作業四手寫

9. Given the following set of data

$\{ f_0 = f(-1) = 1,\ f_1 = f'(-1) = 1,\ f_2 = f'(1) = 2,\ f_3 = f(2) = 1 \}$,

Prove that the Hermite - Birkoff interpolating polynomial H_3 does not exist for them.

[Solution : letting $H_3(f) = a_3 x^3 + a_2 x^2 + a_1 x + a_0$, one must check that the matrix of the linear system $H_3(x_i) = f_i$ for $i = 0, \dots, 3$ is singular.]

$\Rightarrow$ Consider $H_3(f) = a_3 x^3 + a_2 x^2 + a_1 x + a_0$

Since $\{ f_0 = f(-1) = 1,\ f_1 = f'(-1) = 1,\ f_2 = f'(1) = 2,\ f_3 = f(2) = 1 \}$

[Note : $H_3'(f) = 3a_3 x^2 + 2a_2 x + a_1$]

① $f_0 = f(-1) = 1$

$H_3(f(-1)) = a_3(-1)^3 + a_2(-1)^2 + a_1(-1) + a_0$

$= -a_3 + a_2 - a_1 + a_0 = 1$ ※

② $f_1 = f'(-1) = 1$

$H_3(f'(-1)) = 3a_3(-1)^2 + 2a_2(-1) + a_1$

$= 3a_3 - 2a_2 + a_1 = 1$ ※

③ $f_2 = f'(1) = 2$

$H_3(f'(1)) = 3a_3(1)^2 + 2a_2(1) + a_1$

$= 3a_3 + 2a_2 + a_1 = 2$ ※

④ $f_3 = f(2) = 1$

$H_3(f(2)) = a_3(2)^3 + a_2(2)^2 + a_1(2) + a_0$

$= 8a_3 + 4a_2 + 2a_1 + a_0 = 1$ ※

(解聯立)

$$\Rightarrow \begin{cases} 3a_3 - 2a_2 + a_1 = 1 \\ 3a_3 + 2a_2 + a_1 = 2 \end{cases} \quad -)$$

$-4a_2 = -1 \Rightarrow a_2 = \frac{1}{4}$

$$\Rightarrow \begin{cases} 8a_3 + 4a_2 + 2a_1 + a_0 = 1 \\ -a_3 + a_2 - a_1 + a_0 = 1 \end{cases} \quad -)$$

$9a_3 + 3a_2 + 3a_1 = 0$ (代入 a_2)

$$\Rightarrow 9a_3 + 3a_1 = \frac{-3}{4} \rightarrow 3a_3 + a_1 = \frac{-1}{4}$$

$(a_1 = -\frac{1}{4} - 3a_3)$ 代入 ②

$$\Rightarrow 3a_3 - \frac{2}{4} - \frac{1}{4} - 3a_3 = 1$$

But "=" does not hold

Thus the Hermite - Birkoff interpolatin polynomial H_3 does not exist for them. ※

12. Let $f(x) = \cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$; then, consider the following rational approximation

$$r(x) = \frac{a_0 + a_2x^2 + a_4x^4}{1 + b_2x^2}, \qquad (8.75)$$

called the Padé approximation. Determine the coefficients of r in such a way that $f(x) - r(x) = r_8x^8 + r_{10}x^{10} + \cdots$

[Solution : $a_0 = 1$, $a_2 = -7/15$, $a_4 = 1/40$, $b_2 = 1/30$.]

$$\Rightarrow \text{Consider } r(x) = \frac{a_0 + a_2x^2 + a_4x^4}{1 + b_2x^2}$$

$$= (a_0 + a_2x^2 + a_4x^4)(1 - b_2x^2 + b_2^2x^4 - b_2^3x^6 + \cdots)$$

(展開檢查各項係數)

① x^0 : $a_0 = 1$ ※

② x^2 : $-a_0b_2x^2 + a_2x^2 \cdot 1 = -\frac{x^2}{2}$

$\Rightarrow -a_0b_2 + a_2 = -\frac{1}{2}$ ※

③ x^4 : $a_0b_2^2x^4 - a_2b_2x^4 + a_4x^4 \cdot 1 = \frac{x^4}{4!}$

$\Rightarrow a_0b_2^2 - a_2b_2 + a_4 = \frac{1}{24}$ ※

④ x^6 : $-a_0b_2^3x^6 + a_2b_2^2x^6 - a_4b_2x^6 = -\frac{x^6}{6!}$

$\Rightarrow -a_0b_2^3 + a_2b_2^2 - a_4b_2 = -\frac{1}{720}$ ※

(整理一下並代入 $a_0 = 1$)

$$\Rightarrow \begin{cases} -b_2 + a_2 = -\frac{1}{2} \\ b_2^2 - a_2b_2 + a_4 = \frac{1}{24} \\ -b_2^3 + a_2b_2^2 - a_4b_2 = -\frac{1}{720} \end{cases}$$

$$a_2 = -\frac{1}{2} + b_2 \quad \#$$

$$b_2^2 - \left(-\frac{1}{2}+b_2\right) b_2 + a_4 = \frac{1}{24}$$

$$\rightarrow b_2^2 + \frac{1}{2} b_2 - b_2^2 + a_4 = \frac{1}{24}$$

$$\rightarrow a_4 = \frac{1}{24} - \frac{1}{2} b_2 \quad \#$$

$$\Rightarrow -b_2^3 + \left(-\frac{1}{2}+b_2\right) b_2^2 - \left(\frac{1}{24} - \frac{1}{2} b_2\right) b_2 = -\frac{1}{720}$$

$$\rightarrow -b_2^3 - \frac{1}{2} b_2^2 + b_2^3 - \frac{1}{24} b_2 + \frac{1}{2} b_2^2 = -\frac{1}{720}$$

$$\rightarrow -\frac{1}{24} b_2 = -\frac{1}{720}$$

$$\rightarrow b_2 = \frac{1}{30} \quad \#$$

$$\Rightarrow a_2 = -\frac{1}{2} + \frac{1}{30} = -\frac{14}{30} = -\frac{7}{15} \quad \#$$

$$a_4 = \frac{1}{24} - \frac{1}{2} \cdot \frac{1}{30} = \frac{3}{120} = \frac{1}{40} \quad \#$$

Thus $a_0 = 1$, $a_2 = -7/15$, $a_4 = 1/40$, $b_2 = 1/30$. #