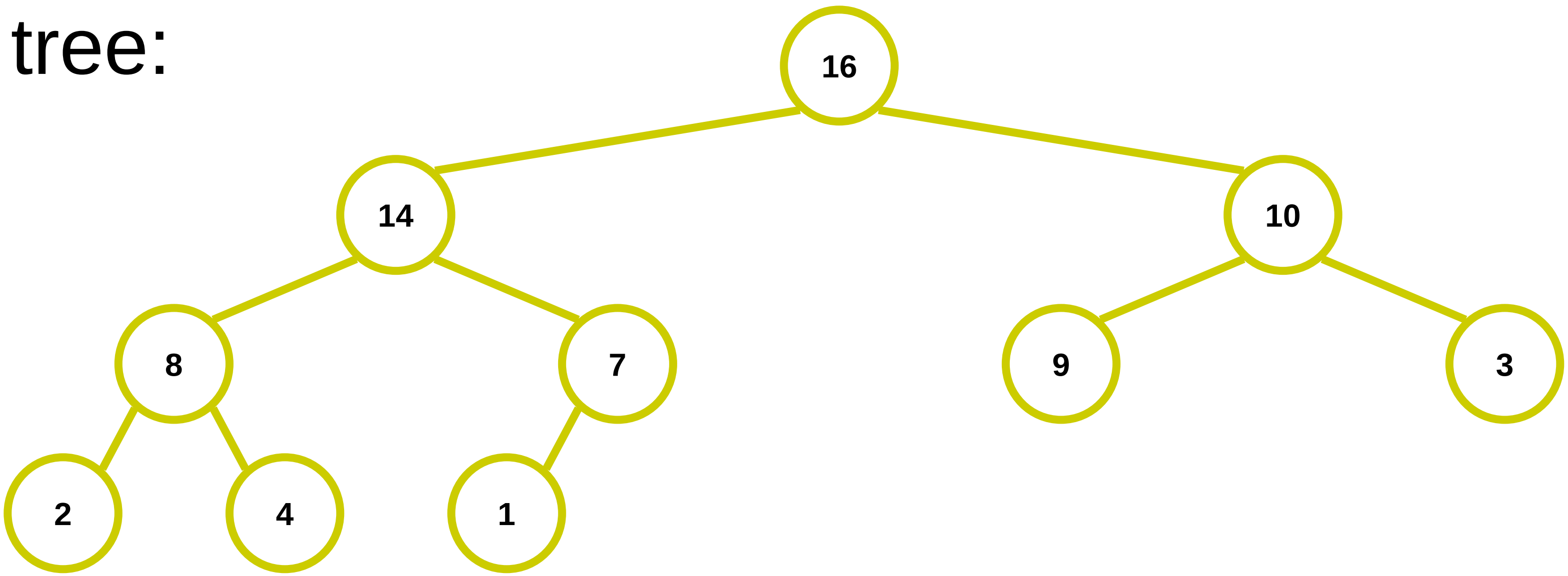
Lecture # 7

- Binary Heaps
- Heap Sort

Heaps

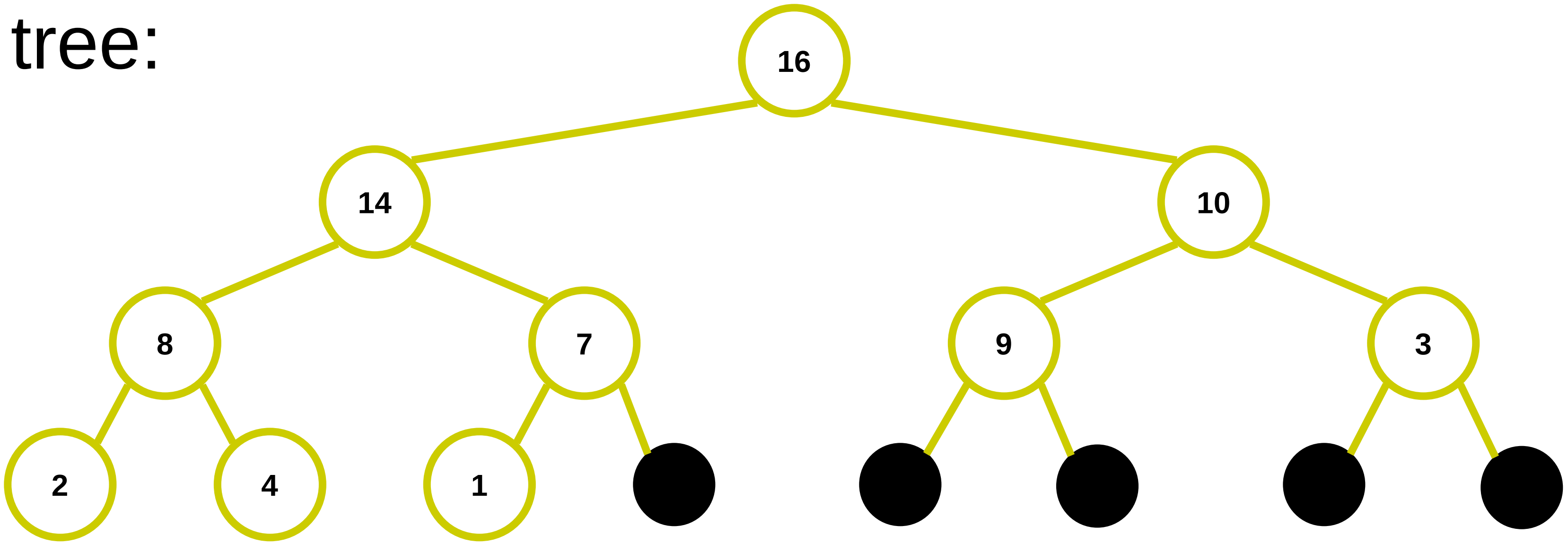
- A *heap* can be seen as a complete binary tree:



