Examen 2

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(5.1)

Problem. Prove by induction on n that, for all positive integers n, $3|n^3 - n$.

Solution. For $n = 1$, $1^3 - 1 = 1 - 1 = 0 = 3 \cdot 0$, so $3|0$. Now suppose for all $n \in \mathbb{Z}^+$ that $3|n^3 = -n$. Then $(n+1)^3 - (n+1) = (n+1)(n+1)(n+1) - (n+1) = (n^2+2n+1)(n+1) - (n+1) = n^3+3n^2+3n+1-n-1 = (n^3-n)+3n^2+3n$. Now by hypothesis, we have $n^3-n = 3m$ for some $m \in \mathbb{Z}$. Then $(n^3-n)+3n^2+3n = 3m+3n^2+3n = 3(m+n^2+n)$. Therefore $3|(n+1)^3 - (n+1)$

(5.2)

Problem. Prove by induction on m that $m^3 \leq 2^m$.

Solution. For $m = 10$, we have $10^3 = 1000$ and $2;10 = 1024$, and $1000 \geq 1024$. Now suppose for $m \geq 10$, that $m^3 \geq 2^m$. Then $(m+1)^3 = (m+1)(m+1)(m+1)(m^2 + 2m+1)(m+1) = m^3 + (3m^2+3m+1) \geq m^3 + m^3 \geq 2^m + 2^m = 2 \cdot 2^m = 2^{m+1}$.

(5.3)

Problem. Prove by induction on n, for all positive integers n, $n \geq 1$.

Solution. For $n = 1$, $1 \geq 1$ is true. Now suppose that for all $n \in \mathbb{Z}^+$ that $n \geq 1$. By the axioms of Peano, we have that the successor of n is $s(n) = n + 1 > n \geq 1$, hence $n + 1 \geq 1$.

(8.2)

Problem. Define functions $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ by $f(x) = x^3$ and $g(x) = 1 - x$. Find the functions:

(1) $f \circ f$.
(2) $f \circ g$.
(3) $g \circ f$.
(4) $g \circ g$.

List the elements of the set $\{x \in \mathbb{R} : f \circ g(x) = g \circ f(x)\}$.

Solution. Let $f, g : \mathbb{R} \to \mathbb{R}$ be defined by $f : x \to x^3$ and $g : x \to 1 - x$. Then $f \circ f = f^2 : \mathbb{R} \to \mathbb{R}$ is defined by $f^2 : x \to x^2 \to (x^3)^3 = x^9$. $f \circ g : \mathbb{R} \to \mathbb{R}$ is defined by $f \circ g : x \to 1 - x \to (1-x)^3 = (1-x)(1-x)(1-x) = (1-2x+x^2)(1-x) = x^3 - x^2$. $g \circ f : \mathbb{R} \to \mathbb{R}$ is defined by $g \circ f : x \to x^3 \to 1 - x^3$, and finally $g \circ g = g^2 : \mathbb{R} \to \mathbb{R}$ is defined by $g^2 : x \to 1 - x \to 1 - (1-x) = x$.

Now consider $f \circ g$ and $g \circ f$, when is $f \circ g = g \circ f$? Consider $x^3 - x^2 = 1 - x^3$. Then $x^2 + 1 = 0$, which has no solutions in $\mathbb{R}$. So $f \circ g \neq g \circ f$ for all $x \in \mathbb{R}$.

Remark. The second part of this problem may be misconstrued, as in my copy of the book it says to list all the elements of the set $\{x \in \mathbb{R} : fg(x) = gf(x)\}$; which may mean something different than function composition for some authors. However, it is most likely that composition is implied by that notation since in general $g \circ f \neq f \circ g$, moreover it is a popular algebraic notation to express composition by fg instead of $f \circ g$.

(8.3)

Problem. Find the functions $f_i : \mathbb{R} \to \mathbb{R}$ with the images as follows:

(1) $f_1(\mathbb{R}) = \mathbb{R}$.
(2) $f_2(\mathbb{R}) = \mathbb{R}^+$.
(3) $f_3(\mathbb{R}) = \mathbb{R}\backslash\mathbb{Z}$.
(4) $f_4(\mathbb{R}) = \mathbb{Z}$.

Solution. (1) Take $f_1 : \mathbb{R} \to \mathbb{R}$ by $f_1 : x \to x^3$. Then $f_1(\mathbb{R}) = \mathbb{R}$.

(2) Take $f_2 : \mathbb{R} \to \mathbb{R}$ by $f_2 : x \to \begin{cases} x, \ 0 < x < 1 \\ 1, \ x \geq 1 \end{cases}$. Then $f_2(\mathbb{R}) = \mathbb{R}^+$.

(3) Take $f_3 : \mathbb{R} \to \mathbb{R}$ by $f(x) = x - \lfloor x \rfloor$; where $\lfloor x \rfloor$ is the greatest interger less or equal to x. Then $f_3(\mathbb{R}) = \mathbb{R}\backslash\mathbb{Z}$.

(4) Take $f_4 : \mathbb{R} \to \mathbb{R}$ by $f_4 : x \to \lfloor x \rfloor$. Then $f_4(\mathbb{R}) = \mathbb{Z}$.