Undetermined coefficients- Superposition approach :

To solve a nonhomogenous linear differential equation

$$a_n \frac{d^n y}{dx^n} + a_{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \dots + a_1 \frac{dy}{dx} + a_0 y = g(x). \quad \text{—(1)}$$

We should need to do two things:

(i) find the complementary function y_c &

(ii) find any particular solution y_p of the nonhomogenous equation.

* The complementary function is a solution of associated homogenous differential equation

$$a_n \frac{d^n y}{dx^n} + a_{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \dots + a_0 y = 0. \quad \text{—— (2)}$$

In order to obtain particular solution of (1) we are using method of undetermined coefficients.

The method of undetermined coefficients would be used if

* The coefficients a_i, $i = 0, 1, \dots, n$ are constants and

** the source function $g(x)$ is a constant k, a polynomial function, an exponential function $e^{\alpha x}$, sine or cosine functions $\sin\beta x$ or $\cos\beta x$, or finite sum & products of these functions.

For example, the functions

$g(x) = 10$, $g(x) = x^2 - 5x$, $g(x) = 15x - 6 + 8e^{-x}$,

$g(x) = \sin 3x - 5x\cos 2x$, $g(x) = xe^{x}\sin x + (3x^2-1)e^{-4x}$.

Example: Solve $y'' + 4y' - 2y = 2x^2 - 3x + 6. \longrightarrow (1)$

Solution:

Step (1): The associated homogenous equation is

$$y'' + 4y' - 2y = 0. \longrightarrow (2)$$

The auxillary equation is

$$m^2 + 4m - 2 = 0$$

$$m = \frac{-4 \pm \sqrt{16 - 4(1)(-2)}}{2(1)} = \frac{-4 \pm \sqrt{24}}{2}$$

$$m = \frac{-4 \pm \sqrt{6 \times 4}}{2} = \frac{-4 \pm 2\sqrt{6}}{2}$$

$$m = -2 \pm \sqrt{6}$$

$m_1 = -2 + \sqrt{6}$, $m_2 = -2 - \sqrt{6}$. Thus,

$$y_c = c_1 e^{(-2+\sqrt{6})x} + c_2 e^{(-2-\sqrt{6})x}. \longrightarrow (3)$$

Step 2: Since $g(x) = 2x^2 - 3x + 6$

which is a quadratic polynomial, thus

we seek a particular solution y_p

in the form

$$\left.\begin{aligned} y_p &= Ax^2 + Bx + C. \\ y_p' &= 2Ax + B. \quad y_p'' = 2A. \end{aligned}\right\} \rightarrow (4)$$

so (4) into (1) we get

$$y_p'' + 4y_p' - 2y_p = 2A + 8Ax + 4B - 2Ax^2 - 2Bx - 2C = 0$$

$$\Longrightarrow -2Ax^2 + (8A - 2B)x + 4B + 2A - 2C = 2x^2 - 3x + 6$$

Comparing the coefficients we have

x^2; $-2A = 2 \Longrightarrow \boxed{A = -1}$ $\boxed{B = -5/2}$

x; $8A - 2B = -3 \Longrightarrow -2B = 8 - 3 \Longrightarrow \boxed{\cancel{B = -4/2}}$

x^0; $4B + 2A - 2C = 6 \Rightarrow \cancel{-16 + (-2) - 2C = 6}$

$\Longrightarrow \cancel{-2C =} \; 6 + 18$

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$4(-5/2)-2-2C=6$

$\Rightarrow -12=6+2C$

$\Rightarrow 2C=-18 \Rightarrow \boxed{C=-9}$

Thus, a particular solution is

$y_p=-x^2-\frac{5}{2}x-9x.$

So the general solution of ① is

$y=c_1e^{(-2+\sqrt{6})x}+c_2e^{(-2-\sqrt{6})x}-x^2-\frac{5}{2}x-9x.$ Ans.

