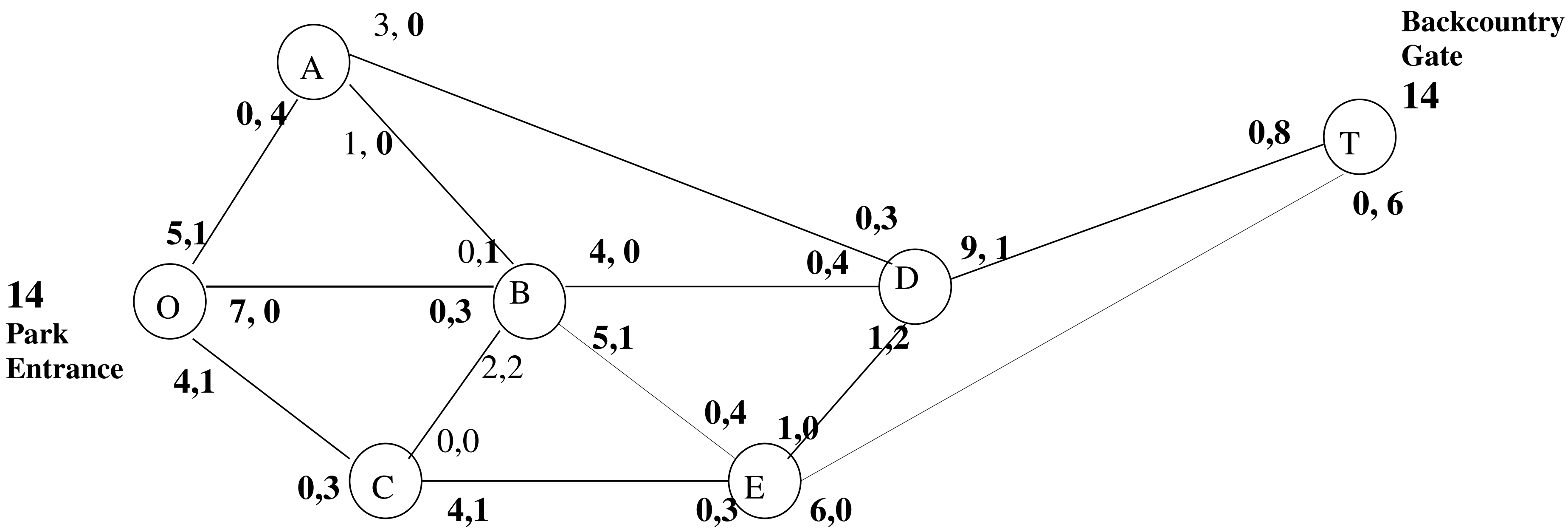
Iteration 7

- ***Iteration 7.*** Assign flow of 1 to O – C – E – B – D – T

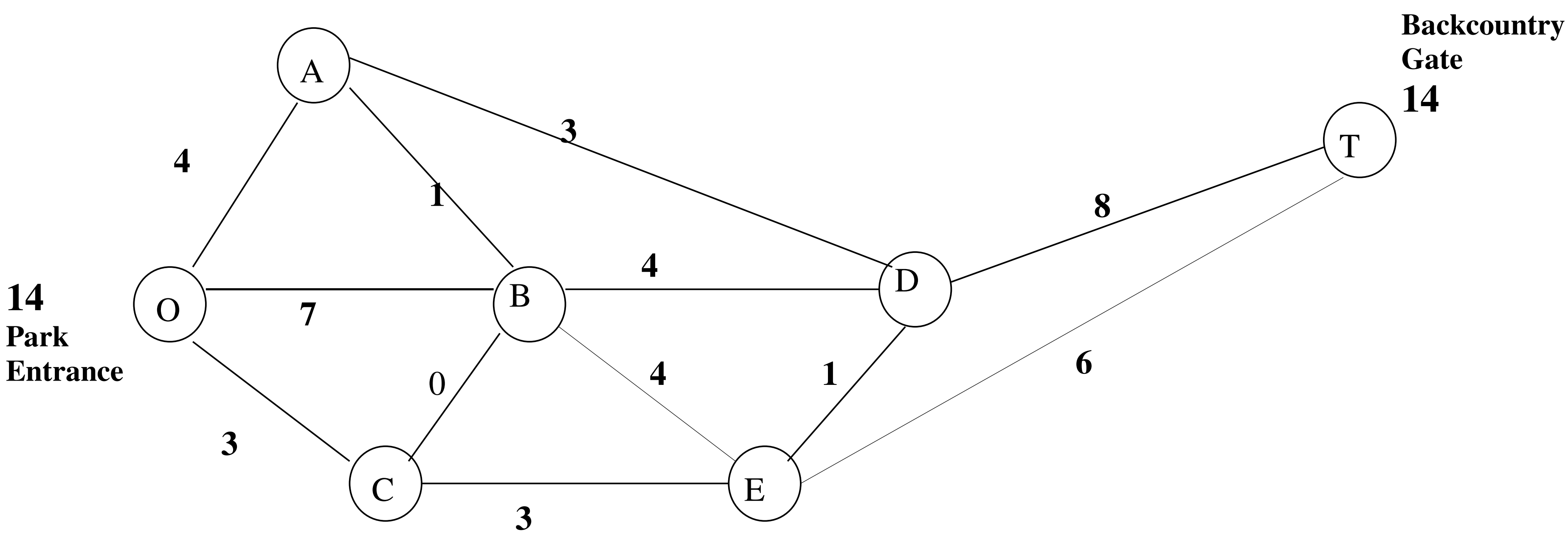


Optimal Pattern

- ***Optimal found by flow differences from original flows:***
  - (**Original capacity**, ending capacity)



