Measure Theory

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Text

Real Analysis (4^{th} edition)

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Contents

Chapter 1

The Real Numbers

1.1 Open Sets, and σ-Algebras

Definition. We call a set U of real numbers **open** provided for any $x \in U$, there is an $r > 0$ such that $(x - r, x + r) \subseteq U$.

Lemma 1.1.1. *The set of real numbers* $\mathbb{R}$*, together with open sets defines a topology on* $\mathbb{R}$*.*

Proof. Notice that both $\mathbb{R}$ and $\emptyset$ are open sets. Moreover, if $\{U_n\}$ is a collection of open sets, then so is thier union. Now, consider the fintie collection $\{U_k\}_k = 1^n$ and let $U = \bigcap_{k=1}^n U_k$. If U is empty, we are done. Otherwise, let $x \in U$. Then $x \in U_k$ for every $1 \leq k \leq n$, and since each U_k is open, choose an $r_k > 0$ for which $(x - r_k, x + r_k) \subseteq U_k$. Then let $r = \min\{r_1, \dots, r_n\}$. Then $r > 0$, and we have $(x - r, x + r) \subseteq U$, which makes U open in $\mathbb{R}$. ■

Lemma 1.1.2. *Every nonempty set is the disjoint union of a countable collection of open sets.*

Proof. Let U be nonempty and open in $\mathbb{R}$. LEt $x \in U$. Then there is a $y > x$ for which $(x, y) \subseteq U$ and there is a $z < x$ for which $(z, x) \subseteq U$. Now, let $a_x = \inf\{z : (z, x) \subseteq U\}$ and $b_x = \sup\{y : (x, y) \subseteq U\}$, and let $I_x = (a_x, b_x)$. Then we have

$$x \in I_x \text{ and } a_x \notin I_x \text{ and } b_x \notin I_x$$

Let $w \in I_x$ such that $x < w < b_x$. Then there is a $y > w$ such that $(x, y) \subseteq U$ so that $w \in U$. Now, if $b_x \in U$, then there is an $r > 0$ for which $(b_x - r, b_x + r) \subseteq U$, in particular, $(x, b_x + r) \subseteq U$. But b_r is the least upperbound of all such numbers, and $b_x < b_x + r$, a contradiction. Thus $b_x \notin U$, and hence $b_x \notin I_x$. A similar argument shows that $a_x \notin I_x$.

Consider now the collection $\{I_x\}_{x \in U}$. Then $U = \bigcup I_x$ and since $a_x, b_x \notin I_x$ for each x, the collection $\{I_x\}$ is a disjoint collection. Lastly, by the density of $\mathbb{Q}$ in $\mathbb{R}$ there is a 1–1 mapping between this collection and $\mathbb{Q}$, making it countable. ■

Definition. Let $E \subseteq \mathbb{R}$ a set. We call a point $x \in \mathbb{R}$ a **point of closure** of E if every open interval containing x also contains a point of E. We call the collection of all such points the **closure** of E, and denote it $\operatorname{cl} E$. If $E = \operatorname{cl} E$, then we say that E is **closed**.

Lemma 1.1.3. *For any set E of real numbers, $\mathrm{cl}\,E$ is closed; i.e. $\mathrm{cl}\,E = \mathrm{cl}\,(\mathrm{cl}\,E)$. Moreover, $\mathrm{cl}\,E$ is the smallest closed set containing E.*

Lemma 1.1.4. *Every set E of rea numbers is open if, and only if $\mathbb{R} \backslash E$ is closed.*

Definition. Let $E \subseteq \mathbb{R}$ a set. We call a collection $\{E_\lambda\}$ a **cover** of E if $E \subseteq \bigcup E_\lambda$. If each E_λ is open, then we call this collection an **open cover** of E.

Theorem 1.1.5 (Heine-Borel)**.** *For any closed and bounded set F of $\mathbb{R}$, every open cover of F has a finite subcover.*

Proof. Suppose first that $F = [a, b]$, for $a \leq b$ real numbers. Then F is closed and bounded. Let $\mathcal{F}$ be an open cover of $[a, b]$, and deifne $E = \{x \in [a, b] : [a, x] \text{ has a finite subcover by sets in } \mathcal{F}\}$. Notice that $a \in E$, so that E is nonempty. Now, since E is bounded by b, by the completeness of $\mathbb{R}$, let $c = \sup\{E\}$. Then $c \in [a, b]$ and there is a set $U \in \mathcal{F}$ with $c \in U$. Since U is open, there exists an $\varepsilon > 0$ such that $(c - \varepsilon, c + \varepsilon) \subseteq U$. Now, $c - \varepsilon$ is not an upperbound of E, so there is an $x \in E$ with $c - \varepsilon < x$, and a finite collection of open sets $\{U_i\}_{i=1}^k$ covering $[a, x]$. Then the collection $\{U_i\}_{i=1}^k \cup U$ covers $[a, x]$ so that $c = b$, and we have found a finite subcover of F.

