Wronskian ⇒

Suppose $f_1, f_2, \dots, f_n$ possesses at least $n-1$ derivatives, then the determinant

$$W(f_1, f_2, \dots, f_n) = \begin{vmatrix} f_1 & f_2 & \cdots & f_n \\ f_1' & f_2' & \cdots & f_n' \\ \vdots & \vdots & & \\ f_1^{(n-1)} & f_2^{(n-1)} & \cdots & f_n^{(n-1)} \end{vmatrix},$$

where the primes denotes differentiation with respect to x, is called Wronskian of the functions.

Theorem ⇒ Let $y_1, y_2, \dots, y_n$ be n solutions of homogenous linear nth-order differential equation

$$a_n(x)\frac{d^n y}{dx^n} + \dots + a_1(x)\frac{dy}{dx} + a_0(x)y = f(x).$$

→ ①

then the set of solutions is linearly independent on interval I if & only if $W(y_1, y_2, \cdots, y_n) \neq 0$, for every x in interval I.

<u>Example:</u> The functions $y_1 = e^{x}$, $y_2 = e^{2x}$, $y_3 = e^{3x}$ are the solutions of $y''' - 6y'' + 11y'' - 6y = 0$.

Since

$$W(y_1, y_2, y_3) = \begin{vmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1'' & y_2'' & y_3'' \end{vmatrix} = \begin{vmatrix} e^{x} & e^{2x} & e^{3x} \\ e^{x} & 2e^{2x} & 3e^{3x} \\ e^{x} & 4e^{2x} & 9e^{3x} \end{vmatrix}$$

$$= e^{x} \begin{vmatrix} 2e^{2x} & 3e^{3x} \\ 4e^{2x} & 9e^{3x} \end{vmatrix} - e^{2x} \begin{vmatrix} e^{x} & 3e^{3x} \\ e^{x} & 9e^{3x} \end{vmatrix} + e^{3x} \begin{vmatrix} e^{x} & 2e^{2x} \\ e^{x} & 4e^{2x} \end{vmatrix}$$

$$= e^{x}\left[18e^{2x+3x} - 12e^{2x+3x}\right] - e^{2x}\left[9e^{x+3x} - 3e^{x+3x}\right] + e^{3x}\left[4e^{x+2x} - 2e^{x+2x}\right]$$

$W(y_1, y_2, y_3) = 6e^{6x} - 6e^{6x} + 2e^{6x} = 2e^{6x} \neq 0$

Thus, y_1, y_2, y_3 are linearly independent & form a fundamental set.

Theorem⇒ General solution - Nonhomogenous equations:

Let y_p be any particular solution of the homogenous linear nth-order differential equation (1) on an interval I and let $y_1, y_2, \ldots, y_n$ be a fundamental set of solutions of associated homogenous equation

$$a_n(x)\frac{d^n y}{dx^n} + \cdots + a_0(x)y = 0 , \quad (2)$$

then the general solution of the

equation (1) is given by

$$y = c_1 y_1(x) + c_2 y_2(x) + \dots + c_n y_n(x) + y_p, \quad (3)$$

where c_i's are arbitrary constants.

Example: $y''' - 6y'' + 11y' - 6y = 3x$, ——— (i)

has a particular solution $y_p = -\frac{11}{2} - \frac{x}{2}$.

The associated homogenous equation

$$y''' - 6y'' + 11y' - 6y = 0$$

has a solution

$$y_c = c_1 e^{x} + c_2 e^{2x} + c_3 e^{3x}.$$

Hence general solution of (i) is

$$y = y_c + y_p = c_1 e^{x} + c_2 e^{2x} + c_3 e^{3x} + \frac{11}{2} - \frac{x}{2}.$$

Theorem ⇒ Superposition principle (Non homogeneous equations).

Let $y_{p_1}, y_{p_2}, \dots, y_{p_k}$ be k particular Solutions of the nonhomogeneous linear nth-order DE $a_n(x)\frac{d^ny}{dx^n} + a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}} + \dots + a_1(x)\frac{dy}{dx} + a_0(x)y = g(x)$ on an interval I Corresponding in turn to k distinct ftns $g_1, g_2, \dots g_k$; that is, Suppose y_{p_i} denotes a particular Solution of the corresponding DE

$$a_n(x)y^n + a_{n-1}(x)y^{n-1} + \dots + a_1(x)y' + a_0(x)y = g_i(x)$$

Where $i = 1, 2, \dots k$ then

$$y_p = y_{p_1}(x) + y_{p_2}(x) + \dots y_{p_k}(x)$$

is a particular solution of

$$a_n(x)y^n + a_{n-1}(x)y^{n-1} + \dots + a_1(x)y' + a_0(x)y$$
$$= g_1(x) + g_2(x) + \dots + g_k(x).$$

Reduction of order ⇒

Suppose that y_1 denotes a nontrivial solution

$$a_2(x)y'' + a_1(x)y' + a_0(x)y = 0. \longrightarrow (1)$$

We seek a second solution, y_2 so that y_1, y_2 is a linearly independent set on I.

