Chapter 9-10

Confidence Intervals and Hypothesis Testing

CI for the Mean when σ is unknown, PI, TL

Statistics

Mehmet Güray Güler, PhD

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CI for the mean – σ UNknown

- We know that T has a t-distribution with n-1 dof:

$$T = \frac{\bar{X} - \mu}{S/\sqrt{n}}$$

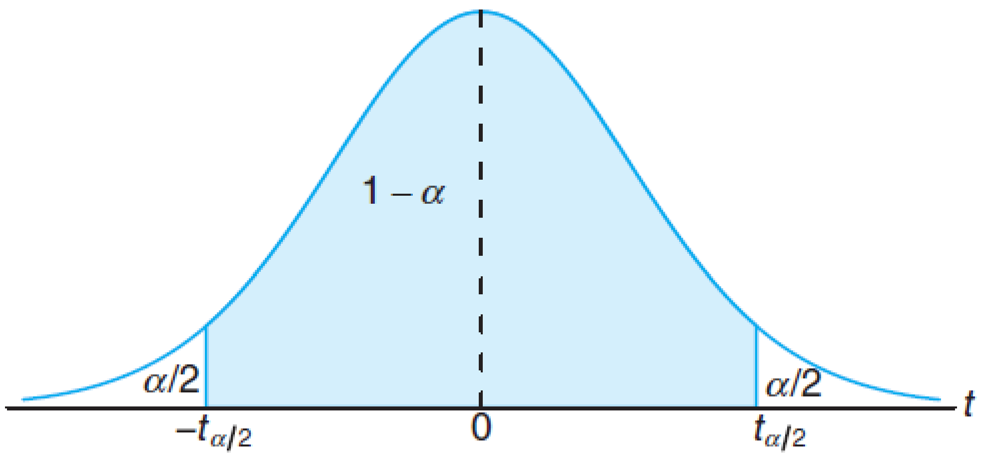

Figure 9.5: $P(-t_{\alpha/2} < T < t_{\alpha/2}) = 1 - \alpha$.

CI for the mean – σ UNknown

- We know that T has a t-distribution with n-1 dof:

$$T = \frac{\bar{X} - \mu}{S/\sqrt{n}}$$

$$P(-t_{\alpha/2} < T < t_{\alpha/2}) = 1 - \alpha$$

$$P\left(-t_{\alpha/2} < \frac{\bar{X} - \mu}{S/\sqrt{n}} < t_{\alpha/2}\right) = 1 - \alpha$$

$$P\left(\bar{X} - t_{\alpha/2}\frac{S}{\sqrt{n}} < \mu < \bar{X} + t_{\alpha/2}\frac{S}{\sqrt{n}}\right) = 1 - \alpha.$$

CI for the mean – σ UNknown

- Hence we can write the confidence interval as follows:

If $\bar{x}$ and s are the mean and standard deviation of a random sample from a normal population with unknown variance σ^2, a $100(1-\alpha)\%$ confidence interval for μ is

$$\bar{x} - t_{\alpha/2}\frac{s}{\sqrt{n}} < \mu < \bar{x} + t_{\alpha/2}\frac{s}{\sqrt{n}},$$

where $t_{\alpha/2}$ is the t-value with $v = n - 1$ degrees of freedom, leaving an area of $\alpha/2$ to the right.

CI for the mean – σ UNknown

- **Example**:
- We have the heights of 10 students from this class as:

171	175	156	151	179
175	170	164	167	162

- Find the %95 confidence interval assuming that the heights follow a normal distribution.

CI for the mean – σ UNknown

Solution:

- We need $t_{\alpha/2} = t_{0.025}$ which can be found from the table as 2.262
- Then we can write the confidence interval as:

$$\bar{x} - t_{0.025}\frac{s}{\sqrt{n}} < \mu < \bar{x} + t_{0.025}\frac{s}{\sqrt{n}}$$

$$= 167 - 2.262\frac{8.87}{\sqrt{10}} < \mu < 167 + 2.262\frac{8.87}{\sqrt{10}}$$

$$= 160.65 < \mu < 173.34$$

Hence, we are %95 confident that the real μ of the is between 160.65 and 173.34.

