Algebraic Geometry.

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Contents

Chapter 1

Affine Algebraic Sets

1.1 Affine n-Space and Algebraic Sets

Definition. Let k be a field. We define **affine n-space** over k to be the cartesian product $\mathbb{A}^n(k) = \underbrace{k \times \cdots \times k}_{n\text{-times}}$. If the field k is understood, we write $\mathbb{A}^n$. We call the elements of $\mathbb{A}^(k)$ **affine points**. We call $\mathbb{A}^1(k)$ and $\mathbb{A}^2(k)$ the **affine line** and **affine plane** over k, respectively.

Definition. Let k be a field, and let $f \in k[x_1, \ldots, x_n]$. We call an affine point $P \in \mathbb{A}^n(k)$ a **zero**, or **root** of f if $f(P) = 0$, where $f(P)$ is understood to be $f(a_1, \ldots, a_n)$, where $P = (a_1, \ldots, a_n)$. We call the set of zeros of f, $V(f)$ the **hypersurface** defined by f. We call hypersurfaces in $\mathbb{A}^2(k)$ **affine plane curves**. If $\deg f = 1$, we call $V(f)$ a **hyperplane**. We call hypersurfaces in $\mathbb{A}^1(k)$ **lines**.

Example 1.1. The following are algebraic curves.

Figure 1.1:

Definition. Let k be a field, and S any set of polynomials in $k[x_1, \ldots, x_n]$. We define the **set of zeros** of S to be the set $V(S) = \{P \in \mathbb{A}^n(k) : f(P) = 0 \text{ for all } f \in S\}$. We call a subset X of $\mathbb{A}^n(k)$ an **affine algebraic set** if $X = V(S)$ for some set S of polynomials.

Lemma 1.1.1. *The following are true for any field k.*

(1) If $\mathfrak{a}$ is an ideal in $k = [x_1, \ldots, x_n]$ generated by a set $S \subseteq k[x_1, \ldots, x_n]$, then $V(\mathfrak{a}) = V(S)$.

(2) If $\{\mathfrak{a}_\alpha\}$ is a collection of ideals of $k[x_1, \ldots, x_n]$, then

$$V\Big(\bigcup V(\mathfrak{a}_\alpha)\Big) = \bigcap V(\mathfrak{a}_\alpha)$$

(3) If $\mathfrak{a} \subseteq \mathfrak{b}$ are idelas, then $V(\mathfrak{b}) \subseteq V(\mathfrak{a})$.

(4) *If* $f, g \in k[x_1, \ldots, x_n]$*, then* $V(fg) = V(f) \cup V(g)$*.*

(5) $V(0) = \mathbb{A}^n(k)$ *and* $V(1) = \emptyset$*.*

Proof. ∎

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