  - *What makes a binary tree complete?*
  - *Is the example above complete?*

Heaps

- A *heap* can be seen as a complete binary tree:

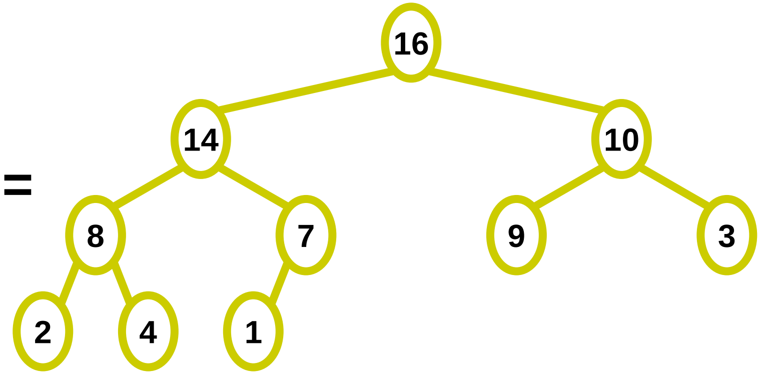


  - Some books calls them "***nearly complete***" binary trees; can think of unfilled slots as null pointers

Heaps

- In practice, heaps are usually implemented as arrays:

$A =$	16	14	10	8	7	9	3	2	4	1	=

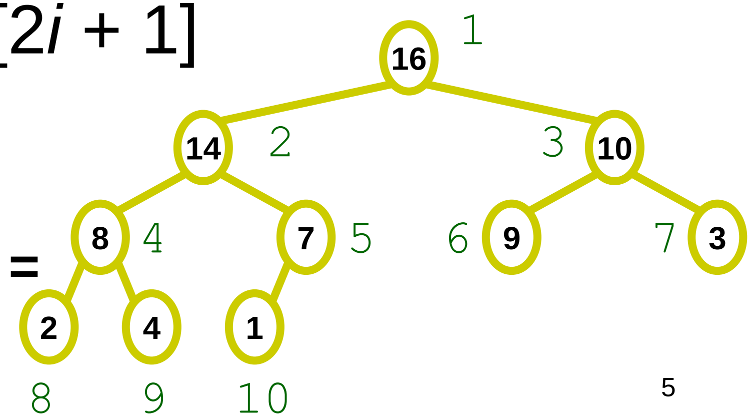


Heaps

- To represent a complete binary tree as an array:
  - The root node is A[1]
  - Node *i* is A[i]
  - The parent of node *i* is A[*i*/2] (note: integer divide)
  - The left child of node *i* is A[2*i*]
  - The right child of node *i* is A[2*i* + 1]

	1	*2*	*3*	*4*						*10*
A =	*16*	*14*	*10*	*8*	*7*	*9*	*3*	*2*	*4*	*1*

=



Referencing Heap Elements

- So…

```
Parent(i) { return ⌊i/2⌋; }
Left(i) { return 2*i; }
right(i) { return 2*i + 1; }
```

The *max-Heap* Property

- max-*heap property says*:

  $A[Parent(i)] \geq A[i]$ for all nodes $i > 1$

  – In other words, the value of a node is at most the value of its parent

  – Where is the largest element in a heap stored?

- Definitions:

  – The *height* of a node in the tree = the number of edges on the longest downward path to a leaf

  – The height of a tree = the height of its root

The *min-Heap* Property

- min-*heap property says*:
  A[*Parent*(i)] $\leq$ A[i]   for all nodes $i > 1$

  – Where is the smallest element in a heap stored?

