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Section:
BS (CS) -2D

Question # 1

$$y'' - (x+1)y' - y = 0$$

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Which has an ordinary point at $x = 0$. If the differential equation ~~α~~ has an ordinary point at $x = x_0$, then it has a power ~~se~~ series solution, with two linearly independent solutions, which has the following form.

$$y = \sum_{n=0}^{\infty} C_n (x - x_0)^n$$

Thus, the given differential equation has a solution of the following form.

$$y = \sum_{n=0}^{\infty} c_n x^n \quad \text{--- (1)} \qquad (\because x_0 = 0).$$

Now we will find the two linearly independent series solution for the given differential equation.
So we will find $y'$ & $y''$, so taking $\frac{d}{dx}$ on b/s of eq (1).

$$y' = \sum_{n=0}^{\infty} c_n n x^{n-1}$$

$$y'' = \sum_{n=0}^{\infty} c_n n(n-1) x^{n-2}$$

Now shifting the summation index to $n=1$ for $y'$ & $n=2$ for $y''$, as the first term of the power series equals to zero. So

$$y' = \sum_{n=1}^{\infty} c_n n (x)^{n-1} \quad \text{--- (2)}$$

$$y'' = \sum_{n=2}^{\infty} c_n n(n-1) x^{n-2} \quad \text{--- (3)}$$

Putting the eqs 1, 2 & 3 in (A)

$$f(y'', y', y) = \sum_{n=2}^{\infty} c_n n(n-1) x^{n-2} - (x+1) \sum_{n=1}^{\infty} c_n n x^{n-1} - \sum_{n=0}^{\infty} c_n x^n$$

$$= \sum_{n=2}^{\infty} C_n n(n-1)x^{n-2} - \sum_{n=1}^{\infty} x \cdot C_n n x^{n-1} - \sum_{n=1}^{\infty} 1 \cdot C_n n x^{n-1} - \sum_{n=0}^{\infty} C_n x^n$$

$$= \underbrace{\sum_{n=2}^{\infty} C_n n(n-1)x^{n-2}}_{a_2 \to x^0} - \underbrace{\sum_{n=1}^{\infty} C n x^n}_{a_1 \to x^1} - \underbrace{\sum_{n=1}^{\infty} C_n n x^{n-1}}_{a_1 \to x^0} - \underbrace{\sum_{n=0}^{\infty} C_n x^n}_{a_0 \to x^0}$$

Now, we will find whether the summation index and the powers of $x$ are in same phase or not, so.

$$= \left[C_2 \cdot 2(2-1)x^{2-2} + \sum_{n=3}^{\infty} C_n n(n-1)x^{n-2}\right] - \sum_{n=1}^{\infty} C_n n x^n -$$

$$\left[C_1 \cdot 1 \cdot x^0 + \sum_{n=2}^{\infty} C_n n x^{n-1}\right] - \left[C_0 x^0 + \sum_{n=1}^{\infty} C_n x^n\right]$$

$$= 2c_2 + \sum_{n=3}^{\infty} C_n n(n-1)x^{n-2} - \sum_{n=1}^{\infty} C_n n x^n - C_1 - \sum_{n=2}^{\infty} C_n n x^{n-1} -$$

$$C_0 - \sum_{n=1}^{\infty} C_n x^n$$

$$= \underbrace{\sum_{n=3}^{\infty} C_n n(n-1)x^{n-2}}_{a_3 \to x^1} - \underbrace{\sum_{n=1}^{\infty} C_n n x^n}_{a_1 \to x^1} - \underbrace{\sum_{n=2}^{\infty} C_n n x^{n-1}}_{a_2 \to x^1} - \underbrace{\sum_{n=1}^{\infty} C_n x^n}_{a_1 \to x^1} +$$

$$2C_2 - C_1 - C_0$$

\# Now to make the summation index and the power of $x$ same we will replace the summation index.

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$$= \underbrace{\sum_{n=3}^{\infty} c_n n(n-1)x^{n-2}}_{\text{to } n+2} - \sum_{n=1}^{\infty} c_n n x^n - \underbrace{\sum_{n=2}^{\infty} c_n n x^{n-1}}_{\text{to } n-1} - \sum_{n=1}^{\infty} c_n x^n +$$

$$2c_2 - c_1 - c_0$$

$$= \sum_{n+2=3}^{\infty} c_{n+2}(n+2)[(n+2)-1]x^{(n+2)-2} - \sum_{n=1}^{\infty} c_n n x^n -$$

$$\sum_{n+1=2}^{\infty} c_{n+1}(n+1)x^{(n+1)-1} - \sum_{n=1}^{\infty} c_n x^n + 2c_2 - c_1 - c_0$$

$$= \sum_{n=1}^{\infty} c_{n+2}(n+2)(n+1)x^n - \sum_{n=1}^{\infty} c_n n x^n - \sum_{n=1}^{\infty} c_{n+1}(n+1)x^n -$$

