Prob. Final Solution, Spring 2022

Problem # 01 **[7]**

(Diagnostics of computer codes): A new computer program consists of two modules. The first module contains an error with probability 0.2. The second module is more complex; it has a probability of 0.4 to contain an error, independently of the first module. An error in the first module alone causes the program to crash with probability 0.5. For the second module, this probability is 0.8. If there are errors in both modules, the program crashes with probability 0.9. Suppose the program crashed. What is the probability of errors in both modules?

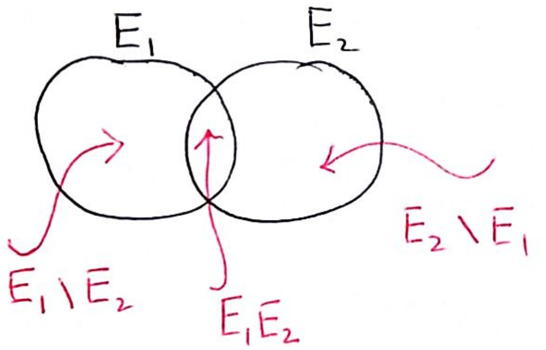
Sol: Let E_1 denote error in module 1 &
E_2 " " " module 2

$P(E_1E_2 | C) = ?$

Then $P(E_1) = 0.2$ & $P(E_2) = 0.4$

Since E_2 & E_2 are independent therefore

$P(E_1E_2) = P(E_1)P(E_2) = 0.2 \times 0.4 \Rightarrow P(E_1E_2) = 0.08$



Let C = Crash

$P(C | E_1 \setminus E_2) = 0.5$ & $P(C | E_2 \setminus E_1) = 0.8$

also $P(C | E_1E_2) = 0.9$.

By Bayes Rule

$$P(E_1E_2 | C) = \frac{P(C | E_1E_2)\, P(E_1E_2)}{P(C)} \quad \text{—— } \star$$

let's first find $P(C)$.

By Law of Total Probability

$$P(C) = P(C \mid E_1 \setminus E_2)\,P(E_1 \setminus E_2) + P(C \mid E_1E_2)\,P(E_1E_2) + P(C \mid E_2 \setminus E_1)\,P(E_2 \setminus E_1) + P(C \mid (E_1 \cup E_2)^c)\,P((E_1 \cup E_2)^c) \quad \text{— } {*}{*}$$

$$P(E_1 \setminus E_2) = P(E_1) - P(E_1E_2)$$

$$\Rightarrow P(E_1 \setminus E_2) = 0.2 - 0.08 = 0.12$$

Similarly $P(E_2 \setminus E_1) = P(E_2) - P(E_1E_2)$

$$\Rightarrow P(E_2 \setminus E_1) = 0.4 - 0.08 = 0.32$$

Also $P(C \mid (E_1 \cup E_2)^c) = 0$

So By $**$ $P(C) = 0.5(0.12) + 0.9(0.08) + 0.8(0.32) + 0$

$$\Rightarrow \boxed{P(C) = 0.388}$$

Substituting values in $*$ we get

$$P(E_1E_2 \mid C) = \frac{(0.9)(0.08)}{0.388}$$

$$\Rightarrow \boxed{P(E_1E_2 \mid C) \simeq 0.1856}$$ Ans

Problem # 02 **[8]**

Sketch a stem and leaf plot of the following data and find Interquartile range (IQR) of the data.

82, 89, 94, 110, 74, 122, 112, 95, 100, 78, 65, 60, 90, 83, 87, 75, 114, 85
69, 94, 124, 115, 107, 88, 97, 74, 72, 68, 83, 91, 90, 102, 77, 125, 108, 65

Sol:

6	0, 5, 5, 8, 9
7	2, 4, 4, 5, 7, 8,
8	2, 3, 3, 5, 7, 8, 9
9	0, 0, 1, 4, 4, 5, 7
10	0, 2, 7, 8
11	0, 2, 4, 5
12	2, 4, 5

$IQR = Q_3 - Q_1$

For Q_1, $P = 1/4$, here $n = 36$

So $np = 36 \cdot \frac{1}{4} = 9$

So $\boxed{Q_1 = \frac{75 + 77}{2} = 76}$

For Q_3, $P = 3/4$, here $n = 36$

So $np = 36 \cdot \frac{3}{4} = 9 \cdot 3 = 27$

So $\boxed{Q_3 = \frac{102 + 107}{2} = 104.5}$

So $\boxed{IQR = 104.5 - 76 = 28.5}$ Ans

Problem # 03 **[10]**

An internet service provider charges its customers for the time of the internet use rounding it up to the nearest hour. The joint distribution of the used time (X, hours) and the charge per hour (Y, cents) is given in the table below.

$p(x, y)$		x = 1	2	3	4
y	1	0	0.06	0.06	0.10
	2	0.10	0.10	0.04	0.04
	3	0.40	0.10	0	0

Each customer is charged $Z = X.Y$ cents, which is the number of hours multiplied by the price of each hour. Find the distribution of Z. (i.e. PMF of Z).

