

MATE6261-0U1
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Analysis I

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Lecture 1: Review

We begin with a review of some preliminary results of set theory and advanced calculus.

Definition. Let A and B be sets, and let $f : A \rightarrow B$ be a map of A into B . We say that f is **1–1** if for every $x, y \in X$, $f(x) = f(y)$ implies $x = y$. We say f is **onto** if for every $y \in B$, there is an $x \in A$ for which $y = f(x)$. We say that f is a **1–1 correspondence** of A **onto** B if f is both 1–1 and onto and onto.

Theorem 1 (Beinstern’s Theorem). *Let A and B be sets. If there is a 1–1 map of A into B , and a 1–1 map of B into A , then there exists a 1–1 correspondence of A onto B .*

Axiom 1 (The Axiom of Choice). *Suppose that $\{A_\alpha\}_{\alpha \in \Lambda}$ is a collection of nonempty sets indexed by the set Λ . Then there exists a map $f : \Lambda \rightarrow \bigcup A_\alpha$ called a **choice function**, defined by the rule $f(\alpha) \in A_\alpha$ for all $\alpha \in \Lambda$.*

Remark. What this axiom says is that given any (not necessarily countable) collection of sets, one can “choose” an element from each set.

Definition. We define an **partial order** on a set A to be a relation $<$ satisfying the following properties for all $a, b, c \in A$:

- (1) $a < a$ (Reflexive).
- (2) If $a < b$ and $b < a$, then $a = b$ (Antisymmetry).
- (3) If $a < b$ and $b < c$, then $a < c$ (Transitivity).

We say that elements $a, b \in A$ are **comparable** under $<$ if either $a < b$ or $b < a$. If every element of A is comparable, then $<$ is called a **total order**.

Definition. Let A be a set with partial order $<$. We call an element $x \in A$ a **maximum** of A if $x < a$ implies $x = a$ for all $a \in A$. If $B \subseteq A$, then we call an element $a \in A$ an **upper bound** of B if $b < a$ for all B in B ; and we say B is **bounded above**. If $b < a'$ implies that $a < a'$ for any $a' \in A$, then we call a a **least upper bound** of B and write $\sup B = a$.

Definition. Let A be a set with partial order $<$. We call an element $x \in A$ a **minimum** of A if $a < x$ implies $x = a$ for all $a \in A$. If $B \subseteq A$, then we call an element $a \in A$ an **lower bound** of B if $a < b$ for all B in B ; and we say B is **bounded below**. If $a' < b$ implies that $a' < a$ for any $a' \in A$, then we call a a **greatest lower bound** of B and write $\inf B = a$.

Theorem 2 (Zorn's Lemma). *Let A be a set with partial order $<$. If every totally ordered subset of A under $<$ has an upperbound, then A has a maximum element.*

Remark. It can be shown that Zorn's lemma and the axiom of choice are equivalent statements. That is you can prove Zorn's lemma from the axiom of choice, and you can prove the axiom of choice from Zorn's lemma.

Theorem 3. *For any sets A and B , there is either a 1-1 map of A into B , or a 1-1 map of B into A .*

Proof. Consider the collection of all triples $C = \{(X, Y, f) : f : X \rightarrow Y \text{ is 1-1 and onto, where } X \subseteq A, Y \subseteq B\}$, and define a partial order $<$ on C by: $(X, Y, f) < (X', Y', g)$ if, and only if $X \subseteq X'$, $Y \subseteq Y'$, and $g|_X = f$. Now suppose that the collection $\{(X_\alpha, Y_\alpha, f_\alpha)\}$ is a totally ordered subset of C . Then define

$$\begin{aligned}\tilde{X} &= \bigcup_{\alpha} X_{\alpha} \\ \tilde{Y} &= \bigcup_{\alpha} Y_{\alpha}\end{aligned}$$

and since for all $x \in \tilde{X}$, there is an α for which $x \in X_\alpha$, define the map $\tilde{f} : \tilde{X} \rightarrow \tilde{Y}$ by the rule $\tilde{f}(x) = f_\alpha(x) \in Y_\alpha$. This map is well defined by the total order on $\{(X_\alpha, Y_\alpha, f_\alpha)\}$.

Now, notice that \tilde{f} is an upperbound of the collection $\{(X_\alpha, Y_\alpha, f_\alpha)\}$, therefore, by Zorn's lemma there is a maximum element (X, Y, f) of C . By definition we get that f is a 1-1 correspondence of X onto Y , and by maximality, we get that either $X = A$ or $Y = B$. Suppose on the contrary that this is not true. Then there is an $a \in A \setminus X$ and a $b \in B \setminus Y$. Letting $X' = X \cup a$ and $Y' = Y \cup b$, and $f' : X' \rightarrow Y'$ defined by $f'|_X = f$ and $f'(a) = b$. Then $(X, Y, f) < (X', Y', f')$, which contradicts the maximality of (X, A, f) .

Therefore, we must have that either $X = A$ or $Y = B$. Now, if $X = A$, then $f : A \rightarrow Y$ is 1-1 and onto, taking the extension of Y to B , $f_B : A \rightarrow B$, we see it must be 1-1 as well. By the same reasoning we can assure there is a 1-1 map $f_A : B \rightarrow A$ of B into A . ■

Lecture 2: Smallest Infinite Cardinality

Definition. A set A is **countably infinite** if there exists a 1-1 map $f : \mathbb{N} \rightarrow A$ of \mathbb{N} onto A .

