Prof. Claudio Landim
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Analysis I

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Lecture 0: A Nonmeasurable Set

Consider the following half-open bounded interval

Figure 1

Then the length of $(a, b]$ is $l((a, b]) = b - a$. We can define the length of an interval as a function $l : 2^{\mathbb{R}} \to \mathbb{R}^+_\infty$, where $\mathbb{R}^+_\infty = \mathbb{R}^+ \cup \{\infty\}$. Notice then that the following hold for our function l:

(1) $l(A + y) = l(A)$, where $A + y = \{a + y : a \in A\}$.

(2) If $A \subseteq B$, then $l(A) \leq l(B)$.

(3) For any collection $\{A_n\}$ of disjoint intervals, we have

$$l(\bigcup A_n) = \sum l(A_n)$$

Now, we wish to define a general function on all subsets of $\mathbb{R}$, $\mu : 2^{\mathbb{R}} \to \mathbb{R}^+_\infty$ such that

(1) $\mu((a, b]) = b - a$

(2) If $A \subseteq B$, then $\mu(A) \leq \mu(B)$.

(3) $\mu(A + y) = \mu(A)$

(4) For any disjoint collection of sets $\{A_n\}$,

$$\mu(\bigcup A_n) = \sum \mu(A_n)$$

We would also like $\mu(\emptyset) = 0$.

Theorem 1. *No such function $\mu : 2^{\mathbb{R}} \to \mathbb{R}^+_\infty$ satisfying (1)–(4) exists.*

Proof. Suppose such a function exists. Define the equivalence relation $\sim$ on $\mathbb{R}$ such that

$$x \sim y \text{ if, and only if } x - y \in \mathbb{Q}$$

Let $\Lambda = {}^{\mathbb{R}}\!/\!_{\sim}$. Then Λ is not countable, as that would make $\mathbb{R}$ countable which is impossible. Now, let $R \subseteq [0,1]$ a collections of representatives of equivalence classes of Λ, choose one representative for each element, by the axiom of choice, that is

$$R = \{x : [x] \in \Lambda \text{ and } x = y \text{ if } [x] = [y]\}$$

Now, let $p, q \in \mathbb{Q}$, then we have that either $R + p = R + q$ or that $R + p$ and $R + q$ are disjoint. Suppose that they are not disjoint, and choose an $x \in (R + p) \cap (R + q)$. Then $x = \alpha + p$ and $x = \beta + q$. Thus $\alpha - \beta = q - p \in \mathbb{Q}$ which makes $\alpha \sim \beta$. By our definition, then $\alpha = \beta$, and we get $R + p = R + q$.

Now consider the collection of all $\{R + q\}_{q \in \mathbb{Q}}$. This collection is disjoint as each q is distinct, and consider

$$\mu(\bigcup R + q) = \sum \mu(R + q) \text{ for } -1 < q < 1$$

Notice that $\bigcup R + q \subseteq (-1, 2)$, and so we get

$$\sum \mu(R + q) \leq \mu((-1, 2)) = 3 \text{ for } -1 < q < 1$$

By (3), we also get that

$$\mu(R + q) = \mu(R)$$

so that

$$\sum \mu(R) \leq \mu((-1, 2)) = 3$$

■

Now, since this sum is bounded, $\mu(R) = 0$. On the other hand, notice that $(0, 1) \subseteq \bigcup R + q$. So that $\mu((0, 1)) = 1 \leq \mu(R) = 0$. This is a contradiction. Therefore no such μ exists.

Definition. We call a set for which no such function μ, satisfying (1)–(4) exists a **nonmeasurable set**.

We would now like to restrict our sets to those that permit such a function μ to exist; the so called "measurable" sets.

Lecture 1: Semi-Algebras, Algebras, and σ-Algebras

Definition. Let R be a set, we call a set $S \subseteq 2^R$ a **semi-algebra** if

(1) $R \in S$.

(2) If $A, B \in S$, then $A \cap B \in S$.

(3) If $A \in S$, then there is a finite collection of disjoints elements $\{E_i\}_{i=1}^n$ for which

$$S \backslash A = \bigcup_{i=1}^{n} E_i$$

Example 1. Consider $\mathbb{R}$ the real numbers, and let S be the collection

$$S = \{(a, b] : a < b, \text{ and } a, b \in \mathbb{R}\} \cup \{(-\infty, b]\}_{b \in \mathbb{R}} \cup \{(a, \infty)\}_{a \in \mathbb{R}} \cup \emptyset$$

Then S is a semi-algebra on $\mathbb{R}$.