Now, let F be closed and bounded. Then it is contained in a closed bounded interval $[a, b]$. Now, let $U = \mathbb{R} \backslash F$ open and $\mathcal{F}$ an open cover of F. Let $\mathcal{F}' = \mathcal{F} \cup U$. Since $\mathcal{F}$ covers F, $\mathcal{F}'$ covers $[a, b]$. By above, there is a finite subcover of $[a, b]$, and hence of F by sets in $\mathcal{F}'$. Removine U from $\mathcal{F}'$, we get a finite subcover of F by sets in $\mathcal{F}$. ■

Theorem 1.1.6 (The Nested Set Theorem)**.** *Let $\{F_n\}$ be a descending collection of nonempty closed sets of $\mathbb{R}$, for which F_1 is bounded. Then*

$$\bigcap F_n \neq \emptyset$$

Proof. Let $F = \bigcap F_n$, and suppose to the contrary that F is empty. Then for all $x \in \mathbb{R}$, there is an $n \in \mathbb{Z}^+$ for which $x \notin F_n$. So that $x \in U_n = \mathbb{R} \backslash F_n$. TYhen $U_n = \mathbb{R}$, and each U_n is open. So $\{U_n\}$ is an open cover of $\mathbb{R}$, and hence F_1. By the theorem of Heine-Borel, there is an $N > 0$ such that $F \subseteq \bigcup_{n=1}^N U_n$. Since $\{F_n\}$ is descending, the collection $\{U_n\}$ is ascending, and hence $\bigcup U_n = U_N = \mathbb{R} \backslash F_N$ which makes $F_1 \mathbb{R} \backslash F_N$, a contradiciton. ■

Definition. Let X be a set. We call a collection $\mathcal{A}$ of subsets of X **σ-algebra** if

(1) $\emptyset \in \mathcal{A}$.

(2) For any $A \in \mathcal{A}$, $X \backslash A \in \mathcal{A}$.

(3) If $\{A_n\}$ is a countable collection of elements of $\mathcal{A}$, then their union is an element of $\mathcal{A}$.

Lemma 1.1.7. *Let $\mathcal{F}$ a collection of subsets of a set X. The intersection of all σ-algebras containing $\mathcal{F}$ is a σ-algebra. Moreover, it is the smallest such σ-algebra.*

Definition. We define the **Borel sets** of $\mathbb{R}$ to be the σ-algebra of $\mathbb{R}$ cotnaining all open sets in $\mathbb{R}$

Lemma 1.1.8. *Every closed set of $\mathbb{R}$ is a Borel set.*

Definition. We call a countable intersection of open sets of $\mathbb{R}$ a **G_δ-set** and we call a countable union of closed sets of $\mathbb{R}$ an **F_σ-set**.

1.2 Sequences of Real Numbers

Definition. A sequence $\{a_n\}$ of real numbers is said to **converge** to a point a, if, for every $\varepsilon > 0$, there is an $N > 0$ such that

$$|a - a_n| < \varepsilon \text{ whenever } n \geq N$$

We call a the **limit** of $\{a_n\}$ and write $\{a_n\} \to a$, or

$$\lim_{n \to \infty} \{a_n\} = a$$

Lemma 1.2.1. *Let $\{a_n\} \to a$ a sequence of real numbers converging to $a \in \mathbb{R}$. Then the limit of $\{a_n\}$ is unique, $\{a_n\}$ is bounded, and for any $c \in \mathbb{R}$, if $a_n \leq c$ for all n, then $a \leq c$.*

Theorem 1.2.2 (The Monoton CVonvergence Theorem)**.** *A monotone sequence of real numbers converges to a point if, and only if it is bounded.*

Proof. Without loss of generality, suppose that the sequence $\{a_n\}$ is increasing. If $\{a_n\} \to a$, by lemma 1.2.1, $\{a_n\}$ is bounded. On the otherhand, suppose that $\{a_n\}$ is bounded. Let $S = \{a_n : n \in \mathbb{Z}^+\}$, then by the completeness of $\mathbb{R}$, let $a = \sup S$. Let $\varepsilon > 0$. Notice that $a_n \leq a$ for all n. Now, since $a - \varepsilon$ is not an upperbound, there exists an $N > 0$ for which $a_N > a - \varepsilon$, then since $\{a_n\}$ is increasing, $a_n > a - \varepsilon$ whenever $n \geq N$. So we get

$$|a - a_n| < \varepsilon \text{ whenever } n \geq N$$

Which makes $\{a_n\} \to a$. ■

Theorem 1.2.3 (Bolzano-Weierstrass)**.** *Every bounded sequence has a convergent subsequence.*

