Homogenous Equations

We use the used homogeneous for two cases in 219

1- $y''+2y'+y=f(x)$ on $\underbrace{\text{ODE}}_{\text{ordinary PDE (not partial)}}$ if $f(x)\equiv 0$ then it is called homogeneous

2- <u>Definition:</u> A 1st order ODE is said to be homogeneous if it can be written as $\frac{dy}{dx}=h\left(\frac{y}{x}\right)$ for some function h

Note: $\frac{dy}{dx}=f(x,y)$ is homogeneous iff $f(x,y)=f(\lambda x,\lambda y)$ $\forall \lambda \in \mathbb{R}$

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ex: $y' = x+y$ check if it is homogeneous

do

$f(\lambda x, \lambda y) = \lambda x + \lambda y = \lambda(x+y) = \lambda f(x,y) \neq f(x,y)$ for most λ

So, it is not homogeneous

Linear: No mixed terms of y''', y'', y', y

$y''y + (y')^2 = x^2$ is not linear.

Ex: $y' = \underbrace{\frac{x+y}{x-y}}_{f(x,y)}$ Check if it is homogeneous

$$f(\lambda x, \lambda y) = \frac{\lambda x + \lambda y}{\lambda x - \lambda y} = \frac{\lambda(x+y)}{\lambda(x-y)} = \frac{x+y}{x-y} = f(x,y)$$

So, it is homogeneous

Ex: $y' = \underbrace{\frac{y^2+2xy}{x^2}}_{f(x,y)}$ Check if it is homogeneous

Solve it.

$$f(\lambda x, \lambda y) = \frac{\lambda^2 y^2 + 2\lambda x \cdot \lambda y}{\lambda^2 x^2} = \frac{\cancel{\lambda^2}(y^2+2xy)}{\cancel{\lambda^2} x^2} = \frac{y^2+2xy}{x^2} = f(x,y)$$

Therefore, it is homogeneous

Solve it by reducing it to a separable equations.

Substitute $u = \frac{y}{x}$ $(y = ux) \Rightarrow y' = u'x + u$, here $u' = \frac{du}{dx}$, $y' = \frac{dy}{dx}$

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$$y' = \frac{\frac{y^2}{x^2} + \frac{2xy}{x^2}}{\frac{x^2}{x^2}} \Rightarrow y' = \frac{y^2}{x^2} + \frac{2y}{x} \Rightarrow u'x + u = u^2 + 2u$$

$$\frac{du}{dx} \cdot x = u^2 + u \Rightarrow (u^2+u)^{-1} \cdot \frac{du}{dx} = \frac{1}{x} \xrightarrow[\text{wrt } x]{\text{Integrate}} \int \frac{1}{u(u+1)} \underbrace{\frac{du}{dx} dx}_{du} = \int \frac{dx}{x}$$

$$\int \frac{du}{u(u+1)} = \int \frac{dx}{x}$$

$$\underbrace{\int \frac{1}{u} du - \int \frac{1}{u+1} du}_{\text{Integration by partial fractions}} = \ln|x| + c \longrightarrow \ln|u| - \ln|u+1| = \ln|x| + C_0 = \ln k$$

$$\ln\left|\frac{u}{u+1}\right| = \ln|x| + \ln|c| \qquad \left|\frac{u}{u+1}\right| = |cx|$$

Assuming everything possible $\Rightarrow \frac{u}{u+1} = cx$ back substitution $u = \frac{y}{x}$

$$cx = \frac{\frac{y}{x}}{\frac{y}{x}+1} \qquad cx = \frac{y}{x} \cdot \frac{x}{x+y} \rightarrow cx = \frac{y}{x+y} \Rightarrow y = \frac{cx^2}{1-cx}$$

Chapter 2: Linear First Order ODE's (and the method of Integrating Factors)

$y' + p(x)y = q(x)$ if $p(x)$ and $q(x)$ are constant functions, then it is called constant coefficient 1st order ODE.

Ex: $y' - y = 0 \Rightarrow$ the general solution is $y = ce^x$

Ex: $y' + 2y = 3$ To solve it, multiplying by e^{2x}

$$e^{2x}y' + 2e^{2x}y = 3e^{2x} \Rightarrow \frac{d}{dx}\left(e^{2x}y\right) = 3e^{2x} \xrightarrow{\text{Integrate wrt } x} \int \frac{d}{dx}\left(e^{2x}y\right)dx = \int 3e^{2x}dx$$

product rule $= \frac{d}{dx}\left(e^{2x}y\right)$

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$$e^{2x}y = \frac{3}{2}e^{2x} + c \Rightarrow y = \frac{3}{2} + ce^{-2x}$$

(method of Int factors summary: We want to see LHS (left hand side) $y'+2y$ as a derivative like $\frac{d}{dx}(\mu(x)y)$)

Ex: $y' + \frac{y}{2} = x+2$ Solve it.

Multiply it by the factor $e^{\frac{x}{2}}$

$$\underbrace{e^{\frac{x}{2}}y' + e^{\frac{x}{2}}\frac{y}{2}}_{=\frac{d}{dx}(e^{\frac{x}{2}}y)} = e^{\frac{x}{2}}x + 2e^{\frac{x}{2}}$$

$$= \frac{d}{dx}(e^{\frac{x}{2}}y) \quad = xe^{\frac{x}{2}} + 2e^{\frac{x}{2}} \Rightarrow \int \frac{d}{dx}(e^{\frac{x}{2}}y)\,dx = \int \underbrace{(xe^{\frac{x}{2}} + 2e^{\frac{x}{2}})}_{=2\frac{d}{dx}(xe^{\frac{x}{2}})}dx$$

$$e^{\frac{x}{2}}y = 2\int \frac{d}{dx}(xe^{\frac{x}{2}})\,dx$$

$$e^{\frac{x}{2}}y = 2xe^{\frac{x}{2}} + c$$

$$y = 2x + ce^{-\frac{x}{2}}$$

This is the general solution

General case $\boxed{y' + p(x)y = q(x)}$

Multiply by the factor $\mu(x) = e^{\int p(x)dx}$

Ex: $xy' + 2y = 4x^2$ Solve it.

$$y' + \underbrace{\frac{2}{x}}_{p(x)}y = \underbrace{4x}_{q(x)} \qquad \mu(x) = e^{\int \frac{2}{x}dx} = x^2$$

Multiply by $\mu(x) = x^2$: $x^2y' + x^2\frac{2}{x}y = 4x^3$

$$\underbrace{x^2y' + 2xy}_{=\frac{d}{dx}(x^2y)} = 4x^3 \Rightarrow x^2y = x^4 + c$$

$$y = x^2 + cx^{-2}$$

General case is solved by the method of Integrating factors and the integrating factor is $\mu(x) = e^{\int p(x)dx}$

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Ex: $xy'+2y=4x^2$ $\quad \underbrace{y(1)=5}_{\text{initial value}}$ IVP (Initial Value Problem)

Previous example general solution is $y(x)=x^2+\frac{c}{x^2}$

$y(1)=5 \Rightarrow 5=1+c$

$c=4$

The solution to the IVP is $y(x)=x^2+\frac{4}{x^2}$

Ex: $y'+\cos x\, y=\sin x$ Solve it.

Multiply by $\mu(x)=e^{\int \cos x\, dx}=e^{\sin x}$

$$e^{\sin x}y'+e^{\sin x}\cos x\, y=\sin x\, e^{\sin x}$$

$$\int \frac{d}{dx}\left(e^{\sin x}y\right)dx=\int \sin x\, e^{\sin x}\, dx$$

$$y=\frac{1}{e^{\sin x}}\left(\int \sin x\, e^{\sin x}\, dx\right)$$

1st Order Equations
(Modeling)

ex: (Newton's Law of Cooling)

The temperature of an object changes at a rate proportional to the difference between its temperature and the temperature of its surrounding.

→ A hard boil egg is at 98°C. It is left in a sink of 18°C water. After 5 min. past, the degree of the egg is 38°C. How long does it take for the egg to cool to 20°C?

$T(t)$: temperature of egg

t: time

Let's write an ODE.

$$\frac{dT}{dt} \sim (T-T_0) \Rightarrow \frac{dT}{dt}=k.(T-T_0)$$

where k is a some constant and T_0 is a temperature of the surroundings.

In our example $T_0=18°C$.

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We must find k.

$$\frac{dT}{dt} = k.(T-18) \Rightarrow \underbrace{\frac{dT}{dt} - kT = -18k}_{1^{st} \text{ linear ODE}}$$

$$\mu(t) = e^{\int -k\,dt} = e^{-kt}$$

Multiply by e^{-kt}

$$e^{-kt}\frac{dT}{dt} - e^{-kt}.kT = -e^{-kt}.18k$$

$$\int \frac{d}{dt}\left(e^{-kt}T\right)dt = 18\int -ke^{-kt}\,dt$$

$$e^{-kt}.T = 18e^{-kt} + C$$

$T(t) = 18 + Ce^{kt}$ is the general solution

at $T(0) = 98$ $\quad 98 = 18 + C.1$

$$C = 80$$

$$T(t) = 18 + 80e^{kt}$$

at $T(5) = 38$

$$38 = 18 + 80e^{5k}$$

$$\frac{1}{4} = e^{5k} \quad k = \frac{1}{5}\ln\left(\frac{1}{4}\right) = -\frac{1}{5}\ln 4$$

$$T(t) = 18 + 80e^{-\frac{1}{5}\ln 4.t}$$

Question is $T(t_1) = 20$ $\quad t_1 = ?$

$$20 = 18 + 80e^{-\frac{1}{5}\ln 4.t_1}$$

$$\frac{1}{40} = e^{-\frac{1}{5}\ln 4.t_1} \quad \ln\frac{1}{40} = -\frac{1}{5}\ln 4.t_1$$

$$t_1 = 5\ln 10 = 11.5 \text{ min}$$

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Ex: (Salty water)

A water tank has volume of 100 Litres. At $t=0$, Q_0 is the amount of salt dissolved in it in kg. Water containing 0.25 kg of salt per litre is flowing into tank at a rate (derivative) of r L/min. Also water draining from the tank at the rate r L/min

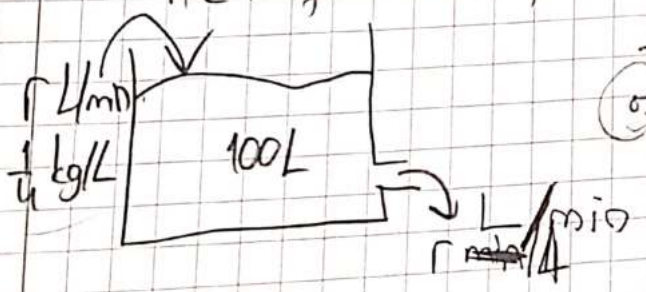
1- Set up an IVP to describe the flow process.

2- Solve the IVP to find the amount of $Q(t)$ of salt in the tank at time t.

3- Find the limited amount Q_L

4- If $r=3$ and $Q_0=2Q_L$, find the time T after which the salt amount is within 0/02 of Q_L

5- Find the flow rate R if T not to exceed 45 min. (0/02 25)



Solution (1) $Q(t)$ = amount of salt in the water in kg wrt t (time)

rate of change in the amount of salt = $\underbrace{\text{rate in}}_{\text{kg/min}} - \underbrace{\text{rate out}}_{\text{kg/min}}$

$$\frac{dQ}{dt} = r\cdot\frac{1}{4} - r\cdot\frac{Q}{100}$$

$$e^{\int \frac{r}{100}dt}$$

$Q(0)=Q_0$

This is the IVP (Q_0 constant, r constant)

(2) $Q' + \frac{r}{100}Q = \frac{r}{4}$ Integrating factor is $e^{\frac{rt}{100}}$

$Q(0)=Q_0$

$Q(t) = 25 + ce^{-rt/100}$ is the general solution

IV: $Q(0)=Q_0 \Rightarrow Q_0 = 25 + c$

$c = Q_0 - 25$ The solution is $Q(t) = 25 + (Q_0 - 25)\cdot e^{-rt/100}$

(3) The limiting amount is $\lim_{t\to\infty} Q(t) = 25 = Q_L$

(4) $r=3$ $Q_0 = 2Q_L = 50$

then $Q(t) = 25 + 25e^{-\frac{3t}{100}}$

Find T after which $Q(t)$ is within 0/02 neighborhood of 25 $(0/02\ 25 = 0.5)$
(for $t>T$)

$25 \cancel{-} \leq Q(t) \leq 25.5$ T is where $Q(T) = 25.5$

$Q(t) = 25 + 25e^{-\frac{3t}{100}}$

$25.5 = 25 + 25e^{-\frac{3T}{100}}$

$\frac{1}{50} = e^{-\frac{3T}{100}}$ $\frac{3T}{100} = \ln 50$

$T = \ln 50 \cdot \frac{100}{3}$

(5) $Q_0 = 50$ and $T = 45$ find R.

$25.5 = 25 + 25e^{-\frac{R\cdot 45}{100}}$ $R = \frac{20}{9}\ln 50 \approx 8.7 kg/min$

2.4

TODAY: Existence And Uniqueness Theorems
(for 1st order linear and nonlinear)

$\boxed{\text{Remind}}$ 1st order linear: $y' + p(t)y = q(t)$
$y = y(t)$

1st order nonlinear equation: $y' = \underbrace{f(t,y)}_{\text{nonlinear in } y}$

Definition: Let $y_1(t)$ and $y_2(t)$ be two functions of t.

A function $c_1y_1(t) + c_2y_2(t)$ (for $c_1, c_2 \in \mathbb{R}$) is called a linear combination of y_1 and y_2.

Two propositions (Principle of Superposition)

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homogeneous equation

Proposition 1 If y_1 and y_2 are solutions to $y' + p(t)y = 0$, then $c_1 y_1 + c_2 y_2$ is also a solution for $c_1, c_2 \in \mathbb{R}$.

Proposition 2: If y_1 and y_2 are solutions of $y' + p(t)y = q(t)$, then $y_1 - y_2$ is a solution to this $y' + p(t)y = 0$.

proof 2: y_1 and y_2 are solutions

$$y_1' + p(t)y_1 = q(t)$$
$$- \quad y_2' + p(t)y_2 = q(t)$$
$$(y_1 - y_2)' + p(t)(y_1 - y_2) = 0$$

ex: (These two propositions are good only for linear ODE's. Nonlinear case does not hold in general)

$y' - y^2 = 0$ Find general solution.

$$\frac{dy}{dt} = y^2 \underset{\text{separable}}{\Rightarrow} \frac{1}{y^2}\frac{dy}{dt} = 1 \quad \text{so}$$

$$\Rightarrow \int \frac{1}{y^2}\frac{dy}{dt}\,dt = \int 1\,dt \qquad \int \frac{dy}{y^2} = \int 1\,dt$$

$$-\frac{1}{y} = t + c$$

$$\Rightarrow y = \frac{-1}{t+c} \quad \text{is the general solution.}$$

$$\left.\begin{array}{l} c=0 \quad y_1 = \frac{-1}{t} \\ c=1 \quad y_2 = \frac{-1}{t+1} \end{array}\right\} \text{are two solutions}$$

Is $y_1 + y_2$ a solution?

$$y = y_1 + y_2 = \frac{-1}{t} + \frac{-1}{t+1} = -\frac{2t+1}{t(t+1)} = \frac{-2t-1}{t(t+1)}$$

$$y' = \frac{2t(t+1) - (2t+1)(2t+1)}{-t^2(t+1)^2}$$

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$y^2 = \frac{(2(t+1))^2}{t^2((t+1)^2)} \Rightarrow y' - y^2 = 0$, Therefore

$y_1 + y_2$ is not a solution.

Existence / Uniqueness Theorems

Theorem 1 (Liner case)

Consider the IVP

$$y' + p(t)y = q(t) \quad , y(t_0) = y_0$$

Assume p and q are continuous on an open interval (a,b) around t_0. Then this IVP has a unique solution $y(t)$ which is valid on (a,b)

proof is based on the method of integrating factors

$$\underbrace{e^{\int p(t)dt}}_{\mu(t)} \cdot y' + e^{\int p(t)dt} \cdot p(t)y = e^{\int p(t)dt} \cdot q(t)$$

The general solution $y(t) = \frac{\int \mu(t)\, q(t)\, dt}{\mu(t)}$

$y(t_0) = y_0 \Rightarrow y_0(t)$ becomes unique.

Theorem 2 (Nonlinear case)

Consider the IVP.

$$y' = f(t,y) \quad , y(t_0) = y_0$$

Assume $f(t,y)$, $\frac{\partial f}{\partial y}$ are both continuous on a rectangle $\underbrace{(a,b)}_{t} \times \underbrace{(c,}$

containing (t_0, y_0) , then this IVP has a unique solution $y(t)$ on a sub (a,b).

ex: ("$\frac{\partial f}{\partial y}$ is cont." not is necessary)

$y' = y^{1/3}$ $y(0) = 0$. Find three solutions to it.

$y_1(t) \equiv 0$ is a solution

$$y^{-1/3} y' = 1 \Rightarrow \int y^{-1/3} \frac{dy}{dt} dt = \int 1 dt$$

$$\Rightarrow \frac{y^{2/3}}{2/3} = t + c \qquad y(t) = \pm \left(\frac{2}{3}(t+c) \right)^{3/2} \text{ is a solution.}$$

$y(0) = 0 \Rightarrow c = 0$

$y_2(t) = -\left(\frac{2}{3} t\right)^{3/2}$ and $y_3(t) = \left(\frac{2}{3} t\right)^{3/2}$ are solutions to the IVP.

Why is there more than one solution to this IVP?

- A superficial answer is because we can not apply Theorem 2. Because $f(t,y) = y^{1/3}$ and $\frac{\partial f}{\partial y} = \frac{1}{3} \frac{1}{\sqrt[3]{y^2}}$ is not cont. at $(0,0) = (t_0, y_0)$

In general, in nonlinear case if have a unique solution ensured by Theorem 2, we cannot know how that solution goes.

ex: $y' = y^2$ $y(0) = \frac{1}{5}$ Find solution and the open interval on which it is valid

$f(t,y) = y^2$, $\frac{\partial f}{\partial y} = 2y$ are cont. on $\mathbb{R}^2$.

By the existence/uniquness Theorem, there is a unique solution $y(t)$ around $0 = t_0$

$y(t) = \frac{-1}{t+c}$ is the general solution

$y(0) = \frac{1}{5} \Rightarrow \frac{1}{5} = \frac{-1}{c}$ $c = -5$

The unique solution is $y(t) = \frac{-1}{t-5}$ around zero. The largest set is $(-\infty,$

$(-\infty, 5)$ $(5, \infty)$

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2.6 Exact Equations and Integrating Factors

ex: Solve $2x+y^2+2xyy'=0$ (Nonlinear 1st order ODE)

Another way to write it $(2x+y^2)dx+(2xy)dy=0$

Theorem (from Math 120)

$M(x,y)+N(x,y)y'=0$. Assume that $M, N, \underbrace{M_y}_{\frac{\partial M}{\partial y}}, \underbrace{N_x}_{\frac{\partial N}{\partial x}}$ are continuous.

There is a function $F(x,y)$ satisfying $F_x=M$ and $F_y=N$ if and only if $M_y=N_x$

<u>Definition</u>: If such an F exists then $M+Ny'=0$ is called <u>exact</u>.

ex: $2x+y^2+2xyy'=0$

Let $M=2x+y^2$ $\quad N=2xy$

$M_y=2y \quad N_x=2y$ then $M_y=2y=N_x$ so this ODE is EXACT.

We must find $F(x,y)$ s.t. $F_x=M$ and $F_y=N$

Integrate M wrt x: $F(x,y)=\int(2x+y^2)dx=x^2+xy^2+\underbrace{h(y)}_{\text{constant for } x}$
(as if y is a constant)

$F_y=N \Rightarrow 2xy+\underbrace{h'(y)}_{\text{derivative wrt } y}=\cancel{2xy} \Rightarrow h'(y)=0 \quad h(y)\to \text{constant} \in \mathbb{R}$

For our purposes, we may take this constant as zero.

$F(x,y)=x^2+xy^2$. What is this F good for?

Take a look at $\underbrace{M}_{F_x}+\underbrace{N}_{F_y}y'=0$. It is $F_x+F_y\frac{dy}{dx}=0$

Remember the chain rule $\quad F \to x, y \to x \qquad \underbrace{\frac{dF}{dx}}_{\left(\frac{dF}{dx}\right)}=F_x+F_y\frac{dy}{dx}=0$ which means

$\frac{dF}{dx}=0 \Rightarrow F(x,y)=c$, so $F(x,y)=c$ is the implicit solution.

$x^2+xy^2=c$ is the general solution which is given implicitly $(y=y(x))$.

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ex: $\underbrace{y\cos x + 2xe^y}_{M} + \underbrace{(\sin x + x^2e^y - 1)}_{N}y' = 0$ Solve it.

Check exactness

$$\left.\begin{aligned} M_y &= \cos x + 2xe^y \\ N_x &= \cos x + 2xe^y \end{aligned}\right\} M_y = N_x \Rightarrow \text{ODE is exact.}$$

Integrate M wrt x to find F: (y is like a constant)

$$F(x,y) = \int M\,dx = \int (y\cos x + 2xe^y)\,dx = y\sin x + x^2e^y + h(y)$$

$$F_y = N \Rightarrow \cancel{\sin x} + \cancel{x^2e^y} + h'(y) = \cancel{\sin x} + \cancel{x^2e^y} - 1$$

we take it to be zero

$$h'(y) = -1 \Rightarrow h(y) = -y + \cancel{C_0}$$

$$F(x,y) = y\sin x + x^2e^y - y$$

$\Rightarrow y\sin x + x^2e^y - y = c$ is the general solution

Ex: Solve $\underbrace{3xy + y^2}_{M} + \underbrace{(x^2 + xy)}_{N}y' = 0$

Check if it is exact

$$\left.\begin{aligned} M_y &= 3x + 2y \\ N_x &= 2x + y \end{aligned}\right\} M_y \neq N_x$$

so this ODE is NOT exact

We cannot solve it as in the previous examples.

$$F = \int M\,dx = \int (3xy + y^2)\,dx = \frac{3}{2}x^2y + xy^2 + h(y)$$

$$F_y = N \Rightarrow \frac{3}{2}x^2 + 2xy + h'(y) = x^2 + xy$$

$$h'(y) = \frac{-x^2}{2} - xy$$

$\Rightarrow h$ is a function of x and y at the same time

There should be no x here!

But there is a trick to make this $M + Ny' = 0$ an exact equation. Multiply it by $\mu(x)$ s.t. $\frac{d\mu}{dx} = \left(\frac{M_y - N_x}{N}\right)\mu$.

This will work if $\frac{M_y - N_x}{N}$ is depending only on x NOT on y.

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$\ln\mu = \ln x \int \frac{d\mu}{\mu dx} dx = \int \frac{1}{x} \frac{d\mu}{\mu dx} = \frac{1}{x}$ $\quad \frac{d\mu}{\mu} = \frac{dx}{x}$ $\ln|\mu| = \ln|x|$

$\int \frac{1}{\mu} \frac{d\mu}{dx} dx = \int \frac{dx}{x}$

In this example, $\frac{M_y - N_x}{N} = \frac{x+y}{x^2+xy} = \frac{1}{x}$

We solve $\frac{d\mu}{dx} = \frac{\mu}{x} \xrightarrow[\text{separate}]{} \int \frac{\mu'}{\mu} dx = \int \frac{dx}{x} \Rightarrow \mu(x) = x$

$\mu M + \mu N y' = 0 \Rightarrow \underbrace{3x^2y + xy^2}_{\mu M} + \underbrace{(x^3+x^2y)}_{\mu N} y' = 0$

Check Exactness / $(\mu M)_y = 3x^2 + 2xy = (\mu N)_x$

$F(x,y) = \int \mu M dx = \int 3x^2y + xy^2 dx = x^3y + \frac{x^2y^2}{2} + h(y)$

$\mu N = F_y = \cancel{x^3} + \cancel{x^2y} + h'(y) = \cancel{x^3} + \cancel{x^2y}$

$h'(y) = 0 \Rightarrow h(y) = 0$

$\Rightarrow F(x,y) = x^3y + \frac{x^2y^2}{2}$ and $x^3y + \frac{x^2y^2}{2} = c$ is the general solution.

Ex: Solve $\underbrace{\left(\frac{4x^3}{y^2} + \frac{3}{y}\right)}_{M} + \underbrace{\left(\frac{3x}{y^2} + 4y\right)}_{N} y' = 0$

$$\left.\begin{aligned} M_y &= \frac{-8x^3}{y^3} - \frac{3}{y^2} \\ N_x &= \frac{3}{y^2} \end{aligned}\right\} M_y \neq N_x \Rightarrow \text{NOT exact}$$

We check $\frac{M_y - N_x}{N} = \frac{\frac{-8x^3}{y^3} - \frac{6}{y^2}}{\frac{3x}{y^2} + 4y}$ is depending on x and y (Not good)

We check $\frac{N_x - M_y}{N} = \frac{\frac{8x^3}{y^3} + \frac{6}{y^2}}{\frac{4x^3}{y^2} + \frac{3}{y}} = \frac{2}{y}$ depending on y and not on x

Then we use $\mu(y)$ s.t $\frac{d\mu}{dy} = \left(\frac{N_x - M_y}{M}\right)\mu$

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Solve $\frac{d\mu}{dy} = \frac{2\mu}{y} \Rightarrow \mu(y) = y^2$ $\longrightarrow$ $\frac{d\mu}{dy} = \frac{2\mu}{y}$ separable 1st order equation

$\frac{d\mu}{dy}\frac{1}{\mu} = \frac{2}{y}$

Integ. wrt y $\int \frac{1}{\mu}\frac{d\mu}{dy}\,dy = \int \frac{2}{y}\,dy$

$= \int \frac{d\mu}{\mu} = \int \frac{2}{y}\,dy \Rightarrow$

$\ln|\mu| = 2\ln|y|$

$\mu = y^2$

Multiply $M + Ny' = 0$ by $\mu(y)$:

$\underbrace{4x^3 + 3y}_{\mu M} + \underbrace{(3x + 4y^3)}_{\mu N}y' = 0$

$(\mu M)_y = 3 = (\mu N)_x \Rightarrow$ exact

$F = \int 4x^3 + 3y\,dx = x^4 + 3xy + h(y)$

$F_y = 3x + h'(y) = 3x + 4y^3 \Rightarrow h(y) = y^4$

$F = x^4 + 3xy + y^4 \Rightarrow x^4 + y^4 + 3xy = c$ is the general solution

Ex: $\left(\frac{y}{x^2+y^2}\right) + \left(\frac{-x}{x^2+y^2}\right)y' = 0$ not exact and cannot be made exact.

(from 120) (exercise is $x^2+y^2 = c^2$ a solution?)

