

作業三手寫

1. Let $E_0(f)$ and $E_1(f)$ be the quadrature errors in (9.6) and (9.12).

Prove that $|E_1(f)| \simeq 2 |E_0(f)|$

Note:

$$(9.6) \quad E_0(f) = \frac{h^3}{3} f''(\xi), \quad h = \frac{b-a}{2}$$
$$(9.12) \quad E_1(f) = -\frac{h^3}{12} f''(\xi), \quad h = b-a$$

$$\Rightarrow \text{Since } E_0(f) = \left(\frac{b-a}{2}\right)^3 \cdot \frac{1}{3} f''(\xi) = (b-a)^3 \cdot \frac{1}{24} f''(\xi)$$

$$\text{and } E_1(f) = -(b-a)^3 \cdot \frac{1}{12} f''(\xi)$$

$$\text{then } |E_1(f)| = \left| -\frac{(b-a)^3}{12} f''(\xi) \right| = \left| \frac{(b-a)^3}{12} f''(\xi) \right|$$

$$\text{and } |E_0(f)| = \left| \frac{(b-a)^3}{24} f''(\xi) \right|$$

$$\text{Thus } |E_1(f)| \simeq 2 |E_0(f)| \quad *$$

3. Let $I_n(f) = \sum_{k=0}^n \alpha_k f(x_k)$ be a Lagrange quadrature formula on $n+1$ nodes.

Compute the degree of exactness r of the formulae:

$$(a) \quad I_2(f) = \left(\frac{2}{3}\right) [2f(-\frac{1}{2}) - f(0) + 2f(\frac{1}{2})],$$

$$(b) \quad I_4(f) = \left(\frac{1}{4}\right) [f(-1) + 3f(-\frac{1}{3}) + 3f(\frac{1}{3}) + f(1)].$$

Which is the order of infinitesimal p for (a) and (b)?

[Solution: $r=3$ and $p=5$ for both $I_2(f)$ and $I_4(f)$.]

⇒ \textcircled{Q} Consider $f \in [a, b]$ where a, b 分別 等於 $1, -1$
it only need to check the basis of polynomials
e.g.: $1, x, x^2, \dots$ e.t.c

(a 的情况) $I_2(f) = \frac{2}{3} [2f(-\frac{1}{2}) - f(0) + 2f(\frac{1}{2})]$

We have $\begin{cases} x_0 = -\frac{1}{2}, & w_0 = 2 \\ x_1 = 0, & w_1 = -1 \\ x_2 = \frac{1}{2}, & w_2 = 2 \end{cases}$ and normalized by $\frac{2}{3}$

① $f(x) = 1$

$\Rightarrow I_2(f) = \frac{2}{3} \cdot [2 \cdot 1 - 1 + 2 \cdot 1] = \frac{2}{3} \cdot 3 = 2$

Since $\int_{-1}^1 1 dx = 1 - (-1) = 2$

then it is correct *

② $f(x) = x$

$\Rightarrow I_2(f) = \frac{2}{3} [2 \cdot (-\frac{1}{2}) - 0 + 2 \cdot \frac{1}{2}] = 0$

Since $\int_{-1}^1 x dx = \frac{1}{2} \cdot 1 - \frac{1}{2} \cdot (-1) = 0$

then it is correct *

③ $f(x) = x^2$

$\Rightarrow I_2(f) = \frac{2}{3} [2 \cdot \frac{1}{4} - 0 + 2 \cdot \frac{1}{4}] = \frac{2}{3}$

Since $\int_{-1}^1 x^2 dx = \frac{1}{3} \cdot 1 - \frac{1}{3} \cdot (-1) = \frac{2}{3}$

then it is correct *

④ $f(x) = x^3$

$\Rightarrow I_2(f) = \frac{2}{3} [2 \cdot (-\frac{1}{8}) - 0 + 2 \cdot \frac{1}{8}] = 0$

