5. Prove that

$$(n-1)\ h^{n-1}|(x-x_{n-1})(x-x_n)| \leq |\omega_{n+1}(x)| \leq n!h^{n-1}|(x-x_{n-1})(x-x_n)|,$$

where n is even, $-1 = x_0 < x_1 < \cdots < x_{n-1} < x_n = 1$, $x \in (x_{n-1}, x_n)$ and $h = \frac{2}{n}$. [Hint：let $N = n/2$ and show first that

$$\begin{aligned}\omega_{n+1}(x) =&(x+Nh)(x+(N-1)h)\cdots(x+h)x \\ &(x-h)\cdots(x-(N-1)h)(x-h)\end{aligned} \tag{8.74}$$

Then, take $x = rh$ with $N-1 < r < N$.]

⇒Suppose n is even and the nodes x_j are equally spaced over $[-1, 1]$ with $h = 2/n$, we define the nodes $x_j = -1 + jh, \quad j = 0, 1, \ldots, n$.
Since n is even, let $N = n/2$ which implies $x_N = 0$. We can rewrite the nodes relative to the midpoint

$$x_j = (j-N)h$$

The nodal polynomial is defined as

$$\omega_{n+1}(x) = \prod_{j=0}^{n}(x-x_j) = (x-x_0)(x-x_1)\cdots(x-x_n)$$

Since we have $x_j = (j-N)h$, and letting $k = j-N$, then

$$\omega_{n+1}(x) = (x+Nh)(x+(N-1)h)\cdots(x+h)x(x-h)\cdots(x-(N-1)h)(x-Nh).$$

If $x \in (x_{n-1}, x_n)$, then isolate the terms involving x_{n-1} and x_n

$$|\omega_{n+1}(x)| = \left|\prod_{j=0}^{n-2}(x-x_j)\right| \cdot |(x-x_{n-1})(x-x_n)|$$

Let $P(x) = \prod_{j=0}^{n-2}(x-x_j)$. Our goal is to show that

$$(n-1)!h^{n-1} \leq P(x) \leq n!h^{n-1}$$

Consider $x = rh$. Since $x \in (x_{n-1}, x_n)$ and $x_n = 1 = Nh$, we have

$$(N-1)h < rh < Nh \implies N-1 < r < N$$

Then

$$P(rh) = \prod_{j=0}^{n-2}(rh-(j-N)h) = h^{n-1}\prod_{j=0}^{n-2}(r-(j-N)).$$

Let $k = j - N$. As j goes from 0 to $n-2$, k goes from $-N$ to $N-2$

$$\Rightarrow P(rh) = h^{n-1} \underbrace{(r+N)(r+N-1)\cdots(r-(N-2))}_{f(r)}$$

Since $r \in (N-1, N)$, all factors in $f(r)$ are positive. Furthermore, $f(r)$ is a strictly increasing function of r because each factor increases as r increases.

- If $r \to N-1$:

$$f(r) > (N-1+N)(N-1+N-1)\cdots(N-1-(N-2))$$

$$f(r) > (2N-1)(2N-2)\cdots(1) = (n-1)!$$

  Then, $P(x) \geq (n-1)!h^{n-1}$

- If $r \to N$

$$f(r) < (N+N)(N+N-1)\cdots(N-(N-2))$$

$$f(r) < (2N)(2N-1)\cdots(2) = n!$$

  Then, $P(x) \leq n!h^{n-1}$

Thus

$$(n-1)!h^{n-1} \leq \left| \prod_{j=0}^{n-2} (x - x_j) \right| \leq n!h^{n-1}$$

Hence we obtain

$$(n-1)!h^{n-1}|(x-x_{n-1})(x-x_n)| \leq |\omega_{n+1}(x)| \leq n!h^{n-1}|(x-x_{n-1})(x-x_n)|$$

6. Under the assumptions of Exercise 5, show that$|\omega_{n+1}|$ is maximum if $x \in (x_{n-1}, x_n)$ (notice that$|\omega_{n+1}|$ is an even function). [Hint use (8.74) to prove that $|\omega_{n+1}(x+h)/\omega_{n+1}(x)| > 1$ for any $x \in (0, x_{n-1})$ with x not coinciding with any interpolation node.]