Systems of 1st order linear ODE's (Chapter 7)

$x_j = x_j(t)$

(1)
$$\begin{aligned}
x_1' &= f_1(t, x_1, x_2, \dots, x_n), \quad x_1(t_0) = x_1^0 \\
x_2' &= f_2(t, x_1, x_2, \dots, x_n), \quad x_2(t_0) = x_2^0 \\
&\vdots \\
x_n' &= f_n(t, x_1, x_2, \dots, x_n), \quad x_n(t_0) = x_n^0
\end{aligned}$$

not necessarily linear (an IVP)

(2) linear
$$\begin{cases}
x_1' = p_{11}(t)x_1 + \dots + p_{1n}(t)x_n + g_1(t) \\
x_2' = p_{21}(t)x_1 + \dots + p_{2n}(t)x_n + g_2(t) \\
\vdots \\
x_n' = p_{n1}(t)x_1 + \dots + p_{nn}(t)x_n + g_n(t)
\end{cases}$$

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$$\left.\begin{aligned} x_1' &= p_{11}x_1 + \dots + p_{1n}x_n \\ &\vdots \\ x_n' &= p_{n1}x_1 + \dots + p_{nn}x_n \end{aligned}\right\} \text{homogeneous linear}$$

Ex: $x_1' = x_1$, $x_2' = 2t$ $\Rightarrow$ $x_1 = c_1 e^t$, $x_2 = t^2 + c_2$

matrix notation

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} c_1 e^t \\ t^2 + c_2 \end{pmatrix}$$

Ex: $x_1' = x_2$, $x_2' = x_1$ $\quad \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = ?$

Later!

Existence / Uniqueness

Thm 1: Let $f_1, f_2, \dots, f_n$ and $\frac{\partial f_1}{\partial x_1}, \dots, \frac{\partial f_1}{\partial x_n}, \frac{\partial f_2}{\partial x_1}, \dots, \frac{\partial f_n}{\partial x_n}$ be continuous on a region $R \subset \mathbb{R}^{n+1}$ around $(t_0, x_1^0, \dots, x_n^0)$

Then, the system (1) has a unique solution (possibly on a smaller subset of R)

Thm 2: If the functions $p_{11}, p_{12}, \dots, p_{1n}, p_{21}, \dots, p_{nn}$ and $g_1, \dots, g_n$ are continuous near t_0, then the system (2) (the IVP) has a unique solution

Section 7.4

Basic Theory of Systems of First Order Linear Equations

$x_1 = x_1(t)$

$x_2 = x_2(t)$

$\vdots$

$x_n = x_n(t)$

$$\vec{x}(t) = x(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{pmatrix} \qquad x'(t) = \begin{pmatrix} x_1' \\ \vdots \\ x_n' \end{pmatrix}$$

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$$x_1' = P_{11}(t)x_1 + \dots + P_{1n}(t) + g_1(t)$$
$$\vdots$$
$$x_n' = P_{n1}(t)x_1 + \dots + P_{nn}(t)x_n + g_n(t)$$

We can write
$$\vec{x}'(t) = P(t)\vec{x}(t) + \vec{g}(t)$$
$$x' = Px + g$$

A homogeneous equation is $x' = Px$

Ex: $x' = \begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix} x$ $\quad x_1' = x_1 + x_2$, $\quad x_2' = 4x_1 + x_2$ $\quad \begin{pmatrix} x_1' \\ x_2' \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$

$X^{(1)} = \begin{pmatrix} e^{3t} \\ 2e^{3t} \end{pmatrix}$ is a solution (given)

$$\frac{d}{dt} X^{(1)} = \begin{pmatrix} 3e^{3t} \\ 6e^{3t} \end{pmatrix}$$

$$\begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix} X^{(1)} = \begin{pmatrix} e^{3t} + 2e^{3t} \\ 4e^{3t} + 2e^{3t} \end{pmatrix} = \begin{pmatrix} 3e^{3t} \\ 6e^{3t} \end{pmatrix} = \frac{dx^{(1)}}{dt} \Rightarrow x^{(1)} \text{ satisfies the system of equation}$$

$X^{(2)} = \begin{pmatrix} e^{-t} \\ -2e^{-t} \end{pmatrix}$ is another solution. A general solution is $x = c_1 x^{(1)} + c_2 x^{(2)}$

$\in \mathbb{R}$ or $\in \mathbb{C}$ (complex numbers)

In this example, there are only two linearly independent solutions $x^{(1)}$ and $x^{(2)}$ which we call fundamental solutions. General solution is a linear combination of the fundamental solutions.

Theorem: If $x^{(1)}$ and $x^{(2)}$ are solutions of $x' = Px$, then $c_1 x^{(1)} + c_2 x^{(2)}$ is also a solution.

$$X(t) = \begin{pmatrix} x^{(1)} & x^{(2)} & \dots & x^{(n)} \end{pmatrix}_{n \times n}, \quad x^{(j)} = \begin{pmatrix} x_{1j} \\ x_{2j} \\ \vdots \\ x_{nj} \end{pmatrix}$$

A matrix of functions

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Define Wronskian of $X(t)$ is $\det X(t) = w(t)$ determinant of $X(t)$ at point t.

Theorem A: If $w(t) = \det X(t)$ is zero for a point on an interval, then it is zero for all points on that interval.

Saying that enough to check $w(t)$ at a single point

Theorem B: $x^{(1)}, \dots, x^{(n)}$ are linearly independent fundamental solutions of $x' = Px$ on an interval iff $\det X(t) \neq 0$ on that interval

Ex: $x^{(1)} = \begin{pmatrix} e^{3t} \\ 2e^{3t} \end{pmatrix}$ and $x^{(2)} = \begin{pmatrix} e^{-t} \\ -2e^{-t} \end{pmatrix}$ are solutions to $x' = \begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix} x$

Are they linearly independent?

$$X(t) = \begin{pmatrix} x^{(1)} & x^{(2)} \end{pmatrix} = \begin{pmatrix} e^{3t} & e^{-t} \\ 2e^{3t} & -2e^{-t} \end{pmatrix}$$

$$w(t) = \det X(t) = -2e^{2t} - 2e^{2t} = -4e^{2t} \neq 0$$

This means that $\{x^{(1)}, x^{(2)}\}$ is a set of fundamental solutions.

Theorem: If $\phi(t)$ is a solution of $x' = Px$, then we can write $\phi(t) = c_1 x^{(1)} + \dots + c_n x^{(n)}$ where $c_1, \dots, c_n$ are unique and $x^{(1)}, \dots, x^{(n)}$ are fundamental solutions.

Question: How to find fundamental solutions?

7.5 Homogeneous Linear Systems with Constant Coefficients

Ex: $x' = \underbrace{\begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix}}_{\text{constant coefficient}} x + 0$

We consider solutions of the form $x = \begin{pmatrix} k_1 \\ k_2 \end{pmatrix} e^{rt}$, $r \in \mathbb{R}$ or $r \in \mathbb{C}$

$\vec{x} = \vec{k} e^{rt}$ $\quad k = \begin{pmatrix} k_1 \\ k_2 \end{pmatrix}$

Let $k e^{rt}$ be the solution of $x' = Px$ (P is a scalar matrix)

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$x' = r\vec{k}e^{rt}$

$r\vec{k}e^{rt} = Pke^{rt}$ that is $\begin{pmatrix} rk_1e^{rt} \\ rk_2e^{rt} \end{pmatrix} = P\begin{pmatrix} k_1e^{rt} \\ k_2e^{rt} \end{pmatrix}$

$\Rightarrow (P - rI)\underbrace{ke^{rt}}_{\text{not zero}} = 0 \quad \Rightarrow (P - rI)k = 0$ (I is identity)

an eigenvalue an eigenvector for r

Ex: $x' = \begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix} x$ find the fundamental solutions

solution: $A = \begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix}$ Find eigenvalues and eigenvectors of A.

$\det(A - rI) = 0 \Rightarrow \begin{vmatrix} 1-r & 1 \\ 4 & 1-r \end{vmatrix} = 0 \quad (1-r)^2 - 4 = 0$

$r^2 - 2r - 3 = 0$

$(r-3)(r+1) = 0$

$r = -1, 3 \quad r_1 = 3, \; r_2 = -1$ eigenvalues

e^{3t} and e^{-t} will be used

For $r_1 = 3$ $A - r_1 I = \begin{pmatrix} -2 & 1 \\ 4 & -2 \end{pmatrix} \longrightarrow \begin{pmatrix} -2 & 1 \\ 0 & 0 \end{pmatrix}$

An eigenvector k for $r_1 = 3$ is a solution vector of $(A - 3I)k = 0$

$\begin{pmatrix} -2 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} k_1 \\ k_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad -2k_1 + k_2 = 0$

choose k_2 as a free variable

All solutions to $(A - 3I)k = 0$ are $k = \begin{pmatrix} 1/2 \\ 1 \end{pmatrix} k_2$ for $k_2 \in \mathbb{R}$

I take $\begin{pmatrix} 1/2 \\ 1 \end{pmatrix}$ as the eigenvector corresponding to $r_1 = 3$ instead of $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$

$\begin{pmatrix} e^{3t} \\ 2e^{3t} \end{pmatrix}$ is a fundamental solution

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For $r_2=-1$ $A-r_2I=\begin{pmatrix}2&1\\4&2\end{pmatrix}\to\begin{pmatrix}2&1\\0&0\end{pmatrix}$ $2k_1+k_2=0$

Let $k_2=1 \Rightarrow \begin{pmatrix}1\\-2\end{pmatrix}$ is the eigenvector for $r_2=-1$

$\left(\begin{pmatrix}e^{-t}\\-2e^{-t}\end{pmatrix}\right.$ is a fundamental solution$\left.\right)$

So $x^{(1)}=\begin{pmatrix}e^{3t}\\2e^{3t}\end{pmatrix}$ and $x^{(2)}=\begin{pmatrix}e^{-t}\\-2e^{-t}\end{pmatrix}$ are fundamental solutions

We already checked that $W(x^{(1)},x^{(2)})\neq 0$. Wronskian is nonzero.

Ex $x'=\begin{pmatrix}0&1&1\\1&0&1\\1&1&0\end{pmatrix}x$ We found the eigenvalues and eigenvectors for P last time

$r_1=2$ $k_1=\begin{pmatrix}1\\1\\1\end{pmatrix}$ $r_2=-1$ $k_2=\begin{pmatrix}1\\0\\-1\end{pmatrix}$ $r_3=-1$ $k_3=\begin{pmatrix}0\\1\\-1\end{pmatrix}$

$x^{(1)}=\begin{pmatrix}e^{2t}\\e^{2t}\\e^{2t}\end{pmatrix}$ $x^{(2)}=\begin{pmatrix}e^{-t}\\0\\-e^{-t}\end{pmatrix}$ $x^{(3)}=\begin{pmatrix}0\\e^{-t}\\-e^{-t}\end{pmatrix}$

$$W=\begin{vmatrix}e^{2t}&e^{-t}&0\\e^{2t}&0&e^{-t}\\e^{2t}&-e^{-t}&-e^{-t}\end{vmatrix}=e^{2t}\cdot e^{3t}+2e^{-t}\cdot e^{-t}+0=e^{5t}+2\neq 0$$

$\Rightarrow \{x^{(1)},x^{(2)},x^{(3)}\}$ give a set of fundamental solutions and any solution can be written as $c_1x^{(1)}+c_2x^{(2)}+c_3x^{(3)}$

$\underbrace{\det(A-rI)=0}_{\text{polynomial}}$ There are three cases:

1- All roots are real and distinct.

2- There are complex roots if $a+ib$ is a root, then $a-ib$ is also root.

3- Eigenvalues are repeated.

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What that complex eigenvalues?

Section 7.6 Complex Eigenvalues

Ex: $x' = \begin{pmatrix} -1/2 & 1 \\ -1 & -1/2 \end{pmatrix} x$

Euler's formula $\quad e^{ib} = \cos b + i \sin b$

$(i^2 = -1) \quad e^{a+ib} = e^a(\cos b + i \sin b)$

Ex: $x' = \underbrace{\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}}_{A} x \qquad \left[x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \quad \begin{matrix} x_1' = x_2 \\ x_2' = -x_1 \end{matrix} \right]$

eigenvalues: $\det(A - rI) = 0 \Rightarrow \begin{vmatrix} -r & 1 \\ -1 & -r \end{vmatrix} = 0 \quad r^2 + 1 = 0 \Rightarrow \left. \begin{matrix} r_1 = i \\ r_2 = -i \end{matrix} \right\}$ eigenvalues

for $r_1 = i \quad (A - rI)k = 0 \quad A - rI = \begin{pmatrix} -i & 1 \\ -1 & -i \end{pmatrix} \xrightarrow{iR_2} \begin{pmatrix} -i & 1 \\ -i & 1 \end{pmatrix} \xrightarrow{R_2 - R_1} \begin{pmatrix} -i & 1 \\ 0 & 0 \end{pmatrix}$

$-ik_1 + k_2 = 0 \Rightarrow k_2 = ik_1$

Let k_1 be the free variable $\begin{pmatrix} 1 \\ i \end{pmatrix}$ is the eigenvector for i

$\begin{pmatrix} -i & 1 \\ -1 & -i \end{pmatrix} \begin{pmatrix} k_1 \\ k_2 \end{pmatrix} = 0$

$-k_1 i + k_2 = 0$

$k_1 i = k_2$

$\underbrace{\text{for } r_2 = -i}_{\text{no need for this one}} \Rightarrow \begin{pmatrix} 1 \\ -i \end{pmatrix}$ is the eigenvector

conjugate $\begin{pmatrix} 1 \\ i \end{pmatrix}$

Fundamental solutions are $x^{(1)} = \begin{pmatrix} 1 \\ i \end{pmatrix} e^{it}$ and $x^{(2)} = \begin{pmatrix} 1 \\ -i \end{pmatrix} e^{-it}$

complex valued solutions $= \begin{pmatrix} e^{it} \\ ie^{it} \end{pmatrix} \qquad = \begin{pmatrix} e^{-it} \\ -ie^{-it} \end{pmatrix}$

for real valued solutions

$e^{it} = \cos t + i \sin t \quad$ Take $x^{(1)} = \begin{pmatrix} e^{it} \\ ie^{it} \end{pmatrix} = \begin{pmatrix} \cos t + i \sin t \\ i \cos t - \sin t \end{pmatrix}$

$= \begin{pmatrix} \cos t \\ -\sin t \end{pmatrix} + i \cdot \underbrace{\begin{pmatrix} \sin t \\ \cos t \end{pmatrix}}_{\text{a number}}$

that is $u(t) = \begin{pmatrix} \cos t \\ -\sin t \end{pmatrix}$ and $v(t) = \begin{pmatrix} \sin t \\ \cos t \end{pmatrix}$ are real valued fundamental solutions.

(check Wronskian for lin. indep.)

Keep in mind that $x^{(1)}_{(t)} = u(t) + v(t) i$

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Ex: $x' = \underbrace{\begin{pmatrix} -1 & -1 \\ 2 & -1 \end{pmatrix}}_{=A} x$ $\quad x(0) = \begin{pmatrix} 2 \\ 2 \end{pmatrix} \rightarrow$ initial value

an IVT

$$\det(A - rI) = 0 \Rightarrow \begin{vmatrix} -1-r & -1 \\ 2 & -1-r \end{vmatrix} = 0 \Rightarrow (1+r)^2 + 2 = 0 \quad \begin{matrix} r_1 = -1+\sqrt{2}i \\ r_2 = -1-\sqrt{2}i \end{matrix}$$

For $r = -1+\sqrt{2}i$ we have system $\begin{pmatrix} -\sqrt{2}i & -1 \\ 2 & -\sqrt{2}i \end{pmatrix} \xrightarrow[-\sqrt{2}i R_1]{} \begin{pmatrix} -2 & +\sqrt{2}i \\ 2 & -\sqrt{2}i \end{pmatrix} \rightarrow \begin{pmatrix} 2 & -\sqrt{2}i \\ 0 & 0 \end{pmatrix}$

the eigenvector is $k = \begin{pmatrix} 1 \\ -\sqrt{2}i \end{pmatrix}$ or we can take $k = \begin{pmatrix} i \\ \sqrt{2} \end{pmatrix}$

$x_1 = c e^{rt} k$

for $r = -1-\sqrt{2}i$ $\bar{k} = \begin{pmatrix} -i \\ \sqrt{2} \end{pmatrix}$ is the eigenvector

$$x^{(1)} = \begin{pmatrix} i \\ \sqrt{2} \end{pmatrix} e^{(-1+\sqrt{2}i)t} = \begin{pmatrix} i e^{(-1+\sqrt{2}i)t} \\ \sqrt{2} e^{(-1+\sqrt{2}i)t} \end{pmatrix}$$

$e^{it} = \cos t + i \sin t$

Euler's formula $x^{(1)} = \begin{pmatrix} i e^{-1}(\cos\sqrt{2}t + i\sin\sqrt{2}t) \\ \sqrt{2} e^{-1}(\cos\sqrt{2}t + i\sin\sqrt{2}t) \end{pmatrix}$

$$= e^{-1} \underbrace{\begin{pmatrix} -\sin\sqrt{2}t \\ \sqrt{2}\cos\sqrt{2}t \end{pmatrix}}_{\operatorname{Re} x^{(1)}} + i e^{-1} \underbrace{\begin{pmatrix} \cos\sqrt{2}t \\ \sqrt{2}\sin\sqrt{2}t \end{pmatrix}}_{\operatorname{Im} x^{(1)}}$$

$u(t) = \begin{pmatrix} -\sin\sqrt{2}t \\ \sqrt{2}\cos\sqrt{2}t \end{pmatrix} e^{-t}$ $\quad v(t) = \begin{pmatrix} \cos\sqrt{2}t \\ \sqrt{2}\sin\sqrt{2}t \end{pmatrix} e^{-t}$ general solution is $x(t) = c_1 u + c_2 v$.

exercise: Check Wronskian for linear independence.

We have initial values $x(0) = \begin{pmatrix} 2 \\ 2 \end{pmatrix}$

$$\begin{pmatrix} 2 \\ 2 \end{pmatrix} = x(0) = c_1 u(0) + c_2 v(0)$$

$$= c_1 \begin{pmatrix} -\sin\sqrt{2}.0 \\ \sqrt{2}\cos\sqrt{2}.0 \end{pmatrix} e^{-1} + c_2 \begin{pmatrix} \cos\sqrt{2}.0 \\ \sqrt{2}\sin\sqrt{2}.0 \end{pmatrix} e^{-1}$$

$$\Rightarrow \begin{pmatrix} 0 \\ c_1\sqrt{2} \end{pmatrix} e^{-1} + c_2 \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{-1} = \begin{pmatrix} 2 \\ 2 \end{pmatrix}$$

$$= \begin{pmatrix} 0 + c_2 e^{-1} \\ c_1\sqrt{2} e^{-1} + 0 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \end{pmatrix} \quad \begin{matrix} c_2 = 2e \\ c_1 = \sqrt{2}e \end{matrix}$$

Keskin Color The unique solution to the IVT is $\sqrt{2}e.u + 2e.v$

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7.4 Fundamental Matrices (Ψ, Φ)

$x' = P(t)x$ Let $x^{(1)}, \dots, x^{(n)}$ be fundamental solutions

The matrix $\Psi(t) = (x^{(1)} \dots x^{(n)})$ is called a fundamental matrix

Ex: $x' = \begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix} x$ we know $x^{(1)} = \begin{pmatrix} e^{3t} \\ 2e^{3t} \end{pmatrix}$ $x^{(2)} = \begin{pmatrix} e^{-t} \\ -2e^{-t} \end{pmatrix}$

$\Psi(t) = (x^{(1)} x^{(2)}) = \begin{pmatrix} e^{3t} & e^{-t} \\ 2e^{3t} & -2e^{-t} \end{pmatrix}$ is a fundamental matrix

$x^{(1)}$ $x^{(1)}$

We can write $\Psi'(t) = P(t)\Psi(t)$

$y^{(1)} = \frac{1}{2} \cdot \begin{pmatrix} e^{3t} \\ 2e^{3t} \end{pmatrix} + \frac{1}{2} \begin{pmatrix} e^{-t} \\ -2e^{-t} \end{pmatrix}$

$c_1 = 1/2 = c_2$

$2|\ c_1 + c_2 = 1$

$2c_1 - 2c_2 = 0$

$c_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ -2 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

Ex: $x' = \begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix} x$ $x^{(1)}(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ $x^{(2)}(0) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ IVP

we can write $\Psi(0) = I$

$x^{(1)}(0) = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

$x^{(2)}(0) = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$

First initial value: $\begin{pmatrix} 1 \\ 0 \end{pmatrix} = c_1 x^{(1)}(0) + c_2 x^{(2)}(0)$

$\Rightarrow \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} c_1 + c_2 \\ 2c_1 - 2c_2 \end{pmatrix} \Rightarrow c_1 = \frac{1}{2} = c_2$

$y^{(1)} = \frac{1}{2} e^{3t} \begin{pmatrix} 1 \\ 2 \end{pmatrix} + \frac{1}{2} e^{-t} \begin{pmatrix} 1 \\ -2 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} e^{3t} + \frac{1}{2} e^{-t} \\ e^{3t} - e^{-t} \end{pmatrix}$

$c_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ -2 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$2|\ c_1 + c_2 = 0$

$2c_1 - 2c_2 = 1$

$c_1 = \frac{1}{4}$

$c_2 = -\frac{1}{4}$

Second initial value: $\begin{pmatrix} 0 \\ 1 \end{pmatrix} = c_1 x^{(1)}(0) + c_2 x^{(2)}(0)$

$\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} c_1 + c_2 \\ 2c_1 - 2c_2 \end{pmatrix} \Rightarrow c_1 = \frac{1}{4}$

$c_2 = -\frac{1}{4}$

$y^{(2)} = \frac{1}{4} e^{3t} \begin{pmatrix} 1 \\ 2 \end{pmatrix} - \frac{1}{4} e^{-t} \begin{pmatrix} 1 \\ -2 \end{pmatrix}$

$\frac{1}{4} \begin{pmatrix} e^{3t} \\ 2e^{3t} \end{pmatrix} \begin{pmatrix} e^{-t} \\ -2e^{-t} \end{pmatrix}$

$= \begin{pmatrix} \frac{1}{4} e^{3t} - \frac{1}{4} e^{-t} \\ \frac{1}{2} e^{3t} - \frac{1}{2} e^{-t} \end{pmatrix}$

A fundamental matrix to $x' = \begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix} x$ is $\Phi(t) = (y^{(1)} y^{(2)})$

$= \begin{pmatrix} \frac{1}{2} e^{3t} + \frac{1}{2} e^{-t} & \frac{1}{4} e^{3t} - \frac{1}{4} e^{-t} \\ e^{3t} - e^{-t} & \frac{1}{2} e^{3t} - \frac{1}{2} e^{-t} \end{pmatrix}$

Now we have two fundamental matrices Ψ and Φ and Φ is more complicated. However, for the special set of initial values $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ this matrix $\Phi(0) = I$, so it is

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the solution matrix of the IVP $X' = PX$ $X^{(1)}(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $X^{(2)}(0) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

If $X^{(1)}(t), \dots, X^{(n)}(t)$ are fundamental solutions of $X'(t) = P(t)X(t)$, then

a fundamental matrix is $(X^{(1)}(t), \dots, X^{(n)}(t))_{n \times n}$

Φ is a fundamental matrix such that $\Phi(t_0) = I$

Ex: $X' = \begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix} X$

A fundamental matrix is $\Psi = \begin{pmatrix} e^{3t} & e^{-t} \\ 2e^{3t} & -2e^{-t} \end{pmatrix}$ found by eigenvalues eigenvectors

Ex: $y' = \begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix} y$ $y^{(1)}(0) = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$ $\dots$ $y^{(n)}(0) = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}_{n \times 1}$

$$\Phi(t) = \begin{pmatrix} \frac{1}{2}e^{3t} + \frac{1}{2}e^{-t} & \frac{1}{4}e^{3t} - \frac{1}{4}e^{-t} \\ e^{3t} - e^{-t} & \frac{1}{2}e^{3t} + \frac{1}{2}e^{-t} \end{pmatrix}$$

How did we find it?

$y(t) = c_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{3t} + c_2 \begin{pmatrix} 1 \\ -2 \end{pmatrix} e^{-t}$

$\begin{pmatrix} 1 \\ 0 \end{pmatrix} = c_1 \dots \Rightarrow (c_1, c_2) = \left(\frac{1}{2}, \frac{1}{2}\right)$

$\begin{pmatrix} 0 \\ 1 \end{pmatrix} \Rightarrow (c_1, c_2) = \left(\frac{1}{4}, -\frac{1}{4}\right)$

What is special about Φ?

First: it makes easier in calculations for different initial values.

Second: Theoritical purposes. For this we need to learn diagonalization.

A square matrix A is diagonalizable if there is invertible matrix T and a diagonal matrix D s.t. $T^{-1}AT = D$

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eigenvalues

roots of $\det(A - rI) = 0$

Ex: $A = \begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix}$ $\quad T = \begin{pmatrix} 1 & 1 \\ 2 & -2 \end{pmatrix}$ $\quad D = \begin{pmatrix} 3 & 0 \\ 0 & -1 \end{pmatrix}$ $\quad \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ -2 \end{pmatrix}$ are corresponding eigenvectors.

$$T^{-1} = \frac{1}{-4}\begin{pmatrix} -2 & -1 \\ -2 & 1 \end{pmatrix} = \begin{pmatrix} 1/2 & 1/4 \\ 1/2 & -1/4 \end{pmatrix}$$

$$\begin{matrix} e^{3t} & 0 \\ 0 & e^{-t} \end{matrix}$$

$$T^{-1}AT = D$$

In our example $X' = \begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix} X$

$$T^{-1}\Phi T = \begin{pmatrix} e^{3t} & 0 \\ 0 & e^{-t} \end{pmatrix}$$

What is this $\begin{pmatrix} e^{3t} & 0 \\ 0 & e^{-t} \end{pmatrix}$? $\quad$ Answer $= \exp(At)$

$$\begin{pmatrix} e^{3t} & e^{-t} \\ 2e^{3t} & -2e^{-t} \end{pmatrix} = T \begin{pmatrix} e^{3t} & 0 \\ 0 & e^{-t} \end{pmatrix}$$

Exponential Matrices ($\exp(P)$)

Remember $a \in \mathbb{R}$

$$e^{at} = \exp(at) = \sum_{j=0}^{\infty} \frac{a^j t^j}{j!} = 1 + at + \frac{a^2t^2}{2} + \frac{a^3t^3}{3!} + \frac{a^4t^4}{4!} + \cdots$$

Similarly, for a matrix P, we have $\exp(P) = I + P + \frac{P^2}{2} + \frac{P^3}{6} + \cdots$

$$= \sum_{j=0}^{\infty} \frac{P^j}{j!}$$

A single ODE $y' = ay$ $\quad y(0) = y_0$ then $y(t) = y_0 e^{at}$ is the solution.

$\vec{X}' = A\vec{x}$ a system of homogeneous constant coefficient linear 1st order.

