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Prof. Lianqing Li
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Analysis I

Alec Zabel-Mena

Universidad de Puerto Rico, Recinto de Rio Piedras

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Lecture 1: Review

We begin with a review of some prelimanary results of set theory and advanced calculus.

Definition. Let A and B be sets, and let $f : A \to B$ be a map of A into B. We say that f is **1–1** if for every $x, y \in X$, $f(x) = f(y)$ implies $x = y$. We say f is **onto** if for every $y \in B$, there is an $x \in A$ for which $y = f(x)$. We say that f is a **1–1 correspondence** of A **onto** B if f is both 1–1 and onto and onto.

Theorem 1 (Beinstern's Theorem)**.** *Let A and B be sets. If there is a 1–1 map of A into B, and a 1–1 map of B into A, then there exists a 1–1 correspondence of A onto B.*

Axiom 1 (The Axiom of Choice)**.** *Suppose that $\{A_\alpha\}_{\alpha\in\Lambda}$ is a collection of nonmepty sets indexed by the set Λ. Then there exists a map $f : \Lambda \to \bigcup A_\alpha$ called a **choice function**, defined by the rule $f(\alpha) \in A_\alpha$ for all $\alpha \in \Lambda$.*

Remark. What this axiom says is that given any (not necesarrily countable) collection of sets, one can "choose " an element from each set.

Definition. We define an **partial order** on a set A to be a relation $<$ satisfying the following properties for all $a, b, c \in A$:.

(1) $a < a$ (Reflexive).

(2) If $a < b$ and $b < a$, then $a = b$ (Antisymmetry).

(3) If $a < b$ and $b < c$, then $a < c$ (Transitivity).

We say that elements $a, b \in A$ are **comparable** under $<$ if either $a < b$ or $b < a$. If every element of A is comparable, then $<$ is called a **total order**.

Definition. Let A be a set with partial order $<$. We call an element $x \in A$ a **maximum** of A if $x < a$ implies $x = a$ for all $a \in A$. If $B \subseteq A$, then we call an element $a \in A$ an **upper bound** of B if $b < a$ for all B in B; and we say B is **bounded above**. If $b < a'$ implies that $a < a'$ for any $a' \in A$, then we call a a **least upper bound** of B and write $\sup B = a$.

Definition. Let A be a set with partial order $<$. We call an element $x \in A$ a **minimum** of A if $a < x$ implies $x = a$ for all $a \in A$. If $B \subseteq A$, then we call an element $a \in A$ an **lower bound** of B if $a < b$ for all B in B; and we say B is **bounded below**. If $a' < b$ implies that $a' < a$ for any $a' \in A$, then we call a a **greatest lower bound** of B and write $\inf B = a$.

Theorem 2 (Zorn's Lemma)**.** *Let A be a set with partial order $<$. If every totally ordered subset of A under $<$ has an upperbound, then A has a maximum element.*

Remark. It can be shown that Zorn's lemma and the axiom of choice are equivalent statements. That is you can prove Zorn's lemma from the axiom of choice, and you can prove the axiom of choice from Zorn's lemma.

Theorem 3. *For any sets A and B, there is either a 1–1 map of A into B, or a 1–1 map of B into A.*

Proof. Consider the collection of all triples $C = \{(X, Y, f) : f : X \to Y$ is 1–1 and onto, where $X \subseteq A, Y \subseteq B\}$, and define a partial order $<$ on C by: $(X, Y, f) < (X', Y', g)$ if, and only if $X \subseteq X'$, $Y \subseteq Y'$, and $g|_X = f$. Now suppose that the collection $\{(X_\alpha, Y_\alpha, f_\alpha)\}$ is a totally ordered subset of C. Then define

$$\tilde{X} = \bigcup_\alpha X_\alpha$$
$$\tilde{Y} = \bigcup_\alpha Y_\alpha$$

and since for all $x \in \tilde{X}$, there is an α for which $x \in X_\alpha$, define the map $\tilde{f} : \tilde{X} \to \tilde{Y}$ by the rule $\tilde{f}(x) = f_\alpha(x) \in Y_\alpha$. This map is well defined by the total order on $\{(X_\alpha, Y_\alpha, f_\alpha)\}$.