Definition. Let R be a set, and let $S \subseteq 2^{\mathbb{R}}$. We call S an **algebra** if

(1) $R \in S$.

(2) If $A, B \in S$, then $A \cap B \in S$.

(3) If $A \in S$, then $S \backslash A \in S$.

Definition. Let R be a set, and let $S \subseteq 2^{\mathbb{R}}$. We call S an **σ-algebra** if

(1) $R \in S$.

(2) If $\{A_j\}$ is a countable collection of elements of S, then $\bigcap A_n \in S$.

(3) If $A \in S$, then $S \backslash A \in S$.

Lemma 2. *For any set R, if S is a σ-algebra, then it is an algebra.*

Lemma 3. *For any set R, if S is an algebra, then it is closed under finite unions.*

Proof. Notice that $A \cup B = S \backslash (S \backslash A \cap S \backslash B)$. ■

Corollary. *algebras are closed under arbitrary unions.*

Lemma 4. *Let R be a set and $\{S_\alpha\}$ a collection (not necessarily countable) of σ-algebras of R. Then the set*

$S{=}\bigcap S_\alpha$

is a σ-algebra.

Proof. Since $R \in S_\alpha$ for all α, $R \in S$, moreover if $A, B \in S$, then $A, B \in S_\alpha$ for all α, so that $A \cap B \in \alpha$, this makes $A \cap B \in S$.

Lastly, let $A \in S$, then $A \in S_\alpha$ for all α, so that $S_\alpha \backslash A \in S_\alpha$, Since σ-algebras are closed under arbitrary unions, we get $\bigcup (S_\alpha \backslash A) = (\bigcap S_\alpha) \backslash A = S \backslash A \in S$. ■

Definition. Let R be a set, and $\mathcal{C} \subseteq 2^R$. We call the algebra S the algebra **generated** by $\mathcal{C}$ if $\mathcal{C} \subseteq S$ and if $\mathbb{B}$ is an algebra containing $\mathcal{C}$, then $S \subseteq B$. That is, S is the smallest algebra that contains $\mathcal{C}$.

Lemma 5. *Let $\{S_\alpha\}$ a collection of algebras on a set R, containing $\mathcal{C} \subseteq 2^R$. Then $S = \bigcap S_\alpha$ is the algebra generated by $\mathcal{C}$.*

Proof. We have that S is an algebra by a similar proof to that of lemma 4, Moreover, by definition, $\mathcal{C} \in S$. Now, let B be an algebra, containing $\mathcal{C}$. Then B is in the collection $\{S_\alpha\}$, so that $S \subseteq B$. ■

Definition. Let R be a set, and $\mathcal{C} \subseteq 2^R$. We call the σ-algebra S the $\sigma-$algebra **generated** by $\mathcal{C}$ if $\mathcal{C} \subseteq S$ and if $\mathbb{B}$ is an σ-algebra containing $\mathcal{C}$, then $S \subseteq B$. That is, S is the smallest σ-algebra that contains $\mathcal{C}$.

Lemma 6. *Let R be a set and S a semi-algebra of R. Let $T(S)$ be the algebra generated by S. Then $A \in T(S)$ if, and only if there exists a finite collection $\{E_i\}_{i=1}^n$ of disjoint elements of S for which*

$$A = \bigcup_{i=1}^n E_i$$

Proof. Observe that if $A = \bigcup E_i$, since $E_i \in S$ for all $1 \leq i \leq n$, then $E_i \in T(S)$. This makes $A \in T(S)$.

Conversely, suppose that $A \in T(S)$. Consider the collection

$$\mathcal{B} = \{\bigcup F_j : F_j \in S \text{ and } F_i \cap F_j = \emptyset \text{ whenever } i \neq j\}$$

Notce that $S \subseteq \mathcal{B}$, and hence, so is R. Now, let $A = \bigcup F_i$ and $B = \bigcup F_j$. Then $A \cap B = (\bigcup F_j) \cap (\bigcup F_i) = \bigcup (F_i \cap F_j)$, which is a disjoint union of elements in S. So $A \cap B \in \mathcal{B}$ whenever $A, B \in \mathcal{B}$.

