作業三手寫

1. Let $E_0(f)$ and $E_1(f)$ be the quadrature errors in (9.6) and (9.12).
Prove that $|E_1(f)| \simeq 2|E_0(f)|$

Note: (9.6) $E_0(f) = \frac{h^3}{3} f''(\xi)$, $h = \frac{b-a}{2}$

(9.12) $E_1(f) = -\frac{h^3}{12} f''(\xi)$, $h = b-a$

$\Rightarrow$ Since $E_0(f) = \left(\frac{b-a}{2}\right)^3 \cdot \frac{1}{3} f''(\xi) = (b-a)^3 \cdot \frac{1}{24} f''(\xi)$

and $E_1(f) = -(b-a)^3 \cdot \frac{1}{12} f''(\xi)$

then $|E_1(f)| = \left|\frac{-(b-a)^3}{12} f''(\xi)\right| = \left|\frac{(b-a)^3}{12} f''(\xi)\right|$

and $|E_0(f)| = \left|\frac{(b-a)^3}{24} f''(\xi)\right|$

Thus $|E_1(f)| \simeq 2|E_0(f)|$ ※

3. Let $I_n(f) = \sum_{k=0}^{n} \alpha_k f(x_k)$ be a Lagrange quadrature formula on $n+1$ nodes.
Compute the degree of exactness r of the formulae:

(a) $I_2(f) = (2/3)\left[2f(-1/2) - f(0) + 2f(1/2)\right]$,

(b) $I_4(f) = (1/4)\left[f(-1) + 3f(-1/3) + 3f(1/3) + f(1)\right]$.

Which is the order of infinitesimal p for (a) and (b)?

[Solution: $r=3$ and $p=5$ for both $I_2(f)$ and $I_4(f)$.]

Consider $f \in [a,b]$ where a,b 分別等於 $1, -1$

$\Rightarrow$ 習: it only need to check the basis of polynomails

e.g.: $1, x, x^2, \ldots$, e.t.c

(a 的場合) $I_2(f) = \frac{2}{3}[2f(-\frac{1}{2}) - f(0) + 2f(\frac{1}{2})]$

We have $\begin{cases} x_0 = -\frac{1}{2}, & w_0 = 2 \\ x_1 = 0, & w_1 = -1 \\ x_2 = \frac{1}{2}, & w_2 = 2 \end{cases}$ and normalized by $\frac{2}{3}$

① $f(x) = 1$

$\Rightarrow I_2(f) = \frac{2}{3} \cdot [2 \cdot 1 - 1 + 2 \cdot 1] = \frac{2}{3} \cdot 3 = 2$

Since $\int_{-1}^{1} 1 \, dx = 1 - (-1) = 2$

then it is correct #

② $f(x) = x$

$\Rightarrow I_2(f) = \frac{2}{3}[2 \cdot (-\frac{1}{2}) - 0 + 2 \cdot \frac{1}{2}] = 0$

Since $\int_{-1}^{1} x \, dx = \frac{1}{2} \cdot 1 - \frac{1}{2} \cdot 1 = 0$

then it is correct #

③ $f(x) = x^2$

$\Rightarrow I_2(f) = \frac{2}{3}[2 \cdot \frac{1}{4} - 0 + 2 \cdot \frac{1}{4}] = \frac{2}{3}$

Since $\int_{-1}^{1} x^2 \, dx = \frac{1}{3} \cdot 1 - \frac{1}{3}(-1) = \frac{2}{3}$

then it is correct #

④ $f(x) = x^3$

$\Rightarrow I_2(f) = \frac{2}{3}[2 \cdot (-\frac{1}{8}) - 0 + 2 \cdot \frac{1}{8}] = 0$

Since $\int_{-1}^{1} x^3 \, dx = \frac{1}{4} \cdot 1 - \frac{1}{4} \cdot 1 = 0$

then it is correct #

⑤ $f(x) = x^4$

$\Rightarrow I_2(f) = \frac{2}{3}[2 \cdot \frac{1}{16} + 0 + 2 \cdot \frac{1}{16}] = \frac{1}{6}$

but $\int_{-1}^{1} x^4 \, dx = \frac{1}{5} \cdot 1 - \frac{1}{5}(-1) = \frac{2}{5}$

So it is not correct. #

Thus the degree of exactness of $I_2(f)$ is 3

also the error is $O(h^5)$

(b 的場合) $I_4(f) = \frac{1}{4}[f(-1) + 3f(-\frac{1}{3}) + 3f(\frac{1}{3}) + f(1)]$

We have $\begin{cases} x_0 = -1, & w_0 = 1 \\ x_1 = -\frac{1}{3}, & w_1 = 3 \\ x_2 = \frac{1}{3}, & w_2 = 3 \\ x_3 = 1, & w_3 = 1 \end{cases}$ and normalized by $\frac{1}{4}$

① $f(x) = 1$

$\Rightarrow I_4(f) = \frac{1}{4}[1 + 3 + 3 + 1] = \frac{1}{4} \cdot 8 = 2$

Since $\int_{-1}^{1} 1 \, dx = 1 - (-1) = 2$

then it is correct. #

② $f(x) = x$

$\Rightarrow I_4(f) = \frac{1}{4}[-1 + 3(-\frac{1}{3}) + 3(\frac{1}{3}) + 1] = 0$

Since $\int_{-1}^{1} x\,dx = \frac{1}{2}\cdot 1 - \frac{1}{2}\cdot 1 = 0$

then it is correct ※

③ $f(x) = x^2$

$\Rightarrow I_4(f) = \frac{1}{4}[1 + 3(\frac{1}{9}) + 3(\frac{1}{9}) + 1] = \frac{1}{4}\cdot\frac{8}{3} = \frac{2}{3}$

