作業二手寫

1. Chapter 8 : #5、#6、#8

#5. Prove that $(n-1)!\, h^{n-1} |(x-x_{n-1})(x-x_n)| \le |\omega_{n+1}(x)| \le n!\, h^{n-1} |(x-x_{n-1})(x-x_n)|$,

where n is even, $-1 = x_0 < x_1 < \cdots < x_{n-1} < x_n = 1$, $x \in (x_{n-1}, x_n)$ and $h = 2/n$.

[Hint: Let $N = n/2$ and show first that

$$\omega_{n+1}(x) = (x+Nh)(x+(N-1)h)\cdots(x+h)x(x-h)\cdots(x-(N-1)h)(x-Nh) \quad (8.74)$$

寫：分上下界做討論

Suppose $N = n/2$ and $x = rh$ where $N-1 < r < N$, and $h = 2/n$.

Since $-1 = x_0 < x_1 < \cdots < x_{n-1} < x_n = 1$,

then $x_k = -1 + kh = -N + kh$, $k = 0, \cdots, n$

Thus $\omega_{n+1}(x) = \prod_{i=0}^{n}(x - x_i) = \prod_{k=-N}^{N}(x - kh)$

Set $x = rh$ where $r \in (N-1, N)$

Since $x \in (x_{n-1}, x_n) = ((N-1)h, Nh)$

$\Rightarrow \omega_{n+1}(x) = h^{2N+1} \prod_{k=-N}^{N}(r-k)$

$$\prod_{k=-N}^{N}(r-h) = (r-N)(r-(N-1)) \cdot \prod_{j=1}^{N-1}(r-j)(r+j)$$

$$= (r-N)(r-(N-1))\, r \cdot \prod_{j=1}^{N-1}(r^2-j^2)$$

$\boxed{\text{下界}}$

Note: $x - x_{n-1} = rh - (N-1)h = (r-(N-1))h$

$x - x_n = rh - Nh = (r-N)h$

$\Rightarrow (n-1)!\, h^{n-1} |(x-x_{n-1})(x-x_n)| = (2N-1)!\, h^{2N-1} |(r-(N-1))(r-N)|\, h^2$

$= (2N-1)!\, h^{2N+1} |(r-(N-1))(r-N)|$

Since $\omega_{n+1}(x) = h^{2N+1}(r-N)(r-(N-1)) \prod_{j=1}^{N-1}(r^2-j^2)$

$\forall j = 1, \cdots, N-1$, since $r \ge N-1$, $r - j \ge (N-1) - j$

$$\begin{cases} \prod_{j=1}^{N-1}(r-j) \ge \prod_{j=1}^{N-1}(N-1-j+1) = (N-1)! & (\because r-j \ge (N-1)-j+1 = N-j) \\ \prod_{j=1}^{N-1}(r+j) \ge \prod_{j=1}^{N-1} j = (N-1)! & (\because r+j \ge j) \end{cases}$$

$$\Rightarrow \prod_{j=1}^{N-1}(r^2-j^2) = \prod_{j=1}^{N-1}(r-j) \prod_{j=1}^{N-1}(r+j) \ge (N-1)!\,(N-1)!$$

$$\Rightarrow |\omega_{n+1}(x)| = h^{2N+1} |(r-N)(r-(N-1))|\, r \prod_{j=1}^{N-1}|r^2-j^2|$$

$$\ge h^{2N+1} |(r-N)(r(N-1))| \cdot 1 \cdot (N-1)!$$

We have $\dfrac{|w_{n+1}(x)|}{h^{2n+1}\,|(r-(N-1))(r-N)|} \geq (N-1)!$

Since $r+j \geq N+j$ and $r-j \geq 1$

$\Rightarrow |w_{n+1}(x)| \geq (2N+1)!\, h^{2N+1}\, |(r-(N+1))(r-N)|$

$\rightarrow (n-1)!\, h^{n-1}\, |(x-x_{n-1})(x-x_n)| \leq |w_{n+1}(x)|$

上界

$\forall j = 1, \dots, N-1$, we have $r+j \leq N+(N-1) = 2N-1$ and $r-j \leq N$

$\Rightarrow \prod_{j=1}^{N-1} (r^2 - j^2) \leq N!\,(2N-1)! \,/\, ((r-(N-1))(r-N)\,r)$