Remark. That is we can put A into countable order and represent it as an increasing sequence $A = \{a_n\}_{n \in \mathbb{N}}$ where each a_n is distinct for all n .

Example 1. Both \mathbb{N} and $2\mathbb{N}$ are countable. Consider the identity map $i : \mathbb{N} \rightarrow \mathbb{N}$ and the map $f : \mathbb{N} \rightarrow 2\mathbb{N}$ defined by $n \rightarrow 2n$.

Theorem 4. A set A is finite if, and only if there is no 1-1 map of A onto a subset of itself.

Corollary. Every infinite set contains a countable subset.

Theorem 5. A countable union of sets is countable.

Proof. Let A and B be nonempty countable sets. Then there exists nonrepeating increasing sequences $A = \{a_n\}$ and $B = \{b_m\}$. Then take $A \cup B = \{a_n, b_m\}$, and delete any repeated members. Then we can say that $A \cup B = \{c_k\}$ where $c_k = a_k$ for k odd, and $c_k = b_k$ for k even. Then $A \cup B$ is countable.

Now suppose that $\{A_n\}$ is a collection of countable sets. Then we have the following:

$$\begin{aligned} A_1 &= a_{11} \ a_{12} \ \dots \ a_{1n} \ \dots \\ A_2 &= a_{21} \ a_{22} \ \dots \ a_{2n} \ \dots \\ A_3 &= a_{31} \ a_{32} \ \dots \ a_{3n} \ \dots \\ &\vdots \\ A_n &= a_{n1} \ a_{n2} \ \dots \ a_{nn} \ \dots \\ &\vdots \end{aligned}$$

Construct the set:

$$A = \{a_{11}, a_{12}, a_{21}, a_{13}, a_{22}, a_{31}, \dots\}$$

Then A is countable and $A = \bigcup A_n$. ■

Example 2. (1) $-\mathbb{N}$ is countable, then $-\mathbb{N} \cup \mathbb{N} = \mathbb{Z}$ is also countable, which makes \mathbb{Z} countable.

(2) Consider the rational numbers $\mathbb{Q} = \{\frac{p}{q} : p, q \in \mathbb{Z}\}$. Let $\frac{\mathbb{Z}}{n} = \{\frac{p}{n} : p \in \mathbb{Z}\}$ for some $n \in \mathbb{N} \setminus \{0\}$. Then notice that $\frac{\mathbb{Z}}{n}$ is countable and that $\mathbb{Q} = \bigcup \frac{\mathbb{Z}}{n}$, which makes \mathbb{Q} countable as well.

Lemma 6. *Let A and B be two countably finite sets. Then $A \times B$ is countable.*

Proof. Take $A \times B = (\bigcup A_n) \times \{b_n\}$, where $b_n \in B$. ■

Example 3. (1) Let P be the set of all polynomials with integer coefficients. Let P_i be the set of all polynomials of degree $\deg = i$. Then we have the following:

$$\begin{aligned} P_0 &\simeq \mathbb{Z} \\ P_1 &\simeq \mathbb{Z} \times \mathbb{Z} \\ &\vdots \\ P_n &\simeq \mathbb{Z} \times \mathbb{Z} \times \cdots \times \mathbb{Z} \\ &\vdots \end{aligned}$$

Then we have that $P \simeq \bigcup P_i$. Since each P_i is countable, so is $\bigcup P_i$, so by isomorphism, P must also be countable.

(2) Let $\mathcal{A} = \{x \in \mathbb{C} : \text{there exists a polynomial with integer coefficients } p \text{ such that } p(x) = 0\}$ be the set of all algebraic numbers over \mathbb{C} . Then we have $\mathbb{Q} \subseteq \mathcal{A}$, since the polynomial $z^2 + 1 = 0$ has complex roots. \mathcal{A} is countable.

Theorem 7. *Let A be a set, then $|A| < |2^A|$.*

Proof. We have that by definition, $|A| > |2^A|$ is impossible, so suppose that $|A| = |2^A|$. Then there exists a 1-1 map $f : A \rightarrow 2^A$ of A onto its powerset. Then we have that for each $a \in A$, $f(a) \subseteq A$. Now, let $A_0 = \{x \in A : x \notin f(x)\}$. Since f is onto, there is an $x_0 \in A$ with $f(x_0) = A_0$. Now, if $x \in f(x_0)$, we get that $x \in A_0$, which means that $x_0 \notin f(x_0)$. This cannot happen; so it must be that $x_0 \notin f(x_0)$, but then we get that $x_0 \in A_0$, which forces $x_0 \in f(x_0)$. A contradiction! Therefore it must be that $|A| < |2^A|$ ■

Example 4. $|\mathbb{N}| < |2^{\mathbb{N}}|$

Now consider the set of real numbers \mathbb{R} . We have the following theorems.

Theorem 8. \mathbb{R} is a field.

Theorem 9. \mathbb{R} is a totally ordered set.

Corollary. The following are true for all $a, b, c \in \mathbb{R}$.

(1) $a > b$ implies that $a + c > b + c$.

(2) $a > b$ and $c > 0$ implies $ac > bc$.

(3) For all $a > 0$ and $b > 0$, there exists an $n \in \mathbb{N}$ such that $an > b$ (Property of Archimedes).

Theorem 10. \mathbb{R} has the least upperbound property. That is, if $\{a_n\}$ is an increasing sequence of real numbers bounded above by b , then $\sup \{a_n\} \in \mathbb{R}$.