$$\sum_{n=1}^{\infty} c_n x^n + 2c_2 - c_1 - c_0.$$

$$= \sum_{n=1}^{\infty} \left[c_{n+2}(n+2)(n+1) - c_n n - c_{n+1}(n+1) - c_n\right]x^n +$$

$$2c_2 - c_1 - c_0$$

$$= \sum_{n=1}^{\infty} \left[c_{n+2}(n+2)(n+1) - c_{n+1}(n+1) - c_n(n+1)\right]x^n + 2c_2 - c_1 - c_0$$

$$= \sum_{n=1}^{\infty} \left[c_{n+2}(n+2) - c_{n+1} - c_n\right](n+1)x^n + \underbrace{2c_2 - c_1 - c_0}_{\text{co-efficients of } x_0}$$

$$= 0$$

Which means.

$$c_{n+2}(n+2) - c_{n+1} - c_n = 0 \quad \& \quad 2c_2 - c_1 - c_0 = 0.$$

$$\Downarrow$$

$$c_2 = \frac{c_1 + c_0}{2} \quad \text{—(4)}$$

Now

$$C_{n+2}(n+2) = C_{n+1} + C_n$$

$$C_{n+2} = \frac{C_{n+1} + C_n}{n+2}.$$

$$C_{n+2} = \frac{C_{n+1}}{n+2} + \frac{C_n}{n+2} \quad — (5)$$

From eq (4) & (5) we will find the co-efficients of the power series in terms of $C_0$ & $C_1$, so

$n = 1$

$$C_{n+2} = \frac{C_{n+1}}{n+2} + \frac{C_n}{n+2}$$

$$C_3 = \frac{C_2}{3} + \frac{C_1}{3}$$

$$= \frac{\frac{C_1+C_0}{2}}{3} + \frac{C_1}{3}$$

$$= \frac{C_1}{6} + \frac{C_0}{6} + \frac{C_1}{3}.$$

$$C_3 = \frac{C_0}{6} + \frac{C_1}{2} \quad — (6)$$

$n = 2$

$$C_4 = \frac{C_3}{4} + \frac{C_2}{4}$$

$$C_4 = \frac{1}{4}\left[\frac{C_0}{6} + \frac{C_1}{2}\right] + \frac{1}{4}\left[\frac{C_0}{2} + \frac{C_1}{2}\right]$$

$$C_4 = \frac{C_0}{6} + \frac{C_1}{4} \quad -(7)$$

Now expand the power series in (1) & substitute 4, 6 & 7 in it, so.

$$y = \sum_{n=0}^{\infty} C_n x^n$$

$$y = C_0 + C_1 x + C_2 x^2 + C_3 x^3 + C_4 x^4 + \cdots\cdots$$

$$y = C_0 + C_1 x + \left[\frac{C_0}{2} + \frac{C_1}{2}\right]x^2 + \left[\frac{C_0}{6} + \frac{C_1}{2}\right]x^3 + \left[\frac{C_0}{6} + \frac{C_1}{4}\right]x^4 + \cdots\cdots$$

$$y = C_0 + C_1 x + \frac{C_0 x^2}{2} + \frac{C_1 x^2}{2} + \frac{C_0 x^3}{6} + \frac{C_1 x^3}{2} + \frac{C_0 x^4}{6} + \frac{C_1 x^4}{4} + \cdots\cdots$$

$$y = C_0\left[1 + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{6}x^4 + \cdots\right] + C_1\left[x + \frac{1}{2}x^2 + \frac{1}{2}x^3 + \frac{1}{4}x^4 + \cdots\right]$$

So, the two power series solution of the given differential equation are.

$$y_1 = C_0\left[1 + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{6}x^4 + \cdots\right]$$

$$y_2 = C_1\left[x + \frac{1}{2}x^2 + \frac{1}{2}x^3 + \frac{1}{4}x^4 + \cdots\right].$$

Question # 2

$$(x^2+2)y'' + 3xy' - y = 0$$

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$$y' = \sum_{n=1}^{\infty} C_n n x^{n-1} \quad — (2)$$

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Putting the eq 1, 2 & 3 in (A)

$$\Rightarrow (x^2+2)\sum_{n=2}^{\infty} C_n n(n-1)x^{n-2} + 3x\sum_{n=1}^{\infty} C_n n x^{n-1} - \sum_{n=0}^{\infty} C_n x^n = 0.$$

$$\Rightarrow x^2\sum_{n=2}^{\infty} C_n n(n-1)x^{n-2} + 2\sum_{n=2}^{\infty} C_n n(n-1)x^{n-2} + 3x\sum_{n=1}^{\infty} C_n n x^{n-1} - \sum_{n=0}^{\infty} C_n x^n = 0$$

$$\Rightarrow \sum_{n=2}^{\infty} C_n n(n-1)x^n + 2\sum_{n=2}^{\infty} C_n n(n-1)x^{n-2} + 3\sum_{n=1}^{\infty} C_n n x^{n} - \sum_{n=0}^{\infty} C_n x^n = 0$$