Proof. Let $\{a_n\}$ be a bounded sequence, and let $M > 0$ such that $|a_n| \geq M$ for all $n \in \mathbb{Z}^+$. Define $E_n = \operatorname{cl}\{a_j : j \geq n\}$. Then $EE \subseteq [-M, M]$. Thus $\{E_n\}$ is a decreasing sequence of closed, bounded, and nonempty sets of $\mathbb{R}$. By the nested set theorem, the intersection $E = \bigcap E_n$ is nonempty. Choose an $a \in E$. THen for every $k \in \mathbb{Z}^+$, a is a point of closure of the set $\{a_j : j \geq k\}$. SO that $a_j \in (a - \frac{1}{k}, a + \frac{1}{k})$ whenever $j \geq k$. By induction, construct a strictly increasing sequence $\{n_k\}$ of natural numbers for which $|a - a_{n_k}| < \varepsilon$. Thenb by the principle of Archimedes, $\{a_{n_k}\} \to a$, and we have a convergent subsequence. ■

Definition. We call a sequence $\{a_n\}$ **Cauchy** if for every $\varepsilon > 0$, there is an $N > 0$ for which

$$|a_m - a_n| < \varepsilon \text{ whenever } m, n \geq N$$

Theorem 1.2.4 (The Cauchy Convergence Criterion)**.** *A sequence of real numbers converges if, and only if it is Cauchy.*

Proof. Suppose that the sequence $\{a_n\} \to a$ converges to $a \in \mathbb{R}$. Then for any $m, n \in \mathbb{Z}^+$, notice that $|a_m - a_n| \leq |a_m - a| + |a - a_n|$. Let $\varepsilon > 0$ and choose $N > 0$ such that $|a - a_n| < \frac{\varepsilon}{2}$, and $|a_m - a| < \frac{\varepsilon}{2}$. Then if $n, m \geq N$, we get $|a_m - a_n| < \varepsilon$, which makes $\{a_n\}$ Cauchy.

Conversely, suppose that $\{a_n\}$ is Cauchy. Let $\varepsilon = 1$ and choose $N > 0$ such that if $m, n \geq N$, then $|a_m - a_n| < 1$. Then we get $|a_n| \leq 1 + |a_N|$ for all $n \geq N$. Define $M = 1 + \max\{|a_1|, \dots, |a_N|\}$. Then $|a_n| \leq M$ for all n. This makes $\{a_n\}$ bounded. By the theorem of Bolzano-Weierstrass, $\{a_n\}$ has a convergent subsequence $\{a_{n_k}\} \to a$. Let $\varepsilon > 0$, since $\{a_n\}$ is Cauchy, choose an $N > 0$ such taht $|a_m - a_n| < \frac{\varepsilon}{2}$ whenever $n, m \geq N$. Likewise, we get $|a - a_{n_k}| < \frac{\varepsilon}{2}$ and $n_k \geq N$. Thus we observe that $|a_n - a| \leq |a_n - a_{n_k}| + |a - a_{n_k}| < \varepsilon$ and so $\{a_n\} \to a$. ∎

Theorem 1.2.5. *Let $\{a_n\} \to a$ and $\{b_n\} \to b$ be convergent sequences. Then for any $\alpha, \beta \in \mathbb{R}$, we have that the sequence $\{\alpha a_n + \beta b_n\}$ converges and that*

$$\lim_{n \to \infty} \{\alpha a_n + \beta b_n\} = \alpha a + \beta b$$

Definition. We say a sequence $\{a_n\}$ of real numbers **converges to infinity** $\infty \in \mathbb{R}_\infty$ if for every $c \in \mathbb{R}$, there is an $N > 0$ such that $a_n \geq c$ whenever $n \geq N$. We write $\{a_n\} \to \infty$, or

$$\lim_{n \to \infty} \{a_n\} = \infty$$

Definition. Let $\{a_n\}$ be a sequence of real numbers. We define the **limit superior** of $\{a_n\}$ to be

$$\limsup \{a_n\} = \lim_{n \to \infty} (\sup \{a_k : k \geq n\})$$

Similarly, we define the **limit inferiro** of $\{a_n\}$ to be

$$\liminf \{a_n\} = \lim_{n \to \infty} (\inf \{a_k : k \geq n\})$$

Theorem 1.2.6. *For any sequences $\{a_n\}$ and $\{b_n\}$ of real numbers, the following are true:*

