Solving a Homogenous DE

Solve $(x^2+y^2)dx + (x^2 - xy)dy = 0.$ $\longrightarrow$ (1)

Solution ⇒ $M(x,y) = x^2+y^2$, $N(x,y) = x^2 - xy$

Clearly both are homogenous functions of the same degree.

let $\boxed{y = ux \quad \text{then} \quad dy = u\,dx + x\,du}$ $\longrightarrow$ (2)

Using (2) in (1) we have

$$(x^2 + u^2x^2)dx + (x^2 - xux)(dx(u) + x\,du) = 0$$

$$\Rightarrow x^2(1+u^2)dx + x^2(1-u)(u\,dx + x\,du) = 0$$

$$\Rightarrow (1+u^2)\,dx + (1-u)(u\,dx + x\,du) = 0$$

$$\Rightarrow [1+u^2+u-u^2]dx + x(1-u)\,du = 0$$

$$\Rightarrow (1+u)dx + x(1-u)\,du = 0$$

$$\Rightarrow (1+u)dx = -x(1-u)du$$

$$\Rightarrow \frac{dx}{x} = \frac{(u-1)\,du}{(u+1)}$$

Integrating on both sides we have

$$\int \frac{dx}{x} = \int \frac{(u-1+1-1)}{(u+1)}du$$

$$\Rightarrow \ln x + \ln c = \int \left(1 - \frac{2}{u+1}\right)du$$

$$\Rightarrow \ln x + \ln c = u - 2\ln|u+1|$$

or $\ln(cx) = u - \ln|u+1|^2$

$$e^{\ln(cx)} = e^{u - \ln|u+1|^2}$$

or $$e^{\ln(cx)} = e^{u} e^{\ln|u+1|^{-2}}$$

or $$cx = e^{u}(u+1)^{-2}$$

or $$cx = \frac{e^{u}}{(u+1)^2} \cdot \longrightarrow (3)$$

Resubstituting $y = ux$ we have

$$cx = \frac{e^{y/x}}{\left(\frac{y}{x}+1\right)^2} \cdot \quad \text{Ans}$$

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Solve $-y\,dx + (x + \sqrt{xy})\,dy = 0 \longrightarrow (1)$

$M = -y$, $N = x + \sqrt{xy}$, clearly both M & N are homogenous of degree one.

let $y = ux$, $dy = u\,dx + x\,du$.

So (1) implies

$-ux\,dx + (x + \sqrt{x}\sqrt{u}\sqrt{x})(u\,dx + x\,du) = 0$.

or $-ux\,dx + x(1+\sqrt{u})(u\,dx + x\,du) = 0$

$\Rightarrow -u\,dx + (1+\sqrt{u})(u\,dx + x\,du) = 0$

$\Rightarrow -u\,dx + u(1+\sqrt{u})\,dx + x(1+\sqrt{u})\,du = 0$

$\Rightarrow (-u + u + u^{3/2})\,dx + x(1+\sqrt{u})\,du = 0$

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$$u^{3/2}\,dx + x(1+\sqrt{u})\,du = 0$$

or $$u^{3/2}\,dx = -x(1+\sqrt{u})\,du$$

or $$\frac{dx}{x} = -\frac{(1+\sqrt{u})}{u^{3/2}}\,du$$

or $$\int\frac{dx}{x} = -\int\left(u^{-3/2} + u^{-1}\right)du$$

or $$\ln x + \ln c = -\frac{u^{-3/2+1}}{\left(-3/2+1\right)} - \ln u$$

or $$\ln cx = 2u^{-1/2} - \ln u. \longrightarrow (2)$$

Reputting $y = ux$ we have

$$\ln cx = \frac{2}{\sqrt{y/x}} - \ln\left(y/x\right).$$ Ans.

Bernoulli's Equation ⇒

The differential equation

$$\frac{dy}{dx} + P(x)y = f(x)y^n, \longrightarrow (1)$$

where n is a real number, is called Bernoulli's equation. Note that for $n=0$ & $n=1$, (1) is linear. For $n \neq 0$ & $n \neq 1$, the substitution $u = y^{1-n}$ reduces any equation of the form (1) to a linear equation.

