作業三手寫

1. Let $E_0(f)$ and $E_1(f)$ be the quadrature errors in (9.6) and (9.12).
Prove that $|E_1(f)| \simeq 2|E_0(f)|$

Note : (9.6) $E_0(f) = \frac{h^3}{3} f''(\xi)$, $h = \frac{b-a}{2}$

(9.12) $E_1(f) = -\frac{h^3}{12} f''(\xi)$, $h = b-a$

$\Rightarrow$ Since $E_0(f) = \left(\frac{b-a}{2}\right)^3 \cdot \frac{1}{3} f''(\xi) = (b-a)^3 \cdot \frac{1}{24} f''(\xi)$

and $E_1(f) = -(b-a)^3 \cdot \frac{1}{12} f''(\xi)$

then $|E_1(f)| = \left| \frac{-(b-a)^3}{12} f''(\xi) \right| = \left| \frac{(b-a)^3}{12} f''(\xi) \right|$

and $|E_0(f)| = \left| \frac{(b-a)^3}{24} f''(\xi) \right|$

Thus $|E_1(f)| \simeq 2|E_0(f)|$ ※

3. Let $I_n(f) = \sum_{k=0}^{n} \alpha_k f(x_k)$ be a Lagrange quadrature formula on $n+1$ nodes.
Compute the degree of exactness r of the formulae:

(a) $I_2(f) = (2/3)[2f(-1/2) - f(0) + 2f(1/2)]$,

(b) $I_4(f) = (1/4)[f(-1) + 3f(-1/3) + 3f(1/3) + f(1)]$.

Which is the order of infinitesimal p for (a) and (b)?

[Solution : $r = 3$ and $p = 5$ for both $I_2(f)$ and $I_4(f)$.]

Consider $f \in [a, b]$ where a, b 分別等於 $1, -1$

$\Rightarrow$ 習 : it only need to check the basis of polynomails

e.g : $1, x, x^2, \ldots,$ e.t.c

(a 的場合) $I_2(f)=\frac{2}{3}\left[2f(-\frac{1}{2})-f(0)+2f(\frac{1}{2})\right]$

We have $\begin{cases} x_0=-\frac{1}{2}, & w_0=2 \\ x_1=0, & w_1=-1 \\ x_2=\frac{1}{2}, & w_2=2 \end{cases}$ and normalized by $\frac{2}{3}$

① $f(x)=1$

$\Rightarrow I_2(f)=\frac{2}{3}\cdot[2\cdot1-1+2\cdot1]=\frac{2}{3}\cdot3=2$

Since $\int_{-1}^{1}1\,dx=1-(-1)=2$

then it is correct ※

② $f(x)=x$

$\Rightarrow I_2(f)=\frac{2}{3}\left[2\cdot(-\frac{1}{2})-0+2\cdot\frac{1}{2}\right]=0$

Since $\int_{-1}^{1}x\,dx=\frac{1}{2}\cdot1-\frac{1}{2}\cdot1=0$

then it is correct ※

③ $f(x)=x^2$

$\Rightarrow I_2(f)=\frac{2}{3}\left[2\cdot\frac{1}{4}-0+2\cdot\frac{1}{4}\right]=\frac{2}{3}$

Since $\int_{-1}^{1}x^2\,dx=\frac{1}{3}\cdot1-\frac{1}{3}(-1)=\frac{2}{3}$

then it is correct ※

④ $f(x)=x^3$

$\Rightarrow I_2(f)=\frac{2}{3}\left[2\cdot(-\frac{1}{8})-0+2\cdot\frac{1}{8}\right]=0$

Since $\int_{-1}^{1}x^3\,dx=\frac{1}{4}\cdot1-\frac{1}{4}\cdot1=0$

then it is correct ※

⑤ $f(x)=x^4$

$\Rightarrow I_2(f)=\frac{2}{3}\left[2\cdot\frac{1}{16}+0+2\cdot\frac{1}{16}\right]=\frac{1}{6}$

but $\int_{-1}^{1}x^4\,dx=\frac{1}{5}\cdot1-\frac{1}{5}(-1)=\frac{2}{5}$

So it is not correct. ※

Thus the degree of exactness of $I_2(f)$ is 3

also the error is $O(h^5)$

(b 的場合) $I_4(f)=\frac{1}{4}\left[f(-1)+3f(-\frac{1}{3})+3f(\frac{1}{3})+f(1)\right]$