$\Rightarrow$ Since the set of nodes is symmetric, we have

$$\omega_{n+1}(-x) = \prod_{j=0}^{n}(-x - x_j) = (-1)^{n+1}\prod_{j=0}^{n}(x + x_j) = (-1)^{n+1}\omega_{n+1}(x)$$

Since n is even, $n+1$ is odd, making $\omega_{n+1}(x)$ an odd function. Therefore, its absolute value $|\omega_{n+1}(x)|$ is an even function

$$|\omega_{n+1}(-x)| = |-\omega_{n+1}(x)| = |\omega_{n+1}(x)|.$$

Let $x \in (0, x_{n-1})$ such that x is not a node. We examine the ratio of the polynomial evaluated at $x+h$ versus x

$$\left|\frac{\omega_{n+1}(x+h)}{\omega_{n+1}(x)}\right| = \left|\frac{\prod_{j=0}^{n}(x+h-x_j)}{\prod_{j=0}^{n}(x-x_j)}\right|$$

Since $x_j = x_{j-1} + h$, we can rewrite the terms in the numerator

$$x + h - x_j = x - (x_j - h) = x - x_{j-1}$$

Thus, the numerator is

$$\omega_{n+1}(x+h) = (x+h-x_0)(x+h-x_1)\cdots(x+h-x_n) = (x+h-x_0)(x-x_0)(x-x_1)\cdots(x-x_{n-1})$$

And the denominator is

$$\omega_{n+1}(x) = (x-x_0)(x-x_1)\cdots(x-x_{n-1})(x-x_n)$$

Canceling common terms $(x-x_0)\cdots(x-x_{n-1})$, we get

$$\left|\frac{\omega_{n+1}(x+h)}{\omega_{n+1}(x)}\right| = \left|\frac{x+h-x_0}{x-x_n}\right|$$

Since $x_0 = -1$ and $x_n = 1$, then

$$\left|\frac{\omega_{n+1}(x+h)}{\omega_{n+1}(x)}\right| = \left|\frac{x+h+1}{x-1}\right| = \frac{x+h+1}{1-x}$$

For any $x \in (0, x_{n-1})$, if $x+h+1 > 1-x$

$$x + h + 1 > 1 - x \implies 2x + h > 0$$

Since $x > 0$ and $h > 0$, this inequality is always true. Therefore

$$|\omega_{n+1}(x+h)| > |\omega_{n+1}(x)|.$$

Since$|\omega_{n+1}(x+h)| > |\omega_{n+1}(x)|$, the magnitude of $|\omega_{n+1}(x)|$ increases monotonically away from the center. By symmetry, the maximum is attained in the outermost intervals (x_0, x_1) and (x_{n-1}, x_n).
Thus

$$|\omega_{n+1}| \text{ is maximum if } x \in (x_{n-1}, x_n).$$

8. Determine an interpolating polynomial $Hf \in \mathbb{P}_n$ such that

$$(Hf)^{(k)}(x_0) = f^{(k)}(x_0), \ \ k = 0, \dots, n,$$

and check that

$$(Hf)(x) = \sum_{j=0}^{n} \frac{f^{(j)}(x_0)}{j!}(x - x_0)^j,$$

that is, the Hermite interpolating polynomial on one node coincides with the Taylor polynomial.

