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Section:
BS(CS) - 3C.

Assignment # 3.

Question # 1

$$\langle f,g\rangle = \int_a^b f(x)g(x)\,dx.$$

We will verify the four axioms

Axiom 1.

$$\langle f,g\rangle = \int_a^b f(x)\,g(x)\,dx.$$

$$= \int_a^b g(x)f(x)\,dx.$$

$$= \langle g,f\rangle$$

Axiom holds.

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Axiom 2:

let $h = h(x)$. so.

$$\langle f+g, h\rangle = \int_a^b (f(x)+g(x))\,h(x)\,dx.$$

$$= \int_a^b f(x)h(x)\,dx + \int_a^b g(x)h(x)\,dx$$

$$= \langle f,h\rangle + \langle g,h\rangle.$$

Axiom holds.

Axiom 3:

$$\langle kf, g\rangle = \int_a^b kf(x)g(x)\,dx$$

$$= k\int_a^b f(x)g(x)\,dx$$

$$= k\langle f,g\rangle.$$

Axiom holds.

Axiom 4:

$$\langle f,f\rangle = \int_a^b f^2(x)\,dx \geq 0$$

Since $f^2(x) \geq 0$ for all x in the interval $[a,b]$, so Axiom holds.

Hence $\langle f,g \rangle = \int_a^b f(x)g(x)\,dx$ become an inner product space on $C[a,b]$.

Question #4

$$\langle p,q \rangle = \int_0^1 p(x)q(x)\,dx \qquad P_2$$

$$S = \{1, x, x^2\}$$

let $u_1 = 1$, $u_2 = x$, $u_3 = x^2$.

So

$$V_1 = u_1$$
$$V_1 = 1$$

$$\|V_1\|^2 = (u_1, u_1)$$

$$= \int_0^1 u_1^2\,dx.$$

$$= \int_0^1 1\,dx.$$

$$= x\Big|_0^1$$

$$\|V_1\| = 1$$

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$$V_2 = u_2 - \frac{\langle u_2, V_1 \rangle}{\|V_1\|^2} \cdot V_1$$

$$V_2 = x - \frac{\int_0^1 u_2 V_1 \, dx}{\|V_1\|^2} \cdot V_1 \qquad (\because V_1 = 1)$$

$$V_2 = x - \int_0^1 x \cdot 1 \, dx = \left(\frac{x^2}{2}\right)_0^1 = \frac{1}{2}.$$

$$V_2 = x - \frac{1}{2}$$

$$\|V_2\|^2 = \left(\sqrt{\int_0^1 V_2^2 \, dx} = \int_0^1 \left(x - \frac{1}{2}\right)^2 dx\right)^2$$

$$= \left(\sqrt{\left(\frac{x^3}{3} + \frac{1}{4}x - \frac{x^2}{2}\right)_0^1}\right)^2$$

$$= \left(\sqrt{\frac{1}{3} + \frac{1}{4} - \frac{1}{2}}\right)^2$$

$$= \left(\sqrt{\frac{1}{12}}\right)^2$$

$$\|V_2\|^2 = \frac{1}{12}$$

$$V_3 = u_3 - \frac{\langle u_3, V_1 \rangle}{\|V_1\|^2} V_1 - \frac{\langle u_3, V_2 \rangle}{\|V_2\|^2} V_2$$

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$$\langle U_3, V_1 \rangle = \int_0^1 x^2\,dx = \left(\frac{x^3}{3}\right)_0^1 = \frac{1}{3}$$

$$\langle U_3, V_2 \rangle = \int_0^1 x\left(x - \frac{1}{2}\right)dx = \left(\frac{x^4}{4} - \frac{1}{2}\frac{x^3}{3}\right)_0^1$$