Sol First we will find possible values of Z and then its corresponding probabilities.

Possible values of $Z = X \cdot Y$ are 1, 2, 3, 4, 6, 8, 9, 12.

$P\{Z = 01\} = P\{X=1, Y=1\} = 0$

$P\{Z=2\} = P\{X=1, Y=2\} + P\{X=2, Y=1\}$

$\Rightarrow P\{Z=2\} = 0.10 + 0.06 = 0.16$

$P\{Z=3\} = P\{X=1, Y=3\} + P\{X=3, Y=1\} = 0.40 + 0.06$

$\Rightarrow P\{Z=3\} = 0.46$

$P\{Z=4\} = p(4,1) + p(2,2) = 0.10 + 0.10 = 0.20$

$P\{Z=6\} = p(2,3) + p(3,2) = 0.10 + 0.04 = 0.14$

$P\{Z=8\} = p(4,2) = 0.04$, $P\{Z=9\} = p(3,3) = 0$

$P\{Z=12\} = p(4,3) = 0$. As a check we can see $\sum_z p(z) = 1$

So the distribution of Z is:

z	$P\{Z=z\}$
1	0
2	0.16
3	0.46
4	0.20

z	$P\{Z=z\}$
6	0.14
8	0.04
9	0
12	0

Problem # 04 **[10*2=20]**

(a) In a certain population, 85% of the people have Rh-positive blood. Suppose that two people from this population marry. What is the probability that they are both Rh-negative, thus making it inevitable that their children will be Rh-negative?

Sol: Given that $P(\text{Rh-positive}) = 85/100 = 0.85$

Since we are interested in Rh-negative

So $p = \text{Prob. of success} = P(\text{Rh-negative}) = 1 - P(\text{Rh-positive})$

$\Rightarrow p = 1 - 0.85 \Rightarrow p = 0.15 \cdot \quad n = 2.$

So $P\{X=2\} = \binom{2}{2}(0.15)^2(1-0.15)^{2-2} = 1 \times (0.15)^2 \times 1$

$\Rightarrow \boxed{P\{X=2\} = 0.0225}$ Ans

(b) The lifetime in hours of electronic tubes is a random variable having a probability density function given by $f(x) = a^2 x e^{-ax}, x \geq 0$. Compute the expected lifetime of such a tube.

Sol

$$E[x] = \int_{-\infty}^{\infty} x f(x) dx$$

$$= \int_0^{\infty} x \cdot a^2 x e^{-ax} dx = a^2 \int_0^{\infty} x^2 e^{-ax} dx$$

$$= a^2 \left[x^2 \frac{e^{-ax}}{-a} \Big|_0^{\infty} - \int_0^{\infty} 2x \cdot \frac{e^{-ax}}{-a} dx \right]$$

I used L-Hospital Rule various times

$$= -a\left(x^2 e^{-ax}\right)\Big|_0^{\infty} + 2a \int_0^{\infty} x e^{-ax} dx$$

$$= -a\left(x^2/e^{ax}\right)\Big|_0^{\infty} + 2a\left[-\frac{x e^{-ax}}{a}\Big|_0^{\infty} + \frac{1}{a}\int_0^{\infty} e^{-ax} dx\right]$$

$$E[x] = -a(0-0) - 2\left(x/e^{ax}\right)\Big|_0^{\infty} + \frac{2}{a}\frac{e^{-ax}}{-a}\Big|_0^{\infty} = \boxed{\frac{2}{a}}$$ Ans

Problem # 05 **[15]**

The life span, in hours, of an electrical component is a random variable with commulative distribution function

$$F(x) = \begin{cases} 1 - e^{-x/75}, & x > 0, \\ 0, & \text{elsewhere.} \end{cases}$$

(a) Determine its probability density function.

(b) Calculate the probability that the life span of such a component will exceed 70 hours.

Sol (a) $f(x) = \frac{d}{dx} F(x)$

$$\Rightarrow f(x) = \frac{d}{dx}\left(1 - e^{-x/75}\right) = 0 - e^{-x/75}\left(-\frac{1}{75}\right)$$

$$\Rightarrow \boxed{f(x) = \frac{1}{75} e^{-x/75}, x > 0}$$ Ans

(b) $P\{X > 70\} = 1 - P\{X \leq 70\}$

$$= 1 - F(70)$$

$$= 1 - \left(1 - e^{-70/75}\right) = \cancel{1} - \cancel{1} + e^{-70/75}$$

$$\Rightarrow \boxed{P\{X > 70\} = e^{-70/75} \simeq 0.3932}$$ Ans

Problem # 06 **[10]**

A randomly chosen IQ test taker obtains a score that is approximately a normal random variable with mean 100 and standard deviation 15. What is the probability that the score of such a person is **(a)** more than 125; **(b)** between 90 and 110?