Heap Height

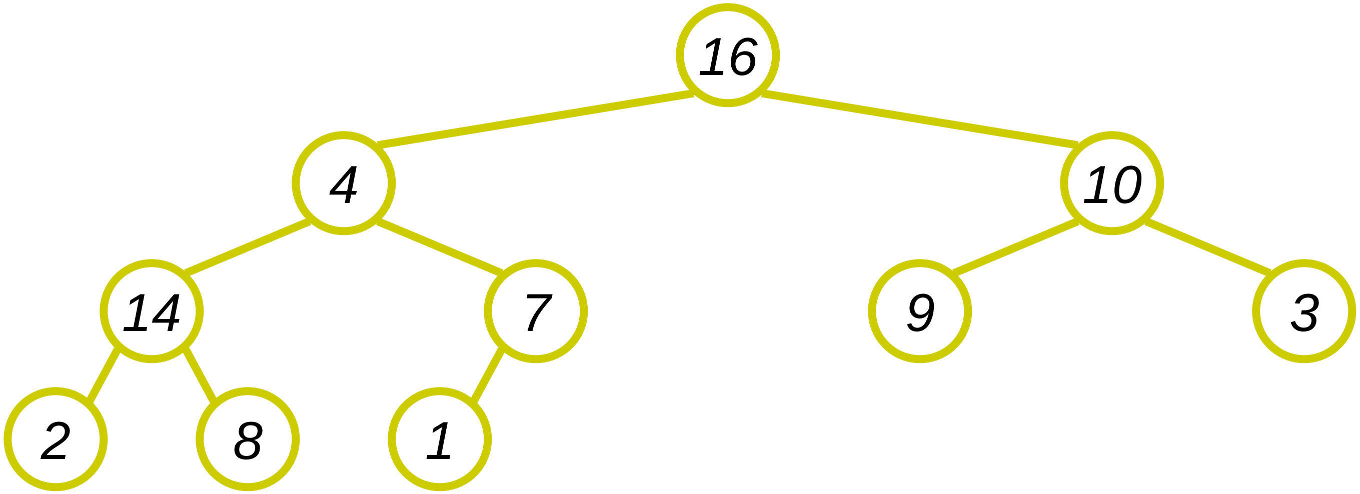
- What is the height of an n-element heap? Why?

- Basic heap operations take at most time proportional to the height of the heap

Heap Operations: Max-Heapify(A,i)

- Max-Heapify(): maintain the heap property
  - Its input is an Array A and an index i of array
  - Left and right sub trees of node i are assumed to be max-heaps.
  - Problem: The sub tree rooted at i may violate the heap property (*How?*)
  - Action: let the value of the parent node "float down" so sub tree at i satisfies the heap property

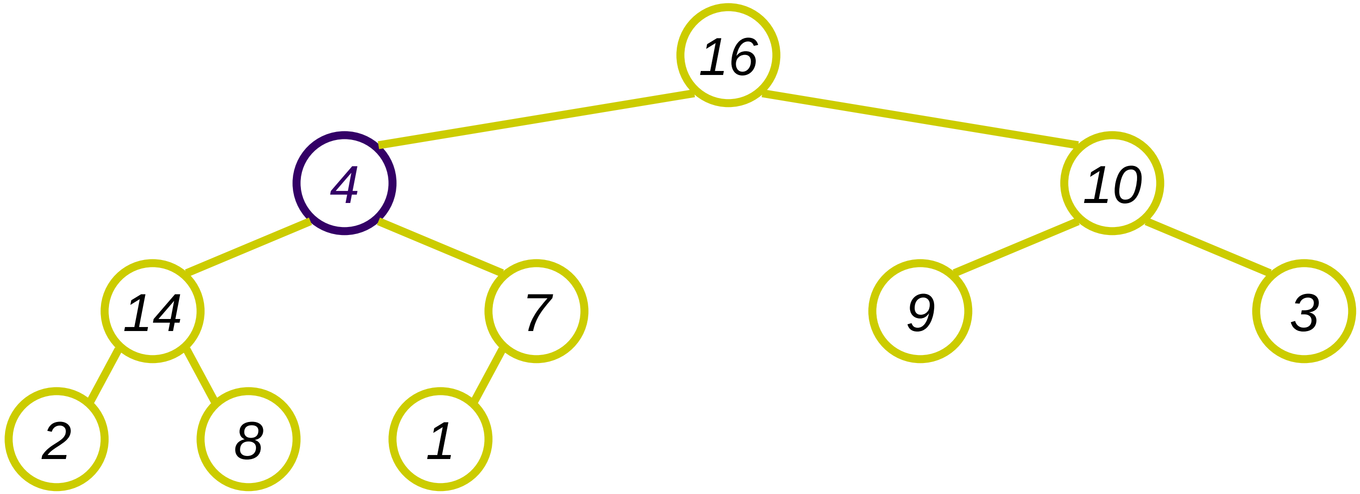
Max-Heapify( ) Example



A =

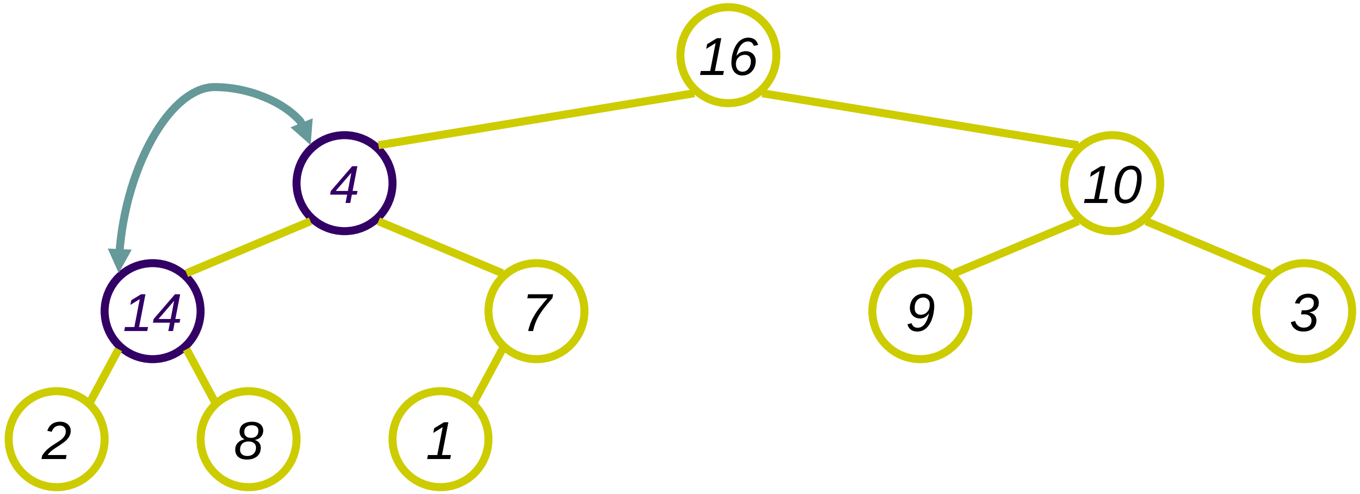
16	*4*	*10*	*14*	*7*	*9*	*3*	*2*	*8*	*1*

