作業八

5. Prove the estimate (12.23).

[Hint: for each internal node x_j, $j=1,\dots,n-1$, integrate by parts (12.21) to get

$$\tau_h(x_j) = -u''(x_j) - \frac{1}{h^2}\left[\int_{x_j-h}^{x_j} u''(t)(x_j-h-t)\,dt - \int_{x_j}^{x_j+h} u''(t)(x_j+h-t)\,dt\right]$$

Then, pass to the squares and $\tau_h(x_j)^2$ for $j=1,\dots,n-1$.

On nothing that $(a+b+c)^2 \le 3(a^2+b^2+c^2)$, for any real numbers a, b, c, and applying the Cauchy-Schwarz inequality yields the desired result.]

Note: (12.23) $\|\tau_h\|_h^2 \le 3\left(\|f\|_h^2 + \|f\|_{L^2(0,1)}^2\right)$

(12.21) $\tau_h(x_j) = \frac{1}{h^2}\left(R_4(x_j+h) + R_4(x_j-h)\right)$

where $R_4(x_j+h) = \int_{x_j}^{x_j+h} (u'''(t) - u'''(x_j)) \frac{(x_j+h-t)^2}{2}\,dt$

and $R_4(x_j-h) = -\int_{x_j-h}^{x_j} (u'''(t) - u'''(x_j)) \frac{(x_j-h-t)^2}{2}\,dt$

Remark 12.3: Let $e = u - u_h$ be the discretization error grid function.
Then $L_h e = L_h u - L_h u_h = L_h u - f_h = \tau_h$

$\Rightarrow$ Let
$$\tau_h(x_j) = \frac{1}{h^2}\Bigg[\underbrace{\int_{x_j}^{x_j+h} (u'''(t) - u'''(x_j)) \frac{(x_j+h-t)^2}{2}\,dt}_{①} \underbrace{- \int_{x_j-h}^{x_j} (u'''(t) - u'''(x_j)) \frac{(x_j-h-t)^2}{2}\,dt}_{②}\Bigg]$$

By integer by parts,

① Let $v = \frac{(x_j+h-t)^2}{2}$, $dg = u'''(t)\,dt$

$dv = -(x_j+h-t)\,dt$, $g = u''(t)$

$$\Rightarrow \int v\,dg = gv - \int g\,dv = u''(t)\frac{(x_j-h-t)^2}{2}\Big]_{x_j}^{x_j+h} + \int_{x_j}^{x_j+h} u''(t)(x_j+h-t)\,dt$$

$$= -\frac{h^2}{2}u''(x_j) + \int_{x_j}^{x_j+h} u''(t)(x_j+h-t)\,dt$$

and

$$\int_{x_j}^{x_j+h} u'''(x_j)\frac{(x_j+h-t)^2}{2}\,dt = u'''(x_j)\frac{-1}{6}(x_j+h-t)^3\Big]_{x_j}^{x_j+h} = \frac{-h^3}{6}u'''(x_j)$$

② Let $v = \frac{(x_j-h-t)^2}{2}$, $dg = u'''(t)\,dt$

$dv = -(x_j-h-t)\,dt$, $g = u''(t)$

$$\Rightarrow -\int v\,dg = -gv + \int g\,dv = -u''(t)\frac{(x_j-h-t)^2}{2}\Big]_{x_j-h}^{x_j} - \int_{x_j-h}^{x_j} u''(t)(x_j-h-t)\,dt$$

$$= -u''(x_j)\cdot\frac{h^2}{2} - \int_{x_j-h}^{x_j} u''(t)(x_j-h-t)\,dt$$

and $\displaystyle\int_{x_j-h}^{x_j} u'''(x_j)\frac{(x_j-h-t)^2}{2}\,dt = -u'''(x_j)\frac{(x_j-h-t)^3}{6}\Bigg]_{x_j-h}^{x_j} = -\frac{h^3}{6}u'''(x_j)$

$$\Rightarrow \tau_h(x_j) = \frac{1}{h^2}\Big[-\frac{h^2}{2}u''(x_j) + \int_{x_j}^{x_j+h} u''(t)(x_j+h-t)\,dt + \frac{h^3}{6}u'''(x_j)$$

$$-\frac{h^2}{2}u''(x_j) - \int_{x_j-h}^{x_j} u''(t)(x_j-h-t)\,dt - \frac{h^3}{6}u'''(x_j)\Big]$$

then $\displaystyle \tau_h(x_j) = -u''(x_j) - \frac{1}{h^2}\Big[\int_{x_j-h}^{x_j} u''(t)(x_j-h-t)\,dt$

