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Section:
BS(CS) - 2D

Question # 1

$$y'' + 3y' + 2y = \sin e^x$$

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The associated homogeneous equation is

$$y'' + 3y' + 2y = 0$$

And the auxillary equation is

$$m^2 + 3m + 2 = 0$$

$$m^2 + m + 2m + 2 = 0$$
$$m(m+1) + 2(m+1) = 0$$
$$(m+1)(m+2) = 0$$

$$m+1 = 0 \quad , \quad m+2 = 0.$$
$$m_1 = -1 \quad , \quad m_2 = -2$$

where m_1 & m_2 are real & distinct. So, according to general solution

$$y_c = C_1 y_1 + C_2 y_2$$

Put $y = e^{mx}$

$$y_c = C_1 e^{m_1 x} + C_2 e^{m_2 x}$$

And put $m_1 = -1$ & $m_2 = -2$

$$y_c = C_1 e^{-x} + C_2 e^{-2x}$$

Now we will find the particular solution

$$y_p = u_1(x) y_1 + u_2(x) y_2$$

Since

$$y_1 = e^{-x}$$

$$y_2 = e^{-2x}$$, then

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$$

$$W = \begin{vmatrix} e^{-x} & e^{-2x} \\ -e^{-x} & -2e^{-2x} \end{vmatrix}$$

$W = (e^{-x}) \times (-2e^{-2x}) - (e^{-2x}) \times (-e^{-x})$

$W = -2e^{-3x} + e^{-3x}$

$W = -e^{-3x}$

Now we will find W_1 & W_2, so

$$W_1 = \begin{vmatrix} 0 & y_2 \\ f(x) & y_2' \end{vmatrix}$$

And $f(x) = \sin e^{x}$ with the help of standard form.

$$W_1 = \begin{vmatrix} 0 & e^{-2x} \\ \sin e^{x} & -2e^{-2x} \end{vmatrix}$$

$W_1 = 0 - e^{-2x} \times (\sin e^{x})$

$= -e^{-2x} \sin e^{x}$

And

$$W_2 = \begin{vmatrix} y_1 & 0 \\ y_1' & f(x) \end{vmatrix}$$

$$W_2 = \begin{vmatrix} e^{-x} & 0 \\ -e^{-x} & \sin e^{x} \end{vmatrix}$$

$W_2 = e^{-x} \times (\sin e^x) - 0$

$W_2 = e^{-x} \sin e^x$

Now we will find the value of u_1' & u_2', so

$$u_1' = \frac{W_1}{W}$$

$$= \frac{-e^{-2x} \sin e^x}{-e^{-3x}}$$

$$= -e^{3x} \times -e^{-2x} \sin e^x$$

$$= e^x \sin e^x$$

And

$$u_2' = \frac{W_2}{W}$$

$$= \frac{e^{-x} \sin e^x}{-e^{-3x}}$$

$$= -e^{+3x} \times e^{-x} \sin e^x$$

$$= -e^{2x} \sin e^x$$

But we need the value of u_1 & u_2 so we will take integral on b/s.

$$u_1 = \int e^x \sin e^x \, dx$$

Let $v = e^x$

$dv = e^x \, dx$

$$u_1 = \int \sin v \, \cancel{dx} \, dv$$

$$u_1 = -\cos v$$

$$u_1 = -\cos e^x$$

Now

$$u_2 = -\int e^{2x} \sin e^x \, dx.$$

$$u_2 = -\int e^x . e^x \sin e^x \, dx.$$

Let $e^x = s$

$ds = e^x \, dx.$

$$u_2 = -\int s \, \cancel{x} \sin s \, ds.$$

Now we will use integration by parts i.e.

$v = s$, $dv = ds$, and let $dn = \sin s \, ds$, then $n = -\cos s$, so.

$u_2 = -(v \times n - \int n\, dv)$ $\quad (\because \int AB\,dx = A\int B\,dx - \int(\int B\,dx)\cdot A'\,dx)$

$u_2 = -(-s\cos s - \int -\cos s\, ds)$

$= -(-s\cos s + \sin s)$

$= s\cos s - \sin s$

$u_2 = e^x \cos e^x - \sin e^x$

Now putting the values in equation

$$y_p = u_1 y_1 + u_2 y_2$$

$y_p = (-\cos e^x) \times (e^{-x}) + (e^x \cos e^x - \sin e^x) \times (e^{-2x})$

$= -e^{-x}\cancel{\cos e^x} + e^{-x}\cancel{\cos e^x} - e^{-2x}\sin e^x$

$= -e^{-2x}\sin e^x$

Now putting the values in general solution

$$y = y_c + y_p$$

$y = C_1 e^{-x} + C_2 e^{-2x} - e^{-2x}\sin e^x$ Ans.

