Complex Analysis

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May 24, 2023

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Chapter 1

The Complex Numbers

1.1 The Field of Complex Numbers

Definition. We define the set of **complex numbers** to be the collection of all ordered pairs of real numbers $\mathbb{C} = \{(a, b) : a, b \in \mathbb{R}\}$ together with the binary operations $+$ and $\cdot$ of **complex addition**, and **complex multiplication**, respectively defined by the rules

$$\begin{aligned}(a, b) + (c, d) &= (a + c, b + d)\\ (a, b)(c, d) &= (ac - bd, bc + ad)\end{aligned}$$

Theorem 1.1.1. *The set of complex numbers $\mathbb{C}$ forms a field together with complex addition and complex multiplication.*

Corollary. *$\mathbb{C}$ is a field extension of the real numbers $\mathbb{R}$.*

Proof. The map $a \to (a, 0)$ from $\mathbb{R} \to \mathbb{C}$ defines an imbedding of $\mathbb{R}$ into $\mathbb{C}$. ∎

Definition. We define the element $i = (0, 1)$ of $\mathbb{C}$ so that $i^2 = -1$, and the polynomial z^2+1 has as root i. We write $(a, b) = a + ib$. If $z = a + ib$, we call a the **real part** of z, and b the **imaginary part** of z and write $\operatorname{Re} z = a$ and $\operatorname{Im} z = z$.

Definition. Let $z = a+ib \in \mathbb{C}$. We define the **norm** (or **modulus**) of z to be $|z| = \sqrt{a^2 + b^2}$. We define the complex **conjugate** of z to be $\overline{z} = a - ib$.

Lemma 1.1.2. *For every $z \in \mathbb{C}$, $|z|^2 = z\overline{z}$.*

Proof. Let $z = a+ib$. Then $\overline{z} = a-ib$, and so $z\overline{z} = (a+ib)(a-ib) = a^2+b^2 = (\sqrt{a^2 + b^2})^2 = |z|^2$. ∎

Corollary. *If $z \neq 0$, then $z^{-1} = \frac{1}{z} = \frac{\overline{z}}{|z|^2}$.*

Proof. The relation follows from above, and it remains to show that it is indeed the inverse element. Notice that if $z \in \mathbb{C}$ is nonzero, then $z\frac{\overline{z}}{|z|^2} = \frac{z\overline{z}}{|z|^2} = \frac{|z|^2}{|z|^2} = 1$. ∎

Example 1.1. (1) Let $z = a + ib$. Then we get that $\frac{1}{z} = \frac{\overline{z}}{|z|}$ has real part $\operatorname{Re}\frac{1}{z} = \frac{a}{a^2+b^2}$ and imaginary part $\operatorname{Im}\frac{1}{z} = -\frac{b}{a^2+b^2}$.

(2) Let $z = a + ib$, and $c \in \mathbb{R}$. Then $\frac{z-c}{z+c} = \operatorname{Re} \frac{z-c}{z+c}$, so $\operatorname{Im} \frac{z-c}{z+c} = 0$.

(3) Let $z = a + ib$, then $z^3 = a^3 - 3ab^2 + i(3a^2b - b^3)$ So that $\operatorname{Re} z^3 = a^3 - 3ab^2$ and $\operatorname{Im} z = 3a^2b - b^3$.

(4) $\frac{3+i5}{1+i7} = \frac{19}{25} - i\frac{18}{25}$.

(5) $(-\frac{1}{2} + i\frac{\sqrt{3}}{2})^3 = 1$, and hence $(-\frac{1}{2} + i\frac{\sqrt{3}}{2})^6 = 1$.

(6) Notice that $i^n = 1, i, -1, -i$ whenever $n \equiv 0 \mod 4$, $n \equiv 1 \mod 4$, $n \equiv 2 \mod 4$, and $n \equiv 3 \mod 4$. respectively.

(7) $|-2+i| = \sqrt{5}$, and $|(2+i)(4+i3)| = |5+i10| = 5\sqrt{5}$.

Lemma 1.1.3. *The following are true for all $z, w \in \mathbb{C}$.*

(1) $\operatorname{Re} z = \frac{1}{2}(z + \overline{z})$ *and* $\operatorname{Im} z = \frac{1}{2i}(z - \overline{z})$.

(2) $\overline{(z + w)} = \overline{z} + \overline{w}$ *and* $\overline{zw} = \overline{z}\,\overline{w}$.

(3) $|\overline{z}| = |z|$.

Proof. Let $z = a + ib$ and $w = c + id$. Then notice that

$$\frac{(a+ib)+(a-ib)}{2} = \frac{2a + (ib - ib)}{2} = \frac{2a}{2} = a = \operatorname{Re} z$$

and

$$\frac{(a+ib)-(a-ib)}{2i} = \frac{(a-a)+2ib}{2} = \frac{2ib}{2i} = b = \operatorname{Im} z$$

Moreover

$$\overline{(a+ib)+(c+id)} = \overline{(a+c)+i(b+d)} = (a+c) - i(b+d) = (a-ib) + (c-id)$$

And

$$\overline{(a+ib)(c+id)} = \overline{(ac-bd)+i(bc+ad)} = (ac-bd) - i(bc+ad) = (a-ib)(c-id)$$

so that $\overline{z+w} = \overline{z} + \overline{w}$ and $\overline{zw} = \overline{z}\,\overline{w}$.

Now, we have that $|zw|^2 = (zw)\overline{zw} = (zw)(\overline{z}\,\overline{w}) = (z\overline{z})(w\overline{w}) = |z|^2|w|^2$. Taking square roots, we get the result

$$|zw| = |z||w|$$

Finally, notice that $|z|^2 = z\overline{z} = \overline{z}\,\overline{\overline{z}} = |\overline{z}|$. ∎

Corollary. *The following are also true; provided $w \neq 0$.*

(1) $\overline{(\frac{z}{w})} = \frac{\overline{z}}{\overline{w}}$.

(2) $|\frac{z}{w}| = \frac{|z|}{|w|}$

Corollary. *If $z = z_1 + \cdots + z_n$, and $w = w_1 \ldots w_n$, with $z_i, w_i \in \mathbb{C}$ for all $1 \leq i \leq n$, then*

(1) $\overline{z} = \overline{z_1} + \cdots + \overline{z_n}$.

(2) $|w| = |w_1| \ldots |w_n|$.