(1) $\limsup \{a_n\} = l \in \mathbb{R}_\infty$ *if, and only if for every $\varepsilon > 0$, there exists infintiely many $n \in \mathbb{Z}^+$ such that $a_n > l - \varepsilon$ and finitely many $n \in \mathbb{Z}^+$ for which $a_n > l + \varepsilon$.*

(2) $\limsup \{a_n\} = \infty$ *if, and only if $\{a_n\}$ is not bounded above.*

(3) $\limsup \{a_n\} = -\liminf \{-a_n\}$

(4) $\{a_n\} \to a \in \mathbb{R}_\infty$ *if, and only if* $\limsup \{a_n\} = \liminf \{a_n\}$.

(5) If $a_n \leq b_n$ for all n, then $\limsup \{a_n\} \leq \limsup \{b_n\}$.

Definition. Let $\{a_n\}$ a sequence of real numbers. We call the series $\sum_{k=1}^{\infty} a_k$ **summable** if the sequence of partial sums $\{s_n = \sum_{k=1}^{n} a_k\} \to s$ converges to a point $s \in \mathbb{R}$.

Lemma 1.2.7. *Let $\{a_n\}$ a sequence of real numbers. Then the following are true.*

(1) The series $\sum a_k$ is summable if, and only if for every $\varepsilon > 0$, there is an $N > 0$ such that

$$|\sum_{k=n}^{n+m} a_k| < \varepsilon \text{ for all } m \in \mathbb{Z}^+ \text{ whenever } n \geq N$$

(2) If $\sum |a_k|$ is summable, then so is $\sum a_k$.

(3) If $a_k \geq 0$, then $\sum a_k$ is summable if, and only if the sequence of partial sums $\{s_n\}$ is bounded.

1.3 Continuous Functions of a Real Variable.

Definition. A realvalued function f on a domain E is said to be **continuous** at a point $x \in E$ provided for any $\varepsilon > 0$ there is a $\delta > 0$ for which

$$|f(x) - f(y)| < \varepsilon \text{ whenever } |x - y| < \delta \text{ for any } y \in E$$

We call f **continuous** on E if it is continuous at every point in E. We call f **Lipschitz continuous** if there is a $c \geq 0$ for which

$$|f(x) - f(y)| \leq c|x - y| \text{ for all } x, y \in E$$

Lemma 1.3.1. *A Lipschitz continuous function on a domain is continuous on that domain.*

Lemma 1.3.2 (The Sequential Criterion)**.** *A realvalued function f defined on a domain E is continous at a point $x \in E$ if, and only if for any ssequence $\{x_n\} \to x$ of points in E, converging to x, that the sequence $\{f(x_n)\} \to f(x)$ converges to $f(x)$.*

Theorem 1.3.3 (The Extreme Value Theorem)**.** *A continuous realvalued function defined on a nonempty, closed and bounded domain takes on a maximum value, and a minimum value on that domain.*

Proof. Let f be a continuous realvalued function defined on the domain E, where E is nonempty, closed, and bounded. Let $x \in E$ and $\delta > 0$ and $\varepsilon = 1$. Define the open interval $I_x = (x - \delta, x + \delta)$. Then if $y \in E \cap I_x$, then $|f(x) - f(y)| < 1$. So that $|f(y)| \leq |f(x)| + 1$. Notice also that the collection $\{I_x\}$ is an open cover of E. By the theorem of Heine-Borel, there is a finite subcover of E, $\{I_{x_k}\}_{k=1}^n$. Define, then, $M = 1 + \max\{|f(x_1)|, \ldots, |f(x_n)|\}$. Then we get that $|f(x)| \leq M$ and f is bounded.

Now, let $m = \sup f(E)$. If f does not take the value m for any points in E, then the function $x \to \frac{1}{f(x)-m}$ is a contoinuous unbounded function on E; which is impossible. So there is an $x \in E$ with $f(x) = m$ and m is a maximum value. Now, since f is continuous, then so is $-f$, and hence $-m$ defines a minimum value on f. ∎

Theorem 1.3.4 (The Intermediate Value Theorem)**.** *If f is a continuous realvalued function on a closed bounded interval $[a, b]$, for which $f(a) < c < f(b)$, then there exists an $x_0 \in (a, b)$ for which $f(x_0) = c$.*

