作業八

5. Prove the estimate (12.23).

[Hint : for each internal node x_j, $j = 1, \dots, n-1$, integrate by parts (12.21) to get

$$\tau_h(x_j) = -u''(x_j) - \frac{1}{h^2}\left[\int_{x_j-h}^{x_j} u''(t)(x_j-h-t)^2\,dt - \int_{x_j}^{x_j+h} u''(t)(x_j+h-t)^2\,dt\right]$$

Then, pass to the squares and $\tau_h(x_j)^2$ for $j = 1, \dots, n-1$.

On nothing that $(a+b+c)^2 \le 3(a^2+b^2+c^2)$, for any real numbers a, b, c, and applying the Cauchy-Schwarz inequality yields the desired result.]

Note : (12.23) $\|\tau_h\|_h^2 \le 3\left(\|f\|_h^2 + \|f\|_{L^2(0,1)}^2\right)$

(12.21) $\tau_h(x_j) = \frac{1}{h^2}\left(R_4(x_j+h) + R_4(x_j-h)\right)$

where $R_4(x_j+h) = \int_{x_j}^{x_j+h}\left(u'''(t) - u'''(x_j)\right)\frac{(x_j+h-t)^2}{2}\,dt$

and $R_4(x_j-h) = -\int_{x_j-h}^{x_j}\left(u'''(t) - u'''(x_j)\right)\frac{(x_j-h-t)^2}{2}\,dt$

Remark 12.3 : Let $e = u - u_h$ be the discretization error grid function.

Then $L_h e = L_h u - L_h u_h = L_h u - f_h = \tau_h$

$\Rightarrow$ Let

$$\tau_h(x_j) = \frac{1}{h^2}\Bigg[\underbrace{\int_{x_j}^{x_j+h}\left(u'''(t) - u'''(x_j)\right)\frac{(x_j+h-t)^2}{2}\,dt}_{①} \underbrace{- \int_{x_j-h}^{x_j}\left(u'''(t) - u'''(x_j)\right)\frac{(x_j-h-t)^2}{2}\,dt}_{②}\Bigg]$$

By integer by parts,

then

$$\tau_h(x_j) = -u''(x_j) - \frac{1}{h^2}\left[\int_{x_j-h}^{x_j} u''(t)(x_j-h-t)^2\,dt - \int_{x_j}^{x_j+h} u''(t)(x_j+h-t)^2\,dt\right]$$

Since $(a+b+c)^2 \le 3(a^2+b^2+c^2)$

$$\Rightarrow [\tau_h(x_j)]^2 \le 3\left[\left(-u''(x_j)\right)^2 + \left(-\frac{1}{h}\int_{x_j-h}^{x_j} u''(t)(x_j-h-t)^2\,dt\right)^2 + \left(-\frac{1}{h^2}\int_{x_j}^{x_j+h} u''(t)(x_j+h-t)^2\,dt\right)^2\right]$$

$\| \tau \|^2 \quad \sum^{n-1} \tau (x_j)^2$

$$+ \left(-\frac{1}{h^2} \int_{x_j} u''(t)(x_j + h - t)\,dt \right) \Big]$$

$$\to \| I_h \|_h^2 = h \sum_{j=1}^{n-1} I_h(x_j)^2$$

$$\leq 3h \sum_{j=1}^{n-1} \left(-u''(x_j) \right)^2 + 3h \sum_{j=1}^{n-1} \left(-\frac{1}{h} \int_{x_j-h}^{x_j} u''(t)(x_j - h - t)^2\,dt \right)^2$$

$$+ 3h \sum_{j=1}^{n-1} \left(-\frac{1}{h^2} \int_{x_j}^{x_j+h} u''(t)(x_j + h - t)^2\,dt \right)^2$$

7. Let $g = 1$ and prove that $T_h g(x_j) = \frac{1}{2} x_j (1 - x_j)$.

[Solution : use the definition (12.25) with $g(x_k) = 1$, $k = 1, \dots, n-1$

and recall that $G^k(x_j) = h\, G(x_j, x_k)$ from the exercise above.

Then $T_h g(x_i) = h \left[\sum_{k=1}^{j} x_k (1 - x_j) + \sum_{k=j+1}^{n-1} x_j (1 - x_k) \right]$

from which, after straightforward computations, one gets the desired result.]

Note : $\boxed{(12.25) \quad w_h = T_h g, \ w_h = \sum_{k=1}^{n-1} g(x_k) G^k}$

8. Prove Young's inequality (12.40)

Note : $\boxed{(12.40) \quad ab \le \varepsilon a^2 + \frac{1}{4\varepsilon} b^2 \quad \forall\, a, b \in \mathbb{R},\ \forall\, \varepsilon > 0.}$

$$\Rightarrow \quad \varepsilon a^2 - ab + \frac{1}{4\varepsilon} b^2 \ge 0$$

$$\rightarrow (\sqrt{\varepsilon}\, a)^2 - 2 \cdot \sqrt{\varepsilon}\, a \cdot \frac{1}{2\sqrt{\varepsilon}} b + \left(\frac{1}{2\sqrt{\varepsilon}} b\right)^2 \ge 0$$

$$\rightarrow \left(\sqrt{\varepsilon}\, a - \frac{1}{2\sqrt{\varepsilon}} b\right)^2 \ge 0$$

9. Show that $\|v_h\|_h \le \|v_h\|_{h,\infty} \quad \forall\, v_h \in V_h$

11. Discretize the fowth-order differential operator $L u(x) = -u^{(iv)}(x)$ using centered finite differences.
[Solution: apply twice the second order centered finite difference operator L_h defined in (12.9).]

Note : (12.9)