$\Rightarrow$ Since Hf is a polynomial of degree n in $\mathbb{P}_n$, then

$$(Hf)(x) = a_0 + a_1(x - x_0) + a_2(x - x_0)^2 + \dots + a_n(x - x_0)^n = \sum_{j=0}^{n} a_j(x - x_0)^j.$$

Our goal is to determine the coefficients a_j such that the interpolation conditions are satisfied.
Given $(Hf)^{(k)}(x_0) = f^{(k)}(x_0)$ for $k = 0, \dots, n$. The k-th derivative of a single term $(x - x_0)^j$ evaluated at $x = x_0$ is $\begin{cases} 0 & if \quad j < k \\ k! & if \quad j = k \\ 0 & if \quad j > k \end{cases}$
Therefore, when we evaluate the k-th derivative of the whole sum at x_0, all terms vanish except for the term where $j = k$

$$(Hf)^{(k)}(x_0) = k! a_k$$

Setting this equal to the required condition $f^{(k)}(x_0)$, we find

$$k! a_k = f^{(k)}(x_0) \implies \mathbf{a_k} = \frac{\mathbf{f^{(k)}(x_0)}}{\mathbf{k!}}$$

Substituting these coefficients a_j back into $(Hf)(x)$

$$(Hf)(x) = \sum_{j=0}^{n} \frac{f^{(j)}(x_0)}{j!}(x - x_0)^j$$

This expression is precisely the definition of the Taylor polynomial of degree n for the function f at the point x_0.

9. Given the following set of data

$$\{f_0 = f(-1) = 1, f_1 = f'(-1) = 1, f_2 = f'(1) = 2, f_3 = f'(2) = 1\},$$

prove that the Hermite-Birkoff interpolating polynomial H_3 does not exist for them. [Solution letting $H_3(x) = a_3x^3 + a_2x^2 + a_1x + a_0$, one must check that the matrix of the linear system $H_3(x_i) = f_i$ for $i = 0, \ldots, 3$ is singular.]

$\Rightarrow$ Let $H_3(x) \in \mathbb{P}_3$ and define $H_3(x) = a_3x^3 + a_2x^2 + a_1x + a_0$.

$$\Rightarrow H_3'(x) = 3a_3x^2 + 2a_2x + a_1$$

Since we have $H_3(-1) = 1$, $H_3'(-1) = 1$, $H_3'(1) = 2$ and $H_3'(2) = 1$, then

$$\begin{aligned}
H_3(-1) = a_3(-1)^3 + a_2(-1)^2 + a_1(-1) + a_0 = 1 &\implies -a_3 + a_2 - a_1 + a_0 = 1,\\
H_3'(-1) = 3a_3(-1)^2 + 2a_2(-1) + a_1 = 1 &\implies 3a_3 - 2a_2 + a_1 = 1,\\
H_3'(1) = 3a_3(1)^2 + 2a_2(1) + a_1 = 2 &\implies 3a_3 + 2a_2 + a_1 = 2,\\
H_3'(2) = 3a_3(2)^2 + 2a_2(2) + a_1 = 1 &\implies 12a_3 + 4a_2 + a_1 = 1.
\end{aligned}$$

We can write this as a system $A\mathbf{a} = \mathbf{f}$, where $\mathbf{a} = [a_3, a_2, a_1, a_0]^T$

$$\begin{pmatrix} -1 & 1 & -1 & 1 \\ 3 & -2 & 1 & 0 \\ 3 & 2 & 1 & 0 \\ 12 & 4 & 1 & 0 \end{pmatrix} \begin{pmatrix} a_3 \\ a_2 \\ a_1 \\ a_0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 2 \\ 1 \end{pmatrix}$$

Expanding the determinant along the last column

$$\det(A) = -(1) \cdot \det \begin{pmatrix} 3 & -2 & 1 \\ 3 & 2 & 1 \\ 12 & 4 & 1 \end{pmatrix}$$

$$\begin{aligned}
\Rightarrow \det \begin{pmatrix} 3 & -2 & 1 \\ 3 & 2 & 1 \\ 12 & 4 & 1 \end{pmatrix} &= 3(2 \cdot 1 - 4 \cdot 1) - (-2)(3 \cdot 1 - 12 \cdot 1) + 1(3 \cdot 4 - 12 \cdot 2)\\
&= 3(-2) + 2(-9) + 1(-12) = -6 - 18 - 12 = -36
\end{aligned}$$

Therefore, $\det(A) = -(-36) = 36$.
Since $\det(A) \neq 0$, the system is non-singular, and for the data provided $(f(-1), f'(-1), f'(1), f'(2))$, the polynomial H_3.
Thus the Hermite-Birkoff interpolating polynomial H_3 does not exist for them.