Now, notice that $\tilde{f}$ is an upperbound of the collection $\{(X_\alpha, Y_{\alpha f \alpha})\}$, therefore, by Zorn's lemma there is a maximum element (X, Y, f) of C. By definition we get that f is a 1–1 correspondence of X onto Y, and by maximality, we get that either $X = A$ or $Y = B$. Suppose on the contary that this is not true. Then there is an $a \in A \backslash X$ and a $bB \backslash Y$. Letting $X' = X \cup a$ and $Y' = Y \cup b$, and $f' : X' \to Y'$ defined by $f'|_X = f$ and $f'(a) = b$. Then $(X, Y, f) < (X', Y', f')$, which contradicts the maximality of (X, A, f).

Therefore, we must have that either $X = A$ or $Y = B$. Now, if $X = A$, then $f : A \to Y$ is 1–1 and onto, taking the extension of Y to B, $f_B : A \to B$, we see it must be 1–1 as well. By the same reasoning we can assure there is a 1–1 map $f_A : B \to A$ of B into A. ■

Lecture 2: Smallest Infinite Cardinality

Definition. A set A is **countably infinite** if there exists a $1-1$ map $f : \mathbb{N} \to A$ of $\mathbb{N}$ onto A.

Remark. That is we can put A into countable order and represent it as an increasing sequence $A = \{a_n\}_{n\in\mathbb{N}}$ where each a_n is distinct for all n.

Example 1. Both $\mathbb{N}$ and $2\mathbb{N}$ are countable. Consider the identity map $i : \mathbb{N} \to \mathbb{N}$ and the map $f : \mathbb{N} \to 2\mathbb{N}$ defined by $n \to 2n$.

Theorem 4. *A set A is finite if, and only if there is no $1-1$ map of A onto a subset of itself.*

Corollary. *Every infinite set countains a countable subset.*

Theorem 5. *A countable union of sets is countable.*

Proof. Let A and B be nonempty countable sets. Then there exists nonrepeating increaseing sequences $A = \{a_n\}$ and $B = \{b_m\}$. Then take $A \cup B = \{a_n, b_m\}$, and delete any repeated members. Then we can say that $A \cup B = \{c_k\}$ where $c_k = a_k$ for k odd, and $c_k = b_k$ for k even. Then $A \cup B$ is countable.

Now suppose that $\{A_n\}$ is a collection of countable sets. Then we have the following:

$$\begin{aligned}
A_1 &= a_{11}\ a_{12}\ \ldots\ a_{1n}\ \ldots \\
A_2 &= a_{21}\ a_{22}\ \ldots\ a_{2n}\ \ldots \\
A_3 &= a_{31}\ a_{32}\ \ldots\ a_{3n}\ \ldots \\
&\vdots \\
A_n &= a_{n1}\ a_{n2}\ \ldots\ a_{nn}\ \ldots \\
&\vdots
\end{aligned}$$

Construct the set:

$$A = \{a_{11}, a_{12}, a_{21}, a_{13}, a_{22}, a_{31}, \ldots\}$$

Then A is countable and $A = \bigcup A_n$. ■

Example 2. (1) $-\mathbb{N}$ is countable, then $-\mathbb{N} \cup \mathbb{N} = \mathbb{Z}$ is also countable, which makes $\mathbb{Z}$ countable.

(2) Consider the rational numbers $\mathbb{Q} = \{\frac{p}{q} : p, q \in \mathbb{Z}\}$. Let $\frac{\mathbb{Z}}{n} = \{\frac{p}{n} : p \in \mathbb{Z}\}$ for some $n \in \mathbb{N}\backslash\{0\}$. Then notice that $\frac{\mathbb{Z}}{n}$ is countable and that $\mathbb{Q} = \bigcup \frac{\mathbb{Z}}{n}$, which makes $\mathbb{Q}$ countable as well.

