FOURTH-ORDER TIME-STEPPING FOR STIFF PDEs*

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1 Introduction and Problem Statement

In this paper, many time-dependent partial differential equations arising in physics and engineering can be written in the semilinear form

$$u_t = Lu + N(u,t)$$

where L is lineraly and N is nonlinearly. Standard explicit time-stepping methods for such problems are severely restricted by stability constraints, whereas fully implicit methods often incur high computational cost due to the repeated solution of large nonlinear systems.

The main problem addressed in this paper is the construction of efficient and accurate time integration schemes for semilinear evolution equations that overcome stiffness induced by the linear operator without sacrificing accuracy in the nonlinear dynamics.
Therefore, this paper employs the concept of integral factors from the IF method, where the linear parts of the equation are precisely processed using matrix exponents to derive an accurate formula for constant variability.
Based on this expression, the author developed an exponential time difference scheme by approximating the remaining nonlinear integral using quadrilateral forms of Taylor and Runge-Kutta types. These methods preserve the stability advantages of the rigid linear term implicit scheme while maintaining the simplicity and efficiency of explicit nonlinear component approaches.
The paper further analyzes the accuracy and performance of the proposed electron emission tomography method and demonstrates its effectiveness in representing rigid semilinear problems.

2 Method Explanation

This paper mentions a modified ETD method, which is similar to the IF method.
From the original assumptions in the paper $u_t = Lu + N(u,t)$, we multiply it by the integrating factor e^{-Lt}.

$$\Rightarrow e^{-Lt}u_t = e^{-Lt}Lu + e^{-Lt}N(u,t)$$

$$\to e^{-Lt}u_t - e^{-Lt}Lu = e^{-Lt}N(u,t)$$

Since $\frac{\mathrm{d}}{\mathrm{d}t}(e^{-Lt}u) = e^{-Lt}u_t - e^{-Lt}Lu$, then $\frac{\mathrm{d}}{\mathrm{d}t}(e^{-Lt}u) = e^{-Lt}N(u,t)$.

(Integrating time part)

$$\Rightarrow \int_{t_n}^{t_n+h} \frac{\mathrm{d}}{\mathrm{d}t} e^{-Lt}u_t \mathrm{d}t = \int_{t_n}^{t_n+h} e^{-Lt}N(u,t)\mathrm{d}t$$

$$\to e^{-L(t_n+h)}u_(t_n+h) - e^{-Lt_n}u(t_n) = \int_{t_n}^{t_n+h} \frac{\mathrm{d}}{\mathrm{d}t} e^{-Lt}N(u,t)\mathrm{d}t$$

$$\Rightarrow u_{n+1} = e^{Lh}u_n + \int_{t_n}^{t_n+h} e^{-Lt}N(u,t)\mathrm{d}t$$

Let $t = t_n + \tau$.

$$\Rightarrow u_{n+1} = e^{Lh}u_n + e^{Lh}\int_0^h e^{-L\tau}N(u(t_n+\tau), t_n+\tau)\mathrm{d}\tau$$

Consider

$$u_{n+1} = e^{Lh}u_n + e^{Lh}\int_0^h e^{-L\tau}N(u(t_n+\tau), t_n+\tau)\mathrm{d}\tau$$

(Take it to become a general form)

$$\Rightarrow u_{n+1} = e^{Lh}u_n + h\sum_{m=0}^{s-1} g_m \sum_{k=0}^{m} (-1)^k \binom{m}{k} N_{n-k}$$

where ($Lhg_0 = e^{Lh} - I$)

$$Lhg_{m+1} + I = g_m + \frac{1}{2}g_{m-1} + \frac{1}{3}g_{m-2} + \cdots + \frac{g_0}{m+1},\ m \geq 0$$