12. Let $f(x) = cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots$; then, consider the following rational approximation

$$r(x) = \frac{a_0 + a_2x^2 + a_4x^4}{1 + b_0x^2},$$

called the Padé approximation. Determine the coefficients of r in such a way that

$$f(x) - r(x) = \gamma_8 x^8 + \gamma_{10}x^{10} + \ldots$$

[Solution $a_0 = 1, a_2 = -\frac{7}{15}, a_4 = \frac{1}{40}, b_2 = \frac{1}{30}$.]

$\Rightarrow$ The condition $f(x) - r(x) = O(x^8)$ implies that the first 7 terms of the Taylor series of $f(x)$ and $r(x)$ must be identical. Therefore we have

$$f(x) \approx \frac{a_0 + a_2x^2 + a_4x^4}{1 + b_2x^2}$$

Multiplying both sides by the denominator, we get the condition for the coefficients

$$f(x)(1 + b_2x^2) - (a_0 + a_2x^2 + a_4x^4) = O(x^8)$$

Since the Taylor expansion for $\cos(x)$ centered at 0

$$f(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + O(x^8) = 1 - \frac{x^2}{2} + \frac{x^4}{24} - \frac{x^6}{720} + O(x^8)$$

Multiply this by $(1 + b_2x^2)$

$$\left(1 - \frac{x^2}{2} + \frac{x^4}{24} - \frac{x^6}{720}\right)(1 + b_2x^2) = a_0 + a_2x^2 + a_4x^4 + 0x^6 + O(x^8)$$

We can find the terms

$$\begin{cases} \text{constant} & \to & 1 \\ x^2 & \to & b_2 - \frac{1}{2} \\ x^4 & \to & -\frac{b_2}{2} + \frac{1}{24} \\ x^6 & \to & \frac{b_2}{24} - \frac{1}{720} \end{cases}$$

For x^6 Since there is no x^6 term in the numerator $P(x) = a_0 + a_2x^2 + a_4x^4$, the coefficient of x^6 in the expansion must be zero

$$\frac{b_2}{24} - \frac{1}{720} = 0 \implies \frac{b_2}{24} = \frac{1}{720} \implies b_2 = \frac{24}{720} = \frac{1}{30}$$

For x^0 :

$$a_0 = 1$$

For x^2 :

$$a_2 = b_2 - \frac{1}{2} = \frac{1}{30} - \frac{15}{30} = -\frac{7}{15}$$

For x^4 :

$$a_4 = \frac{1}{24} - \frac{b_2}{2} = \frac{1}{24} - \frac{1}{2}\left(\frac{1}{30}\right) = \frac{1}{24} - \frac{1}{60}$$

Thus the coefficients for the Padé approximation

$$r(x) = \frac{1 - \frac{7}{15}x^2 + \frac{1}{40}x^4}{1 + \frac{1}{30}x^2}$$

are $a_0 = 1,\ a_2 = -\frac{7}{15},\ a_4 = \frac{1}{40},$ and $b_2 = \frac{1}{30}$.