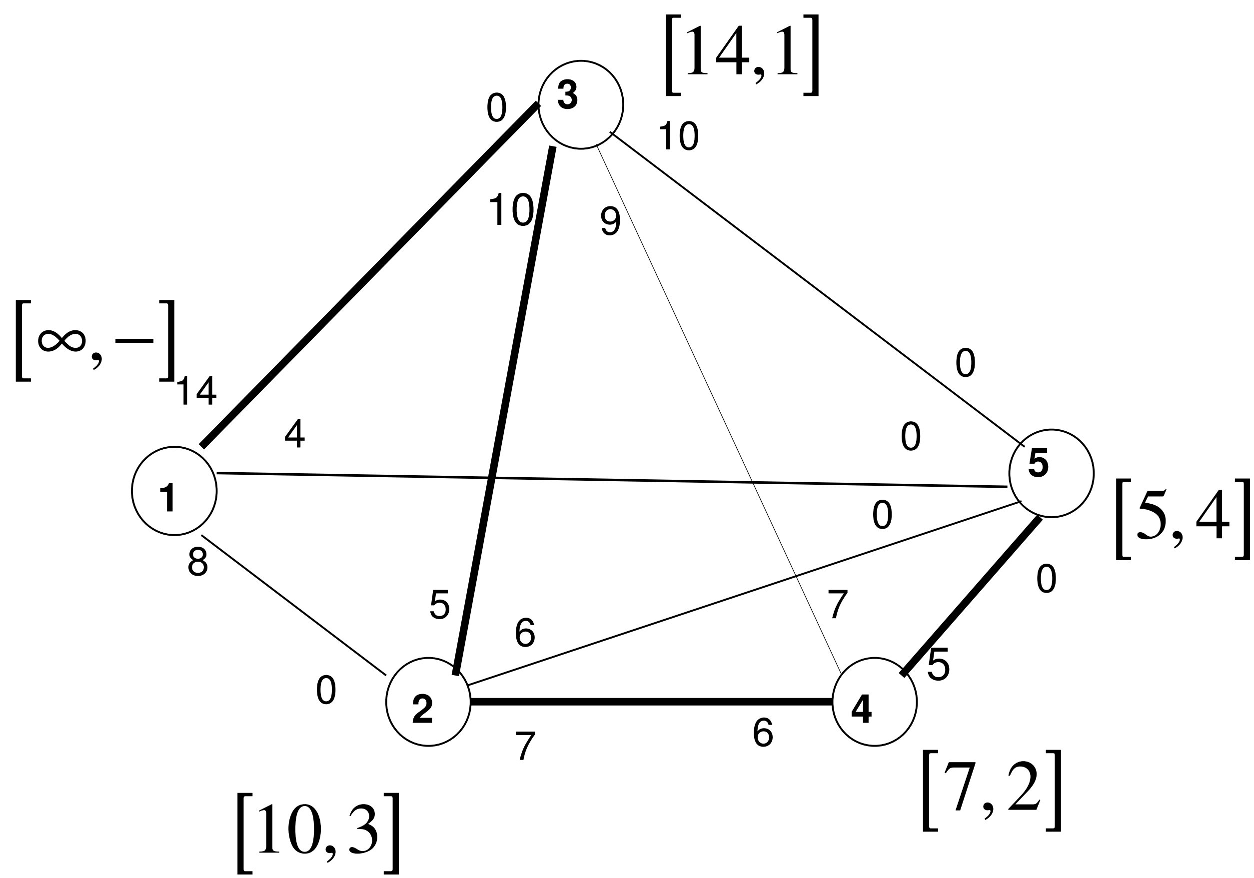
Optimal Pattern



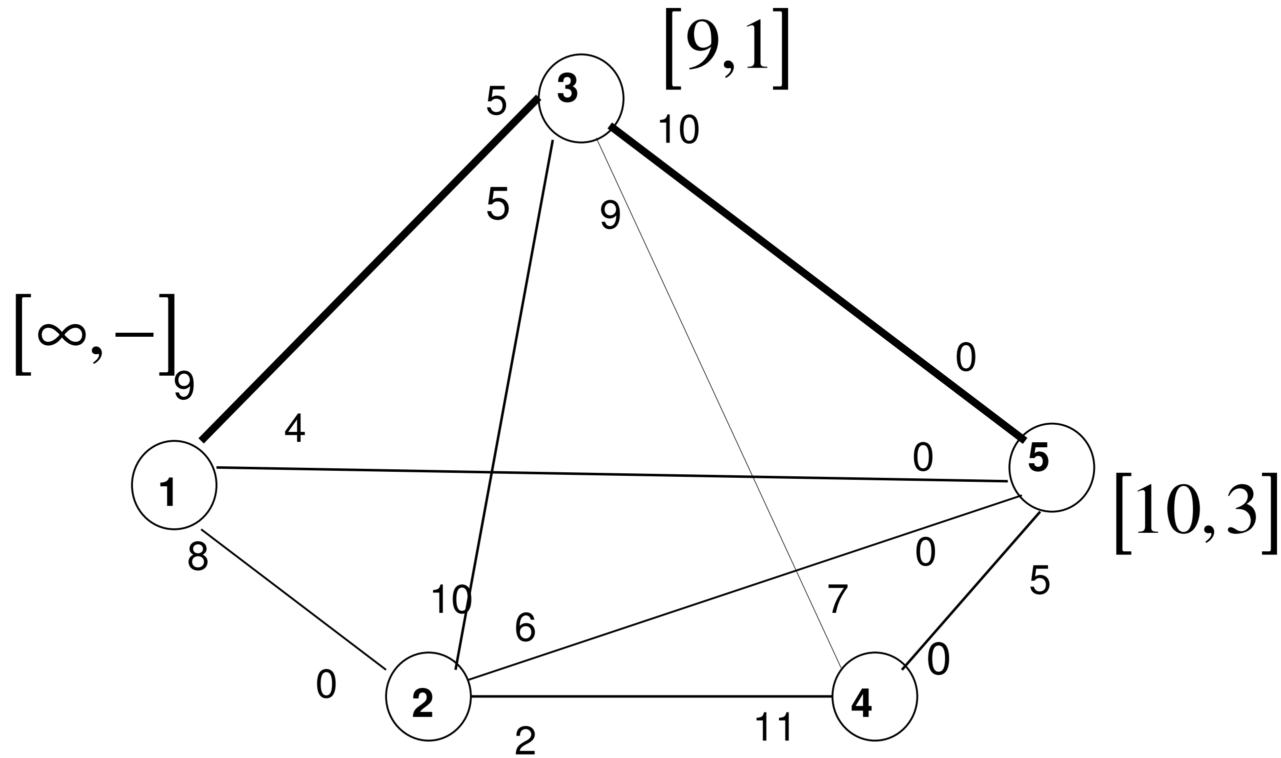
Example:

Determine the maximal flow and the optimum flow in each arc for the network in the following figure

