作業五手寫

1. Prove that Heun's method has order 2 with respect to h.

[Hint: notice that $h\tau_{n+1} = y_{n+1} - y_n - h\Phi(t_n, y_n; h) = E_1 + E_2$, where

$$E_1 = \int_{t_n}^{t_{n+1}} f(s, y(s))\,ds - \frac{h}{2}\left[f(t_n, y_n) + f(t_{n+1}, y_{n+1})\right]$$

and

$$E_2 = \frac{h}{2}\left\{\left[f(t_{n+1}, y_{n+1}) - f(t_{n+1}, y_n + hf(t_n, y_n))\right]\right\},$$

where E_1 is the error due to numerical integration with the trapezoidal method and E_2 can be bounded by the error due to using the forward Euler method]

Note: Heun's method

$$u_{n+1} = u_n + \frac{h}{2}\left[f_n + f(t_{n+1}, u_n + hf_n)\right].$$

$\Rightarrow$ Consider $y_{n+1} = y_n + \frac{h}{2}\left[f_n + f(t_{n+1}, y_n + hf_n)\right]$

$= y_n + h\Phi(t_n, f_n, y_n; h)$ where $f_n = f(t_n, y_n)$

$\Rightarrow h\tau_{n+1} = \boxed{y_{n+1} - y_n} - \frac{h}{2}\left[f_n + f(t_{n+1}, y_n + hf_n)\right]$

Since $y(t_{n+1}) = y(t_n) + \int_{t_n}^{t_{n+1}} f(s, y(s))\,ds$

then $y(t_{n+1}) - y(t_n) = \int_{t_n}^{t_{n+1}} f(s, y(s))\,ds$

then $h\tau_{n+1} = \boxed{\int_{t_n}^{t_{n+1}} (f(s, y(s))\,ds - \frac{h}{2}\left[f(t_n, y_n) + f(t_{n+1}, y_{n+1})\right]} \to E_1$

$\boxed{+ \frac{h}{2}\left[f(t_{n+1}, y_{n+1}) - f(t_{n+1}, y_n + hf(t_n, y_n))\right]} \to E_2$

(計算 error)

E_1: $= -\frac{h^3}{12} f''(\xi, y(\xi)) = O(h^3)$ for some $\xi \in [t_n, t_{n+1}]$

(E_1 is the error due to numerical integration with the trapezoidal method)

$\Rightarrow |E_1| \le C_1 h^3$

E_2: Suppose there exists a Lipschitz constant L s.t.

$|E_2| \le \frac{h}{2} L \,|y_{n+1} - (y_n + hf(t_n, y_n))| \approx O(h^3)$

$\Rightarrow h\tau_{n+1} = O(h^3) \to \tau_{n+1} = O(h^2)$

Thus the Heun's method has order 2 with respect to h. ※