Proof. We prove both results by induction on n. For $n = 2$, we have already shown that $\overline{z} = \overline{z_1} + \overline{z_2}$ and $|w| = |w_1||w_2|$. Now, for all $n \geq 2$, suppose that both

$$\begin{aligned} \overline{z} &= \overline{z_1} + \cdots + \overline{z_n} \\ |w| &= |w_1| \ldots |w_n| \end{aligned}$$

Then let $z' = z + z_{n+1}$ and $w' = ww_{n+1}$ for $z_{n+1}, w_{n+1} \in \mathbb{C}$. Then we have that

$$\begin{aligned} z' &= z + z_{n+1} = z_1 + \cdots + z_n + z_{n+1} \\ w' &= ww_{n+1} = w_1 \ldots w_n w_{n+1} \end{aligned}$$

so by the induction hypothesis, we have

$$\overline{z'} = \overline{(z + z_{n+1})} = \overline{z} + \overline{z_{n+1}} = \overline{z_1} + \cdots + \overline{z_n} + \overline{z_{n+1}}$$

and that

$$|w'| = |ww_{n+1}| = |w||w_{n+1}| = |w_1| \ldots |w_n||w_{n+1}|$$

which completes the proof. ■

Lemma 1.1.4. *Let $z \in \mathbb{C}$. Then z is a real number if, and only if $z = \overline{z}$.*

Proof. If z is real, then $z = a + i0$, for some $a \in \mathbb{R}$, and hence $\overline{z} = a - i0 = z$. COnversely, suppose that $z = \overline{z}$. Then we have

$$\operatorname{Re} z = \frac{1}{2}(z + \overline{z}) = \frac{1}{2}(z + z) = z$$

so z has only a real part, and hence must be a real number. ■

Lemma 1.1.5. *The following are true for all $z, w \in \mathbb{C}$.*

(1) $|z + w|^2 = |z|^2 + 2 \operatorname{Re} z\overline{w} + |w|^2$.

(2) $|z - w|^2 = |z|^2 - 2 \operatorname{Re} z\overline{w} + |w|^2$.

(3) $|z + w|^2 + |z - w|^2 = 2(|z|^2 + |w|^2)$.

Proof. We first notice that $|z+w|^2 = (z+w)\overline{(z + w)} = (z+w)(\overline{z}+\overline{w}) = z\overline{z}+z\overline{w}+w\overline{z}+w\overline{w} = |z|^2 + z\overline{w} + w\overline{z} + |w|^2$. Now, let $z = a + ib$ and $w = c + id$. Then we have

$$\begin{aligned} (a + ib)(c - id) &= (ac + bd) - i(ad - bc) \\ (c + id)(a - ib) &= (ac + bd) + i(ad - bc) \end{aligned}$$

so that $z\overline{w} + w\overline{z} = 2(ac + bd) = 2\operatorname{Re} z\overline{w}$, and we are done. To get the identity for $|z - w|^2$, we simply replace w by $-w$, and use the above argument.

Now, we have that $|z + w|^2 = |z^2| + 2\operatorname{Re} z\overline{w} + |w|^2$, and $|z - w|^2 = |z^2| - 2\operatorname{Re} z\overline{w} + |w|^2$, so that adding them together, the terms $2\operatorname{Re} z\overline{w}$ cancel out and we are left with

$$|z + w|^2 + |z - w|^2 = 2(|z|^2 + |w|^2)$$

■

Lemma 1.1.6. *Let $R(z) \in \mathbb{C}(z)$ a rational function in z. Then if R has coefficients in $\mathbb{R}$, then $\overline{R(z)} = R(\overline{z})$.*

Proof. We first observe the polynomial $f \in \mathbb{C}[z]$, of finite degree $\deg f = n$, and of the form

$$f(z) = a_0 + a_1 z + \cdots + a_n z^n$$

Then if f has all coefficients in $\mathbb{R}$; i.e. $f \in \mathbb{R}[z]$, where $z \in \mathbb{C}$ is treated as indeterminant, then we have that since each $a_i \in \mathbb{R}$, then $\overline{a_i z^i} = \overline{a_i}\overline{z^i} = a_i \overline{z}^i$. So that

$$\overline{f(z)} = \overline{(a_0 + a_1 z + \cdots + a_n z^n)} = a_0 + a_1 \overline{z} + \cdots + a_n \overline{z}^n$$

which makes $\overline{f(z)} = f(\overline{z})$. Now, one can also extend f to a polynomial of infinite degree by taking $n \to \infty$, and the same holds.

Now, let $R(z) \in \mathbb{C}(z)$ a rational function. Recall that $R(z)$ is of the form

$$R(z) = \frac{f(z)}{g(z)} \text{ with } g \neq 0$$

for some polynomials $f, g \in \mathbb{C}[z]$. Then if R has all real coefficients, so do f and g, and hence we get

$$\overline{R(z)} = \frac{\overline{f(z)}}{\overline{g(z)}} = \frac{f(\overline{z})}{g(\overline{z})} = R(\overline{z})$$

which completes the proof. ■

1.2 The Complex Plane

Definition. We define the **complex plane** to be the space of points (x, y) of $\mathbb{R}^2$ for which $z = x + iy$.

Lemma 1.2.1. *For every $z, w \in \mathbb{C}$ $|z + w| \leq |z| + |w|$.*

Proof. Observe that $-|z| \leq \operatorname{Re} z \leq |z|$ for all $z \in \mathbb{C}$, so that $\operatorname{Re} z\overline{w} \leq |z\overline{w}| = |z||w|$. So we get

$$|z + w|^2 = |z|^2 + \operatorname{Re} z\overline{w} + |\overline{w}| \leq |z|^2 + |z||w| + |\overline{w}| = (|z| + |w|)^2$$

Taking square roots gives us the result. ■

Corollary. $|z + w| = |z| + |w|$ *if $z = tw$ for some $t \geq 0$.*

Corollary. *If $z_1, \ldots, z_n \in \mathbb{C}$, then $|z_1 + \cdots + z_n| \leq |z_1| + \cdots + |z_n|$.*

Proof. By induction on n. ■

Corollary. *For all $z, w \in \mathbb{C}$, $\big||z| - |w|\big| \leq |z - w|$.*

Proof. We have that $|z| \leq |z-w| + |w|$, and $|w| \leq |z-w| + |z|$. So we get $|z| - |w| \leq |z-w|$ and $-|z-w| \leq |w| - |z|$, so that $\big||z| - |w|\big| \leq |z - w|$. ■

Definition. We define the **polar form** of a complex number $z \in \mathbb{C}$ to be the polar coordinates (r, θ) where $r = |z|$ and θ is the angle between the line segment from 0 to z and the positive real axis. We call r the **modulus** of z, and θ the **argument** of z. We write $\theta = \arg z$.

Lemma 1.2.2. *Let $z = r_1 \cos\theta_1 + r_1 i \sin\theta_1$ and $w = r_2\cos\theta_2 + r_2 i \sin\theta_2$. Then*

$$zw = r_1 r_2 \cos(\theta_1 + \theta_2) + r_1 r_2 i \sin(\theta_1 + \theta_2)$$

so that $\arg zw = \arg z + \arg w$.