Since $\int_{-1}^1 x^3 dx = \frac{1}{4} \cdot 1 - \frac{1}{4} \cdot (-1) = 0$

then it is correct *

⑤ $f(x) = x^4$

$\Rightarrow I_2(f) = \frac{2}{3} [2 \cdot \frac{1}{16} - 0 + 2 \cdot \frac{1}{16}] = \frac{1}{6}$

but $\int_{-1}^1 x^4 dx = \frac{1}{5} \cdot 1 - \frac{1}{5} \cdot (-1) = \frac{2}{5}$

so it is not correct. *

Thus the degree of exactness of $I_2(f)$ is 3

also the error is $O(h^5)$

(b 的情况) $I_4(f) = \frac{1}{4} [f(-1) + 3f(-\frac{1}{3}) + 3f(\frac{1}{3}) + f(1)]$

We have $\begin{cases} x_0 = -1, & w_0 = 1 \\ x_1 = -\frac{1}{3}, & w_1 = 3 \\ x_2 = \frac{1}{3}, & w_2 = 3 \\ x_3 = 1, & w_3 = 1 \end{cases}$ and normalized by $\frac{1}{4}$

① $f(x) = 1$

$\Rightarrow I_4(f) = \frac{1}{4} [1 + 3 + 3 + 1] = \frac{1}{4} \cdot 8 = 2$

Since $\int_{-1}^1 1 dx = 1 - (-1) = 2$

then it is correct. *

② $f(x) = x$

$$\Rightarrow I_4(f) = \frac{1}{4} \left[-1 + 3\left(-\frac{1}{3}\right) + 3\left(\frac{1}{3}\right) + 1 \right] = 0$$

$$\text{Since } \int_{-1}^1 x \, dx = \frac{1}{2} \cdot 1 - \frac{1}{2} \cdot (-1) = 0$$

then it is correct. *

③ $f(x) = x^2$

$$\Rightarrow I_4(f) = \frac{1}{4} \left[1 + 3\left(\frac{1}{9}\right) + 3\left(\frac{1}{9}\right) + 1 \right] = \frac{1}{4} \cdot \frac{8}{3} = \frac{2}{3}$$

$$\text{Since } \int_{-1}^1 x^2 \, dx = \frac{1}{3} \cdot 1 - \frac{1}{3}(-1) = \frac{2}{3}$$

then it is correct. *

④ $f(x) = x^3$

$$\Rightarrow I_4(f) = \frac{1}{4} \left[-1 + 3\left(-\frac{1}{27}\right) + 3\left(\frac{1}{27}\right) + 1 \right] = 0$$

$$\text{Since } \int_{-1}^1 x^3 \, dx = \frac{1}{4} \cdot 1 - \frac{1}{4} \cdot (-1) = 0$$

then it is correct. *

⑤ $f(x) = x^4$

$$\Rightarrow I_4(f) = \frac{1}{4} \left[1 + 3\left(\frac{1}{81}\right) + 3\left(\frac{1}{81}\right) + 1 \right] = \frac{1}{4} \cdot \frac{56}{27} = \frac{14}{27}$$

$$\text{but } \int_{-1}^1 x^4 \, dx = \frac{1}{5} \cdot 1 - \frac{1}{5}(-1) = \frac{2}{5}$$

So it is not correct. *

Thus the degree of exactness of $I_2(f)$ is 3

also the error is $O(h^5)$

5. Let $I_w(f) = \int_0^1 w(x) f(x) \, dx$ with $w(x) = \sqrt{x}$, and consider the quadrature formula $Q(f) = a f(x_1)$.

Find a and x_1 in such a way that Q has maximum degree of exactness r .

[Solution: $a = 2/3$, $x_1 = 3/5$ and $r = 1$.]

6. Let us consider the quadrature formula $Q(f) = \alpha_1 f(0) + \alpha_2 f(1) + \alpha_3 f'(0)$

for the approximation on $I(f) = \int_0^1 f(x) \, dx$, where $f \in C^1([0, 1])$.

Determine the coefficients α_j , for $j = 1, 2, 3$ in such a way that Q

has degree of exactness $r = 2$.

[Solution: $\alpha_1 = 2/3$, $\alpha_2 = 1/3$ and $\alpha_3 = 1/6$.]