Solve $3xy'' + y' - y = 0. \longrightarrow (1)$

$$y'' + \frac{1}{3x} y' - \frac{1}{3x} y = 0.$$

Clearly $x = 0$ is a regular singular point.

Thus, we seek a solution of the form

$$\left.\begin{aligned} y &= \sum_{n=0}^{\infty} c_n x^{n+r} \\ y' &= \sum_{n=0}^{\infty} c_n (n+r) x^{n+r-1} \\ y'' &= \sum_{n=0}^{\infty} c_n (n+r)(n+r-1) x^{n+r-2} \end{aligned}\right\} \longrightarrow (2)$$

Putting (2) in (1) we have

$$3x \sum_{n=0}^{\infty} c_n (n+r)(n+r-1) x^{n+r-2} + \sum_{n=0}^{\infty} c_n (n+r) x^{n+r-1}$$

$$- \sum_{n=0}^{\infty} c_n x^{n+r} = 0$$

2

$$\Rightarrow 3\sum_{n=0}^{\infty} c_n (n+r)(n+r-1) x^{\overbrace{n+r-1}} + \sum_{n=0}^{\infty} c_n (n+r) x^{n+r-1} - \sum_{n=0}^{\infty} c_n x^{n+r} = 0.$$

$$\underbrace{\quad}_{n-1=k} \qquad \underbrace{\quad}_{n-1=k} \qquad \underbrace{\quad}_{n=k}$$

$$\Rightarrow 3\sum_{k=-1}^{\infty} c_{k+1}(k+r+1)(k+r) x^{k+r} + \sum_{k=-1}^{\infty} C_{k+1}(k+r+1) x^{k+r} - \sum_{k=0}^{\infty} C_k x^{k+r} = 0$$

$$\Rightarrow x^r \left[3\sum_{k=-1}^{\infty} C_{k+1} (k+r+1)(k+r) x^k + \sum_{k=-1}^{\infty} C_{k+1}(k+r+1) x^k - \sum_{k=0}^{\infty} C_k x^k \right] = 0$$

$$\Rightarrow x^{r}\left[3C_0 r(r-1)x^{-1} + 3\sum_{k=0}^{\infty} C_{k+1}(k+r+1)(k+r)x^{k}\right.$$

$$\left. + C_0 r x^{-1} + \sum_{k=0}^{\infty} C_{k+1}(k+r+1)x^{k} - \sum_{k=0}^{\infty} C_k x^{k}\right] = 0$$

$$\Rightarrow x^{r}\left[C_0 r x^{-1}(3r-3+1) + \sum_{k=0}^{\infty}\left(3C_{k+1}(k+r+1)(k+r)\right.\right.$$

$$\left.\left. + C_{k+1}(k+r+1) - C_k\right)x^{k}\right] = 0$$

$$\Rightarrow x^{r}\left[r(3r-2)C_0 x^{-1} + \sum_{k=0}^{\infty}\left\{C_{k+1}(k+r+1)(3k+3r+1)\right.\right.$$

$$\left.\left. - C_k\right\}x^{k}\right] = 0 \quad \rightarrow (3)$$

Applying identity property we have

$$x^{-1}: \quad r(3r-2) = 0 \quad \longrightarrow (4)$$

$$C_{k+1} = \frac{C_k}{(k+r+1)(3k+3r+1)}, \quad k=0,1,2\ldots \qquad (5)$$

Eq (4) is called indicial equation & has two roots $\boxed{r_1 = 0 \ \& \ r_2 = \frac{2}{3}}$.

So for $r=0$ we get from (5)

$$C_{k+1} = \frac{C_k}{(k+1)(3k+1)}, \quad k \geq 0. \qquad (6)$$

& for $r = 2/3$ we have

$$C_{k+1} = \frac{C_k}{(k+5/3)(3k+3)}, \quad k \geq 0. \qquad (7)$$

5

first consider eq(6), so for $k=0$

$$C_1 = \frac{C_0}{(1)(1)} = C_0$$

for $k=1$ $\quad C_2 = \frac{C_1}{(3+1)(1+1)} = \frac{C_1}{8} = \frac{C_0}{8}$

for $k=2$, $\quad C_3 = \frac{C_2}{(3)(7)} = \frac{C_0}{(8)(7)(3)}$

for $k=3$, $\quad C_4 = \frac{C_3}{(4)(10)} = \frac{C_0}{(10)(8)(7)(4)(3)}$

and so on.

Now since $\quad y = \sum_{n=0}^{\infty} C_n x^{n+r}$

$$\Rightarrow \quad y = x^r \left[C_0 + C_1 x + C_2 x^2 + C_3 x^3 + \cdots \right]$$

⑥

$$\Rightarrow y_1(x) = x^r\left[c_0 + c_0 x + \frac{c_0}{8}x^2 + \frac{c_0}{8\times7\times3}x^3 + \cdots\right]$$

$$= c_0 x^r\left(1 + x + \frac{x^2}{8} + \frac{x^3}{8\times7\times3} + \cdots\right).$$

choosing $c_0 = 1$ we get

$$y_1(x) = x^r\left(1 + x + \frac{x^2}{8} + \frac{x^3}{8\times7\times3} + \cdots\right). \longrightarrow (8)$$

Now consider the recurrence relation (7)

$$C_{k+1} = \frac{C_k}{(k+5/3)(3k+3)} = \frac{C_k}{(3k+5)(k+1)} \quad ; \; k \geq 0.$$

for $k=0$,

$$c_1 = \frac{c_0}{5}.$$

for $k=1$,

$$c_2 = \frac{c_1}{8(2)} = \frac{c_0}{8(5)(2)}.$$

for $k=2$,

$$C_3 = \frac{C_2}{(11)(3)} = \frac{C_0}{(11)(8)(5)(3)(2)}$$

and so on.

Now since $y = \sum_{n=0}^{\infty} C_n x^{n+r}$

$$\Rightarrow y = x^r\left[C_0 + C_1 x + C_2 x^2 + C_3 x^3 + \cdots\right]$$

$$y = C_0 x^r\left[1 + \frac{x}{5} + \frac{x^2}{8\times5\times2} + \frac{x^3}{11\times8\times5\times3\times2} + \cdots\right].$$

choosing $C_0 = 1$ we get the second series solution

$$y_2(x) = \left(1 + x/5 + x^2/8\times5\times2 + \frac{x^3}{11\times8\times5\times3\times2} + \cdots\right)x^r. \rightarrow (9)$$

and the general solution of Eq (1) is

$$y(x) = c_1 y_1(x) + c_2 y_2(x), \rightarrow (10)$$

where $y_1(x)$ & $y_2(x)$ are given by (9) & (10) respectively.