Proof. We multiply the expanded forms of z and w together and use the trigonometric identities to get the result. ■

Corollary. *If $z_k = r_k \cos\theta_k + r_k i \sin\theta_k$, then*

$$z_1 \ldots z_n = (r_1 \ldots r_n)\cos(\theta_1 + \cdots + \theta_n) + (r_1 \ldots r_n) i \sin(\theta_1 + \cdots + \theta_n)$$

Proof. By induction on n. ■

Theorem 1.2.3 (DeMoivre's Theorem)**.** *For all integers $n \geq 0$, if $z = \cos\theta + i\sin\theta$, then*

$$z^n = \cos n\theta + i \sin n\theta$$

Proof. We use the corollory to lemma 1.2.2 recursively on z^n. ■

Lemma 1.2.4. *FOr each nonzero $a \in \mathbb{C}$, and integer $n \geq 2$, the polynomial $z^n - a$ has has roots all z of the form*

$$z = |a|^{\frac{1}{n}}(\cos\frac{\alpha + 2k\pi}{n} + i\sin\frac{\alpha + 2k\pi}{n}) \text{ for all } 0 \leq k \leq n-1$$

where $a = |a|\cos\alpha + |a| i \sin\alpha$

Proof. Let $a = |a|\cos\alpha + |a| i \sin\alpha$. Then we have $z^n - a = 0$ has as solution

$$z' = |a|^{\frac{1}{n}}(\cos\frac{\alpha + 2\pi}{n} + i\sin\frac{\alpha + 2\pi}{n})$$

The rest of the solutions are obtained by noting that $(z')^n - a = 0$. ■

Definition. Let $a \in \mathbb{C}$ a nonzero complex number. We call the roots of the the polynomial $z^n - a \in \mathbb{C}[z]$ the n**-th roots** of a. We call the roots of $z^n - 1 \in \mathbb{C}[z]$ the n**-th roots of unity**.

Example 1.2. The n-th roots of unity are all complex numbers of the form

$$z = \cos\frac{2k\pi}{n} + i\sin\frac{2k\pi}{n} \text{ for all } 0 \leq k \leq n-1$$

Lemma 1.2.5. *Let $L \subseteq \mathbb{C}$ a straight line in $\mathbb{C}$. Then $L = \{z \in \mathbb{C} : \operatorname{Im}\frac{z-a}{b} = 0\}$, where $z = a + tb$ for some $t \in \mathbb{R}$.*

Proof. Let a be any point in L, and b the direction vector of L. Then if $z \in L$ $z = a + tb$ for some $t \in \mathbb{R}$. Since $b \neq 0$, $\operatorname{Im}\frac{z-a}{b} = 0$, since $t = \frac{z-a}{b}$, and $t \in \mathbb{R}$. ∎

Corollary. *Let $H_a = \{z \in \mathbb{C} : \operatorname{Im}\frac{z-a}{b} > 0\}$ and $K_a = \{z \in \mathbb{C} : \operatorname{Im}\frac{z-a}{b} < 0\}$. Then $H_a = a + H_0$ and $K_a = a - K_0$.*

Proof. Suppose that $|b| = 1$, and let $a = 0$, then $H_0 = \{z \in \mathbb{C} : \operatorname{Im}\frac{z}{b} > 0\}$. Now, $b = \cos\beta + i\sin\beta$. If $z = r\cos\theta + ri\sin\theta$, then $\frac{z}{b} = r\cos(\theta - \beta) + ri\sin(\theta - \beta)$. So $z \in H_0$ if, and only if $\sin(\theta - \beta) > 0$; that is $\beta < \theta < \pi + \beta$, which makes H_0 the upper half plane about L.

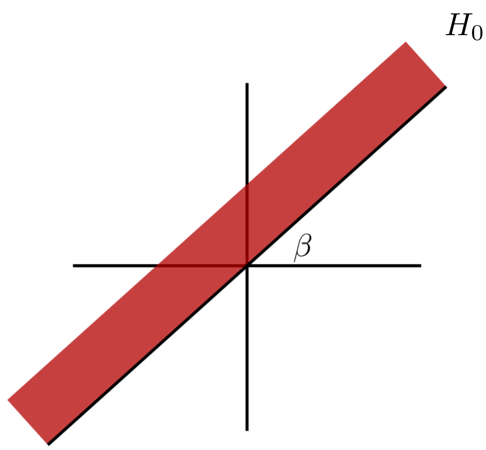

Figure 1.1:

Putting $H_a = \{z \in \mathbb{C} : \operatorname{Im}\frac{z-a}{b} > 0\}$, we get $H_a = a + H_0$. By similar reasoning, we get $K_a = a - K_0$, where $K_a = \{z \in \mathbb{C} : \operatorname{Im}\frac{z-a}{b} < 0\}$. ∎

1.3 The Extended Complex Numbers

Definition. We define the **extended complex numbers** to be the set $\mathbb{C}_\infty = \mathbb{C} \cup \{\infty\}$.

Lemma 1.3.1. *$\mathbb{C}_\infty$ is homeomorphic to the unit sphere S^2 of $\mathbb{R}^3$.*

Proof. Identify $\mathbb{C}$ with the plane $\mathbb{R}^2$ as a subset of $\mathbb{R}^3$. Then $\mathbb{C}$ cuts the sphere S^2 along the equator. Now, let $N = (0, 0, 1)$ be the noth pole of S^2. For $z \in \mathbb{C}$, let L_z the line passing through z and N, and hence cuts S^3 at exactly one point $Z \neq N$. If $|z| > 1$, Z is in the northern hemisphere of S^2, and if $|z| < 1$, then Z is in the southern hemisphere. If $|z| = 1$, then $Z = z$. Then notice that as $|z| \to \infty$, then $Z \to N$; and so identify N with ∞ in $\mathbb{C}_\infty$.

Now, let $z = x+iy$ and $Z = (x_1, x_2, x_3)$ a point on S^2. Then $L_z = \{tN+(1-t)z : t \in \mathbb{R}\}$. Observe then that

$$L_z = \{((1-t)x, (1-t)y, t) : t \in \mathbb{R}\}$$

Then we get

$$1 = (1-t)^2|z|^2 + t^2$$

Taking $t \neq 1$ so that $z \neq \infty$

$$Z = \Big(\frac{2x}{|z|^2+1}, \frac{2y}{|z|^2+1}, \frac{|z|^2-1}{|z|^2+1}\Big)$$

additionally

$$Z = \Big(\frac{z+\overline{z}}{|z|^2+1}, -i\frac{z-\overline{z}}{|z|^2+1}, \frac{|z|^2-1}{|z|^2+1}\Big)$$

Taking $Z \neq N$ and $t = x_1$, we also get by definition of L_z, that

$$z = \frac{x_1 + ix_2}{1 - x_3}$$

Define now, the metric d on $\mathbb{C}_\infty$ by $d(z, w)$ is the distance between the points $Z = (x_1, x_2, x_3)$ and $W = (y_1, y_2, y_3)$ on S^2. Then we get

$$d(z, w) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2 + (x_3 - y_3)^2}$$

Squaring both sides we ovserve tha

$$d(z, w) = 2 - 2(x_1y_1 + x_2y_2 + x_3y_3)$$

Using the previous derivations of the components of Z, we finally obtain

$$d(z, w) = \frac{z|z-w|}{\sqrt{(|z|^2+1)(|w|^2+1)}} \text{ for all } z, w \in \mathbb{C}$$

When $w = \infty$, we have

$$d(z, \infty) = \frac{z}{\sqrt{|z|^2+1}}$$

Then d is the required homeomorphism. ■

Definition. We call the correspondence between S^2 and $\mathbb{C}_\infty$ the **stereographic projection** of S^2 onto $\mathbb{C}_\infty$.