Lastly, let $A = \bigcap F_j$, then $\mathcal{B} \backslash A = \bigcap (\mathcal{B} \backslash F_j) = \bigcap \bigcap_{k_i}^{l_i} F_{i,k_i}$. Since $F_{i,k_i} \in S$, we get $\mathcal{B} \backslash A \in \mathcal{B}$, so that $\mathcal{B}$ is an algebra. This makes $A = T(S) \subseteq \mathcal{B}$. ■

Lecture 2: Set Functions

Definition. Let $\mathcal{C} \subseteq 2^R$ for some set R. We call a function $\mu : \mathcal{C} \to \mathbb{R}^+_\infty$ **additive** if for any finite disjoint collection $\{E_i\}_{i=1}^n$, we have

$$\mu(\bigcup_{i=1}^n E_i) = \sum_{i=1}^n \mu(E_i)$$

Suppose there is an $A \in \mathcal{C}$ for which $\mu(A)$ is finite. By additivity, we have that

$$\mu(A) = \mu(A \cup \emptyset) = \mu(A) + \mu(\emptyset)$$

Subtracting $\mu(A)$ from both sides of the equations then yields

$$\mu(\emptyset) = 0$$

Moreover, if $A \subseteq B$, then notice that $B = A \cup B \backslash A$, so that

$$\mu(B) = \mu(A \cup B \backslash A) = \mu(A) + \mu(B \backslash A)$$

So that

$$\mu(A) \leq \mu(B) \tag{1}$$

Now, in the case where $\mu(B)$ is infinite, we must also have that $\mu(A)$ is infinite, in which case we still get the inequality.

Example 2. Let R a set, and $\{x_n\}$ a sequence of points of R. Let $\{p_n\}$ a sequence of nonnegative numbers, and let $\mathcal{C} \subseteq 2^R$. Define the function $\mu : \mathcal{C} \to \mathbb{R}^+_\infty$ by $\mu(x) = \sum p_n x_n$ Then μ is additive.

Definition. Let R a set and $\mathcal{C} \subseteq 2^R$ with $\emptyset \in \mathcal{C}$. We say that a function $\mu : \mathcal{C} \to \mathbb{R}^+_\infty$ is **σ-additive** if for any countable disjoint collection $\{E_n\}$, we have

$$\mu(\bigcup E_n) = \sum \mu(E_n)$$

Example 3. Let $R = (0, 1)$ and define

$$\mathcal{C} = \{(a, b] : 0 \leq a < b < 1\} \cup \{\emptyset\}$$

and define $\mu : \mathcal{C} \to \mathbb{R}^+_\infty$ by

$$\mu((a, b]) = \begin{cases} b - a, \text{ if } a = 0 \\ \infty, \text{ if } a > 0 \end{cases}$$

Then μ is additive, but it is not σ-additive. Let $\{x_n\} \to 0$ a decreasing sequence, with $x_1 = \frac{1}{2}$. Then take

$$(0, \frac{1}{2}] = \bigcup (x_{i+1}, x_i]$$

Definition. Let $\mathcal{C}$ a collection of subsetes of a set R. We say a function $\mu : \mathcal{C} \to \mathbb{R}^+_\infty$ is **continuous from below** at E provided that for any increasing seqeuence $\{E_n\}$ of subsets of R, we have $\{\mu(E_n)\} \to \mu(E)$ whenever $\{E_n\} \to E$.

Definition. Let $\mathcal{C}$ a collection of subsetes of a set R. We say a function $\mu : \mathcal{C} \to \mathbb{R}^+_\infty$ is **continuous from above** at E provided that for any decreasing seqeuence $\{E_n\}$ of subsets of R, we have $\{\mu(E_n)\} \to \mu(E)$ whenever $\{E_n\} \to E$ and where $\mu(E_{n_0})$ if finite for some $n_0 \in \mathbb{Z}^+$.

Definition. Let $\mathcal{C}$ a collection of subsetes of a set R. We say a function $\mu : \mathcal{C} \to \mathbb{R}^+_\infty$ is **continuous** at E provided it is continuous from below at E, and continuous from above at E.

