

# 作業八

5. Prove the estimate (12.23).

[ Hint: for each internal node  $x_j$ ,  $j = 1, \dots, n-1$ , integrate by parts (12.21) to get

$$T_h(x_j) = -u''(x_j) - \frac{1}{h^2} \left[ \int_{x_j-h}^{x_j} u''(t)(x_j - h - t) dt - \int_{x_j}^{x_j+h} u''(t)(x_j + h - t) dt \right]$$

Then, pass to the squares and  $T_h(x_j)^2$  for  $j = 1, \dots, n-1$ .

Observe nothing that  $(a+b+c)^2 \leq 3(a^2 + b^2 + c^2)$ , for any real numbers  $a, b, c$ , and applying the Cauchy-Schwarz inequality yields the desired result.]

Note: (12.23)  $\|T_h\|_h^2 \leq 3(\|f\|_h^2 + \|f\|_{L^2(0,1)}^2)$

(12.21)  $T_h(x_j) = \frac{1}{h^2} (R_4(x_j+h) + R_4(x_j-h))$

where  $R_4(x_j+h) = \int_{x_j}^{x_j+h} (u'''(t) - u''(x_j)) \frac{(x_j + h - t)^2}{2} dt$

and  $R_4(x_j-h) = - \int_{x_j-h}^{x_j} (u'''(t) - u''(x_j)) \frac{(x_j - h - t)^2}{2} dt$

Remark 12.3: Let  $e = u - u_h$  be the discretization error grid function.

$$\text{Then } L_h e = L_h u - L_h u_h = L_h u - f_h = T_h$$

$$\Rightarrow \text{Let } T_h(x_j) = \frac{1}{h^2} \left[ \underbrace{\int_{x_j}^{x_j+h} (u'''(t) - u''(x_j)) \frac{(x_j + h - t)^2}{2} dt}_{\text{①}} \right. \\ \left. - \underbrace{\int_{x_j-h}^{x_j} (u'''(t) - u''(x_j)) \frac{(x_j - h - t)^2}{2} dt}_{\text{②}} \right]$$

By integrate by parts,

① Let  $v = \frac{(x_j + h - t)^2}{2}$ ,  $dg = u'''(t) dt$

$$dv = -(x_j + h - t) dt, g = u''(t)$$

$$\Rightarrow \int v dg = g v - \int g dv = u''(t) \frac{(x_j - h - t)^2}{2} \Big|_{x_j}^{x_j+h} + \int_{x_j}^{x_j+h} u''(t) (x_j + h - t) dt \\ = -\frac{h^2}{2} u''(x_j) + \int_{x_j}^{x_j+h} u''(t) (x_j + h - t) dt$$

and  $\int_{x_j}^{x_j+h} u''(x_j) \frac{(x_j + h - t)^2}{2} dt = u''(x_j) \frac{-1}{6} (x_j + h - t)^3 \Big|_{x_j}^{x_j+h} = -\frac{h^3}{6} u''(x_j)$

② Let  $v = \frac{(x_j - h - t)^2}{2}$ ,  $dg = u'''(t) dt$

$$dv = -(x_j - h - t) dt, g = u''(t)$$

$$\Rightarrow \int v dg = -g v + \int g dv = -u''(t) \frac{(x_j - h - t)^2}{2} \Big|_{x_j-h}^{x_j} - \int_{x_j-h}^{x_j} u''(t) (x_j - h - t) dt \\ = -u''(x_j) \cdot \frac{h^2}{2} - \int_{x_j-h}^{x_j} u''(t) (x_j - h - t) dt$$

$$\text{and } \int_{x_j-h}^{x_j} u'''(x_j) \frac{(x_j-h-t)^2}{2} dt = -u'''(x_j) \frac{(x_j-h-t)^3}{6} \Big|_{x_j-h}^{x_j} = -\frac{h^3}{6} u'''(x_j)$$