Question # 3

$$3y'' - 6y' + 6y = e^x \sec x$$

$$3y'' - 6y' + 6y = e^x \sec x$$

Dividing both sides with 3 to obtain standard form

$$y'' - 2y' + 2y = \frac{e^x \sec x}{3}$$

The associated homogeneous equation is

$$y'' - 2y' + 2y = 0$$

And the auxillary equation is

$$m^2 \mp 2m + 2 = 0$$

Now we will find the roots, i.e.

$$m = \frac{2 \pm \sqrt{4 - 4(1)(2)}}{2(1)}.$$

$$m = \frac{2 \pm \sqrt{-4}}{2} \Rightarrow \frac{2 \pm 2\sqrt{-1}}{2} \Rightarrow \frac{2(1 \pm \sqrt{-1})}{2}$$

$$m = \cancel{1 \pm \sqrt{-4}}\ 1 \pm i$$

So

$$m_1 = 1 + \cancel{\sqrt{-4}}\, i \quad \& \quad m_2 = 1 - \cancel{\sqrt{-4}}\, i$$

Where m_1 & m_2 are conjugate and complex roots.

So, according to general solution.

$$y_c = C_1 y_1 + C_2 y_2$$

Put $y = e^{mx}$

$$y_c = C_1 e^{m_1 x} + C_2 e^{m_2 x}$$

And put $m_1 = 1 + i$ & $m_2 = 1 - i$

$$y_c = C_1 e^{(1+i)x} + C_2 e^{(1-i)x}$$

$$y_c = C_1 e^{x} e^{ix} + C_2 e^{x} e^{-ix} \quad \text{— (1)}$$

Now from Euler's formula.

$$e^{i\theta} = \cos\theta + i\sin\theta, \text{ so.}$$

$$e^{ix} = \cos x + i \sin x \quad \&$$

$$e^{-ix} = \cos x - i \sin x.$$

Putting the values in (1).

$$y_c = C_1 e^{x} (\cos x + i \sin x) + C_2 e^{x} (\cos x - i \sin x)$$

$$= C_1 e^{x} \cos x + C_1 e^{x} i \sin x + C_2 e^{x} \cos x - C_2 e^{x} i \sin x.$$

$$= C_1 e^{x} \cos x + C_2 e^{x} \cos x + C_1 e^{x} i \sin x - C_2 e^{x} i \sin x$$

$y_c = C_1 e^x \cos x + C_2 e^x \cos x + C_1 e^x i \sin x - C_2 e^x i \sin x$

$y_c = (C_1 + C_2) e^x \cos x + i(C_1 - C_2) e^x \sin x.$

$y_c = C_1 e^x \cos x + C_2 e^x \sin x$

Now we will find the particular solution

$$y_p = u_1 y_1 + u_2 y_2$$

Since

$$y_1 = e^x \cos x$$

$$y_2 = e^x \sin x \text{ , then}$$

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$$

$$W = \begin{vmatrix} e^x \cos x & e^x \sin x \\ e^x(\cos x - \sin x) & e^x(\cos x + \sin x) \end{vmatrix}$$

$W = (e^x \cos x)(e^x[\cos x + \sin x]) - (e^x \sin x)(e^x(\cos x - \sin x))$

$W = e^{2x} \cos^2 x + \cancel{e^{2x} \sin x \cos x} - \cancel{e^{2x} \sin x \cos x} + e^{2x} \sin^2 x.$

$W = e^{2x} (\cos^2 x + \sin^2 x)$

$W = e^{2x}(\cos^2 x + \sin^2 x)$

$W = e^{2x}(1)$ $\quad(\because \sin^2\theta + \cos^2\theta = 1)$

$W = e^{2x}$

Now we will find W_1 & W_2, so.

$$W_1 = \begin{vmatrix} 0 & y_2 \\ f(x) & y_2' \end{vmatrix}$$

Where $f(x) = \frac{e^x \sec x}{3}$, with the help of standard form.