Q//

Solve $x\dfrac{dy}{dx} + y = x^2y^2$. $\longrightarrow$ (1)

We first rewrite the equation in standard form as

$$\frac{dy}{dx} + \frac{y}{x} = xy^2. \longrightarrow (2)$$

Comparing with Bernoulli's equation we have

$\boxed{n = 2.}$

So let $u = y^{1-n} = y^{1-2} = y^{-1}$. or $\boxed{y = u^{-1}}$

$$\Rightarrow \boxed{\frac{dy}{dx} = -u^{-2}\frac{du}{dx} = -\frac{1}{u^2}\frac{du}{dx}} \quad (3)$$

Using (3) into (2) we have

$$-\frac{1}{u^2}\frac{du}{dx}+\frac{1}{x}u^{-1}=xu^{-2}$$

or $$-\frac{1}{u^2}\frac{du}{dx}+\frac{1}{xu}=\frac{x}{u^2}$$

or $$\boxed{\frac{du}{dx}-\frac{u}{x}=-x} \rightarrow (4)$$

which is linear equation in u. Thus, the integrating factor is

$$\mu(x)=e^{\int P(x)\,dx}=e^{\int\left(-\frac{1}{x}\right)dx}=e^{-\ln x}=e^{\ln x^{-1}}=x^{-1}$$

$$\mu(x)=x^{-1}. \rightarrow (5)$$

xing (4) & (5) we have

$$x^{-1}\frac{du}{dx} - x^{-1}\frac{u}{x} = -\frac{x}{x}$$

$$\Rightarrow \frac{d}{dx}\left[u x^{-1}\right] = -1$$, integration yields

$$u x^{-1} = -x + c$$

or $\boxed{u = -x^2 + cx}$ $\longrightarrow$ 6

Re putting $y = u^{-1}$ or $u = y^{-1}$ we get

$$y^{-1} = -x^2 + cx$$

or $\boxed{y = \frac{1}{cx - x^2}}$

required solution.

Q. $y^{1/2}\frac{dy}{dx} + y^{3/2} = 1$, $y(0) = 4$. $\rightarrow$ (1)

Solution $\Rightarrow$ $\frac{dy}{dx} + \frac{y^{3/2}}{y^{1/2}} = y^{-1/2}$

$$\Rightarrow \frac{dy}{dx} + y = y^{-1/2}. \longrightarrow (2)$$

$\boxed{\text{Here } n = -\tfrac{1}{2}, \text{ so } u = y^{1-n}}$,

Let $u = y^{1-(-1/2)} = y^{3/2}$ or $u^{2/3} = y^{\frac{3}{2}\times\frac{2}{3}}$

or $\boxed{y = u^{2/3}}$ or $y^{3/2} = u$.

$$\frac{dy}{dx} = \frac{2}{3}u^{\frac{2}{3}-1}\frac{du}{dx} = \frac{2}{3}u^{-1/3}\frac{du}{dx}$$

$$\rightarrow \boxed{\frac{dy}{dx} = \frac{2}{3}u^{-1/3}\frac{du}{dx}}$$

Therefore (2) implies

$$\frac{2}{3}u^{-1/3}\frac{du}{dx} + u^{2/3} = \left(u^{\frac{2}{3}}\right)^{-1/2}$$

or $$\frac{2}{3}u^{-1/3}\frac{du}{dx} + u^{2/3} = u^{-1/3}$$

or $$\frac{2}{3}\frac{du}{dx} + \frac{u^{2/3}}{u^{-1/3}} = \frac{u^{-1/3}}{u^{-1/3}}$$

or $$\frac{2}{3}\frac{du}{dx} + u = 1$$

or $$\frac{du}{dx} + \frac{3}{2}u = \frac{3}{2}. \longrightarrow (3)$$

$$\mu(x) = e^{\int \frac{3}{2}dx} = e^{3/2\,x}. \longrightarrow (4)$$

xing (3) by (4) gives

$$e^{\frac{3}{2}x}\frac{du}{dx} + \frac{3}{2}e^{\frac{3}{2}x}u = \frac{3}{2}e^{\frac{3}{2}x}$$

