Solutions about singular points ⟹

$$\frac{d^2y}{dx^2} + P(x)\frac{dy}{dx} + Q(x)y = 0 \longrightarrow (1)$$

A $x = x_0$ is called singular point of DE (1) if ~~both~~ either $P(x)$ or $Q(x)$ are not analytic at $x = x_0$.

Regular & Irregular singular points ⟹

A singular point x_0 is called regular singular point of DE (1) if the functions $(x-x_0)P(x) = p(x)$ & $(x-x_0)^2 Q(x) = q(x)$ are both analytic at x_0.

A point that is not regular singular point is called irregular singular point.

Example:

$$(x^2-4)^2 y'' + 3(x-2)y' + 5y = 0 \rightarrow (1)$$

$$\Rightarrow y'' + \frac{3(x-2)}{(x^2-4)^2} y' + \frac{5y}{(x^2-4)^2} = 0 \rightarrow (2)$$

$$P(x) = \frac{3(x-2)}{(x-2)^2(x+2)^2} = \frac{3}{(x-2)(x+2)^2}$$

$$Q(x) = \frac{5}{(x-2)^2(x+2)^2}$$

Clearly $x=2,-2$ are singular points.

$$p(x) = (x-x_0)P(x)$$

$$q(x) = (x-x_0)^2 Q(x)$$

for $x_0 = 2$

$$p(x) = (x-2)P(x)$$

$$= \cancel{(x-2)} \frac{3}{\cancel{(x-2)}(x+2)^2} = \frac{3}{(x+2)^2}$$

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$$q(x) = \cancel{(x-2)^2}\left[\frac{5}{\cancel{(x-2)^2}(x+2)^2}\right]$$

$$= \frac{5}{(x+2)^2}$$

So ~~both~~ $p(x)$ & $q(x)$ are both analytic at $x=2$, thus $x=2$ is regular singular point.

For $x_0=-2$

$$p(x) = \cancel{(x+2)}\left\{\frac{3}{(x-2)(x+2)^{\cancel{2}}}\right\}$$

$$= \frac{3}{(x-2)(x+2)} \quad \left(\frac{3}{(-2-2)(-2+2)}\right) = \frac{3}{-4(0)} = \frac{3}{0} = \infty$$

$\Rightarrow p(x) \to \infty$ at $x=-2$

So $p(x)$ is not analytic at $x=-2$, so singular point $x=-2$ is irregular singular point of DE 1.

Frobenius' theorem =)

Statement =) If $x = x_0$ is a regular singular point of DE (1) then there exists at least one solution of the form

$$y = (x-x_0)^r \sum_{n=0}^{\infty} c_n (x-x_0)^n$$

$$= \sum_{n=0}^{\infty} c_n (x-x_0)^{n+r}$$

where the number r is constant to be determined. The series will converges at least on some interval $0 < x - x_0 < R$

$$\boxed{y'' + \ln x \, y = 0} \rightarrow (*)$$

Frobenius theorem does not allowed on (*)