$$\Rightarrow T_h(x_j) = \frac{1}{h^2} \left[ -\frac{h^2}{2} u''(x_j) + \int_{x_j}^{x_j+h} u''(t) (x_j+h-t) dt + \frac{h^3}{6} u'''(x_j) \right. \\ \left. - \frac{h^2}{2} u''(x_j) - \int_{x_j-h}^{x_j} u''(t) (x_j-h-t) dt - \frac{h^3}{6} u'''(x_j) \right]$$

then  $T_h(x_j) = -u''(x_j) - \frac{1}{h^2} \left[ \int_{x_j-h}^{x_j} u''(t) (x_j-h-t) dt \right. \\ \left. - \int_{x_j}^{x_j+h} u''(t) (x_j+h-t) dt \right]$

$$\text{Since } (a+b+c)^2 \leq 3(a^2 + b^2 + c^2)$$

$$\Rightarrow [T_h(x_j)]^2 \leq 3 \left[ (-u''(x_j))^2 + \left( -\frac{1}{h^2} \int_{x_j-h}^{x_j} u''(t) (x_j-h-t) dt \right)^2 \right. \\ \left. + \left( \frac{1}{h^2} \int_{x_j}^{x_j+h} u''(t) (x_j+h-t) dt \right)^2 \right]$$

$$\rightarrow \|T_h\|_h^2 = h \sum_{j=1}^{n-1} T_h(x_j)^2 \\ \leq 3h \sum_{j=1}^{n-1} (-u''(x_j))^2 + 3h \sum_{j=1}^{n-1} \left( -\frac{1}{h^2} \int_{x_j-h}^{x_j} u''(t) (x_j-h-t) dt \right)^2 \rightarrow ① \\ + 3h \sum_{j=1}^{n-1} \left( \frac{1}{h^2} \int_{x_j}^{x_j+h} u''(t) (x_j+h-t) dt \right)^2 \rightarrow ② \\ \Rightarrow = 3\|f\|_h^2 \quad (\because -u''(x_j) = f(x_j))$$

① By Cauchy - Schwarz inequality,

$$\begin{aligned} & \Rightarrow \left[ -\frac{1}{h^2} \int_{x_j-h}^{x_j} u''(t) (x_j - h - t) dt \right]^2 \\ & \leq \frac{1}{h^4} \left( \int_{x_j-h}^{x_j} |u''(t)|^2 dt \right) \left( \int_{x_j-h}^{x_j} (x_j - h - t)^2 dt \right) \\ & \text{Let } \xi = t - x_{j-1} = t - (x_j - h) = -(x_j - h - t) \\ & \text{then } (x_j - h - t)^2 = \xi^2 \\ & \leq \frac{1}{h^4} \left( \int_{x_{j-1}}^{x_j} |u''(t)|^2 dt \right) \left( \int_0^h \xi^2 dt \right) \\ & = \frac{1}{h^4} \left( \int_{x_{j-1}}^{x_j} |u''(t)|^2 dt \right) \cdot \left[ \frac{\xi^3}{3} \Big|_0^h \right] = \frac{h^3}{3} \\ & = \frac{1}{3h} \int_{x_{j-1}}^{x_j} |u''(t)|^2 dt \end{aligned}$$