Proof. Define $a_1 = a$ and $b_1 = b$ and let m_1 be the midpoint of the interval $[a_1, b_1]$. If $c < f(m_1)$, define $a_2 = a_1$ and $b_2 = m_1$, otherwise define $a_2 = m_1$ and $b_2 = m_1$, so that in either case we get $f(a_2) \leq c \leq f(b_2)$ and $b_2 - a_2 = \frac{b-a}{2}$. By induction, construct the collection of closde bounded intervals $\{[a_n, b_n]\}$ such that $f(a_n) \leq c \leq f(b_n)$ and $b_n - a_n = \frac{b-a}{2^{n-1}}$. This collection is a descending collection, so by the nested set theorem, the intersection $I = \bigcap [a_n, b_n]$ is nonempty. Choose an $x_0 \in I$, and observe that

$$|a_n - x_0| \leq b_n - a_n = \frac{b - a}{2^{n-1}}$$

So the sequence $\{a_n\} \to x_0$. By the sequential criterion, since f is continuous at x_0, we get the sequence $\{f(a_n)\} \to f(x_0)$. Since $f(a_n) \leq c$, and $(-\infty, c]$ is closed, we also get $f(x_0) \leq c$.

By similar reasoning to the argument provided above, we also get that $f(x_0) \geq c$ so that equality is established. ∎

Definition. A realvalued function f on a domain E is said to be **uniformly continuous** if for everyu $\varepsilon > 0$ there is a $\delta > 0$ such that

$$|f(x) - f(y)| < \varepsilon \text{ whenever } |x - y| < \delta \text{ for all } x, y \in E$$

Lemma 1.3.5. *If f is a uniformly continuous function on a domain E, then it is continuous on E.*

Theorem 1.3.6. *A continuous realvalued function on a closed and bounded domain is unifomrly continuous.*

Proof. Let f be continuous on E, and E a closed and bounded domain. Let $\varepsilon > 0$. For every $x \in E$, there is a $\delta_x > 0$ for which $|f(x) - f(y)| < \varepsilon$ whenever $|x - y| < \delta_x$ for some $y \in E$. Define $I_x = (x - \frac{\delta_x}{2}, x + \frac{\delta_x}{2})$. Then $\{I_x\}$ is an open cover for E, so that by the theorem of Heine-Borel, there is a finite subcover $\{I_{x_k}\}_{k=1}^n$ of E. Define $\delta = \frac{1}{2} \min\{\frac{\delta_{x_1}}{2}, \dots, \frac{\delta_{x_n}}{2}\}$. Then $\delta > 0$ moreover, if $x, y \in E$, with $|x - y| < \delta$, then asince $\{I_{x_k}\}$ covers E, there is a $k > 0$ such that

$$|x - x_k| < \frac{\delta_{x_k}}{2} \text{ and } |x_{x_k} - y| < \frac{\delta_{x_k}}{2}$$

Then we have $|f(x) - f(x_k)| < \frac{\varepsilon}{2}$ and $|f(x_k) - f(y)| < \frac{\varepsilon}{2}$ so that $|f(x) - f(y)| < \varepsilon$, which makes f unifomrly continuous. ■

Chapter 2

Lebesgue Measure

2.1 Lebesgue Outermeasure

Definition. Let I be a nonempty interval of $\mathbb{R}$. We define the **lenght** of I, denoted $l(I)$, to be the difference of its endpoints, if I is bounded, and ∞ otherwise.

Definition. Let A a subset of $\mathbb{R}$. We define the **Lebesgue outer measure** of A to be

$$m^*(A) = \inf\{\sum l(I_k)\}$$

Where $\{I_k\}$ is a countable collection of bounded open sets, covering A.

Lemma 2.1.1. *The emptyset has Lebesgue outermeasure* 0*. Moreover, the Lebesgue outermeasure is monotone; that is, if* $A, B \subseteq \mathbb{R}$ *such that* $A \subseteq B$*, then* $m^*(A) \leq m^*(B)$*.*

Proof. Notice that the singleton $\{a\} = [a, a]$ covers the emptyset . Moreover $l([a, a]) = a - a = 0$, so by definition $m^*(\emptyset) = 0$.

Now, let A, B subsets of $\mathbb{R}$ such that $A \subseteq B$. Then if $\{I_k\}$ is a countable collection of bounded open sets covering B, they also cover A, hence by definition, we get $m^*(A) \leq m^*(B)$. ∎

Corollory. *Lebesgue outermeasure is nonnegative. That is,* $0 \leq m^*(E)$ *for any set* $E \subseteq \mathbb{R}$*.*

Proof. Notice the length of any interval I is nonnegative. ∎

Example 1. Countable sets have measure 0. Let C be a countable set with enumeration $\{c_k\}$. Let $\varepsilon > 0$ and define $I_k = (c_k - \frac{\varepsilon}{2^{k+1}}, c_k + \frac{\varepsilon}{2^{k+1}})$. Then $\{I_k\}$ is a countable collection of bounded open sets covering $C = \{c_k\}$. Hence we get that

$$0 \leq m^*(C) \leq \sum I_k \leq \sum \frac{\varepsilon}{2^k} = 0$$

So that $m^*(C) = 0$.