CI for the mean – σ UNknown

Example:

- Assume we know that $\sigma = 9$.
- Now construct the CI again

Solution:

- Since σ is known, we can use ***z*** now.
- $\bar{x} - z_{0.025}\frac{\sigma}{\sqrt{n}} < \mu < \bar{x} + z_{0.025}\frac{\sigma}{\sqrt{n}}$
- $167 - 1.96\frac{9}{\sqrt{10}} < \mu < 167 + 1.96\frac{9}{\sqrt{10}}$
- $161.4 < \mu < 172.58$

Some Notes

1. When the population variance is known we can rely on the CLTand construct an approximate $(1-\alpha)$ confidence interval for μ, based on the normal distribution.
2. However, when the population variance is unknown, CLT does not help, and under the assumption that the distribution is (nearly) normal, we can use the t-distribution to construct a $(1-\alpha)$ confidence interval for μ.
3. In many applications, provided that the population distribution is not far from symmetric, and nearly bell-shaped we may use the t-distribution to compute confidence intervals with approximate but satisfactory results.

Prediction Intervals

- Up to now, we deal with what the value of μ is…
- Sometimes we will be interested in predicting the possible value of a *single* observation.
- Remember the difference:
  - $P(X > 180)$ = Probability that a single student is taller than 180
  - $P(\bar{X} > 180)$ = Probability that the average of the class is taller than 180
- Prediction Interval!

Prediction Intervals

- For any normal RV X with μ and σ, we can write the following:

$$P\left(\mu - z_{\alpha/2}\sigma < X < \mu + z_{\alpha/2}\sigma\right) = 1 - \alpha$$

- For example for $\mu = 170\ and\ \sigma = 10\ and\ z_{0.05} = 1.96$ we have:

$$P(170 - (1.96)(10) < X < 170 + (1.96)(10))$$
$$= P(150.4 < X < 189.6) = 0.90$$

- What does this mean in terms of single observation?

Prediction Intervals

- If we don't know σ, is the following true?

$$P\left(\mu - z_{\alpha/2}\sigma < X < \mu + z_{\alpha/2}\sigma\right)$$
$$= P\left(\bar{X} - z_{\alpha/2}\sigma < X < \bar{X} + z_{\alpha/2}\sigma\right)$$
$$= 1 - \alpha$$

- Now, we have $\bar{X}$ instead of μ which has its own standard deviation!
- It can be adjusted as follows

$$P\left(\bar{X} - z_{\alpha/2}\sigma\sqrt{1+\frac{1}{n}} < X < \bar{X} - z_{\alpha/2}\sigma\sqrt{1+\frac{1}{n}}\right) = 1 - \alpha$$

- This is the a $(1-\alpha)$ 100% prediction interval of a future observation X

Prediction Intervals, σ Unknown

Now we assume that a random **sample comes from a normal population** with unknown mean μ and unknown variance σ^2.

Then a $(1-\alpha)\,100\%$ prediction interval of a future observation, say x_0 , can be calculated from:

$$\bar{x} - t_{a/2}\, s\sqrt{1+\frac{1}{n}} < x_0 < \bar{x} + t_{a/2}\, s\sqrt{1+\frac{1}{n}}.$$

Tolerance Limits

- Recall $\mu = 170\ and\ \sigma = 10\ and\ z_{0.05} = 1.96$ we have:

$$P(170 - (1.96)(10) < X < 170 + (1.96)(10))$$
$$= P(150.4 < X < 189.6) = 0.90$$

- What does this mean in terms of entire population?
  - I am 100% sure that 90% of the population is in [150.4,189.6]
- What happens if we don't know $\mu\ and\ \sigma$?
  - Construct an interval so that I am 95% sure that 90% of the population is inside.

Tolerance Limits

- When the engineering specifications for the output of a process is tight, the manager will be concerned about the long-range performance rather than the next observation only.

- If we can determine bounds that *cover a large proportion of the population* values with a high probability, that would be very useful to the manager.

Tolerance Limits

- For a normal distribution of measurements with unknown mean μ and unknown variance σ^2 tolerance limits are given by: $\bar{x} \pm ks$
- Here k is determined such that we can claim with $\underline{(1-\gamma)\ 100\%}$ confidence that the given limits contain at least $(1-\alpha)$ of the measurements.

- Table A.7 gives values of k for
- $1-\alpha = 0.90, 0.95, 0.99$ and $1-\gamma = 0.95, 0.99$,
- for selected values of n from 2 to 300.