② By Cauchy - Schwarz inequality,

$$\begin{aligned} & \Rightarrow \left[ \frac{1}{h^2} \int_{x_j}^{x_j+h} u''(t) (x_j + h - t) dt \right]^2 \\ & \leq \frac{1}{h^4} \left( \int_{x_j}^{x_j+h} |u''(t)|^2 dt \right) \left( \int_{x_j}^{x_j+h} (x_j + h - t)^2 dt \right) \\ & \text{Let } \xi = t - x_{j+1} = t - (x_j + h) = -(x_j + h - t) \\ & \text{then } (x_j + h - t)^2 = \xi^2 \\ & \leq \frac{1}{h^4} \left( \int_{x_j}^{x_{j+1}} |u''(t)|^2 dt \right) \left[ \left( \int_0^h \xi^2 dt \right) \right] = \frac{h^3}{3} \\ & = \frac{1}{3h} \int_{x_j}^{x_{j+1}} |u''(t)|^2 dt \\ & \leq 3 \|f\|_h^2 + 3h \underbrace{\sum_{j=1}^{n-1} \left( \frac{1}{3h} \int_{x_{j-1}}^{x_j} |u''(t)|^2 dt \right)}_{= 3 \|f\|_h^2} + 3h \sum_{j=1}^{n-1} \left( \frac{1}{3h} \int_{x_j}^{x_{j+1}} |u''(t)|^2 dt \right) \\ & = 3 \|f\|_h^2 + 3h \cdot \frac{1}{3h} \int_0^{x_{n-1}} |u''(t)|^2 dt + 3h \cdot \frac{1}{3h} \int_{x_n}^1 |u''(t)|^2 dt \\ & \leq 3 \|f\|_h^2 + \int_0^1 |u''(t)|^2 dt + \int_0^1 |u''(t)|^2 dt \\ & \leq 3 \|f\|_h^2 + 2 \|u''(t)\|_{L^2(0,1)}^2 \leq 3 \|f\|_h^2 + 3 \|u''(t)\|_{L^2(0,1)}^2 \\ & (\because -u''(x_j) = f(x_j)) \end{aligned}$$

Thus  $\|I_n\|_h^2 \leq 3 \|f\|_h^2 + 3 \|f\|_{L^2(0,1)}^2 *$

7. Let  $g=1$  and prove that  $T_h g(x_j) = \frac{1}{2} x_j (1 - x_j)$ .

[ Solution : use the definition (12.25) with  $g(x_k) = 1$ ,  $k = 1, \dots, n-1$

and recall that  $G^k(x_j) = h G(x_j, x_k)$  from the exercise above.

$$\text{Then } T_h g(x_j) = h \left[ \sum_{k=1}^j x_k (1 - x_j) + \sum_{k=j+1}^{n-1} x_j (1 - x_k) \right]$$

from which, after straightforward computations, one gets the desired result.]

Note : (12.25)  $\omega_h = T_h g$ ,  $\omega_h = \sum_{k=1}^{n-1} g(x_k) G^k$

$\Rightarrow$  Consider  $g(x_k) = 1$ ,  $k = 1, \dots, n-1$ , and  $G^k = h G(x_j, x_k)$

$$\text{Since } \omega_h = T_h g = \sum_{k=1}^{n-1} g(x_k) G^k$$

$$\text{then } T_h g = \sum_{k=1}^{n-1} 1 \cdot h G(x_j, x_k)$$

$$\text{Suppose } G(x, s) = \begin{cases} s(1-x) & \text{if } 0 \leq s \leq x \\ x(1-s) & \text{if } x \leq s \leq 1 \end{cases}$$

$$\text{then } G(x_j, x_k) = \begin{cases} x_k(1-x_j) & \text{if } k \leq j \\ x_j(1-x_k) & \text{if } k > j \end{cases}$$

$$\Rightarrow T_h g = h \left[ \sum_{k=1}^j x_k(1-x_j) + \sum_{k=j+1}^{n-1} x_j(1-x_k) \right]$$

$$= h \left[ (1-x_j) \underbrace{\sum_{k=1}^j x_k}_{\text{(since } x_k = k h \text{ and } x_j = j h \text{)}} + x_j \underbrace{\sum_{k=j+1}^{n-1} (1-x_k)}_{\text{)}} \right]$$

$$= h \left[ (1-x_j) \sum_{k=1}^j kh + x_j \sum_{k=j+1}^{n-1} (1-kh) \right]$$

$$= h \left[ (1-x_j) h \frac{j(j+1)}{2} + x_j \left( (n-1-j) - h \sum_{k=j+1}^{n-1} k \right) \right]$$