$\vec{X}(t) = \exp(At)\vec{X_0}$

In other words,

$$\exp(At) = \sum_{j=0}^{\infty} \frac{A^j t^j}{j!}$$ is the fundamental matrix Φ

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Fact: Let A be diagonalizable. Let $T^{-1}AT = D$. Then,

$$\exp(D) = \exp(T^{-1}AT)$$
$$= T^{-1}\exp(A)\,T$$

Consider the system $\vec{x}' = A\vec{x}$ where A is diagonalizable ($T^{-1}AT = D$)

Let $\vec{x} = T.\vec{y}$ $\Rightarrow \underbrace{Ty'}_{x'} = \underbrace{ATy}_{= Ax}$

Multiply by T^{-1}: $\underbrace{T^{-1}T}_{I}y' = \underbrace{T^{-1}AT}_{D}y$ $\quad x' = Ax$

$\Rightarrow y' = Dy$ a new system

If $r_1, r_2, \dots, r_n$ are eigenvalues of A, then,

$$y(t) = \begin{pmatrix} e^{r_1 t} & & 0 \\ & \ddots & \\ 0 & & e^{r_n t} \end{pmatrix} \text{ is a solution to } y' = Dy$$

So $T\begin{pmatrix} e^{r_1 t} & & 0 \\ & \ddots & \\ 0 & & e^{r_n t} \end{pmatrix}$ is a solution to $x' = Ax$ (A fundamental matrix)

And $\boxed{\Phi(t) = TyT^{-1}}$

Ex: $x' = \underbrace{\begin{pmatrix} 3 & -2 \\ 2 & -2 \end{pmatrix}}_{A} x$ Find the fundamental matrix Φ

solution: characteristic polynomial $= \det(A - rI) = \begin{vmatrix} 3-r & -2 \\ 2 & -2-r \end{vmatrix}$

$$= r^2 - r - 6 + 4 = r^2 - r - 2$$
$$= (r-2)(r+1) \quad \underbrace{r_1 = 2 \text{ and } r_2 = -1}_{\text{eigenvalues}}$$

eigenvectors are $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$ for $r = 2$

$\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ for $r = -1$

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$$T = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \quad T^{-1} = \frac{1}{3}\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$$

$$D = \begin{pmatrix} 2 & 0 \\ 0 & -1 \end{pmatrix} \qquad T^{-1}AT = D$$

$$\Phi(t) = T\begin{pmatrix} e^{2t} & 0 \\ 0 & e^{-t} \end{pmatrix}T^{-1}$$

$$= \frac{1}{3}\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} e^{2t} & 0 \\ 0 & e^{-t} \end{pmatrix}\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$$

$$= \begin{pmatrix} 2e^{2t} & e^{-t} \\ e^{2t} & 2e^{-t} \end{pmatrix}\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} = \frac{1}{3}\begin{pmatrix} 4e^{2t}-e^{-t} & -2e^{2t}+2e^{-t} \\ 2e^{2t}-2e^{-t} & -e^{2t}+4e^{-t} \end{pmatrix}$$

7.8 Repeated Eigenvalues

We see $\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$ with repeated eigenvalues r_1 and r_2 with multiplicities 1 and 2, respectively eigenvectors 1 and 2, resp.

So, A is diagonalizable and some methods applies.

But what if A were not diagonalizable?

Ex: $X' = \begin{pmatrix} 1 & -1 \\ 1 & 3 \end{pmatrix} X$ characteristic polynomial $= \begin{vmatrix} r-1 & -1 \\ 1 & 3-r \end{vmatrix}$

$\begin{vmatrix} 1-r & -1 \\ 1 & 3-r \end{vmatrix}$ $\quad 3-4r+r^2+1=0$

$r^2-4r+4=0$

$(r-2)^2=0 \quad r=2$ double root

$= r^2-4r+4 = (r-2)^2$

$\Rightarrow r=2$ is the eigenvalue with multiplicity 2.

Eigenvector(s): $\begin{pmatrix} -1 & -1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} k_1 \\ k_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$\begin{pmatrix} -1 & -1 \\ 1 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \quad k_1 = -k_2 \quad \begin{pmatrix} 1 \\ -1 \end{pmatrix}$ is the eigenvector for 2.

$$\begin{pmatrix} -1 & -1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} k_1 \\ k_2 \end{pmatrix} = 0 \qquad k\begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$-k_1-k_2=0$

$k_1=-k_2$

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For a second (generalized) eigenvector $m=\begin{pmatrix} m_1 \\ m_2 \end{pmatrix}$ We solve the system

$$\begin{pmatrix} -1 & -1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} m_1 \\ m_2 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \qquad (A-2I)m=k$$

$$\left(\begin{array}{cc|c} -1 & -1 & 1 \\ 1 & 1 & -1 \end{array}\right) \longrightarrow \left(\begin{array}{cc|c} 1 & 1 & -1 \\ 0 & 0 & 0 \end{array}\right)$$

$m_1+m_2=-1$

$-m_1-m_2=1$

$m_1+1=-m_2$

$k_1=\begin{pmatrix} 1 \\ -1 \end{pmatrix}$

$k_2=\begin{pmatrix} 0 \\ -1 \end{pmatrix}$

$m_1=-1-m_2$

Let $m_1=0$ and $\begin{pmatrix} 0 \\ -1 \end{pmatrix}$ is an eigenvector

Let $T=\begin{pmatrix} 1 & 0 \\ -1 & -1 \end{pmatrix}$ $T^{-1}=\begin{pmatrix} 1 & 0 \\ -1 & -1 \end{pmatrix}$

$\quad\quad\quad\ \ k \quad m$

$x^{(1)}=e^{2t}\begin{pmatrix} 1 \\ -1 \end{pmatrix}=\begin{pmatrix} e^{2t} \\ -e^{2t} \end{pmatrix}$

$x^{(2)}=t\cdot e^{2t}\begin{pmatrix} 1 \\ -1 \end{pmatrix}+e^{2t}\begin{pmatrix} 0 \\ -1 \end{pmatrix}$

$$T^{-1}AT=\begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}=J$$

This the Jordan form the nondiagonalizable matrix A.

A new system is $y'=Jy$ which is same as $y_1'=2y_1+y_2$

$y_2'=2y_2$

$y_2=c_2e^{2t}$

$y_1'=2y_1+c_2e^{2t}$

The trick is $y_1(t)=c_1te^t$ and

$\exp(At)=\begin{pmatrix} e^{2t} & te^{2t} \\ 0 & e^{2t} \end{pmatrix}$ and $X=T\exp(At)$ is the general solution.

Last Time Repeated Eigenvalues, Jordan Forms

Ex: $x'=\begin{pmatrix} 1 & 9 \\ -1 & -5 \end{pmatrix}x$ Find the general solution.

solution: characteristic polynomial: $\det(A-rI)=\begin{vmatrix} 1-r & 9 \\ -1 & -5-r \end{vmatrix}=r^2+4r+4$

$=(r+2)^2 \Rightarrow r=2$ is only eigenvalue with multiplicity 2

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$k_1 + 3k_2 = 0$

Eigenvector for -2: $\begin{pmatrix} 3 & 9 \\ -1 & -3 \end{pmatrix}\begin{pmatrix} k_1 \\ k_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$: $\begin{pmatrix} 3 & 9 \\ -1 & -3 \end{pmatrix} \to \begin{pmatrix} 1 & 3 \\ 0 & 0 \end{pmatrix}$

So, $\begin{pmatrix} k_1 \\ k_2 \end{pmatrix} = \begin{pmatrix} 3 \\ -1 \end{pmatrix} \Rightarrow x^{(1)} = \begin{pmatrix} 3 \\ -1 \end{pmatrix} e^{-2t}$ is a solution.

If A is not diagonalizable, solutions of $x' = Ax$ are in the form $x^{(1)} = \vec{k} e^{-2t}$ and $x^{(0)} = \vec{k} t e^{-2t} + \vec{m} e^{-2t}$ (-2 is the eigenvalue) such that $(A - rI)k = 0$ (A is a 2×2 matrix) and $(A - rI)m = k$

We already have $k = \begin{pmatrix} 3 \\ -1 \end{pmatrix}$ for $r = -2$ $x^{(1)} = \begin{pmatrix} 3 \\ -1 \end{pmatrix} e^{-2t}$

For $x^{(2)}(t)$ we need m. We find m by solving $\begin{pmatrix} 3 & 9 \\ -1 & -3 \end{pmatrix}\begin{pmatrix} m_1 \\ m_2 \end{pmatrix} = \begin{pmatrix} 3 \\ -1 \end{pmatrix}$

$\left(\begin{array}{cc|c} 3 & 9 & 3 \\ -1 & -3 & -1 \end{array}\right) \to \left(\begin{array}{cc|c} 1 & 3 & 1 \\ 0 & 0 & 0 \end{array}\right) \Rightarrow m_1 + 3m_2 = 1$, $m_1 = 1 - 3m_2$, if $m_2 = 0$, $m_1 = 1$ / $3m_1 + 9m_2 = 3$, $3/ -m_1 - 3m_2 = -1$

$m = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ is OK.

$\Rightarrow x^{(2)}(t) = \begin{pmatrix} 3 \\ -1 \end{pmatrix} t e^{-2t} + \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{-2t}$ $\quad W(x^{(1)} x^{(2)}) \neq 0$

$x^{(1)}$ and $x^{(2)}$ are linearly independent

The general solution is $x(t) = c_1 x^{(1)} + c_2 x^{(2)}$

$= c_1 \begin{pmatrix} 3 \\ 1 \end{pmatrix} e^{-2t} + c_2 \left[\begin{pmatrix} 3 \\ 1 \end{pmatrix} t e^{-2t} + \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{-2t} \right]$

A fundamental matrix is $\Psi(t) = \left(x^{(1)} \; x^{(2)} \right) = \begin{pmatrix} 3e^{-2t} & 3te^{-2t} + e^{-2t} \\ -e^{-2t} & -te^{-2t} \end{pmatrix}$

Alternatively, we use JORDAN FORM of A.

A is not diagonalizable in this example, so no T and no D (diagonal) s.t. $TAT^{-1} = D$

But if we choose $T = \begin{pmatrix} 3 & 1 \\ -1 & 0 \end{pmatrix}$ and $J = \begin{pmatrix} -2 & 1 \\ 0 & -2 \end{pmatrix}$ then, we have

$T^{-1} A T = J$ $\left(T^{-1} = \begin{pmatrix} 0 & -1 \\ 1 & 3 \end{pmatrix} \right)$

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This J is the Jordan form of A.

$T = \begin{pmatrix} 3 & 1 \\ -1 & 0 \end{pmatrix}$ $\quad J = \begin{pmatrix} -2 & 1 \\ 0 & -2 \end{pmatrix}$ → upper triangular

(generalized) eigenvectors $\quad$ eigenvalues

diagonalizable $\quad$ almost diagonal $\begin{pmatrix} \cdot & 1 \\ 0 & \cdot \end{pmatrix}$ 3x3

diagonalization $\begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}$ eigenvalues

If the system is $x' = Ax$ then, we can write another system $\underbrace{y' = Jy}_{\text{easier to solve}}$ where $x = Ty$

Solution to $y' = Jy$ is $\begin{pmatrix} e^{-2t} & te^{-2t} \\ 0 & e^{-2t} \end{pmatrix} = \exp(Jt)$

$y(1) \quad y(2)$

$$x = Ty \Rightarrow \underset{\text{A fundamental matrix}}{\Psi} = T\exp(Jt) = \begin{pmatrix} 3 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} e^{-2t} & te^{-2t} \\ 0 & e^{-2t} \end{pmatrix}$$

$$= \begin{pmatrix} 3e^{-2t} & 3te^{-2t} + e^{-2t} \\ -e^{-2t} & -te^{-2t} \end{pmatrix}$$

Ex: $x' = \begin{pmatrix} 3 & -4 \\ 1 & -1 \end{pmatrix} x$ solve it.

<u>solution:</u> $\begin{vmatrix} 3-r & -4 \\ 1 & -1-r \end{vmatrix} = 0 \Rightarrow r = 1$ is the eigenvalue (multiplicity 2)

$\begin{pmatrix} 2 & -4 \\ 1 & -2 \end{pmatrix} \to \begin{pmatrix} 1 & -2 \\ 0 & 0 \end{pmatrix} \Rightarrow k = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$

$\exp(Jt) = \begin{pmatrix} e^{rt} & te^{rt} \\ 0 & e^{rt} \end{pmatrix}$

$\left(\begin{array}{cc|c} 2 & -4 & 2 \\ 1 & -2 & 1 \end{array}\right) \to \left(\begin{array}{cc|c} 1 & -2 & 1 \\ 0 & 0 & 0 \end{array}\right) \Rightarrow m = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

k

$T = \begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix}$, $J = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $T^{-1}AT = J$

A fundamental matrix is $\Psi = T\begin{pmatrix} e^{t} & te^{t} \\ 0 & e^{t} \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} e^{t} & te^{t} \\ 0 & e^{t} \end{pmatrix} = \begin{pmatrix} 2e^{t} & 2te^{t} + e^{t} \\ e^{t} & te^{t} \end{pmatrix}$

$y(1) \quad y(2)$

$x^{(1)} = Ty^{(1)} \quad x^{(2)} = Ty^{(2)}$

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To general solution $x(t) = c_1\begin{pmatrix}2\\1\end{pmatrix}e^t + c_2\left[\begin{pmatrix}2\\1\end{pmatrix}te^t + \begin{pmatrix}1\\0\end{pmatrix}e^t\right]$

exercise: Solve the IVP $x' = \begin{pmatrix}3 & -4\\1 & -1\end{pmatrix}x$ and $x(0) = \begin{pmatrix}1\\1\end{pmatrix}$

In general, A Jordan block is $\begin{pmatrix} r & & 0\\ & \ddots & \\ 0 & & r\end{pmatrix}$

ex: $\begin{pmatrix} r & 1 & 0\\ 0 & r & 1\\ 0 & 0 & r\end{pmatrix}$ ex: $\begin{pmatrix} r & 1 & 0 & 0\\ 0 & r & 1 & 0\\ 0 & 0 & r & 1\\ 0 & 0 & 0 & r\end{pmatrix}$

A jordan form is

$$\begin{pmatrix} \boxed{\begin{matrix} r_1 & 1\\ 0 & r_1\end{matrix}} & & & 0\\ & \boxed{\begin{matrix} r_2 & 1 & 0\\ 0 & r_2 & 1\\ 0 & 0 & r_2\end{matrix}} & & \\ & & \boxed{\begin{matrix} r_3 & 1\\ 0 & r_3\end{matrix}} & \\ 0 & & & \ddots\end{pmatrix}$$

If A is not diagonalizable.

If A is diagonalizable then Jordan form is a diagonal matrix $\begin{pmatrix} r & & 0\\ & r & \\ 0 & & r\end{pmatrix}$

Ex: $A = \begin{pmatrix} 5 & 4 & 2 & 1\\ 0 & 1 & -1 & -1\\ -1 & -1 & 3 & 0\\ 1 & 1 & -1 & 2\end{pmatrix}$ not diagonalizable

eigenvalues are $r = 1, 2, 4, 4$

$$J = \begin{pmatrix} 1 & 0 & 0 & 0\\ 0 & 2 & 0 & 0\\ 0 & 0 & 4 & 1\\ 0 & 0 & 0 & 4\end{pmatrix}$$

$$\exp\begin{pmatrix} r & 1 & & 0\\ & r & 1 & \\ & & \ddots & 1\\ 0 & & & r\end{pmatrix}_{n\times n} = e^{rt}\begin{pmatrix} 1 & t & \frac{t^2}{2} & \frac{t^3}{3!} & \cdots & \frac{t^{n-1}}{(n-1)!}\\ 0 & 1 & t & \frac{t^2}{2} & \cdots & \frac{t^{n-2}}{(n-2)!}\\ & & 1 & & & \\ & & & \ddots & & \\ & 0 & & & 1 & t\\ & & & & & 1\end{pmatrix}$$

remember $\exp(A) = I + A + \frac{A^2}{2} + \frac{A^3}{3!} + \cdots$

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$x' = Ax \to y' = Jy$ where $x = Ty$

solution to $y' = Jy$ is $\exp(Jt)$ and $x = T\exp(Jt)$

Ex: $A = \begin{pmatrix} 1 & 9 \\ -1 & -5 \end{pmatrix}$ $r = -2$ $J = \begin{pmatrix} -2 & 1 \\ 0 & -2 \end{pmatrix}$ $\exp(Jt) = \begin{pmatrix} e^{-2t} & te^{-2t} \\ 0 & e^{-2t} \end{pmatrix}$

Ex: $A = \begin{pmatrix} 5 & 4 & 2 & 1 \\ 0 & 1 & -1 & -1 \\ -1 & -1 & 3 & 0 \\ 1 & 1 & -1 & 2 \end{pmatrix}_{4\times4}$ $J = \begin{pmatrix} 1 & & & \\ & 2 & & \\ & & 4 & 1 \\ & & & 4 \end{pmatrix}$ $\exp(Jt) = \left(\begin{array}{c|c} \begin{matrix} e^{t} & 0 \\ 0 & e^{2t} \end{matrix} & 0 \\ \hline 0 & \begin{matrix} e^{4t} & te^{4t} \\ 0 & e^{4t} \end{matrix} \end{array}\right)$

$y' = Jy$ is $\exp(Jt)$ a fundamental matrix

Ex: If A is 3×3 matrix where only eigenvalue is r with multiplicity 3 then,

(nondiagonalizable with a single eigenvector) $J = \begin{pmatrix} r & 1 & 0 \\ 0 & r & 1 \\ 0 & 0 & r \end{pmatrix}$ $\exp(Jt) = \begin{pmatrix} e^{rt} & te^{rt} & \frac{t^2e^{rt}}{2} \\ 0 & e^{rt} & te^{rt} \\ 0 & 0 & e^{rt} \end{pmatrix}$

solutions of $y' = Jy$ will be of the form $e^{rt}, te^{rt}, t^2e^{rt}$

$\exp(Jt)$ is calculated as $Jt = \begin{pmatrix} rt & t & 0 \\ 0 & rt & t \\ 0 & 0 & rt \end{pmatrix}$

$Jt^2 = (\ldots)$ $Jt^3 = (\ldots)$

$$I + Jt + \frac{(Jt)^2}{2} + \frac{(Jt)^3}{3!} + \frac{(Jt)^4}{4!} + \ldots = \exp(Jt)$$

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Burayı kaçır

7.9 Nonhomogeneous Linear Systems (with variation of parameter!)

$$\vec{x}'(t) = P(t)\vec{x}(t) + \underbrace{g(t)}_{\text{nonhomogeneous part}}$$

$$x' = Px + g$$

A solution of the corresponding homogeneous system $x' = Px$ is called the fundamental (complementary solution) solution of $x' = Px + g$

General solution of $x' = Px + g$ is $X_g = X_h + X_p$ where X_h is the fundamental solution and X_p is the particular solution.

(the general solution of $x' = Px$)

Ex: $x' = \underbrace{\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}}_{A} x + \underbrace{\begin{pmatrix} 2e^{4t} \\ e^{4t} \end{pmatrix}}_{g(t)}$ Find the general solution $X_g = X_h + X_p$

Solution: Firstly, find X_h: general solution to $x' = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} x$

Secondly, find X_p by variation of parameters method.

① $\begin{vmatrix} 1-r & 2 \\ 2 & 1-r \end{vmatrix} = (r-1)^2 - 4 = r^2 - 2r - 3 = (r-3)(r+1) = 0 \Rightarrow \begin{matrix} r_1 = 3 \\ r_2 = -1 \end{matrix}$ } eigenvalues

$\det(A - rI) = 0$

for $r_1 = 3$: $\begin{pmatrix} -2 & 2 \\ 2 & -2 \end{pmatrix} \to \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix}$ eigenvector is $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$

for $r_2 = -1$: $\begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix} \to \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$ " is $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$

Fundamental solution is $c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{3t} + c_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-t}$

② Variation of parameters: Assume $\Psi(t)u(t)$ is a particular solution with $u(t) = \begin{pmatrix} u_1(t) \\ u_2(t) \end{pmatrix}$

and $\Psi(t) = \begin{pmatrix} e^{3t} & e^{-t} \\ e^{3t} & -e^{-t} \end{pmatrix}$

We try to find $u(t)$

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Ex: $X' = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} X + \begin{pmatrix} 2e^{4t} \\ e^{4t} \end{pmatrix}$ Find the general solution. (X_g)

Firstly, find X_h.

$X_h : \Psi(t) = \begin{pmatrix} e^{3t} & e^{-t} \\ e^{3t} & -e^{-t} \end{pmatrix}$

$\left[\begin{array}{cc} \text{eigenvalues are} & \text{, eigenvectors} \\ 3, -1 & \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \end{pmatrix} \end{array} \right]$

$\left[X_h = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{3t} + c_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-t} \right]$

general solution of $X' = AX$

Second, find X_p by "Variation of parameters"

We assume $X = \Psi(t) u(t)$ where $u = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$, Then

$X' = Ax + g \Rightarrow \Psi u' = g$

$x = \Psi u \Rightarrow x' = \Psi' u + \Psi u'$

$x' = Ax + g \Rightarrow \Psi' u + \Psi u' = A\Psi u + g \Big\} \Psi u' = g$

$\Psi u' = g \Rightarrow \begin{pmatrix} e^{3t} & e^{-t} \\ e^{3t} & -e^{-t} \end{pmatrix} \begin{pmatrix} u_1' \\ u_2' \end{pmatrix} = \begin{pmatrix} 2e^{4t} \\ e^{4t} \end{pmatrix}$

$\Rightarrow e^{3t} u_1' + e^{-t} u_2' = 2e^{4t}$

$+ \quad e^{3t} u_1' - e^{-t} u_2' = e^{4t}$

$2e^{3t} u_1' = 3e^{4t} \Rightarrow u_1' = \frac{3}{2} e^{t}$

$\frac{3}{2} e^{t} \cdot e^{3t} - e^{-t} u_2' = e^{4t}$

$\frac{\frac{3}{2} e^{t} \cdot e^{3t} - e^{4t}}{e^{-t}} = u_2'$

$+\frac{1}{2} e^{5t}$

e^{-t}

$\Rightarrow u_1 = \frac{3}{2} e^{t} + c_1 \qquad u_2' = +\frac{1}{2} e^{5t}$

$u_2 = \frac{1}{10} e^{5t} + c_2$

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$$X = \Psi u = \begin{pmatrix} e^{3t} & e^{-t} \\ e^{3t} & -e^{-t} \end{pmatrix} \begin{pmatrix} \frac{3}{2}e^{4t} + c_1 \\ \frac{1}{10}e^{5t} + c_2 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{3}{2}e^{4t} + c_1 e^{3t} + \frac{1}{10}e^{4t} + c_2 e^{-t} \\ \frac{3}{2}e^{4t} + c_1 e^{3t} - \frac{1}{10}e^{4t} - c_2 e^{-t} \end{pmatrix}$$

$$= \underbrace{\underbrace{c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{3t} + c_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-t}}_{X_h} + \underbrace{\begin{pmatrix} 16/10 \\ 14/10 \end{pmatrix} e^{4t}}_{X_p}}_{X_g}$$

A remark about repeated eigenvalues

If $A_{2\times 2}$ is a matrix with a single eigenvalue (k) and a single eigenvector (m) (means not diagonalized) remember that a second eigenvector is produced by solving $\boxed{(A-rI)m = k}$. But be warned that we cannot take any multiple of m as the second eigenvector, because it depends on the first eigenvector k.

If the A is a matrix with complex eigenvalues or repeated eigenvalues then the only difference for a nonhomogeneous system is finding X_h. After that, everything is same.

Ex: $y''' - 3y'' + 3y' - y = g(t)$

nonhomogeneous part

e^t
te^{2t}
$at+b$
$\cos t + 2\sin t$

We can use the above method.

But if $g(t)$ is like $\tan x$ or $\sec x$ then this method doesnot apply. We use Variation of Parameters (summarized in 3.7 and 4.4)
(We rely on systems for this method)

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1- Solve the following equations.

a) $2x^2y.y' + y^2 = 2$

b) $xy' = y + xe^{y/x}$

c) $(2y+1)dx + \frac{(x^2-y)}{x} dy = 0$

d) $ty' + (t+1)y = 2te^{-t}$

a) $2x^2yy' + y^2 = 2$

$2x^2yy' = 2 - y^2$

$y' = \frac{2-y^2}{2yx^2}$ (separable eqn.)

$\int \frac{2y\,dy}{2-y^2} = \int \frac{dx}{x^2}$ $\quad y^2 - 2 = 0$

$-\ln|2-y^2| = -\frac{1}{x} + c$

$\ln|2-y^2| = \frac{1}{x} + c$ $\quad e^{\frac{1}{x}+c}$

$2 - y^2 = c.e^{1/x}$

$y^2 = 2 - ce^{1/x}$

Also, if $y^2 - 2 = 0 \Rightarrow y = \pm\sqrt{2}$

Substitute $y = \sqrt{2}$ and $y = -\sqrt{2}$ into the eqn.

$2xy.0 + 2 = 2$ $\quad 2 = 2$ true

$y = \sqrt{2}$ and $y = -\sqrt{2}$ also solution

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b) $xy' = y + xe^{y/x}$

$y' = \frac{y}{x} + e^{y/x}$ (homogeneous eqn.)

$y = u.x \quad u = \frac{y}{x}$

$y' = u'x + u$

$u'x + u = u + e^{u}$

$u'x = e^{u}$ (separable) $\quad u'.e^{-u} \quad \int u.e^{-u} du$

$\int e^{-u} du = \int \frac{dx}{x}$

$-e^{-u} = \ln|x| + c$ (return to y)

$-e^{-\frac{y}{x}} = \ln|x| + c$

c) $(2y+1)dx + \frac{x^2-y}{x} dy = 0$

$(2yx+x)dx + (x^2-y)dy = 0$

Exact $\quad M = 2yx+x \quad , N = x^2-y$

$M_y = 2x \quad N_x = 2x$

$M_y = N_x$ (Exact equation)

To solve find $F(x,y)$ s.t $F_x = M$ and $F_y = N$

$F_x = 2yx + x$

$F(x,y) = \int (2yx+x)dx = yx^2 + \frac{1}{2}x^2 + h(y)$

$F_y = N$

$x^2 + h'(y) = x^2 - y$

$h'(y) = -y \quad \Rightarrow h(y) = -\frac{y^2}{2} + A$ (put $A=0$)

$F(x,y) = yx^2 + \frac{x^2}{2} - \frac{y^2}{2}$

Solution $F(x,y) = C \quad \Rightarrow yx^2 + \frac{x^2-y^2}{2} = C$

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1) $ty' + (t+1)y = 2te^{-t}$ (Linear eqn.)

$y' + \frac{t+1}{t} y = 2e^{-t}$

Integrating factor $e^{\int \frac{t+1}{t} dt} = e^{t+\ln(t)} = \mu = te^{t}$

Multiply by μ

$te^{t} \cdot y' + (te^{t} + e^{t}) y = 2t$

$(te^{t} y)' = 2t$

$te^{t} y = t^2 + c$

$y = te^{-t} + ct^{-1}e^{-t}$

2) Find a differential equation of first order that has following general solution

$t^3 \cdot y + e^{t} + y^2 = c$

for exact equation

$f_t dt + f_y dy = 0$

$f = c$ is a solution

$f(t,y) = t^3 y + e^{t} + y^2$

$f_t = 3t^2 y + e^{t}$, $f_y = t^3 + 2y$

The equation

$(3t^2 y + e^{t}) dt + (t^3 + 2y) dy = 0$ has general solution

$t^3 y + e^{t} + y^2 = c$

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3) For what values of y_0

We can guarantee the existence and uniqueness of solution of

$$y' = 2 + \sqrt[3]{y-2t} \qquad y(1) = y_0$$

Put $f(t,y) = 2 + \sqrt[3]{y-2t}$

The solution exists and is unique if f, f_y are continuous around $(1, y_0)$

$f(t,y) = 2 + \sqrt[3]{y-2t}$ continuous everywhere

$$f_y = (y-2t)^{\frac{1}{3}} = \frac{1}{3}(y-2t)^{-\frac{2}{3}} = \frac{1}{3\sqrt[3]{y-2t}}$$ is continuous for all (t,y) s.t $y-2t \neq 0$

$\boxed{y_0 \neq 2}$

For $(\underset{t_0}{1}, y_0)$ if $y_0 - 2.1 \neq 0$ the conditions of uniqueness and are satisfied

and we can guarantee the existence of unique solution

for all $y_0 \neq 2$, we have unique solution.