Since $\int_{-1}^{1} x^2\,dx = \frac{1}{3}\cdot 1 - \frac{1}{3}(-1) = \frac{2}{3}$

then it is correct. ※

④ $f(x) = x^3$

$\Rightarrow I_4(f) = \frac{1}{4}[-1 + 3(-\frac{1}{27}) + 3(\frac{1}{27}) + 1] = 0$

Since $\int_{-1}^{1} x^3\,dx = \frac{1}{4}\cdot 1 - \frac{1}{4}\cdot 1 = 0$

then it is correct. ※

⑤ $f(x) = x^4$

$\Rightarrow I_4(f) = \frac{1}{4}[1 + 3(\frac{1}{81}) + 3(\frac{1}{81}) + 1] = \frac{1}{4}\cdot\frac{56}{27} = \frac{14}{27}$

but $\int_{-1}^{1} x^4\,dx = \frac{1}{5}\cdot 1 - \frac{1}{5}(-1) = \frac{2}{5}$

So it is not correct. ※

Thus the degree of exactness of $I_2(f)$ is 3

also the error is $O(h^5)$

5. Let $I_w(f) = \int_0^1 w(x) f(x)\,dx$ with $w(x) = \sqrt{x}$, and consider the quadrature formula $Q(f) = a f(x_1)$.

Find a and x_1 in such a way that Q has maximum degree of exactness r.

[Solution : $a = 2/3$, $x_1 = 3/5$ and $r = 1$.]

答: 直接測試!

$\Rightarrow$ Consider $I_w(f) = \int_0^1 w(x) f(x)\,dx$ where $w(x) = \sqrt{x}$, and $Q(f) = a f(x_1)$

① $f(x) = 1$

Since $Q(f) = a\cdot 1 = a$

$\Rightarrow I_w(f) = \int_0^1 \sqrt{x}\cdot 1\,dx = \frac{2}{3} x^{\frac{3}{2}} \Big|_0^1 = \frac{2}{3}$

Thus $a = 2/3$ ※

② $f(x) = x$

Since $Q(f) = a \cdot x_1 = \frac{2}{3} x_1$

$\Rightarrow I_w(f) = \int_0^1 \sqrt{x} \cdot x \, dx = \frac{2}{5} x^{\frac{5}{2}} \Big|_0^1 = \frac{2}{5}$

Set $\frac{2}{3} x_1 = \frac{2}{5}$ (if "=" holds)

then $x_1 = 3/5$ ※

③ $f(x) = x^2$

Since $Q(f) = a \cdot x_1^2 = \frac{2}{3} \cdot \left(\frac{3}{5}\right)^2 = \frac{6}{25}$

but $I_w(f) = \int_0^1 \sqrt{x} \cdot x^2 \, dx = \frac{2}{7} x^{\frac{7}{2}} \Big|_0^1 = \frac{2}{7}$

therefore it is not exactness to 2

Thus $r = 1$. ※

6. Let us consider the quadrature formula $Q(f) = \alpha_1 f(0) + \alpha_2 f(1) + \alpha_3 f'(0)$ for the approximation on $I(f) = \int_0^1 f(x)\,dx$, where $f \in C^1([0,1])$. Determine the coefficients α_j, for $j = 1, 2, 3$ in such a way that Q has degree of exactness $r = 2$.

[Solution: $\alpha_1 = 2/3$, $\alpha_2 = 1/3$ and $\alpha_3 = 1/6$.]

$\Rightarrow$ Consider $Q(f) = \alpha_1 f(0) + \alpha_2 f(1) + \alpha_3 f'(0)$ and $I(f) = \int_0^1 f(x)\,dx$

① $f(x) = 1$

Since $Q(f) = \alpha_1 \cdot 1 + \alpha_2 \cdot 1 + \alpha_3 \cdot 0 = \alpha_1 + \alpha_2$

$\Rightarrow I(f) = \int_0^1 1 \, dx = x \Big|_0^1 = 1$

then we got $\alpha_1 + \alpha_2 = 1$ ※

② $f(x) = x$

Since $Q(f) = \alpha_1 \cdot 0 + \alpha_2 \cdot 1 + \alpha_3 \cdot 1 = \alpha_2 + \alpha_3$

$\Rightarrow I(f) = \int_0^1 x \, dx = \frac{1}{2} x^2 \Big|_0^1 = \frac{1}{2}$

then we got $\alpha_2 + \alpha_3 = \frac{1}{2}$ ※

③ $f(x) = x^2$

Since $Q(f) = \alpha_1 \cdot 0 + \alpha_2 \cdot 1 + \alpha_3 \cdot 2 \cdot 0 = \alpha_2$

$\Rightarrow I(f) = \int_0^1 x^2 \, dx = \frac{1}{3} x^3 \Big|_0^1 = \frac{1}{3}$

thus $\alpha_2 = \frac{1}{3}$ ※

Since $\alpha_1 + \alpha_2 = 1$, then $\alpha_1 = \frac{2}{3}$

Since $\alpha_2 + \alpha_3 = \frac{1}{2}$, then $\alpha_3 = \frac{1}{6}$

(But we need to check that Q has degree of $r=2$)

④ $f(x)=x^3$

Since $Q(f)=\alpha_1\cdot 0+\alpha_2\cdot 1+\alpha_3\cdot 3\cdot 0=\alpha_2=\frac{1}{3}$

$\Rightarrow I(f)=\int_0^1 x^3\,dx=\frac{1}{4}x^4\Big|_0^1=\frac{1}{4}$

therefore it is not correct to $r=3$

Thus Q has degree of exactness of $r=2$. ※