$$W_1 = \begin{vmatrix} 0 & e^x \sin x \\ \frac{e^x \sec x}{3} & e^x(\cos x + \sin x) \end{vmatrix}$$

$$W_1 = 0 - (e^x \sin x)\left(\frac{e^x \sec x}{3}\right)$$

$$W_1 = 0 - \frac{1}{3} e^{2x} \sin x \times \frac{1}{\cos x}$$

$$W_1 = -\frac{1}{3} e^{2x} \tan x.$$

And

$$W_2 = \begin{vmatrix} y_1 & 0 \\ y_1' & f(x) \end{vmatrix}$$

$$W_2 = \begin{vmatrix} e^x \cos x & 0 \\ e^x(\cos x - \sin x) & \frac{e^x \sec x}{3} \end{vmatrix}$$

$$W_2 = (e^x \cos x)\left(\frac{e^x \sec x}{3}\right) - 0$$

$$W_2 = \frac{1}{3} e^{2x} \cancel{\cos x} \times \frac{1}{\cancel{\cos x}}$$

$$W_2 = \frac{1}{3} e^{2x}.$$

Now we will find the value of u_1' & u_2', so

$$u_1' = \frac{W_1}{W}$$

$$u_1' = \frac{-1/3\, \cancel{e^{2x}} \tan x}{\cancel{e^{2x}}}$$

$$u_1' = -\frac{1}{3} \tan x$$

And

$$u_2' = \frac{W_2}{W}$$

$$= \frac{1/3\,\cancel{e^{2x}}}{\cancel{e^{2x}}}$$

$$u_2' = \frac{1}{3}$$

But we need the value of u_1 & u_2, so we will take integral on b/s.

$$u_1 = \int -\frac{1}{3} \tan x \, dx$$

$$u_1 = \frac{1}{3} \int -\frac{\sin x}{\cos x} \, dx$$

$$u_1 = \frac{1}{3} \ln(\cos x).$$

And

$$u_2 = \int \frac{1}{3} \, dx.$$

$$u_2 = \frac{1}{3} \int 1 \, dx$$

$$u_2 = \frac{1}{3} x.$$

Now putting the values in equation

$$y_p = u_1 y_1 + u_2 y_2$$

$$y_p = \left(\frac{1}{3}\ln(\cos x)\right) \times (e^x \cos x) + \left(\frac{1}{3}x\right) \times (e^x \sin x)$$

$$y_p = \frac{1}{3} e^x \cos x \ln(\cos x) + \frac{1}{3} x e^x \sin x$$

Now putting the values in general solution

$$y = y_c + y_p$$

$$y = C_1 e^x \cos x + C_2 e^x \sin x + \frac{1}{3} e^x \cos x \ln \cos x + \frac{1}{3} x e^x \sin x \quad \text{Ans}$$

Question # 4

$$4y'' - y = x e^{1/2 x} \quad , \quad y(0) = 0 , \; y'(0) = 0$$

$$4y'' - y = x e^{1/2 x}$$

Dividing both sides by 4 to get standard form.

$$y'' - \frac{1}{4} y = \frac{x e^{1/2 x}}{4}$$

The associated homogeneous equation is

$$y'' - \frac{1}{4} y = 0$$

And the auxillary equation is

$$m^2 - \frac{1}{4} = 0.$$

$$\left(m - \frac{1}{2}\right)\left(m + \frac{1}{2}\right) = 0$$

$$m_1 = \frac{1}{2}, \quad m_2 = -\frac{1}{2}$$

where m_1 & m_2 are real and distinct. So according to general solution.

$$y_c = C_1 y_1 + C_2 y_2$$

Put $y = e^{mx}$

$$y_c = C_1 e^{m_1 x} + C_2 e^{m_2 x}$$

And put $m_1 = \frac{1}{2}$ & $m_2 = -\frac{1}{2}$

$$y_c = C_1 e^{x/2} + C_2 e^{-x/2}$$

Now we will find partial particular solution

$$y_p = u_1 y_1 + u_2 y_2$$

Since

$$y_1 = e^{x/2}$$

$$y_2 = e^{-x/2}$$

then

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$$

$$= \begin{vmatrix} e^{x/2} & e^{-x/2} \\ \frac{1}{2}e^{x/2} & -\frac{1}{2}e^{-x/2} \end{vmatrix}$$