$$-\int_{x_j}^{x_j+h} u''(t)(x_j+h-t)\,dt\Big]$$

Since $(a+b+c)^2 \le 3(a^2+b^2+c^2)$

$$\Rightarrow [\tau_h(x_j)]^2 \le 3\Big[(-u''(x_j))^2 + \Big(-\frac{1}{h^2}\int_{x_j-h}^{x_j} u''(t)(x_j-h-t)\,dt\Big)^2$$

$$+\Big(\frac{1}{h^2}\int_{x_j}^{x_j+h} u''(t)(x_j+h-t)\,dt\Big)^2\Big]$$

$$\rightarrow \|\tau_h\|_h^2 = h\sum_{j=1}^{n-1}\tau_h(x_j)^2$$

$$\le \underline{3h\sum_{j=1}^{n-1}(-u''(x_j))^2} + 3h\sum_{j=1}^{n-1}\underline{\Big(-\frac{1}{h^2}\int_{x_j-h}^{x_j} u''(t)(x_j-h-t)\,dt\Big)^2} \rightarrow ①$$

$$+3h\sum_{j=1}^{n-1}\underline{\Big(\frac{1}{h^2}\int_{x_j}^{x_j+h} u''(t)(x_j+h-t)\,dt\Big)^2} \rightarrow ②$$

$$= 3\|f\|_h^2 \quad (\because -u''(x_j) = f(x_j))$$

① By Cauchy - Schwarz inequalized,

$$\Rightarrow \left[-\frac{1}{h^2} \int_{x_j - h}^{x_j} u''(t)\,(x_j - h - t)\, dt \right]^2$$

$$\leq \frac{1}{h^4} \left(\int_{x_j-h}^{x_j} |u''(t)|^2 dt \right) \left(\int_{x_j-h}^{x_j} (x_j - h - t)^2 dt \right)$$

Let $\xi = t - x_{j-1} = t - (x_j - h) = -(x_j - h - t)$

then $(x_j - h - t)^2 = \xi^2$

$$\leq \frac{1}{h^4} \left(\int_{x_{j-1}}^{x_j} |u''(t)|^2 dt \right) \left(\int_0^h \xi^2 dt \right)$$

$$= \frac{1}{h^4} \left(\int_{x_{j-1}}^{x_j} |u''(t)|^2 dt \right) \cdot \left[\frac{\xi^3}{3} \Big|_0^h \right] = \frac{h^3}{3}$$

$$= \frac{1}{3h} \int_{x_{j-1}}^{x_j} |u''(t)|^2 dt$$

② By Cauchy - Schwarz inequalized,

$$\Rightarrow \left[\frac{1}{h^2} \int_{x_j}^{x_j + h} u''(t)\,(x_j + h - t)\, dt \right]^2$$

$$\leq \frac{1}{h^4} \left(\int_{x_j}^{x_j+h} |u''(t)|^2 dt \right) \left(\int_{x_j}^{x_j+h} (x_j + h - t)^2 dt \right)$$

Let $\xi = t - x_{j+1} = t - (x_j + h) = -(x_j + h - t)$

then $(x_j + h - t)^2 = \xi^2$

$$\leq \frac{1}{h^4} \left(\int_{x_j}^{x_{j+1}} |u''(t)|^2 dt \right) \left[\left(\int_0^h \xi^2 dt \right) \right] = \frac{h^3}{3}$$

$$= \frac{1}{3h} \int_{x_j}^{x_{j+1}} |u''(t)|^2 dt$$

$$\leq 3\|f\|_h^2 + 3h \underline{\sum_{j=1}^{n-1} \left(\frac{1}{3h} \int_{x_{j-1}}^{x_j} |u''(t)|^2 dt \right)} + 3h \underline{\sum_{j=1}^{n-1} \left(\frac{1}{3h} \int_{x_j}^{x_{j+1}} |u''(t)|^2 dt \right)}$$

$$= 3\|f\|_h^2 + 3h \cdot \frac{1}{3h} \int_0^{x_{n-1}} |u''(t)|^2 dt + 3h \cdot \frac{1}{3h} \int_{x_1}^1 |u''(t)|^2 dt$$

$$\leq 3\|f\|_h^2 + \int_0^1 |u''(t)|^2 dt + \int_0^1 |u''(t)|^2 dt$$

$$\leq 3\|f\|_h^2 + 2\|u''(t)\|_{L^2(0,1)}^2 \leq 3\|f\|_h^2 + 3\|u''(t)\|_{L^2(0,1)}^2$$

$(\because -u''(x_j) = f(x_j))$

Thus $\|I_h\|_h^2 \leq 3\|f\|_h^2 + 3\|f\|_{L^2(0,1)}^2$ ※

7. Let $g=1$ and prove that $T_h g(x_j) = \frac{1}{2} x_j (1-x_j)$.

[Solution : use the definition (12.25) with $g(x_k) = 1$, $k=1, \dots, n-1$

and recall that $G^k(x_j) = h\, G(x_j, x_k)$ from the exercise above.