Heapify() Example



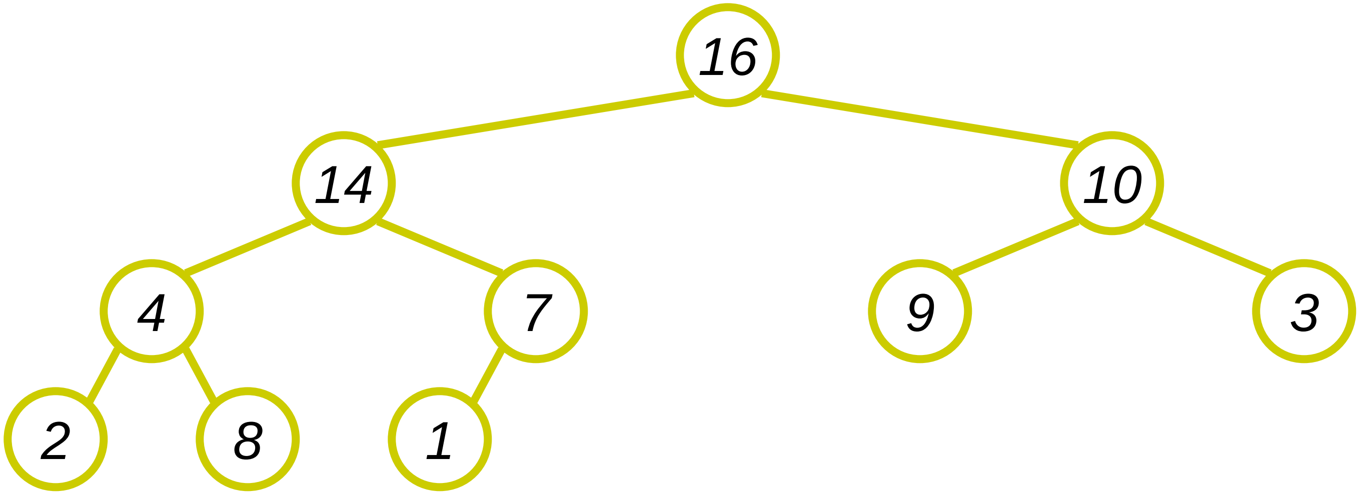
A =	16	4	10	14	7	9	3	2	8	1

Heapify() Example



A =	*16*	*4*	*10*	*14*	*7*	*9*	*3*	*2*	*8*	*1*

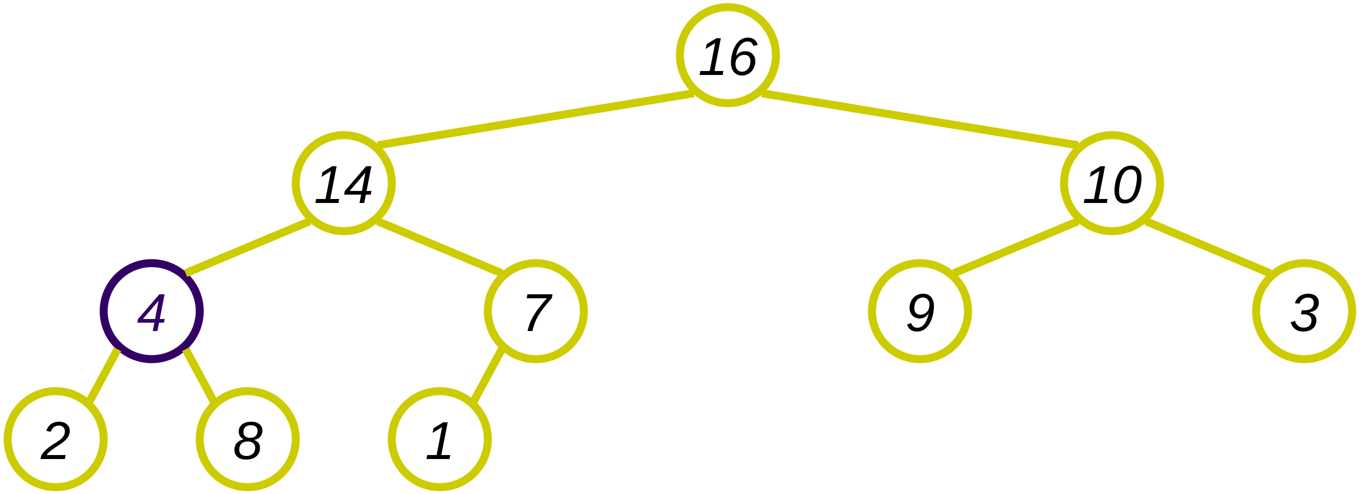
Heapify() Example



A =

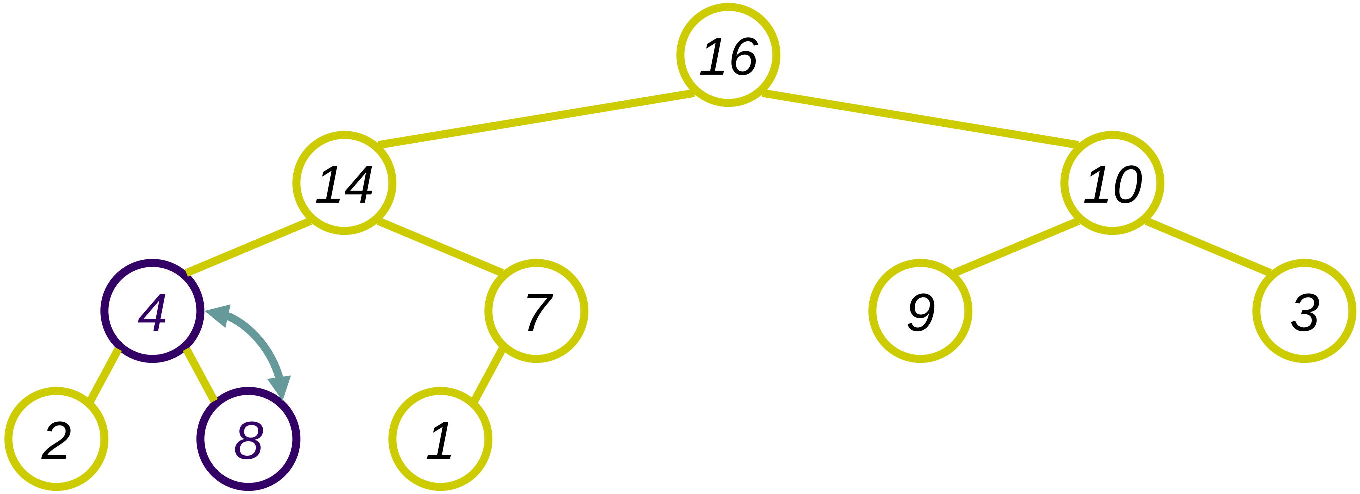
16	14	10	4	7	9	3	2	8	1