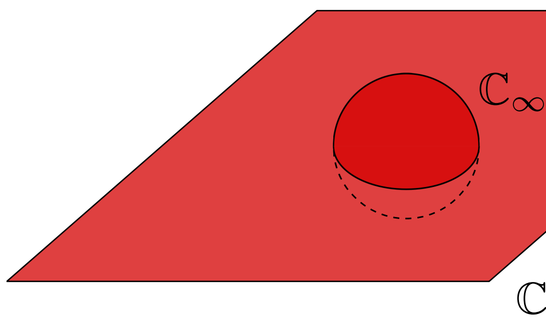

Figure 1.2: The Extended Complex Numbers.

Chapter 2

The Topology of $\mathbb{C}$.

2.1 Metric Spaces

Definition. A **metric space** is a set X together with a map $d : X \times X \to \mathbb{R}$ such that for all $x, y, z \in X$

(1) $d(x, y) \geq 0$ and $d(x, y) = 0$ if, and only if $x = y$.

(2) $d(x, y) = d(y, x)$.

(3) $d(x, y) \leq d(x, z) + d(z, y)$ (The Triangle Inequality).

We call d a **metric** on X. If $x \in X$, and $r > 0$, we define the **open ball** centered about x of **radius** r to be the set $B(x, r) = \{y \in X : d(x, y) < r\}$. We define the **closed ball** centered about x of radius r to be the set $\overline{B}(x, y) = \{y \in X : d(x, y) \leq r\}$.

Example 2.1. (1) The metric $d(x, y) = \|z - w\|$ defined by $\|z - w\| = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)}$, where $z = x_1 + iy_1$ and $w = x_2 + iy_2$ makes $\mathbb{R}$ and $\mathbb{C}$ into metric spaces. In fact, d defines the norm on $\mathbb{C}$, i.e. $\|z\| = d(z, 0)$. In $\mathbb{R}$, $\|\cdot\|$ is the absolute value. We denote howerver $\|\cdot\| = |\cdot|$ is $\mathbb{C}$ as well, when we are talking about the norm of a complex number.

(2) If X is a metric space with metric d, and $Y \subseteq X$, then d makes Y into a metric space.

(3) Define $d(x + iy, a + ib) = |x - a| + |y - b|$. Then $(\mathbb{C}, d)$ is a metric space. We call d the **taxicab metric**.

(4) Define $d(x + iy, a + ib) = \max\{|x - a|, |y - b|\}$. Then $(\mathbb{C}, d)$ is a metric space. We call d the **square metric**.

(5) Let X be any set, and define $d : X \times X \to \mathbb{R}$ by

$$d(x, y) = \begin{cases} 1, & \text{if } x \neq y \\ 0, & \text{if } x = y \end{cases}$$

Then d is a metric on X. Notice also that for any $\varepsilon > 0$, that $B(x, \varepsilon) = \{x\}$ if $\varepsilon \leq 1$, and $B(x, \varepsilon) = X$ if $\varepsilon > 1$.

(6) Define d on $\mathbb{R}^n$ by

$$d(x,y) = \sqrt{\sum_{i=1}^{n} (x_i - y_i)^2}$$

Then d is a metric on $\mathbb{R}^n$ defining the general norm. That is $\|x\| = d(x,0)$.

(7) Let S and let $B(S)$ the set of all complex valued functions $f : S \to \mathbb{C}$ such that $\|f\|_\infty = \sup\{|f(s|) : s \in S\}$ is finite. That is, $B(S)$ is the set of all complex valued functions whose image is contained within a disk of finite radius. Define d on $B(S)$ by $d(f,g) = \|f - g\|_\infty$. Let $f, g, h \in B(S)$. Then

$$\|f(s) - g(s)\| = \|(f(s) - h(s)) - (h(s) - g(s))\| \leq \|f(s) - h(s)\| + \|h(s) - g(s)\|$$

taking least upper bounds, we get

$$\|f - g\|_\infty \leq \|f - h\|_\infty + \|h - g\|_\infty$$

Definition. Let X be a metric space together with metric d. We call a subset U of X **open** if there exists an $\varepsilon > 0$ for which $B(x, \varepsilon) \subseteq U$ for every $x \in U$.

Example 2.2. $U = \{z \in \mathbb{C} : a < \operatorname{Re} z < b\}$ is open in $\mathbb{C}$, but $U = \{z \in \mathbb{C} : \operatorname{Re} z < 0\}$ is not, as $B(0, \varepsilon) \notin U$ no matter how small we make ε.

Theorem 2.1.1. *Let X be a metric space with metric d. Then X is a topological space whose open sets are those subsets of X containing ε-balls for every element, and for $\varepsilon > 0$.*

Definition. We call a subset V of a metrix space (X,d) **closed** if $X \backslash V$ is open in X.

Lemma 2.1.2. *If (X,d) is a metric space, then it is a topology by closed sets.*

Definition. Let $A \subseteq X$ where X is a metric space. We define the **interior** of A to be the union of all open sets contained in A, and write $\operatorname{int} A$. We define the **closure** of A to be the intersection of all closed sets containing A and write $\operatorname{cl} A$. We define the **boundry** of A to be $\partial A = \operatorname{cl} A \cap \operatorname{cl}(X \backslash A)$.

Example 2.3. We have $\operatorname{int} \mathbb{Q}(i) = \emptyset$ and $\operatorname{cl} \mathbb{Q}(i) = \mathbb{C}$.

Lemma 2.1.3. *Let X be a metric space and A, BX. Then the following are true*

(1) A is open if, and only if $A = \operatorname{int} A$.

(2) A is closed if, and only if $A = \operatorname{cl} A$.

(3) $\operatorname{int} A = X \backslash \operatorname{cl}(X \backslash A)$, $\operatorname{cl} A = X \backslash \operatorname{int}(X \backslash A)$, and $\partial A = \operatorname{cl} A \backslash \operatorname{int} A$.

(4) $\operatorname{cl}(A \cup B) = \operatorname{cl} A \cup \operatorname{cl} B$.

(5) $x_0 \in \operatorname{int} A$ if, and only if there is an $\varepsilon > 0$ for which $B(x_0, \varepsilon) \subseteq A$.

(6) $x_0 \in \operatorname{cl} A$ if, and only if for every $\varepsilon > 0$, $B(x_0, \varepsilon) \cap A = \emptyset$.

Definition. A subset A of a metric space X is **dense** in X if $\operatorname{cl} A = X$.

Example 2.4. $\mathbb{Q}$ is dense in $\mathbb{R}$, notice that $\operatorname{cl} \mathbb{Q} = \mathbb{R}$. Moreover, $\mathbb{Q}(i)$ is dense in $\mathbb{C}$.

2.2 Connectedness in $\mathbb{C}$

Definition. We say a metric space X is connected provided there are no disjoint nonempty open sets $A, B \subseteq X$ for which $X = A \cup B$.

Lemma 2.2.1. *A metric space X is connected if its only closed and open sets are the emtpyset and itself.*

Example 2.5. Consider the space $X = \{z \in \mathbb{C} : |z| < 1\} \cup \{z \in \mathbb{C} : |z - 3| < 1\}$. Let $A = \{z \in \mathbb{C} : |z| < 1\}$ and $B = \{z \in \mathbb{C} : |z - 3| < 1\}$. Then then both A and B are open in X. Moreover, A is also closed in X as $B = X \backslash A$. So X is not connected.