$$= \frac{1}{4} - \frac{1}{6}$$

$$= \frac{1}{12}$$

$$V_3 = x^2 - \frac{1}{3} - \frac{1}{12(1/12)}\left(x - \frac{1}{2}\right)$$

$$= x^2 - x + \frac{1}{6}$$

$$\|V_3\| = \int_0^1 \left(x^2 - x + \frac{1}{6}\right)^2 dx$$

$$= \int_0^1 \left(x^4 + x^2 + \frac{1}{36} - 2x^3 - \frac{1}{3}x + \frac{1}{3}x^2\right)dx$$

$$= \frac{x^5}{5} + \frac{x^3}{3} + \frac{1}{36}x - 2\frac{x^4}{4} - \frac{1}{3}\frac{x^2}{2} + \frac{1}{3}\frac{x^3}{3}\Big|_0^1$$

$$= \frac{1}{5} + \frac{1}{3} + \frac{1}{36} - \frac{1}{2} - \frac{1}{6} + \frac{1}{9}$$

$$= \frac{8}{15} + \frac{5}{36} - \frac{2}{3}$$

$$= \frac{8}{15} + \frac{1}{36}(5-24)$$

$$= \frac{8}{15} - \frac{19}{36}$$

$$= \frac{288-285}{(15)36.}$$

$$= \frac{1}{5(36)}$$

$$\|V_3\| = \frac{1}{6\sqrt{5}}$$

Thus

$$V_1 = 1, \; V_2 = x - \frac{1}{2}, \quad V_3 = x^3 - x + \frac{1}{6}$$

Form and orthogonal basis for P_2
The norms of these vectors are.

$$\|V_1\| = 1, \; \|V_2\| = \frac{1}{\sqrt{3}} \quad \& \; \|V_3\| = \frac{1}{6\sqrt{5}}.$$

$$\frac{V_1}{\|V_1\|}, \frac{V_2}{\|V_2\|}, \frac{V_3}{\|V_3\|} = \left(1, \frac{1}{(1/\sqrt{12})}\left(x - \frac{1}{2}\right), \frac{1}{(1/6\sqrt{5})}\left(x^2 - x + \frac{1}{6}\right)\right)$$

$= 1, \sqrt{3}(2x-1), \sqrt{5}(6x^2-6x+1)$

So an orthonormal basis for P_2 is

$1, \sqrt{3}(2x-1), \sqrt{5}(6x^2-6x+1)$

Question # 5

$$A = \begin{bmatrix} 1 & 2 \\ 1 & 1 \\ 2 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$$

$A^T A x = A^T b$.

$$A^T A = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 1 & 0 \end{bmatrix} \times \begin{bmatrix} 1 & 2 \\ 1 & 1 \\ 2 & 0 \end{bmatrix}.$$

$$= \begin{bmatrix} 6 & 3 \\ 3 & 5 \end{bmatrix}$$

$$A^T b = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 1 & 0 \end{bmatrix} \times \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 7 \\ 8 \end{bmatrix}$$

$A^TAx = A^Tb$.

$$\begin{bmatrix} 6 & 3 \\ 3 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 7 \\ 8 \end{bmatrix}.$$

$$\left[\begin{array}{cc|c} 6 & 3 & 7 \\ 3 & 5 & 8 \end{array}\right].$$

$$\left[\begin{array}{cc|c} 1 & 1/2 & 7/6 \\ 3 & 5 & 8 \end{array}\right] \quad 1/6R_1$$

$$\left[\begin{array}{cc|c} 1 & 1/2 & 7/6 \\ 0 & 7/2 & 9/2 \end{array}\right] \quad R_2 - 3R_1$$

$$\left[\begin{array}{cc|c} 1 & 1/2 & 7/6 \\ 0 & 1 & 9/7 \end{array}\right] \quad \frac{2}{7}R_2$$

$$\left[\begin{array}{cc|c} 1 & 0 & 11/21 \\ 0 & 1 & 9/7 \end{array}\right] \quad -1/2R_2 + R_1$$

