Reduction to constant Coefficients

$$x^2y'' - xy' + y = \ln x \longrightarrow (1)$$

$$\boxed{x = e^t \quad \text{or} \quad t = \ln x} \longrightarrow (2)$$

$$y' = \frac{dy}{dx} = \frac{dy}{dt}\frac{dt}{dx} = \frac{1}{x}\frac{dy}{dt}$$

$$\Rightarrow \boxed{xy' = \frac{dy}{dt}}$$

$$y'' = \frac{d}{dx}\left(\frac{1}{x}\frac{dy}{dt}\right) = \frac{1}{x}\frac{d}{dx}\left(\frac{dy}{dt}\right) + \frac{dy}{dt}\frac{d}{dx}\left(\frac{1}{x}\right)$$

$$= \frac{1}{x}\frac{d^2y}{dt^2}\frac{dt}{dx} + \frac{dy}{dt}\left(-\frac{1}{x^2}\right)$$

$$= \frac{1}{x^2}\frac{d^2y}{dt^2} - \frac{1}{x^2}\frac{dy}{dt}$$

$$x^2y'' = \frac{d^2y}{dt^2} - \frac{dy}{dt}, \text{ So } (1) \Rightarrow$$

$$\frac{d^2y}{dt^2} - \frac{dy}{dt} - \frac{dy}{dt} + y = t$$

$$\frac{d^2y}{dt^2} - 2\frac{dy}{dt} + y = t \longrightarrow (3)$$

The associated homogeneous eqtn is

$$\frac{d^2y}{dt^2} - 2\frac{dy}{dt} + y = 0$$

The auxillary eqtn is

$$m^2 - 2m + 1 = 0$$

$$\Rightarrow (m-1)^2 = 0$$

$$\Rightarrow m = 1 = m_1 = m_2$$

$$y_c = C_1 e^t + C_2 t e^t$$

$$y_p = At + B$$

$$y_p' = A, \quad y_p'' = 0$$

putting into eq (3) we have

$0 - 2A + At + B = t$

$\Rightarrow (B - 2A) + At = t$

Comparing the co-efficients we have

$\boxed{A = 1}$

$B - 2A = 0 \Rightarrow \boxed{B = 2A = 2}$

$y_p = t + 2$

$y = C_1 e^t + C_2 t e^t + t + 2$

Again using $x = e^t$

So $y = c_1 x + c_2 x \ln x + \ln x + 2$ Ans

Q2 $x^2 y'' + 9xy' - 20y = 0 \longrightarrow (*)$

let $x = e^t$, $t = \ln x$

$y' = \frac{dy}{dx} = \frac{dy}{dt}\frac{dt}{dx} = \frac{1}{x}\frac{dy}{dx}$

$xy' = \frac{dy}{dt}$

$$y'' = \frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{1}{x}\frac{dy}{dt}\right)$$

$$= -\frac{1}{x^2}\frac{dy}{dt} + \frac{1}{x}\frac{d}{dx}\left(\frac{dy}{dt}\right)$$

$$= -\frac{1}{x^2}\frac{dy}{dt} + \frac{1}{x}\frac{d}{dt}\left(\frac{dy}{dt}\right)\frac{dt}{dx}$$

$$= -\frac{1}{x^2}\frac{dy}{dt} + \frac{1}{x}\frac{d}{dt}\left(\frac{dy}{dt}\right)\frac{dt}{dx}$$

$$= -\frac{1}{x^2}\frac{dy}{dt} + \frac{1}{x^2}\frac{d^2y}{dt^2}$$

$$\Rightarrow x^2y'' = \frac{d^2y}{dt^2} - \frac{dy}{dt}$$

$$\frac{d^2y}{dt^2} - 8\frac{dy}{dt} - 20y = 0$$

$$m^2 - 8m - 20 = 0$$

$$m = \frac{8 \pm \sqrt{64 - 4(1)(20)}}{2(1)}$$

$$m = \frac{8 \pm 12}{2} = 4 \pm 6$$

$m_1 = 10, \quad m_2 = -2$

$y = c_1 e^{10t} + c_2 e^{-2t}$

$\boxed{y = c_1 x^{10} + c_2 x^{-2}}$ Ans

Q3 $x^2 y'' + 10xy' + 8y = x^2 \longrightarrow (1)$

let $x = e^t, \quad t = \ln x$

$y' = \frac{dy}{dx} = \frac{dy}{dt}\frac{dt}{dx} = \frac{1}{x}\frac{dy}{dt}$

$\Rightarrow \boxed{xy' = \frac{dy}{dt}}$

$\frac{d^2y}{dx^2} = y'' = \frac{d}{dx}\left(\frac{1}{x}\frac{dy}{dt}\right)$

$= \frac{1}{x^2}\frac{d^2y}{dt^2} - \frac{1}{x^2}\frac{dy}{dt}$

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$$\Rightarrow \boxed{x^2\frac{d^2y}{dx^2} = \frac{d^2y}{dt^2} - \frac{dy}{dt}}$$

$$\frac{d^2y}{dt^2} - \frac{dy}{dt} + 10\frac{dy}{dt} + 8y = e^{2t}$$

$$\Rightarrow \frac{d^2y}{dt^2} + 9\frac{dy}{dt} + 8y = e^{2t} \longrightarrow (3)$$

$$m^2 + 9m + 8 = 0$$

$$\Rightarrow m^2 + 8m + m + 8 = 0$$

$$\Rightarrow m(m+8) + 1(m+8) = 0$$

$$\Rightarrow (m+1)(m+8) = 0$$

$$\Rightarrow m_1 = -1, \quad m_2 = -8$$

$$y_c = c_1 e^{-t} + c_2 e^{-8t} \longrightarrow (4)$$

$$y_p = Ae^{2t}, \quad y_p' = 2Ae^{2t}$$

$$y_p'' = 4Ae^{2t}$$

$$4Ae^{2t} + 18Ae^{2t} + 8Ae^{2t} = e^{2t}$$

$$\boxed{A = \frac{1}{30}}$$

$$y_p = \frac{1}{30}e^{2t} \longrightarrow 5$$

$$y = y_c + y_p = c_1e^{-t} + C_2e^{-8t} + \frac{1}{30}e^{2t}$$

$$y = c_1\left(\frac{1}{x}\right) + C_2\frac{1}{x^8} + \frac{1}{30}x^2$$