4) Find the largest rectangle on ty plane where conditions of existence and uniqueness theorem satisfied for

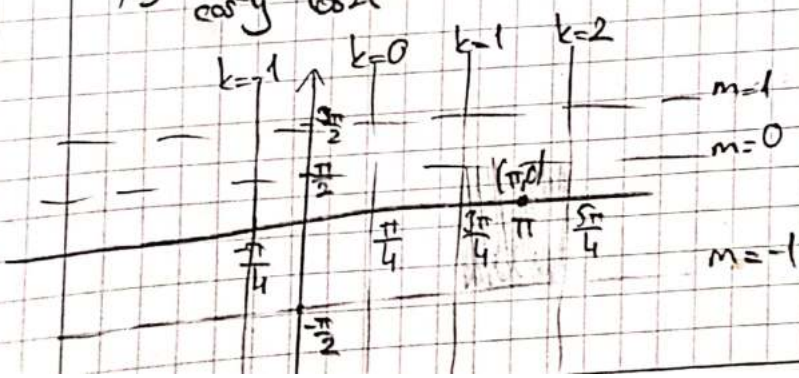
$$(\cos 2t)y' = 2 + \tan y \qquad y(\pi) = 0$$

$$y' = \frac{2}{\cos 2t} + \frac{\sin y}{\cos y} \cdot \frac{1}{\cos 2t}$$

$f(t,y) = \frac{2}{\cos 2t} + \frac{\sin y}{\cos y \sin 2t}$ $\quad f(t,y)$ cont. for (t,y) s.t $\cos 2t \neq 0$, $2t = \frac{\pi}{2} + \pi k$ $\quad k \in \mathbb{Z}$

$\cos y \neq 0 \quad y \neq \frac{\pi}{2} + \pi m \quad m \in \mathbb{Z}$

$f_y = \frac{1}{\cos^2 y} \cdot \frac{1}{\cos 2t} \Rightarrow f_y$ cont. for all (t,y) $\quad \cos 2t \neq 0$, $\cos y \neq 0$ same as before



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The conditions of existence and uniqueness theorem are satisfied in the rectangle $R: \frac{3\pi}{4} < t < \frac{5\pi}{4}$, $-\frac{\pi}{2} < y < \frac{\pi}{2}$ around $(\pi, 0)$

5) A tank contains 200 L of solution with concentration of salt 1g/L. Fresh water flows in the tank at a rate 2L/min and well stirred solution flows out at the same rate. At what time the concentration of salt will reach 0.1% of the original value.

Let $Q(t)$: amount of salt at time t

$Q(0) = 1g/L \cdot 200L = 200g$

$t_0 = ?$

$Q(t_0) = 1 \cdot \frac{1}{1000} g/L \cdot 200L = 0.2g$

Find expression for $Q(t)$

$\frac{dQ}{dt} = (\text{rate in}) - (\text{rate out})$

$\text{rate in} = 0g/L \cdot 2L/min = 0$

$\text{rate out} = \frac{Q}{200} g/L \cdot 2L/min = \frac{Q}{100}$

$\frac{dQ}{dt} = \frac{-Q}{100} \qquad Q(0) = 200$

$\int \frac{dQ}{Q} = \int \frac{-dt}{100}$

$\ln|Q| = -\frac{1}{100} t + c$

$Q(t) = ce^{\frac{-t}{100}}$

$Q(0) = 200 \Rightarrow 200 = c$

$Q(t) = 200 e^{\frac{-t}{100}}$

$t_0 = ? \qquad Q(t_0) = 0.2 \qquad 0.2 = 200 e^{\frac{-t_0}{100}}$

$\frac{1}{1000} = e^{\frac{-t_0}{100}} \qquad -\ln 1000 = \frac{-t_0}{100}$

$\boxed{t_0 = 100 \ln 1000}$

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6) Solve $y' = \frac{1}{e^y - t}$ $\quad y(1)=0 \rightarrow t=1, y=0$

Hint: consider t as function of y

$$\frac{dy}{dt} = \frac{1}{e^y - t} \Rightarrow \frac{dt}{dy} = e^y - t$$

$$t' + t = e^y \qquad t = \text{dependent}$$

$$t(0) = 1 \qquad y = \text{independent}$$

Integrating factor $e^{\int 1\,dy} = e^y$

Multiply by e^y: $e^y \cdot t' + e^y \cdot t = e^{2y}$

$$(te^y)' = e^{2y} \qquad te^y = \frac{1}{2}e^{2y} + c$$

$$t = \frac{1}{2}e^y + ce^{-y} \qquad t(0) = 1$$

$$\frac{1}{2} + c = 1$$

$$c = \frac{1}{2}$$

7) Find a fundamental matrix for $x_1' = 4x_1 - x_2$
$x_2' = x_1 + 2x_2$

Recall $\Phi(t) = [\vec{x_1}, \vec{x_2}]$

We solve the system

$$\vec{x}' = \begin{pmatrix} 4 & -1 \\ 1 & 2 \end{pmatrix} \vec{x}$$

Find eigenvalues

$\det(A - rI) = 0$

$$\begin{vmatrix} 4-r & -1 \\ 1 & 2-r \end{vmatrix} = 0$$

$$8 - 6r + r^2 + 1 = 0$$

$$r^2 - 6r + 9 = 0$$

$$(r-3)^2 = 0$$

$r_{1,2} = 3$ is double root multiplicity 2

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Eigenvectors

$(A - rI)\vec{\varepsilon} = 0$

$$\begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} \varepsilon_1 \\ \varepsilon_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \begin{matrix} \varepsilon_1 - \varepsilon_2 = 0 \\ \varepsilon_1 - \varepsilon_2 = 0 \end{matrix} \Big\} \varepsilon_1 - \varepsilon_2 = 0$$

2 variables one equation means 1 variable free

$$\begin{cases} \varepsilon_1 = c \\ \varepsilon_2 = c \end{cases} \Rightarrow \varepsilon = \begin{pmatrix} c \\ c \end{pmatrix} = c\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

Put $c = 1$ $\vec{\varepsilon}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ eigenvector

We have only one eigenvector so we have one solution but we need two.

We look for generalised eigenvectors

$(A - rI)\varepsilon = \vec{\varepsilon}_1$

$$\begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} \varepsilon_1 \\ \varepsilon_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$\varepsilon_1 - \varepsilon_2 = 1$ $\qquad \varepsilon = \begin{pmatrix} c \\ c+1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} + c\begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$\varepsilon_1 = c$, $\varepsilon_2 = c - 1$ $\qquad$ Put $c = 0$

$\vec{\varepsilon}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

We have chain $\vec{\varepsilon}_1 \to \vec{\varepsilon}_2$

Solutions $\vec{\varepsilon}_1$ $\quad e^{3t}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \vec{x}_1$

$\varepsilon_1 \to \varepsilon_2$ $\quad e^{3t}\left(\begin{pmatrix} 0 \\ 1 \end{pmatrix} + t\begin{pmatrix} 1 \\ 1 \end{pmatrix}\right) = \vec{x}^2$

$x^{(1)} = e^{3t}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} e^{3t} \\ e^{3t} \end{pmatrix}$

$x^{(2)} = t \cdot e^{3t}\begin{pmatrix} 1 \\ 1 \end{pmatrix} + e^{3t}\begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$x^{(2)} = \begin{pmatrix} te^{3t} \\ te^{3t} + e^{3t} \end{pmatrix}$

A fundamental matrix

$$\Phi = \begin{pmatrix} e^{3t} & te^{3t} \\ e^{3t} & e^{3t} + te^{3t} \end{pmatrix}$$

$$\begin{pmatrix} e^{3t} & te^{3t} \\ e^{3t} & te^{3t} + e^{3t} \end{pmatrix}$$

$$\left| \begin{matrix} e^{rt}\vec{v}_1 \\ e^{rt}(\vec{v}_2 + t\vec{v}_1) \\ e^{rt}\left(\vec{v}_3 + t\vec{v}_2 + \frac{t^2}{2}\vec{v}_1\right) \end{matrix} \right.$$

Recall chain

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8) Find the general solution

$$\vec{x}' = \begin{pmatrix} 2 & -5 \\ 1 & -2 \end{pmatrix} \vec{x} + \begin{pmatrix} -\cos t \\ \sin t \end{pmatrix}$$

Nonhomogeneous system

General solution $\vec{x} = \vec{x}_h + \vec{x}_p$

$\vec{x}_h \to$ general solution of associated hom.

$\vec{x}_p \to$ particular sol. (any)

$\vec{x}_h = ?$

$$\vec{x}' = \begin{pmatrix} 2 & -5 \\ 1 & -2 \end{pmatrix} \vec{x}$$

Eigenvalues

$$\begin{vmatrix} 2-r & -5 \\ 1 & -2-r \end{vmatrix} = 0 \qquad r = \mp i$$

Eigenvectors

$r = i$

$$\begin{pmatrix} 2-i & -5 \\ 1 & -2-i \end{pmatrix} \begin{pmatrix} \varepsilon_1 \\ \varepsilon_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$(2-i)\varepsilon_1 - 5\varepsilon_2 = 0$$

Put $\varepsilon_1 = c$ put $c = 5$

$$\varepsilon_2 = \frac{(2-i)c}{5}$$

$$\vec{\varepsilon}_1 = \begin{pmatrix} 5 \\ 2-i \end{pmatrix}$$

Complex solution

$$\vec{\Psi} = e^{it} \begin{pmatrix} 5 \\ 2-i \end{pmatrix} = (\cos t + i \sin t) \begin{pmatrix} 5 \\ 2-i \end{pmatrix}$$

$$\vec{x}^1 = \operatorname{Re} \vec{\Psi}$$

$$\vec{x}^2 = \operatorname{Im} \vec{\Psi}$$

$$\vec{x}^1 = \begin{pmatrix} 5\cos t \\ 2\cos t + \sin t \end{pmatrix}$$

$$\vec{x}^2 = \begin{pmatrix} 5\sin t \\ -\cos t + 2\sin t \end{pmatrix}$$

$$\vec{x}_h = c_1 \begin{pmatrix} 5\cos t \\ 2\cos t + \sin t \end{pmatrix} + c_2 \begin{pmatrix} 5\sin t \\ -\cos t + 2\sin t \end{pmatrix}$$

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$\vec{X}_p = ?$

Variation of parameters

$\vec{X}_p = u_1(t)\,\vec{x}^1 + u_2(t)\,\vec{x}^2$ where u_1, u_2 satisfy

$u_1'\vec{x}^1 + u_2'\vec{x}^2 = 0$

$u_1'(5\cos t) + u_2'(5\sin t) = -\cos t$

$u_1'(2\cos t + \sin t) + u_2'(-\cos t + 2\sin t) = \sin t$

$(\text{eqn}2) - \frac{2}{5}(\text{eqn}1)$

$$\begin{cases} u_1'(\cos t) + u_2'(\sin t) = -\frac{1}{5}\cos t & / -\sin t \\ u_1'(\sin t) + u_2'(-\cos t) = \sin t + \frac{2}{5}\cos t & / \cos t \end{cases}$$

$u_1' = -\frac{1}{5}\cos^2 t + \sin^2 t + \frac{2}{5}\cos t \sin t$

$u_2' = -\frac{1}{5}\cos t \sin t - \cos t \sin t - \frac{2}{5}\cos^2 t$

$u_1 = \int \left(-\frac{1}{5}\cos^2 t + \sin^2 t + \frac{1}{5}\sin 2t\right) dt$

$u_2 = \int \left(-\frac{3}{5}\sin 2t - \frac{2}{5}\cos^2 t\right) dt$

find u_1, u_2 $\Rightarrow$ find X_p and X_g

$\Psi u' = g$

$\Psi u = X_g$

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Last Time: Homogeneous, High Order, Linear, Constant coeff ODE's

Ex: $y'' - 3y' - 4y = 0$

general solution y is found by characteristic polynomial: $r^2 - 3r - 4 = 0$

$(r-4)(r+1) = 0 \Rightarrow r = -1$ or $r = 4$

$y = c_1 e^{-t} + c_2 e^{4t}$

complex, repeated roots etc.

Nonhomogeneous, High Order, Linear, Constant Coefficient ODES (Section 4.3)

The general solution is $y_g = y_c + y_p$ → particular solution

y_c: (fundamental) complementary solution

general solution to the corresponding homogeneous equation

Ex: $y'' - 3y' - 4y = \underbrace{2e^{2t}}_{\text{nonhomogeneous part}}$

What is y_g?

solution: $y_g = y_c + y_p$

$y_c = c_1 e^{-t} + c_2 e^{4t}$ solution to $y'' - 3y' - 4y = 0$

$\boxed{\text{Suppose } y_p = Ae^{2t} \rightarrow \text{Assumption}}$

Then, $y_p'' - 3y_p' - 4y_p = 2e^{2t}$

$y_p'' = 4Ae^{2t}$, $y_p' = 2Ae^{2t}$

$\Rightarrow 4Ae^{2t} - 6Ae^{2t} - 4Ae^{2t} = 2e^{2t}$

$-6Ae^{2t} = 2e^{2t}$

$A = -\frac{1}{3}$

$y_p = -\frac{1}{3}e^{2t}$

$\Rightarrow \boxed{y_g = c_1 e^{-t} + c_2 e^{4t} - \frac{1}{3}e^{2t}}$

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Ex: $y'' - 3y' - 4y = 4t^2$ $\quad y_g = ?$

solution:

$$y_g = y_c + y_p$$

$$y_c = c_1 e^{-t} + c_2 e^{4t}$$

Suppose, $\boxed{y_p = A + Bt + Ct^2 \rightarrow \text{Assumption}}$

$$y_p'' - 3y_p' - 4y_p = 4t^2$$

$$y_p' = 2Ct + B \quad , y_p'' = 2C$$

$$\Rightarrow 2C - 6Ct - 3B - 4A - 4Bt - 4Ct^2 = 4t^2$$

$$A = \frac{-13}{8}, B = \frac{3}{2}, C = -1$$

$$y_p = \frac{-13}{8} + \frac{3}{2}t - t^2$$

Thus, $y_g = c_1 e^{-t} + c_2 e^{4t} - \frac{13}{8} + \frac{3}{2}t - t^2$

In general all particular solutions are of the form

nonhomogeneous part	y_p
$a + bt + ct^2 + \dots$	$A + Bt + Ct^2 + \dots$
$t^n . e^{st}$	$Ae^{st} + Bte^{st} + Ct^2 e^{st} + \dots$
$t^n . e^{at} . \cos bt$ or $t^n . e^{at} . \sin bt$	$A_1 e^{at} \cos bt + B_1 e^{at} \sin bt + A_2 t e^{at} \cos bt + B_2 t e^{at} \sin bt + \dots$

Notation: $y^{(n)} = D^n$ i.e $\frac{dy}{dt^n} = D^n$

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Ex: $y''-3y'-4y=2e^{2t} \leftrightarrow (D^2-3D-4)y=2e^{2t}$

Let $L(D)y=g(t)$ be a linear nonhomogeneous constant coefficient ODE and $H(D)g=0$

then, $H(D).L(D)y=0$. Solution to this homogeneous ODE is $y_g=y_c+y_p$ to $L(D)y=g$

Ex: $y''-3y'-4y=e^{-t}$ $\quad y_g=?$

Solution:

$(D^2-3D-4)y=e^{-t}$

$(D+1)(D-4)y=e^{-t}$ and $(D+1)e^{-t}=0$

$-t$'nin katsayısını pozit. $-t=-1$

$D-(-1)=D+1$

$\left(\underline{(D-a)e^{at}=0}\right)$

$\left[\begin{array}{l}\frac{d}{dt}e^{-t}+e^{-t}=0\\ -e^{-t}+e^{-t}=0 \checkmark\end{array}\right]$

$\frac{d}{dt}=D$

y_g is the solution to $L(D)H(D)y=0$

$(D+1)(D-4) \quad (D+1)$

$\Rightarrow (D+1)^2(D-4)y=0$

$y_g=\underbrace{c_1e^{-t}+\underbrace{Ate^{-t}}_{y_p}+c_2e^{4t}}_{y_c}$

$y_c=c_1e^{-t}+c_2e^{4t}$

$y_p=Ate^{-t}$

We must find A

$y_p''-3y_p'-4y_p=e^{-t}$

$y_p'=Ae^{-t}-Ate^{-t}$

$y_p''=-Ae^{-t}-Ae^{-t}+Ate^{-t}$

$\Rightarrow -2Ae^{-t}+\cancel{Ate^{-t}}-3Ae^{-t}+\cancel{3Ate^{-t}}-\cancel{4Ate^{-t}}=e^{-t}$

$A=-\frac{1}{5}$. Thus, $y_g=c_1e^{-t}+c_2e^{4t}-\frac{1}{5}te^{-t}$

This is the method of "undetermined coefficients"

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The method using $\underbrace{(D+1)}_{\text{[called the annihilator of } e^{-t}]}e^{-t}=0$ makes us see that y_p is equal to Ate^{-t} but not equal to Ae^{-t}

Ex: Find y_p and y_g for $y''-3y'-4y=-8e^{t}\cos 2t$

Solution:

$$(D^2-3D-4)y=-8e^{t}\cos 2t$$

What is $H(D)$ s.t. $H(D)\underbrace{e^{t}}_{\text{[constant (-8) is not important]}}\cos 2t$

$$\boxed{\begin{array}{l}\text{Euler's Formula: } e^{(a+ib)t}=e^{at}(\cos bt+i\sin bt)\\ (D-(a+ib))(D-(a-ib))=(D-a)^2+b^2\end{array}}$$

In this question

$(D-1)^2+4$: annihilator of $-8e^{t}\cos 2t$

If $e^{(a+ib)t}$ is a solution, also the $e^{(a-ib)t}$ is solution

$e^{at}\cos bt$ and $e^{at}\sin bt$ are linear combination of the former two.

i.e $(D-1)^2+4$ annihilates $e^{t}\cos 2t$ and $e^{t}\sin 2t$

$$(D-1)^2+4=D^2-2D+5$$

$$H(D)L(D)y=0 \Leftrightarrow (D^2-2D+5)(D+1)(D-4)y=0$$

Thus, $y_g=\underbrace{c_1e^{-t}+c_2e^{4t}}_{y_c}+\underbrace{Ae^{t}\cos 2t+Be^{t}\sin 2t}_{y_p}$

must be found explicitly

$$(D^2-3D-4)y_p=-8e^{t}\cos 2t \Rightarrow A=\frac{10}{13} \text{ and } B=\frac{2}{13}$$

And Thus, $y_g=c_1e^{-t}+c_2e^{4t}+\frac{10}{13}e^{t}\cos 2t+\frac{2}{13}e^{t}\sin 2t$

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(Solution in the book)

Exercise: $(D^4+2D^2+1)y = 3\sin t - 5\cos t$

Ex: $y''' - 3y'' + 3y' - y = g(t)$ ← nonhomogeneous part

e^t
te^{2t}
$at+b$
$\cos t + 2\sin t$

We can use the above method.

But if $g(t)$ is like $\tan x$ or $\sec x$ then this method does not apply. We use

Variation of Parameters (summarized in 3.7 and 4.4)

(We rely on systems for this method)

Ex: $y'' + 4y = 3\operatorname{cosec} t$ $\quad y_g = ?$ (What is the general solution)

We may say $y_g = \underbrace{y_c}_{\text{complementary}} + \underbrace{y_p}_{\text{particular}}$

characteristic polynomial of $y''+4y=0$ is $r^2+4=0 \Rightarrow r = \pm 2i$ $\quad r_1 = 2i \quad r_2 = -2i$

$$y_c = c_1\cos 2t + c_2 \sin 2t$$

This step of finding y_c is redundant if we rely on systems to apply the method of variation of parameters.

Solution: Replace the ODE with the system below:

$$\begin{matrix} x_1 = y \\ x_2 = y' \end{matrix} \Rightarrow \begin{matrix} x_1' = x_2 \\ x_2' = -4x_1 + 3\operatorname{cosec} t \end{matrix} \Rightarrow \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}' = \begin{pmatrix} 0 & 1 \\ -4 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 0 \\ 3\operatorname{cosec} t \end{pmatrix}$$

$$\vec{x}' = \begin{pmatrix} 0 & 1 \\ -4 & 0 \end{pmatrix}\vec{x} + \vec{g}(t)$$

Remember section 7.9 (variation of parameters for systems): Find $\Psi(t)$ and use it to find particular solution.

$|A - rI| = 0 \Rightarrow \begin{vmatrix} -r & 1 \\ -4 & -r \end{vmatrix} = 0 \Rightarrow r^2+4=0 \quad r = \mp 2i$ (eigenvalues)

for $r = 2i$: $A - rI = \begin{pmatrix} -2i & 1 \\ -4 & -2i \end{pmatrix} \xrightarrow{iR_1} \begin{pmatrix} +2 & i \\ -4 & -2i \end{pmatrix} \xrightarrow{2R_1+R_2} \begin{pmatrix} 2 & i \\ 0 & 0 \end{pmatrix} \Rightarrow \begin{matrix} 2k_1 + ik_2 = 0 \\ 2k_1 = -ik_2 \end{matrix} \quad \begin{pmatrix} 1 \\ 2i \end{pmatrix}$

$$\Rightarrow \begin{pmatrix} 1 \\ 2i \end{pmatrix} e^{2it} = \begin{pmatrix} 1 \\ 2i \end{pmatrix}\left[\cos 2t + \sin 2t\right] = \begin{pmatrix} \cos 2t + i\sin 2t \\ -2\sin 2t + 2i\cos 2t \end{pmatrix} = \begin{pmatrix} \cos 2t \\ -2\sin 2t \end{pmatrix} + i\begin{pmatrix} \sin 2t \\ 2\cos 2t \end{pmatrix}$$

real valued solution

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$\int -dx = x$

$\mu = e$

$\Psi = \begin{pmatrix} \cos 2t & \sin 2t \\ -2\sin 2t & 2\cos 2t \end{pmatrix}$

Variation of parameters: $X = \Psi u$ where $u = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$ then $\Psi \vec{u} = \vec{g}$

$e^{2\ln t} = t^2$

$\int e^{2dt}$

$$\Rightarrow \begin{pmatrix} \cos 2t & \sin 2t \\ -2\sin 2t & 2\cos 2t \end{pmatrix} \begin{pmatrix} u_1' \\ u_2' \end{pmatrix} = \begin{pmatrix} 0 \\ 3\operatorname{cosec} t \end{pmatrix}$$

$$\Rightarrow \cos 2t . u_1' + \sin 2t . u_2' = 0 \qquad \Rightarrow u_1' = -\frac{\sin 2t}{\cos 2t} u_2'$$

$$-2\sin 2t\, u_1' + 2\cos 2t . u_2' = 3\operatorname{cosec} t$$

$$\Rightarrow -2\sin 2t . \frac{-\sin 2t}{\cos 2t} u_2' + 2\cos 2t . u_2' = 3\operatorname{cosec} t$$

$$\Rightarrow \left(\frac{2\sin^2 2t}{\cos 2t} + 2\cos 2t\right) u_2' = 3\operatorname{cosec} t$$

$$\Rightarrow 2\left(\frac{\sin^2 2t + \cos^2 2t}{\cos 2t}\right) u_2' = 3\operatorname{cosec} t$$

$$\Rightarrow u_2' = \frac{3}{2}\operatorname{cosec} t \cos 2t \qquad \Rightarrow u_2' = \frac{3}{2}\operatorname{cosec} t\,(1 + 2\cos^2 t)$$

$$u_2' = \frac{3}{2}\left(\frac{1}{\sin t} + \frac{2\cos^2 t}{\sin t}\right) \qquad u_2 = \underbrace{\int \frac{3}{2}\operatorname{cosec} t}_{\frac{3}{2}\ln \operatorname{cosec} t - \cot t} + 3\underbrace{\int \frac{\cos^2 t}{\sin t} dt}_{\int \cos t \cot t\, dt}$$

$u = \cot t \qquad du = -\operatorname{cosec}^2 t$

$du = \cos t\, dt \qquad v = \sin t$

$$u_2 = \frac{3}{2}\ln|\operatorname{cosec} t - \cot t| + 3\sin t \cot t - 3\ln|\operatorname{cosec} t - \cot t| + c_2$$

$$u_2 = \frac{3}{2}\ln|\operatorname{cosec} t - \cot t| + 3\sin t \cot t + c_2$$

$$u_1' = -3\cos t$$

$$u_1 = -3\int \cos t\, dt = -3\sin t + c_1$$

$$u = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \qquad \Rightarrow X = \begin{pmatrix} \cos 2t & \sin 2t \\ -2\sin 2t & 2\cos 2t \end{pmatrix} \begin{pmatrix} -3\sin t + c_1 \\ \frac{3}{2}\ln|\operatorname{cosec} t - \cot t| + 3\sin t \cot t + c_2 \end{pmatrix}$$

$X = \Psi u$

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$$X=\begin{pmatrix}x_1\\x_2\end{pmatrix}=\begin{pmatrix}y\\y'\end{pmatrix} \Rightarrow y_g = \underbrace{-3\cos 2t\sin t}_{y_p} + \underbrace{c_1\cos 2t}_{y_c} + \underbrace{\tfrac{1}{2}\sin 2t\ln|\sec t-\cot t| + 3\sin 2t\tan t}_{y_p} + \underbrace{c_2\sin 2t}_{y_c}$$

Ex: $y'''+y'=\tan t \qquad y_g=?$

$x_3' = y''' = -y' + \tan t = -x_2 + \tan t$

solution: Introduce the system:

$$\begin{aligned} x_1 &= y & x_1' &= x_2 \\ x_2 &= y' \Rightarrow & x_2' &= x_3 \\ x_3 &= y'' & x_3' &= -x_2 + \tan t \end{aligned}$$

$$\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}' = \begin{pmatrix}0&1&0\\0&0&1\\0&-1&0\end{pmatrix}\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix} + \begin{pmatrix}0\\0\\\tan t\end{pmatrix}$$

$$\vec{X}' = \underbrace{\begin{pmatrix}0&1&0\\0&0&1\\0&-1&0\end{pmatrix}}_{A}\vec{X} + \underbrace{\begin{pmatrix}0\\0\\\tan t\end{pmatrix}}_{g}$$

First find Ψ (a fundamental matrix)