Sol $\mu = 100$ & $\sigma = 15$

(a) $P\{X > 125\} = P\left\{\frac{X-100}{15} > \frac{125-100}{15}\right\}$

$= P\{Z > 1.67\}$

$= 1 - P\{Z \le 1.67\}$

$= 1 - \Phi(1.67)$

$= 1 - 0.9525$

$\Rightarrow \boxed{P\{X > 125\} = 0.0475}$ Ans

(b) $P\{90 < X < 110\}$

$= P\left\{\frac{90-100}{15} < \frac{X-100}{15} < \frac{110-100}{15}\right\}$

$= P\{-0.67 < Z < 0.67\}$

$= \Phi(0.67) - \Phi(-0.67)$

$= 0.7486 - 0.2514 = \boxed{0.4972}$

Problem 7 Solution

[15 Marks]

(a)

$$P\{X > 1, Y < 1\} = \int_0^1 \int_1^\infty 2e^{-x}e^{-2y}\,dx\,dy$$

$$= \int_0^1 2e^{-2y}(-e^{-x}|_1^\infty)\,dy$$

$$= e^{-1}\int_0^1 2e^{-2y}\,dy$$

(7.5 Marks)

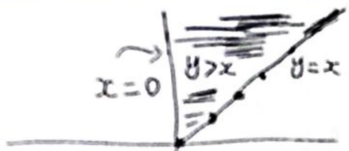
$$\Longrightarrow P\{X > 1, Y < 1\} = e^{-1}(1 - e^{-2})$$

299

299

(b)

$$P\{X < Y\} = \iint_{(x,y):x<y} 2e^{-x}e^{-2y}\,dx\,dy$$



$$= \int_0^\infty \int_0^y 2e^{-x}e^{-2y}\,dx\,dy = \int_0^\infty 2e^{-2y}(1 - e^{-y})\,dy$$

$$= \int_0^\infty 2e^{-2y}\,dy - \int_0^\infty 2e^{-3y}\,dy \quad = 1 - \frac{2}{3}$$

$$\Longrightarrow P\{X < Y\} = \frac{1}{3}$$

(7.5 Marks)

300

300

$$E[X_1] = 0\cdot P\{X_1=0\} + 1\cdot P\{X_1=1\} + 2\cdot P\{X_1=2\} + 3\cdot P\{X_1=3\}$$

X_1 \ X_2	1	2	$P\{X_1=x_1\}$
0	$\frac{1}{8}$	$\frac{1}{16}$	$\frac{3}{16}$
1	$\frac{1}{16}$	$\frac{1}{16}$	$\frac{2}{16}$
2	$\frac{3}{16}$	$\frac{1}{8}$	$\frac{5}{16}$
3	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{3}{8}$
$P\{X_2=x_2\}$	$\frac{8}{16}$	$\frac{8}{16}$	

So $E[X_1] = 0 + 1\left(\frac{1}{8}\right) + \cancel{2}^{1}\cdot\left(\frac{5}{\cancel{16}_{8}}\right) + 3\left(\frac{3}{8}\right) = \frac{15}{8} \Rightarrow \boxed{E[X_1] = \frac{15}{8} = 1.875}$ Ans

$$Var(X_2) = E[X_2^2] - (E[X_2])^2$$

Now $E[X_2^2] = 1^2 P\{X_2 = 1\} + 2^2 P\{X_2 = 2\}$

$$= 1 \cdot \frac{1}{2} + 4 \cdot \frac{1}{2} = \frac{5}{2}$$

$$\Rightarrow E[X_2^2] = \frac{5}{2}$$

and $E[X_2] = 1 \cdot \frac{1}{2} + 2 \cdot \frac{1}{2} = \frac{3}{2}$

So $Var(X_2) = \frac{5}{2} - \left(\frac{3}{2}\right)^2 = \frac{5}{2} - \frac{9}{4} = \frac{1}{4}$

$\Rightarrow \boxed{Var(X_2) = \frac{1}{4}}$ Ans

and

$$Cov(X_1, X_2) = E[X_1 X_2] - E[X_1] E[X_2]$$

$$E[X_1 X_2] = (0)(1)\frac{1}{8} + (0)(2)\frac{1}{16} + (1)(1)\frac{1}{16} + (1)(2)\frac{1}{16}$$

$$+ (2)(1)\frac{3}{16} + (2)(2)\frac{1}{8} + (3)(1)\left(\frac{1}{8}\right) + (3)(2)\left(\frac{1}{4}\right)$$

$$= \frac{1}{16} + \frac{2}{16} + \frac{6}{16} + \frac{4}{8} + \frac{3}{8} + \frac{6}{4} = \frac{1+2+6+8+6+24}{16}$$

$$E[X_1 X_2] = \frac{47}{16}$$

So $Cov(X_1, X_2) = \frac{47}{16} - \frac{15}{8} \cdot \frac{3}{2} = \frac{47}{16} - \frac{45}{16} = \frac{2}{16}$

$\Rightarrow \boxed{Cov(X_1, X_2) = \frac{1}{8} = 0.125}$ Ans