Lemma 2.1.2. *For any nonempty interval* I*,* $m^*(I) = l(I)$*.*

Proof. Consider first, the closed bounded interval $[a, b]$, where $a < b$. Let $\varepsilon > 0$. Notice that $[a, b] \subseteq (a-\varepsilon, b+\varepsilon)$, so that $m^*([a, b]) \leq l((a-\varepsilon, b+\varepsilon)) = b-a+2\varepsilon$. Hence $m^*([a, b]) \leq b-a$. It remains to show that $b - a \leq m^*([a, b])$.

Let $\{I_k\}$ a countable collection of open bounded intervals covering $[a, b]$. By the theorem of Heine-Borel, there is a finite subcover $\{I_k\}_{k=1}^n$ of $[a, b]$. Notice that since $a \in \bigcup I_k$, at least one I_k contains a. Hence choose an interval (a_1, b_1) in this cover for which $a_1 < a < b_1$. Now, if $b < b_1$, we are done as

$$\sum_{k=1}^{n} l(I_k) \geq b_1 - a_1 > b - a$$

Otherwise, $b_1 \in [a, b_1)$. In this case, choose an interval (a_2, b_2), distinct from (a_1, b_1) for which $a_2 < b_1 < b_2$. If $b_2 \geq b$, then we are done by similar reasoing as above. Otherwise, continue the process of choosing intervals. This process terminates as we eventually exhaust the endpoints of each I_k in the open cover. Thus, we get a subcollection $\{(a_k, b_k)\}_{k=1}^N$ for which $a_1 < a$ and $a_{k+1} < b_k$ for all $1 \leq k \leq N-1$. We also have a $b_N > b$. Then we have

$$\sum l(I_k) \geq \sum_{k=1}^{N} l((a_k, b_k)) = (b_N - a_N) + \cdots + (b_1 - a_1) \geq b - a$$

so that we get $b - a \leq m^*([a, b])$.

Now, let I be any bounded interval. Notice that there exist closed bounded intervals J_1 and J_2 for which

$$J_1 \subseteq I \subseteq J_2$$

and for some $\varepsilon > 0$,

$$l(I) - \varepsilon < l(J_1) \leq l(I) \leq l(J_1) < l(I) + \varepsilon$$

Then since J_1 and J_2 are closed and bounded intervals, and by monotonicity of m^*, we have

$$l(I) - \varepsilon < m^*(J_1) \leq m^*(I) \leq m^*(J_1) < l(I) + \varepsilon$$

so that $l(I) - \varepsilon < m^*(I) < l(I) + \varepsilon$ for all $\varepsilon > 0$. This establishes equality. ■

Lemma 2.1.3. *The Lebesgue outermeasure is translation invariant. That is, if $A \subseteq \mathbb{R}$, and $y \in \mathbb{R}$, then $m^*(A) = m^*(A + y)$.*

Proof. Notice that a countable collection of open bounded intervals $\{I_k\}$ covers A if, and only if the collection $\{I_k + y\}$ of open bounded intervals covers $A + y$. Moreover, notice that $l(I_k) = l(I_k + y)$, so that we get

$$\sum l(I_k) = \sum l(I_k + y)$$

the rest follows from definition. ■

Lemma 2.1.4. *The Lebesgue outermeasure is countable subadditive; that is, if $\{E_k\}$ is a collection of subsets of $\mathbb{R}$, then*

$$m^*(\bigcup E_k) \leq \sum m^*(E_k)$$

Proof. Let $\{E_k\}$ a countable collection of sets, and let $E = \bigcup E_k$. Notice that if atleast one E_k has infinite measure, then we are done. Suppose then that for all k, $m^*(E_k)$ is finite. Let $\varepsilon > 0$. Then for all k, there exists a countable collection of open bounded intervals $\{I_{k,i}\}$ covering E_k, and $\sum_i l(I_{k,i}) < m^*(E_k) + \frac{\varepsilon}{2^k}$. By definition, we get

$$m^*(E) \leq \sum l(I_{k,i}) = \sum_k \sum_i l(I_{k,i}) < \sum_k (m^*(E_k) + \frac{\varepsilon}{2^k}) = \sum_k m^*(E_k) + \varepsilon$$