Heapify() Example



A =									
16	14	10	4	7	9	3	2	8	1

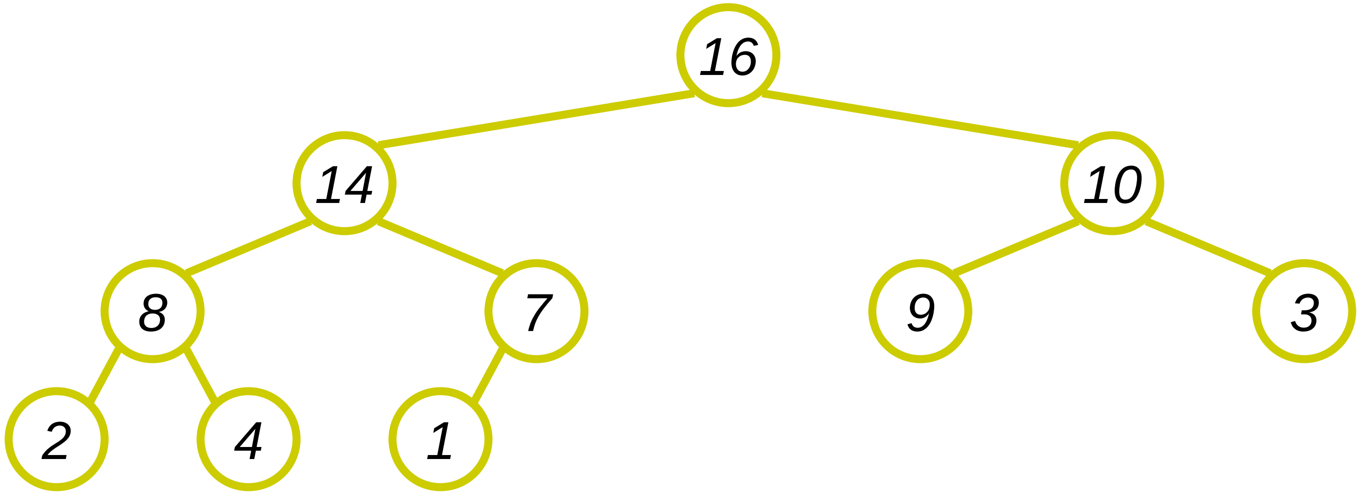
Heapify() Example



$A =$

16	14	10	4	7	9	3	2	8	1

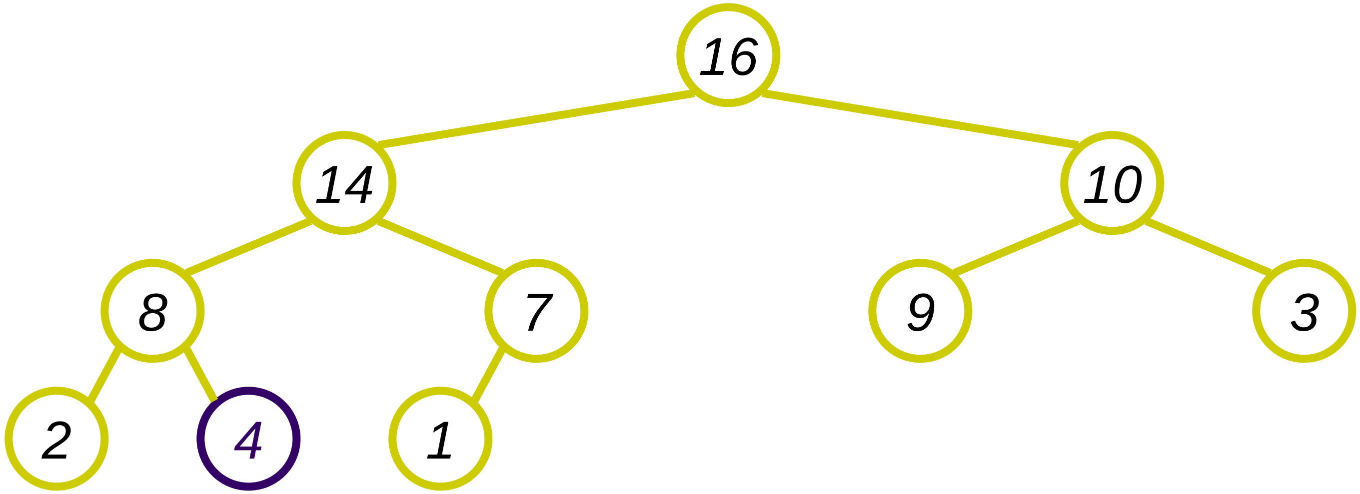
Heapify() Example



A =

16	*14*	*10*	*8*	*7*	*9*	*3*	*2*	*4*	*1*

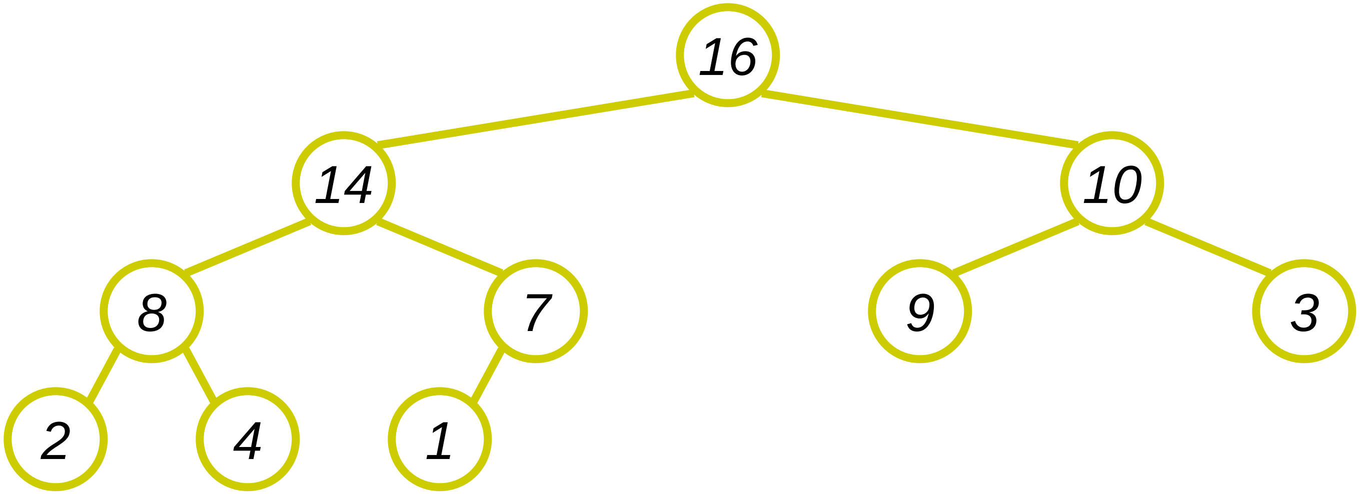
Heapify() Example



A =

16	14	10	8	7	9	3	2	4	1

Heapify() Example



A =

16	14	10	8	7	9	3	2	4	1

Analyzing Heapify( )

- The number of nodes in a complete binary tree of height h is

$$n = 2^0 + 2^1 + 2^2 + \cdots + 2^h = \sum_{i=0}^{h} 2^i = 2^{h+1} - 1$$

- So h in terms of n is:

$$h = (\log(n + 1)) - 1 \approx \log n \in \Theta(\log n)$$

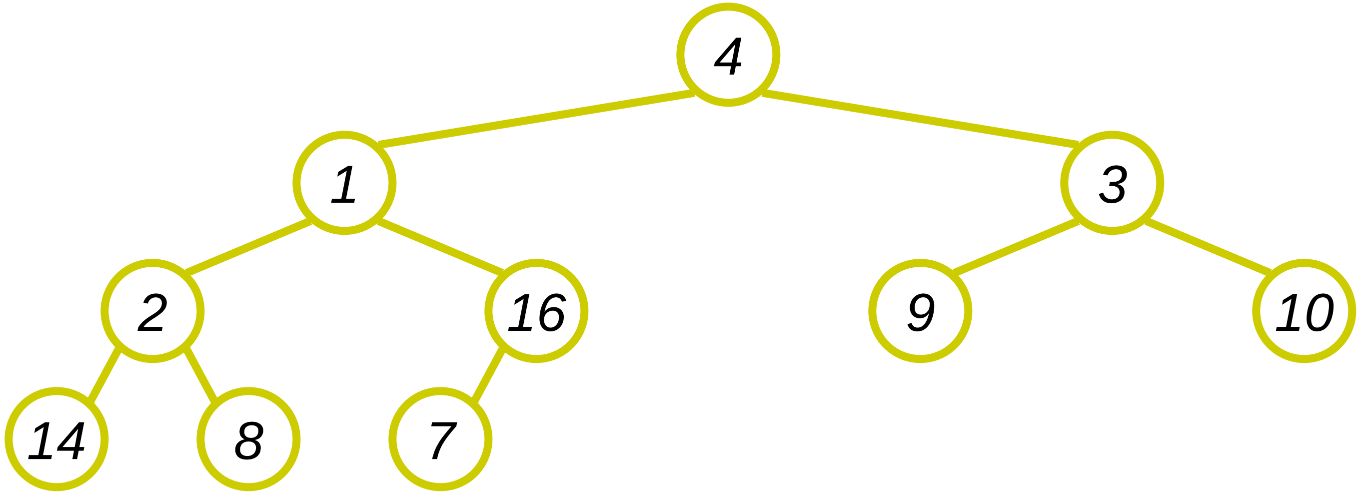
- Running time of Max-Heapify( ) on a node of height h will be O(lg *n*) or O(h).

Heap Operations: BuildHeap()

- We can build a heap in a bottom-up manner by running **`Heapify()`** on successive subarrays
  - Fact: for array of length *n*, all elements in range A[$\lfloor n/2 \rfloor$ + 1 .. n] are heaps (*Why?*)
  - Ans: Because sub array A[$\lfloor n/2 \rfloor$ + 1 .. n] are all leaves.
  - So:
    - Walk backwards through the array from n/2 to 1, calling **`Heapify()`** on each node.
    - Order of processing guarantees that the children of node *i* are heaps when *i* is processed

BuildHeap() Example

- Work through example
  A = {4, 1, 3, 2, 16, 9, 10, 14, 8, 7}



BuildHeap( )

- For array of length *n*, all elements in range A[$\lfloor n/2 \rfloor$ + 1 .. n] are leaves of the tree.

- The procedure BuildHeap( ) goes through the remaining each and every node of the tree and apply Heapify( ) on every node so that the whole tree is in the form of heap, having all heap properties.

Analyzing BuildHeap( )

- Each call to Max-Heapify() costs O(lg *n*) or O(h) time, and
- There would be total of O(*n*) such calls (specifically, $\lfloor n/2 \rfloor$) for complete set of nodes in a heap.
- Thus the running time is O(n.lg *n*). This upper bound is correct but not *asymptotically tight*.

Analyzing BuildHeap(): Tight bound

- The time required by max-heapify() when called on single node of height h is O(h).

- Fact: an *n*-element heap has height $\lfloor \lg n \rfloor$ and at most $\lceil n/2^{h+1} \rceil$ nodes of any height h.

  (leaves are at height 0)

- We have the formula:

$$\sum_{k=0}^{\infty} k.x^k = x / (1-x)^2 \text{ for } |x| < 1$$

Put $x = ½ < 1$

$$\sum_{k=0}^{\infty} k/2^k = 2$$

- So we can express the total cost of BuildHeap() as follows.

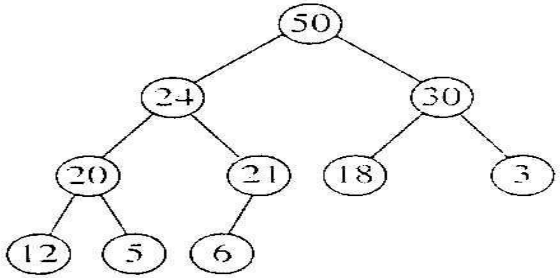
$$\sum_{h=0}^{\lfloor \lg n \rfloor} \lceil n/2^{h+1} \rceil . O(h) = O\left(n . \sum_{h=0}^{\lfloor \lg n \rfloor} \lceil h/2^{h} \rceil \right)$$

As $$\sum_{h=0}^{\infty} h/2^{h} = 2$$

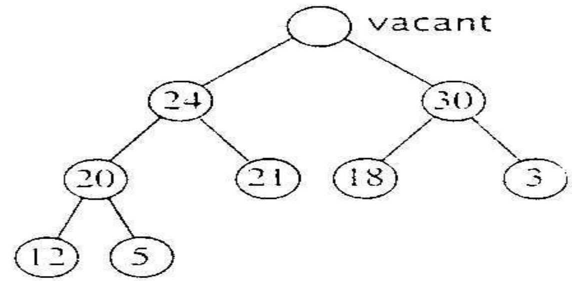
- Thus running time of BuildHeap() can be O(n).

