作業六

7. Prove that the gamma function

$$\Gamma(z) = \int_0^\infty e^{-t} t^{z-1} dt \ , \ z \in C \ , \ \text{Re } z > 0$$

is the solution of the difference equation $\Gamma(z+1) = z\Gamma(z)$

[Hint: integrate by parts.]

$\Rightarrow$ Consider $\Gamma(z) = \int_0^\infty e^{-t} t^{z-1} dt$

then $\Gamma(z+1) = \int_0^\infty e^{-t} t^{(z+1-1)} dt = \int_0^\infty \underbrace{e^{-t} \overset{u}{t^{z}} dt}_{dv}$

By integrate by parts ($\int u\,dv = uv - \int v\,du$)

Set $u = t^z$, $dv = e^{-t} dt$

$du = z t^{z-1} dt$, $v = -e^{-t}$

$$= t^z(-e^{-t}) - \int_0^\infty (-e^{-t}) z t^{z-1} dt$$

$$= \boxed{-t^z e^{-t} \Big|_0^\infty} + \int_0^\infty e^{-t} z t^{z-1} dt$$

取極限

since Re $z > 0$

$$\Rightarrow \lim_{t\to\infty} (-t^z e^{-t}) = \lim_{t\to\infty} \frac{-t^{z}}{e^t} \overset{L'H}{=} \lim_{t\to\infty} \frac{-z t^{z-1}}{e^t} = \frac{1}{\infty} = 0$$

$$-0^z e^{-0} = 0$$

$$= \int_0^\infty e^{-t} z t^{z-1} dt = z\int_0^\infty e^{-t} t^{z-1} dt = z\Gamma(z)$$

Thus $\Gamma(z+1) = z\Gamma(z)$ ※

9. Consider the following family of one-step methods deepending on the real parameter α

$$u_{n+1} = u_n + h\left[\left(1-\frac{\alpha}{2}\right)f(x_n,u_n) + \frac{\alpha}{2}f(x_{n+1},u_{n+1})\right].$$

Study their consistency as a function of α ; then.

take $\alpha = 1$ and use the corresponding method to solve the Cauchy problem

$$\begin{cases} y'(x) = -10y(x), & x>0 \\ y(0) = 1 \end{cases}$$

Determine the values of h in correspondance of which the method is absolutely stable.

[Solution : the family of methods is consistent for any value of α.

The method of highest order (equal to two) is obtained

for $\alpha = 1$ and coincides with the Crank - Nicolson method.]

$\Rightarrow$ Consider $y_{n+1} = y_n + h\left[\left(1-\frac{\alpha}{2}\right)f(x_n,y_n) + \frac{\alpha}{2}f(x_{n+1},y_{n+1})\right]$

$$\Rightarrow h\left[\left(\frac{1}{2}+\frac{1-\alpha}{2}\right)f(x_n,y_n) + \left(\frac{1}{2}-\frac{1-\alpha}{2}\right)f(x_{n+1},y_{n+1})\right]$$

$$= \frac{h}{2}\left[f(x_n,y_n)+f(x_{n+1},y_{n+1})\right] + \frac{h(1-\alpha)}{2}\left[f(x_n,y_n)-f(x_{n+1},y_{n+1})\right]$$

Since the exact solution $y_{n+1} = y_n + \frac{h}{2}\left[f(x_n,y_n)+f(x_{n+1},y_{n+1})\right] + O(h^3)$

Since $\left[f(x_n,y_n)-f(x_{n+1},y_{n+1})\right] = O(h)$

Therefore $\frac{h(1-\alpha)}{2}\left[f(x_n,y_n)-f(x_{n+1},y_{n+1})\right] \approx O(h^2)$

Then it consistent for any value of α. *

Set $\alpha = 1$

$$\Rightarrow y_{n+1} = y_n + \frac{h}{2}\left[f(x_n,y_n)+f(x_{n+1},y_{n+1})\right]$$

(which is the Crank - Nicolson method.)

Let $\begin{cases} y'(x) = -10y(x), & x>0 \\ y(0) = 1 \end{cases}$ which imples $\lambda = -10$

$$\Rightarrow y_{n+1} = y_n + h\left[\left(1-\frac{\alpha}{2}\right)\lambda y_n + \frac{\alpha}{2}\lambda y_{n+1}\right]$$

$$\rightarrow y_{n+1}\left(1-\frac{h\alpha}{2}\lambda\right) = y_n\left(1+h\left(1-\frac{\alpha}{2}\right)\lambda\right)$$

$$\rightarrow \frac{y_{n+1}}{y_n} = \frac{1 + h\lambda\left(1 - \frac{\alpha}{2}\right)}{1 - h\lambda\frac{\alpha}{2}}$$

If $\alpha = 1$: $\frac{y_{n+1}}{y_n} = \frac{1 + h\lambda\frac{1}{2}}{1 - h\lambda\frac{1}{2}} = \frac{1 - 5h}{1 + 5h}$

It's stabily iff $\left|\frac{y_{n+1}}{y_n}\right| < 1$