Assignment 1

Submitted by:-

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BCS - 3C

Linear Algebra

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Q1) for what values of h and k is the following system consistent?

$$A = \begin{bmatrix} 2 & -1 \\ -6 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} h \\ k \end{bmatrix}$$

with X above $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ and B above $\begin{bmatrix} h \\ k \end{bmatrix}$.

$2x_1 - x_2 = h$

$-6x_1 + 3x_2 = k$

The Augmented Matrix for given system is:

$$\ddot{A} = \left[\begin{array}{cc|c} 2 & -1 & h \\ -6 & 3 & k \end{array}\right]$$

∴ Gauss elimination.

$$\ddot{A} = \left[\begin{array}{cc|c} 2 & -1 & h \\ 0 & 0 & \underbrace{}_{3h+k} \end{array}\right] \therefore 3(R_1) + R_2$$

$$\ddot{A} = \left[\begin{array}{cc|c} 2 & -1 & h \\ 0 & 0 & 3h+k \end{array}\right]$$

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$\therefore$ if $k+3h \neq 0$ then system is inconsistent

$\therefore$ if $k+3h = 0$, then we have infinitely many possibilities of h & k and system is consistent

Q2) Solve the system.

$$x_1 - 3x_2 + 4x_3 = -4$$

$$3x_1 - 7x_2 + 7x_3 = -8$$

$$-4x_1 + 6x_2 - x_3 = 7$$

$$A = \begin{bmatrix} 1 & -3 & +4 \\ 3 & -7 & 7 \\ -4 & +6 & -1 \end{bmatrix}, \quad X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \quad b = \begin{bmatrix} -4 \\ -8 \\ 7 \end{bmatrix}$$

$$\tilde{A} = \left[\begin{array}{ccc|c} 1 & -3 & +4 & -4 \\ 3 & -7 & 7 & -8 \\ -4 & +6 & -1 & 7 \end{array}\right] \Rightarrow \text{Augmented Matrix.}$$

i.e. $AX = B$, $X = A^{-1}B$

(i) Gauss Elimination method.

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(i) Gauss Elimination Method

$\therefore$ Applying Row Operations on the Augmented Matrix

$$\tilde{A} = \left[\begin{array}{ccc|c} 1 & -3 & 4 & -4 \\ 3 & -7 & 7 & -8 \\ -4 & 6 & -1 & 7 \end{array}\right]$$

$$\tilde{A} = \left[\begin{array}{ccc|c} 1 & -3 & 4 & -4 \\ 0 & +2 & -5 & 4 \\ -4 & 6 & -1 & 7 \end{array}\right] \quad (-3R_1)+R_2$$

$$\tilde{A} = \left[\begin{array}{ccc|c} 1 & -3 & 4 & -4 \\ 0 & 2 & -5 & 4 \\ 0 & -6 & 15 & -9 \end{array}\right] \quad (4R_1)+R_3$$

$$\cancel{\tilde{A} = \left[\begin{array}{ccc|c} 1 & -3 & 4 & -4 \\ 0 & 2 & -5 & 4 \\ 0 & 0 & 0 & 3 \end{array}\right] \quad (3R_2)+R_3}$$

$$\cancel{\tilde{A} = \left[\begin{array}{ccc|c} 1 & -3 & 4 & -4 \\ 0 & 2 & 5 & 4 \\ 0 & 0 & 14 & \end{array}\right] \quad (R_2+4)+R_3}$$

$$\hat{A} = \left[\begin{array}{ccc|c} 1 & -3 & 4 & -4 \\ 0 & 2 & -5 & 4 \\ 0 & 0 & 0 & -9 \end{array}\right] \quad 3R_2 + R_3$$

⇒ ~~~~ solution

↳ As the last row is zero thus the system has no solution.