Then $T_h g(x_i) = h\left[\sum_{k=1}^{j} x_k (1-x_j) + \sum_{k=j+1}^{n-1} x_j (1-x_k) \right]$

from which, after straightforward computations, one gets the desired result.]

$$\boxed{\text{Note : } (12.25) \quad \omega_h = T_h g, \ \omega_h = \sum_{k=1}^{n-1} g(x_k) G^k}$$

$\Rightarrow$ Consider $g(x_k) = 1$, $k=1, \dots, n-1$, and $G^k = h\, G(x_j, x_k)$

Since $\omega_h = T_h g = \sum_{k=1}^{n-1} g(x_k) G^k$

then $T_h g = \sum_{k=1}^{n-1} 1 \cdot h\, G(x_j, x_k)$

Suppose $G(x,s) = \begin{cases} s(1-x) & \text{if } 0 \le s \le x \\ x(1-s) & \text{if } x \le s \le 1 \end{cases}$

then $G(x_j, x_k) = \begin{cases} x_k(1-x_j) & \text{if } k \le j \\ x_j(1-x_k) & \text{if } k > j \end{cases}$

$$\Rightarrow T_h g = h\left[\sum_{k=1}^{j} x_k (1-x_j) + \sum_{k=j+1}^{n-1} x_j (1-x_k) \right]$$

$$= h\left[(1-x_j) \underline{\sum_{k=1}^{j} x_k} + x_j \underline{\sum_{k=j+1}^{n-1} (1-x_k)} \right]$$

(since $x_k = kh$ and $x_j = jh$)

$$= h\left[(1-x_j) \sum_{k=1}^{j} kh + x_j \sum_{k=j+1}^{n-1} (1-kh) \right]$$

$$= h\left[(1-x_j)\, h \frac{j(j+1)}{2} + x_j \left((n-1-j) - h \underbrace{\sum_{k=j+1}^{n-1} k}_{= \frac{(n-1)n}{2} - \frac{j(j+1)}{2}} \right) \right]$$

$$= h\left[(1-x_j)\, h \frac{j(j+1)}{2} + x_j \left((n-1-j) - h\left(\frac{(n-1)n}{2} - \frac{j(j+1)}{2} \right) \right) \right]$$

$$= \frac{hj(1-hj)}{2} \qquad (\because x_j = jh)$$

$$= \frac{1}{2} x_j (1-x_j)$$

Thus $T_h g(x_j) = \frac{1}{2} x_j (1-x_j)$ ※

8. Prove Young's inequality (12.40)

Note : $\boxed{(12.40) \quad ab \le \varepsilon a^2 + \frac{1}{4\varepsilon} b^2 \quad \forall a,b \in \mathbb{R},\ \forall \varepsilon > 0.}$

$$\Rightarrow \quad \varepsilon a^2 - ab + \frac{1}{4\varepsilon} b^2 \ge 0$$

$$\rightarrow (\sqrt{\varepsilon}\, a)^2 - 2 \cdot \sqrt{\varepsilon}\, a \cdot \frac{1}{2\sqrt{\varepsilon}} b + \left(\frac{1}{2\sqrt{\varepsilon}} b\right)^2 \ge 0$$

$$\rightarrow \left(\sqrt{\varepsilon}\, a - \frac{1}{2\sqrt{\varepsilon}} b\right)^2 \ge 0$$

9. Show that $\|v_h\|_h \le \|v_h\|_{h,\infty} \quad \forall v_h \in V_h$

$\Rightarrow$ Define $v_j = v_h(x_j) \quad j = 1, \cdots, n-1$

then $\|v_h\|_h = \left(h \sum_{j=1}^{n-1} v_j^2 \right)^{1/2}$, $\|v_h\|_{h,\infty} = \max_{1 \le j \le n-1} |v_j|$

Since $|v_j| \le \|v_h\|_{h,\infty}^2 \quad \forall j$,

then $\sum_{j=1}^{n-1} v_j^2 \le (n-1) \|v_h\|_\infty^2$

$$\Rightarrow \|v_h\|_h = \left(h \sum_{j=1}^{n-1} v_j^2 \right)^{1/2} \le (h(n-1))^{1/2} \|v_h\|_{h,\infty}$$

$(\because h = \frac{1}{n}$, then $h(n-1) \le 1)$

$$\le \|v_h\|_{h,\infty} \quad \#$$

11. Discretize the fourth-order differential operator $Lu(x) = -u^{(iv)}(x)$ using centered finite differences.

[Solution : apply twice the second order centered finite difference operator L_h defined in (12.9).]

Note : $\boxed{(12.9) \quad (L_h w_h)(x_j) = -\dfrac{w_{j+1} - 2w_j + w_{j-1}}{h^2}, \ j = 1, \cdots, n-1}$