$x_1 = 11/21$
$x_2 = 9/7$

$b - A\hat{x}$

$$\begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} - \begin{bmatrix} 1 & 2 \\ 1 & 1 \\ 2 & 0 \end{bmatrix} \begin{bmatrix} 11/21 \\ 9/7 \end{bmatrix}.$$

$$\begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} - \begin{bmatrix} 65/21 \\ 38/21 \\ 22/21 \end{bmatrix}$$

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$$\begin{bmatrix} -2/21 \\ 4/21 \\ -1/21 \end{bmatrix} . \quad \left\{ \begin{aligned} \|b - A\hat{x}\| &= \sqrt{(-2/21)^2 + (4/21)^2 + (-1/21)^2} \\ &= \sqrt{\frac{1}{21}} \end{aligned} \right.$$

Question # 2

a) $d(f, g)$

$$d(f,g) = \sqrt{\langle f-g, f-g \rangle}$$

$$= \sqrt{\int_a^b (1-x)^2 \, dx}$$

$$= \sqrt{\int_a^b (x^2 - 2x + 1) \, dx}$$

$$= \sqrt{\frac{x^3}{3} - x^2 + x \Big|_{-1}^{1}}$$

$$= \sqrt{\left(\frac{1}{3} - 1 + 1\right) - \left(\frac{-1}{3} - 1 - 1\right)}$$

$$= \sqrt{\frac{1}{3} + \frac{7}{3}}$$

$$= \sqrt{\frac{8}{3}}$$

b) $\|f\|$

$$\|f\| = \sqrt{\langle f, f \rangle}$$

$$= \sqrt{\int_a^b (f(x))^2 \, dx}$$

$$= \sqrt{\int_a^b (1)^2 \, dx}$$

$$= \sqrt{\int_a^b 1 \, dx}$$

$$= \sqrt{x \Big|_{-1}^{1}}$$

$$= \sqrt{1-(-1)}$$

$$\|f\| = \sqrt{2}$$

Question # 3.

$V_1 = (1, -2, 3, -4)$
$V_2 = (2, 1, -4, -3)$
$V_3 = (-3, 4, 1, -2)$
$V_4 = (4, 3, 2, 1)$

$$\langle V_1, V_2 \rangle = \langle (1, -2, 3, -4), (2, 1, -4, -3) \rangle$$

$$= (1 \times 2) + (-2 \times 1) + (3 \times -4) + (-4 \times -3)$$
$$= 0$$

$$\langle V_1, V_3 \rangle = \langle (1, -2, 3, -4); (-3, 4, 1, -2) \rangle$$
$$= (1 \times -3) + (-2 \times 4) + (3 \times 1) + (-4 \times -2)$$
$$= 0$$

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$$\langle V_1, V_4 \rangle = \langle (1,-2,3,-4), (4,3,2,1) \rangle$$
$$= (1\times4)+(-2\times3)+(3\times2)+(-4\times1)$$
$$= 0$$

$$\langle V_2, V_3 \rangle = \langle (2,1,-4,-3), (-3,4,1,-2) \rangle$$
$$= (2\times-3)+(1\times4)+(-4\times1)+(-3\times-2)$$
$$= 0$$

$$\langle V_2, V_4 \rangle = \langle (2,1,-4,-3), (4,3,2,1) \rangle$$

$$= (2\times4)+(1\times3)+(-4\times2)+(-3\times1)$$
$$= 0$$

$$\langle V_3, V_4 \rangle = \langle (-3,4,1,-2), (4,3,2,1) \rangle$$

$$= (-3\times4)+(4\times3)+(1\times2)+(-2\times1)$$
$$= 0.$$

Since $\langle V_1, V_2 \rangle = \langle V_1, V_3 \rangle = \langle V_1, V_4 \rangle = \langle V_2, V_3 \rangle = \langle V_2, V_4 \rangle = \langle V_3, V_4 \rangle = 0$,

So the vectors form an orthogonal basis for R^4.