作業五手寫

1. Prove that Heun's method has order 2 with respect to h.

[Hint: notice that $h\tau_{n+1} = y_{n+1} - y_n - h\Phi(t_n, y_n; h) = E_1 + E_2$, where

$$E_1 = \int_{t_n}^{t_{n+1}} f(s, y(s))\,ds - \frac{h}{2}\left[f(t_n, y_n) + f(t_{n+1}, y_{n+1})\right]$$

and

$$E_2 = \frac{h}{2}\left\{\left[f(t_{n+1}, y_{n+1}) - f(t_{n+1}, y_n + h f(t_n, y_n))\right]\right\},$$

where E_1 is the error due to numerical integration with the trapezoidal method and E_2 can be bounded by the error due to using the forward Euler method]

Note: Heun's method

$$u_{n+1} = u_n + \frac{h}{2}\left[f_n + f(t_{n+1}, u_n + h f_n)\right].$$

$\Rightarrow$ Consider $y_{n+1} = y_n + \frac{h}{2}\left[f_n + f(t_{n+1}, y_n + hf_n)\right]$

$= y_n + h\Phi(t_n, f_n, y_n; h)$ where $f_n = f(t_n, y_n)$

$\Rightarrow h\tau_{n+1} = \boxed{y_{n+1} - y_n} - \frac{h}{2}\left[f_n + f(t_{n+1}, y_n + hf_n)\right]$

Since $y(t_{n+1}) = y(t_n) + \int_{t_n}^{t_{n+1}} f(s, y(s))\,ds$

then $y(t_{n+1}) - y(t_n) = \int_{t_n}^{t_{n+1}} f(s, y(s))\,ds$

then $h\tau_{n+1} = \boxed{\int_{t_n}^{t_{n+1}} (f(s, y(s))\,ds - \frac{h}{2}\left[f(t_n, y_n) + f(t_{n+1}, y_{n+1})\right]} \rightarrow E_1$

$\boxed{+ \frac{h}{2}\left[f(t_{n+1}, y_{n+1}) - f(t_{n+1}, y_n + hf(t_n, y_n))\right]} \rightarrow E_2$

(計算 error)

E_1: $= -\frac{h^3}{12} f''(\xi, y(\xi)) = O(h^3)$ for some $\xi \in [t_n, t_{n+1}]$

(E_1 is the error due to numerical integration with the trapezoidal method)

$\Rightarrow |E_1| \le C_1 h^3$

E_2: Suppose there exists a Lipschitz constant L s.t.

$|E_2| \le \frac{h}{2} L \left| y_{n+1} - (y_n + h f(t_n, y_n)) \right| \approx O(h^3)$

$\Rightarrow h\tau_{n+1} = O(h^3) \rightarrow \tau_{n+1} = O(h^2)$

Thus the Heun's method has order 2 with respect to h. ※

2. Prove that the Crank - Nicoloson method has order 2 with respect to h

[Solution : using (9.12) we get, for a suitable ξ_n in (t_n, t_{n+1})

$$y_{n+1} = y_n + \frac{h}{2}\left[f(t_n, y_n) + f(t_{n+1}, y_{n+1})\right] - \frac{h^3}{12} f''(\xi_n, y(\xi_n))$$

or equivalently,

$$\frac{y_{n+1} - y_n}{h} = \frac{1}{2}\left[f(t_n, y_n) + f(t_{n+1}, y_{n+1})\right] - \frac{h^2}{12} f''(\xi_n, y(\xi_n)) \quad (11.90)$$

Therefore, relation (11.9) coincides with (11.90) up to an infinitesimal of order 2 with respect to h, provided that $f \in C^2(I)$.]

Note : ① If $f \in C^2([a,b])$, the quadrature error is given by

$$E_1(f) = -\frac{h^3}{12} f''(\xi), \quad h = b - a \qquad (9.12)$$

② trapezoidal (or Crank - Nicolson) method

$$u_{n+1} = u_n + \frac{h}{2}\left[f_n + f_{n+1}\right] \qquad (11.9)$$

$\Rightarrow$ Consider $y_{n+1} = y_n + \frac{h}{2}\left[f(t_n, y_n) + f(t_{n+1}, y_{n+1})\right] \to (*)$

Since $y_{n+1} = y_n + \int_{t_n}^{t_{n+1}} f(s, y(s))\, ds$, then $\exists$ some $\xi \in (t_n, t_{n+1})$

s.t. $\int_{t_n}^{t_{n+1}} f(s, y(s))\, ds = \frac{h}{2}\left[f(t_n, y_n) + f(t_{n+1}, y_{n+1})\right] - \frac{h^3}{12} f''(\xi, y(\xi))$

$$\Rightarrow y_{n+1} = y_n + \frac{h}{2}\left[f(t_n, y_n) + f(t_{n+1}, y_{n+1})\right] - \frac{h^3}{12} f''(\xi, y(\xi))$$

$$\to \frac{y_{n+1} - y_n}{h} = \frac{1}{2}\left[f(t_n, y_n) + f(t_{n+1}, y_{n+1})\right] \boxed{- \frac{h^2}{12} f''(\xi, y(\xi))}$$

(與真解相比) → (*)

$$\Rightarrow \frac{y_{n+1} - y_n}{h} = \frac{1}{2}\left[f(t_n, y_n) + f(t_{n+1}, y_{n+1})\right] + O(h^2)$$

Thus the Crank - Nicoloson method has order 2 with respect to h #