Lemma 7. *Let S an algebra on R and $\mu : S \to \mathbb{R}^+_\infty$ be additive. Then the following are true*

(1) If μ is σ-additive, then it is continuous at E for all $E \in S$.

(2) If μ is continuous from below, then μ is σ-additive.

(3) If μ is continuous from above at $\emptyset$, with $\mu(R)$ finite, then it is σ-additive.

Proof. (1) Suppose that μ is sigma additive. Let $\{E_n\}$ an increasing sequence of subsets of R convergine to E. Let $F_1 = E_1$, $F_2 = E_2 \backslash E_1$, $\ldots$, $F_n = E_n \backslash E_{n-1}$, in accordance to figure 2. Notice then that F_k is a disjoint collection with $\bigcup F_k = \bigcup E_n = E$. Then

$$\mu(E) = \sum \mu(F_k) = \lim_{n \to \infty} \sum_{k=1}^{n} \mu(F_k) = \lim \mu(F_k) = \lim \mu(E_n)$$

Thus μ is continuous from below at E.

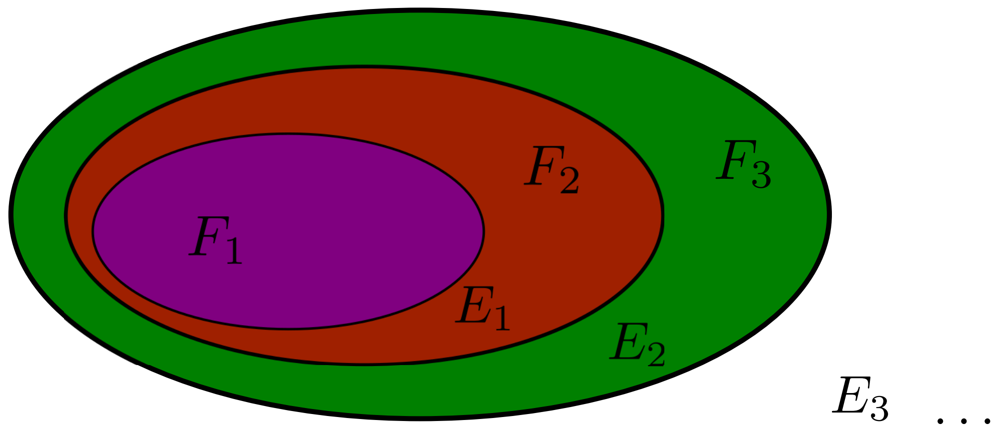

Figure 2

Now let $E \in S$ and let $\{E_n\}$ a decreasing sequence converging to E, and which $\mu(E_{n_0})$ is finite for some $n_0 \in \mathbb{Z}^+$. In similar manner to what was defined for figure 2, define $G_1 = E_{n_0} \backslash E_{n_0+1}, \ldots, G_k = E_{n_0} \backslash E_{n_0+k}$. Then $\{G_k\}$ is an increasing sequence of disjoint subsets of R with $\{G_k\} \to E_{n_0} \backslash E$. Now, since μ is σ-additive, and continuous from below, we get that $\{\mu(G_k)\} \to \mu(E_{n_0} \backslash E)$. So we have that

$$\begin{aligned} \mu(E_{n_0} \backslash E) &= \lim \mu(E_{n_0} \backslash E_{n_0+k}) \\ &= \lim \mu(E_{n_0}) - \lim \mu(E_{n_0+k}) \end{aligned}$$

then

$$\mu(E_{n_0}) - \mu(E) = \mu(E_{n_0}) - \lim \mu(E_{n_0+k})$$

Subtracting and rearranging terms gives us the result.

(2) Suppose that μ is continuous from below. Let $E = \bigcup E_k$ a disjoint union. Then notice that

$$\bigcup_{k=1}^{n} \subseteq E$$

so that

$$\begin{aligned} &\mu(\bigcup_{k=1}^{n} E_k) \leq \mu(E), \text{ so that} \\ &\sum_{k=1}^{n} \mu(E_k) \leq \mu(E), \text{ then as } n \to \infty \\ &\sum \mu(E_k) \leq \mu(E) \end{aligned}$$

Now, let $F = \bigcup_{k=1}^{n} E_k \in S$. Then $\{F_n\} \to E$ is an increasing sequence, and since μ is continuous from below, we have that $\{\mu(F_n)\} \to \mu(E)$ so that

$$\sum_{k=1}^{n} \mu(F_k) \to \mu(E)$$

making $\mu(E) \leq \sum \mu(E_k)$, making μ σ-additive.