$$\frac{d}{dx}\left[ue^{\frac{3}{2}x}\right] = \frac{3}{2}e^{\frac{3}{2}x}$$

$$ue^{\frac{3}{2}x} = \frac{3}{2}\int e^{\frac{3}{2}x}\,dx$$

$$\Rightarrow \quad ue^{\frac{3}{2}x} = \frac{3}{2}\frac{e^{\frac{3}{2}x}}{\left(\frac{3}{2}\right)} + C \quad \text{or} \quad u = \frac{e^{\frac{3}{2}x}}{e^{\frac{3}{2}x}} + \frac{C}{e^{\frac{3}{2}x}}$$

$$\Rightarrow \quad \boxed{u = 1 + Ce^{-\frac{3}{2}x}}$$

Substituting again $u = y^{3/2}$ we have

$$y^{3/2} = 1 + Ce^{-\frac{3}{2}x}$$

$$\text{or} \quad y = \left(1 + Ce^{-\frac{3}{2}x}\right)^{2/3} \quad \text{Ans}$$

$y(0)=4$ implies $4 = (1 + Ce^{0})^{2/3}$ or $1 + C = (4)^{3/2}$ or $\boxed{C = -1 + 4^{3/2}}$

Redution to Separation of variables ⇒

~~Solve~~ $\frac{dy}{dx}$ A differential equation of the form $\frac{dy}{dx} = f(Ax+By+C)$, $\rightarrow$ (*)

can always be redued to separable equation by means of substitution

$\boxed{u = Ax+By+C}$. $\longrightarrow$ (**)

Example: Solve $\frac{dy}{dx} = (-2x+y)^2 - 7$. —— (1)

Let $u = -2x+y$, $\frac{du}{dx} = -2 + \frac{dy}{dx}$

$$\frac{dy}{dx} = \frac{du}{dx} + 2 \longrightarrow (2)$$

Using (2) in (1) we have

$$\frac{du}{dx} + 2 = u^2 - 7$$

or $\frac{du}{dx} = u^2 - 9$ or $\frac{du}{u^2-9} = dx$

$$\int \frac{du}{(u-3)(u+3)} = \int dx \longrightarrow (3)$$

$$\frac{1}{(u-3)(u+3)} = \frac{A}{(u-3)} + \frac{B}{(u+3)}$$

or $1 = A(u+3) + B(u-3)$.

For $u = -3$ we have $1 = 0 + B(-6)$

$$\Rightarrow \boxed{B = \frac{-1}{6}}$$

And for $u=3$ we have $\boxed{A=\frac{1}{6}}$

Thus $\frac{1}{(u-3)(u+3)} = \frac{1}{6(u-3)} - \frac{1}{6(u+3)}$

So (3) implies

$$\frac{1}{6}\int \frac{du}{(u-3)} - \frac{1}{6}\int \frac{du}{u+3} = x + C$$

or $\frac{1}{6}\ln(u-3) - \frac{1}{6}\ln(u+3) = x + C$

$$\Rightarrow \frac{1}{6}\ln\left(\frac{u-3}{u+3}\right) = x + C$$

$\left(\because \ln\left(\frac{A}{B}\right) = \ln A - \ln B\right)$

or $\left(\frac{u-3}{u+3}\right)^{1/6} = c_1 e^{x}$

or $\boxed{\left(\frac{-2x+y-3}{-2x+y+3}\right)^{1/6} = c_1 e^{x}}$ Ans

Q2. $\dfrac{dy}{dx} = 2 + \sqrt{y-2x+3}$. $\longrightarrow$ (1)

Let $y-2x+3 = u$ or $\dfrac{dy}{dx} - 2 = \dfrac{du}{dx}$

or $\dfrac{dy}{dx} = \dfrac{du}{dx} + 2$, so (1) implies

$\dfrac{du}{dx} + 2 = 2 + \sqrt{u}$ or $\dfrac{du}{dx} = \sqrt{u}$

or $\dfrac{du}{u^{1/2}} = dx$ or $\int u^{-1/2} du = \int dx$

or $\dfrac{u^{1/2}}{1/2} = x + c$ or $2u^{1/2} = x + c$

or $2\sqrt{y-2x+3} = x + c$

or $4(y-2x+3) = (x+c)^2$

or $y-2x+3 = \frac{1}{4}(x+c)^2$ or $\boxed{y = 2x - 3 + \frac{1}{4}(x+c)^2}$

Ans.