We have $\begin{cases} x_0=-1, & w_0=1 \\ x_1=-\frac{1}{3}, & w_1=3 \\ x_2=\frac{1}{3}, & w_2=3 \\ x_3=1, & w_3=1 \end{cases}$ and normalized by $\frac{1}{4}$

① $f(x)=1$

$\Rightarrow I_4(f)=\frac{1}{4}[1+3+3+1]=\frac{1}{4}\cdot8=2$

Since $\int_{-1}^{1}1\,dx=1-(-1)=2$

then it is correct. ※

② $f(x) = x$

$\Rightarrow I_4(f) = \frac{1}{4}[-1 + 3(-\frac{1}{3}) + 3(\frac{1}{3}) + 1] = 0$

Since $\int_{-1}^{1} x\,dx = \frac{1}{2}\cdot 1 - \frac{1}{2}\cdot 1 = 0$

then it is correct ✱

③ $f(x) = x^2$

$\Rightarrow I_4(f) = \frac{1}{4}[1 + 3(\frac{1}{9}) + 3(\frac{1}{9}) + 1] = \frac{1}{4}\cdot\frac{8}{3} = \frac{2}{3}$

Since $\int_{-1}^{1} x^2\,dx = \frac{1}{3}\cdot 1 - \frac{1}{3}(-1) = \frac{2}{3}$

then it is correct. ✱

④ $f(x) = x^3$

$\Rightarrow I_4(f) = \frac{1}{4}[-1 + 3(-\frac{1}{27}) + 3(\frac{1}{27}) + 1] = 0$

Since $\int_{-1}^{1} x^3\,dx = \frac{1}{4}\cdot 1 - \frac{1}{4}\cdot 1 = 0$

then it is correct. ✱

⑤ $f(x) = x^4$

$\Rightarrow I_4(f) = \frac{1}{4}[1 + 3(\frac{1}{81}) + 3(\frac{1}{81}) + 1] = \frac{1}{4}\cdot\frac{56}{27} = \frac{14}{27}$

but $\int_{-1}^{1} x^4\,dx = \frac{1}{5}\cdot 1 - \frac{1}{5}(-1) = \frac{2}{5}$

So it is not correct. ✱

Thus the degree of exactness of $I_2(f)$ is 3

also the error is $O(h^5)$

5. Let $I_w(f) = \int_0^1 w(x) f(x)\,dx$ with $w(x) = \sqrt{x}$, and consider the quadrature formula $Q(f) = a f(x_1)$.

Find a and x_1 in such a way that Q has maximum degree of exactness r.

[Solution : $a = 2/3$, $x_1 = 3/5$ and $r = 1$.]

寫：直接測試！

$\Rightarrow$ Consider $I_w(f) = \int_0^1 w(x) f(x)\,dx$ where $w(x) = \sqrt{x}$, and $Q(f) = a f(x_1)$

① $f(x) = 1$

Since $Q(f) = a \cdot 1 = a$

$\Rightarrow I_w(f) = \int_0^1 \sqrt{x}\cdot 1\,dx = \frac{2}{3} x^{\frac{3}{2}} \Big|_0^1 = \frac{2}{3}$

Thus $a = 2/3$ ✱

② $f(x) = x$

Since $Q(f) = a \cdot x_1 = \frac{2}{3} x_1$

$\Rightarrow I_w(f) = \int_0^1 \sqrt{x} \cdot x \, dx = \frac{2}{5} x^{\frac{5}{2}} \Big|_0^1 = \frac{2}{5}$

Set $\frac{2}{3} x_1 = \frac{2}{5}$ (if "=" holds)

then $x_1 = 3/5$ #

③ $f(x) = x^2$

Since $Q(f) = a \cdot x_1^2 = \frac{2}{3} \cdot \left(\frac{3}{5}\right)^2 = \frac{6}{25}$

but $I_w(f) = \int_0^1 \sqrt{x} \cdot x^2 \, dx = \frac{2}{7} x^{\frac{7}{2}} \Big|_0^1 = \frac{2}{7}$

therefore it is not exactness to 2

Thus $r = 1$. #

6. Let us consider the quadrature formula $Q(f) = \alpha_1 f(0) + \alpha_2 f(1) + \alpha_3 f'(0)$ for the approximation on $I(f) = \int_0^1 f(x)\,dx$, where $f \in C^1([0,1])$. Determine the coefficients α_j, for $j = 1, 2, 3$ in such a way that Q has degree of exactness $r = 2$.

[Solution: $\alpha_1 = 2/3$, $\alpha_2 = 1/3$ and $\alpha_3 = 1/6$.]