Lemma 6. *Let A and B be two countably finite sets. Then $A \times B$ is countable.*

Proof. Take $A \times B = (\bigcup A_n) \times \{b_n\}$, where $b_n \in B$. ■

Example 3. (1) Let P be the set of all polynomials with integer coefficients. Let P_i be the set of all polynomials of degree $\deg = i$. Then we have the following:

$$\begin{aligned} &P_0 \simeq \mathbb{Z} \\ &P_1 \simeq \mathbb{Z} \times \mathbb{Z} \\ &\quad\vdots \\ &P_n \simeq \mathbb{Z} \times \mathbb{Z} \times \cdots \times \mathbb{Z} \\ &\quad\vdots \end{aligned}$$

Then we have that $P \simeq \bigcup P_i$. Since each P_i is countable, so is $\bigcup P_i$, so by isomorphism, P must also be countable.

(2) Let $\mathcal{A} = \{x \in \mathbb{C} :$ there exists a polynomial with integer coefficients p such that $p(x) = 0\}$ be the set of all algebraic numbers over $\mathbb{C}$. Then we have $\mathbb{Q} \subseteq \mathcal{A}$, since the polynomial $z^2 + 1 = 0$ has complex roots. $\mathcal{A}$ is countable.

Theorem 7. *Let A be a set, then $|A| < |2^A|$.*

Proof. We have that by definition, $|A| > |2^A|$ is impossible, so suppose that $|A| = |2^A|$. Then there exists a $1 - 1$ map $f : A \to 2^A$ of A onto its powerset. Then we have that for each $a \in A$, $f(a) \subseteq A$. Now, let $A_0 = \{x \in A : x \notin f(x)\}$. Since f is onto, there is an $x_0 \in A$ with $f(x_0) = A_0$. Now, if $x \in f(x_0)$, we get that $x \in A_0$, which means that $x_0 \notin f(x_0)$. This cannot happen; so it must be that $x_0 \notin f(x_0)$, but then we get that $x_0 \in A_0$, which forces $x_0 \in f(x_0)$. A contradiction! Therefore it must be that $|A| < |2^A|$ ■

Example 4. $|\mathbb{N}| < |2^{\mathbb{N}}|$

Now consider the set of real numbers $\mathbb{R}$. We have the following theorems.

Theorem 8. $\mathbb{R}$ *is a field.*

Theorem 9. $\mathbb{R}$ *is a totally ordered set.*

Corollary. *The following are true for all $a, b, c \in \mathbb{R}$.*

(1) $a > b$ impl ies that $a + c > b + c$.

(2) $a > b$ and $c > 0$ implies $ac > bc$.

(3) For all $a > 0$ and $b > 0$, there exists an $n \in \mathbb{N}$ such that $an > b$ (Property of Archimedes).

Theorem 10. $\mathbb{R}$ *has the least upperbound property. That is, if $\{a_n\}$ is an increasing sequence of real numbers bounded above by b, then* $\sup \{a_n\} \in \mathbb{R}$.

Lecture 3

Theorem 11. *If $A \subseteq \mathbb{R}$ is a nonempty set bounded below, then A has a greatest lowerbound in $\mathbb{R}$.*

Lemma 12. *For every A nonempty,* $\inf A \leq \sup A$

Example 5. For $A = \emptyset$, $\inf A = \infty$ and $\sup A = -\infty$.

Definition. We define the **extended real numbers** to be $\mathbb{R}$ together with the symbols $-\infty$ and ∞.

Theorem 13. *The following are true for the extended real numbers:*

(1) For any $a \in \mathbb{R}$, $a + \infty = \infty$ and $a - \infty = -\infty$.

(2) $\infty + \infty = \infty$.

(3) $\infty \cdot \infty = \infty$ and $\infty \cdot (-\infty) = \infty$.