(3) Lastly, suppose that μ is continuous from below at $\emptyset$, with $\mu(R)$ finite. Let $\{E_k\}$ be an increasing sequence and $F_n = \bigcup_{k \geq n} E_k$ which defines a decreasing sequence $\{F_n\} \to \emptyset$. By finiteness of $\mu(R)$, we have that $\mu(F_1)$ is also finite, and by continuity, that $\{m(F_n)\} \to 0$. Now,

$$\begin{aligned} \mu(E) &= \mu(\bigcup_{k=1}^{n} E_k \cup \bigcup_{k \geq n} E_k) \\ &= \sum_{k=1}^{n} \mu(E_k) + \mu(F_{n+1}) \end{aligned}$$

Now, since $\{F_n\} \to 0$, we get that

$$\mu(E) = \sum \mu(E_k)$$

■

Example 4. Refering to example 3, we have that μ is additive, but not σ-additive. Take $\{E_n\} \to \emptyset$ decreasign, with $E_n = (a_{n_1}, b_{n_1}] \cup (a_{n_k}, b_{n_k}]$, with $a_{n_j} < a_{n_{j+1}}$. If $a_{n_1} = 0$, for all n, then we have $\mu((a_{n_1}, b_{n_1}])$ is infinite, and there is nothing to prove. Now, if $a_{n_{k_0}} > 0$ for all k_0, then μ is continuous from below at $\emptyset$, but μ is infinite.

Theorem 8. *Let S be a semialgebra on R and $\mu : S \to \mathbb{R}^+_\infty$ additive. Then there exists a function $\nu : Q(S) \to \mathbb{R}^+_\infty$ which is additive such that*

(1) $\nu(A) = \mu(A)$ for all $A \in S$.

(2) ν is unique.

where $Q(S)$ is the algebra generated by S.

Proof. Recall that if $A \in Q(S)$, then

$$A = \bigcup_{k=1}^{n} E_k$$

where $\{E_k\}_{k=1}^{n}$ is a finite disjoint collection of elements of S.

Define $\nu : Q(S) \to \mathbb{R}_\infty^+$ by

$$\nu(A) = \sum \nu(E_n) \text{ where } \nu(E_n) = \mu(S_n) \text{ for all } E_n \in S$$

Let $A = \bigcup_{i=1}^{m} F_i$, then $\nu(A) = \sum_{i=1}^{m} \mu(F_k) = \sum_{k=1}^{n} \mu(E_k)$. Since μ is additive on S, and $E_k \subseteq S$, then

$$E_k = E_k \cap \bigcup_{i=1}^{m} F_i = \bigcup_{i=1}^{m} (E_k \cap F_i)$$

Then

$$\mu(E_k) = \sum \mu(E_k \cap F_i)$$

So that

$$\nu(A) = \sum_{k=1}^{n} \sum_{i=1}^{m} \mu(E_k \cap F_i)$$

which makes $\sum \mu(F_i) = \sum \mu(E_k)$. Therefore ν is well defined.

Now let $A, B \in Q(S)$. Then $A = \bigcup_{i=1}^{n} E_i$ and $B = \bigcup_{j=1}^{m} F_j$ with A and B disjoint. Then we have that

$$\nu(A \cup B) = \mu(A \cup B) = \mu(A) + \mu(B) = \nu(A) + \nu(B)$$

So that ν is additive.

Lastly, suppose that $\nu' : Q(S) \to \mathbb{R}_\infty^+$ is additive such that $\nu'(A) = \mu(A)$ for any $A \in S$. Since $\nu(A) = \mu(A)$, take $B \in Q(S)$ so that $B = \bigcup_{k=1}^{n} E_k$. Then $\nu(B) = \sum \nu(E_k) = \sum \mu(E_k)$ and $\nu'(B) = \sum \nu'(E_k) = \sum \mu(E_k)$ so that $\nu(B) = \nu'(B)$ for all $B \in S$. ■

Corollary. *If μ is σ-additive, then so is ν.*

Lecture 3: Caratheodory's Theorem