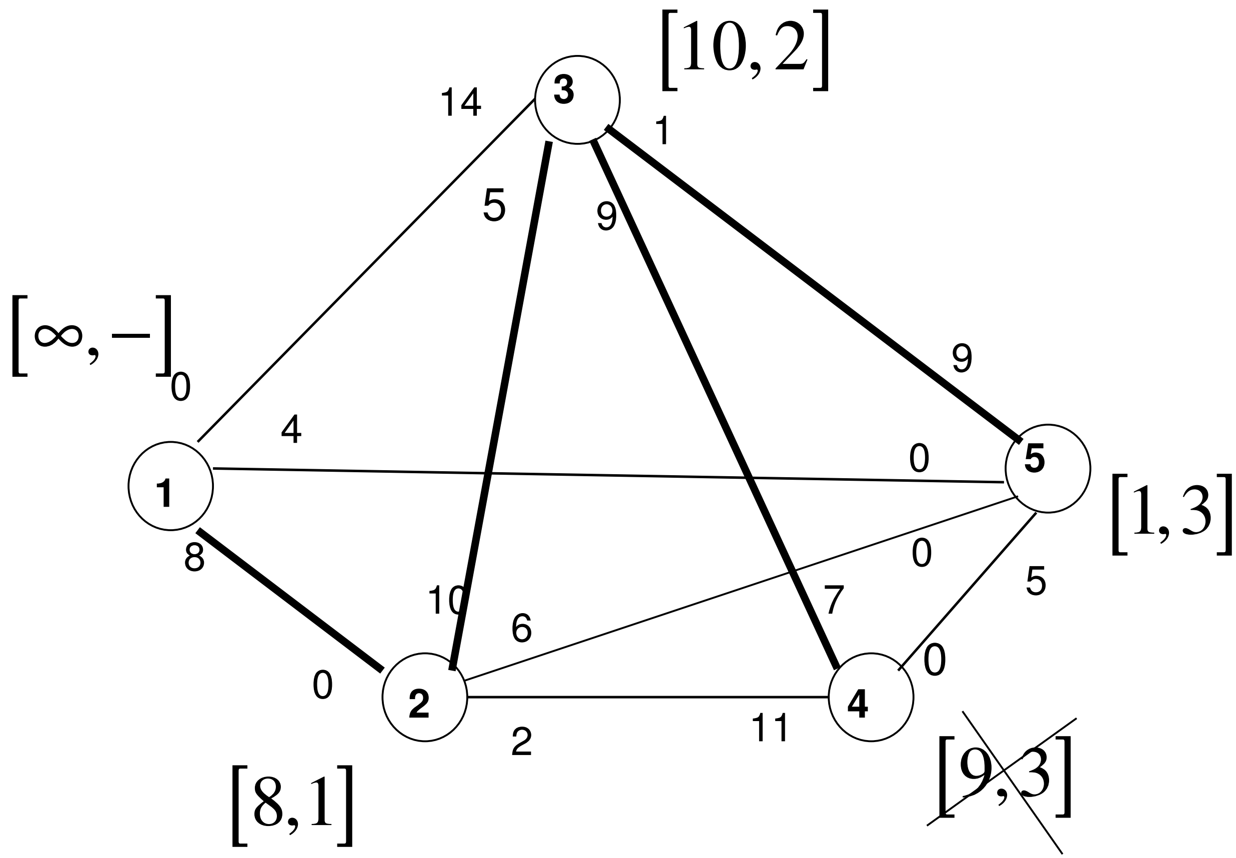
f_1=5 Units



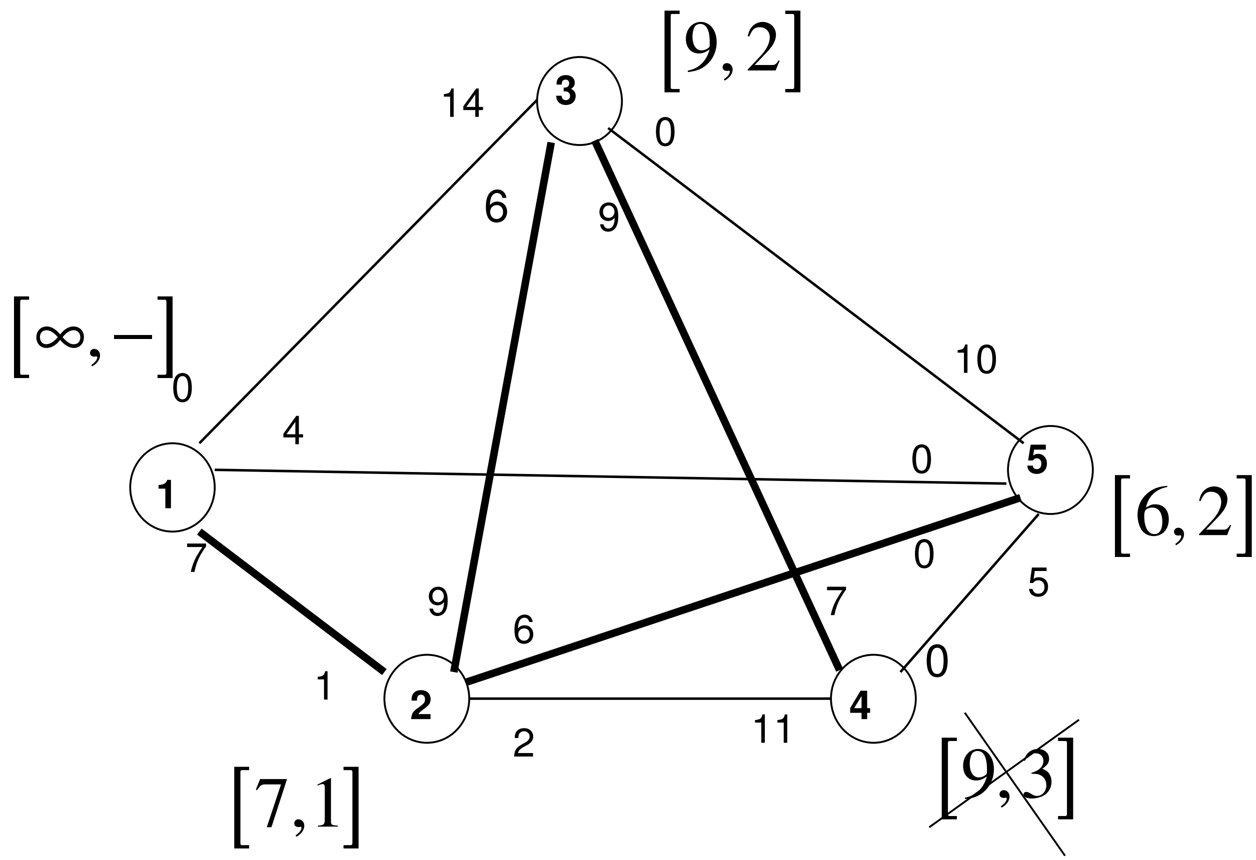
f_2=9 Units



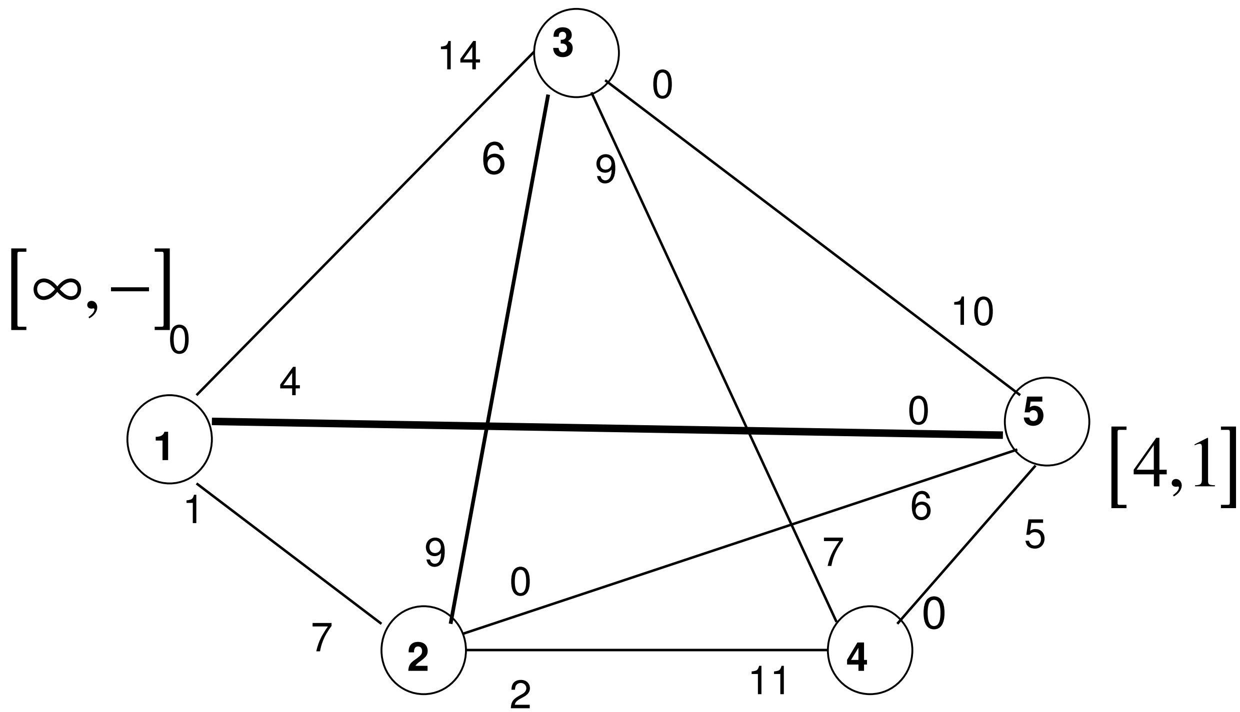
f_3=1 Units

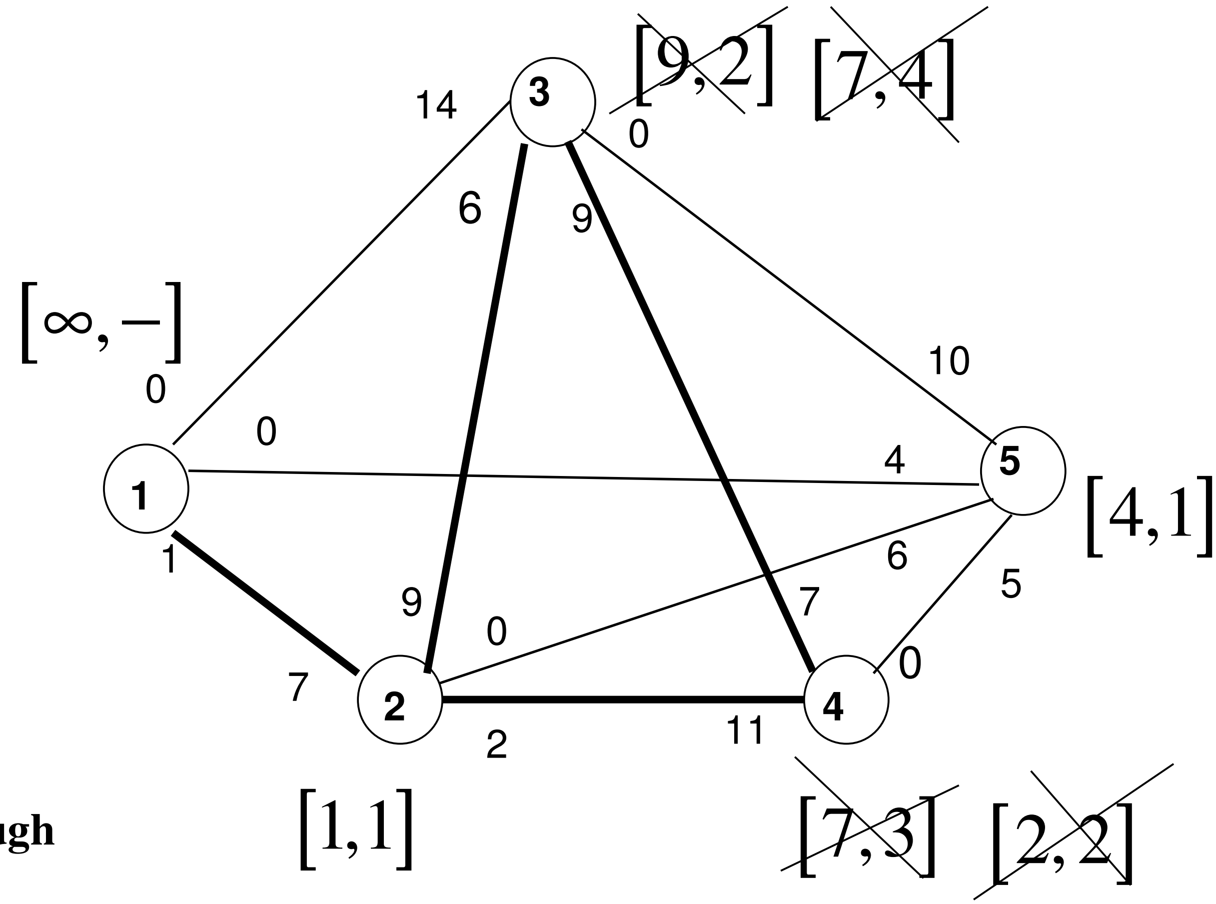


f_4=6 Units



f_5=4 Units

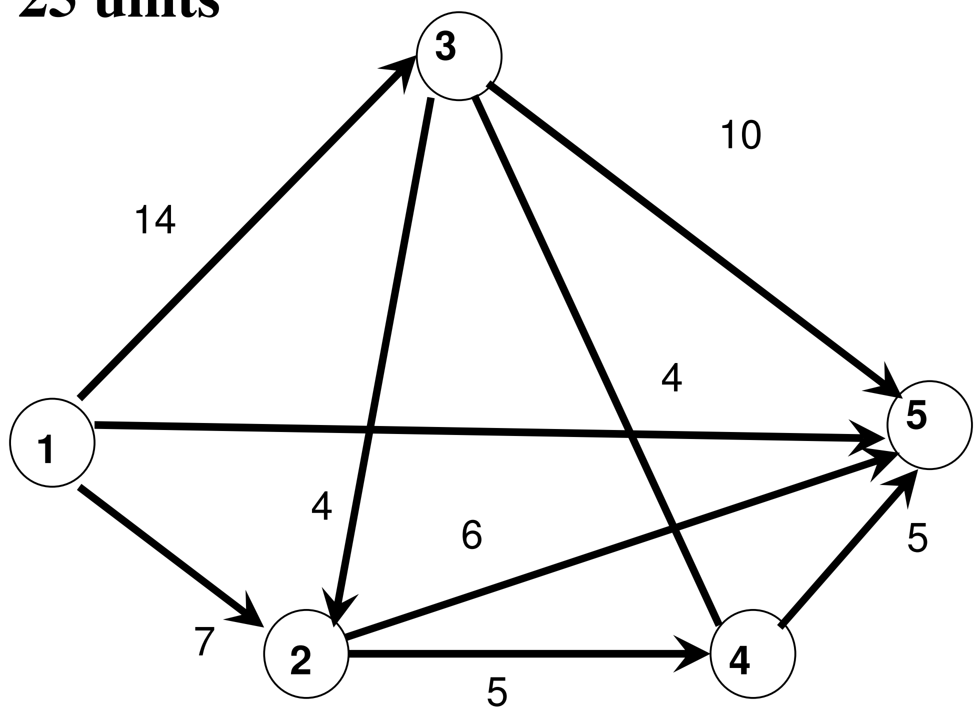




No breakthrough

Solution

Maximum flow 25 units



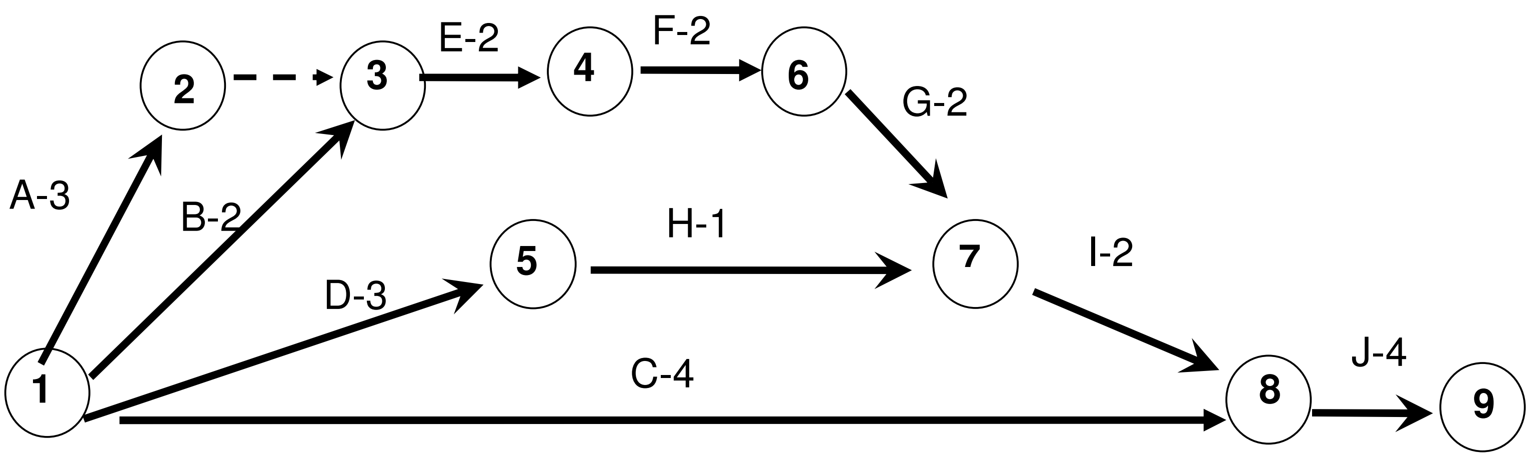
Network Representation

A publisher has a contract with an Author to publish a text book. The activities associated with the production of the text book are given subsequently. Develop the associated network for the project.