Riccati's Equation ⇒

The differential equation

$$\frac{dy}{dx} = P(x) + Q(x)y + R(x)y^2 \quad \text{—(1)}$$ is known as Riccati's equation.

* A Riccati's equation can be solved by a succession of two substitutions provided that we know a particular solution y_1 of the equation.

Substitution first: Let $\boxed{y = y_1 + u}$ → (2)

then $$\frac{dy}{dx} = \frac{dy_1}{dx} + \frac{du}{dx}$$

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$$\frac{dy_1}{dx} + \frac{du}{dx} = P(x) + Q(x)y_1 + Q(x)u + R(x)(y_1^2 + u^2 + 2y_1 u)$$

$$\Rightarrow \frac{dy_1}{dx} + \frac{du}{dx} = P(x) + Q(x)y_1 + Q(x)u + R(x)y_1^2 + R(x)u^2 + 2R(x)y_1 u$$

$$\Rightarrow \frac{du}{dx} + \frac{dy_1}{dx} = \underbrace{\left(P(x) + Q(x)y_1 + R(x)y_1^2\right)}_{\to \frac{dy_1}{dx}} + Q(x)u + R(x)u^2 + 2y_1 R(x)u$$

$$\text{or } \frac{du}{dx} = \left(Q(x) - 2y_1 R\right)u + R(x)u^2$$

$$\Rightarrow \frac{du}{dx} + (2y_1 R + Q)u = R(x)u^2. \longrightarrow (3)$$

which is Bernoulli's equation & can be solved with substitution $\boxed{u = y^{1-n} = y^{1-2}}$

<u>Example:</u> Solve $\dfrac{dy}{dx} = -\dfrac{4}{x^2} - \dfrac{1}{x}y + y^2$ — (1), where

$y_1 = \dfrac{2}{x}$ is a known solution.

<u>Solution:</u> Let $y = y_1 + u = \dfrac{2}{x} + u$ or $\boxed{u = y - \frac{2}{x}}$

$$\Rightarrow \boxed{\frac{dy}{dx} = -\frac{2}{x^2} + \frac{du}{dx}}$$ so (1) implies

$$-\frac{2}{x^2} + \frac{du}{dx} = -\frac{4}{x^2} - \frac{1}{x}\left(\frac{2}{x} + u\right) + \left(\frac{2}{x} + u\right)^2$$

$$\Rightarrow -\frac{2}{x^2} + \frac{du}{dx} = -\frac{4}{x^2} - \frac{2}{x^2} - \frac{u}{x} + \frac{4}{x^2} + \frac{4u}{x} + u^2$$

$$\Rightarrow \frac{du}{dx} = -\frac{u}{x} + \frac{4}{x}u + u^2$$

$$\Rightarrow \frac{du}{dx} - \left(\frac{4}{x} - \frac{1}{x}\right)u = u^2$$

$$\frac{du}{dx} - \frac{3}{x}u = u^2, \longrightarrow (2)$$

which is Bernoulli's equation with $n=2$. So

let $\cancel{u = z^{1-n}}$ $z = u^{1-n} = u^{1-2} = u^{-1}$

$$\left.\begin{aligned} &\boxed{z = u^{-1} \text{ or } u = z^{-1}} \\ &\frac{du}{dx} = \frac{-1}{z^2}\frac{dz}{dx} \end{aligned}\right\} \longrightarrow (3)$$