(ii) <u>Gauss Jordan Elimination:-</u>

↳ Zeroes above and below 1 in Augmented matrix

$$\hat{A} = \left[\begin{array}{ccc|c} 1 & -3 & 4 & -4 \\ 3 & -7 & 7 & -8 \\ -4 & 6 & 1 & 7 \end{array}\right]$$

$$\hat{A} = \left[\begin{array}{ccc|c} 1 & -3 & 4 & -4 \\ 0 & 2 & -5 & 4 \\ -4 & 6 & -1 & 7 \end{array}\right] (-3R_1) + R_2$$

$$\hat{A} = \left[\begin{array}{ccc|c} 1 & -3 & 4 & -4 \\ 0 & 2 & -5 & 4 \\ 0 & -6 & 15 & -9 \end{array}\right] (4R1) + R_3$$

$$\hat{A} = \left[\begin{array}{ccc|c} 1 & -3 & 4 & -4 \\ 0 & 2 & -5 & 4 \\ 0 & 0 & 0 & 9 \end{array}\right] \quad 3R_2 + R_3$$

$\Rightarrow$ no solution.

Since the final Row $= 0$ thus it has no solution.

(iii) Matrix Inversion Method :-

By Matric Inversion method first we calulate if A is invertible

thus,
$$A = \begin{bmatrix} 1 & -3 & 4 \\ 3 & -7 & 7 \\ -4 & 6 & -1 \end{bmatrix}$$

$\therefore$ Expansion by R_1

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a) $\det(A) = \begin{vmatrix} 1 & -3 & 4 \\ 3 & -7 & 7 \\ -4 & 6 & -1 \end{vmatrix}$

$\therefore \det(A) = 1\begin{vmatrix} -7 & 7 \\ 6 & -1 \end{vmatrix} + 3\begin{vmatrix} 3 & 7 \\ -4 & -1 \end{vmatrix} + 4\begin{vmatrix} 3 & -7 \\ -4 & 6 \end{vmatrix}$

$= (7 - 42) + 3(-3 + 28) + 4(18 - 28)$

$= (-35) + 3(25) + 4(-10)$

$= -35 + 75 - 40$

$= -75 + 75$

$\boxed{\det(A) = 0} \rightarrow$ thus this matrix is ~~non~~ singular

thus has no solutions.

/ invertible matrix.

(iii) Cramer's Rule

$$A = \begin{bmatrix} 1 & -3 & 4 \\ 3 & -7 & 7 \\ -4 & 6 & -1 \end{bmatrix}$$

from previous method i.e matrix inversion method

↳ we calculate $|A| = 0$, so Cramer's Rule will be undefined thus solution is not possible.

Q3 :- Verify $\det A = \det B + \det C$.

$$A = \begin{bmatrix} a_{11} & a_{12} & u_1+v_1 \\ a_{21} & a_{22} & u_2+v_2 \\ a_{31} & a_{32} & u_3+v_3 \end{bmatrix}.$$

$$\therefore B = \begin{bmatrix} a_{11} & a_{12} & u_1 \\ a_{21} & a_{22} & u_2 \\ a_{31} & a_{32} & u_3 \end{bmatrix} \quad C = \begin{bmatrix} a_{11} & a_{12} & v_1 \\ a_{21} & a_{22} & v_2 \\ a_{31} & a_{32} & v_3 \end{bmatrix}$$

$\therefore$ L.H.S

$$\text{L.H.S} \Rightarrow \det(A) = \begin{vmatrix} a_{11} & a_{12} & u_1+v_1 \\ a_{21} & a_{22} & u_2+v_2 \\ a_{31} & a_{32} & u_3+v_3 \end{vmatrix}$$

$\therefore$ Expand by C_1

$$\det(A) = a_{11}\begin{vmatrix} a_{22} & u_2+v_2 \\ a_{32} & u_3+v_3 \end{vmatrix} - a_{21}\begin{vmatrix} a_{12} & u_1+v_1 \\ a_{32} & u_3+v_3 \end{vmatrix} + a_{31}\begin{vmatrix} a_{12} & u_1+v_1 \\ a_{22} & u_2+v_2 \end{vmatrix}$$