Activity	Predecessor	Duration (weeks)
(A) Manuscript proofreading by editor	-----	3
(B) Sample pages prepared by typesetter	-----	2
(C) Book cover design	-----	4
(D) Preparation of artwork for book figures	-----	3
(E) Author's approval of edited manuscript and sample page	A, B	2
(F) Book typesetting	E	4
(G) Author checks typeset pages	F	2
(H) Author checks artwork	D	1
(I) Production of printing plates	G, H	2
(J) Book production and binding	C, I	4

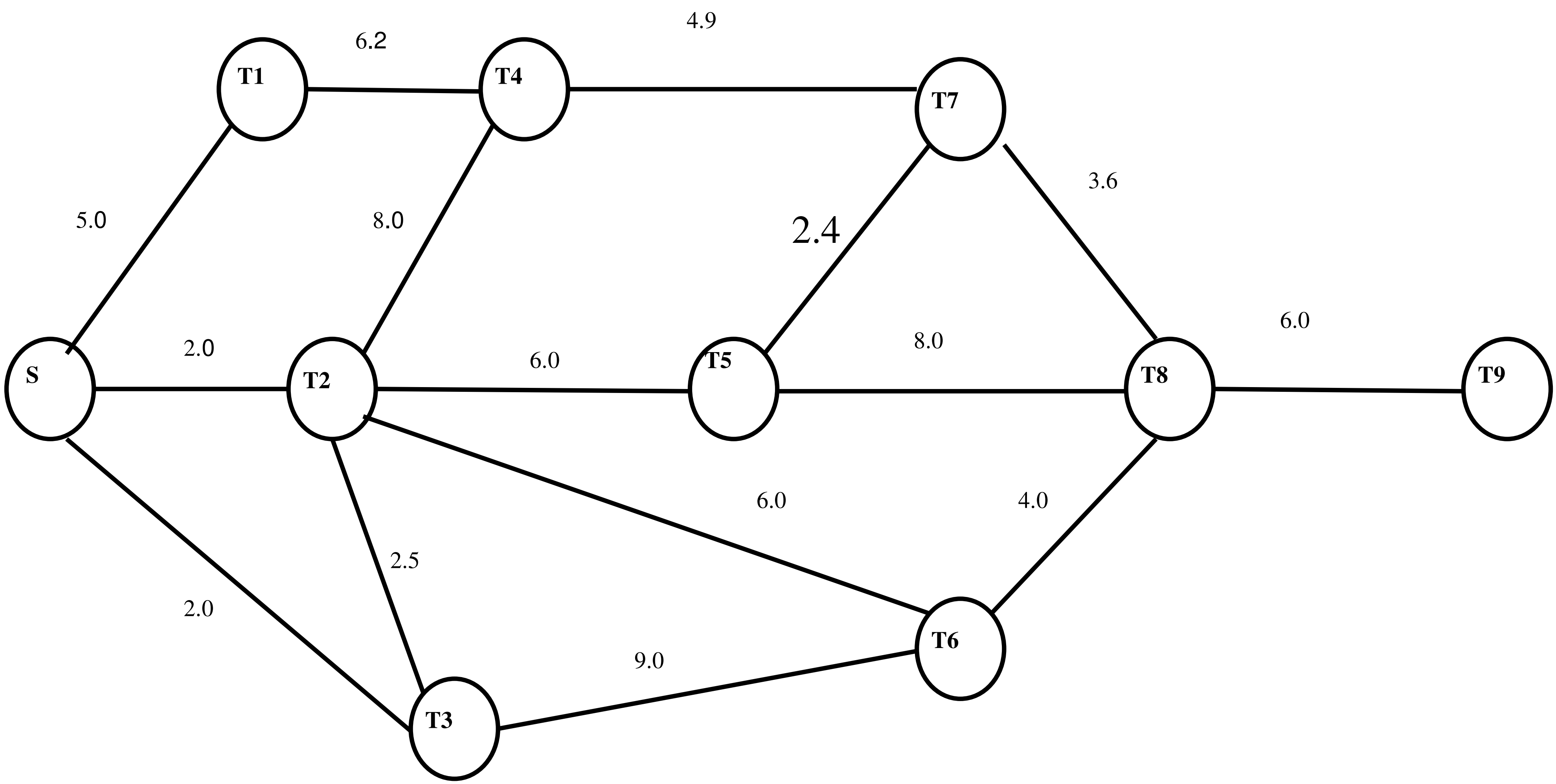
Network Representation

Network describes the precedence relationships among the different activities.