$$\Rightarrow \sum_{n=2}^{\infty} C_n n(n-1)x^n + 2\cdot 2\cdot 1 C_2 x^0 + 2\cdot 3\cdot 2 C_3 x + \sum_{n=4}^{\infty} C_n n(n-1)x^{n-2} +$$

$$3\cdot 1 C_1 x + 3\sum_{n=2}^{\infty} C_n n x^n - C_0 - C_1 x - \sum_{n=2}^{\infty} C_n x^n = 0$$

$$\Rightarrow \sum_{n=2}^{\infty} C_n n(n-1)x^n + 4C_2 + 12C_3 x + 2\sum_{n=4}^{\infty} C_n n(n-1)x^{n-2} + 3C_1 x +$$

$$3\sum_{n=2}^{\infty} C_n n x^n - C_0 - C_1 x - \sum_{n=2}^{\infty} C_n x^n = 0$$

$$\Rightarrow 4C_2 - C_0 + (12C_3 + 3C_1 - C_1)x + \sum_{n=2}^{\infty} C_n n(n-1)x^n +$$

$$2\sum_{n=4}^{\infty} C_n n(n-1)x^{n-2} + 3\sum_{n=2}^{\infty} C_n n x^{n-1} - \sum_{n=2}^{\infty} C_n x^n = 0$$

Now for first series, let $k=n$, so.

$$\sum_{k=2}^{\infty} C_k k(k-1)x^k$$

And for second series, let $k=n-2$, so.

$$2\sum_{k=2}^{\infty} C_{k+2}(k+2)(k+1)x^k$$

And for third and forth series, let $k=n$, So

$$3x\sum_{k=2}^{\infty} C_k k x^{k-1},$$

$$\sum_{k=2}^{\infty} C_k x^k$$

We are doing this so that, the summation index and the power of $x$ ~~x~~ become same, Hence

$$\rightarrow 4c_2 - C_0 + (12C_3 + 2c_1)x + \sum_{k=2}^{\infty} C_k[k(k-1)]x^k + 2\sum_{k=2}^{\infty}(k+2)(k+1)C_{k+2}x^k +$$
$$3\sum_{k=2}^{\infty} kC_k x^k - \sum_{k=2}^{\infty} C_k x^k = 0$$

$$\rightarrow 4c_2 - c_0 + (12c_3 + 2c_1)x + \sum_{k=2}^{\infty} x^k\left[k(k-1)c_k + (k+2)(k+1)c_{k+2} +\right.$$
$$\left. kC_k + C_k\right] = 0$$

Now using identity property.

$$4c_2 - c_0 = 0$$

$$c_2 = \frac{c_0}{4}$$

$$12c_3 + 2c_1 = 0.$$

$$c_3 = -\frac{c_1}{6}$$

$$k(k-1)c_k + 2(k+2)(k+1)c_{k+2} + 3kc_k - c_k = 0$$

$$c_{k+2} = \frac{c_k(-k^2-2k+1)}{2(k+2)(k+1)}$$

Now,

Let $k=2 \Rightarrow c_4 = \frac{c_2(-7)}{4\cdot 3\cdot 2} = \frac{-7c_0}{4\cdot 4!}$

$k=3 \Rightarrow c_5 = -\frac{14c_3}{2\cdot 5\cdot 4} = \frac{14c_1}{2\cdot 5!}$

$k=5 \Rightarrow c_6 = -\frac{23c_4}{2\cdot 6\cdot 5} = \frac{23\cdot 7c_1}{8\cdot 6!}$

$k=6 \Rightarrow c_7 = \frac{-34c_5}{2\cdot 7\cdot 6} = -\frac{34\cdot 14c_1}{4\cdot 7!}$

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Now expand the power series in 1 & substitute the values in it, so.

$$y = C_0 + C_1 x + \frac{1}{4} C_0 x^2 - \frac{1}{6} C_1 x^3 - \frac{7}{4.4!} C_0 x^4 + \frac{14}{2.5!} C_1 x^5 + \frac{23.7}{8.6!} C_1 x^6 - \frac{34.14}{4.7!} C_1 x^7 + \cdots\cdots$$

$$y = C_0\left[1 + \frac{1}{4}x^2 - \frac{7}{4.4!}x^4 + \frac{23.7}{8.6!}x^6 - \cdots\right] + C_1\left[x - \frac{1}{6}x^3 + \frac{14}{2.5!}x^5 - \frac{34.14}{4.7!}x^7 + \cdots\right]$$

$$y_1 = C_0\left[1 + \frac{1}{4}x^2 - \frac{7}{4.4!}x^4 + \frac{23.7}{8.6!} + \cdots\right]$$

$$y_2 = C_1\left[x - \frac{1}{6}x^3 + \frac{14}{2.5!}x^5 - \frac{34.14}{4.7!}x^7 + \cdots\right]$$ Ans