Lemma 2.2.2. *A space $X \subseteq X$ is connected if, and only if it is an interval.*

Proof. Suppose that $X = [a, b]$, where $a, b \in \mathbb{R}$ and $a < b$. Let $A \subseteq X$ be open, with $a \in A$ and $b \in B$ and where $X \neq A$. Then there is an $\varepsilon > 0$ for which $[a, a + \varepsilon) \subseteq A$. Let $r = \sup\{\varepsilon : [a, a + \varepsilon) \subseteq A\}$. If $a \leq x < a + r$, putting $h = a + (r - x) > 0$ there is an $\varepsilon > 0$ for which $r - h < \varepsilon < r$ and $[a, a + \varepsilon) \subseteq A$. However, $a \leq a + (r - h) < a + \varepsilon$ putting $x \in A$. So that $[a, a + r) \subseteq A$. Now, if $a + r \in A$, then by the openness of A, there is a $\delta > 0$ with $[a + r, a + r + \delta] \subseteq A$, which puts $[a + r, a + r + \delta) \subseteq A$. But that contradicrts that r is a least upper boundl; so $a + r \notin A$.

Now, if A were closed, then $a + r \in B = X \backslash A$, which is open, so that there is a $\delta > 0$ such that $(a + r - \delta, a + r) \subseteq B$, which contradicts that $[a, a + r) \subseteq A$. ∎

Remark. Note that the first part of this proof lacks the proof for the other types of intervals.

Definition. Let $z, w \in \mathbb{C}$. We define the **staight line segment** $[z, w]$ from z to w to be the set

$$[z, w] = \{tw + (1 - t)z : 0 \leq t \leq 1\}$$

A **polygon** from z to w is defined to be the set

$$P[z, w] = \bigcup_{k=1}^{n} [z_k, w_k]$$

where $z_1 = z$, $w_n = w$, and $z_{k+1} = w_k$ for all $1 \leq k \leq n - 1$. When the endpoints of the polygon are understood, we may simply just write P, or we enumerate the points of P as $P = [z, z_2, \ldots, z_n, w]$.

Theorem 2.2.3. *An open set U of $\mathbb{C}$ is connected if, and only if for all $z, w \in U$, there exists a polygon $P[z, w]$ from z to w contained in U.*

Proof. Let $P[z, w] \subseteq U$ be the given polygon. Suppose that U were not connected. Then there exist disjoint nonempty open sets Z and W of U (as a subspace of $\mathbb{C}$) for which $U = Z \cup W$. Let $z \in Z$ and $w \in W$. Consider the case for when $P[z, w] = [z, w]$. Define $S = \{s \in [0, 1] : sw + (1 - s)z \in A\}$ and $T = \{s \in [0, 1] : sw + (1 - s)z \in B\}$. Then notice that S and T are disjoint, and that $S \cup T = [0, 1]$. Moreover, they are open subsets of the interval $[0, 1] \subseteq \mathbb{R}$; but $[0, 1]$ is connected in $\mathbb{R}$, which is a contradiction. Therefore U must be connected.

On the otherhand, let $w \in Z$ and let $P = [z, z_2, \cdots z_n, w] \subseteq U$ SInce U is open, there is an $\varepsilon > 0$ such that $B(w, \varepsilon) \subseteq U$. Now, if $u \in B(w, \varepsilon)$, then $[w, u] \subseteq B(w, \varepsilon) \subseteq U$, so the polygon $Q = P \cup [w, u] \subseteq U$. Hence $B(w, \varepsilon) \subseteq Z$, which makes Z oepn. On the otherhand, consider $u \in U \backslash Z$, and let $\varepsilon > 0$ such that $B(u, \varepsilon) \subseteq U$. Then there is a $w \in Z \cap B(u, \varepsilon)$. Construct, then a polygon $P[z, u]$ so that $B(u, \varepsilon) \cap Z$ is empty. That is, $B(u, \varepsilon) \subseteq U \backslash Z$ making $U \backslash Z$ open, and hence Z closed. ■

Corollary. *If $U \subseteq \mathbb{C}$ is an open and connected set, then for all $z, w \in U$, there is a polygon $P[z, w]$ in U made up of straight line segments parallel to either the real axis, or the imaginary axis.*

Definition. Let X be a metric space. We call a subset $C \subset X$ a **connected component** if it is maximally connected in X.

Example 2.6. (1) A and B in example 2.5 are connected components.

(2) Let $X = \{\frac{1}{k} : k \in \mathbb{Z}^+\} \cup \{0\}$. Then every connected component is a point of x, and vise versa; with, the exception of 0.

Lemma 2.2.4. *Let X be a metric space with $x_0 \in X$. If $\{D_j\}$ is a collection of connected subsets of X, such that $x_0 \in D_j$, then the union $D = \bigcup D_j$ is connected.*

Proof. Let $A \subseteq D$, which is a metric space, for which A is both open and closed, and nonempty. Then $A \cap D_j$ is open and closed for all j. Now, since D_j is connected, either $A \cap D_j =$, or $A \cap D_j = D_j$. Since A is nonempty, we must have the latter case. Then there exists at least one index k for which $A \cap D_k = D_k$. Then if $x_0 \in A$, $x_0 \in A \cap D_k$ so that $x_0 \in D_k$ making $A \cap D_j = D_j$ for all j or $D_j \subseteq A$. In either case, we get $D = A$. ■

Theorem 2.2.5. *The connected components of a metric space partition the space.*

Proof. Let $\mathcal{D}$ the collection of all connected subsets of X containing a point $x_0 \in X$. Then $\mathcal{D}$ is nonempty by definition, and by hypothesis, we have that $C = \bigcup D_j$ is connected, and that $x_0 \in C$.

Now, suppose that $C \subseteq D$ for some connected st D. Then $x_0 \in D$ so that $D \in \mathcal{D}$, and hence $D \subseteq C$. This makes $C = D$, and hence C is a connected component of X. This then implies that $X = \bigcup C_j$ where $\{C_j\}$ is the collection of connected components of X.

Now, consider $\{C_j\}$, and suppose that for distinct components C_1 and C_2, that there is an $x_0 \in C_1 C_2$. Then $x_0 \in C_1$, and $x_0 \in C_2$ so that $C_1 = C_1 \cup C_2 = C_2$, which is a contradiction. Therefore the connected components are pairwise disjoint. ■

Lemma 2.2.6. *If X is a connected metric space with $A \subseteq X$, and $A \subseteq B \subseteq \operatorname{cl} A$, then B is also connected.*

Corollary. *Connected components of a metric space are closed.*

Theorem 2.2.7. *If U is open in $\mathbb{C}$, then U has countably many connected components; each of which is open.*

Proof. Let $C \subseteq U$ a connected component, with $x_0 \in C$. Since U is open, there is an $\varepsilon > 0$ for which $B(x_0, \varepsilon) \subseteq U$. Then $B(x_0, \varepsilon) \cup C$ is connected so that $B(x_0, \varepsilon) \cup C = C$, so that $B(x_0, \varepsilon) \subseteq C$. This makes each C open.