for all $\varepsilon > 0$. This inequality also holds for $\varepsilon = 0$. ■

Corollory. *The Lebesgue outermeasure is finitely subadditive.*

Proof. Recall that finite collections are also countable collectuions. ■

2.2 Lebesuge Measurable Sets

Definition. We call a set E of $\mathbb{R}$ **Lebesuge measurable**, provided for any subset A of $\mathbb{R}$,

$$m^*(A) = m^*(A \cap E) + m^*(A \cap \mathbb{R} \backslash E)$$

Lemma 2.2.1. *A set E is Lebesuge measurable if, and only if for any subset A of $\mathbb{R}$,*

$$m^*(A) \geq m^*(A \cap E) + m^*(A \cap \mathbb{R} \backslash E)$$

Proof. We have $A = (A \cap E) \cup (A \cap \mathbb{R} \backslash E)$, so by finite subadditivity, $m^*(A) \leq m^*(A \cap E) + m^*(A \cap \mathbb{R} \backslash E)$. ■

Lemma 2.2.2. *Any set of Lebesuge outer measure 0 is Lebesgue measurable.*

Proof. Let E have $m^*(E) = 0$ and let $A \subseteq \mathbb{R}$. Notice that $A \cap E \subseteq E$ and $A \cap \mathbb{R} \backslash E \subseteq E$, so that $m^*(A \cap E) \leq m^*(E) = 0$ and $m^*(A \cap \mathbb{R} \backslash E) \leq m^*(A)$. Then we have

$$m^*(A) \geq m^*(A \cap \mathbb{R} \backslash E) = 0 + m^*(A \cap \mathbb{R} \backslash E) = m^*(A \cap E) + m^*(A \cap \mathbb{R} \backslash E)$$

■

Corollory. *Countable sets are measurable.*

Lemma 2.2.3. *The union of two measurable sets is measurable.*

Proof. Let E_1 and E_2 be measurable sets and $A \subseteq \mathbb{R}$. Then $m^*(A) = m^*(A \cap E_1) + m^*(A \cap \mathbb{R} \backslash E_1) = m^*(A \cap E_1) + m^*((A \cap \mathbb{R} \backslash E_1) \cap E_2) + m^*((A \cap \mathbb{R} \backslash E_1) \cap \mathbb{R} \backslash E_2)$. Moreover, notice that

$$(A \cap \mathbb{R} \backslash E_1) \cap \mathbb{R} \backslash E_2 = A \cap \mathbb{R} \backslash (E_1 \cup E_2) \text{ and } (A \cap E_1) \cup (A \cap \mathbb{R} \backslash E_1 \cap E_2) = A \cap (E_1 \cup E_2)$$

Then we get

$$m^*(A) = m^*(A \cap E_1) + m^*((A \cap \mathbb{R} \backslash E_1) \cap E_2) + m^*(A \cap \mathbb{R} \backslash (E_1 \cup E_2)) \geq m^*(A \cap (E_1 \cup E_2)) + m^*(A \cap \mathbb{R} \backslash (E_1 \cup E_2))$$

which makes E_1 and E_2 measurable. ■

Corollory. *The union of a finite collection of measurable sets is measurable.*

Proof. Let $\{E_k\}_{k=1}^n$ a finite collection of measurable sets. By induction on n, we showed that this is true for $n = 1$ and $n = 2$. Now, consider the collections $\{E_k\}_{k=1}^{n+1}$ and suppose that the union $E = \bigcup_{k=1}^n E_k$ is measurable. Notice, then that

$$\bigcup_{k=1}^{n+1} E_k = E \cup E_{n+1}$$

both of which are measurable. Hence measurability of the union of $\{E_k\}_{k=1}^{n+1}$ follows by above. ∎

Lemma 2.2.4. *Let A a subset of $\mathbb{R}$ and $\{E_k\}_{k=1}^n$ a finite, disjoint collection of measurable sets. Then*

$$m^*(A \cap \bigcup_{k=1}^n E_k) = \sum_{k=1}^n m^*(A \cap E_k)$$

Proof. By induction on n, for $n = 1$ it is true. Now, suppose that it is true for n, and consider the collection $\{E_k\}_{k=1}^{n+1}$ of disjoint measurable sets. Then we have $A \cap (\bigcup_{k=1}^n E_k) \cap E_{n+1} = A \cap E_{n+1}$ and $A \cap (\bigcup_{k=1}^n) \cap \mathbb{R}\backslash E_{n+1} = A \cap \bigcup_{k=1}^n E_k$. SInce E_{n+1} is measurable we get

$$m^*(A \cap \bigcup_{k=1}^{n+1} E_k) = m^*(A \cap E_{n+1}) + m^*(A \cap \bigcup_{k=1}^n E_k) = \sum_{k=1}^{n+1} m^*(A \cap E_k)$$

∎

Definition. We call a collection of subsets of $\mathbb{R}$ an **algebra** if it contains $\mathbb{R}$ and it is closed under complements (with respect to $\mathbb{R}$) and finite unions.

