FOURTH-ORDER TIME-STEPPING FOR STIFF PDEs∗

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1 Introduction and Problem Statement

In this paper, many time-dependent partial differential equations arising in physics and engineering can be written in the semilinear form

$$u_t = Lu + N(u, t)$$

where L is lineraly and N is nonlinearly. Standard explicit time-stepping methods for such problems are severely restricted by stability constraints, whereas fully implicit methods often incur high computational cost due to the repeated solution of large nonlinear systems.

The main problem addressed in this paper is the construction of efficient and accurate time integration schemes for semilinear evolution equations that overcome stiffness induced by the linear operator without sacrificing accuracy in the nonlinear dynamics.
To this end, the paper adopts the integrating factor (IF) framework, in which the linear part of the equation is treated exactly through matrix exponentials, leading to an exact variation-of-constants formula.
Based on this formulation, the authors develop exponential time differencing (ETD) schemes by approximating the remaining nonlinear integral using Taylor and Runge–Kutta–type quadratures. These methods retain the stability advantages of implicit schemes for stiff linear terms while maintaining the simplicity and efficiency of explicit methods for the nonlinear components. The paper further analyzes the accuracy and performance of the proposed ETD methods and demonstrates their effectiveness on representative stiff semilinear problems.

2 Method Explanation

This paper mentions a modified ETD method, which is similar to the IF method.
From the original assumptions in the paper $u_t = Lu + N(u, t)$, we multiply it by the integrating factor e^{-Lt}.

$$\Rightarrow e^{-Lt}u_t = e^{-Lt}Lu + e^{-Lt}N(u, t)$$

$$\rightarrow e^{-Lt}u_t - e^{-Lt}Lu = e^{-Lt}N(u, t)$$

Since $\frac{\mathrm{d}}{\mathrm{d}t}(e^{-Lt}u) = e^{-Lt}u_t - e^{-Lt}Lu$, then $\frac{\mathrm{d}}{\mathrm{d}t}(e^{-Lt}u) = e^{-Lt}N(u,t)$.

(Integrating time part)

$$\Rightarrow \int_{t_n}^{t_n+h} \frac{\mathrm{d}}{\mathrm{d}t} e^{-Lt}u_t \mathrm{d}t = \int_{t_n}^{t_n+h} e^{-Lt}N(u,t)\mathrm{d}t$$

$$\to e^{-L(t_n+h)}u(t_n+h) - e^{-Lt_n}u(t_n) = \int_{t_n}^{t_n+h} \frac{\mathrm{d}}{\mathrm{d}t} e^{-Lt}N(u,t)\mathrm{d}t$$

$$\Rightarrow u_{n+1} = e^{Lh}u_n + \int_{t_n}^{t_n+h} e^{-Lt}N(u,t)\mathrm{d}t$$

Let $t = t_n + \tau$.

$$\Rightarrow u_{n+1} = e^{Lh}u_n + e^{Lh}\int_0^h e^{-L\tau}N(u(t_n+\tau), t_n+\tau)\mathrm{d}\tau$$

Consider

$$u_{n+1} = e^{Lh}u_n + e^{Lh}\int_0^h e^{-L\tau}N(u(t_n+\tau), t_n+\tau)\mathrm{d}\tau$$

(Take it to become a general form)

$$\Rightarrow u_{n+1} = e^{Lh}u_n + h\sum_{m=0}^{s-1} g_m \sum_{k=0}^{m} (-1)^k \binom{m}{k} N_{n-k}$$

where $(Lhg_0 = e^{Lh} - I)$

$$Lhg_{m+1} + I = g_m + \frac{1}{2}g_{m-1} + \frac{1}{3}g_{m-2} + \cdots + \frac{g_0}{m+1},\ m \geq 0$$

Due to some issues with the ETDRK4 formula, this article has made some changes.
Let $g(z) = \frac{e^z-1}{z}$, which is approximated using Taylor polynomials.
But it is still imperfect, and the truncation errors still exist.
Therefore, it take the new coefficients

$$\begin{cases} \alpha &= h^{-2}L^{-3}[-4 - Lh + e^{Lh}(4 - 3Lh + (Lh^2))] \\ \beta &= h^{-2}L^{-3}[2 + Lh + e^{Lh}(-2 + Lh)] \\ \gamma &= h^{-2}L^{-3}[-4 - 3Lh - (Lh^2) + e^{Lh}(4 - Lh)] \end{cases}$$

Regarding accuracy issues, the answer is obtained by solving f(z), where Γ is on the complex plane.

$$f(z) = \frac{1}{2\pi i}\int_\Gamma \frac{f(t)}{(t-z)}\mathrm{d}t$$

If L is the matrix

$$\Rightarrow f(L) = \frac{1}{2\pi i}\int_{\Gamma} f(t)(tI - L)^{-1}\mathrm{d}t$$

Consider Γ as a circle, typically 32 or 64 equidistant points are sufficient.

3 Experiment

Under the condition of fixing everything and only changing the nonlinear integral approximation method, we attempted to compare error calculations using Euler, linear Taylor, and ETDRK4.
*：Please refer to the "ETDRK4_test" file for the code section.

Figure 1: The errror

4 Conclusion

This work shows that exponential time differencing methods based on integrating factors offer an effective balance between stability and efficiency for stiff semilinear problems. By treating the linear stiffness exactly and approximating the nonlinear effects with high-order quadrature, ETD schemes achieve accurate and stable time integration without the cost of fully implicit methods.