Second find $X=\Psi u$ (Variation of parameters)

$$\det(A-rI)=0 \Rightarrow \begin{vmatrix} r&1&0\\0&r&1\\0&-1&r\end{vmatrix}=0 \Rightarrow r^3+r=0 \Rightarrow \underbrace{r=0}_{=0} \text{ eigenvalues}$$

or

$r=\mp i \longrightarrow \cos t$, $\sin t$

for $r=0$: $\begin{pmatrix}0&1&0\\0&0&1\\0&-1&0\end{pmatrix} \to \begin{pmatrix}0&1&0\\0&0&1\\0&0&0\end{pmatrix} \Rightarrow \begin{pmatrix}1\\0\\0\end{pmatrix} \to$ a fundamental solution

for $r=-i$ $\begin{pmatrix}i&1&0\\0&i&1\\0&-1&i\end{pmatrix} \xrightarrow{iR_2} \begin{pmatrix}i&1&0\\0&-1&i\\0&-1&i\end{pmatrix} \to \begin{pmatrix}1&-i&0\\0&1&i\\0&0&0\end{pmatrix} \to \begin{pmatrix}1&0&1\\0&1&i\\0&0&0\end{pmatrix}$

$$\begin{aligned} k_1&=-k_3\\ k_2&=-ik_3\end{aligned} \Rightarrow \begin{pmatrix}1\\i\\-1\end{pmatrix}e^{-it} = \begin{pmatrix}1\\i\\-1\end{pmatrix}(\cos -t + i\sin(-t)) = \begin{pmatrix}1\\i\\-1\end{pmatrix}(\cos t - i\sin t) = \begin{pmatrix}\cos t - i\sin t\\ \sin t + i\cos t\\ -\cos t + i\sin t\end{pmatrix}$$

$$= \begin{pmatrix}\cos t\\ \sin t\\ -\cos t\end{pmatrix} + i\begin{pmatrix}-\sin t\\ \cos t\\ \sin t\end{pmatrix}$$ Thus, A fundamental matrix is $\Psi = \begin{pmatrix}1&\cos t&-\sin t\\0&\sin t&\cos t\\0&-\cos t&\sin t\end{pmatrix}$

To find y_g apply variation of parameters $X=\Psi u$ is the solution

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$\Rightarrow \Psi u' = g$ where $u' = \begin{pmatrix} u_1' \\ u_2' \\ u_3' \end{pmatrix}$ and $g = \begin{pmatrix} 0 \\ 0 \\ \tan t \end{pmatrix}$

$u_1' + \cos t\, u_2' - \sin t\, u_3' = 0$

$\sin t\, u_2' + \cos t\, u_3' = 0 \quad \Rightarrow u_1' = \tan t \quad \rightarrow u_1 = \ln|\sec t| + c_1$

$-\cot t\, u_2' + \sin t\, u_3' = \tan t$

Multiply 2nd equation with $\cos t$ $\quad \cos t \sin t\, u_2' + \cos^2 t\, u_3' = 0$

3rd equation with $\sin t$ $\quad -\cos t \sin t\, u_2' + \sin^2 t\, u_3' = \dfrac{\sin^2 t}{\cos t}$

$$u_3' = \frac{\sin^2 t}{\cos t} = \sin t \cdot \tan t$$

$$u_3 = \int \sin t \tan t\, dt = -\cos t \tan t + \ln|\sec t + \tan t| + c_3$$

$u = \tan t \quad du = \sec^2 t\, dt$

$dv = \sin t\, dt \quad v = -\cos t$

$\sin t\, u_2' = -\cos t \sin t \tan t$

$u_2' = -\cos t \tan t = -\sin t \Rightarrow u_2 = \cos t + c_2$

$$X = \Psi u \Rightarrow X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} y \\ y' \\ y'' \end{pmatrix} = \begin{pmatrix} 1 & \cos t & -\sin t \\ 0 & \sin t & \cos t \\ 0 & -\cos t & \sin t \end{pmatrix} \begin{pmatrix} \ln|\sec t| + c_1 \\ \cos t + c_2 \\ -\sin t + \ln|\sec t + \tan t| + c_3 \end{pmatrix}$$

$$y = \underbrace{\ln|\sec t| + c_1 + \cos^2 t + c_2 \cos t + \sin^2 t - \sin t \cdot \ln|\sec t + \tan t|}_{y_p} \underbrace{- c_3 \sin t}_{y_c}$$

Exercise: Solve $y''' + y' = 0$ and see $y = c_1 + c_2 \cos t + c_3 \sin t$

Ex: $y'' + y' = t^2 + e^{2t} \quad y_g = ?$

It is more convenient to use the method of "undetermined coeffs." to the solve problem

Solution: $(D^3 + D)y = t^2 + e^{2t}$ annihilator of t^2 is D^3 $\quad D^3 t^2 = 2?$ (c) $\quad$ $t^{0,1,2}$

annihilator of e^{2t} is $D-2$ $\quad (D-2)e^{2t} = 0$

$L(D) = D^3 + D \qquad H(D) = D^3(D-2)$

$= (D^2+1)D$

$L(D)H(D)y = 0 \Rightarrow D^4(D-2)(D^2+1)y = 0$

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The solution -- be of the form $\left[\begin{array}{l} L(D)y=0 \\ D(D^2+1)y=0 \\ c_1+c_2\cos t+c_3\sin t \end{array}\right]$

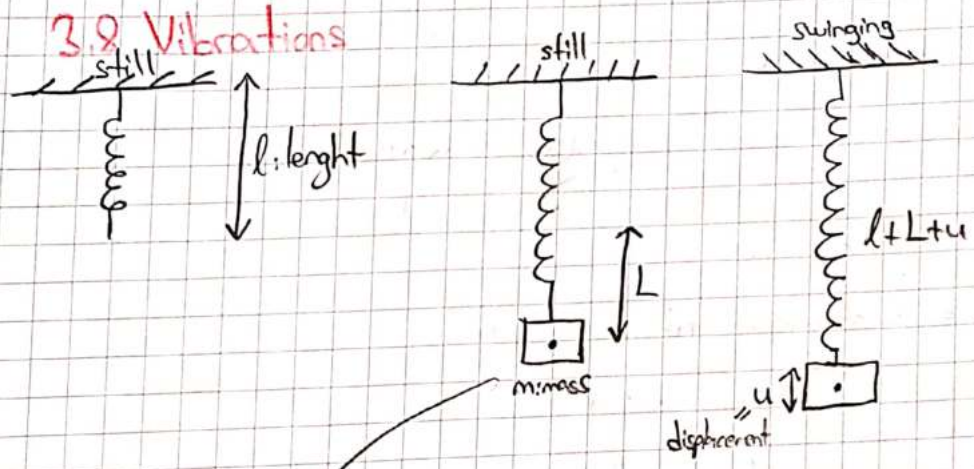
$y_g = c_1 + At + Bt^2 + Ct^3 + Ee^{2t} + c_2\cos t + c_3\sin t$ where A, B, C, E are undetermined coefficients

To find A, B, C, E we plug $y_p = At + Bt^2 + Ct^3 + Ee^{2t}$ into $y_p''' + y_p' = t^2 + e^{2t}$

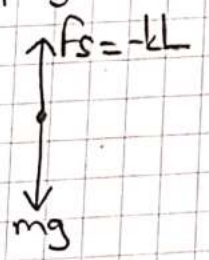
$$6C + 8Ee^{2t} + A + 2Bt + 3Ct^2 + 2Ee^{2t} = t^2 + e^{2t}$$

$$8E + 2E = 1$$
$$E = 1/10, \quad C = 1/3, \quad B = 0, \quad A = -2 \quad (\text{SON!})$$

3.8 Vibrations



Force diagram of still spring-mass



k = spring constant

g = gravity constant

and $mg - kL = 0$

When spring is $\boxed{\text{swinging}}$

$f_s = -k(L+u)$

Inner resistence → damping force $f_d = -\gamma u'$ (velocity)

External force $= F$

<u>Newton:</u> Net force $= ma$

u'' is acceleration, so $mu'' = \overbrace{mg - \underbrace{k(L+u)}_{F_s}}^{0} \underbrace{-\gamma u'}_{F_d} + F$

$\Rightarrow mu'' + \gamma u' + ku = F \qquad m, k > 0 \quad \gamma \geq 0$

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free vibration if $F=0$ (homogeneous)

$$mu'' + \gamma u' + ku = 0$$

undamped if $\gamma = 0$ $\quad mu'' + ku = F$

underdamped if $\gamma < 2\sqrt{mk}$

critically damped if $\gamma = 2\sqrt{mk}$

overdamped if $\gamma > 2\sqrt{mk}$ (nothing to see no vibrations)

Ex: (free undamped vibrations)

$\gamma = 0 \quad F = 0$

$mu'' + ku = 0 \Rightarrow u(t) = A\cos(\omega_0 t) + B\sin(\omega_0 t)$

where $\omega_0^2 = \frac{k}{m} > 0 \quad (k, m > 0)$

Make a change of notation:

$A = R\cos\delta$ and $B = R\sin\delta$

Then, $u(t) = R\cos\delta\cos(\omega_0 t) + R\sin\delta\sin(\omega_0 t)$

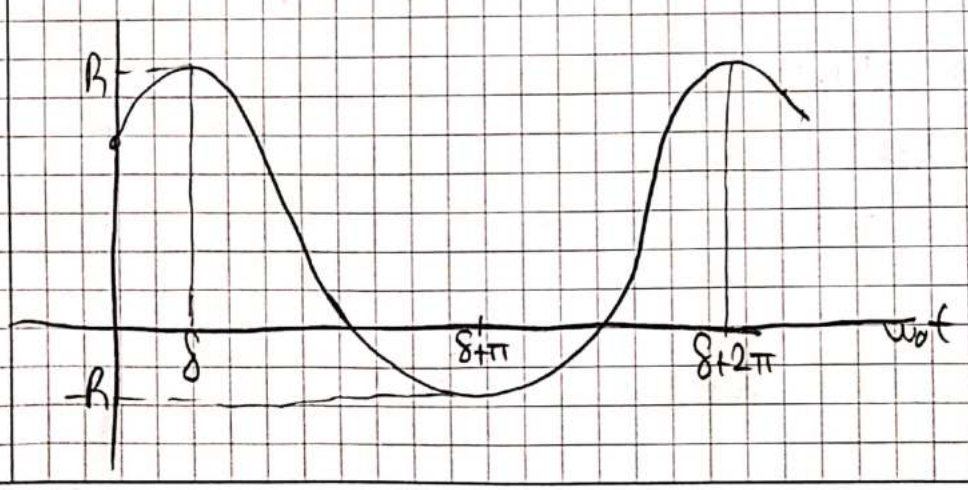
$u(t) = R\cos(\omega_0 t - \delta)$

We call $\omega_0 = \sqrt{\frac{k}{m}}$ the frequency

$T = \frac{2\pi}{\omega_0}$ the period

$R = \sqrt{A^2 + B^2}$ the amplitude

δ the phase



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Ex: (free damped vibrations)

$F=0 \qquad m,k,\gamma>0$

$mu''+\gamma u'+ku=0$

char. poly. $mr^2+\gamma r+k=0 \Rightarrow r=\dfrac{-\gamma\mp\sqrt{\gamma^2-4mk}}{2m}$

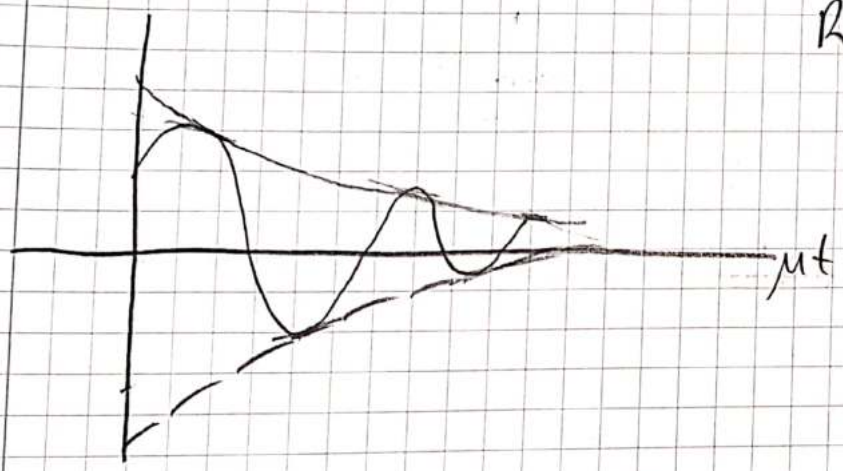
<u>underdamped $\gamma^2<4mk$</u>

for "small" damping

$u(t)=Re^{-\frac{\gamma}{2m}t}\cdot\cos(\mu t-\delta)$

$R\cos(\omega t-\delta)$

$R\cos(\mu t-\delta)\cdot e^{-\frac{\gamma}{2m}t}$

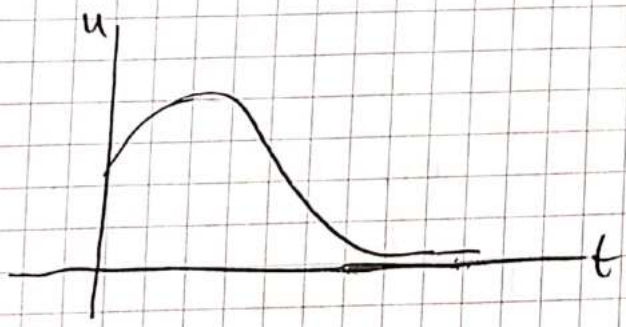


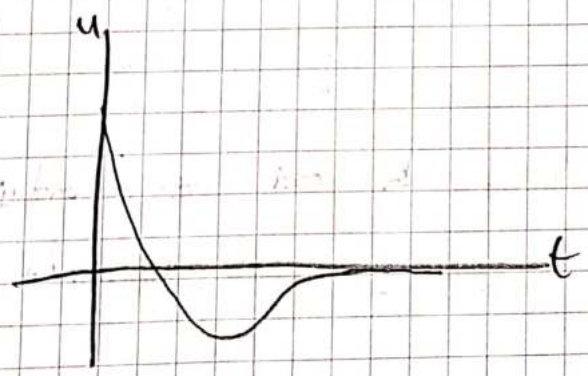
$u(t)=(A+Bt)e^{\frac{\gamma}{2m}t}$ (critical)

$u(t)=Ae^{r_1t}+Be^{r_2t}$ (over)

<u>critically damped $\gamma^2=4mk$</u>

$u(t)=(A+Bt)e^{-\frac{\gamma}{2m}t}$





$r=\dfrac{-\gamma\pm\sqrt{\gamma^2-4mk}}{2m}$

$r=\dfrac{-\gamma}{2m} \Rightarrow e^{\frac{-\gamma}{2m}t}$

$c_1e^{\frac{-\gamma}{2m}t}+c_2te^{\frac{-\gamma}{2m}t}$

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Ex: (free small damped vibration)

$$u'' + 0.125u' + u = 0 \qquad u(0) = 2, \quad u'(0) = 0$$

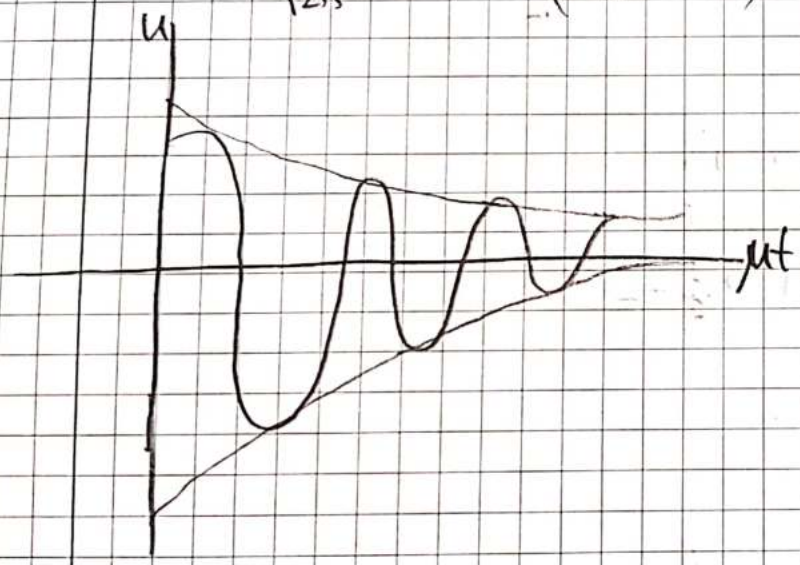
exercise: Show that the solution is

$$u(t) = e^{-\frac{t}{16}}\left(A\cos\frac{\sqrt{255}}{16}t + B\sin\frac{\sqrt{255}}{16}t\right)$$

Initial Value $\Rightarrow A = 2$ and $B = \frac{2}{\sqrt{255}}$

$$\Rightarrow u(t) = \frac{32}{\sqrt{255}} e^{-\frac{t}{16}} \cos\left(\frac{\sqrt{255}}{16}t - \delta\right) \quad \text{where } \delta \cong 0.06254$$

$$\left(\tan\delta = \frac{1}{\sqrt{255}}\right)$$



where $\mu = \frac{\sqrt{255}}{16}$

3.9 Forced Vibrations (nonhomogeneous)

Consider $mu'' + \gamma u' + ku = F_0\cos(\omega t)$ where $\omega > 0$, $F_0 \in \mathbb{R}$

If $\omega_0 = \sqrt{\frac{k}{m}}$ and $\omega \neq \omega_0$ and if $\gamma = 0$ then,

$$u(t) = c_1\cos(\omega_0 t) + c_2\sin(\omega_0 t) + \frac{F_0}{m(\omega_0^2 - \omega^2)}\cos(\omega t)$$ is the general solution to

$$\boxed{mu'' + ku = F_0\cos(\omega t)}$$

$\gamma = 0$, so $\gamma u' = 0$

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Ex: (Beats) $mu'' + ku = f_0\cos(\omega t)$

$u(0)=0$ and $u'(0)=0$

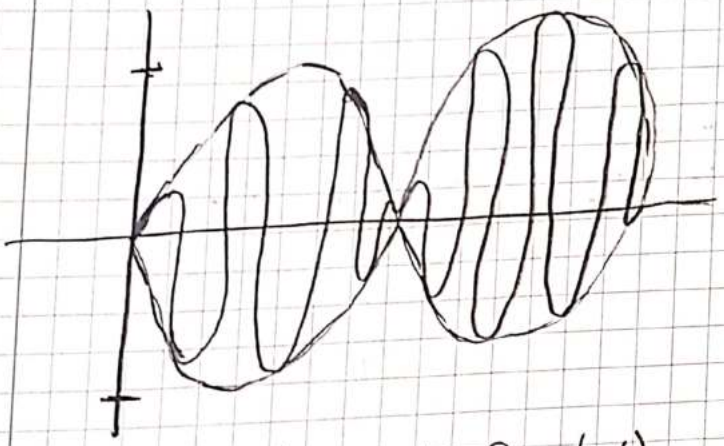
$(\omega \neq \omega_0)$ $u(t) = c_1\cos(\omega_0 t) + c_2\sin(\omega_0 t) + \dfrac{f_0}{m(\omega_0^2-\omega^2)}\cos(\omega t)$

IVs $\Rightarrow c_1 = -\dfrac{f_0}{m(\omega_0^2-\omega^2)}$ $\quad c_2 = 0$

$\Rightarrow u(t) = \dfrac{f_0}{m(\omega_0^2-\omega^2)}\left(\cos(\omega t) - \cos(\omega_0 t)\right)$

By trigonometric identities

$= \left(\dfrac{2f_0}{m(\omega_0^2-\omega^2)} \underbrace{\sin\dfrac{(\omega_0-\omega)t}{2}}\right) \underbrace{\sin\dfrac{(\omega_0+\omega)t}{2}}$

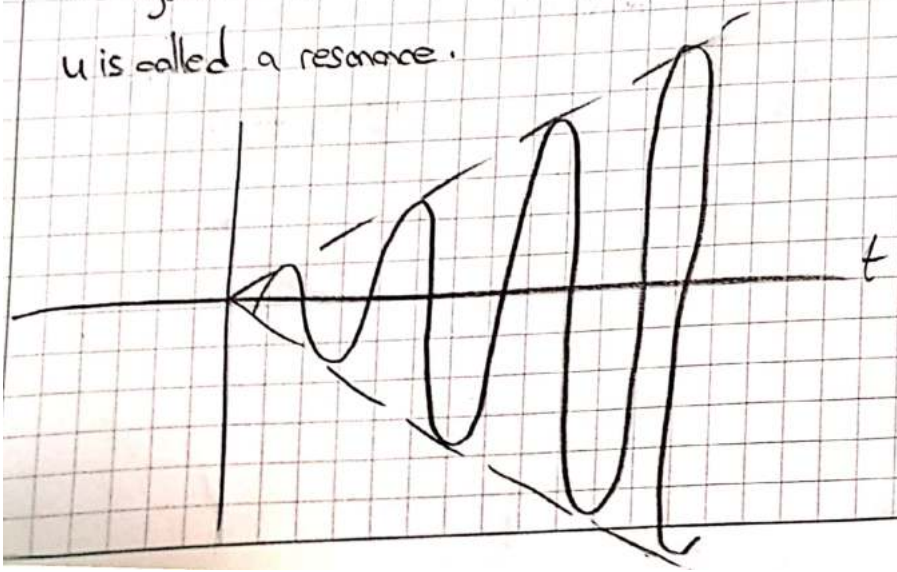


Ex: (Resonances) $mu'' + ku = f_0\cos(\omega t)$

$\omega = \sqrt{\dfrac{k}{m}} = \omega_0$ $\quad u(t) = c_1\cos(\omega_0 t) + c_2\sin(\omega_0 t) + \dfrac{f_0}{2m\omega_0} t\sin(\omega_0 t)$

$\lim_{t\to\infty} |u| = \infty$ means that

the system will vibrate indefinitely with increasing amplitude, and in this case, u is called a resonance.



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Last Time: Some applications to second order linear ODE's with constant coefficients

Vibrations: $mu'' + \gamma u' + ku = F$

Today: Sections 5.1, 5.2, 5.3

Power Series Solutions (CHAPTER 5) (Good for nonconstant coeffient) (polynomial)

From 120

$\sum_{n=0}^{\infty} a_n (x-x_0)^n$ is a power series around x_0

Facts: If $\sum_{n=0}^{\infty} |a_n| |x-x_0|^n$ is convergent then $\sum_{n=0}^{\infty} a_n (x-x_0)^n$ is convergent

Absolute convergence $\Rightarrow$ convergence

Ratio Test for convergence for $\sum_{n=0}^{\infty} a_n (x-x_0)^n$

$$\lim_{n\to\infty} \left| \frac{a_{n+1}(x-x_0)^{n+1}}{a_n (x-x_0)^n} \right| = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| |x-x_0| < 1 \text{ iff } |x-x_0| < \frac{1}{\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right|}$$

$\Downarrow$

$\sum_{n=0}^{\infty} a_n (x-x_0)^n$ is convergent absolutely

$\frac{1}{\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right|}$ is called the radius of convergence R, which is allowed to be 0 or ∞ or any positive number.

Ratio Test $\Rightarrow \sum_{n=0}^{\infty} a_n (x-x_0)^n$ is absolute convergent on $\underbrace{(x_0 - R, x + R)}_{\text{Interval of convergence}}$

End points are not told in ratio test, should be checked separetly.

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Ex: $\sum_{n=1}^{\infty} \frac{(2x+1)^n}{n2^n}$ Find the interval of convergence

solution: Ratio Test: $\lim_{n\to\infty} \left| \frac{(2x+1)^{n+1}}{(n+1)2^{n+1}} \Big/ \frac{(2x+1)^n}{n2^n} \right| = \underbrace{\lim_{n\to\infty} \left|\frac{n}{n+1}\right|}_{1} \cdot \frac{1}{2}|2x+1|$

$= \frac{|2x+1|}{2}$

The serie is convergent absolutely if $\frac{|2x+1|}{2} < 1 \Rightarrow |2x+1| < 2 \rightarrow -2 < 2x+1 < 2$

$-\frac{3}{2} < x < \frac{1}{2}$

The interval of convergence is $\left(-\frac{3}{2}, \frac{1}{2}\right)$ or $\left(-\frac{3}{2}, \frac{1}{2}\right]$ or $\left[-\frac{3}{2}, \frac{1}{2}\right)$ or $\left[-\frac{3}{2}, \frac{1}{2}\right]$.

We need to check the endpoints to see which one is the interval of convergence.

When $x = -\frac{3}{2}$, the series becomes $\sum_{n=1}^{\infty} \frac{\left(2\left(-\frac{3}{2}\right)+1\right)^n}{n2^n} = \sum_{n=1}^{\infty} \frac{(-2)^n}{n2^n}$

$= \sum_{n=0}^{\infty} \frac{(-1)^n}{n}$ is convergent by Alternating Series Test

When $x = \frac{1}{2}$, the series is $\sum_{n=1}^{\infty} \frac{\left(2\cdot\frac{1}{2}+1\right)^n}{n2^n} = \sum_{n=1}^{\infty} \frac{1}{n}$ is divergent by p-test.

The interval of convergence is $\left[-\frac{3}{2}, \frac{1}{2}\right)$. The radius of convergence is $R = 1$

$\left[-\frac{3}{2} - 1, -\frac{1}{2} + 1\right)$ $x_0 = -\frac{1}{2}$

Ex: $\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n!} x^{2n}$, $\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1}$ on $\mathbb{R}$

Ex: $e^x = \sum_{n=0}^{\infty} \frac{1}{n!} \cdot x^n$ Ex: $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$ for $|x| < 1$

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Taylor series of a function f around x_0:

$$\sum_{n=0}^{\infty} a_n (x-x_0)^n \quad \text{where} \quad a_n = \frac{f^{(n)}(x_0)}{n!} \implies \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n$$

Def: If $f(x)$ = its Taylor series at some point x, then we say f is analytic at x.

Ex: Find the power series solution to $y'+y=0$ around 0.

power of x
ie $x_0=0$

solution: $y(x) = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \dots$

$$y' = 0 + a_1 + 2a_2 x + 3a_3 x^2 + \dots$$
$$= \sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n$$

$$y'+y=0 \implies \sum_{n=1}^{\infty} n a_n x^{n-1} + \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\implies \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n + \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\implies \sum_{n=0}^{\infty} \left[(n+1)a_{n+1} + a_n\right] x^n = 0$$

$$(n+1)a_{n+1} + a_n = 0 \quad (n \geq 0) \quad \text{(A recurrence relation)}$$

$$a_1 = \frac{-a_0}{1} = -a_0, \quad a_2 = \frac{-a_1}{2} = \frac{+a_0}{2}, \quad a_3 = \frac{-a_2}{3} = \frac{-a_0}{3.2}$$

$$a_4 = \frac{-a_3}{4} = \frac{a_0}{4.3.2} = \frac{a_0}{4!}, \quad a_n = \frac{(-1)^n a_0}{n!}$$

$$\implies y = \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} a_0 x^n = \underbrace{a_0}_{\text{arbitrary constant}} \underbrace{\sum_{n=0}^{\infty} \frac{(-1)^n}{n!} x^n}_{\text{analytical solution}}$$

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Ratio Test $\Rightarrow$ Radius of convergence $R = \dfrac{1}{\lim\limits_{n\to\infty}\left|\frac{n!}{(n+1)!}\right|} = \infty$

The solution is good on $(-\infty, \infty)$ (we already know $y = ce^{-x}$ is the general solution)

Ex: $y'' + y = 0$ Find it around 0.