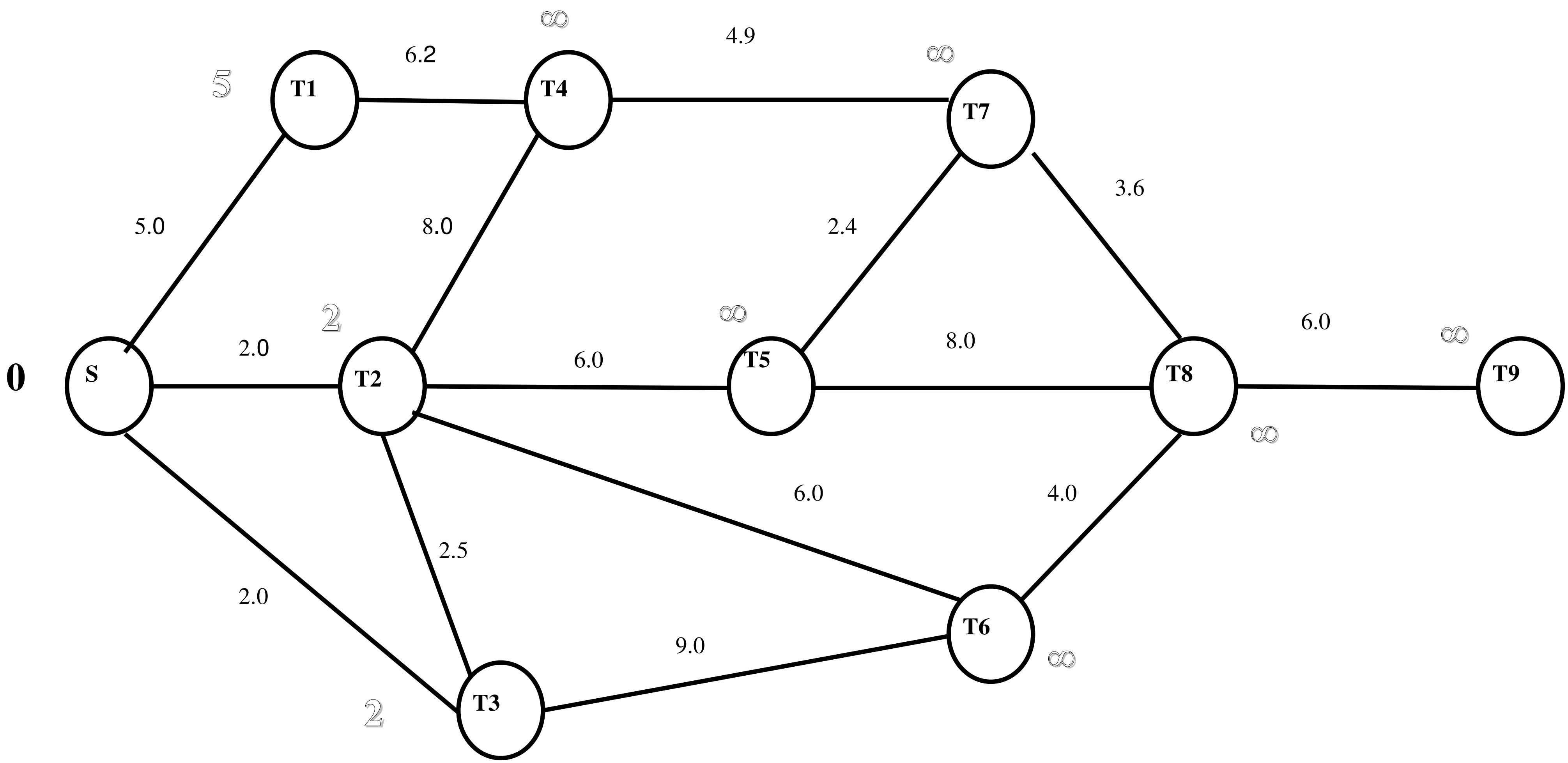
Example

In the following figure find the **shortest** path between node S and 9 by using LP and Dijkstra method.