$$\frac{(n-1)n}{2} - \frac{j(j+1)}{2}$$

$$= h \left[ (1-x_j) h \frac{j(j+1)}{2} + x_j \left( (n-1-j) - h \left( \frac{(n-1)n}{2} - \frac{j(j+1)}{2} \right) \right) \right]$$

$$= \frac{h j (1-hj)}{2} \quad (\because x_j = j h)$$

$$= \frac{1}{2} x_j (1 - x_j)$$

$$\text{Thus } T_h g(x_j) = \frac{1}{2} x_j (1 - x_j) *$$

8. Prove Young's inequality (12.40)

Note: (12.40)  $ab \leq \varepsilon a^2 + \frac{1}{4\varepsilon} b^2 \quad \forall a, b \in \mathbb{R}, \forall \varepsilon > 0.$

$$\Rightarrow \varepsilon a^2 - ab + \frac{1}{4\varepsilon} b^2 \geq 0$$

$$\rightarrow (\sqrt{\varepsilon} a)^2 - 2 \cdot \sqrt{\varepsilon} a \cdot \frac{1}{2\sqrt{\varepsilon}} b + \left(\frac{1}{2\sqrt{\varepsilon}} b\right)^2 \geq 0$$

$$\rightarrow (\sqrt{\varepsilon} a - \frac{1}{2\sqrt{\varepsilon}} b)^2 \geq 0$$

9. Show that  $\|\nabla_h u\|_h \leq \|\nabla_h u\|_{h,\infty} \quad \forall u_h \in V_h$

$\Rightarrow$  Define  $v_j = \nabla_h(x_j) \quad j=1, \dots, n-1$

$$\text{then } \|\nabla_h u\|_h = \left( h \sum_{j=1}^{n-1} v_j^2 \right)^{1/2}, \quad \|\nabla_h u\|_{h,\infty} = \max_{1 \leq j \leq n-1} |v_j|$$

$$\text{Since } |v_j| \leq \|\nabla_h u\|_{h,\infty}^2 \quad \forall j,$$

$$\text{then } \sum_{j=1}^{n-1} v_j^2 \leq (n-1) \|\nabla_h u\|_{h,\infty}^2$$

$$\Rightarrow \|\nabla_h u\|_h = \left( h \sum_{j=1}^{n-1} v_j^2 \right)^{1/2} \leq \left( h(n-1) \right)^{1/2} \|\nabla_h u\|_{h,\infty}$$

$$(\because h = \frac{1}{n}, \text{ then } h(n-1) \leq 1)$$

$$\leq \|\nabla_h u\|_{h,\infty} *$$

11. Discretize the fourth-order differential operator  $L u(x) = -u^{(iv)}(x)$   
using centered finite differences.

[Solution: apply twice the second order centered finite difference  
operator  $L_h$  defined in (12.9).]

Note : (12.9)  $(L_h \omega_h)(x_j) = -\frac{\omega_{j+1} - 2\omega_j + \omega_{j-1}}{h^2}, j=1, \dots, n-1$

$$\Rightarrow \text{Since } (L_h \omega_h)(x_j) = -\frac{\omega_{j+1} - 2\omega_j + \omega_{j-1}}{h^2}, j=1, \dots, n-1$$

$$\text{then } (L_h \omega_h)(x_{j+1}) = -\frac{\omega_{j+2} - 2\omega_{j+1} + \omega_j}{h^2}$$

$$\text{and } (L_h \omega_h)(x_{j-1}) = -\frac{\omega_j - 2\omega_{j-1} + \omega_{j-2}}{h^2}$$

$$\Rightarrow (L_h^2 \omega_h)(x_j) = \frac{(L_h \omega_h)(x_{j+1}) - 2(L_h \omega_h)(x_j) + (L_h \omega_h)(x_{j-1})}{h^4}$$

$$= \frac{\omega_{j+2} - 4\omega_{j+1} + 6\omega_j - 4\omega_{j-1} + \omega_{j-2}}{h^4}$$