Now if y_1 & y_2 are linearly independent then their quotient y_2/y_1 is nonconstant on I.

Suppose $y_2/y_1 = u(x)$ or $\boxed{y_2 = u(x)y_1}$

The function $u(x)$ can be found by substituting $y_2 = u(x)y_1$ into the DE (1).

This method is called reduction of order.

Example:- Given that $y_1 = e^x$ is a solution of $y'' - y = 0$. Use reduction of order to find a second solution.

Solution:- Given that $y'' - y = 0$, —— ①

Let $y_2 = u(x)y_1$ or $y_2(x) = u(x)e^x$. ⟶ ②

Then $y' = u(x)e^x + u'(x)e^x$

& $y'' = u(x)e^x + u'(x)e^x + u'(x)e^x + u''(x)e^x$ ⟶ ③

Putting ② & ③ in ① we have

$u(x)e^x + 2u'(x)e^x + u''(x)e^x - u(x)e^x = 0$

$\Rightarrow (u'' + 2u'(x))e^x = 0$

$\Rightarrow u'' + 2u' = 0.$ ⟶ ④

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Let $U'(x) = \omega$
then $U''(x) = \frac{d\omega}{dx}$ $\longrightarrow$ (5)

Using (5) in (4) we get

$\frac{d\omega}{dx} + 2\omega = 0$. $\longrightarrow$ (6)

$I.F = \mu(x) = e^{\int P(x)dx} = e^{\int (2) dx} = e^{2x}$ $\longrightarrow$ (7)

Xing (7) by (6) we get

$e^{2x} \frac{d\omega}{dx} + 2\omega e^{2x} = 0$

$\Rightarrow$ $\frac{d}{dx}[e^{2x}\omega] = 0$, integrating we have

$e^{2x}\omega = c_1$

$\Rightarrow$ $\omega = c_1 e^{-2x}$. $\longrightarrow$ (8)

Using (5) again we get

$$U'(x) = C_1 e^{-2x}$$

Integrating we get

$$U(x) = \frac{C_1 e^{-2x}}{(-2)} + C_2$$

$$\Rightarrow U(x) = -\frac{C_1}{2} e^{-2x} + C_2. \longrightarrow 9$$

Choosing $C_2 = 0$ & $C_1 = -2$ we have

$$U(x) = e^{-2x}$$

Thus 2 implies $y_2 = e^{-2x} y_1 = e^{-2x} e^{x}$

$$\Rightarrow \boxed{y_2(x) = e^{-x}}$$

Thus, the general solution of 1 is

$$y = C_1 e^{x} + C_2 e^{-x}.$$ Ans.

Example: $y'' - 6y' + 9y = 0 \longrightarrow (1)$, $y_1 = e^{3x}$

Let $y(x) = u(x)\, y_1 = u(x) e^{3x}$

$$y' = 3u(x) e^{3x} + u'(x) e^{3x}$$

$$y'' = 9u(x) e^{3x} + 3u'(x) e^{3x} + u''(x) e^{3x} + 3u'(x) e^{3x}$$

$$y'' = 9u e^{3x} + 3u' e^{3x} + 3u' e^{3x} + u'' e^{3x}$$

$$y'' = 9u e^{3x} + 6u' e^{3x} + u'' e^{3x}.$$

Thus (1) implies

$$9u e^{3x} + 6u' e^{3x} + u'' e^{3x} - 18u e^{3x} - 6u' e^{3x} + 9u(x) e^{3x} = 0$$

$$\Rightarrow u'' e^{3x} = 0$$

$$\Rightarrow u'' = 0$$

$$\Rightarrow \boxed{u(x) = c_1 x + c_2}$$

Choosing $c_2 = 0$ & $c_1 = 1$ we have

$$u(x) = x$$

Thus $\boxed{y_2 = xe^{3x}}$.

GS is $y = c_1 e^{3x} + c_2 x e^{3x}$ Ans.

* Reduction of order formula ⇒

$$y_2(x) = y_1(x) \int \frac{e^{-\int p(x)dx}}{y_1^2(x)} dx. \longrightarrow (A)$$

Example: Solve $\underbrace{y'' + 36y = 0}_{(1)}$. $\underbrace{y_1 = \cos 6x}_{(2)}$.