<u>solution:</u> $y = \sum\limits_{n=0}^{\infty} a_n x^n$ $\quad y'' = \sum\limits_{n=2}^{\infty} (n-1).n a_n x^{n-2} = \sum\limits_{n=0}^{\infty} (n+2)(n+1) a_{n+2}.x^n$

$$y' = \sum_{n=1}^{\infty} n a_n x^{n-1}$$

$$y'' + y = 0 \Rightarrow \sum_{n=0}^{\infty} \left((n+2)(n+1)a_{n+2} + a_n\right) x^n = 0$$

$$\Rightarrow (n+2)(n+1)a_{n+2} + a_n = 0 \quad (n \geq 0) \quad \text{(A recurrence relation)}$$

$$a_2 = \frac{-a_0}{2.1} = \frac{-a_0}{2}, \quad a_4 = \frac{-a_2}{4.3} = \frac{+a_0}{4.3.2.1}, \quad a_6 = \frac{-a_4}{6.5} = \frac{-a_0}{6!}$$

$$a_{2k} = \frac{(-1)^k}{(2k)!} . a_0$$

$$a_3 = \frac{-a_1}{3.2}, \quad a_5 = \frac{-a_3}{5.4} = \frac{a_1}{5.4.3.2}, \quad a_{2k+1} = \frac{(-1)^k . a_1}{(2k+1)!}$$

$$y = \sum_{k=0}^{\infty} \frac{(-1)^k a_0}{(2k)!} x^{2k} + \sum_{k=0}^{\infty} \frac{(-1)^k . a_1}{(2k+1)!} . x^{2k+1} = \sum_{n=0}^{\infty} a_n x^n$$

$$y = a_0 \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} x^{2k} + a_1 \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} x^{2k+1}$$

a constant — linear independent solution — a constant — linear independent solution

$$y = a_0 y_1 + a_1 y_2$$

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$$R = \frac{1}{\lim_{n\to\infty}\left|\frac{(2l+1)!}{(2l+3)!}\right|} = \infty$$ interval of convergence is $\mathbb{R}$

$$R = \frac{1}{\lim_{n\to\infty}\left|\frac{(2l)!}{(2l+2)!}\right|} = \infty$$

Consider $P(x)y'' + Q(x)y' + R(x)y = 0$

Assume P, Q and R are polynomials.

Define $p(x) = \frac{Q(x)}{P(x)}$ and $q(x) = \frac{R(x)}{P(x)}$

The equation becomes $y'' + p(x)y' + q(x)y = 0$

Let p and q be analytical on some interval

Definition ① If p and q are analytical at x_0 then x_0 is called an ordinary point of the ODE

② If p and q are not analytical at x_0 then x_0 is called a singular point of the ODE

③ If x_0 is a singular and $\lim_{x\to x_0}(x-x_0)p(x) = \alpha < \infty$

and $\lim_{x\to x_0}(x-x_0)^2 q(x) = \beta < \infty$

then x_0 is called a regular singular point of the ODE.

Theorem: $Py'' + Qy' + Ry = 0$ Assume x_0 is an ordinary point of this ODE.
(i.e $\frac{Q}{P}$ and $\frac{R}{P}$ are analytical at x_0)

Then, $y = \sum_{n=0}^{\infty} a_n (x-x_0)^n = a_0 y_1 + a_1 y_2$ is the general solution of the ODE with a radius of convergence at least as big as the smaller of the radius of convergence of $\frac{Q}{P}$ and $\frac{R}{P}$

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So, to find the radius of convergence for $y''+py'+qy=0$, check radius of convergence for p and q and take the smaller one.

Ex: $y''+y=0$ Find the interval of convergence.

Solution: $p=0$ and $q=1$ are analytical on $\mathbb{R}$ i.e radius of convergence are ∞.
So, $y''+y=0$ has a power series solution on $\mathbb{R}$.

Ex: $(1+x^2)y''+y=0$ Find the interval of convergence.

Solution: $y''+\frac{1}{1+x^2}y=0$

Note that $\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$ on $(-1,1)$, $R=1$

$p=0$ and $q=\frac{1}{1+x^2}$

one way to find R for $\frac{1}{1+x^2}$ is write it like $\frac{1}{1-(-x^2)}$

$=\sum_{n=0}^{\infty}(-x^2)^n = \sum_{n=0}^{\infty}(-1)^n(x^{2n})$ The interval of convergence is $(-1,1)$

for $|-x^2|<1$ for $|-x^2|<1 \Rightarrow |x|<1 \Rightarrow R=1$ around 0

Another way: $1+x^2=0 \Rightarrow x=\mp i$ $\overset{(zero)}{0}$ is 1 unit away from $\pm i$

so, $R=1$. On real line $0\mp 1=\mp 1$ so interval is $(-1,1)$.
Check the end points and see it's divergent there.

Ex: Find the interval of convergence of the solution of $(x^2-2x+2)y''+y'=0$ around 0 and around 1

solution: $y''+\frac{1}{x^2-2x+2}y'=0$ $p=\frac{1}{x^2-2x+2}$ and $q=0$

$R=?$ around 0, around 1 $R=\infty$ around 0, around 1 $-1<x-1$

$$p=\frac{1}{(x^2-2x+1)+1}=\frac{1}{1+(x-1)^2}=\frac{1}{1-[-(x-1)^2]}=\sum_{n=0}^{\infty}(-1)^n(x-1)^{2n}$$

for $|(-1)^n(x-1)^{2n}|<1$

The interval of convergence around 1 is $(0,2)$, the radius of convergence around 1 is 1. $\Leftarrow$ for $|x-1|<1$

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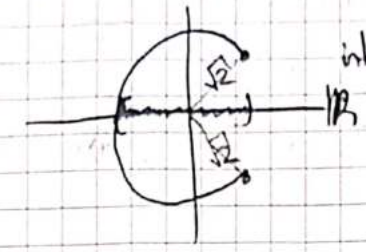
For around 0 we see that $x^2-2x+2=0 \Rightarrow x=1\mp i$ which are $\sqrt{2}$ units away from zero(0)

So, $R=\sqrt{2}$



interval of convergence is $(-\sqrt{2},\sqrt{2})$

Ex: (Airy's equation) $y''-xy=0$ Find the general solution and interval of convergence.

<u>solution:</u> $p=0$ and $q=-x$ which are analytical everywhere.

find the general solution the interval of convergence will be $\mathbb{R}$.

$$y=\sum_{n=0}^{\infty} a_n x^n \qquad y''=\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} = \sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n$$

$$=2.1a_2+\sum_{n=1}^{\infty}(n+2)(n+1)a_{n+2}x^n$$

$$xy=x\sum_{n=0}^{\infty}a_n x^n$$

$$=x(a_0+a_1x+a_2x^2+\dots)$$

$$=a_0x+a_1x^2+a_2x^3+\dots$$

$$=\sum_{n=0}^{\infty}a_n x^{n+1} = \sum_{n=1}^{\infty}a_{n-1}x^n$$

$\to 0$

$\to 1$

$$y''-xy=0 \Rightarrow 2a_2+\sum_{n=1}^{\infty}\left((n+2)(n+1)a_{n+2}-a_{n-1}\right)x^n=0$$

$$(n+2)(n+1)a_{n+2}-a_{n-1}=0 \quad (n\geq 1)$$

$$\text{and } a_2=0$$

$$3.2.a_3=a_0 \Rightarrow a_3=\frac{a_0}{6}, \quad a_6=\frac{a_0}{6.5.3.2}, \quad a_9=\frac{a_0}{9.8.6.5.3.2}$$

$$a_{3k}=\frac{a_0}{(3k)(3k-1)(3k-3)(3k-4)\dots 3.2}$$

$$\sum_{k=0}^{\infty}\frac{a_0}{(3k)(3k-1)(3k-3)(3k-4)\dots 3.2}\cdot x^{3k}$$

$$a_{3k+1}=\frac{a_1}{(3k+1)(3k)(3k-2)(3k-3)\dots 4.3}$$

$$a_{3k+2}=0$$

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The general solution is $y = a_0\left(1 + \frac{x^3}{2.3} + \frac{x^6}{2.3.5.6} + \cdots\right)$

$$+ a_1\left(x + \sum_{n=1}^{\infty} \frac{x^{3k+1}}{(3k+1)(3k)\cdots 4.3}\right) \quad \text{on } (-\infty, \infty)$$

exercise: Check this solution carefully.

Last Week: Power Series Solutions Around ordinary points

Today: Euler equations

Power series solutions for regular singular pts.

Euler equations

$$\boxed{x^2y'' + \alpha xy' + \beta y = 0}$$

Make the substitution $t = \ln|x|$, we get equation (indicial eqn.)

$$r(r-1) + \alpha r + \beta = 0$$

If roots of this equation are r_1 and r_2, then there are 3 cases:

Case 1: r_1, r_2 real and distinct

$$y(x) = c_1 x^{r_1} + c_2 x_2^{r_2}$$

e^{r_1} e^{r_2}

$r = 2$

e^{2t}

$r = 1+2i$
$1-2i$

$e^{1t}(\cos 2t) + e^{1t}(\sin 2t)$

Case 2: $r_1 = r_2$ real

$$y(x) = c_1 x^{r_1} + c_2 x^{r_1} \ln|x|$$

Case 3: $r = a \mp bi$ (complex)

$$y(x) = c_1 x^a \cos(b \ln|x|) + c_2 x^a \sin(b \ln|x|)$$

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Ex: $x^2y'' + \underset{\alpha}{5}xy' + \underset{\beta}{4}y = 0$ Solution indicial equation is $r(r-1)+5r+4=0$

$r^2+4r+4=0$

$(r+2)^2=0 \Rightarrow r_1=-2 \quad r_2=-2$ $\quad r(r-1)$

The general solution is $y(x)=c_1x^{-2}+c_2x^{-2}\ln|x| \quad x\neq 0$

Solutions for Regular Singular Points

$P(x)y''+Q(x)y'+R(x)y=0$

Remember the definition If $P(x_0)=0$ and $\lim_{x\to x_0}(x-x_0)\frac{Q}{P}=\alpha_0<\infty$ and

$\lim_{x\to x_0}(x-x_0)^2\frac{R}{P}=\beta_0<\infty$

then x_0 is called a regular singular point for this equation.

If this is the case then we have a indicial equation

$r(r-1)+\alpha_0 r+\beta_0=0$

Let r_1 and r_2 be two <u>real</u> roots of this equation s.t $r_1\geq r_2$

Then there is a solution of the ODE which is of the form $y_1(x)=|x-x_0|^{r_1}\cdot\left(1+\sum_{n=1}^{\infty}a_n(x-x_0)^n\right)$

$\left(\text{If } \underset{x_0=0}{x>0} \text{ then } y_1(x)=\sum_{n=0}^{\infty}a_n x^n\right)$ $\quad x^{r_1}\left(1+\sum_{n=1}^{\infty}a_n x^n\right)$

For the second solution, there are 3 cases: $\quad y_1(x)=x^{r_1}\sum_{n=0}^{\infty}a_n x^n$

Case 1: $r_1-r_2\notin\mathbb{Z}$

$y_2(x)=|x-x_0|^{r_2}\cdot\left(1+\sum_{n=1}^{\infty}b_n(x-x_0)^n\right)$ $\quad y_2(x)=x^{r_2}\sum_{n=0}^{\infty}a_n x^n$

Case 2: $r_1=r_2$

$y_2(x)=|x-x_0|^{r_1}\cdot\left(1+\sum_{n=1}^{\infty}b_n(x-x_0)^n\right)+\ln|x-x_0|\cdot y_1(x)$

$y_1=x^{r_1}\sum_{n=0}^{\infty}a_n x^n$

$y_2=y_1(x)\ln x+x^{r_1}\sum_{n=0}^{\infty}b_n x^n$

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Case 3: $r_1 - r_2$ is a positive integer

$$y_2(x) = |x-x_0|^{r_2}\left(1+\sum_{n=1}^{\infty} b_n (x-x_0)^n\right) + c\ln|x-x_0|\cdot y_1(x)$$

c may be zero

$$y_1(x) = x^{r_1}\sum_{n=0}^{\infty} a_n x^n$$

$$y_2(x) = c\,y_1(x)\ln x + x^{r_2}\sum_{n=0}^{\infty} b_n x^n$$

In each case, y_1 and y_2 are linearly independent and they are good on an interval with lenght at least smaller of the radii of convergence for $x\frac{Q}{P}$ and $x^2\frac{R}{P}$

Ex: $2xy'' + y' - y = 0$ around zero

$P = 2x \quad Q = 1 \quad R = -1$

$x^2y'' + \alpha x y' + \beta y = 0$

Solution:

$P(0) = 0$

$$\lim_{x\to 0} x\frac{1}{2x} = \frac{1}{2} = \alpha_0 < \infty$$

$$\lim_{x\to 0} x^2\frac{-1}{2x} = 0 < \infty$$

α $\quad x \cdot \frac{x}{x^2}$

0(zero) is a regular singular point

Radii of convergence for $x\frac{1}{2x}$ and $x^2\frac{-1}{2x}$ are both ∞

So, the solutions we're going to find will be good except $x=0$

For simplicity let's assume $x>0$,

the indicial equation is $r(r-1)+\frac{1}{2}r+0=0$

$\Rightarrow r(r-\frac{1}{2})=0 \quad r_1=\frac{1}{2} \quad r_2=0$ we see $r_1-r_2=\frac{1}{2}\notin\mathbb{Z}$

Then, There are two linearly independent solutions of the form $\sum_{n=0}^{\infty} a_n x^{n+r}$ where r is 1/2 or 0

We must find a_n for both of $r_1=\frac{1}{2}$ and $r_2=0$

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

$$y_1(x) = x^{r_1}\sum_{n=0}^{\infty} a_n x^n$$

$$y' = \sum_{n=0}^{\infty}(n+r)a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty}(n+r)(n+r-1)a_n x^{n+r-2}$$

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$$2xy''+y'-y=0 \Rightarrow \sum_{n=0}^{\infty} 2(n+r)(n+r-1)a_n x^{n+r-1} + \sum_{n=0}^{\infty}(n+r)a_n x^{n+r-1} - \sum_{n=0}^{\infty} a_n x^{n+r} = 0$$

$$\Rightarrow \sum_{n=-1}^{\infty} 2(n+r+1)(n+r)a_{n+1}x^{n+r} + \sum_{n=-1}^{\infty}(n+r+1)a_{n+1}x^{n+r} - \sum_{n=0}^{\infty} a_n x^{n+r} = 0$$

$$\Rightarrow 2r(r-1)a_0x^{r-1} + ra_0x^{r-1} + \sum_{n=0}^{\infty}\left([2(n+r-1)(n+r)+(n+r+1)]a_{n+1} - a_n\right)x^{n+r} = 0$$

$$\Rightarrow \left(2r(r-1)+r\right)a_0 = 0 \qquad (n+r+1)(2n+2r+1)a_{n+1} - a_n = 0 \qquad n \geq 0$$

We assume $a_0 \neq 0$ $\quad a_{n+1} = \dfrac{1}{(n+r+1)(2n+2r+1)}a_n$ for $n \geq 0$

For $r = \frac{1}{2}$ $\quad a_{n+1} = \dfrac{1}{(n+\frac{3}{2})(2n+2)}a_n \quad n \geq 0$

$$a_{n+1} = \frac{1}{(n+1)(2n+3)}a_n$$

$$a_1 = \frac{1}{1.3}a_0$$

$$a_2 = \frac{1}{2.5}a_1 = \frac{1}{1.2.3.5}a_0$$

$$a_3 = \frac{1}{3.7}a_2 = \frac{1}{(1.2.3)(3.5.7)}a_0$$

$$a_n = \frac{1}{n!(1.3.5\ldots(2n-1)}a_0 \Rightarrow y_1(x) = a_0\sum_{n=0}^{\infty}\frac{1}{n!(1.3.5\ldots(2n+1)}x^{n+\frac{1}{2}}$$

For $r=0$

$$a_{n+1} = \frac{1}{(n+1)(2n+1)}a_n \quad n \geq 0$$

$$\mathcal{L}(y'') = s^2\mathcal{L}(y) - sy(0) - y'(0)$$

$$\mathcal{L}(y') = s\mathcal{L}(y) - y(0)$$

$$\mathcal{L}(u_1 - u_2) + u_2 e^{t}$$

$$a_1 = a_0 \qquad a_2 = \frac{1}{2.3}a_1 = \frac{1}{2.3}a_0$$

$$a_3 = \frac{1}{3.5}a_2 = \frac{1}{2.3.3.5}a_1 = \frac{1}{(2.3)(3.5)}a_0$$

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$$a_4 = \frac{1}{(2.3.4)(3.5.7)} a_0$$

$$\Rightarrow a_n = \frac{1}{n!(1.3.5\ldots(2n-1))} a_0 \quad \Rightarrow y_2(x) = b_0 \sum_{n=0}^{\infty} \frac{1}{n!(1.3.5\ldots 2n-1)} x^n$$

The general solution is $y(x) = a_0 y_1(x) + b_0 y_2(x)$

good for any interval not including zero

Ex: $2(x-1)y'' + (4-x)y' - y = 0$ about $x_0 = 1$ $\longrightarrow$ answer is

$$y = a_0\left(\sum_{n=0}^{\infty} \frac{1}{1.3.5\ldots(2n+1)}(x-1)^n\right) + b_0\left(\sum_{n=1}^{\infty} \frac{1}{2.4.6\ldots 2n}(x-1)^{n-\frac{1}{2}}\right)$$

(exercise)

Ex: $x^2y'' + xy' + (x^2-1)y = 0 \qquad x_0 = 0$

solution: Check $x_0 = 0$ is a regular singular point

$x^2|_0 = 0 \quad \lim_{x\to 0} x\frac{x}{x^2} = 1 = \alpha_0 \qquad \lim_{x\to 0} x^2 \cdot \frac{x^2-1}{x^2} = -1 = \beta_0$

The indicial equation is $r(r-1) + r - 1 = 0$

$r^2 - 1 = 0 \quad r_1 = 1 \quad r_2 = -1 \quad r_1 - r_2 = 2 \in \mathbb{Z}$ [case 3]

for $r_1 = 1$ $\quad y_1(x) = a_0\left(x + \sum_{n=1}^{\infty} \frac{(-1)^n}{(2.4\ldots 2n)(4.6\ldots 2n+2)} x^{2n+1}\right) \quad x > 0$

the other solution is $y_2(x) = A y(x) \ln x + \sum_{n=0}^{\infty} b_n x^{n-1}$ this b_n to be determined

Bessel Equations (5.7 Chapter)

$x^2y'' + xy' + (x^2 - a^2)y = 0 \qquad a \in \mathbb{R}$

$x = 0$ is a regular singular point

$$\lim_{x\to 0} \underbrace{x\frac{x}{x^2}}_{x\frac{Q}{P}} = 1 \quad \text{and} \quad \lim_{x\to 0} \underbrace{x^2\frac{(x^2-a^2)}{x^2}}_{x^2\frac{R}{P}} = -a^2$$

for today for simplicity assume $x > 0$

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Ex $x^2y''+xy'+x^2y=0$ (Bessel eqn. with $a=0$)

exercise: See that the roots of indicial eqn. are 0 and 0. It is a double and find y_1 and y_2, the solutions of the equation to the end.

Ex $x^2y''+xy'+(x^2-1)y=0$ (Bessel eqn. with $a=1$)

$(x>0)$ From last time we know it has a solution $y_1(x)$ which is good for $x\neq 0$ $(x>0)$.

Now we are going to find this y_1. $x=0$ is a regular singular point.

$$\alpha_0=\lim_{x\to 0} x\frac{x}{x^2}=1 \quad \text{and} \quad \beta_0=\lim_{x\to 0} x^2\frac{x^2-1}{x^2}=-1$$

The initial equation is $r(r-1)+r-1=0$

$$\Rightarrow r^2-r+r-1=0$$

$$r^2-1=0 \qquad r_1=1 \text{ and } r_2=-1$$

We know $y_1(x)=\sum_{n=0}^{\infty} a_n x^{n+r_1}=\sum_{n=0}^{\infty} a_n x^{n+1}$

$$r_1-r_2=2\in\mathbb{Z} \Rightarrow y_2(x)=cy_1(x)\ln|x|+\sum_{n=0}^{\infty} b_n x^{n+r_2}$$

the set of integers

$$y_1=\sum_{n=0}^{\infty} a_n x^{n+1}$$

$$y_1'=\sum_{n=0}^{\infty}(n+1)a_n x^n \Rightarrow xy_1'=\sum_{n=0}^{\infty}(n+1)a_n x^{n+1}$$

$$y_1''=\sum_{n=0}^{\infty} n(n+1)a_n x^{n-1} \Rightarrow x^2y_2''=\sum_{n=0}^{\infty} n(n+1)a_n x^{n+1}$$

$$x^2y''+xy'+(x^2-1)y=0$$

$$\Rightarrow \sum_{n=0}^{\infty} n(n+1)a_n x^{n+1}+\sum_{n=0}^{\infty}(n+1)a_n x^{n+1}+\underbrace{\sum_{n=0}^{\infty} a_n x^{n+3}}_{x^2y_1}-\underbrace{\sum_{n=0}^{\infty} a_n x^{n+1}}_{y_1}=0$$

$$\underbrace{\qquad\qquad}_{(x^2-1)y_1}$$

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$$\Rightarrow \sum_{n=1}^{\infty} n(n+1)a_n x^{n+1} + \sum_{n=0}^{\infty}(n+1)a_n x^{n+1} + \sum_{n=2}^{\infty} a_{n-2} x^{n+1} - \sum_{n=0}^{\infty} a_n x^{n+1} = 0$$

$$\Rightarrow 1.2.a_1.x^2 + \cancel{1.a_0 x^1} + 2a_1 x^2 \cancel{- a_0 x} - a_1 x^2 + \sum_{n=2}^{\infty}\left(n(n+1)a_n x^{n+1} + (n+1)a_n x^{n+1} + a_{n-2}x^{n+1} - a_n x^{n+1}\right) = 0$$

$$\Rightarrow 3a_1 x^2 + \sum_{n=2}^{\infty}\left(\{n(n+1) + (n+1) - 1\}a_n + a_{n-2}\right)x^{n+1} = 0$$

All the coefficients of all power of x should be zero(0). Therefore,

$$\left.\begin{array}{l} 3a_1 = 0 \Rightarrow a_1 = 0 \\ n(n+2)a_n + a_{n-2} = 0 \end{array}\right\} a_{2k+2} = 0 \quad \forall k \geq 0$$

For even $n \geq 2$ $\quad a_n = \dfrac{-a_{n-2}}{n(n+2)}$

For simplicity assume $a_0 = 1$ Then $a_2 = \dfrac{-1}{2.4}$ $\quad a_4 = \dfrac{-a_2}{4.6} = \dfrac{(-1)^2}{(2.4)(4.6)}$

$$a_6 = \frac{-a_4}{6.8} = \frac{(-1)^3}{(2.4.6)(4.6.8)} \Rightarrow a_{2k} = \frac{(-1)^k}{2^{2k}k!(k+1)!}$$

$$\Rightarrow y_1(x) = \sum_{n=0}^{\infty} \frac{(-1)^k}{2^{2k}k!(k+1)!} x^{2k+1} \quad \text{for } x > 0$$

$$= \sum_{n=0}^{\infty} a_n x^{n+1}$$

The second solution: $y_2(x) = c y_1(x) \ln x + \sum_{n=0}^{\infty} b_n x^{n-1}$ $\quad \boxed{r_2 = -1}$

constant $\longrightarrow$ to be found

For $y_2(x)$:

$$y_2 = c y_1 \ln x + \sum_{n=0}^{\infty} b_n x^{n-1}$$

$$y_2' = c y_1' \ln x + c y_1 \frac{1}{x} + \sum_{n=0}^{\infty}(n-1)b_n x^{n-2}$$

$$y_2'' = c y_1'' \ln x + c y_1' \frac{1}{x} + c y_1' \frac{1}{x} - c y_1 \frac{1}{x^2} + \sum_{n=0}^{\infty}(n-2)(n-1)b_n x^{n-3}$$

$$x^2 y'' + x y' + (x^2 - 1)y = 0 \Rightarrow$$

$$c \ln x \underbrace{(x^2 y_1'' + x y_1' + (x^2-1)y)}_{=0} + c x^2 y_1' \frac{1}{x} + c x^2 y_1' \frac{1}{x} - c x^2 y_1 \frac{1}{x^2} + c x y_1 \frac{1}{x} + \ldots$$

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$$\sum_{n=0}^{\infty}(n-2)(n-1)b_n x^{n-1} + \sum_{n=0}^{\infty}(n-1)b_n x^{n-1} + \sum_{n=0}^{\infty} b_n x^{n+1} - \sum_{n=0}^{\infty} b_n x^{n-1} = 0$$

$$\Rightarrow 2cxy_1' + \sum_{n=0}^{\infty}\left(n(n-2)b_n + b_{n-2}\right)x^{n-1} = 0$$

$$y_1 = \sum_{n=0}^{\infty} a_n x^{n+1} \Rightarrow 2cxy_1' = \sum_{n=0}^{\infty} 2c(n+1)a_n x^{n+1}$$

$$\Rightarrow 2c(n+1)a_n + (n+2)n\,b_{n+2} + b_n = 0 \quad n \geq 0$$

$$\left.\begin{aligned} a_0 &= 1 \\ a_{2k+1} &= 0 \\ a_{2k} &= \frac{(-1)^k}{2^{2k}\,k!\,(k+1)!} \end{aligned}\right\}$$ Find b_n and write the solution ...

CHAPTER 6 Laplace Transforms

<u>Definition:</u> The Laplace transform of $f(t)$ is defined as

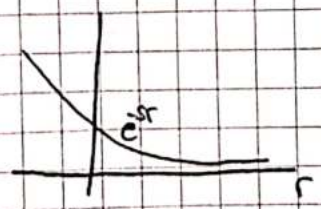
$$\mathcal{L}\{f(t)\} = F(s) = \int_0^{\infty} e^{-st} f(t)\,dt \qquad (s>0)$$

provided that the integral is convergent.

