# 作業六

7. Prove that the gamma function

$$\Gamma(z) = \int_0^\infty e^{-t}\, t^{z-1}\, dt \ , \ z \in \mathbb{C} \ , \ \operatorname{Re} z > 0$$

is the solution of the difference equation $\Gamma(z+1) = z\,\Gamma(z)$

[Hint : integrate by parts.]

$\Rightarrow$ Consider $\Gamma(z) = \int_0^\infty e^{-t}\, t^{z-1}\, dt$

then $\Gamma(z+1) = \int_0^\infty e^{-t}\, t^{(z+1-1)}\, dt = \int_0^\infty \underbrace{e^{-t}\, \overset{u}{t^{z}}\, dt}_{dv}$

By integrate by parts $\left(\int u\,dv = uv - \int v\,du\right)$

Set $u = t^z$ , $dv = e^{-t}dt$

$du = z\,t^{z-1}dt$ , $v = -e^{-t}$

$$= t^z(-e^{-t}) - \int_0^\infty (-e^{-t})\, z\, t^{z-1}\, dt$$

$$= \boxed{-t^z e^{-t}\Big|_0^\infty} + \int_0^\infty e^{-t}\, z\, t^{z-1}\, dt$$

$\downarrow$

取極限

since $\operatorname{Re} z > 0$

$$\Rightarrow \lim_{t\to\infty} (-t^z e^{-t}) = \lim_{t\to\infty} \frac{-t^{z}}{e^{t}} \overset{L'H}{=} \lim_{t\to\infty} \frac{-z\,t^{z-1}}{e^{t}} = \frac{1}{\infty} = 0$$

$$-0^z e^{-0} = 0$$

$$= \int_0^\infty e^{-t}\, z\, t^{z-1}\, dt = z\int_0^\infty e^{-t}\, t^{z-1}\, dt = z\,\Gamma(z)$$

Thus $\Gamma(z+1) = z\,\Gamma(z)$ ※

9. Consider the following family of one-step methods deepending on the real parameter $\alpha$

$$u_{n+1} = u_n + h\left[ \left(1 - \frac{\alpha}{2}\right) f(x_n, u_n) + \frac{\alpha}{2} f(x_{n+1}, u_{n+1}) \right].$$

Study their consistency as a function of $\alpha$ ; then.

take $\alpha = 1$ and use the corresponding method to solve the Cauchy problem

$$\begin{cases} y'(x) = -10\,y(x) & , \ x > 0 \\ y(0) = 1 \end{cases}$$

Determine the values of $h$ in correspondance of which the method is absolutely stable.

[Solution : the family of methods is concistent for any value of $\alpha$.

The method of highest order (equal to two) is obtained

for $\alpha = 1$ and coincides with the Crank - Nicolson method. ]