MATE6261-0U1
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Analysis I

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Lecture 1: Review

We begin with a review of some prelimanary results of set theory and advanced calculus.

Definition. Let A and B be sets, and let $f : A \to B$ be a map of A into B. We say that f is **1–1** if for every $x, y \in X$, $f(x) = f(y)$ implies $x = y$. We say f is **onto** if for every $y \in B$, there is an $x \in A$ for which $y = f(x)$. We say that f is a **1–1 correspondence** of A **onto** B if f is both 1–1 and onto and onto.

Theorem 1 (Beinstern's Theorem)**.** *Let A and B be sets. If there is a 1–1 map of A into B, and a 1–1 map of B into A, then there exists a 1–1 correspondence of A onto B.*

Axiom 1 (The Axiom of Choice)**.** *Suppose that $\{A_\alpha\}_{\alpha \in \Lambda}$ is a collection of nonmepty sets indexed by the set Λ. Then there exists a map $f : \Lambda \to \bigcup A_\alpha$ called a **choice function**, defined by the rule $f(\alpha) \in A_\alpha$ for all $\alpha \in \Lambda$.*

Remark. What this axiom says is that given any (not necesarrily countable) collection of sets, one can "choose " an element from each set.

Definition. We define an **order** on a set A to be a relation $<$ satisfying the following properties for all $a, b, c \in A$:.

(1) $a < a$ (Reflexive).

(2) If $a < b$ and $b < a$, then $a = b$ (Antisymmetry).

(3) If $a < b$ and $b < c$, then $a < c$ (Transitivity).

We say that elements $a, b \in A$ are **comparable** under $<$ if either $a < b$ or $b < a$. If every element of A is comparable, then $<$ is called a **total order**.

Definition. Let A be a set with order $<$. We call an element $x \in A$ a **maximum** of A if $x < a$ implies $x = a$ for all $a \in A$. If $B \subseteq A$, then we call an element $a \in A$ an **upper bound** of B if $b < a$ for all B in B; and we say B is **bounded above**. If $b < a'$ implies that $a < a'$ for any $a' \in A$, then we call a a **least upper bound** of B and write $\sup B = a$.

Definition. Let A be a set with order $<$. We call an element $x \in A$ a **minimum** of A if $a < x$ implies $x = a$ for all $a \in A$. If $B \subseteq A$, then we call an element $a \in A$ an **lower bound** of B if $a < b$ for all B in B; and we say B is **bounded below**. If $a' < b$ implies that $a' < a$ for any $a' \in A$, then we call a a **greatest lower bound** of B and write $\inf B = a$.

Theorem 2 (Zorn's Lemma)**.** *Let A be a set with order $<$. If every totally ordered subset of A under $<$ has an upperbound, then A has a maximum element.*

Remark. It can be shown that Zorn's lemma and the axiom of choice are equivalent statements. That is you can prove Zorn's lemma from the axiom of choice, and you can prove the axiom of choice from Zorn's lemma.

Theorem 3. *For any sets A and B, there is either a 1–1 map of A into B, or a 1–1 map of B into A.*

Proof. Consider the collection of all triples $C = \{(X, Y, f) : f : X \to Y$ is 1–1 and onto, where $X \subseteq A, Y \subseteq B\}$, and define an order $<$ on C by: $(X, Y, f) < (X', Y', g)$ if, and only if $X \subseteq X'$, $Y \subseteq Y'$, and $g|_X = f$. Now suppose that the collection $\{(X_\alpha, Y_\alpha, f_\alpha)\}$ is a totally ordered subset of C. Then define

$$\tilde{X} = \bigcup_\alpha X_\alpha$$
$$\tilde{Y} = \bigcup_\alpha Y_\alpha$$

and since for all $x \in \tilde{X}$, there is an α for which $x \in X_\alpha$, define the map $\tilde{f} : \tilde{X} \to \tilde{Y}$ by the rule $\tilde{f}(x) = f_\alpha(x) \in Y_\alpha$. This map is well defined by the total order on $\{(X_\alpha, Y_\alpha, f_\alpha)\}$.

Now, notice that $\tilde{f}$ is an upperbound of the collection $\{(X_\alpha, Y_{\alpha f \alpha})\}$, therefore, by Zorn's lemma there is a maximum element (X, Y, f) of C. By definition we get that f is a 1–1 correspondence of X onto Y, and by maximality, we get that either $X = A$ or $Y = B$. Suppose on the contary that this is not true. Then there is an $a \in A \backslash X$ and a $bB \backslash Y$. Letting $X' = X \cup a$ and $Y' = Y \cup b$, and $f' : X' \to Y'$ defined by $f'|_X = f$ and $f'(a) = b$. Then $(X, Y, f) < (X', Y', f')$, which contradicts the maximality of (X, A, f).

Therefore, we must have that either $X = A$ or $Y = B$. Now, if $X = A$, then $f : A \to Y$ is 1–1 and onto, taking the extension of Y to B, $f_B : A \to B$, we see it must be 1–1 as well. By the same reasoning we can assure there is a 1–1 map $f_A : B \to A$ of B into A. ■

Lecture 2: Review