Ex: $f(t) = 1$ $\quad \mathcal{L}\{1\} = \int_0^{\infty} e^{-st}\,1\,dt = \int_0^{\infty} e^{-st}\,dt \qquad (s>0)$

By definition $\int_0^{\infty} e^{-st}\,dt = \lim_{r\to\infty}\int_0^{r} e^{-st}\,dt = \lim_{r\to\infty} \left.\frac{-e^{-st}}{s}\right|_{t=0}^{t=r}$

$$= \lim_{r\to\infty} \frac{-e^{-sr}}{s} + \frac{e^{-s\cdot 0}}{s} = 0 + \frac{1}{s} = \frac{1}{s}$$



$$\Rightarrow F(s) = \frac{1}{s} \text{ for } f(t) = 1 \quad (s>0)$$

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Chapter 6 Laplace Transform (Notation: $\mathcal{L}\{f(t)\} = L(f(t)) = F(s) = Y(s)$)

Definition

$$F(s) = \int_0^\infty e^{-st} f(t)\,dt \qquad (s>a \text{ for some } a)$$

when the integral is convergent

Theorem: Suppose that (A, M, K, a numbers)

1- f is piecewise continuous on $[0,A]$ for some $A>0$

2- $|f(t)| \leq Ke^{at}$ for $t \geq M > 0$ and for some $K>0$ $a \geq 0$

Then the Laplace transform of f exists for $s>a$

Ex: $f(t)=1 \Rightarrow F(s)=\frac{1}{s}$ $s>0$

Ex: $f(t)=e^{at}$

$$F(s)=\int_0^\infty e^{-st}\cdot e^{at}\,dt = \int_0^\infty e^{(a-s)t}\,dt = \lim_{r\to\infty}\int_0^r e^{(a-s)t}\,dt = \frac{1}{s-a} \text{ for } s>a$$

Ex: $f(t)=t$, $F(s)=\int_0^\infty e^{-st}t\,dt$

$u=t \quad du=dt$

$dv=e^{-st}dt \quad v=\frac{e^{-st}}{-s}$

$$F(s)=\lim_{r\to\infty}\left(\frac{t\cdot e^{-st}}{-s}\Big|_0^r - \int_0^r \frac{e^{-st}}{-s}\,dt\right) \qquad \lim_{r\to\infty}\left(\underbrace{\frac{r\cdot e^{-sr}}{-s}}_{=0 \text{ by L'H}} - 0 + \frac{1}{s}\underbrace{\int_0^r e^{-st}\,dt}_{\frac{1}{s}}\right.$$

$$= \frac{1}{s^2} \text{ for } s>0$$

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Formula:

Ex: $\mathcal{L}(t^n) = \frac{n!}{s^{n+1}}$ $s>0$

Ex: $\mathcal{L}\{\sin at\} = \frac{a}{s^2+a^2}$ $s>0$

Ex: $f(t) = \sin 2t$

$F(s) = \int_0^\infty e^{-st}\sin 2t\,dt$ (Integration by parts with $u = e^{-st}$, $du = -se^{-st}dt$, $dv = \sin 2t\,dt$, $v = \frac{-\cos 2t}{2}$)

$$e^{-st}\cdot\frac{(-\cos 2t)}{2}\Big|_0^\infty - \int_0^\infty \frac{-\cos 2t}{2}\cdot(-se^{-st})\,dt$$

$u = e^{-st}$

$dv = \cos 2t\,dt$

$$= \underbrace{-\frac{e^{-st}\cos 2t}{2}\Big|_0^\infty}_{=\frac{1}{2}} - \underbrace{\frac{s}{2}\cdot\frac{\sin 2t}{2}e^{-st}\Big|_0^\infty}_{=0} - \frac{s}{2}\int\frac{s}{2}e^{-st}\sin 2t\,dt$$

$$\Rightarrow \int_0^\infty e^{-st}\sin 2t\,dt = \frac{1}{2} - \frac{s^2}{4}\int_0^\infty e^{-st}\sin 2t\,dt$$

$$\Rightarrow F(s) = \frac{1}{2} - \frac{s^2}{4}F(s) \Rightarrow F(s) = \frac{2}{s^2+4} \quad s>0$$

Ex: $f(t) = \cos at$ $s>0$

$F(s) = \frac{s}{s^2+a^2}$

Fact: $\mathcal{L}(c_1f_1(t) + c_2f_2(t)) = c_1\mathcal{L}(f_1(t)) + c_2\mathcal{L}(f_2(t))$

Theorem: Assume $\mathcal{L}(f)$ exists and f' is piecewise continuous on $[0, t]$ for some $A>0$

Then $\mathcal{L}(f'(t)) = s\mathcal{L}(f(t)) - \underbrace{f(0)}_{\text{constant}}$

a function of s

proof by at by parts with $u = e^{-st}$

$dv = f'dt$

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Similarly, $\mathcal{L}(f'') = s^2\mathcal{L}(f) - s\cdot f(0) - f'(0)$

(if everything is nice) $\mathcal{L}(f''') = s^3\mathcal{L}(f) - s^2 f(0) - s f'(0) - f''(0)$

$$\mathcal{L}(f^{(n)}) = s^n\mathcal{L}(f) - s^{n-1}f(0) - \dots - s f^{(n-2)}(0) - f^{(n-1)}(0)$$

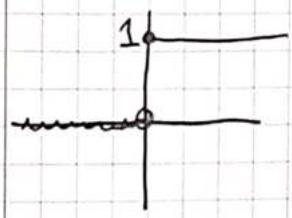
Ex: If $F(s) = \mathcal{L}(f(t))$ then, $\mathcal{L}(e^{at}f(t)) = F(s-a)$

proof: $\mathcal{L}(e^{at}f(t)) = \int_0^\infty e^{-st}e^{at}f(t)\,dt = \int_0^\infty e^{-(s-a)t}f(t)\,dt \underset{s=s-a}{=} \int_{-a}^\infty e^{-st}f(t)\,dt$

$= F(s) = F(s-a)$

Step Function

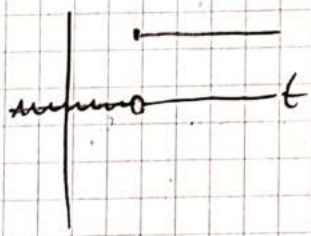
Ex: (Heaviside function, $u_0(t)$)



$$u_0(t) = \begin{cases} 0 & t<0 \\ 1 & t\geq 0 \end{cases}$$

Ex: $u_c(t) = \begin{cases} 0 & t<c \\ 1 & t\geq c \end{cases}$

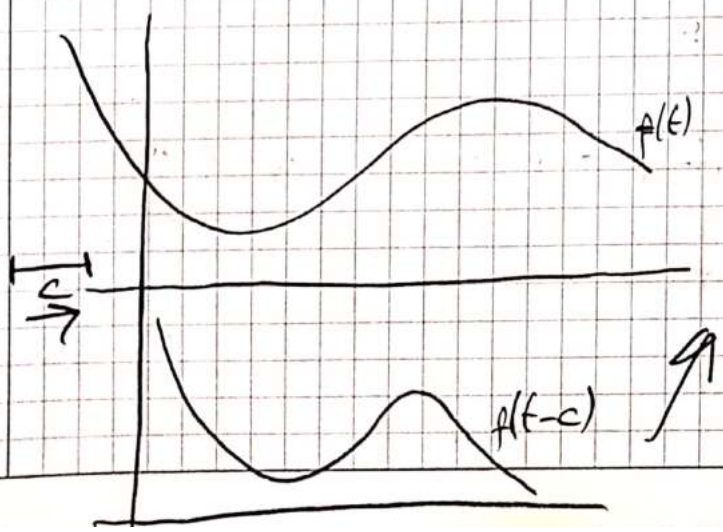
Let $c \in \mathbb{R}$

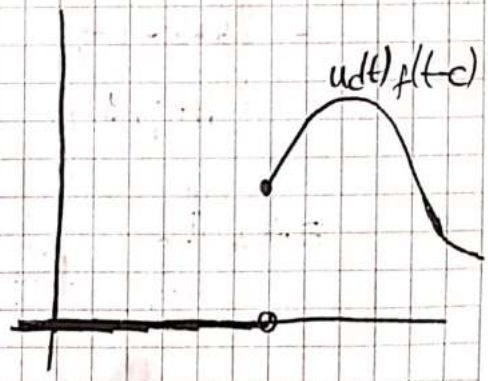


Ex: $f(t) = u_c(t)$ then $F(s) = ?$

$$F(s) = \int_0^\infty e^{-st}u_c(t)\,dt$$

$$= \int_c^\infty e^{-st}\,dt = \lim_{r\to\infty}\int_c^r e^{-st}\,dt = \lim_{r\to\infty}\frac{e^{-st}}{-s}\Big|_c^r = \frac{e^{-cs}}{s} \qquad s>0$$





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Ex $\mathcal{L}(u_c(t)f(t-c)) = e^{-cs}\mathcal{L}(f(t))$

proof: $\int_0^\infty e^{-st}u_c(t)f(t-c)dt = \int_c^\infty e^{-st}f(t-c)dt$

$$= \int_0^\infty e^{-s(t+c)}f(t)dt = \int_0^\infty e^{-st}e^{-sc}f(t)dt = e^{-sc}\int_0^\infty e^{-st}f(t)dt = e^{-sc}\mathcal{L}(f(t))$$

page 304 of the textbook 7th ed. has a useful list of Laplace transforms we are going to use

Ex $F(s) = \frac{4}{(s-1)^3}$ what is $f(t) = ?$

We learned $\mathcal{L}(t^2) = \frac{2}{s^3}$ $\Rightarrow$ $\mathcal{L}(e^{at}f) = F(s-a)$

$\mathcal{L}(e^{t}f) = F(s-1)$

$F(s) = \frac{4}{(s-1)^3} = 2\frac{2}{(s-1)^3}$ $\quad \mathcal{L}(e^t t^2) = \frac{2}{(s-1)^3}$

$\mathcal{L}(2f) = 2\mathcal{L}(f)$

$\Rightarrow f(t) = 2e^t t^2$

Ex $F(s) = \frac{2s-3}{s^2+2s+10}$ Let's find $f(t) = \mathcal{L}^{-1}(F(s))$

$F(s) = \frac{2s-3}{s^2+2s+1+9} = \frac{2s-3}{(s+1)^2+9}$ $\quad$ we learned that $\mathcal{L}(\sin 3t) = \frac{3}{s^2+9}$

$\mathcal{L}(\cos 3t) = \frac{s}{s^2+9}$

$\mathcal{L}(e^{-t}f) = F(s+1)$

$$= \frac{2s+2-2-3}{(s+1)^2+9} = \frac{2(s+1)}{(s+1)^2+9} - \frac{5}{(s+1)^2+9}$$

$$= 2\frac{s+1}{(s+1)^2+9} - \frac{5}{3}\cdot\frac{3}{(s+1)^2+9} \Rightarrow f(t) = 2e^{-t}\cos 3t - \frac{5}{3}e^{-t}\sin 3t$$

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Ex: $f(t) = (t-3)u_2(t) - (t-2)\cdot u_3(t)$ We learned $\mathcal{L}(u_2(t)) = \frac{e^{-2s}}{s}$

$\mathcal{L}(u_2(t) f(t-2)) = e^{-2s} \cdot F(s)$

$= (t-2)u_2(t) - u_2(t) - (t-3)u_3(t) - u_3(t)$ $t-3 = t-2-1$

$\mathcal{L}(t) = \frac{1}{s^2}$

$F(s) = e^{-2s}\frac{1}{s^2} - \frac{e^{-2s}}{s} - \frac{e^{-3s}}{s^2} - \frac{e^{-3s}}{s}$ $\mathcal{L}(1) = \frac{1}{s}$

Ex: $f(t) = \begin{cases} 0 & t<1 \\ t^2-2t+2 & t \geq 1 \end{cases}$ we learned $\mathcal{L}(u_1(t) f(t-1)) = e^{-s} F(s)$

$= \begin{cases} 0 & t<1 \\ (t-1)^2+1 & t \geq 1 \end{cases}$ $\mathcal{L}(t^2+1) = \mathcal{L}(t^2) + \mathcal{L}(1) = \frac{2}{s^3} + \frac{1}{s}$

$\Rightarrow f(t) = u_1(t)\left((t-1)^2+1\right) \Rightarrow F(s) = e^{-s}\left(\frac{2}{s^3} + \frac{1}{s}\right)$

Ex: $F(s) = \frac{e^2 e^{-4s}}{2s-1} = \frac{e^2}{2} \frac{e^{-4s}}{s-\frac{1}{2}}$ We learned $\mathcal{L}(u_4(t) f(t-4)) = e^{-4s}\mathcal{L}(f(t))$

$\mathcal{L}(e^{+\frac{1}{2}t}) = \frac{1}{s-\frac{1}{2}}$

$= \frac{e^2}{2} u_4(t) e^{\frac{1}{2}(t-4)} = \frac{\cancel{e^2}}{2} u_4(t) \cdot e^{\frac{t}{2}} \cdot \cancel{e^{-2}} = \frac{1}{2} u_4(t) e^{\frac{t}{2}}$

<u>Last Time:</u> Laplace transforms (Chapter 6)

Def: $\mathcal{L}f(t) = \int_0^\infty e^{-st} f(t)\,dt$

$f(t) = \mathcal{L}^{-1}(F(s))$

ex: $\mathcal{L}(t^n) = \frac{n!}{s^{n+1}}$ $s>0$ Ex: $\mathcal{L}(\sin at) = \frac{a}{s^2+a^2}$ $s>0$

ex: $\mathcal{L}(u_c(t)) = \frac{e^{-cs}}{s}$ $s>0$ Ex: $\mathcal{L}(u_c(t) f(t-c)) = e^{-cs} F(s)$

ex: $\mathcal{L}(e^{ct} f(t)) = F(s-c)$ ex: $\mathcal{L}(f'') = s^2\mathcal{L}(f) - sf(0) - f'(0)$

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Initial Value Problems (solutions with Laplace Transform)

Ex: $y''+y=\cos 2t$ Find a solution using Laplace

$y(0)=2$, $y'(0)=1$

solution: If a solution y exists then

$\cos t$ $\quad \frac{s}{s^2+2^2}$ $\quad \frac{d}{ds}$ $\quad \frac{d}{ds}\mathcal{L}(\cos t)$

$$\mathcal{L}(y'') = s^2 Y(s) - s y(0) - y'(0) \quad \text{where } Y(s) = \mathcal{L}(y)$$

Laplace

$$= s^2 Y(s) - 2s - 1$$

$$\mathcal{L}(\cos 2t) = \frac{s}{s^2+4}$$

$$\Rightarrow s^2 Y(s) - 2s - 1 + Y(s) = \frac{s}{s^2+4} \Rightarrow s^2 Y(s) = \frac{s}{s^2+4} + 2s + 1$$

$$\Rightarrow (s^2+1)Y(s) = \frac{s}{s^2+4} + 2s + 1 \Rightarrow (s^2+1)Y(s) = \frac{s + 2s^3 + s^2 + 8s + 4}{s^2+4}$$

$$\Rightarrow Y(s) = \frac{2s^3 + s^2 + 9s + 4}{(s^2+1)(s^2+4)} \Rightarrow y(t) = \mathcal{L}^{-1}(Y(s))$$

Remember partial functions

$$\frac{2s^3 + s^2 + 9s + 4}{(s^2+1)(s^2+4)} = \frac{As+B}{s^2+1} + \frac{Cs+D}{s^2+4}$$

$$\Rightarrow (As+B)(s^2+4) + (Cs+D)(s^2+1) = 2s^3 + s^2 + 9s + 4$$

$$\Rightarrow \cancel{As^3} + 4\cancel{As} + \cancel{Bs^2} + 4B + \cancel{Cs^3} + \cancel{Cs} + \cancel{Ds^2} + D = 2s^3 + s^2 + 9s + 4$$

$$(A+C)s^3 + (B+D)s^2 + (4A+C)s + 4B + D = 2s^3 + s^2 + 9s + 4$$

$A+C=2$ $\quad A=\frac{7}{3}$, $C=\frac{-1}{3}$ $\quad B=1$ $\quad D=0$

$B+D=1$

$4A+C=9$

$4B+D=4$

$$Y(s) = \frac{\frac{7}{3}s + 1}{s^2+1} + \frac{-\frac{1}{3}s}{s^2+4} = \frac{7}{3}\frac{s}{s^2+1} + \frac{1}{s^2+1} - \frac{1}{3}\frac{s}{s^2+4}$$

$$y(t) = \frac{7}{3}\cos t + \sin t - \frac{1}{3}\cos 2t$$

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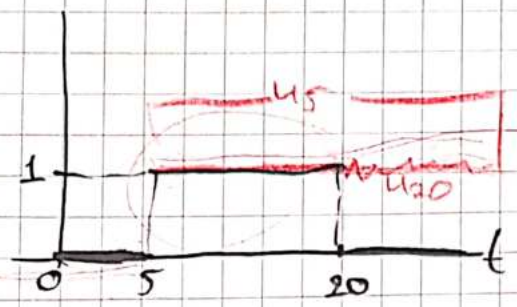
Ex: $2y''+y'+2y=g(t)$ where $g(t)=\begin{cases}1 & 5\leq t\leq 20\\ 0 & 0\leq t\leq 5 \text{ and } t\geq 20\end{cases}$

$y(0)=0 \quad y'(0)=0$



Consider $u_5(t)$

$u_{20}(t)$ $\quad g(t)=u_5-u_{20}$

$$\mathcal{L}(y'')=s^2Y(s)-sy(0)-y'(0) = s^2Y(s)$$

$$\mathcal{L}(y')=sY(s)-y(0) = sY(s)$$

$$\mathcal{L}(g(t))=\mathcal{L}(u_5)-\mathcal{L}(u_{20})$$

$$=\frac{e^{-5s}}{s}-\frac{e^{-20s}}{s}$$

$$\Rightarrow 2s^2Y(s)+sY(s)+2Y(s)=\frac{e^{-5s}-e^{-20s}}{s}$$

$$Y(s)(2s^2+s+2)=\frac{e^{-5s}-e^{-20s}}{s}$$

$$Y(s)=e^{-5s}-e^{-20s}\cdot\underbrace{\frac{1}{s(2s^2+s+2)}}_{H(s)}$$

Then, $Y(s)=(e^{-5s}-e^{-20s})\cdot H(s)$ because $\mathcal{L}(u_c h(t-c))=e^{-cs}H(s)$

$$y(t)=u_5h(t-5)-u_{20}h(t-20)$$

$H(s)=\dfrac{1}{s(2s^2+s+2)}$ $\quad h(t)=?$ where $\mathcal{L}(h(t))=H(s)$

$$H(s)=\frac{1}{s2\left(s^2+\frac{s}{2}+1\right)}=\frac{1}{2s\left(s^2+\frac{s}{2}+\frac{1}{16}+\frac{15}{16}\right)}=\frac{1}{2s\left(\left(s+\frac{1}{4}\right)^2+\frac{15}{16}\right)}$$

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partial fractions: $H(s) = \frac{1}{2s} - \frac{s+\frac{1}{2}}{2\left(s+\frac{1}{4}\right)^2+\frac{15}{16}} \cdot \frac{\left(s+\frac{1}{4}\right)+\frac{1}{4}}{\left(s+\frac{1}{4}\right)^2+\frac{15}{16}}$

$$\Rightarrow H(s) = \frac{1}{2s} - \frac{s+\frac{1}{2}}{2\left[\left(s+\frac{1}{4}\right)^2+\frac{15}{16}\right]} - \frac{1}{4}\,\frac{1/4}{\left(s+\frac{1}{4}\right)^2+\frac{15}{16}} \cdot \frac{\frac{\sqrt{15}}{4}}{\frac{\sqrt{15}}{4}}$$

$$h(t) = \frac{1}{2} - \frac{1}{2}e^{-\frac{t}{4}}\cos\frac{\sqrt{15}}{4}t - \frac{1}{\sqrt{15}}e^{-\frac{t}{4}}\sin\frac{\sqrt{15}}{4}t$$

$\mathcal{L}^{-1}\left(\frac{1}{s}\right)=1$

$\mathcal{L}\left(e^{at}f(t)\right)=F(s-a)$

$\mathcal{L}(\sin at)=\frac{a}{s^2+a^2}$

$\mathcal{L}(\cos at)=\frac{s}{s^2+a^2}$

$\mathcal{L}\left(e^{ct}\cos at\right)=\frac{s-c}{(s-c)^2+a^2}$

So, $y(t) = u_5 \cdot \frac{1}{2}e^{-\frac{(t-5)}{4}}\cos\frac{\sqrt{15}}{4}(t-5) - \frac{1}{\sqrt{15}}e^{-\frac{(t-5)}{4}}\sin\frac{\sqrt{15}}{4}(t-5) + \frac{1}{2}u_{20}e^{-\frac{(t-20)}{4}}\cdot\cos\frac{\sqrt{15}}{4}(t-20)$

$+ \frac{1}{\sqrt{15}}u_{20}e^{-\frac{(t-20)}{4}}\sin\frac{\sqrt{15}}{4}(t-20)$

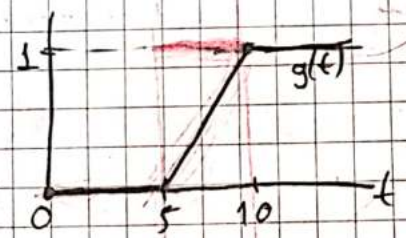
$+ \frac{1}{2}(u_5-u_{20})$

ex: $y''+4y=g(t)$ where $g(t)=\begin{cases} 0 & 0\le t\le 5 \\ \frac{t-5}{5} & 5\le t<10 \\ 1 & t\ge 10 \end{cases}$

$y(0)=0 \quad y'(0)=0$

Write the Laplace transform of this IVP.

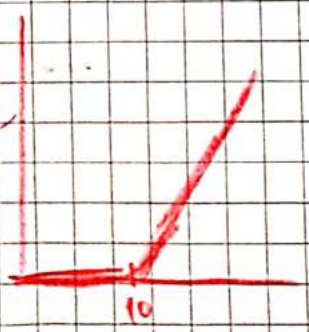
Solution: What is $\mathcal{L}(g(t))=?$ Write it in terms of step functions.



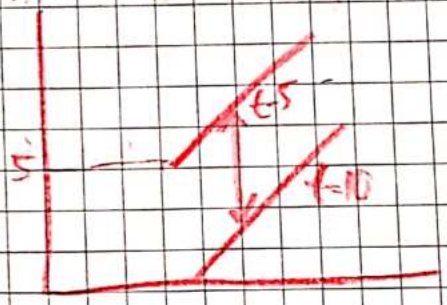
$u_5(t)(t-5)$

$u_{10}(t)$

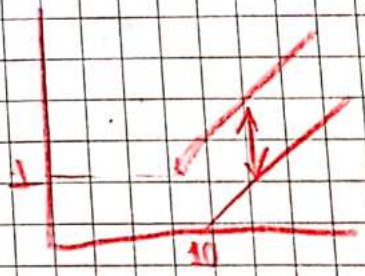
$u_{10}(t)(t-10)$



$(t-5)-(t-10) \quad t\ge 10$



$\frac{(t-5)-(t-10)}{5} \quad t\ge 0$



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$$g(t) = \frac{u_5(t)(t-5) - u_{10}(t-10)}{5}$$

$$\mathcal{L}(g(t)) = \frac{1}{5}\mathcal{L}(u_5(t)(t-5)) - \frac{1}{5}\mathcal{L}(u_{10}(t)(t-10))$$

$$= \frac{1}{5}\frac{e^{-5s}}{s} - \frac{1}{5}\frac{e^{-10s}}{s}$$

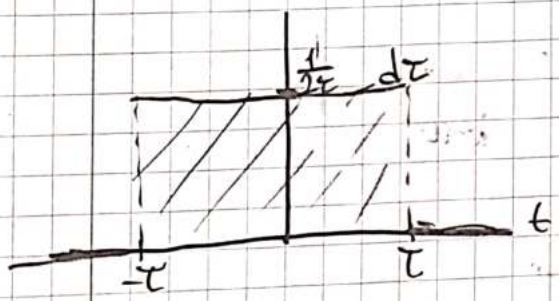
$$\mathcal{L}(y'') = s^2 Y(s)$$

$$\mathcal{L}(u_c(t) f(t-c)) = e^{-cs} F(s)$$

$$s^2 Y(s) + 4Y(s) = \frac{1}{5s}\left(e^{-5s} - e^{-10s}\right) \Rightarrow Y(s) = \left(e^{-5s} - e^{-10s}\right)\frac{1}{5s(s^2+4)}$$

$\Rightarrow y(t) = \mathcal{L}^{-1}(Y(s))$ exercise: find $y(t)$

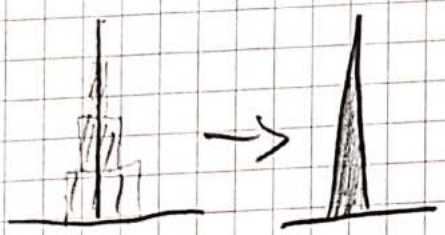
Impulse Function



$$\text{Area} = \int_{-\infty}^{\infty} d_\tau(t)\,dt = \int_{-\tau}^{\tau} d_\tau(t)\,dt$$

Area = 1

$\lim_{\tau \to 0} d_\tau(t)$ is not a function but it is called impulse



$$\lim_{\tau \to 0} \int_{-\infty}^{\infty} d_\tau(t)\,dt = 1$$

So the integral of the impulse is 1 even though it is not a function. The impulse function $\lim_{\tau \to 0} d_\tau$ is denoted $\delta(t)$ (Dirac's delta) $\delta(t) = 0$ for $t \neq 0$ and $\int_{-\infty}^{\infty} \delta(t)\,dt = 1$

And $\delta(t-t_0) = 0$ for $t \neq t_0$ and $\int_{-\infty}^{\infty} \delta(t-t_0)\,dt = 1$

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What is $\mathcal{L}(\delta(t-t_0))$

$$\mathcal{L}(\delta(t-t_0)) = \int_0^\infty e^{-st}\delta(t-t_0)\,dt = \lim_{\tau\to 0}\int_0^\infty e^{-st}\,d_\tau(t-t_0)\,dt = e^{-st_0}$$

$$\mathcal{L}(\delta(t-t_0)) = e^{-t_0 s} \quad \text{and} \quad \mathcal{L}(\delta(t)) = 1$$

However, by the MVT for integrals we have $\int_{-\infty}^{\infty}\delta(t-t_0)f(t)\,dt = f(t_0)$

Ex. $2y''+y'+2y=\delta(t-5)$ $\quad y(0)=0 \quad y'(0)=0$

<u>solutions:</u> $\mathcal{L}(\delta(t-5)) = e^{-5s}$

$$2s^2Y(s)+sY(s)+2Y(s) = e^{-5s} \Rightarrow Y(s) = \frac{e^{-5s}}{2s^2+s+2}$$

$$Y(s) = \frac{e^{-5s}}{2}\cdot\frac{1}{(s+\frac{1}{4})^2+\frac{15}{16}} \Rightarrow y(t) = \frac{4}{\sqrt{15}}u_5(t)e^{-\frac{(t-5)}{4}}\cdot\sin\frac{\sqrt{15}}{4}(t-5)$$

Ex. $2y''+y'+4y=\delta(t-\frac{\pi}{6})\sin t$ $\quad y(0)=0 \quad y'(0)=0$

<u>Last time:</u> Laplace transforms

ex: $\mathcal{L}(\delta(t)) = 1$

ex: $\mathcal{L}(\delta(t-c)) = e^{-cs}$

ex: $\mathcal{L}(u_c(t)f(t-c)) = e^{-cs}\mathcal{L}(f(t)) = e^{-cs}F(s)$ where $\mathcal{L}(f(t)) = F(s)$

ex: $\mathcal{L}(e^{ct}f(t)) = F(s-c)$

ex: $\mathcal{L}(e^{at}\cos bt) = \frac{s-a}{(s-a)^2+b^2}$

ex: $\int_{-\infty}^{\infty}\delta(t-c)f(t)\,dt = f(c)$

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ex: $\mathcal{L}(e^{at}\sin bt) = \frac{b}{(s-a)^2+b^2}$