Putting (2) in (A) we have

$$\therefore \quad y'' + p(x)y' + Q(x)y = 0. \longrightarrow (A_1)$$

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$$y_2(x) = \cos 6x \int \frac{e^{-\int (0)\,dx}}{\cos^2 6x}\,dx$$

$$= \cos 6x \int \frac{1}{\cos^2 6x}\,dx = \cos 6x \int \sec^2 6x\,dx$$

$$= \cos 6x \int \sec^2 6x\,dx = \cos 6x \tan 6x$$

$$= \cos 6x \frac{\sin 6x}{\cos 6x} = \sin 6x .$$

Thus GS of (1) is

$$y = c_1 \cos 6x + c_2 \sin 6x . \quad \text{Ans.}$$

Q2. $x^2 y'' - 3x y' + 5y = 0, \rightarrow (1) \quad y_1 = x^2 \cos(\ln x) . \rightarrow (2)$

Transfering ~~Comparing~~ (1) into ~~with~~ standard form we have

$$y'' - \frac{3}{x} y' + \frac{5}{x^2} y = 0 . \longrightarrow (3)$$

Comparing 3 with

$$y'' + p(x)y' + Q(x)y = 0$$

we get $p(x) = -\frac{3}{x}$

so $$y_2(x) = y_1(x) \int \frac{e^{-\int p(x)dx}}{y_1^2(x)} dx$$

$$y_2(x) = x^2 \cos(\ln x) \int \frac{e^{-\int \left(-\frac{3}{x}\right) dx}}{x^4 \cos^2(\ln x)} dx$$

$$= x^2 \cos(\ln x) \int \frac{e^{3\ln x}}{x^4 \cos^2(\ln x)} dx$$

$$= x^2 \cos(\ln x) \int \frac{x^3}{x^4 \cos^2(\ln x)} dx$$

(14)

$$y_2(x) = x^2\cos(\ln x)\int \frac{\cancel{x}}{x\cos^2(\ln x)}\,dx$$

Let $\ln x = z$ then $\frac{1}{x}dx = dz \Rightarrow e^{-z}dx = dz$

$\Rightarrow dx = e^{z}dz$

or $x = e^{z}$

$\frac{1}{x} = e^{-z}$

$$y_2(x) = x^2\cos(\ln x)\int \frac{e^{-z}\,e^{z}dz}{\cos^2 z}$$

$$= x^2\cos(\ln x)\int \frac{dz}{\cos^2 z} = x^2\cos(\ln x)\int \sec^2 z\,dz$$

$$= x^2\cos(\ln x)\tan z = x^2\cos(\ln x)\tan(\ln z)$$

$$= x^2\sin(\ln x).$$

Thus $y = c_1x^2\cos(\ln x) + c_2x^2\sin(\ln x)$. $\longrightarrow$ Ans

Homogenous Linear equations with constant coefficients:

Auxillary Equation

Consider the second order equation

$$ay'' + by' + cy = 0, \longrightarrow (1)$$

where a, b & c are constants.

Suppose $y = e^{mx}$ is a solution of (1). Then

$$y' = me^{mx} \quad \& \quad y'' = m^2 e^{mx}. \longrightarrow (2)$$

Putting (2) in (1) we get

$$am^2 e^{mx} + bme^{mx} + ce^{mx} = 0$$

$$\Rightarrow (am^2 + bm + c)e^{mx} = 0$$

$$\Rightarrow am^2 + bm + c = 0 \,. \longrightarrow (3)$$

Eq. (3) is called auxillary equation or characteristic equation.

Eq (3) is a quadratic equation, so by quadratic formula

$$m = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$$

Thus $m_1 = \dfrac{-b+\sqrt{b^2-4ac}}{2a}$ & $m_2 = \dfrac{-b-\sqrt{b^2-4ac}}{2a}$

are the two roots of (3).

Case (1) Distinct Real roots $\Rightarrow$

In this case $b^2-4ac>0$, so (3) possesses two real and distinct roots m_1 & m_2.

So the two solutions are $y_1 = e^{m_1 x}$, $y_2 = e^{m_2 x}$

Thus general solution of (1) is

$$y = c_1 e^{m_1 x} + c_2 e^{m_2 x}.$$

Case (2): Repeated Real roots:

In this case the two roots of (3) are real & equal so $m_1 = m_2 = m$, b/c $b^2 - 4ac = 0$

Thus one solution is $y_1 = e^{mx}$.

