Amelia Zhao
Math 179 HW #1

1.) For some random vector $y = Ax + b$:

$$\mathbb{E}[y] = \mathbb{E}[Ax+b]$$
$$= \int_{-\infty}^{\infty} [Ax+b]\, P(y = Ax+b)\, dx$$
$$= \int_{-\infty}^{\infty} [Ax+b]\, P(x)\, dx$$
$$= A\int_{-\infty}^{\infty} x P(x)\, dx + b\int_{-\infty}^{\infty} P(x)\, dx$$

By definition of the expectation value,
$$\int_{-\infty}^{\infty} xP(x)\, dx = \mathbb{E}[x]$$
Probability integrated from $-\infty \to \infty = 1$:
$$\int_{-\infty}^{\infty} P(x)\, dx = 1$$
Thus, $\boxed{\mathbb{E}[y] = A\mathbb{E}[x] + b}$

$$\text{cov}[y] = \text{cov}[Ax+b]$$
covariance matrix Σ of a random vector x
$$\Sigma = \mathbb{E}[(x-\mathbb{E}(x))(x-\mathbb{E}(x))^T]$$
$$\text{cov}[y] = \mathbb{E}[(Ax+b-\mathbb{E}(Ax+b))(Ax+b-\mathbb{E}(Ax+b))^T$$
$$= \mathbb{E}[(Ax+\cancel{b}-A\mathbb{E}(x)-\cancel{b})(Ax+\cancel{b}-A\mathbb{E}(x)-\cancel{b})^T]$$
$$= \mathbb{E}[A(x-\mathbb{E}[x])(x-\mathbb{E}[x])^T A^T]$$
$\mathbb{E}[A]$ and $\mathbb{E}[A^T]$ are constants
$$= A\,\mathbb{E}[(x-\mathbb{E}[x])(x-\mathbb{E}[x])^T]A^T$$
$$= A\,\text{cov}[x]A^T$$
$$= \boxed{A\Sigma A^T} \quad \text{where } \Sigma = \text{cov}[x]$$

2.) a.) $y = \theta^T x$

$D = \{(x, y)\} = \{(0,1), (2,3), (3,6), (4,8)\}$

x_i	y_i	x_i^2	$x_i y_i$
0	1	0	0
2	3	4	6
3	6	9	18
4	8	16	32
$\Sigma x_i = 9$	$\Sigma y_i = 18$	$\Sigma x_i^2 = 29$	$\Sigma x_i y_i = 56$

Cramer's Rule:

$$m = \frac{n\sum_{i=1}^{n} x_i y_i - \left(\sum_{i=1}^{n} x_i\right)\left(\sum_{i=1}^{n} y_i\right)}{n\sum_{i=1}^{n} x_i^2 - \left(\sum_{i=1}^{n} x_i\right)^2} = \frac{4(56) - 9(18)}{4(29) - 9^2} = \frac{62}{35}$$

$$b = \frac{\left(\sum_{i=1}^{n} x_i^2\right)\left(\sum_{i=1}^{n} y_i\right) - \left(\sum_{i=1}^{n} x_i\right)\left(\sum_{i=1}^{n} x_i y_i\right)}{n\sum_{i=1}^{n} x_i^2 - \left(\sum_{i=1}^{n} x_i\right)^2} = \frac{29(18) - 9(56)}{4(29) - 9^2}$$

$$= \frac{18}{35}$$

$$\boxed{y = \theta^T x = \frac{62}{35}x + \frac{18}{35}} \qquad \theta = \begin{bmatrix} \frac{18}{35} \\ \frac{62}{35} \end{bmatrix}$$

b.) $\theta = (X^T X)^{-1} X^T \vec{y}$

$$X = \begin{bmatrix} 1 & 0 \\ 1 & 2 \\ 1 & 3 \\ 1 & 4 \end{bmatrix} \qquad \vec{y} = \begin{bmatrix} 1 \\ 3 \\ 6 \\ 8 \end{bmatrix}$$

↳ represents the b value

$$X^T X = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 2 & 3 & 4 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 2 \\ 1 & 3 \\ 1 & 4 \end{bmatrix} = \begin{bmatrix} 4 & 9 \\ 9 & 29 \end{bmatrix}$$

$$(X^T X)^{-1} = \frac{1}{4(29) - 81} \begin{bmatrix} 29 & -9 \\ -9 & 4 \end{bmatrix} \qquad \begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{\det} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$$\theta = \frac{1}{35} \begin{bmatrix} 29 & -9 \\ -9 & 4 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 2 & 3 & 4 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \\ 6 \\ 8 \end{bmatrix}$$

$$= \frac{1}{35} \begin{bmatrix} 29 & 11 & 2 & -7 \\ -9 & -1 & 3 & 7 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \\ 6 \\ 8 \end{bmatrix} = \frac{1}{35} \begin{bmatrix} 18 \\ 62 \end{bmatrix} = \boxed{\begin{bmatrix} \frac{18}{35} \\ \frac{62}{35} \end{bmatrix}}$$

This is the same result as (a).