Ex: $y''+4y=\delta(t)$ $\quad y(0)=0 \quad y'(0)=0$

solution:

$\mathcal{L}(y'') = s^2Y(s) - s\overset{\to 0}{y(0)} - \overset{\to 0}{y'(0)} = s^2Y(s)$

$\mathcal{L}(\delta(t)) = 1$

$s^2Y(s)+4Y(s)=1 \Rightarrow Y(s) = \frac{1}{s^2+4} = \frac{1}{2}\cdot\frac{2}{s^2+4} \Rightarrow \frac{1}{2}\sin 2t = y(t)$

Ex: $y''+4y=\delta(t-\frac{\pi}{6})\sin t$ $\quad y(0)=0 \quad y'(0)=0$

$\boxed{\mathcal{L}(u_c(t)f(t-c)) = e^{-cs}Y(s)}$

solution: $\mathcal{L}(y'') = s^2Y(s)$

$\mathcal{L}(\delta(t-\frac{\pi}{6})\sin t) = ?$

$= \int_0^\infty e^{-st}\delta(t-\frac{\pi}{6})\sin t\, dt$

by def:

$= \int_{-\infty}^{\infty} e^{-st}\delta(t-\frac{\pi}{6})\sin t\, dt = e^{-s\frac{\pi}{6}}\sin\frac{\pi}{6} \Rightarrow \mathcal{L}(\delta(t-\frac{\pi}{6})\sin t) = \frac{e^{-s\frac{\pi}{6}}}{2}$

$s^2Y(s)+4Y(s) = \frac{e^{-s\frac{\pi}{6}}}{2} \Rightarrow Y(s) = \frac{1}{2}\frac{e^{-s\frac{\pi}{6}}}{s^2+4} = \frac{1}{2}\cdot e^{-s\frac{\pi}{6}}\cdot\frac{1}{s^2+4} = \frac{1}{4}e^{-s\frac{\pi}{6}}\cdot\frac{2}{s^2+4}$

$\downarrow$

$\sin 2t$

$\sin 2(t$

$y(t) = \frac{1}{4}u_{\frac{\pi}{6}}(t)\sin(2(t-\frac{\pi}{6}))$

Convolution Integral (Section 6.6)

Theorem: Let $F(s)=\mathcal{L}(f(t))$ and $G(s)=\mathcal{L}(g(t))$ $s>a>0$, then $H(s)=F(s)G(s)$ is equal to $\mathcal{L}(h(t))$ $s>a>0$ where $h(t)=\int_0^t f(t-\tau)g(\tau)\,d\tau$

also equal to $=\int_0^t f(t)g(t-\tau)\,d\tau$

$h(t)$ is called the convolution of f and g. $h(t)$ is denoted by $(f*g)(t)$

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Facts: ① $f \star g = g \star f$

② $f \star (g_1 + g_2) = f \star g_1 + f \star g_2$

③ $f \star (g \star h) = (f \star g) \star h$

④ $f \star 0 = 0$

Ex: $1 \star f = \int_0^t f(\tau)\,d\tau = \int_0^t f(t-\tau)\,d\tau$

Ex: Let $f(t) = \cos t$

$1 \star f = \int_0^t \cos(\tau)\,d\tau = \sin t$

We see $1 \star f \neq f$

$\mathcal{L}(f \star g) = \mathcal{L}(f)\,\mathcal{L}(g)$

Ex: Find the inverse transform of $\frac{1}{s(s^2+16)}$

solution: We see $\frac{1}{s(s^2+16)} = \underbrace{\frac{1}{s}}_{1} \cdot \underbrace{\frac{1}{4} \cdot \frac{4}{s^2+16}}_{\frac{1}{4}\sin 4t}$

$\Rightarrow \frac{1}{s(s^2+16)}$ is the Laplace transform of $1 \star \frac{\sin 4t}{4} = \int_0^t \frac{\sin 4\tau}{4}\,d\tau$

$= -\frac{1}{16}\cos 4t \Big|_0^t = \frac{1}{16} - \frac{\cos 4t}{16}$

One application of convolution is that if you want to understand a physical system

$$g \rightarrow \overset{f}{\boxed{\text{system}}} \xrightarrow{f \star g}$$

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If system is f and input g then the output is $f*g$, which is independent of initial conditions.

In particular if one gives an impulse to the system the output will be $f(t)$.

Because $f*\delta = f$

REVIEW

1- Can the following functions $2t+3$, $2t^2-1$, $2t^2+2t$ form a fundamental set of solutions for a linear equations.

Check linear independence

Wronskian

$$W = \begin{vmatrix} f_1 & f_2 & f_3 \\ f_1' & f_2' & f_3' \\ f_1'' & f_2'' & f_3'' \end{vmatrix} = \begin{vmatrix} 2t+3 & 2t^2-1 & 2t^2+2t \\ 2 & 4t & 4t+2 \\ 0 & 4 & 4 \end{vmatrix}$$

$$2t+3 \cdot \begin{vmatrix} 4t & 4t+2 \\ 4 & 4 \end{vmatrix} - (2t^2-1) \cdot \begin{vmatrix} 2 & 4t+2 \\ 0 & 4 \end{vmatrix} + 2t^2+2t \cdot \begin{vmatrix} 2 & 4t \\ 0 & 4 \end{vmatrix}$$

$$(2t+3)(-8) - (2t^2-1) \cdot 8 + (2t^2+2t) \cdot 8 = -16 \neq 0$$

Wronskian non-zero hence functions are linearly independent and they can form a fundamental set of solutions.

1- Assume that a linear equation with constant coefficients has following solutions

x^2e^{2x}, $e^{3x}\sin 5x$

What other linearly independent solutions we can guarantee for the equation

$x^2e^{2x} \Rightarrow$ characteristic equation has a root $r=2$ of multiplicity $m=3$

So we have also e^{2x}, xe^{2x}

$e^{3x}\sin 5x \Rightarrow$ there is a root $r=3+5i$ and hence a root $r=3-5i$

So we have also $e^{3x}\cos 5x$

We can guarantee $e^{2x}, xe^{2x}, e^{3x}\cos 5x$

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3- Solve the equations.

$y'' - 2y' + y = x^{-1}e^{x}$

x^{-1} is not a polynomial, so undetermined coefficient do not work.

Variation of parameters

$y'' - 2y' + y = 0$

$r^2 - 2r + 1 = 0$

$(r-1)^2 = 0$

$r = 1$ double root

$y_h = c_1 e^{x} + c_2 x e^{x}$

$y_p = u_1(x) e^{x} + u_2(x) x e^{x}$

u_1, u_2 satisfy

$u_1' e^{x} + u_2' x e^{x} = 0$

$u_1' e^{x} + u_2' (e^{x} + x e^{x}) = x^{-1} e^{x}$

$u_2' e^{x} = x^{-1} e^{x}$ $\quad u_1' = -1$ $\quad u_1 = -x$

$u_1' e^{x} + e^{x} = 0$ $\quad u_2' = x^{-1}$ $\quad u_2 = \ln x$

So, $y_p = -x \cdot e^{x} + \ln x \, x e^{x}$

General solution

$y = c_1 e^{x} + c_2 x e^{x} - x e^{x} + \ln x \, x e^{x}$

4- For equation

$y''' - 6y'' + 9y' = x e^{3x} + e^{3x} \cos x$

determine the form of particular solution (DO not evaluate coefficients)

$y''' - 6y'' + 9y' = 0$

$r^3 - 6r^2 + 9r = 0$

$r(r^2 - 6r + 9) = 0$

$r(r-3)^2 = 0$ $\quad r = 0, r = 3, r = 3$

$y_h = c_1 + c_2 e^{3x} + c_3 x e^{3x}$

y_p and y_h have NO similar terms

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$y_p = ?$

$g = xe^{3x} + e^{3x}\cos x$

$xe^{3x} \to x^2(Ax+B)e^{3x} = Ax^3e^{3x} + Bx^2e^{3x}$

$e^{3x}\cos x \to (C\cos x + D\sin x)e^{3x}$

$y_p = Ax^3e^{3x} + Bx^2e^{3x} + Ce^{3x}\cos x + De^{3x}\sin x$

(To find coeff. we substitute y_p into the equation)

4- $mu'' + \gamma u' + ku = F$

u - displacement from equilibrium

γ - damping coeff $F_d = \gamma . v$

k - spring coeff $F_s = k . l$

A certain spring-mass system is described;

$3u'' + 4u = 0 \quad u(0) = 2 \quad u'(0) = 0$

a) Find amplitude, natural frequency and phase of the motion

b) If an external force $F = \cos^2 \omega t$ is applied, find ω for which resonance occurs.

Find solution (undamped motion)

$y = A\cos\omega_0 t + B\sin\omega_0 t$

$y = R\cos(\omega_0 t - \delta)$

$R = \sqrt{A^2 + B^2}$ amplitude

$\omega_0 \to$ natural frequency

$\delta \to$ phase $\Rightarrow \tan\delta = \frac{B}{A}$

$3u'' + 4u = 0 \quad u(0) = 2 \quad u'(0) = 0$

$3r^2 + 4 = 0$

$r_{1,2} = \pm \frac{2}{\sqrt{3}} i$

$u = A\cos\frac{2}{\sqrt{3}}t + B\sin\frac{2}{\sqrt{3}}t$

$u(0) = 2 \to A = 2$

$u'(0) = 0 \to u' = -\frac{2}{\sqrt{3}}A\sin\frac{2}{\sqrt{3}}t + \frac{2}{\sqrt{3}}B\cos\frac{2}{\sqrt{3}}t \quad \frac{2}{\sqrt{3}}B = 0 \quad B = 0$

Thus, $u = 2\cos\frac{2}{\sqrt{3}}t$

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$u = 2\cos\frac{2}{\sqrt{3}}t$

$R = 2$ amplitude

$\omega_0 = \frac{2}{\sqrt{3}}$ natural frequency

$\delta = 0$ phase

b) $F = \cos^2 \omega t = \frac{1+\cos 2\omega t}{2}$

The resonance occurs if $2\omega = \omega_0$

$\omega = \frac{\omega_0}{2}$

frequency of the force same as natural frequency

5- For equation

$$x^2y'' - x(x+1)y' + y = 0 \quad , \quad x > 0$$

a) Determine the form of general solution

b) Find the general solution about $x_0 = 0$

$$y'' - \frac{x(x+1)}{x^2}y' + \frac{1}{x^2}y = 0$$

$x = 0$ singular point because $p(x) = -\frac{x(x+1)}{x^2}$ and $q(x) = \frac{1}{x^2}$ undefined at $x = 0$

$x\,p(x) = -(x+1)$ analytic so $x_0 = 0$ is regular singular point

$x^2 q(x) = 1$

$\lim_{x \to 0} -x-1 = -1$

$\lim_{x \to 0} 1 = 1$

We look for

$r(r-1) + \alpha r + \beta = 0$

$r^2 - 2r + 1 = 0$

$(r-1)^2 = 0$

$r = 1$

$r^2 - 1 = 0$

$r = 1$

$r = -1$

$$y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

$$y' = \sum_{n=0}^{\infty} (n+r)a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r-1)(n+r)a_n x^{n+r-2}$$

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$$x^2\sum_{n=0}^{\infty}(n+r)(n+r-1)a_n x^{n+r-2} - x^2\sum_{n=0}^{\infty}(n+r)a_n x^{n+r-1} - x\sum_{n=0}^{\infty}(n+r)a_n x^{n+r-1} + \sum_{n=0}^{\infty}a_n x^{n+r} = 0$$

$$\sum_{n=0}^{\infty}(n+r)(n+r-1)a_n x^{n+r} - \sum_{n=0}^{\infty}(n+r)a_n x^{n+r+1} - \sum_{n=0}^{\infty}(n+r)a_n x^{n+r} + \sum_{n=0}^{\infty}a_n x^{n+r} = 0$$

$$\sum_{n=0}^{\infty}(n+r)(n+r-1)a_n x^{n+r} - \sum_{n=1}^{\infty}(n+r-1)a_{n-1} x^{n+r} - \sum_{n=0}^{\infty}(n+r)a_n x^{n+r} + \sum_{n=0}^{\infty}a_n x^{n+r} = 0$$

$n=0$ (indicial equation)

$(r(r-1)a_0 - ra_0 + a_0) = 0$

$a_0(r^2-2r+1) = 0$

$(r-1)^2 = 0 \quad r=1$ double root the indicial eqn.

$$y_1(x) = x^1\left[1 + \sum_{n=1}^{\infty}a_n x^n\right]$$

$$y_2(x) = y_1(x)\ln x + x\sum_{n=0}^{\infty}b_n x^n$$

Find $y_1(x)$

put $r=1$

$n \geq 1$

$(n+1)na_n - na_{n-1} - (n+1)a_n + a_n = 0$

$n^2 a_n - na_{n-1} = 0$

$a_n = \frac{1}{n}a_{n-1}$

$a_1 = a_0 \quad a_2 = \frac{1}{2}a_0 \quad a_3 = \frac{1}{2.3}a_0 \quad a_n = \frac{1}{n!}a_0$

$$y_1(x) = a_0 x\left(1 + \sum_{n=1}^{\infty}\frac{1}{n!}x^n\right) = a_0\sum_{n=0}^{\infty}\frac{x^{n+1}}{n!}$$

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Next find $y_2(x)$

$$y_2 = y_1(x)\ln x + x\sum_{n=1}^{\infty} b_n x^n \quad \text{subst.}$$

$$y_2' = y_1'\ln x + \frac{1}{x}y_1 + \sum_{n=0}^{\infty}(n+1)b_n x^n$$

$$y_2'' = y_1''\ln x + \frac{1}{x}y_1' - \frac{1}{x^2}y_1 + \frac{1}{x}y_1' + \sum_{n=0}^{\infty}(n+1)(n)b_n x^{n-1}$$

Since y_1 is a solution the logarithm terms disappear

$$x^2\left(\frac{2}{x}y_1' - \frac{1}{x^2}y_1 + \sum_{n=0}^{\infty} n(n+1)b_n x^{n-1}\right) - x(x+1)\left(\frac{1}{x}y_1 + \sum_{n=0}^{\infty}(n+1)b_n x^n\right)$$

$$+\sum_{n=0}^{\infty} b_n x^{n+1} = 0$$

$$2xy' - y_1 + \sum_{n=0}^{\infty} n(n+1)b_n x^{n+1} - xy_1 - y_1 - \sum_{n=0}^{\infty}(n+1)b_n x^{n+2} - \sum_{n=0}^{\infty}(n+1)b_n x^{n+1} + \sum_{n=0}^{\infty} b_n x^{n+1}$$

$$2\sum_{n=0}^{\infty}(n+1)\frac{x^{n+1}}{n!} - 2\sum_{n=0}^{\infty}\frac{x^{n+1}}{n!} + \sum_{n=0}^{\infty} n(n+1)b_n x^{n+1} - \sum_{n=0}^{\infty}\frac{x^{n+2}}{n!} - \sum_{n=0}^{\infty}(n+1)b_n x^{n+2}$$

$$-\sum_{n=0}^{\infty}(n+1)b_n x^{n+1} + \sum_{n=0}^{\infty} b_n x^{n+1}$$

Shift index so every terms is $n+1$

then evaluate coefficients for $n=0,1,2,\ldots$ to zero and we get relation for b_n

$b_0 = -1$ $\quad n^2 b_n - n b_{n-1} + \frac{1}{(n+1)!} = 0$

$n \geq 1$

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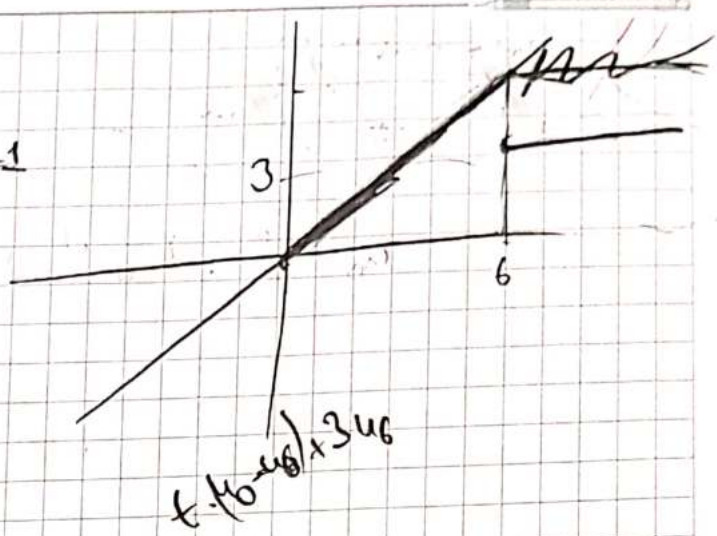
6- Solve the initial value problem

$$y''+y=g(t) \qquad y(0)=0 \quad y'(0)=1$$

$$g(t)=\begin{cases} t & 0\le t\le 6 \\ 3 & t\geq 6 \end{cases}$$



$$g(t)=t(1-u_6(t))+3u_6(t)$$

$$t\cdot(u_0-u_6)+3u_6$$

$$y''+y=t-tu_6(t)+3u_6(t)$$

Laplace transform

$$\mathcal{L}(y)=Y$$

$$\mathcal{L}(y')=sY-y(0)=sY$$

$$\mathcal{L}(y'')=s^2Y-sy(0)-y'(0)=s^2Y-1$$

$$\mathcal{L}(t-tu_6(t)+3u_6(t)) \Rightarrow \mathcal{L}(t-(t-6)u_6(t)+6u_6(t)+3u_6(t))$$

$$=\frac{1}{s^2}-e^{-6s}\cdot\frac{1}{s^2}+9e^{-6s}\frac{1}{s}$$

$$s^2Y-1+Y=\frac{1}{s^2}-e^{-6s}\frac{1}{s^2}+9e^{-6s}\frac{1}{s}$$

$$Y=\frac{1}{1+s^2}+\frac{1}{s^2(1+s^2)}-e^{-6s}\frac{1}{s^2(1+s^2)}+9e^{-6s}\frac{1}{s(s^2+1)}$$

$$\frac{1}{s^2(1+s^2)}=\frac{1}{s^2}-\frac{1}{s^2+1}$$

$$\frac{1}{s(1+s^2)}=\frac{1}{s}-\frac{s}{1+s^2}$$

$$\mathcal{L}^{-1}\left(\frac{1}{1+s^2}\right)=\sin t$$

$$\mathcal{L}^{-1}\left(\frac{s}{1+s^2}\right)=\cos t$$

$$Y=\frac{1}{s^2}-e^{6s}\frac{1}{s^2}+e^{-6s}\frac{1}{1+s^2}+9e^{-6s}\frac{1}{s}-9e^{-6s}\frac{s}{1+s^2}$$

$$y=\mathcal{L}^{-1}(Y)$$

$$y=t-tu_6(t)+u_6(t)\sin(t-6)+9u_6(t)-9u_6(t)\cos(t-6)$$

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$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{N} \left[a_n \cos\left(\frac{n\pi}{L}x\right) + b_n \sin\left(\frac{n\pi}{L}x\right) \right]$$

$$a_0 = \frac{1}{L}\int_{-L}^{L} f(x)\,dx \quad \text{where } P = 2L$$

$$\left.\begin{aligned} a_n &= \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi}{L}x\right)dx \\ b_n &= \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi}{L}x\right)dx \end{aligned}\right\} \text{with } n = 1, 2, 3, \ldots$$

odd → sin
even → cos

<u>Last Time</u> Chapter 10 "PDE's" (Partial Differential Equations)

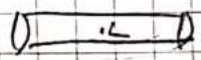
Today: 10.5 Introduction To Heat Eqn. and Separation of Variables

$u = u(x,t)$ $\quad \alpha^2 u_{xx} = u_t \quad 0 < x < L$ (length of a rod) $\quad t > 0$

$u(0,t) = 0 \quad u(L,t) = 0 \quad t > 0$

$u(x,0) = f(x)$

The function $u(x,t)$ is the heat distribution on a metal rod.



To find a solution, we assume that $u(x,t) = X(x)T(t)$
(seperation)

Then, $u_{xx} = X''(x)T(t)$ and $u(0,t) = 0 \Rightarrow X(0)T(t) = 0 \Rightarrow X(0) = 0$
$u_t = X(x)T'(t)$ $\quad u(L,t) = 0 \Rightarrow X(L)T(t) = 0 \Rightarrow X(L) = 0$
$(T(t) \neq 0)$

Then $\alpha^2 X''(x)T(t) = X(x)T'(t)$

$$\Rightarrow \frac{X''}{X} = \frac{1}{\alpha^2}\frac{T'}{T} = -\lambda \in \mathbb{R} \Rightarrow X'' + \lambda X = 0, \quad X(0) = 0 = X(L)$$

$$T' + \alpha^2 \lambda T = 0$$

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The 1st eqn has solutions only for $\lambda = \frac{\pi^2}{L^2}, \frac{4\pi^2}{L^2} = \frac{3\pi^2}{L^2} \dots, \frac{n^2\pi^2}{L^2}$

(eigenvalues)

and the solutions are $\sin\left(\frac{n\pi}{L}x\right)$ for $n=1,2,3,\dots$

The 2nd eqn : $T' + \alpha^2 \frac{n^2\pi^2}{L^2} T = 0 \Rightarrow T$ is $e^{-\frac{\alpha^2 n^2 \pi^2}{L^2} t}$ for $n=1,2,3,\dots$

So $u(x,t) = X(x)T(t) = e^{-\frac{\alpha^2 n^2 \pi^2}{L^2} t} \sin\left(\frac{n\pi}{L}x\right)$ for $n=1,2,3,\dots$

Formally, we write, $u(x,t) = \sum_{n=1}^{\infty} c_n e^{-\frac{\alpha^2 n^2 \pi^2}{L^2}} \sin\left(\frac{n\pi}{L}x\right)$ [a sine series]

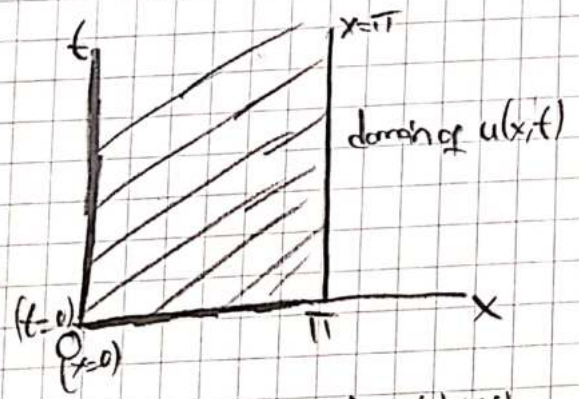
where $c_n = \frac{1}{L}\int_{-L}^{L} f(x) \sin\left(\frac{n\pi}{L}x\right) dx$ where $u(x,0) = f(x) \rightarrow = c_n = \frac{2}{L}\int_0^L f(x) \sin\left(\frac{n\pi}{L}x\right) dx$

because $u(x,0) = \sum_{n=1}^{\infty} c_n \cdot 1 \cdot \sin\left(\frac{n\pi}{L}x\right)$ [a sine series for an odd periodic function which we obtain as the odd periodic extension of $f(x)$

Ex: $7u_{xx} = u_t \quad 0 < x < \pi, t > 0$

$\boxed{u(0,t) = 0 = u(\pi,t) \quad t > 0}$ homogeneous

$u(x,0) = 3\sin 2x - 6\sin 5x$ for $0 < x < \pi$



solution: Assume $u(x,t) = X(x)T(t)$

Then, $7X''T = XT' \quad X(0) = 0 = X(\pi)$

$\Rightarrow \frac{X''}{X} = \frac{1}{7}\frac{T'}{T} \quad (= -\lambda)$

$\Rightarrow X'' + \lambda X = 0 \quad X(0) = 0, X(\pi) = 0$

$T' + 7\lambda T = 0$

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$X(x) = \sin(nx)$ and $\lambda = 1, 4, 9, \dots, n^2 \dots$

$\Rightarrow T' + 7n^2 T = 0 \Rightarrow T = e^{-7n^2 t}$

$\Rightarrow u(x,t) = XT = e^{-7n^2 t} \cdot \sin(nx) \qquad n = 1, 2, 3, \dots$

$\Rightarrow u(x,t) = \sum_{n=1}^{\infty} c_n e^{-7n^2 t} \sin(nx)$ (formal series)

$u(x,0) = 3\sin 2x - 6\sin 5x$

$\Rightarrow 3\sin 2x - 6\sin 5x = \sum_{n=1}^{\infty} c_n \sin(nx)$ [sine series]

$\Rightarrow c_n = \frac{2}{\pi}\int_0^{\pi} 3\sin 2x \sin nx \, dx - \frac{2}{\pi}\int_0^{\pi} 6\sin 5x \sin nx \, dx$

$\left.\begin{matrix} n \neq 2 \\ n \neq 5 \end{matrix}\right\} \Rightarrow c_n = 0$

$n = 2, \; c_2 = \frac{2}{\pi}\int_0^{\pi} 3(\sin 2x)^2 dx = 3$

$n = 5, \; c_5 = -6$

$\Rightarrow u(x,t) = 3e^{-28t} \cdot \sin 2x - 6e^{-175t} \sin 5x$

Ex: $u_{xx} = (t^2+1)u_t \qquad 0 < x < \pi \qquad t > 0$

$u(0,t) = 0 \qquad u(\pi,t) = 0 \qquad t > 0$

$u(x,0) = 1 \qquad 0 < x < \pi$

solution: $u(x,t) = X(x)T(t)$

$\Rightarrow X''T = (t^2+1)XT'$

$X(0) = 0 \qquad X(\pi) = 0$

$\Rightarrow \frac{X''}{X} = -\lambda = (t^2+1)\frac{T'}{T}$

$\Rightarrow X'' + \lambda X = 0 \qquad X(0) = 0 = X(\pi)$

$\frac{T'}{T} = \frac{-\lambda}{t^2+1}$

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$X(x) = \sin(nx)$ for $n = 1, 2, 3 \ldots$

$\lambda = 1, 4, 9 \ldots$

$$\Rightarrow \frac{T'}{T} = \frac{-n^2}{(t^2+1)} = \int \frac{T'}{T}\, dT = \int \frac{-n^2}{t^2+1}\, dt$$

$$\Rightarrow \ln|T| = -n^2 \arctan t$$

$$\Rightarrow T = e^{-n^2 \arctan t} \qquad n = 1, 2 \ldots$$

$$\Rightarrow u(x,t) = \sum_{n=1}^{\infty} c_n e^{-n^2 \arctan t} \cdot \sin(nx) \qquad \text{where } c_n = \frac{2}{\pi}\int_0^{\pi} \sin(nx)\,dx \quad \text{because } u(x,0)=1$$

$$\Rightarrow c_n = \frac{2}{\pi} \frac{-1}{n} \cos(nx) \Big|_0^{\pi}$$

$$\Rightarrow c_n = \frac{2}{n\pi}(1 - \cos n\pi)$$

$$= \begin{cases} 0 & \text{if } n \text{ is even} \\ \frac{4}{n\pi} & \text{if } n \text{ is odd } (n = 2k-1) \end{cases}$$

$$\Rightarrow u(x,t) = \sum_{k=1}^{\infty} \frac{4}{(2k-1)\pi} \cdot e^{-(2k-1)^2 \arctan t} \cdot \sin(2k-1)x$$

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