Lemma 2.2.5. *Any algebra of $\mathbb{R}$ is closed under finite intersections.*

Proof. By DeMorgan's laws. ∎

Theorem 2.2.6. *The collection of all measurable sets of $\mathbb{R}$ forms an algebra.*

Lemma 2.2.7. *The union of a countable collection of measurable sets is measurable.*

Proof. Without loss of generality, let $\{E_k\}$ a countable disjoint collection of measurable sets, and let $E = \bigcup E_k$. Let A a subset of $\mathbb{R}$ and define $F_n =_{k=1}^n E_k$. Then F_n is measurable by lemma 2.2.3, and $\mathbb{R}\backslash E_n \subseteq \mathbb{R}\backslash F_n$. Then

$$m^*(A) = m^*(A \cap F_n) + m^*(A \cap \mathbb{R}\backslash F_n) \geq m^*(A \cap F_n) + m^*(A \cap \mathbb{R}\backslash E_n)$$

hence $m^*(A \cap F_n) = \sum_m^*(A \cap E_k)$ so that

$$m^*(A) \geq \sum m^*(A \cap E_k) + m^*(A \cap \mathbb{R}\backslash E)$$

By countable subadditivity of m^* we have

$$m^*(A) \geq m^*(A \cap E) + m^*(A \cap \mathbb{R}\backslash E)$$

∎

Definition. We call a collection of subsets of $\mathbb{R}$ a **σ-algebra** if it forms an algebra, and it is closed under countable unions.

Lemma 2.2.8. *Any σ-algebra of $\mathbb{R}$ is closed under countable intersections.*

Theorem 2.2.9. *The collection of measurable sets of $\mathbb{R}$ forms a σ-algebra.*

Lemma 2.2.10. *Every interval of $\mathbb{R}$ is measurable.*

Proof. Consider an interval of the form (a, ∞), for any $a \in \mathbb{R}$. Let $A \subseteq \mathbb{R}$, such that $A \notin A$; otherwise, just take $A \backslash \{a\}$. Then since $m^*(A)$ is a greatest lower bound, it is sufficient to show that for any countable collection $\{I_k\}$ of open, bounded intervals covering A, that

$$m^*(A_1) + m^*(A_2) \leq \sum l(I_k)$$

where

$$A_1 = A \cap (-\infty, a) \text{ and } A_2 = A \cap (a, \infty)$$

Indeed, let $\{I_k\}$ be such a collection, and define

$$I_{k,1} = I_k \cap (-\infty, a) \text{ and } I_{k,2} = I_k \cap (a, \infty)$$

Then $\{I_{k,1}\}$ and $\{I_{k,2}\}$ are collections of open, bounded intervals which cover A_1 and A_2 respectively, Hence, by definition of m^*, we have $m^*(A_1) \leq \sum l(I_{k,1})$ and $m^*(A_2) \leq \sum l(I_{k,2})$; moreover, notice that $l(I_k) = l(I_{k,1}) + l(I_{k,2})$. Therefore, we get

$$m^*(A_1) + m^*(A_2) \leq \sum l(I_{k,1}) + \sum l(I_{k,1}) = \sum l(I_k)$$

and we are done. ■

Corollory. *Open sets, and closed sets of $\mathbb{R}$ are measurable.*

Definition. We define the intersection of all σ-algebras of $\mathbb{R}$ to be the **Borel σ-algebra**, and call its elements **Borel sets**.

Theorem 2.2.11. *The σ-algebra of all measurable sets of $\mathbb{R}$ contains the Borel σ-algebra of $\mathbb{R}$. Moreover, it contains every interval of $\mathbb{R}$, open and closed sets, as well as G_δ and F_σ sets.*

Lemma 2.2.12. *Lebesgue measurable sets are translation invariant. That is, if E is Lebesuge measurable, and $y \in \mathbb{R}$, then $E + y$ is Lebesuge measurable.*

Proof. Let E be measurable, $y \in \mathbb{R}$, and $A \subseteq \mathbb{R}$ Then

$$m^*(A) = m^*(A \backslash y) = m^*(A \backslash y \cap E) + m^*(A \backslash y \cap \mathbb{R} \backslash E) = m^*(A \cap (E + y)) + m^*(A \cap \mathbb{R} \backslash (E + y))$$

■

2.3 Inner and Outer Approximations

Bibliography

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