$$= \left(e^{x/2}\right) \times \left(\frac{-1}{2}e^{-x/2}\right) - \left(e^{-x/2}\right) \times \left(\frac{1}{2}e^{x/2}\right)$$

$$= -\frac{1}{2}e^{0} - \frac{1}{2}e^{0}$$

$W = -1.$

Now we will find W_1 & W_2, so

$$W_1 = \begin{vmatrix} 0 & y_2 \\ f(x) & y_2' \end{vmatrix}$$

And $f(x) = \frac{xe^{x/2}}{4}$, with the help of standard form.

$$W_1 = \begin{vmatrix} 0 & e^{-x/2} \\ \frac{xe^{x/2}}{4} & -\frac{1}{2}e^{-x/2} \end{vmatrix}$$

$$W_1 = 0 - (e^{-x/2}) \times \left(\frac{xe^{x/2}}{4}\right)$$

$$= \frac{-xe^{0}}{4}.$$

$$= \frac{-x}{4}.$$

And

$$W_2 = \begin{vmatrix} y_1 & 0 \\ y_1' & f(x) \end{vmatrix}$$

$$W_2 = \begin{vmatrix} e^{x/2} & 0 \\ \frac{1}{2}e^{x/2} & \frac{xe^{x/2}}{4} \end{vmatrix}$$

$$W_2 = (e^{x/2}) \times \left(\frac{xe^{x/2}}{4}\right) - 0$$

$$= \frac{xe^{x}}{4}$$

Now we will find the value of u_1' & u_2', so

$$u_1' = \frac{W_1}{W}$$

$$= \frac{-x/4}{-1}$$

$$u_1' = \frac{x}{4}$$

And

$$u_2' = \frac{W_2}{W}$$

$$= \frac{xe^x/4}{-1}$$

$$u_2' = -\frac{xe^x}{4}$$

But we need the value of u_1 & u_2, so we will take integral on b/s.

$$u_1 = \int \frac{x}{4}\, dx$$

$$u_1 = \frac{1}{4}\int x\, dx$$

$$u_1 = \frac{1}{8}x^2$$

And

$$u_2 = \int -\frac{xe^x}{4}\, dx$$

$$u_2 = \frac{-1}{4}\int xe^x\, dx.$$

Now we will use integration by parts i.e.

Let $s = x$, then we have $ds = dx$, and if we set $dv = e^x$, then we have $v = e^x$, so.

$$u_2 = \frac{-1}{4}\left(s \times v - \int v\,ds\right) \quad \left(\because \int AB\,dx = A\int B\,dx - \int\left(\left(\int B\,dx\right) \cdot A'\,dx\right)\right)$$

$$u_2 = \frac{-1}{4}\left(xe^x - \int e^x\,dx\right)$$

$$u_2 = \frac{-1}{4}(xe^x - e^x)$$

$$u_2 = -\frac{1}{4}xe^x + \frac{1}{4}e^x$$

Now putting the values in equation

$$y_p = u_1 y_1 + u_2 y_2$$

$$y_p = \left(\frac{1}{8}x^2\right) \times (e^{x/2}) + \left(\frac{-1}{4}xe^x + \frac{1}{4}e^x\right) \times (e^{-x/2}).$$

$$y_p = \frac{1}{8}x^2 e^{x/2} - \frac{1}{4}xe^{x/2} + \frac{1}{4}e^{x/2}$$

Now putting the values in general solution

$$y = y_c + y_p$$

$$y = C_1 e^{x/2} + C_2 e^{-x/2} + \frac{1}{8}x^2 e^{x/2} - \frac{1}{4}xe^{x/2} + \frac{1}{4}e^{x/2}$$

$$y = \left(C_1 + \frac{1}{4}\right)e^{x/2} + C_2 e^{-x/2} + \frac{1}{8}x^2 e^{x/2} - \frac{1}{4}xe^{x/2}$$

$$y = C_1 e^{x/2} + C_2 e^{-x/2} + \frac{1}{8} x^2 e^{x/2} - \frac{1}{4} x e^{x/2} \quad \text{— (A)}$$

Now we will apply the point $(x, y) = (0, 1)$ in above equation.