Now, let $S = \{a + ib \in \mathbb{Q}(i) : a + ib \in U\}$. Then S is countable by the density of $\mathbb{Q}(i)$ in $\mathbb{C}$, and each connected component of U contains a point of S. This implies there are countably many such components. ■

2.3 Completeness in $\mathbb{C}$

Definition. We say a sequence $\{x_n\}$ of points of a metric space X **converges** to a point $x \in X$ if for every $\varepsilon > 0$, there is anb $N \in \mathbb{Z}^+$ for which

$$d(x, x_n) < \varepsilon \text{ whenever } n \geq N$$

If $\{x_n\}$ converges to x, we write $\{x_n\} \to x$, or $\lim x_n = x$.

Lemma 2.3.1. *Let X be a metric space. A set $V \subseteq X$ is closed if, and only if for every sequece $\{x_n\}$ of points in V, $\{x_n\}$ converges to a point $x \in V$.*

Proof. If V is closed, and $\{x_n\} \to x$, then for every $\varepsilon > 0$ and $x_n \in B(x, \varepsilon)$, we get that $B(x, \varepsilon) \cap V \neq \emptyset$ so that $x \in \operatorname{cl} F = F$.

Conversly, suppose that V is not closed. Then there exists a point $x_0 \in \operatorname{cl} V \backslash V$. Then we get that for every $\varepsilon > 0$, the set $B(x_0, \varepsilon) \cap F \neq \emptyset$ so that for all $n \in \mathbb{Z}^+$, there is an $x_n \in B(x_0, \frac{1}{n}) \cap F$. This makes $d(x_0, x_n) < \frac{1}{n}$, so that $\{x_n\} \to x_0$. Since $x_0 F$, the condition fails. ■

Definition. We call a point $x \in X$ of a metric space X a **limit point** of a subset $A \subseteq X$ if there exists a sequence of points $\{x_n\}$ in A such that $\{x_n\} \to x$.

Example 2.7. Consider $\mathbb{C}$ and let $A = [0, 1] \cup \{i\}$. Then each point of $[0, 1]$ is a limit point of A, but i is not a limit point of A.

Lemma 2.3.2. *A subset of a metric space is closed if, and only if it contains all its limit points. Moreover, if A is a subset of a metric space X, then $\operatorname{cl} A = A \cup A'$, where A' is the collection of all limit points of A.*

Definition. We call a sequence $\{x_n\}$ of points of a metric space **Cauchy** if for every $\varepsilon > 0$ there is an $N \in \mathbb{Z}^+$ for which

$$d(x_m, x_m) < \varepsilon \text{ for all } m, n \geq N$$

If X is a metric space in which every Cauchy sequence converges in to a point in X, then we say X is **complete**.

Theorem 2.3.3. *The field $\mathbb{C}$ of complex numbers is complete.*

Proof. Let $\{z_n\}$ a Cauchy sequence of complex numbers with $z_n = x_n + iy_n$. Then the sequences $\{x_n\}$ and $\{y_n\}$ are Cauchy in $\mathbb{R}$. Since $\mathbb{R}$ is a complete metric space, we observe that there exist $x, y \in \mathbb{R}$ for which $\{x_n\} \to x$ and $\{y_n\} \to y$. This makes $\{z_n\} \to z$ with $z = x + iy \in \mathbb{C}$. ■

Definition. Let X be a metric space and $A \subseteq X$. We define the **diameter** of A to be the least upper bound:

$$\operatorname{diam} A = \sup \{d(x, y) : x, y \in A\}$$

of all distances of points in A.

Theorem 2.3.4 (Cantor's theorem)**.** *A metric space X is complete if, and only if for every decreasing sequence $\{F_n\}$ of nonempty closed sets, with $\operatorname{diam} F_n \to 0$ for all n, then the intersection*

$$F = \bigcap F_n$$

consists of a single point.

Proof. Suppose that X is complete. Let $\{F_n\}$ a sequence of closed sets such that

(1) $F_{n+1} \subseteq F_n$; i.e. $\{F_n\}$ is a decreasing sequence.

(2) $\lim \operatorname{diam} F_n \to 0$.

Let $x_n \in F_n$. If $n, m \geq N$ then $x_m, x_n \in F_N$ so that $d(x_m, x_n) \leq \operatorname{diam} F_n$ by definition. By hypothesis, choose an N large enough such that $\operatorname{diam} F_N < \varepsilon$ for some $\varepsilon > 0$. This makes the sequence $\{x_n\}$ Cauchy. Then by the completeness of X $\{x_n\} \to x$ for some $x \in X$. Since $x_n \in F_n$ for all $n \geq N$, we get that $F_n \subseteq F_N$ and hence $x \in F_N$ which puts

$$x \in F = \bigcap F_n$$

Now, if $y \in F$, then $x, y \in F_n$ for all n which gives $d(x, y) \leq \operatorname{diam} F_n \to 0$. So $d(x, y) = 0$ which makes $x = y$ and so $F = \{x\}$.

Conversely, let $\{x_n\}$ be Cauchy in X, and take $F_n = \operatorname{cl} \{x_n, x_{n+1}, \dots\}$. Then $F_{n+1} \subseteq F_n$, making $\{F_n\}$ decreasing sequence. If $\varepsilon > 0$, choose an $N > 0$ such that $d(x_m, x_n) < \varepsilon$ for any $m, n \geq N$. Then $\operatorname{diam} F_n \leq \varepsilon$. By hypothesis, there is an $x_0 \in X$ such that $F = \bigcap F_n = \{x_0\}$. Moreover, $x_0 \in F_n$ so that $d(x_0, x_m) \leq \operatorname{diam} F_n \to 0$, which puts $\{x_n\} \to x \in X$ which makes X complete. ■

Lemma 2.3.5. *If X is a complete metric space, and $Y \subseteq X$, then Y is complete if, and only if Y is closed in X.*

Proof. Suppose that Y is complete and let y a limit point of Y. Then there exists a sequence $\{y_n\}$ of points of Y for which $\{y_n\} \to y$. This makes $\{y_n\}$ Cauchy, and so $\{y_n\} \to x_0 \in Y$. It follows that $y = x_0$, so that $Y' \subseteq Y$ and hence Y is closed. ■

2.4 Compactness in $\mathbb{C}$

Definition. Let X be a metric space. We say an collection $\{U_n\}$ of open sets of X **covers** a subset K of X if $K \subseteq \bigcup U_n$. We call $\{U_n\}$ an **open cover** of K. We call K **compact** if every open cover of K has a finite open subcover.

Lemma 2.4.1. *If K is compact in a metric space X, then K is closed. Moreover, if $F \subseteq K$ is closed, then F is also compact.*

Proof. Certainly, we have $K \subseteq \operatorname{cl} K$. Now, let $x_0 \in \operatorname{cl} K$, then $B(x_0, \varepsilon) \cap K$ is nonempty for every $\varepsilon > 0$. Let $G_n = X \backslash \overline{B}(x_0, \frac{1}{n})$, and suppose that $x_0 \notin K$. Then each G_n is open in X, and $K \subseteq \bigcup G_n$. Since K is compact, then ther is an $m \in \mathbb{Z}^+$ for whcih $K \subseteq \bigcup_{n=1}^{m} G_n$. Notice, however that $G_1 \subseteq G_2 \subseteq \cdots \subseteq G_m \subseteq \dots$ so that $K \subseteq G_m = X \backslash \overline{B}(x_0, \frac{1}{m})$, so that $B(x_0, \frac{1}{n}) \cap K = \emptyset$; a contradiction! Therefore $x_0 \in K$ and $K = \operatorname{cl} K$. ∎

Definition. Let X be a set. We say a collection $\{F_n\}$ of subsets of X has the **finite intersection property** (**FIP**) if the intersection of any finite subcollection of $\{F_n\}$ is nonempty.