Using the ETDRK4 formula to approximate integral terms,

$$N_n = \int_0^h e^{-L\tau}N(u(t_n+\tau), t_n+\tau)\mathrm{d}\tau$$

$$\begin{cases} a_n &= e^{Lh/2}u_n + L^{-1}(e^{Lh/2} - I)N(u_n, t_n) \\ b_n &= e^{Lh/2}u_n + L^{-1}(e^{Lh/2} - I)N(a_n, t_n + \frac{h}{2}) \\ c_n &= e^{Lh/2}a_n + L^{-1}(e^{Lh/2} - I)(2N(b_n, t_n + \frac{h}{2})) - N(u_n, t_n) \\ u_{n+1} &= e^{Lh}u_n + h^{-2}L^{-3}\{[-4 - Lh + e^{Lh}(4 - 3Lh + (Lh)^2)]N(u_n, t_n)\} \\ & +2[2 + Lh + e^{Lh}(-2 + Lh)]N(a_n, t_n + \frac{h}{2}) + N(b_n, t_n + \frac{h}{2}) \\ & +[-43Lh + (Lh)^2) + e^{Lh}(4 - Lh)]N(c_n, t_n + h) \end{cases}$$

Due to some issues (numerical instability) with the ETDRK4 formula, this article has made some changes.
Let $g(z) = \frac{e^z - 1}{z}$, which is approximated using Taylor polynomials.
But it is still imperfect, and the truncation errors still exist.
Therefore, it take the new coefficients

$$\begin{cases} \alpha &= h^{-2}L^{-3}[-4 - Lh + e^{Lh}(4 - 3Lh + (Lh^2))] \\ \beta &= h^{-2}L^{-3}[2 + Lh + e^{Lh}(-2 + Lh)] \\ \gamma &= h^{-2}L^{-3}[-4 - 3Lh - (Lh^2) + e^{Lh}(4 - Lh)] \end{cases}$$

Regarding accuracy issues, the answer is obtained by solving f(z), where Γ is on the complex plane.

$$f(z) = \frac{1}{2\pi i}\int_\Gamma \frac{f(t)}{(t-z)}\mathrm{d}t$$

If L is the matrix

$$\Rightarrow f(L) = \frac{1}{2\pi i}\int_\Gamma f(t)(tI - L)^{-1}\mathrm{d}t$$

Consider Γ as a circle, typically 32 or 64 equidistant points are sufficient.

3 Experiment

Under the condition of fixing everything and only changing the nonlinear integral approximation method, we attempted to compare error calculations using Euler, linear Taylor, and ETDRK4.[1]

[1] Please refer to the "ETDRK4_test" file for the code section

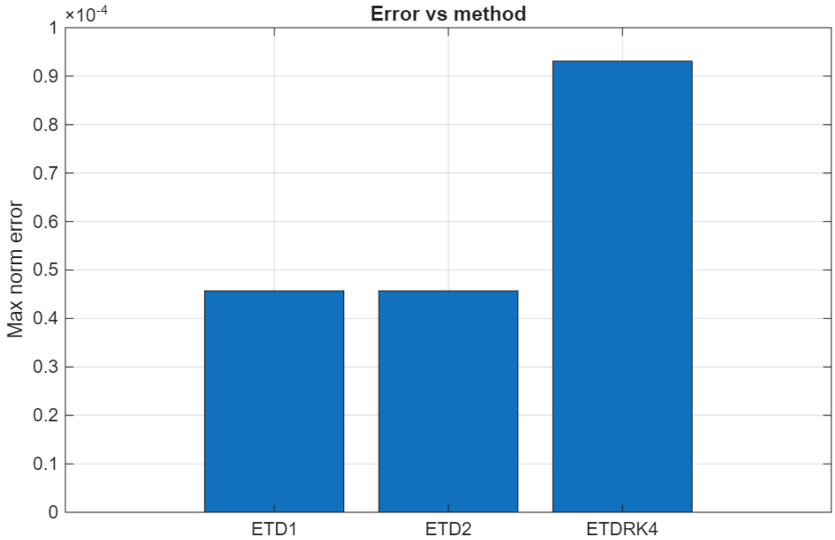

Figure 1: The errror

4 Conclusion

This work shows that exponential time differencing methods based on integrating factors offer an effective balance between stability and efficiency for stiff semilinear problems. By treating the linear stiffness exactly and approximating the nonlinear effects with high-order quadrature, ETD schemes achieve accurate and stable time integration without the cost of fully implicit methods.