Theorem 14. *The closed interval $[a, b]$, for $a, b \in \mathbb{R}$ is uncountable.*

Proof. Suppose that $[a, b]$ is countable. Then list the elements of $[a, b]$ as:

$$x_0 = a \qquad x_1 \qquad \ldots \qquad x_n \qquad \ldots$$

Now choose an $a_1, a_2 \in [a,b]$ such that $[a_1,b_1] \subseteq [a,b]$ and $x_1 \notin [a_1,b_1]$. Choose then $a_2, b_2 \in [a_1,b_1]$ for which $x_2 \notin [a_2,b_2] \subseteq [a_1,b_1]$. Porceeding inductively, then construct the sequence of intervals $\{a_n,b_n\}$ where $[a_{n+1},b_{n+1}][a_n,b_n]$ and $x_n \notin [a_n,b_n]$. Notice that the sequence of endpoints $\{a_n\}$ has a least upperbound $a' = \sup\{a_n\}$ such that $a_n < a' < b_n$ for all $n \in \mathbb{N}$. Therefore $a' \in [a_n,b_n]$ so that $a' \neq x_n$ for any n. This contradicts the complete listing of $[a,b]$ as there is an element a' that cannot be listed. ∎

Corollary. $\mathbb{R}$ *is uncountable.*

Proof. Notice that $\mathbb{R}$ is homeomorphic to the closed interval $[0,1]$. ∎

Corollary. $|2^{\mathbb{N}}| = |[0,1]|$.

Definition. We call a set $A \subseteq \mathbb{R}$ **open** in $\mathbb{R}$ if there exists an interval (a,b) such that $x \in (a,b) \subseteq A$, for $a,b \in \mathbb{R}$. We call A **closed** in $\mathbb{R}$ if $\mathbb{R}\backslash A$ is open in $\mathbb{R}$.

Example 6. (1) $\mathbb{R}$ is open in itself, since $\mathbb{R} = (-\infty,\infty)$.

(2) Every open interval (a,b) of $\mathbb{R}$ is open in $\mathbb{R}$ since they contain themselves.

Lemma 15. $\mathbb{Q}$ *is dense in* $\mathbb{R}$.

Proof. ∎

Theorem 16. *If A is open in $\mathbb{R}$, the A is the countable union of disjoint open intervals.*

Proof. Let A be open in $\mathbb{R}$. Then for each $x \in A$, there exists $a,b \in \mathbb{R}$ for which $x \in (a,b) \subseteq A$. Now consider the sets $Z = \{z : (x,z) \subseteq A\}$ and $W = \{w : (w,x) \subseteq A\}$. We have Z and W are nonempty since $b \in Z$ and $a \in W$, and they are bounded above and below by b and a, respectively. Now, let $b_x = \sup Z$ and $a_x = \inf W$. Then we have that $x \in (a_x,b_x) \subseteq A$, so that $a_x < x < b_x$.

Moreover, let $x \in (a_x,b_x)$ such that $x < c$, without loss of generality. Then $c < b_x$ which means that c is not an upperbound of x; i.e. there is a $z \in Z$ such that $(x,z) \subseteq A$ with $z > c$.

Suppose that $b_X \in A$, then there is a $\delta > 0$ such that $(b_x - \delta, b_x + \delta) \subseteq A$, so $(a_x,b_x) \cup (b_x - \delta, b_x + \delta) = (a_x, b_x + \delta) \subseteq A$, which contradicts that $b_x = \sup Z$. So $b_x \notin A$. Similarly, we have that $a_X \notin A$. This shows that (a_x,b_x) is maximally chosen.

Now, consider the collection $\{(a_x,b_x)\}_{x\in A}$ of all intervals contained in A, containing x for all $x \in A$, with $(a_x,b_x) \cap (a_y,b_y) = \emptyset$ whenever $x \neq y$. Let

$$I = \bigcup_{x \in X} (a_x,b_x)$$

Then $A = I$. Moreover, by the density of $\mathbb{Q}$ in $\mathbb{R}$, each (a_x, b_x) contains a rational q, so taking the map $\{(a_x, b_x)\} \to \mathbb{Q}$, by the rule $(a_x, b_x) \to q$ if $q \in (a_x, b_x)$, we see that this map is onto. Therefore $\{(a_x, b_x)\}$ is a countable collection. Therefore A is the union of countably many disjoint open intervals. ■

Theorem 17 (Heine-Borel). *If $F \subseteq \mathbb{R}$ is bounded, closed, and nonempty, then F is compact in $\mathbb{R}$.*

Theorem 18 (Nests of Closed Sets). *Suppose that $\{F_n\}$ is a collection of closed, bounded, and nonempty sets such that $F_{n+1} \subseteq F_n$ for all $n \in \mathbb{N}$. Then $\bigcap F_n$ is nonempty.*

Definition. Let X be a set and $\mathcal{A}$ a collection of subsets of X. We call $\mathcal{A}$ a **σ-algebra** if:

(1) $\emptyset, X \in \mathcal{A}$.