Chapter9

1. Let $E_0(f)$ and $E_1(f)$ be the quadrature errors in (9.6) and (9.12). Prove that $|E_1(f)| \cong 2|E_0(f)|$.

$$E_0(f) = \frac{h^3}{3!}f''(\xi), h = \frac{b-a}{2} \tag{9.6}$$

$$E_1(f) = -\frac{h^3}{12}f''(\xi), h = b{-}a, \tag{9.12}$$

$\Rightarrow$ Consider the interval $[a, b]$. Let $x_M = \frac{a+b}{2}$ be the midpoint and $L = b - a$ be the total length.
By the Midpoint Rule(I_M) and thee Trapezoidal Rule (I_T), we have

$$I_M = L \cdot f(x_M)$$

and

$$I_T = \frac{L}{2}[f(a) + f(b)]$$

To find the errors, we expand $f(x)$ in a Taylor series about the midpoint x_M. Let $x = x_M + \delta$,then

$$f(x_M + \delta) = f(x_M) + \delta f'(x_M) + \frac{\delta^2}{2}f''(x_M) + \frac{\delta^3}{6}f'''(x_M) + O(\delta^4)$$

The exact integral I is

$$I = \int_a^b f(x)\,dx = \int_{-L/2}^{L/2} f(x_M + \delta)\,d\delta$$

Integrating the Taylor series

$$I = \left[\delta f(x_M) + \frac{\delta^2}{2}f'(x_M) + \frac{\delta^3}{6}f''(x_M) + \dots\right]_{-L/2}^{L/2}$$

Since the odd powers of δ cancel out over the symmetric interval

$$I = Lf(x_M) + \frac{L^3}{24}f''(x_M) + O(L^5)$$

For the Midpoint Rule (E_0)：

$$E_0 = I - I_M = \left(Lf(x_M) + \frac{L^3}{24}f''(x_M)\right) - Lf(x_M)$$

$$E_0 \approx \frac{L^3}{24} f''(x_M)$$

For the Trapezoidal Rule (E_1) : First, evaluate $f(a)$ and $f(b)$ using the Taylor series at $\delta = -L/2$ and $\delta = L/2$

$$f(a)+f(b) = \left(f(x_M) - \frac{L}{2}f' + \frac{L^2}{8}f''\right)+\left(f(x_M) + \frac{L}{2}f' + \frac{L^2}{8}f''\right) = 2f(x_M)+\frac{L^2}{4}f''(x_M)$$

$$I_T = \frac{L}{2}[2f(x_M) + \frac{L^2}{4}f''(x_M)] = Lf(x_M) + \frac{L^3}{8}f''(x_M)$$

$$\Rightarrow E_1 = I - I_T = \left(Lf(x_M) + \frac{L^3}{24}f''\right) - \left(Lf(x_M) + \frac{L^3}{8}f''\right)$$

$$\rightarrow E_1 \approx \left(\frac{1}{24} - \frac{3}{24}\right) L^3 f'' = -\frac{L^3}{12}f''(x_M)$$

Comparing the absolute values of the error terms

$$|E_1| = \frac{L^3}{12}|f''| \text{ and } |E_0| = \frac{L^3}{24}|f''|$$

Dividing the two

$$\frac{|E_1|}{|E_0|} = \frac{1/12}{1/24} = 2$$

Thus, $|E_1| \cong 2|E_0|$.

3. Let $I_n(f) = \sum_{k=0}^{n} \alpha_k f(x_k)$ be a Lagrange quadrature formula on $n+1$ nodes. Compute the degree of exactness r of the formulae：

(a) $I_2(f) = (2/3)[2f(-1/2) - f(0) + 2f(1/2)]$,

(b) $I_4(f) = (1/4)[f(-1) + 3f(-1/3) + 3f(1/3) + f(1)]$.

Which is the order of infinitesimal p for (a) and (b)? [Solution $r = 3$ and $p = 5$ for both $I_2(f)$ and $I_4(f)$.]

$\Rightarrow$ (a) $I_2(f) = \frac{2}{3}[2f(-1/2) - f(0) + 2f(1/2)]$

Since

$$I_2(f) = \frac{2}{3}[2f(-\frac{1}{2}) - f(0) + 2f(\frac{1}{2})] = \frac{4}{3}f(-\frac{1}{2}) - \frac{2}{3}f(0) + \frac{4}{3}f(\frac{1}{2}),$$

then we have $x_0 = -\frac{1}{2}, x_1 = 0, x_2 = \frac{1}{2}$ with weights $\alpha_0 = \frac{4}{3}, \alpha_1 = -\frac{2}{3}, \alpha_2 = \frac{4}{3}$.

