作業五手寫

1. Prove that Heun's method has order 2 with respect to h.
[*Hint*: notice that $h\tau_{n+1} = y_{n+1} - y_n - h\Phi(t_n, y_n; h) = E_1 + E_2$, where

$$E_1 = \int_{t_n}^{t_{n+1}} f(s, y(s))ds - \frac{h}{2}[f(t_n, y_n) + f(t_{n+1}, y_{n+1})]$$

and

$$E_2 = \frac{h}{2}\{[f(t_{n+1}, y_{n+1}) - f(t_{n+1}, y_n + hf(t_n, y_n))]\},$$

where E_1 is the error due to numerical integration with the trapezoidal method and E_2 can be bounded by the error due to using the forward Euler method.]

Note : Heun's method

$$u_{n+1} = u_n + \frac{h}{2}[f_n + f(t_{n+1}, u_n + hf_n)].$$

$\Rightarrow$ Consider $h\tau_{n+1} = y_{n+1} - y_n - h\Phi(t_n, y_n ; h) = E_1 + E_2$

$$= \int_{t_n}^{t_{n+1}} f(s, y(s))ds - \frac{h}{2}[f(t_n, y_n) + f(t_{n+1}, y_{n+1})]$$

$$+ \frac{h}{2}\{[f(t_{n+1}, y_{n+1}) - f(t_{n+1}, y_n + hf(t_n, y_n))]\}$$

寫 : 想法

① $E_1 = -\frac{h^3}{12} f_{tt}(\xi, y(\xi)) = O(h^3)$

$\Rightarrow |E_1| \leq C_1 h^3$

② Assume f for y have Lipschitz L,

then $|E_2| \leq \frac{h}{2} L \underbrace{|y_{n+1} - (y_n + hf(t_n, y_n))|}_{= O(h^2)}$

$\Rightarrow |E_2| = O(h) \times O(h^2)$

Since $E_1 = O(h^3)$ and $E_2 = O(h^3)$

then $\tau_{n+1} = E_1 + E_2 = O(h^3)$

$\Rightarrow O(h^{p+1}) \rightarrow p = 2$

2. Prove that the Crank-Nicoloson method has order 2 with respect to h.
[*Solution*: using (9.12) we get, for a suitable ξ_n in (t_n, t_{n+1})

$$y_{n+1} = y_n + \frac{h}{2}[f(t_n, y_n) + f(t_{n+1}, y_{n+1})] - \frac{h^3}{12} f''(\xi_n, y(\xi_n))$$

or, equivalently,

$$\frac{y_{n+1} - y_n}{h} = \frac{1}{2}[f(t_n, y_n) + f(t_{n+1}, y_{n+1})] - \frac{h^2}{12} f''(\xi_n, y(\xi_n)). \qquad (11.90)$$

Therefore, relation (11.9) coincides with (11.90) up to an infinitesimal of order 2 with respect to h, provided that $f \in C^2(I)$.]

Note : $E_1(f) = -\frac{h^3}{12} f''(\xi)$, $h = b - a$ (9.12)

$\Rightarrow$ Consider