作業八

5. Prove the estimate (12.23).

[Hint : for each internal node x_j, $j = 1, \dots, n-1$, integrate by parts (12.21) to get

$$\tau_h(x_j) = -u''(x_j) - \frac{1}{h^2}\left[\int_{x_{j-h}}^{x_j} u''(t)(x_j - h - t)^2\,dt - \int_{x_j}^{x_{j+h}} u''(t)(x_j + h - t)^2\,dt\right]$$

Then, pass to the squares and $\tau_h(x_j)^2$ for $j = 1, \dots, n-1$.

On nothing that $(a+b+c)^2 \le 3(a^2+b^2+c^2)$, for any real numbers a, b, c, and applying the Cauchy-Schwarz inequality yields the desired result.]

7. Let $g = 1$ and prove that $T_h g(x_j) = \frac{1}{2} x_j(1-x_j)$.

[Solution : use the definition (12.25) with $g(x_k) = 1$, $k = 1, \dots, n-1$ and recall that $G^k(x_j) = h\,G(x_j, x_k)$ from the exercise above.

Then $$T_h g(x_j) = h\left[\sum_{k=1}^{j} x_k(1-x_j) + \sum_{k=j+1}^{n-1} x_j(1-x_k)\right]$$

from which, after straightforward computations, one gets the desired result.]

8. Prove Young's inequality (12.40)

9. Show that $\|v_h\|_h \le \|v_h\|_{h,\infty} \quad \forall v_h \in V_h$

11. Discretize the fourth-order differential operator $Lu(x) = -u^{(iv)}(x)$ using centered finite differences.

[Solution : apply twice the second order centered finite difference operator L_h defined in (12.9).]