Using (3) in (2) we have

$$-\frac{1}{z^2}\frac{dz}{dx} - \frac{3}{x}z^{-1} = z^{-2}$$

or $$\frac{dz}{dx} - \frac{3}{x}\frac{z^{-1}}{(-1/z^2)} = \frac{z^{-2}}{(-1/z^2)}$$

or $$\frac{dz}{dx} + \frac{3}{x}\frac{z^{-1}}{z^{-2}} = -\frac{z^{-2}}{z^{-2}}$$

$$\frac{dz}{dx} + \frac{3}{x} z = -1 . \longrightarrow (4)$$

$$I.F = \mu(x) = e^{\int \frac{3}{x} dx} = e^{3\int \frac{1}{x} dx} = e^{3\ln x} = e^{\ln x^3} = x^3$$

$$\mu(x) = x^3 . \longrightarrow (5)$$

xing (4) by (5) we get

$$x^3 \frac{dz}{dx} + 3x^2 z = -x^3$$

$$\frac{d}{dx}\left[z x^3\right] = -x^3 \text{ , integrating we have}$$

$$z x^3 = \frac{-x^4}{4} + c_1$$

$$\Rightarrow \boxed{z = -\frac{x}{4} + \frac{c_1}{x^3}} \longrightarrow (6)$$

Putting $z = u^{-1}$ in (6) we get

$$\boxed{u^{-1} = -\frac{x}{4} + \frac{c_1}{x^3}} \quad (7)$$

, putting $\boxed{u = y - \frac{2}{x}}$ in (7)

$$\Rightarrow \left(y - \frac{2}{x}\right)^{-1} = \frac{c_1}{x^3} - \frac{x}{4}$$

$$\Rightarrow y - \frac{2}{x} = \left(\frac{c_1}{x^3} - \frac{x}{4}\right)^{-1}$$

$$\Rightarrow y = \frac{2}{x} + \left(\frac{c_1}{x^3} - \frac{x}{4}\right)^{-1}.$$ Ans

Initial- Value problem

In Sec 1.2 we defined an initial- Value problem for a general nth-Order DE. For a linear DE, an nth-Order initial- Value problem is

Solve :

$$a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1} y}{dx^{n-1}} + \dots$$

$$\dots + a_1(x)\frac{dy}{dx} + a_0(x)y$$

$$= g(x)$$

Subject to :

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$y(x_0) = y_0, \; y'(x_0) = y_1, \ldots$

$\ldots, y^{(n-1)}(x_0) = y_{n-1}$."

"Boundary-value Problem

Another type of problem consists

Solving a linear DE of order two or greater in which the dependent variable y or its derivatives are specified at different points.

A problem such as

Solve:

$$a_2(x)\frac{d^2y}{dx^2} + a_1(x)\frac{dy}{dx} + a_0(x)y = g(x).$$

Subject to:

$$y(a) = y_0, \quad y(b) = y_1$$

is called BVP.

The prescribed values $y(a) = y_0$ and $y(b) = y_2$ are called boundary conditions. A solution of the foregoing problem is a ftn satisfying the DE on some interval I, containing a and b."

A linear DE of the form

$$a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_1(x)\frac{dy}{dx} + a_0(x)y = 0$$

is said to be homogeneous, whereas an equation

$$a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_1(x)\frac{dy}{dx} + a_0(x)y = g(x)$$

With $g(x)$ not identically zero, is said to be nonhomogeneous.

For example

$2y'' + 3y' - 5y = 0$ is

a homogeneous

linear Second-Order

DE. Whereas

$x^3 y''' + 6y' + 10y = e^x$

is a nonhomogeneous

linear 3rd order DE.

Theorem:

Superposition Principle - Homogene-ous Equations.

"Let $y_1, y_2, \ldots y_k$ be

Solutions of the homogeneous nth-order DE

$$a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}} + \cdots$$

$$\cdots + a_1(x)\frac{dy}{dx} + a_0(x)y = 0$$

on an interval I.

then The linear

Combination

$$y = C_1 y_1(x) + C_2 y_2(x) + \cdots$$

$$\cdots + C_k y_k(x),$$

Where the c_i, $i = 1, 2, \ldots k$ are arbitrary constants is also a solution on the interval I."

"The ftns $y_1 = x^2$ and $y_2 = x^2 \ln x$ are both solutions of homogeneous linear equation

$$x^3 y''' - 2x y' + 4y = 0$$

(35)

On the interval I $(0,\infty)$. By the Superposition Principle the linear combination

$$y = C_1 x^2 + C_2 x^2 \ln x.$$

is also a solution of the equation on the interval."

Definition

Linear Dependence / Independence.