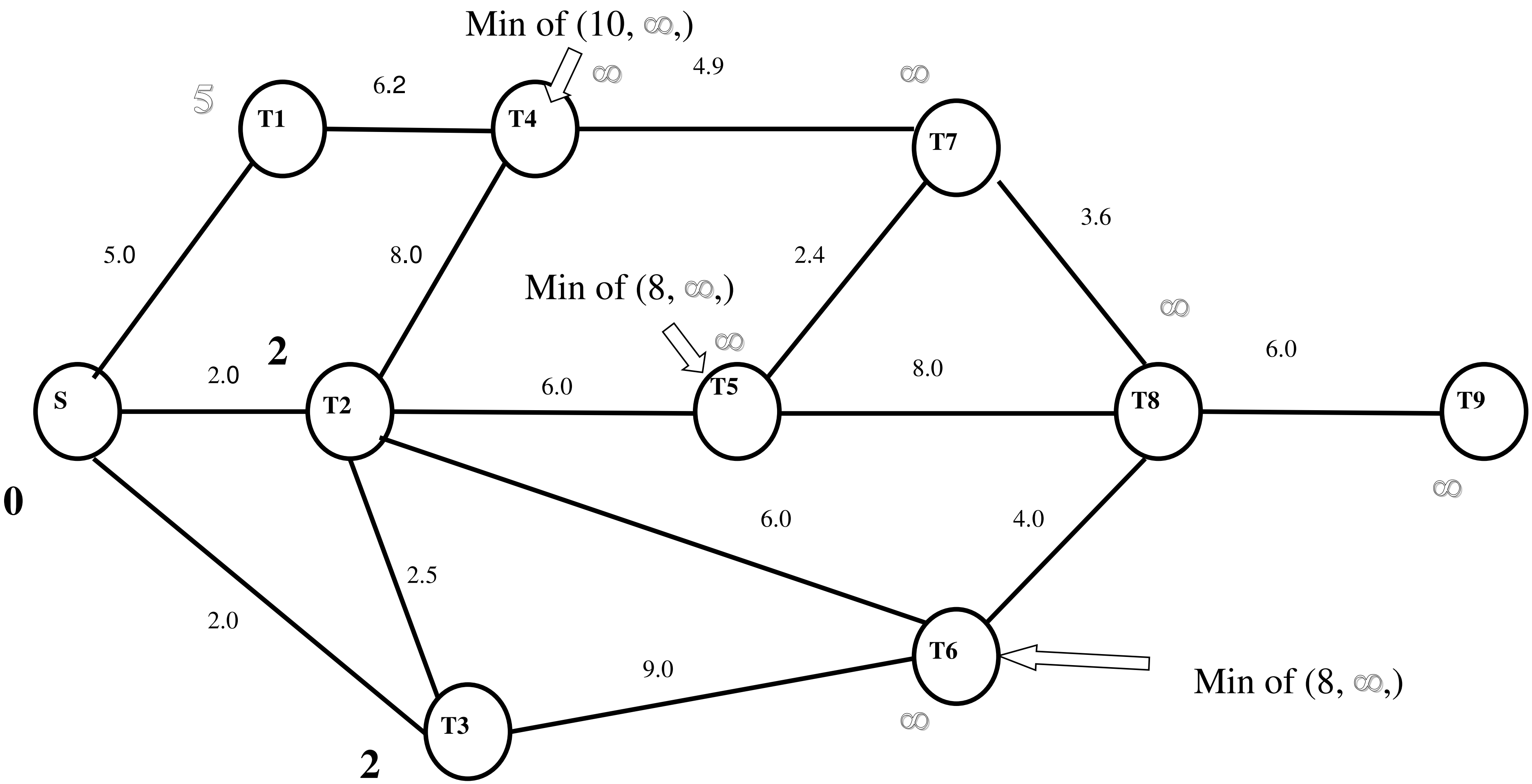
Example

Dijkstra's Algorithm

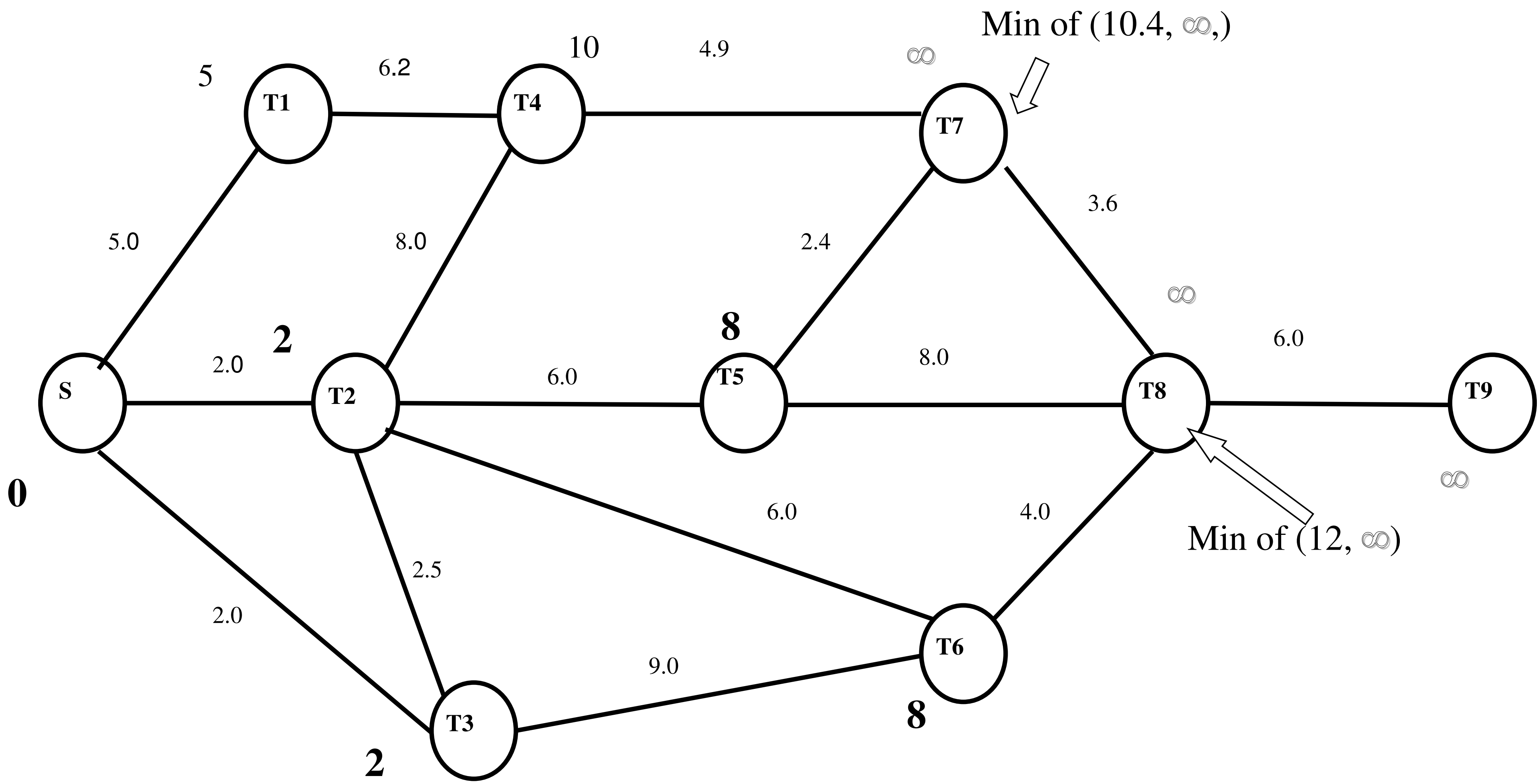


Example

Dijkstra's Algorithm

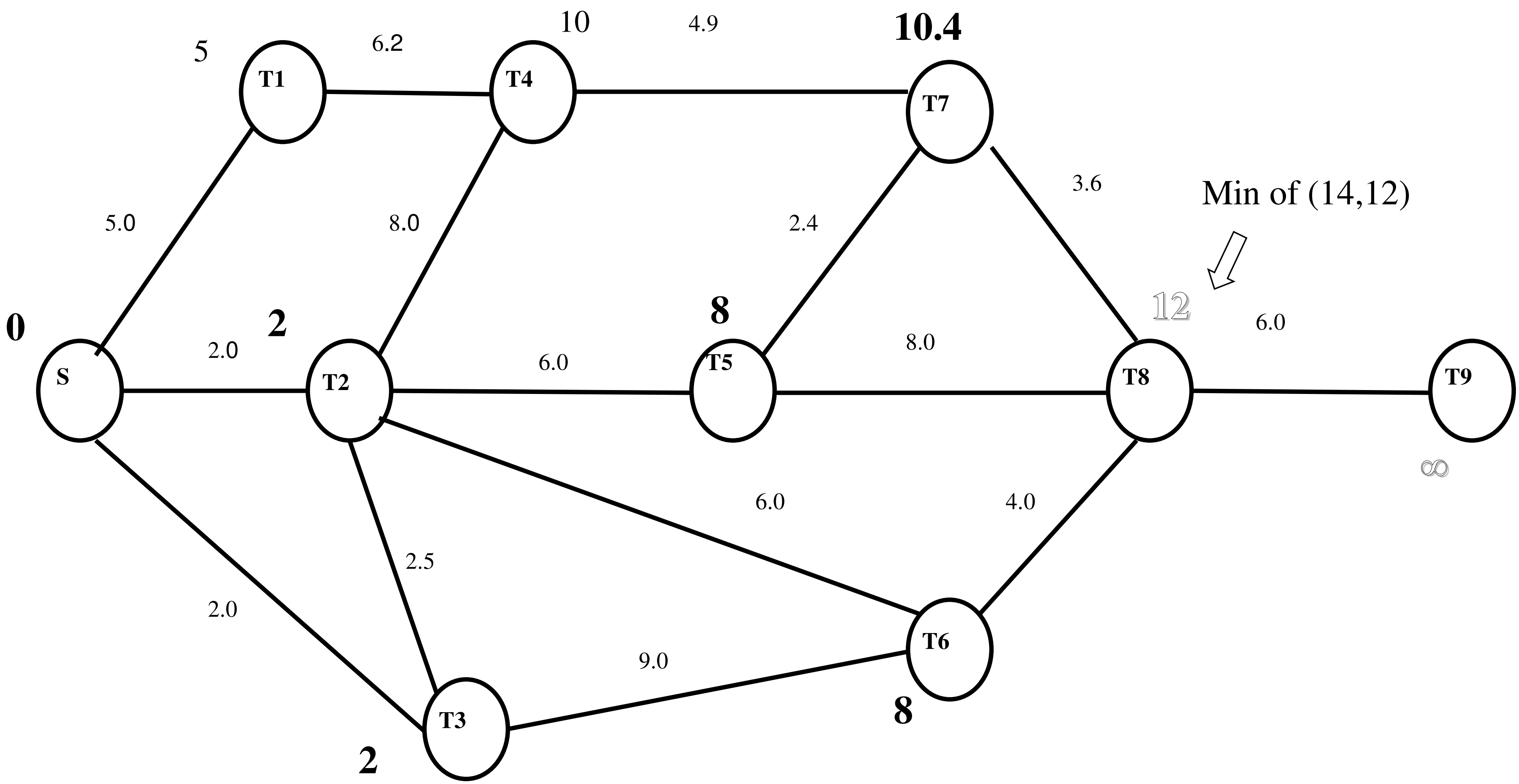


Dijkstra’s Algorithm



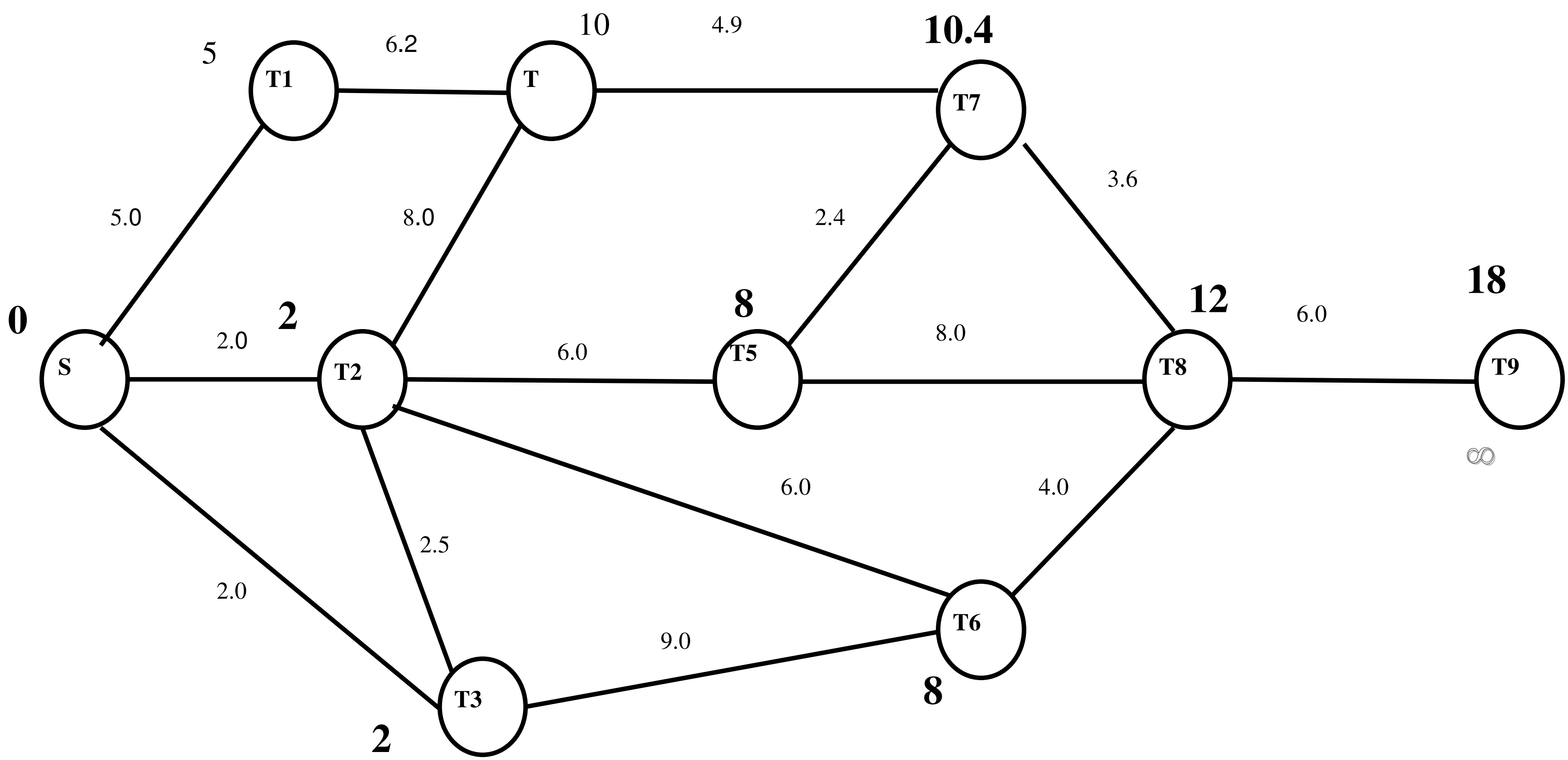
Example

Dijkstra's Algorithm



Example

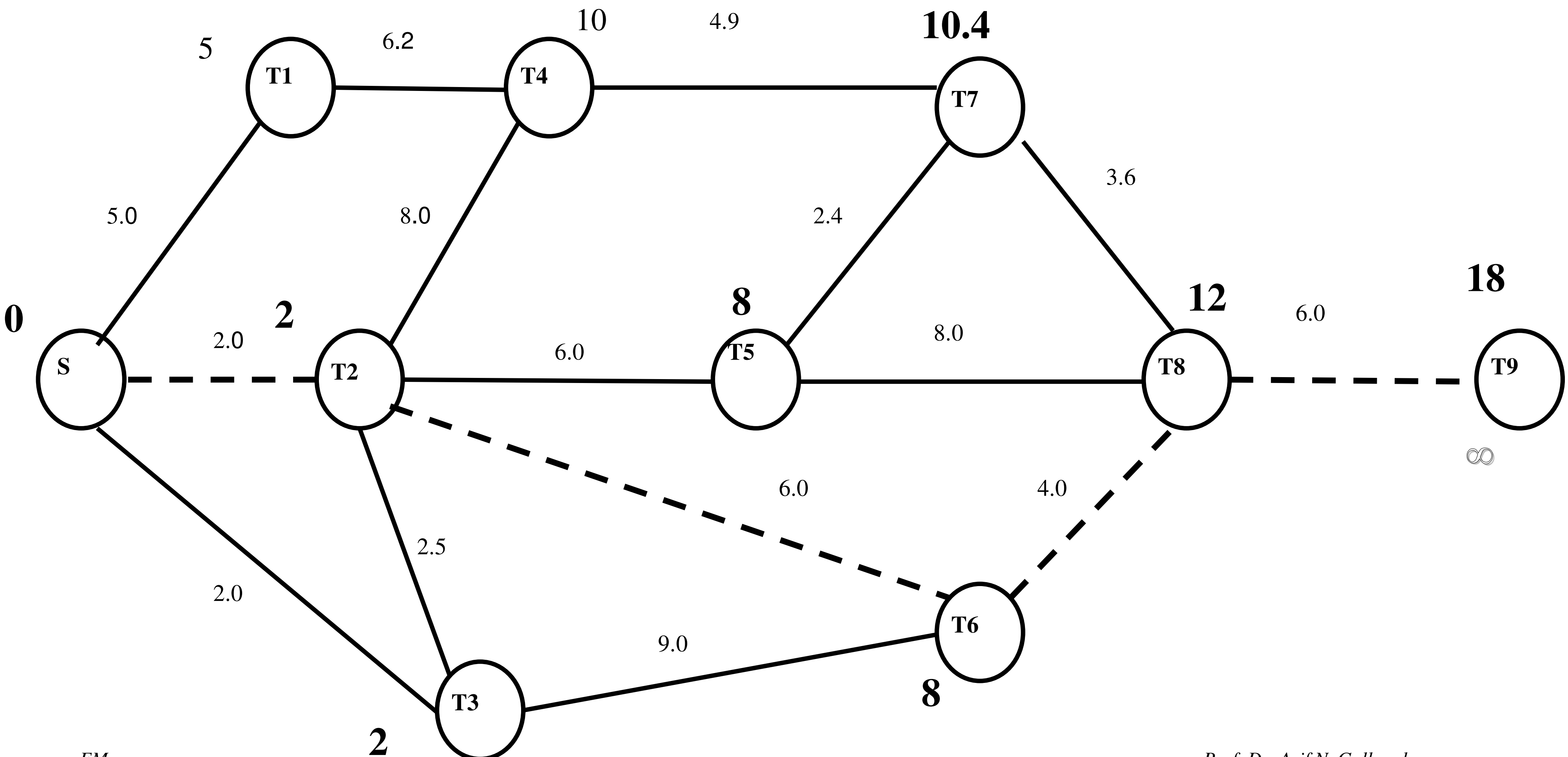
Dijkstra's Algorithm



Example

Dijkstra's Algorithm

Tracing Back thru: Start at last node (j) and go to node (i) which is equal to node j's label minus arc (i,j)



Example

LP Problem :

- We can reduce this problem to find shortest path through network by giving Node S an arbitrary unit of 1 (flow out), and giving Node 9 an unit of –1 (flow in).
- Nodes between S and 9 will have value of 1 or –1 depending on flow in or flow out at that node.
- Problem can be set to minimize the following objective function:

Minimize:

$$6.0T_{98}+3.6T_{87}+8.0T_{85}+4.0T_{86}+3.6T_{78}+2.4T_{75}+4.9T_{74}+4.0T_{68}+6.0T_{62}+9.0T_{63}+8.0T_{58}+2.4T_{57}+6.0T_{52}+4.9T_{47}+8.0T_{42}+6.2T_{41}+9.0T_{36}+2.5T_{32}+2.0T_{30}+6.0T_{26}+6.0T_{25}+8.0T_{24}+2.5T_{23}+2.0T_{20}+6.2T_{14}+5.0T_{10}$$

Example

LP Problem :

Subject to:

T9	T_{98}																										=1
T8	$-T_{98}$	$+T_{87}$	$+T_{85}$	$+T_{86}$	$-T_{78}$			$-T_{68}$			$-T_{58}$																=0
T7		$-T_{87}$			$+T_{78}$	$+T_{75}$	$+T_{74}$					$-T_{75}$		$-T_{47}$													=0
T6				$-T_{86}$				$+T_{68}$	$+T_{62}$	$+T_{63}$							$-T_{36}$			$-T_{26}$							=0
T5			$-T_{85}$			$-T_{75}$					$+T_{58}$	$+T_{57}$	$+T_{52}$								$-T_{25}$						=0
T4							$-T_{74}$							$+T_{47}$	$+T_{42}$	$+T_{41}$						$-T_{24}$			$-T_{14}$		=0
T3										$-T_{63}$							$+T_{36}$	$+T_{32}$	$+T_{30}$				$-T_{23}$				=0
T2									$-T_{62}$				$-T_{52}$		$-T_{42}$			$-T_{32}$		$+T_{26}$	$+T_{25}$	$+T_{24}$	$+T_{23}$	$+T_{20}$			=0
T1																$-T_{41}$									$+T_{14}$	$+T_{10}$	=0
S																			$-T_{30}$					$-T_{20}$		$-T_{10}$	=-1

Microsoft Excel 9.0 Yanıt Raporu
Çalışma Sayfası: [Blocks_Roads_Network_LP.xls]Sheet1
Yaratılan Rapor: 07.12.2003 13:46:07

Hedef Hücre (En Küçük)

Hücre	Ad	İlk Değer	Son Değer
AB6	COST/DISTANCE Total	0	18

Ayarlanabilir Hücreler

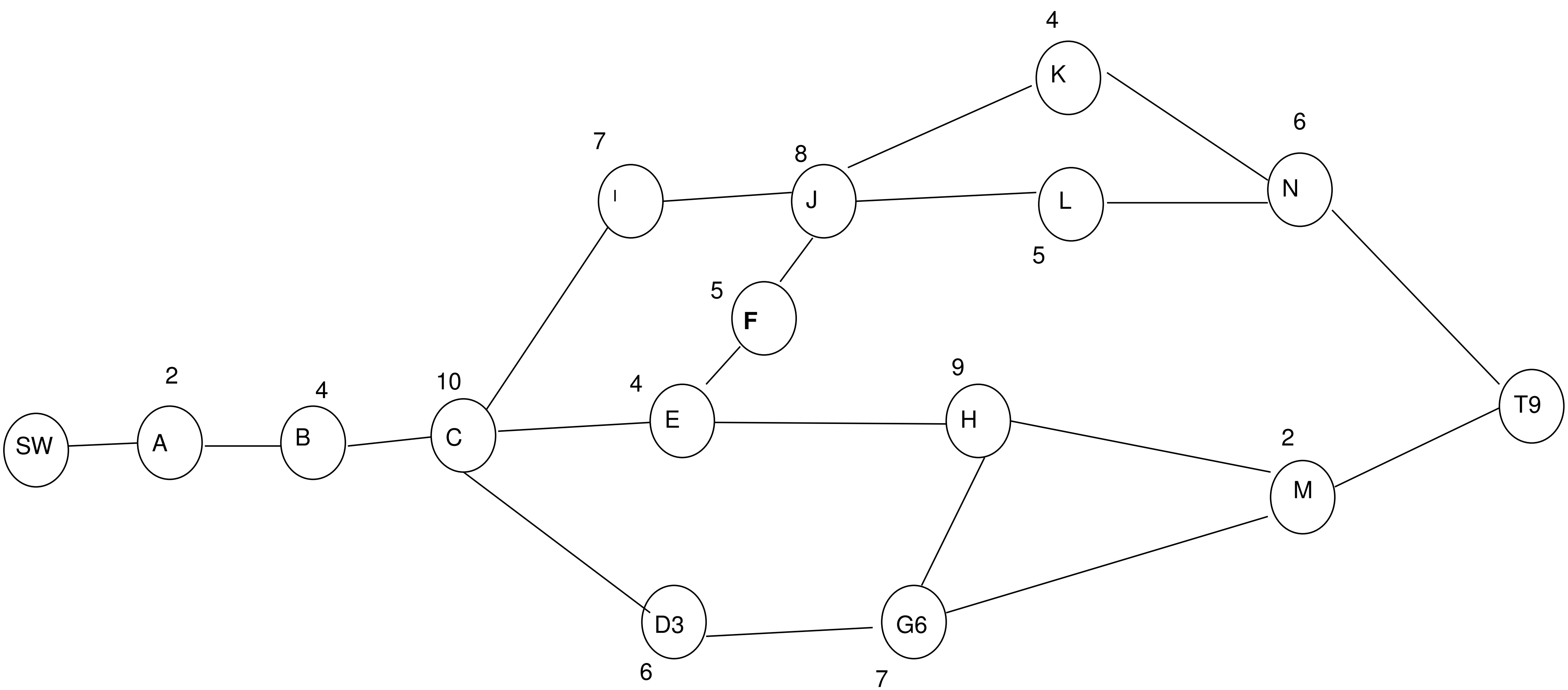
Hücre	Ad	İlk Değer	Son Değer
B5	ANSWER T9-T8	0	1
C5	ANSWER T8-T7	0	0
D5	ANSWER T8-T5	0	0
E5	ANSWER T8-T6	0	1
F5	ANSWER T7-T8	0	0
G5	ANSWER T7-T5	0	0
H5	ANSWER T7-T4	0	0
I5	ANSWER T6-T8	0	0
J5	ANSWER T6-T2	0	1
K5	ANSWER T6-T3	0	0
L5	ANSWER T5-T8	0	0
M5	ANSWER T5-T7	0	0
N5	ANSWER T5-T2	0	0
O5	ANSWER T4-T7	0	0
P5	ANSWER T4-T2	0	0
Q5	ANSWER T4-T1	0	0
R5	ANSWER T3-T6	0	0
S5	ANSWER T3-T2	0	0
T5	ANSWER T3-HW	0	0
U5	ANSWER T2-T6	0	0
V5	ANSWER T2-T5	0	0
W5	ANSWER T2-T4	0	0
X5	ANSWER T2-T3	0	0
Y5	ANSWER T2-HW	0	1
Z5	ANSWER T1-T4	0	0
AA5	ANSWER T1-HW	0	0

Example Problem

The activities in the following table describe the construction of a new house, Develop the associated project network.

Node Description	**Predecessor**	**Duration**
(A)	Start	2
(B)	Node A	4
(C)	Node B	10
(D) Roof	Node C	6
(E) Exterior Plumbing	Node C	4
(F) Interior Plumbing	Node E	5
(G) Exterior siding	Node D	7
(H) Exterior painting	Node E, Node G	9
(I) Electrical work	Node C	7
(J) Wallboard	Node F, Node I	8
(K) flooring	Node J	4
(L) Interior painting	Node J	5
(M) Exterior fixtures	Node H	2
(N) Interior fixtures	Node K, Node L	6
	Node M, Node N	0

Network overview - the construction of a new house project.



Variable	Start	A	B	C	D	E	F	G	H	I	J	K	L	M	N	Finish		
A-Start>=0	1																=	0
B-A>=2		-1	1														>=	2
C-B>=4			-1	1													>=	4
D-C>=10				-1	1												>=	10
E-C>=10				-1		1											>=	10
F-E>=4						-1	1										>=	4
G-D>=6					-1			1									>=	6
H-G>=7								-1	1								>=	7
H-E>=4						-1			1								>=	4
I-C>=10				-1						1							>=	10
J-F>=5							-1				1						>=	5
J-I>=7										-1	1						>=	7
K-J>=8											-1	1					>=	7
L-J>=8											-1		1				>=	8
M-H>=9									-1					1			>=	9
N-K>=4												-1			1		>=	4
N-L>=5													-1		1		>=	5
FINISH-N>=6															-1	1	>=	6
FINISH-M>=2														-1		1	>=	2

Microsoft Excel 9.0 Yanıt Raporu
Çalışma Sayfası: [cpm_small_LP_example.xls]Minimum LP
Yaratılan Rapor: 10.12.2003 17:10:00

Hedef Hücre (En Küçük)

Hücre	Ad	İlk Değer	Son Değer
S7	ES sum	0	337

Ayarlanabilir Hücreler

Hücre	Ad	İlk Değer	Son Değer
C7	ES Start	0	0
D7	ES NodeA	0	0
E7	ES NodeB	0	2
F7	ES NodeC	0	6
G7	ES NodeD	0	16
H7	ES NodeE	0	16
I7	ES NodeF	0	20
J7	ES NodeG	0	22
K7	ES NodeH	0	29
L7	ES NodeI	0	16
M7	ES NodeJ	0	25
N7	ES NodeK	0	32
O7	ES NodeL	0	33
P7	ES NodeM	0	38
Q7	ES NodeN	0	38
R7	ES Finish	0	44