Lemma 2.4.2. *A set K of a metric space X is compact if, and only if for every collection of closed sets $\{F_n\}$ satisfying the finite intersection property, the intersection*

$$F = \bigcap F_n$$

is nonempty.

Proof. Let K be compact in X, and $\{F_n\}$ a collection of closed sets of X with the FIP. Suppose that $F = \bigcap F_n = \emptyset$. Now, take $\mathcal{G} = \{X \backslash F_n\}$ the collecton of open sets. Then observe that

$$\bigcup X \backslash F_n = X \backslash \bigcap F_n = X \backslash F = X$$

by hypothesis. SInce $K \subseteq K$, $\mathcal{G}$ covers K, and since K is compact, there is a finite subcover $\{X \backslash F_i\}_{i=1}^{n}$ of K. That is

$$K \subseteq \bigcup_{i=1}^{n} X \backslash F_i = X \backslash \bigcap_{i=1}^{n} F_i \subseteq X$$

since $\bigcap_{i=1}^{n} F_n \neq \emptyset$. But then $\bigcap_{i=1}^{n} F_i \subseteq X \backslash K$, and since $F_i \subseteq K$ for all $1 \leq i \leq n$, this makes $\bigcap_{i=1}^{n} F_i =$; a contradiction! ∎

Corollary. *Compact metric spaces are complete.*

Corollary. *If X is compact, then every infinite set in X has a limit point in X.*

Proof. Let $S \subseteq X$ infinite, and suppose the set of all limit points of S in X, S', is empty. Consider the sequence $\{a_n\}$ of distinct points of S, and take $F_n = \{a_n, a_{n+1}, \dots\}$. Then F_n has no limit points in X so that $F_n' = \emptyset$. Then $F_n' \subseteq F_n$ so that F_n is closed. Thus $\{F_n\}$ has the finite intersection property. But since $a_1 \neq \ldots \neq a_n \neq$, we get $\bigcap F_n = \emptyset$; which contradicts the above. Therefore S' is nonempty. ∎

Definition. We call a metric space **sequentially compact** if every sequence of point in the space has a convergent subsequence.

Lemma 2.4.3 (Lebesgue's Covering Lemma)**.** *If X is a sequentially compact metric space, and $\mathcal{G}$ is an open cover of X, then there is an $\varepsilon > 0$ such that if $x \in X$ there is a $G \in \mathcal{G}$ with $B(x, \varepsilon) \subseteq G$.*

Proof. Suppose by contradiction that for every open cover $\mathcal{G}$ of X there is no ε for which the statement holds. Then for every $n \in \mathbb{Z}^+$, there is an $x_n \in X$ for which $B(x_n, \frac{1}{n}) \not\subseteq G$. Now, since X is sequentially compact, there is a point $x_0 \in X$ and s subsequence $\{x_{n_k}\}$ of a sequence $\{x_n\}$ for which $\{x_{n_k}\} \to x_0$. Let $G_0 \in \mathcal{G}$ such that $x_0 \in G_0$. Choose $\varepsilon > 0$ such that $n_k \geq N$ and $n_k > \frac{1}{\varepsilon}$. Let $y \in B(x_{n_k}, \frac{1}{n_k})$. THen $d(x_0, y) \leq d(x_0, x_{n_k}) + d(x_{n_k}, y) < \frac{\varepsilon}{2} + \frac{1}{n_k} < \varepsilon$. So that $B(x_{n_k}, \frac{1}{n_k}) \subseteq B(x_0, \varepsilon)$. But that contradicts our choice of $\{x_{n_k}\}$. ∎

Definition. We say a subset K of a metric space X is **totally bounded** if for any $\varepsilon > 0$ there exist a sequence $\{x_n\}$ of points of X for which $K = \bigcup_{k=1}^n B(x_k, \varepsilon)$.

Theorem 2.4.4. *The following are equivalent in every metric space X.*

(1) X is compact.

(2) Every infinite set of X has a limit point in X.

(3) X is sequentially compact.

(4) X is complete, and totally bounded.

Proof. We have that if X is compact, then every infinite set of X has their limit points in X, by the above corollory.

Suppopse every infinite set of X has a limit point in X. Let $\{x_n\}$ a sequence, and suppose without loss of generality, that all the points are distinct. Then $\{x_n\}$ has a limit point x_0. Then there exist an $x_{n_1} \in B(x_0, 1)$. Similarly, there is an $n_2 > n_1$ with $x_{n_2} \in B(x_0, \frac{1}{2})$. Continuiing in this manner, we get for some $n_k > n_{k-1}$, that $x_{n_k} \in B(x_0, \frac{1}{k})$, so that $\{x_{n_k}\} \to x_0$; and so X is sequentially compact.

Suppose now that X is sequentially compact, and let $\{x_n\}$ be a Cauchy sequence. By the sequential compactness of $\{x_n\}$, it has a convergent subsequence, which makes X complete. Now, let $\varepsilon > 0$ and fix $x_1 \in X$. If $X = B(x_1, \varepsilon)$, we are done. Otherwise, choose an $x_2 \in X \backslash B(x_1, \varepsilon)$. If $X = B(x_1, \varepsilon) \cup B(x_2, \varepsilon)$ we are done. Otherwise, continuing in this manner, we find a sequence $\{x_n\}$ of points with $x_{n+1} \in X \backslash \bigcup_{k=1}^n B(x_k, \varepsilon)$. Which implies for $m \neq n$, that $d(x_m, x_n) \geq \varepsilon > 0$. Contradictiong that X is sequentially compact. So we have that X must be totally bounded.

Conversely, suppose that X is complete and totally bounded. Let $\{x_n\}$ a sequence of distint points. Then there is a $y_1 \in X$ and a subsequence $\{x_n^{(1)}\}$ of $\{x_n\}$ for which $\{x_n^{(1)}\} \subseteq B(y_1, 1)$. There also exists a $y_2 \in X$ and s a subsequence $\{x_n^{(2)}\}$ of $\{x_n^{(1)}\}$ such that $\{x_n^{(2)}\} \subseteq B(y_2, \frac{1}{2})$. Continuing in this manner, for all $k \geq 2$, there is a $y_k \in X$ and a subsequence $\{x_n^{(k)}\}$ of $\{x_n^{(k-1)}\}$ for which $\{x_n^{(k)}\} \subseteq B(y_k, \frac{1}{k})$. Take $K_k \operatorname{cl} \{x_n^{(k)}\}$. Then

$$\operatorname{diam} F_k \leq \frac{1}{k}$$

and $\{F_k\}$ is a decreasing collection of closed sets. Thus the intersection $F = \{x_0\}$ is a single point. So $x_0 \in F_k$, so that

$d(x_0, x_n^{(k)}) \leq F_k \leq \frac{1}{k}$ so that $\{x_n^{(k)}\} \to x_0$, making X sequentially compact.