Q2. $y''-y'+y=2\sin 3x. \longrightarrow$ ①.

The associated homogenous equation is $y''-y'+y=0.$

The auxillary equation is

$m^2-m+1=0$

$m=\frac{-(-1)\pm\sqrt{(-1)^2-4(1)(1)}}{2(1)}=\frac{1\pm\sqrt{1-4}}{2}=\frac{1\pm i\sqrt{3}}{2}$

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$\alpha = \frac{1}{2}$, $\beta = \frac{\sqrt{3}}{2}$

So $y_c = c_1 e^{\frac{\alpha}{2}x} \cos\frac{\sqrt{3}}{2}x + c_2 e^{\frac{\alpha}{2}x} \sin\frac{\sqrt{3}}{2}x \longrightarrow (2)$

$$\left.\begin{aligned} y_p &= A\sin 3x + B\cos 3x \\ y_p' &= 3A\cos 3x - 3B\sin 3x \\ y_p'' &= -9A\sin 3x - 9B\cos 3x \end{aligned}\right\} \longrightarrow (3)$$

Using (3) in (1) we get

$-9A\sin 3x - 9B\cos 3x - 3A\cos 3x + 3B\sin 3x + A\sin 3x + B\cos 3x = 2\sin 3x$

or $(-8A + 3B)\sin 3x + (-8B - 3A)\cos 3x = 2\sin 3x$

Equating the coefficients of equal like terms we have

$\sin 3x: \; 3B - 8A = 2 \longrightarrow (4)$

$\cos 3x: \; -8B - 3A = 0 \longrightarrow (5)$

From (5) we get $B = \frac{3}{8}A \longrightarrow (6)$

Using (6) in (4) we have $3\left(\frac{3}{8}\right)A - 8A = 2$

$\Rightarrow \frac{(9-64)}{8} A = 2$

$-55A = 16$

$\Rightarrow \boxed{A = -16/55}$

so (6) $\Rightarrow$ $B = -\frac{16}{55} \times \frac{3}{8} = -6/55$

$\boxed{B = -6/55}$, so particular solution is

$y_p = -16/55 \sin 3x - 6/55 \cos 3x$. Thus, general solution is

$$\boxed{y = c_1 e^{\alpha/2\, x} \cos \frac{\sqrt{3}}{2} x + c_2 e^{\alpha/2\, x} \sin \frac{\sqrt{3}}{2} x - \frac{16}{55} \sin 3x - 6/55 \cos 3x}$$ Ans

Example: ~~7~~ 8

$$y'' - 2y' - 3y = 4x - 5 + 6xe^{2x}$$ ①

The associated homogeneous eqtn is

$$y'' - 2y' - 3y = 0 \longrightarrow ②$$

The corresponding auxillary eqtn is given by

$$m^2 - 2m - 3 = 0$$

$$\Rightarrow m^2 - 3m + m - 3 = 0$$

$$\Rightarrow m(m-3) + 1(m-3) = 0$$

$$\Rightarrow (m+1)(m-3) = 0$$

$$m_1 = -1, \quad m_2 = 3$$

$$y_c = c_1 e^{-x} + c_2 e^{3x} \longrightarrow ①$$

Here $g(x) = 4x - 5 + 6xe^{2x}$

$g(x) = g_1(x) + g_2(x)$

$g_1(x) = 4x-5$, $g_2(x) = 6xe^{2x}$

$y_p = y_{p_1} + y_{p_2}$

Then $y_{p_1} = Ax+B$, $y_{p_2} = (Cx+E)e^{2x}$

$y_p = Ax+B+(Cx+E)e^{2x} \longrightarrow (2)$

$y'_p = 4e^{2x}(Cx+E) + 2ce^{2x} + ce^{2x}$

$y''_p = 4e^{2x}x(c) + 3(e^{2x} + 4Ee^{2x}$

$-3Ax - 2A - 3B - 3Cxe^{2x} + C_2(-3E)e^{2x}$

$= 4x - 5 + 6xe^{2x} \longrightarrow (3)$

Comparing the co-efficients, we have

x ; $-3A = 4$, $\longrightarrow$ (i)

x^0 ; $-2A - 3B = -5$ or $2A + 3B = 5$, (ii)

e^{2x} ; $(2C - 3E) = 0 \longrightarrow$ (iii)