$$0 = C_1 e^0 + C_2 e^0 + 0 + 0$$

$$C_1 + C_2 = 0 \quad \text{— (1)}$$

Now we will take the first derivative of general solution.

$$y' = \frac{1}{2} C_1 e^{x/2} - \frac{1}{2} C_2 e^{-x/2} + \cancel{\frac{1}{4} x e^{x/2}} + \frac{1}{16} x^2 e^{x/2} - \cancel{\frac{1}{4} e^{x/2}} - \frac{1}{8} x e^{x/2}$$

$$y' = \frac{1}{2} C_1 e^{x/2} - \frac{1}{2} C_2 e^{-x/2} + \frac{1}{16} x^2 e^{x/2} + \frac{1}{8} x e^{x/2} - \frac{1}{4} e^{x/2}$$

Now we will apply the point $(x, y') = (0, 0)$ in above equation.

$$0 = \frac{1}{2} C_1 e^0 - \frac{1}{2} C_2 e^0 + 0 + 0 - \frac{1}{4} e^0$$

$$\frac{1}{2} C_1 - \frac{1}{2} C_2 = \frac{1}{4} \quad \text{— (2)}$$

Solving eq 1 & 2 we get

$$C_1 = \frac{1}{4} \quad \& \quad C_2 = \frac{3}{4}.$$

Putting the value of C_1 & C_2 in eq (A).

$$y = \frac{1}{4}e^{x/2} + \frac{3}{4}e^{-x/2} + \frac{1}{8}x^2 e^{x/2} - \frac{1}{4}xe^{x/2}$$ Ans

Question # 5

$$y'' + 2y' - 8y = 2e^{-2x} - e^{-x} \quad , \quad y(0) = 1 , y'(0) = 0$$

$$y'' + 2y' - 8y = 2e^{-2x} - e^{-x}$$

The associated homogeneous equation is

$$y'' + 2y' - 8y = 0$$

And the auxillary equation is

$$m^2 + 2m - 8 = 0$$

$$m^2 + 4m - 2m - 8 = 0$$
$$m(m+4) - 2(m+4) = 0$$
$$(m-2)(m+4) = 0$$
$$m_1 = 2 \quad , \quad m_2 = -4$$

Where m_1 & m_2 are real and distinct, so according to general solution.

$$y_c = C_1 y_1 + C_2 y_2$$

Put $y = e^{mx}$

$$y_c = C_1 e^{m_1 x} + C_2 e^{m_2 x}$$

And put $m_1 = 2$ & $m_2 = -4$

$$y_c = C_1 e^{2x} + C_2 e^{-4x}$$

Now we will find particular solution.

$$y_p = u_1 y_1 + u_2 y_2$$

Since

$$y_1 = e^{2x}$$

$$y_2 = e^{-4x}$$

then

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$$

$$W = \begin{vmatrix} e^{2x} & e^{-4x} \\ 2e^{2x} & -4e^{-4x} \end{vmatrix}$$

$$W = (e^{2x})(-4e^{-4x}) - (e^{-4x})(2e^{2x})$$

$$= -4e^{-2x} - 2e^{-2x}$$

$$= -6e^{-2x}$$

Now we will find W_1 & W_2, so

$$W_1 = \begin{vmatrix} 0 & y_2 \\ f(x) & y_2' \end{vmatrix}$$

$$W_1 = \begin{vmatrix} 0 & e^{-4x} \\ 2e^{-2x} - e^{-x} & -4e^{-4x} \end{vmatrix}$$

$f(x) = 2e^{-2x} - e^{-x}$, with the help of standard form.

$$W_1 = 0 - (e^{-4x}) \times (2e^{-2x} - e^{-x})$$

$$W_1 = -2e^{-6x} + e^{-5x}$$

And

$$W_2 = \begin{vmatrix} y_1 & 0 \\ y_1' & f(x) \end{vmatrix}$$

$$W_2 = \begin{vmatrix} e^{2x} & 0 \\ 2e^{2x} & 2e^{-2x} - e^{-x} \end{vmatrix}$$

$$W_2 = (e^{2x})(2e^{-2x} - e^{-x}) - 0$$

$$= 2 - e^{+x}$$

Now we will find the value of u_1' & u_2', So.