$$= a_{11}\big(a_{22}(u_3+v_3) - a_{32}(u_2+v_2)\big)$$
$$- a_{21}\big(a_{12}(u_3+v_3) - a_{32}(u_1+v_1)\big)$$
$$+ a_{31}\big(a_{12}(u_2+v_2) - a_{22}(u_1+v_1)\big)$$

$$= a_{11}(a_{22}(u_3+v_3) - a_{32}(u_2+v_2))$$
$$- a_{21}(a_{12}(u_3+v_3) - a_{32}(u_1+v_1))$$
$$+ a_{31}(a_{12}(u_2+v_2) - a_{22}(u_1+v_1))$$

$$= a_{11}(a_{22}u_3 + a_{22}v_3 - a_{32}u_2 - a_{32}v_2)$$
$$- a_{21}(a_{12}u_3 + a_{12}v_3 - a_{32}u_1 - a_{32}v_1)$$
$$+ a_{31}(a_{12}u_2 + a_{12}v_2 - a_{22}u_1 - a_{22}v_1)$$

L.H.S $= a_{11}a_{22}u_3 + a_{11}a_{22}v_3 - a_{11}a_{32}u_2 - a_{11}a_{32}v_2$
$$- a_{21}a_{12}u_3 - a_{21}a_{12}v_3 + a_{21}a_{32}u_1 + a_{21}a_{32}v_1$$
$$+ a_{31}a_{12}u_2 + a_{31}a_{12}v_2 - a_{31}a_{22}u_1 - a_{31}a_{22}v_1$$

$\downarrow$ (1)

R.H.S

R.H.S ⇒ det B + det C

∴ Expanding by C_3

$$\det(B) = \begin{vmatrix} a_{11} & a_{12} & u_1 \\ a_{21} & a_{22} & u_2 \\ a_{31} & a_{32} & u_3 \end{vmatrix}$$

$$= u_1 \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} - u_2 \begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix} + u_3 \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$$

$$= u_1(a_{21}a_{32} - a_{31}a_{22}) - u_2(a_{11}a_{32} - a_{31}a_{12}) + u_3(a_{11}a_{22} - a_{21}a_{12})$$

$\det(B)$

$$= u_1a_{21}a_{32} - u_1a_{31}a_{22} - u_2a_{11}a_{32} + u_2a_{31}a_{12} + u_3a_{11}a_{22} - u_3a_{21}a_{12}$$

$$\det(C) = \begin{vmatrix} a_{11} & a_{12} & v_1 \\ a_{21} & a_{22} & v_2 \\ a_{31} & a_{32} & v_3 \end{vmatrix}$$

∴ Expand by C_3.

$$= v_1 \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} - v_2 \begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix} + v_3 \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$$

$$= v_1a_{21}a_{32} - v_1a_{31}a_{22} - v_2a_{11}a_{32} + v_2a_{31}a_{12} + v_3a_{11}a_{22} - v_3a_{21}a_{12}$$

$\det B + \det C$

$= \{ u_1 a_{21} a_{32} - u_1 a_{31} a_{22} - u_2 a_{11} a_{32}$

$+ u_2 a_{31} a_{12} + u_3 a_{11} a_{22} - u_3 a_{21} a_{12} \}$

$+ \{ v_1 a_{21} a_{32} - v_1 a_{31} a_{22} - v_2 a_{11} a_{32} + v_2 a_{31} a_{12}$

$+ v_3 a_{11} a_{22} - v_3 a_{21} a_{12}$

$\therefore$ Rearranging

(2)

$\det B + \det C = a_{11} a_{22} u_3 + a_{11} a_{22} v_3 - a_{11} a_{32} u_2 - a_{11} a_{32} v_2$

$- a_{21} a_{12} u_3 - a_{21} a_{12} v_3 + a_{21} a_{32} u_1 + a_{21} a_{32} v_1$

$+ a_{31} a_{12} u_2 + a_{31} a_{12} v_2 - a_{31} a_{22} u_1 - a_{31} a_{22} v_1$

Thus, from (1) and (2) it is verified that $\det(A) = \det B + \det C$ and that

L.H.S = R.H.S