$\Rightarrow |w_{n+1}(x)| \leq n!\, h^{2N+1}\, |(r-(N-1))(r-N)|$

$\rightarrow |w_{n+1}(x)| \leq n!\, h^{n-1}\, |(x-x_{n-1})(x-x_n)|$

Thus $(n-1)!\, h^{n-1}\, |(x-x_{n-1})(x-x_n)| \leq |w_{n+1}(x)| \leq n!\, h^{n-1}\, |(x-x_{n-1})(x-x_n)|$ ※

6. Under the assumptions of Exercise 5, show that $|w_{n+1}|$ is maximum if $x \in (x_{n-1}, x_n)$ (notice that $|w_{n+1}|$ is an even function)

[Hint : use (8.74) to prove that $|w_{n+1}(x+h) / w_{n+1}(x)| > 1$ for any $x \in (0, x_{n-1})$ with x not coinciding with any interpolation node.]

Note : $w_{n+1}(x) = (x+Nh)(x+(N-1)h)\cdots(x+h)x$
$(x-h)\cdots(x-(N-1)h)(x-Nh)$ (8.74)

: prove $\left|\dfrac{w_{n+1}(x+h)}{w_{n+1}(x)}\right| = \prod_{k=0}^{n} \left|\dfrac{x+h-x_k}{x-x_k}\right| > 1 \quad \forall x \in (x_{n-1}, x_n)$

From Exercise 5, Since n is even

consider $N = n/2$; $x_k = -1 + kh = -N + kh$, $k = 0, \dots, n$ and $h = 2/n$.

Let $x = rh$ where $r \in (N-1, N)$

and $j = -N, -N+1, \dots, N$

Then $\dfrac{x+h-x_k}{x-x_k} = \dfrac{(r+1-k)h}{(r-k)h} = \dfrac{r+1-k}{r-k}$

$$\Rightarrow \prod_{k=-N}^{N} \left| \frac{r+1-k}{r-k} \right|$$

If $k = 0$: $\left| \frac{r+1-0}{r-0} \right| = \frac{r+1}{r} > 0$ $(\because r > 0)$

$k = j > 0$ and $k = -j < 0$:

$$\Rightarrow \left| \frac{r+1-j}{r-j} \right| \cdot \left| \frac{r+1+j}{r+j} \right| = \left| \frac{(r+1-j)(r+1+j)}{(r-j)(r+j)} \right| = \left| \frac{\boxed{(r+1)^2 - j^2}}{r^2 - j^2} \right|$$

(*)

(*) : ① If $r^2 - j^2 \geq 0$

$$|(r+1)^2 - j^2| = |r^2 + 2r + 1 - j^2| = |r^2 - j^2| + (2r+1)$$

② If $r^2 - j^2 < 0$

$$|(r+1)^2 - j^2| = \big|\, |r^2 - j^2| - (2r+1) \big|$$

$$\Rightarrow |(r+1)^2 - j^2| > |r^2 - j^2|$$

Thus $\prod_{k=-N}^{N} \left| \frac{r+1-k}{r-k} \right| > 1$ ※

#8. Determine an interpolating polynomial $Hf \in \mathbb{P}$

$$(Hf)^{(k)}(x_0) = f^{(k)}(x_0), \quad k = 0, \dots, n$$

and check that $Hf(x) = \sum_{j=0}^{n} \frac{f^{(j)}(x_0)}{j!} (x - x_0)^j$,

that is, the Hermite interpolating polynomial on one node coincides with the Taylor polynomial.

Note : $H_{N-1}(x) = \sum_{i=0}^{n} \sum_{k=0}^{m_i} y_i^{(k)} L_{ik}(x)$, where $y_i^{(k)} = f^{(k)}(x_i)$

Consider $T(x) := \sum_{j=0}^{n} \frac{f^{(j)}(x_0)}{j!} (x - x_0)^j$

which is the taylor expansion of f at x_0

① 0-th derivative

$$\Rightarrow T(x) = \sum_{j=0}^{n} \frac{f^{(j)}(x_0)}{j!} (x - x_0)^j = \frac{f^{(0)}(x_0)}{0!} (x - x_0)^0 = f(x_0) \cdot 1 \cdot 1 = f(x_0)$$