$$u_1' = \frac{W_1}{W}$$

$$= \frac{-2e^{-6x} + e^{-5x}}{-6e^{-2x}}$$

$$u_1' = \frac{1}{3}e^{-4x} - \frac{1}{6}e^{-3x}$$

And

$$u_2' = \frac{W_2}{W}$$

$$= \frac{2 - e^{x}}{-6e^{-2x}}$$

$$u_2' = -\frac{1}{3}e^{2x} + \frac{1}{6}e^{3x}$$

But we need the value of u_1 & u_2, so we will take integral on b/s.

$$u_1 = \int \left(\frac{1}{3}e^{-4x} - \frac{1}{6}e^{-3x} \right) dx$$

$$u_1 = \int \frac{1}{3}e^{-4x}\,dx - \int \frac{1}{6}e^{-3x}\,dx$$

$$u_1 = \int \frac{1}{3} e^{-4x} dx - \int \frac{1}{6} e^{-3x} dx.$$

$$u_1 = \frac{1}{3} e^{-4x} \times \frac{-1}{4} - \frac{1}{6} e^{-3x} \times \frac{-1}{3}$$

$$u_1 = \frac{-1}{12} e^{-4x} + \frac{1}{18} e^{-3x}$$

And

$$u_2 = \int \left(\frac{-1}{3} e^{2x} + \frac{1}{6} e^{3x} \right) dx$$

$$u_2 = -\int \frac{1}{3} e^{2x} dx + \int \frac{1}{6} e^{3x} dx$$

$$u_2 = -\frac{1}{3} e^{2x} \times \frac{1}{2} + \frac{1}{6} e^{3x} \times \frac{1}{3}$$

$$u_2 = -\frac{1}{6} e^{2x} + \frac{1}{18} e^{3x}$$

Now putting the values in equation

$$y_p = u_1 y_1 + u_2 y_2$$

$$y_p = \left(-\frac{1}{12} e^{-4x} + \frac{1}{18} e^{-3x} \right) . (e^{2x}) + \left(-\frac{1}{6} e^{2x} + \frac{1}{18} e^{3x} \right) (e^{-4x})$$

$$y_p = -\frac{1}{12}e^{-2x} + \frac{1}{18}e^{-x} - \frac{1}{6}e^{-2x} + \frac{1}{8}e^{-x}$$

$$y_p = -\frac{1}{4}e^{-2x} + \frac{1}{9}e^{-x}$$

Now putting the values is general solution

$$y = y_c + y_p$$

$$y = C_1 e^{2x} + C_2 e^{-4x} - \frac{1}{4}e^{-2x} + \frac{1}{9}e^{-x} \quad — (A)$$

Now we will apply the point $(x, y) = (0, 1)$ in eq (A).

$$1 = C_1 e^0 + C_2 e^0 - \frac{1}{4}e^0 + \frac{1}{9}e^0$$

$$C_1 + C_2 = \frac{41}{36} \quad — (1)$$

Now we will take the first derivative of general solution.

$$y' = 2c_1 e^{2x} - 4c_2 e^{-4x} + \frac{1}{2}e^{-2x} - \frac{1}{9}e^{-x} \quad — (B)$$

Now we will apply the point $(x, y') = (0, 0)$ in eq (B)

$$0 = 2c_1 e^0 - 4c_2 e^0 + \frac{1}{2}e^0 - \frac{1}{9}e^0$$

$$2c_1 - 4c_2 = -\frac{7}{18} \quad — (2)$$

Solving eq ① & ②, we get:

$$C_1 = \frac{25}{36} \quad \& \quad C_2 = \frac{4}{9}$$

Putting the values of C_1 & C_2 in eq Ⓐ.

$$y = \frac{25}{36}e^{2x} + \frac{4}{9}e^{-4x} - \frac{1}{4}e^{-2x} + \frac{1}{9}e^{-x} \quad \text{Ans}$$

Question # 2

$$y'' + 2y' + y = e^{-t}\ln t$$

$$y'' + 2y' + y = e^{-t}\ln t$$

The associated homogeneous equation is

$$y'' + 2y' + y = 0.$$

And the auxillary equation is

$$m^2 + 2m + 1 = 0.$$

$$(m+1)^2 = 0$$

$$(m+1)(m+1) = 0$$

$$m_1 = m_2 = -1$$

Where m_1 & m_2 are repeated real roots, so according to general ~~s~~ solution.