(10)

$x e^{2x}$; $-3C = 6 \Rightarrow \boxed{C = -2}$

From (i) $\boxed{A = -4/3}$ Putting in (ii)

we have

$2A + 3B = 2\left(-\frac{4}{3}\right) + 3B = 5$

$\Rightarrow -8/3 + 3B = 5$

$\boxed{B = \frac{23}{9}}$

From (iii) $3E = 2C$

$\Rightarrow E = \frac{2}{3}C = -4/3$

$\boxed{E = -4/3}$

$\boxed{y_P = -\frac{4}{3}x + \frac{23}{9} - 2xe^{2x} - 4/3\, e^{2x}}$

$\boxed{y = y_c + y_P}$ Ans

Undetermined Co-efficients (Annihilator Approach)

Annihilator operator : An operator L is a differential annihilator operator if

$$L[f(x)] = 0$$

L is annihilator of $f(x)$.

$f(x) = x^2 + 1$

$L = D^3$

$D^3(x^2+1) = 0$

$f(x) = c$

$L = D$, $D(c) = 0$

$f(x) = e^{\alpha x}$, $L = (D-\alpha)$

$f(x) = x^{n-1} e^{\alpha x}$

$L = (D-\alpha)^n$

$f(x) = e^{5x}$, $L = (D-5)$

$f(x) = xe^{5x}$, $L = (D-5)^2$

$f(x) = e^{\alpha x} \cos \beta x$

$L = [D^2 - 2\alpha D + (\alpha^2 + \beta^2)]$

$f(x) = x^{n-1} e^{\alpha x} \cos \beta x$

$L = [D^2 - 2\alpha D + (\alpha^2 + \beta^2)]^n$

$f(x) = xe^{3x} \sin 4x$

$\alpha = 3$, $\beta = 4$

$L = [D^2 - 6D + (9+16)]^2$

y_1 , y_2

$L_1(y_1) = 0$, $L_2(y_2)$

$L_1 L_2 (c_1 y_1 + c_2 y_2) = 0$

$f_1(x) = x^3$, $f_2(x) = e^{\alpha x}$

$L_1 = D^4$, $L_2 = (D-\alpha)$

$D^4(D-\alpha) = (c_1x^3 + c_2e^{\alpha x}) = 0$

Q// $f(x) = e^{-3x}$

Solution

$$L = (D-(-3)) = (D+3)$$

Q// $f(x) = 5e^{-x}\cos 2x + 9e^{-x}\sin 2x$

$\alpha = -1$, $\beta = 2$, $n = 1$

$$L = (D^2 - 2(-1)D + ((-1)^2 + (2)^2))^1$$

$$L = D^2 + 2D + 3$$

$$L = (D^2 + 2D + 3)^2$$

Example

$$y'' + 3y' + 2y = 4x^2$$

Solution

$$y'' + 3y' + 2y = 4x^2 \longrightarrow (1)$$

The associated homogeneous eqtn is

$y'' + 3y' + 2y = 0 \longrightarrow (2)$

Then the auxillary eqtn is

$m^2 + 3m + 2 = 0$

$\Rightarrow m^2 + 2m + m + 2 = 0$

$\Rightarrow m(m+2) + 1(m+2) = 0$

$\Rightarrow (m+1)(m+2) = 0$

$\Rightarrow m_1 = -1, \quad m_2 = -2$

$y_c = C_1 e^{-x} + C_2 e^{-2x} \longrightarrow (3)$

Here $g(x) = 4x^2$

So $L = D^3 = \frac{d^3}{dx^3}$

Now eq (2) can be written in the form

$$\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + 2y = 4x^2$$

Suppose $D = \frac{d}{dx}$

(15).