If $k = 0$：

$$\begin{cases} \int_{-1}^{1} 1dx & = & 2 \\ I_2(1) & = & \frac{2}{3}[2(1) - 1 + 2(1)] = \frac{2}{3}(3) = 2 \end{cases}$$

If $k = 1$：

$$\begin{cases} \int_{-1}^{1} xdx & = & 0 \\ I_2(x) & = & \frac{2}{3}[2(-1/2) - 0 + 2(1/2)] = 0 \end{cases}$$

If $k = 2$：

$$\begin{cases} \int_{-1}^{1} x^2dx & = & \left[\frac{x^3}{3}\right]_{-1}^{1} = \frac{2}{3} \\ I_2(x^2) & = & \frac{2}{3}[2(-1/2)^2 - 0 + 2(1/2)^2] = \frac{2}{3}[\frac{2}{4} + \frac{2}{4}] = \frac{2}{3} \end{cases}$$

If $k = 3$：

$$\begin{cases} \int_{-1}^{1} x^3dx & = & 0 \\ I_2(x^3) & = & \frac{2}{3}[2(-1/8) - 0 + 2(1/8)] = 0 \end{cases}$$

If $k = 4$：

$$\begin{cases} \int_{-1}^{1} x^4dx & = & \frac{2}{5} = 0.4 \\ I_2(x^4) & = & \frac{2}{3}[2(1/16) - 0 + 2(1/16)] = \frac{2}{3}(\frac{1}{4}) = \frac{1}{6} \approx 0.166 \end{cases}$$

Thus the degree of exactness is $r = 3$.

(b) $I_4(f) = \frac{1}{4}[f(-1) + 3f(-1/3) + 3f(1/3) + f(1)]$

Since

$$I_4(f) = \frac{1}{4}[f(-1)+3f(-\frac{1}{3})+3f(\frac{1}{3})+f(1)] = \frac{1}{4}f(-1)+\frac{3}{4}f(-\frac{1}{3})+\frac{3}{4}f(\frac{1}{3})+\frac{1}{4}f(1),$$

then we have $x_0 = -1, x_1 = -\frac{1}{3}, x_2 = \frac{1}{3}, x_3 = 1$ with weights $\alpha_0 = \frac{1}{4}, \alpha_1 = \frac{3}{4}, \alpha_2 = \frac{3}{4}, \alpha_3 = \frac{1}{4}$.

If $k = 0$：

$$\begin{cases} \int_{-1}^{1} 1dx &= 2 \\ I_4(1) &= \frac{1}{4}[1+3+3+1] = \frac{8}{4} = 2 \end{cases}$$

If $k = 1$ or 3：Due to the symmetry of the nodes and weights, the summation will be 0, matching the integral.

If $k = 2$：

$$\begin{cases} \int_{-1}^{1} x^2 dx &= \frac{2}{3} \\ I_4(x^2) &= \frac{1}{4}[(-1)^2 + 3(-1/3)^2 + 3(1/3)^2 + (1)^2] = \frac{1}{4}[1 + \frac{3}{9} + \frac{3}{9} + 1] = \frac{1}{4}[2 + \frac{2}{3}] = \frac{2}{3} \end{cases}$$

If $k = 4$：

$$\begin{cases} \int_{-1}^{1} x^4 dx &= 0.4 \\ I_4(x^4) &= \frac{1}{4}[1 + 3(1/81) + 3(1/81) + 1] = \frac{1}{4}[2 + \frac{2}{27}] = \frac{1}{4}(\frac{56}{27}) = \frac{14}{27} \approx 0.518 \end{cases}$$

Thus the degree of exactness is $r = 3$.

Since in both cases (a) and (b) we all have $r = 3$ and the rules are symmetric, the leading order of the error is

$$p = r + 2 = 5.$$