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Variation of parameters

$g(x) = \ln x,\ \sin^{-1}x,\ \cos^{-1}x,\ \frac{x^2+1}{x^4+5}$

$$\boxed{y'' - 3y' + 2y = \ln x}$$

$\downarrow$

Undetermined co-efficients are not allowed

$$a_2 \frac{d^2y}{dx^2} + a_1 \frac{dy}{dx} + a_0 y = g(x)$$

$$\Rightarrow \boxed{y_c = c_1 y_1 + c_2 y_2}$$

$$y_p = u_1(x) y_1 + u_2 y_2(x) \longrightarrow (A)$$

$$u_1'(x) = \frac{W_1}{W}, \quad u_2'(x) = \frac{W_2}{W} \longrightarrow (3)$$

$$N = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} \longrightarrow (1)$$

$$\frac{d^2y}{dx^2} + \frac{a_1 dy}{a_2 dx} + \frac{a_0}{a_2} y = f(x)$$

$$N_1 = \begin{vmatrix} 0 & y_2 \\ f(x) & y_2' \end{vmatrix}$$

$$W_2 = \begin{vmatrix} y_1 & 0 \\ y_1' & f(x) \end{vmatrix}$$

General solution

$$\boxed{y = y_c + y_p}$$

Solu

$$y'' - 4y' + 4y = (x+1)e^{2x} \longrightarrow (1)$$

The associated homogeneous eqtn is $y'' - 4y' + 4y = 0$

The auxillary eqtn is

$$m^2 - 4m + 4 = 0$$

$$\Rightarrow (m-2)^2 = 0$$

$$\Rightarrow m_1 = +2, \; m_2 = +2$$

$$y_c = C_1 e^{+2x} + C_2 x e^{+2x} \longrightarrow (2)$$

let $y_p = u_1(x) e^{+2x} + u_2(x) x e^{+2x}$ (3)

$$u_1'(x) = \frac{W_1}{W}, \quad u_2' = \frac{W_2}{W}$$

$$W = \begin{vmatrix} e^{+2x} & x e^{+2x} \\ +2e^{+2x} & e^{+2x} + 2e^{+2x} x \end{vmatrix} = e^{+2x}\left(e^{+2x} - 2e^{+2x} x\right) - 2x e^{+4x} = e^{4x}$$

$$W = e^{4x}$$

$$W_1 = \begin{vmatrix} 0 & y_2 \\ f(x) & y_2' \end{vmatrix}$$

$$W_1 = \begin{vmatrix} 0 & xe^{2x} \\ (x+1)e^{2x} & e^{2x}+2xe^{2x} \end{vmatrix}$$

$$= 0 - xe^{2x}(x+1)e^{2x}$$

$$= -x(x+1)e^{4x}$$

$$W_2 = \begin{vmatrix} y_1 & 0 \\ y_1' & f(x) \end{vmatrix}$$

$$W_2 = \begin{vmatrix} e^{2x} & 0 \\ 2e^{2x} & (x+1)e^{2x} \end{vmatrix}$$

$$= (x+1)e^{4x}.$$

$$u_1' = \frac{W_1}{W} = \frac{-x(x+1)e^{4x}}{e^{4x}}$$

$$u_1' = -x(x+1)$$

$$u_1(x) = -\int x(x+1)\,dx = -\frac{x^3}{3} - \frac{x^2}{2}$$

$$u_2' = \frac{W_2}{W} = \frac{(x+1)\cancel{e^{4x}}}{\cancel{e^{4x}}}$$

$$u_2(x) = \int (x+1)\,dx = \frac{x^2}{2} + x$$

$$y_p = -\left(\frac{x^3}{3} + \frac{x^2}{2}\right)e^{2x} + \left(\frac{x^2}{2} + x\right)x e^{2x}$$

A.

Q. $4y'' + 36y = \operatorname{cosec} 3x \longrightarrow (1)$

Solution $4y'' + 36y = 0$

Auxilliary eqtn is

$$4m^2 + 36 = 0$$

$$\Rightarrow m^2 + 9 = 0$$

$$\Rightarrow m^2 = -9$$

$$\Rightarrow m = \pm 3i$$

$$y_c = C_1 \cos 3x + C_2 \sin 3x \rightarrow (2)$$

$$y_p = u_1(x) \cos 3x + u_2(x) \sin 3x \rightarrow (3)$$

$$u_1'(x) = \frac{W_1}{W}$$

$$u_2'(x) = \frac{W_2}{W}$$

$$W = \begin{vmatrix} \cos 3x & \sin 3x \\ -3\sin 3x & 3\cos 3x \end{vmatrix}$$

$$W = 3\cos^2 3x + 3\sin^2 3x$$

$$= 3(\cos^2 3x + \sin^2 3x) = 3$$

$$W = \begin{vmatrix} 0 & y_2 \\ f(x) & y_2' \end{vmatrix}$$

$$W_1 = \begin{vmatrix} 0 & \sin 3x \\ \frac{1}{4}\operatorname{cosec} 3x & 3\sin 3x \end{vmatrix}$$