$y_c = C_1 y_1 + C_2 y_2$

Put

$y_1 = e^{m_1 x}$ & $y_2 = x e^{m_2 x}$

$y_c = C_1 e^{m_1 t} + C_2 t e^{m_2 t}$

Put m_1 & $m_2 = -1$

$y_c = C_1 e^{-t} + C_2 t e^{-t}$

Now we will find the particular solution

$y_p = u_1 y_1 + y_2 u_2$

Since

$y_1 = e^{-t}$, $y_2 = te^{-t}$, then

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$$

$$W = \begin{vmatrix} e^{-t} & te^{-t} \\ -e^{-t} & e^{-t} - te^{-t} \end{vmatrix}$$

$W = e^{-2t} - \cancel{te^{-2t}} + \cancel{te^{-2t}}$

$W = e^{-2t}$

Now we will find W_1 & W_2, so

$$W_1 = \begin{vmatrix} 0 & y_2 \\ f(x) & y_2' \end{vmatrix}$$

And $f(x) = e^{-t}\ln t$ with the help of standard form.

$$W_1 = \begin{vmatrix} 0 & te^{-t} \\ e^{-t}\ln t & e^{-t} - te^{-t} \end{vmatrix}$$

$$W_1 = 0 - (e^{-t}\ln t)(te^{-t})$$

$$= -te^{-2t}\ln t.$$

And

$$W_2 = \begin{vmatrix} y_1 & 0 \\ y_1' & f(x) \end{vmatrix}$$

$$W_2 = \begin{vmatrix} e^{-t} & 0 \\ -e^{-t} & e^{-t}\ln t \end{vmatrix}$$

$$W_2 = (e^{-t})(e^{-t}\ln t) - 0$$

$$= e^{-2t}\ln t.$$

Now we will find the value of u_1' & u_2', so

$$u_1' = \frac{W_1}{W}$$

$$= \frac{-t\cancel{e^{-2t}}\ln t}{\cancel{e^{-2t}}}$$

$$u_1' = -t\ln t$$

And

$$u_2' = \frac{W_2}{W}$$

$$= \frac{\cancel{e^{-2t}}\ln t}{\cancel{e^{-2t}}}$$

$$u_2' = \ln t$$

But we need the value of u_1 & u_2, so we will take integral on b/s.

$$u_1 = \int -t\ln t \, dt$$

$$u_1 = -\int t\ln t \, dt$$

Now we will use integration by parts.

$$u_1 = -\ln t \int t\,dt + \int \left(\int t\,dt\right) \cdot \frac{d}{dt} \ln t \; dt$$

$$u_1 = -\ln t \cdot \frac{t^2}{2} + \int \frac{1}{t} \cdot \frac{t^2}{2} \, dt$$

$$u_1 = -\frac{t^2 \ln t}{2} + \int \frac{t}{2} \, dt$$

$$u_1 = \frac{-t^2 \ln t}{2} + \frac{t^2}{4}$$

Now

$$u_2 = \int \ln t \, dt.$$

$$u_2 = \int \ln t \times 1 \, dt.$$

Now we will use integration by parts.

$$u_2 = \ln t \int dt - \int \left(\int dt\right) \cdot \frac{d}{dt}(\ln t) \, dt.$$

$$u_2 = \ln t \cdot t - \int \frac{1}{t} \cdot t \, dt.$$

$$u_2 = t \ln t - t.$$

Now putting the values in equation

$$y_p = u_1 y_1 + u_2 y_2$$

$$y_p = \left(\frac{-t^2 \ln t}{2} + \frac{t^2}{4}\right) . (e^{-t}) + (t \ln t - t) . (t e^{-t})$$

$$y_p = \left(\frac{-t^2 \ln t}{2} + \frac{t^2}{4}\right) . (e^{-t}) + (t^2 \ln t - t) . (e^{-t})$$

$$y_p = \left(\frac{-t \ln t}{2} + \frac{t^2}{4} + t^2 \ln t - t^2\right) . (e^{-t})$$

$$y_p = \left(\frac{t^2 \ln t}{2} - \frac{3t^2}{4}\right) . (e^{-t})$$

Now putting the values in general solution

$$y = y_c + y_p$$

$$y = c_1 e^{-t} + c_2 t e^{-t} + \left(\frac{t^2 \ln t}{2} - \frac{3t^2}{4}\right) . (e^{-t}) \quad \text{Ans}$$