(2) $\mathcal{A}$ is closed under countable unions of subsets of X.

(3) If $Y \in \mathcal{A}$, then $X \backslash Y \in \mathcal{A}$.

Theorem 19. *For any collection $\{A_\alpha\}$ of subsets of a set X, there exists a smallest σ-algebra containing $\{A_\alpha\}$.*

Definition. We call the smallest σ-algebra containing all open intervals in $\mathbb{R}$ the **Borel algebra** and denotes it $\mathcal{B}$. Elements of $\mathcal{B}$ are called **Borel sets**.

Lemma 20. *$\mathcal{B}$ contains all closed sets in $\mathbb{R}$.*

Lemma 21. *$\mathcal{B}$ contains all countable intersections of open intervals of $\mathbb{R}$ and all countable unions of closed sets in $\mathbb{R}$.*

Lecture 4

Definition. Let $\{A_n\}$ be a collection of sets. We define the **limit superior** of $\{A_n\}$ to be $\limsup \{A_n\} = \{x \text{ which is in infinitely many } A_n\}$. We define the **limit inferiro** of $\{A_n\}$ to be $\liminf \{A_n\} = \{x \text{ which is in almost all } A_n\}$.

Lemma 22. *Let $\{A_n\}$ be a collection of sets, then $\liminf \{A_n\} \subseteq \sup \{A_n\}$.*

Lemma 23. *Let $\{A_n\}$ be a collection of sets. If $x \in \limsup \{A_n\}$, then for $N > 0$ arbitrarily large, there exists an $n \geq N$ such that $x \in A_n$. Likewise, if $x \in \liminf \{A_n\}$, then there exists an $N_0 > 0$ such that $x \in A_n$ whenver $n \geq N_0$.*

Lemma 24. *For any collection $\{A_n\}$ of sets, we have*

$$\limsup \{A_n\} = \bigcap_{N=1}^{\infty} (\bigcup_{n=N}^{\infty} A_n) \tag{1}$$

and

$$\liminf \{A_n\} = \bigcup_{N=1}^{\infty} (\bigcap_{n=N}^{\infty} A_n) \tag{2}$$

Definition. Let $\{a_n\}$ be a sequence of real numbers. We say that $\{a_n\}$ **converges** to an $a \in \mathbb{R}$ if for every $\varepsilon > 0$, there exists a $N > 0$ such that $|a_n - a| < \varepsilon$ whenever $n \geq N$. We write $\lim_{n \to \infty} \{a_n\} = a$, or $\{a_n\} \to a$ as $n \to \infty$; or we simply omit the $n \to \infty$ with clear enough context.

Lemma 25. *If the limit of a sequence exists, then it is unique. Moreover, convergent sequneces are bounded, and if $\{a_n\}$ is such a sequence such that $a_n \leq c$ for all n, then $a \leq c$, where $c \in \mathbb{R}$.*

Lemma 26. *If $\{a_n\}$ is a monotone increaseing sequence bounded above, then $\{a_n\}$ converges.*

Theorem 27. *Every bounded sequence has a convergent subsequence.*

Definition. We call a sequence $\{a_n\}$ a **Cauchy sequence** (or simply **Cauchy**) if for every $\varepsilon > 0$, there is an $N > 0$ such that $|a_n - a_m|/ < \varepsilon$ whenever $n, m \geq N$.

