On Optimal Substructures of the Rod-Cutting Problem

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Equivalence of one- and two-piece decomposition

In CLRS (3ed, p362), it was proposed that the following two formulations of the rod-cutting problem are equivalent:

$$r_n = max(p_n, r_i + r_{n-i}) \; i = 1, 2, \ldots, n-1 \tag{1}$$
$$r_n = max(p_i + r_{n-i}) \; i = 1, \ldots, n \tag{2}$$

The first formulation (1) allows for further cutting in both left and right pieces. The second formulation (2), the length of the left piece is fixed. Only the right piece is cut further.

We will argue the two formations are equivalent.

There are a total of 2^{n-1} ways to cut a rod of length n at positions i=1,2,...,n-1.

Formulation (1) corresponds to all the 2^{n-1} ways of cutting the rod.

From these 2^{n-1} cuttings, select those with left piece of length i=1. The optimal revenue from this set of cuttings is $p_1 + r_{n-1}$ where r_{n-1} is the optimal revenue from the right piece, a rod of length n-1.

Repeat this step with i=2. The optimal revenue from this set of cuttings is $p_2 + r_{n-2}$ where r_{n-2} is the optimal revenue from the right piece, a rod of length n-2.

Repeat this step for i=3,4,...,n-1. At each step, the optimal revenue is $p_i + r_{n-i}$

The overall optimal revenue is given by

$$r_n = max(p_i + r_{\text{n-i}}), i = 1, 2, \ldots, n - 1 \tag{3}$$

If we include the case of no cutting (i=0), in the above equation, we recover equation (2).

We can conclude formulations (1) and (2) are equivalent.