作業七

3. Consider the boundary value problem (12.1) – (12.2) with $f(x) = 1/x$.

Using (12.3) prove that $u(x) = -x\log(x)$. This shows that $u \in C^2(0,1)$ bus $u(0)$ is not defined and u', u'' do not exist at $x = 0$

($\Rightarrow$: if $f \in C^0(0,1)$, but not $f \in C^0(0,1)$, but not $f \in C^0([0,1])$, then u does not belong to $C^0([0,1])$).

Note: (12.1) $-u''(x) = f(x)$, $0 < x < 1$

(12.2) $u(0) = u(1) = 0$

(12.3) $u(x) = \int_0^1 G(x,s)f(s)\,ds$

$\Rightarrow$ Consider $\begin{cases} -u''(x) = f(x), & 0 < x < 1 \\ u(0) = u(1) = 0 \end{cases}$ and $f(x) = 1/x$.

Since the Green function for this B.V.P. is

$$G(x,s) = \begin{cases} s(1-x) & \text{if } 0 \le s \le x \\ x(1-s) & \text{if } x \le s \le 1 \end{cases}$$

Set $u(x) = C_1 + C_2 x - \int_0^x F(s)\,ds$ where $F(s) = \int_0^s f(t)\,dt$

by intergration by parts, $\int_0^x F(s)\,ds = \int_0^x (x-s)f(s)\,ds$

$$\Rightarrow u(x) = \int_0^1 G(x,s)f(s)\,ds$$

$$= \underbrace{\int_0^x s(1-x)\frac{1}{s}\,ds}_{①} + \underbrace{\int_x^1 x(1-s)\frac{1}{s}\,ds}_{②}$$

① $\Rightarrow \int_0^x (1-x)\,ds = (1-x)s\Big|_0^x = (1-x)x$

② $\Rightarrow \int_x^1 \frac{x}{s} - x\,ds = x\ln s\Big|_x^1 - sx\Big|_x^1$

$= x(\ln 1 - \ln x) - x(1-x)$

$= -x\ln x - (1-x)x$

$$= (1-x)x - x\ln x - (1-x)x = -x\ln x$$

Since $u(1) = u(0) = 0$, then $\begin{cases} u(0) = -0 \cdot \ln 0 = 0 \\ u(1) = -\ln 1 = 0 \end{cases}$

Since $-u''(x) = f(x)$ where $0 < x < 1$

$\Rightarrow u'(x) = -\ln x + (-x \cdot \frac{1}{x}) = -\ln x - 1$

($\rightarrow$ if $x = 0$, then $u'(0) = -\ln 0 - 1$ does not exist)

$\rightarrow u''(x) = -\frac{1}{x}$

($\rightarrow$ if $x = 0$, then $u''(0) = -\frac{1}{0}$ does not exist)

then $-u''(x) = \frac{1}{x}$

(The both condition holds)

Thus $u(x) = -x \ln x$ ※

4. Cerify ① the summation by parts formula

$$\sum_{j=0}^{n-1} (w_{j+1} - w_j) v_j = w_n v_n - w_0 v_0 - \sum_{j=0}^{n-1} (v_{j+1} - v_j) w_{j+1},$$

and show that, ② for $v_h \in V_h^0$, $(L_h v_h, v_h) = h^{-1} \sum_{j=0}^{n-1} (v_{j+1} - v_j)^2$.

①

$$\Rightarrow \sum_{j=0}^{n-1} (w_{j+1} - w_j) v_j = \underline{\sum_{j=0}^{n-1} w_{j+1} v_j} - \underline{\sum_{j=0}^{n-1} w_j v_j}$$

Let $k = j+1 \Rightarrow \sum_{k=1}^{n} w_k v_{k-1}$

Since $k = j+1 \Rightarrow \left[\sum_{k=1}^{n-1} w_k v_k \right] + w_0 v_0$

$$= \underline{\sum_{k=1}^{n} w_k v_{k-1} - \sum_{k=1}^{n-1} w_k v_k} - w_0 v_0$$

$$= \left[\sum_{k=1}^{n-1} w_k (v_{k-1} - v_k) \right] + w_n v_{n-1}$$

$$= w_n v_{n-1} - w_0 v_0 + \underline{\sum_{k=1}^{n-1} w_k (v_{k-1} - v_k)}$$ (換回原本的 j)

$$= w_n v_{n-1} - w_0 v_0 - \sum_{j=0}^{n-1} w_{j+1} (v_{j+1} - v_j)$$ ※

② $\Rightarrow$ Since $(w_h, v_h)_h = h\sum_{k=0}^{n} w_k v_k$ and $(L_h v_h)_h = \dfrac{v_{j+1} - 2v_j + v_{j-1}}{h^2}$, $1 \le j \le n-1$

then $$(L_h v_h, v_h)_h = h\sum_{j=0}^{n}\left[\frac{v_{j+1} - 2v_j + v_{j-1}}{h^2}\right] v_j$$

$$= \frac{1}{h}\sum_{j=0}^{n}\left[(v_{j+1} - v_j) - (v_j - v_{j-1})\right] v_j$$

By the summation by parts and set $w_{-1} = v_{-1} = 0$

$$\Rightarrow \sum_{j=0}^{n-1}(v_{j+1} - v_j)v_j = v_n v_n - v_0 v_0 - \sum_{j=0}^{n-1}(v_{j+1} - v_j)v_{j+1}$$

$$= -\sum_{j=0}^{n-1}(v_{j+1} - v_j)v_{j+1}$$

$$\sum_{j=0}^{n-1}(v_{j+1} - v_j)v_j - \sum_{j=0}^{n-1}(v_{j+1} - v_j)v_{j+1} = \sum_{j=0}^{n-1}(v_{j+1} - v_j)(v_j - v_{j+1}) = \sum_{j=0}^{n-1}(v_{j+1} - v_j)^2$$

Thus, $(L_h v_h, v_j)_h = \dfrac{1}{h}\displaystyle\sum_{j=0}^{n-1}(v_{j+1} - v_j)^2$

5. Prove that $G^k(x_j) = hG(x_j, x_k)$, where G is Green's function introduced in (12.4) and G^k is its corresponding discrete counterpart solution of (12.4).

[Solution: we prove the result by verifying that $L_h G = he^k$.

Indeed, for a fixed x_k the function $G(x_k, s)$ is a straight line on the intervals $[0, x_k]$ and $[x_k, 1]$ so that $L_h G = 0$ at every node x_l with $l = 0, \dots, k-1$ and $l = k+1, \dots, n+1$.

Finally, a direct computation shows that $(L_h G)(x_k) = 1/h$ which concludes the proof.]

Note : $G(x,s) = \begin{cases} s(1-x) & \text{if } 0 \le s \le x \\ x(1-s) & \text{if } x \le s \le 1 \end{cases}$