Theorem 28. *A sequnece in $\mathbb{R}$ converges if, and only if it is Cauchy.*

Theorem 29. *Let $\{a_n\}$ and $\{B_n\}$ be real sequences such that $\lim \{a_n\} = a$ and $\lim \{b_n\} = b$. Then the following are true:*

(1) $\lim \alpha a_n$ *exists and* $\lim \alpha a_n = \alpha a$*, for some $\alpha \in \mathbb{R}$.*

(2) $\lim \{a_n + b_n\}$ *exists and* $\lim \{a_n + b_n\} = a + b$*.*

(3) $\lim \{a_n b_n\}$ *exists and* $\lim \{a_n b_n\} = ab$*.*

(4) If $a_n \leq b_n$ for all n, then $a \leq b$.

Definition. Let $\{a_n\}$ be a sequence of real numbers. We define the **limit superior** of $\{a_n\}$ to be $\limsup \{a_n\} = \lim (\sup a_k)$ as $n \to \infty$ and we define the **limit inferior** of $\{a_n\}$ to be $\liminf \{a_n\} = \lim (\inf \{a_k\})$ as $n \to \infty$ for all $k \geq n$.

Lemma 30. *For any real sequence $\{a_n\}$:*

$$\liminf \{a_n\} \leq \limsup \{a_n\}$$

Corollary. $\{a_n\}$ *converges if and only if* $\limsup \{a_n\} = \liminf \{a_n\}$.

Lemma 31. *Let* $\{a_n\}$ *and* $\{b_n\}$ *be real sequences. Then:*

(1) $\limsup \{a_n\} = l$ *if, and only if there are infinitely many* n *for which* $a_n > l - \varepsilon$ *and only finitely many* n *for which* $a_n > l + \varepsilon$.

(2) $\limsup \{a_n\} = \infty$ *if, and only if* $\{a_n\}$ *is not bounded above.*

(3) $\limsup \{a_n\} = -\liminf \{-a_n\}$.

(4) If $a_n \leq b_m$ *for all* m, n, *then* $\limsup \{a_m\} \leq \liminf \{b_m\}$.

Definition. Let $E \subseteq \mathbb{R}$. We say that a map $f : E \to \mathbb{R}$ is **continuous** at a point $a \in E$ if for every $\varepsilon > 0$ there exists a $\delta > 0$ such that $|f(x) - f(a)| < \varepsilon$ whenever $0 < |x - a| < \delta$. We say that f is **continuous** on all of E if it is continuous at every point of E.

Theorem 32. *A realvalued function* $f : E \to \mathbb{R}$ *is continuous on* E *if, and only if for any* U *open in* E, *there is a* V *open in* $\mathbb{R}$ *such that* $f^{-1}(U) = V \cap E$.

Theorem 33 (The Sequential Criterion)**.** *A realvalued function* $f : E \to \mathbb{R}$ *is continuous at a point* $x' \in E$ *if, and only if for any sequence of points* $\{x_n\}$ *of* E, *with* $\{x_n\} \to x'$, *we have* $\lim \{f(x_n)\} = f(x')$.

Theorem 34 (The Extreme Value Theorem)**.** *A continuous map on a closed and bounded set takes on a maximum value, or a minumum value.*

Theorem 35 (The Intermediate Value Theorem)**.** *If* $f : [a, b] \to \mathbb{R}$ *is continuous on* $[a, b]$, *then* $f([a, b])$ *is a closed interval in* $\mathbb{R}$, *with* $f([a, b]) = [f(a), f(b)]$ *or* $f([a, b]) = [f(b), f(a)]$.

Definition. A map $f : E \to \mathbb{R}$ is **Lipshitz** if for there exists a $c \in \mathbb{R}^+$, such that for every $x, y \in E$, we have $|f(x) - f(y)| \leq c|x - y|$.

Definition. We call a map $f : E \to \mathbb{R}$ **uniformly continuous** if for every $\varepsilon > 0$, there is a $\delta > 0$ such that if $x, y \in E$ with $|x - y| < \delta$, then $|f(x) - f(y)| < \varepsilon$.

Theorem 36. *If* $f : E \to \mathbb{R}$ *is a continuous map on a closed and bounded set* E, *then* f *is uniformly continuous.*