Table A.7 Tolerance Factors for Normal Distributions

	Two-Sided Intervals						One-Sided Intervals					
	$\gamma = 0.05$			$\gamma = 0.01$			$\gamma = 0.05$			$\gamma = 0.01$		
	$1-\alpha$			$1-\alpha$			$1-\alpha$			$1-\alpha$		
n	**0.90**	**0.95**	**0.99**	**0.90**	**0.95**	**0.99**	**0.90**	**0.95**	**0.99**	**0.90**	**0.95**	**0.99**
2	32.019	37.674	48.430	160.193	188.491	242.300	20.581	26.260	37.094	103.029	131.426	185.617
3	8.380	9.916	12.861	18.930	22.401	29.055	6.156	7.656	10.553	13.995	17.170	23.896
4	5.369	6.370	8.299	9.398	11.150	14.527	4.162	5.144	7.042	7.380	9.083	12.387
5	4.275	5.079	6.634	6.612	7.855	10.260	3.407	4.203	5.741	5.362	6.578	8.939
6	3.712	4.414	5.775	5.337	6.345	8.301	3.006	3.708	5.062	4.411	5.406	7.335
7	3.369	4.007	5.248	4.613	5.488	7.187	2.756	3.400	4.642	3.859	4.728	6.412
8	3.136	3.732	4.891	4.147	4.936	6.468	2.582	3.187	4.354	3.497	4.285	5.812
9	2.967	3.532	4.631	3.822	4.550	5.966	2.454	3.031	4.143	3.241	3.972	5.389
10	2.839	3.379	4.433	3.582	4.265	5.594	2.355	2.911	3.981	3.048	3.738	5.074
11	2.737	3.259	4.277	3.397	4.045	5.308	2.275	2.815	3.852	2.898	3.556	4.829
12	2.655	3.162	4.150	3.250	3.870	5.079	2.210	2.736	3.747	2.777	3.410	4.633
13	2.587	3.081	4.044	3.130	3.727	4.893	2.155	2.671	3.659	2.677	3.290	4.472
14	2.529	3.012	3.955	3.029	3.608	4.737	2.109	2.615	3.585	2.593	1.189	4.337
15	2.480	2.954	3.878	2.945	3.507	4.605	2.068	2.566	3.520	2.522	3.102	4.222

Example

- **Machine Quality**: A machine produces metal pieces that are cylindrical in shape.
- A sample of these pieces is taken and the diameters are found to be 1.01, 0.97, 1.03, 1.04, 0.99, 0.98, 0.99, 1.01, and 1.03 centimeters.
- Use these data to calculate three interval types and draw interpretations that illustrate the distinction between them in the context of the system. For all computations, assume an approximately normal distribution.
- The sample mean and standard deviation for the given data are $\bar{x}_{obs}$= 1.0056 and s = 0.0246.
- (a) Find a 99% confidence interval on the mean diameter.
- (b) Compute a 99% prediction interval on a measured diameter of a single metal piece taken from the machine.
- (c) Find the 99% tolerance limits that will contain 95% of the metal pieces produced by this machine.

Example

(a) The 99% confidence interval for the mean diameter is given by

$$\bar{x} \pm t_{0.005} s/\sqrt{n} = 1.0056 \pm (3.355)(0.0246/3) = 1.0056 \pm 0.0275.$$

(b) The 99% prediction interval for a future observation is given by

$$\bar{x} \pm t_{0.005} s\sqrt{1 + 1/n} = 1.0056 \pm (3.355)(0.0246)\sqrt{1 + 1/9},$$

with the bounds being 0.9186 and 1.0926.

Example

From Table A.7, for $n = 9$, $1 - \gamma = 0.99$, and $1 - \alpha = 0.95$, we find $k = 4.550$ for two-sided limits. Hence, the 99% tolerance limits are given by

$$\bar{x} + ks = 1.0056 \pm (4.550)(0.0246),$$

with the bounds being 0.8937 and 1.1175. We are 99% confident that the tolerance interval from 0.8937 to 1.1175 will contain the central 95% of the distribution of diameters produced.

When to Use CI's, PI's and TI's

- The computations for the three types of intervals we studied are rather straightforward, but the interpretation can be confusing.
- For a population parameter (like mean μ) we use a confidence interval. The purpose is to estimate the value of μ for a given confidence level.
- A tolerance interval is needed if we want to find where a large proportion of population values will fall.
- To predict the value of a single population member (observation) we calculate a prediction interval.