② k-th derivative

$$\Rightarrow T(x)^{(k)}(x) = \sum_{j=k}^{n} \frac{f^{(j)}(x_0)}{j!} (x - x_0)^{j-k} \frac{j!}{(j-k)!}$$

$$= \frac{f^{(k)}(x_0)}{k!} k! = f^{(k)}(x_0)$$

"=" holds only when $x = x_0$, $j = k$ 那項不為 0

Thus $Hf(x) = \sum_{j=0}^{n} \frac{f^{(j)}(x_0)}{j!} (x - x_0)^j$ ※

2. Show that, for $n+1$ Chebyshev points of the second kind, the barycentric weights are (after rescaling)

$$\omega_i = (-1)^i, \quad i = 1, \dots, n-1 \quad \text{and} \quad \omega_0 = 1/2, \quad \omega_n = (-1)^n/2$$

Note: the barycentric weights $\omega_j = \dfrac{1}{\prod_{k \neq j} (x_j - x_k)}$

Let $\theta_j = \dfrac{j\pi}{n}$, $x_j = \cos\theta_j$

$$\Rightarrow P_j := \prod_{\substack{k=0 \\ k \neq j}}^{n} (x_j - x_k) = \prod_{\substack{k=0 \\ k \neq j}}^{n} (\cos\theta_j - \cos\theta_k)$$

$\left(\because \cos a - \cos b = -2 \sin \frac{a+b}{2} \sin \frac{a-b}{2}\right)$

$$= -2 \sin \frac{\theta_j + \theta_k}{2} \sin \frac{\theta_j - \theta_k}{2}$$

$$\Rightarrow P_j = \prod_{k \neq j} (-2) \prod_{k \neq j} \sin \frac{\theta_j + \theta_k}{2} \prod_{k \neq j} \sin \frac{\theta_j - \theta_k}{2}$$

$$= (-2)^n \underbrace{\boxed{\prod_{k \neq j} \sin \frac{\theta_j + \theta_k}{2}}}_{(*)} \underbrace{\boxed{\prod_{k \neq j} \sin \frac{\theta_j - \theta_k}{2}}}_{(**)}$$

$(*)$: $$\prod_{k \neq j} \sin \frac{\theta_j + \theta_k}{2} = \prod_{k \neq j} \sin \frac{\left(\frac{j\pi}{n} + \frac{k\pi}{n}\right)}{2} = \prod_{k \neq j} \sin \frac{(j+k)\pi}{2n}$$

(Let $m = j-k$)

$m \in \{-j, \dots, -1, 1, \dots, n-j\}$

$$\Rightarrow \prod_{m=1}^{n} \sin \frac{m\pi}{2n} \Big/ \sin \frac{j\pi}{2n}$$

$(**)$: $$\prod_{k \neq j} \sin \frac{\theta_j - \theta_k}{2} = \prod_{\substack{k=0 \\ k \neq j}}^{n} \sin \frac{\left(\frac{j\pi}{n} - \frac{k\pi}{n}\right)}{2} = \prod_{\substack{k=0 \\ k \neq j}}^{n} \sin \frac{(j-k)\pi}{2n}$$

(Let $m = j-k$)

$m \in \{-j, \dots, -1, 1, \dots, n-j\}$

$$\Rightarrow \prod_{m=1}^{j} \sin \frac{m\pi}{2n} \times \prod_{m=1}^{n-j} \sin \frac{m\pi}{2n}$$

$$\Rightarrow P_j = (-2)^n \prod_{m=1}^{n} \sin \frac{m\pi}{2n} \Big/ \sin \frac{j\pi}{2n} \prod_{m=1}^{j} \sin \frac{m\pi}{2n} \times \prod_{m=1}^{n-j} \sin \frac{m\pi}{2n}$$

Note: $$\prod_{m=1}^{n-1} \sin \frac{m\pi}{n} = \frac{n}{2^{n-1}}$$

Thus $$P_j = (-1)^j \, 2^{-n} \cdot n \cdot \frac{1}{\sin\left(\frac{j\pi}{n}\right)}$$

If $j = 0$: $$P_0 = \prod_{k=1}^{n} (x_0 - x_k) = \prod_{k=1}^{n} \left(1 - \cos \frac{k\pi}{n}\right)$$