"A set of functions $f_1(x), f_2(x), \dots f_n(x)$ is said to be linearly dependent on interval I if there exist constants $c_1, c_2, \dots c_n$, not all zero, such that

$c_1 f_1(x) + c_2 f_2(x) + \dots + c_n f_n(x) = 0$ for every x in the interval. If the set of fnts is not linearly dependent on the interval. it is said to be linearly independent."

"The set of ftns

$f_1(x) = \cos^2 x$, $f_2(x) = \sin^2 x$, $f_3(x) = \sec^2 x$, $f_4(x) = \tan^2 x$

is linearly dependent on the interval $(-\pi/2, \pi/2)$.

because $C_1\cos^2 x + C_2\sin^2 x + C_3\sec^2 x + C_4\tan^2 x = 0$ when $C_1 = C_2 = 1$, $C_3 = -1$, $C_4 = 1$. We used here $\cos^2 x + \sin^2 x = 1$ and $1 + \tan^2 x = \sec^2 x$

A set of ftns $f_1(x), f_2(x), \ldots f_n(x)$ is linearly dependent on an interval if at least one ftn can be expressed as a linear combination of the remaining ftns.

"the set of ftns $\cancel{f_4(x)=}$ $f_1(x) = \sqrt{x} + 5$, $f_2(x) = \sqrt{x} + 5x$, $f_3(x) = x-1$, $f_4(x) = x^2$ is linearly dependent on the interval $(0, \infty)$ because f_2 can be written as a linear combination of f_1, f_3 and f_4. Observe that

$f_2(x) = 1 \cdot f_1(x) + 5 \cdot f_3(x) + 0 \cdot f_4(x)$

for every x in the interval $(0, \infty)$"

Definition Wronskian

Suppose ~~that~~ each of the ftns $f_1(x), f_2(x), \dots f_n(x)$ possesses at least $n-1$ derivatives. the determinant

$$W(f_1, f_2, \dots f_n) = \begin{vmatrix} f_1 & f_2 & \cdots & f_n \\ f_1' & f_2' & \cdots & f_n' \\ \vdots & \vdots & & \vdots \\ f_1^{(n-1)} & f_2^{(n-1)} & \cdots & f_n^{(n-1)} \end{vmatrix}$$

Where the primes denote derivatives, is called the Wronskian of the ftns."

Theorem Criterion for Linearly Independent Solutions

Let $y_1, y_2, \ldots, y_n$ be n solutions of the homogeneous linear n^{th} order DE

$$a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_1(x)\frac{dy}{dx} + a_0(x)y = 0$$

on an interval I.

then the set of solutions is linearly independent on I if and only if

$W(y_1, y_2, \ldots y_n) \neq 0$ for every x in the interval.

Definition

Fundamental set of Solutions

Any set $y_1, y_2, \ldots y_n$ of n linearly independent Solutions of the homogeneous linear nth - order DE

$$a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_1(x)\frac{dy}{dx} + a_0(x)y = 0$$

on an interval I is said to be a fundamental set of solutions on the interval."

Theorem

General Solution - Homogeneous Equations.

Let $y_1, y_2, \ldots y_n$ be a fundamental set of solutions of the homogeneous linear n^{th}-order DE

$$a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_1(x)\frac{dy}{dx} + a_0(x)\,y = 0$$

On an interval I. Then the general solution of the equation on the interval is

$$y = c_1 y_1(x) + c_2 y_2(x) + \cdots + c_n y_n(x)$$

Where c_i, $i = 1, 2, \ldots n$ are arbitrary constants."

" General solution of a homogeneous DE.

The ftns $y_1 = e^{3x}$ and $y_2 = e^{-3x}$ are both solutions of the homogeneous linear equation $y'' - 9y = 0$ on the interval

$(-\infty, \infty)$. By inspection the solutions are linearly independent on the x-axis. this fact can be corroborated by observing that the wronskian

$$W(e^{3x}, e^{-3x}) = \begin{vmatrix} e^{3x} & e^{-3x} \\ 3e^{3x} & -3e^{-3x} \end{vmatrix} = -6 \neq 0$$

for every x. We conclude that y_1 and y_2 form a fundamental set of solutions and consequently $y = c_1 e^{3x} + c_2 e^{-3x}$ is the general solution of the equation on the interval.