Heapsort

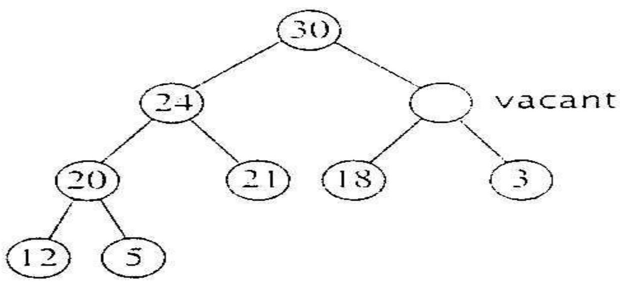
- Given **`BuildHeap()`,** an in-place sorting algorithm is easily constructed:
  – Maximum element is at A[1]
  – Discard by swapping with element at A[n]
    - Decrement heap_size[A]
    - A[n] now contains correct value
  – Restore heap property at A[1] by calling **`Heapify()`**
  – Repeat, always swapping A[1] for A[heap_size(A)]



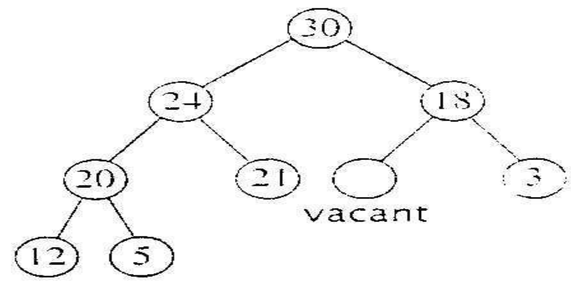
(a) The heap



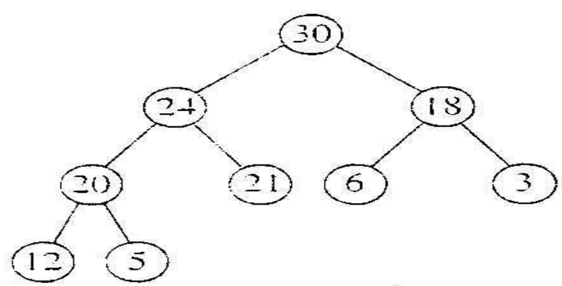
(b) The key at the root has been removed; the rightmost leaf at the bottom level has been removed. $K = 6$ must be reinserted.



(c) The larger child of vacant, 30, is greater than K, so it moves up and vacant moves down.

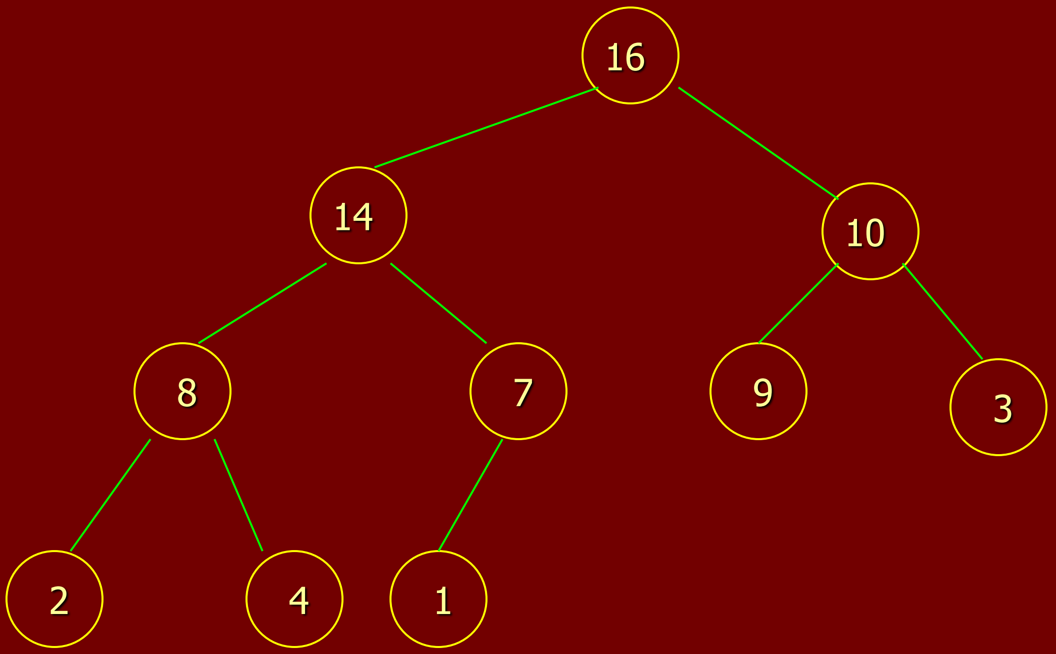


(d) The larger child of vacant, 18, is greater than K, so it moves up and vacant moves down.



(e) Finally, since vacant is a leaf, $K = 6$ is inserted.

- View book (Cormen) page number 137 figure 6.4 for better understanding of heap sort mechanism for following Tree.



Analyzing Heapsort

- The call to **`BuildHeap()`** takes $O(n)$ time
- Each of the $n - 1$ calls to **`Heapify()`** takes $O(\lg n)$ time
- Thus the total time taken by **`HeapSort()`**
  $= O(n) + (n - 1)\, O(\lg n)$
  $= O(n) + O(n \lg n)$
  $= O(n \lg n)$