The second solution can be obtained with the help of reduction of order. Hence,

$$y_2 = y_1(x) \int \frac{e^{-\int p(x)\,dx}}{y_1^2(x)}\,dx. \quad \longrightarrow (5)$$

The standard form of (1) is

$$y'' + \frac{b}{a} y' + \frac{c}{a} y = 0.$$

So $p(x) = b/a$. Thus (5) $\Rightarrow$

$$y_2(x) = e^{mx} \int \frac{e^{-\int b/a\,dx}}{e^{2mx}}\,dx = e^{mx} \int \frac{e^{-b/a\,x}}{e^{2mx}}\,dx. \quad \longrightarrow (6)$$

Since $m = \frac{-b \pm \sqrt{b^2-4ac}}{2a} = -b/2a$

$m = -b/2a$ or $\boxed{b/a = -2m}$

Thus, (6) $\Rightarrow y_2 = e^{mx} \int \frac{e^{+2mx}}{e^{2mx}} dx$

$\Rightarrow y_2(x) = x e^{mx}$

Therefore, GS is $\boxed{y = C_1 e^{mx} + C_2 x e^{mx}}$.

Case (3): Conjugate Complex roots $\Rightarrow$

If m_1 & m_2 are complex, then we can write $m_1 = \alpha + i\beta$ & $m_2 = \alpha - i\beta$,

where α & $\beta > 0$ are real & $i^2 = -1$.

Thus, in this case we can write

$$y = C_1 e^{(\alpha+i\beta)x} + C_2 e^{(\alpha-i\beta)x}.$$

$$\Rightarrow \quad y = C_1 e^{\alpha x} e^{i\beta x} + C_2 e^{\alpha x} e^{-i\beta x}. \longrightarrow (*)$$

Now, from Euler's formula $e^{i\theta} = \cos\theta + i\sin\theta$, so

$$e^{i\beta x} = \cos\beta x + i\sin\beta x, \quad e^{-i\beta x} = \cos\beta x - i\sin\beta x.$$

Therefore $(*)$ implies

$$y = e^{\alpha x}\left[C_1(\cos\beta x + i\sin\beta x) + C_2(\cos\beta x - i\sin\beta x)\right]$$

$$y = e^{\alpha x}\left[(C_1 + C_2)\cos\beta x + i(C_1 - C_2)\sin\beta x\right]$$

$$\Rightarrow \quad y = e^{\alpha x}(c_1\cos\beta x + c_2\sin\beta x), \longrightarrow (**)$$

where $c_1 = C_1 + C_2$ & $c_2 = i(C_1 - C_2)$.

Example: $2y'' - 5y' - 3y = 0 \longrightarrow (1)$

Suppose $y = e^{mx}$, then ~~(1) implies~~ $y' = me^{mx}$, $y'' = m^2 e^{mx}$

so (1) $\Rightarrow$

$$2m^2 e^{mx} - 5me^{mx} - 3e^{mx} = 0$$

$$\Rightarrow (2m^2 - 5m - 3)e^{mx} = 0$$

Thus, the auxillary equation is

$$2m^2 - 5m - 3 = 0$$

$$\cancel{\Rightarrow 2m^2 - 2m - 3m - 3 = 0}$$

$$\cancel{\Rightarrow 2m(m-1) - 3(m-1)}$$

$$\Rightarrow 2m^2 - 6m + m - 3 = 0$$

$$\Rightarrow 2m(m-3) + 1(m-3) = 0$$

$(2m+1)(m-3)=0$

$\Rightarrow m_1=-\frac{1}{2},\ m_2=3$

So $y=c_1e^{-x/2}+c_2e^{3x}$. Ans

Example: $y''-10y'+25y=0$.

The auxillary equation is $m^2-10m+25=0$

$\Rightarrow m^2-2(5)m+(5)^2=0$

$\Rightarrow (m-5)^2=0$ $\qquad \because (a\pm b)^2=a^2+b^2\pm 2ab$

$\Rightarrow m_1=5,\ m_2=5$

So $y=c_1e^{5x}+c_2xe^{5x}$. Ans

Q. $y'' + 4y' + 7y = 0$.

The auxillary equation is

$m^2 + 4m + 7 = 0$

$$m = \frac{-4 \pm \sqrt{16 - 4(1)(7)}}{2(1)} = -2 \pm \sqrt{3}\,i$$

so $m_1 = -2 + \sqrt{3}i = \alpha + \beta i$

$m_2 = -2 - \sqrt{3}i = \alpha - i\beta$

Thus, $\alpha = -2$, $\beta = \sqrt{3}$

so $y = e^{-2x}\left[c_1 \cos\sqrt{3}x + c_2 \sin\sqrt{3}x\right]$. Ans.