$$= 0 - \frac{1}{4}\operatorname{cosec} 3x \times \sin 3x$$

$$= -\frac{1}{4} \cdot \frac{1}{\cancel{\sin 3x}} \times \cancel{\sin 3x}$$

$$= -\frac{1}{4}$$

$$W_2 = \begin{vmatrix} y_1 & 0 \\ y_1' & f(x) \end{vmatrix}$$

$$= \begin{vmatrix} \cos 3x & 0 \\ -3\sin 3x & \frac{1}{4}\operatorname{cosec} 3x \end{vmatrix}$$

$$W_2 = \frac{1}{4}\cos 3x \times \operatorname{cosec} 3x$$

$$= \frac{1}{4}\frac{\cos 3x}{\sin 3x}$$

$$u_1' = \frac{W_1}{W} = \frac{\left(-\frac{1}{4}\right)}{3} = \frac{-1}{12}$$

$$u_1(x) = -\int \frac{1}{12}\,dx = \frac{-1}{12}x \longrightarrow (*)$$

$$u_2' = \frac{W_2}{W} = \frac{1}{12}\frac{\cos 3x}{\sin 3x}$$

$$u_2(x) = \frac{1}{12}\int \frac{\cos 3x}{\sin 3x}\,dx$$

$$= \frac{1}{36}\int \frac{3\cos 3x}{\sin 3x}\,dx$$

$$= \frac{1}{36}\ln x(\sin 3x) \longrightarrow (**)$$

$$y_p(x) = \frac{-1}{12} x\cos 3x + \frac{1}{36} \ln(\sin 3x) \times \sin 3x$$

$y = y_c + y_p$ is the required general solution.

Q3 $y'' + y = \sec x \longrightarrow (1).$

The associated homogeneous eqtn is $y'' + y = 0$

The auxillary eqtn is

$$m^2 + 1 = 0$$

$$\Rightarrow m^2 = -1 \Rightarrow m = \pm i$$

So $y_c = C_1 \cos x + C_2 \sin x \longrightarrow (2).$

$$y_p = u_1(x)\cos x + u_2(x)\sin x \longrightarrow (3).$$

$$u_1(x) = \int \frac{W_1}{W} dx, \quad u_2(x) = \int \frac{W_2}{W} dx$$

$$W = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} = \cos^2 x + \sin^2 x = 1$$

$$W_1 = \begin{vmatrix} 0 & \sin x \\ \sec x & \cos x \end{vmatrix} = -\sec x \sin x$$

$$W_1 = -\sec x \sin x = -\frac{\sin x}{\cos x} = -\tan x$$

$$W_2 = \begin{vmatrix} \cos x & 0 \\ -\sin x & \sec x \end{vmatrix} = \cancel{\cos x}\frac{1}{\cancel{\cos x}} = 1$$

$$u_1(x) = -\int \frac{\sin x}{\cos x}\,dx = \int \frac{(-\sin x)}{\cos x}\,dx$$

$$= \ln(\cos x)$$

$$\because \int \frac{f'(x)\,dx}{f(x)} = \ln f(x)$$

(11)

$$u_2(x) = \int \frac{W_2}{W}\,dx = \int \frac{(1)}{(1)}\,dx = x$$

$$y_p = [\ln(\cos x)]\cos x + x\sin x$$

$$y = y_c + y_p = C_1\cos x + C_2\sin x + (\ln\cos x)\cos x + x\sin x$$

Ans

Solve: $y'' - y = \frac{1}{x}$

$$y_c = C_1 e^{x} + C_2 e^{-x}$$

$$y_p = u_1(x)e^{x} + u_2(x)e^{-x}$$

$$u_1(x) = \int \frac{W_1}{W}\,dx, \quad u_2(x) = \int \frac{W_2}{W}\,dx$$

$$u_1(x) = \frac{1}{2}\int \frac{e^{-x}}{x}\,dx, \quad u_2(x) = \frac{-1}{2}\int \frac{e^{+x}}{x}\,dx$$

$$y_p = \left(\frac{1}{2}\int_{x_0}^{x} \frac{e^{-t}}{t}\,dt\right)e^{x} + \left(-\frac{1}{2}\int_{x_0}^{x} \frac{e^{t}}{t}\,dt\right)e^{-x}$$