$D^2y + 3Dy + 2y = 4x^2$

$\Rightarrow (D^2+3D+2)y = 4x^2 \longrightarrow (2)$

Applying D^3 on both sides of (2).

we have $D^3(D^2+3D+2)y = D^3(4x^2) = 0$

$D^3(D^2+3D+2)y = 0 \longrightarrow (4)$

The auxillary eqtn is

$m^3(m^2+3m+2) = 0$

$m^3(m+1)(m+2) = 0$

$m_1 = m_2 = m_3 = 0, \quad m_4 = -1, m_5 = -2$

Then $\boxed{y = C_1e^{0x} + C_2xe^{0x} + C_3x^2e^{0x} + C_4e^{-x} + C_5e^{-2x}} \longrightarrow 5$

Then $y_p = C_1 + C_2x + C_3x^2$

$\boxed{y_p = A + Bx + Cx^2}$

$y'_p = B + 2Cx$

$y''_p = 2C$ putting in 1

$2C + 3B + 6Cx + 2A + 2Bx + 2Cx^2 = 4x^2$

Comparing the co-efficients we have

x^2; $2C = 4$, $\boxed{C = 2}$

x; $6C + 2B = 0 \Rightarrow 2B = -6C$

$\Rightarrow \boxed{B = -3C = 6}$

x^0; $2C + 3B + 2A = 0$

Then $\boxed{A = 7}$

$\boxed{y_p = 7 + (-6)x + 2x^2}$

Thus General Solution of 1 is

$y = y_c + y_p$

$\boxed{y = C_1 e^{-x} + C_2 e^{-2x} + 7 - 6x + 2x^2}$ Ans

$D^4(D-\alpha) = (C_1 x^3 + C_2 e^{\alpha x}) = 0$

Q2 $y'' - 3y' = 8e^{3x} + 4\sin x \longrightarrow (1)$

The associated homogeneous eqtn is $y'' - 3y' = 0$, The auxillary eqtn is

$m^2 - 3m = 0$

$m(m-3) = 0$

$\Rightarrow m_1 = 0, m_2 = 3$

$y_c = C_1 e^{0x} + C_2 e^{3x} = C_1 + C_2 e^{3x} \longrightarrow (2)$

Now $g(x) = 8e^{3x} + 4\sin x$

$= g_1(x) + g_2(x)$

$g_1(x) = e^{3x}$, $g_2(x) = 4\sin x$

So $L_1 = (D-3)$, $L_2 = [D^2 - 2(0)D + (0^2 + 1^2)]$

$L_2 = (D^2 + 1)$

Thus $L = (D-3)(D^2+1)$ will annihilates $g(x)$

Now eqtn (1) can be written in the form

$$(D^2 - 3D)y = 8e^{3x} + 4\sin x \longrightarrow (*)$$

So applying L on both sides we have

$$(D-3)(D^2+1)(D^2-3D)y = (D-3)(D^2+1)(8e^{3x} + 4\sin x)$$

$$\Rightarrow (D-3)(D^2+1)(D^2-3D)y = 0 \longrightarrow 4$$

Now the auxillary eqtn is

$$(m-3)(m^2+1)m(m-3) = 0$$

$$\Rightarrow m_1 = m_2 = 3, \quad m_3 = 0$$

$$m_4 = i, \quad m_5 = -i$$

$$y = \boxed{c_1 e^{3x}} + c_2 x e^{3x} + \boxed{c_3 e^{0x}}$$

$$+ c_4 \cos x + c_5 \sin x \longrightarrow 5$$

Then

$$y_p = Axe^{3x} + B\cos x + C\sin x \longrightarrow 6$$

$$y = y_c + y_p$$

$$y = C_1 + C_2 e^{3x} + Axe^{3x} + B\cos x + C\sin x$$