Finally, if X is sequentially compact, and $\mathcal{G}$ is an open cover of X, then there exists an $\varepsilon > 0$ such that for every $x \in X$, there is a $G \in \mathcal{G}$, with $B(x, \varepsilon) \subseteq G$. Hence there is a sequence $\{x_n\}$ of points of X for which $X = \bigcup B(x_n, \varepsilon)$ (i.e. X is totally bounded). Then there is a $G_n \in \mathcal{G}$ for all $1 \leq k \leq n$ for which $B(x_k, \varepsilon) \subseteq G_k$. So tghat $X = \bigcup G_k$ which makes X compact. ■

Theorem 2.4.5 (Heine-Borel)**.** *A subset K of $\mathbb{R}^n$ is compact if, and only if it is closed and bounded.*

Proof. Suppose that K is compact, then K is closed by lemma 2.4.1, and K is also totally bounded, which makes K bounded. So K is closed and bounded in $\mathbb{R}^n$.

Conversely, suppose that K is closed and bounded. Then there are sequences $\{a_k\}_{k=1}$ and $\{b_k\}_{k=1}^n$ for which $K \subseteq [a_1, b_1] \times [a_n, b_n]$. Now, since $\mathbb{R}^n$ is complete, and K is closed, K is also complete. Hence it remains to show that K is totally bounded. Let $\varepsilon > 0$, and write K as the union of n-dimensional rectangles of diameters less than ε. Then $K \subseteq \bigcup_{k=1}^m B(x_k, \varepsilon)$ where x_k is contained in one of the rectangles, for all $1 \leq k \leq m$. This makes K totally bounded, and therefore, compact. ■

2.5 Continuity and Uniform Convergence in $\mathbb{C}$

Definition. Let (X, d) and (Y, ρ) be metric spaces, and $f : X \to Y$ a function. We say that f is **continuous** at a point $a \in X$ if for every $\varepsilon > 0$, there is a $\delta > 0$ for which

$$\rho(f(x), y) < \varepsilon \text{ whenever } 0 < d(x, a) < \delta$$

for some $y \in Y$ and we write $\lim_{x \longrightarrow a} f(x) = y$, or simply $f \to y$. If f is continuous at every point in X, we say that f is **continuous** on X (or simply that f is **continuous**).

Lemma 2.5.1. *Let X and Y be metric spaces. If $f : X \to Y$ is a function, then the following statements are equivalent for any $a \in X$ with $y = f(a)$.*

(1) f is continuous at a.

(2) For any $\varepsilon > 0$ $f^{-1}(B(y, \varepsilon))$ contains a ball centered about a.

(3) If $\{x_n\}$ is a sequence of points of X converging to a, then the sequence $\{f(x_n)\}$ converges to y.

Lemma 2.5.2. *Let X and Y be metric spaces, and $f : X \to Y$ a function. The following statements are equivalent.*

(1) f is continuous on X.

(2) For any open set U of Y, $f^{-1}(U)$ is open in X.

(3) For any closed set V of Y, $f^{-1}(V)$ is closed in X.

Lemma 2.5.3. *Let $f : X \to \mathbb{C}$ and $g : X \to \mathbb{C}$ be complex-valued functions. If f and g are continuous, then for every $\alpha, \beta \in \mathbb{C}$, we have*

(1) $\alpha f + \beta g$ is continuous.

(2) fg is continuous, and $\frac{f}{g}$ is continuous provided $g(z) \neq 0$ for all $z \in X$.

Lemma 2.5.4. *If $f : X \to Y$ and $g : Y \to Z$ are continuous, then $g \circ f : X \to Z$ is continuous.*

Definition. We call a function $f : X \to Y$ **uniformly continuous** if for every $\varepsilon > 0$, there is a $\delta > 0$, depending on ε, such that

$$\rho(f(x), f(y)) < \varepsilon \text{ whenever } d(x, y) < \delta$$

We call f **Lipschitz continuous** if there exists an $M > 0$ such that

$$\rho(f(x), f(y)) = Md(x, y) \text{ for all } x, y \in X$$

Lemma 2.5.5. *Lipschitz continuous functions are uniformly continuous, and uniformly contiuous functions are continuous.*

Definition. Let X be a metric space, and $A \subseteq X$ a nonempty subset. We define the **distance** from a point $x \in X$ to A to be

$$d(x, A) = \inf \{d(x, a) : a \in A\}$$

Lemma 2.5.6. *Let X a metric space, and $A \subseteq X$ nonempty. The following are true.*

(1) $d(x, A) = d(x, \operatorname{cl} A)$.

(2) $d(x, A) = 0$ if, and only if $x \in \operatorname{cl} A$.

(3) $|d(x, A) - d(y, A)| \leq d(x, y)$ for all $x, y \in X$.

Proof. Let $A \subseteq B$. Then by definition, $d(x, B) \leq d(x, A)$, so that $d(x, \operatorname{cl} A) \leq d(x, A)$. Now, if $\varepsilon > 0$, there is a $y \in \operatorname{cl} A$ for which $d(x, y) \leq d(x, \operatorname{cl} A) + \frac{\varepsilon}{2}$, and there exists an $a \in A$ with $d(y, a) < \frac{\varepsilon}{2}$. Then

$$|d(x, y) - d(x, a)| < d(y, a) < \frac{\varepsilon}{2}$$

by the triangle inequality. Then $d(x, a) < d(x, y) + \frac{\varepsilon}{2}$ so that $d(x, A) < d(x, \operatorname{cl} A) + \frac{\varepsilon}{2}$. That is $d(x, A) \leq d(x, \operatorname{cl} A)$.

Now, if $x \in \operatorname{cl} A$, then $d(x, \operatorname{cl} A) = d(x, A) = 0$. Conversly, if $d(x, A) = 0$, then consider the decreasing sequence $\{a_n\}$ of A such that $\lim d(x, a_n) = d(x, A)$. Then $\lim d(x, a_n) = 0$ so that $\lim a_n = x$, so that $x \in \operatorname{cl} A$.

Finally, we have for $a \in A$ that $d(x, a) \leq d(x, y) + d(y, a)$, so that $d(x, A) \leq \inf \{d(x, y) + d(y, a) : a \in A\}$ $d(x, y) + d(y, A)$. This gives $d(x, A) - d(y, A) \leq d(x, y)$. Similar reasoning also gives $d(y, A) - d(x, A) \leq d(x, y)$ so that

$$|d(x, A) - d(y, A)| \leq d(x, y) \text{ for all } x, y \in X$$

∎

Corollary. *The function $f : X \to \mathbb{R}$ defined by $f(x) = d(x, A)$ is Lipschitz continuous.*

Theorem 2.5.7. *Let $f : X \to Y$ be continuous. Then following are true.*

(1) If X is compact, then so is $f(X)$.

(2) If X is connected, so is $f(X)$.

Proof. Without loss of generality, suppose $f(X) = Y$. If X is compact, et $\{y_n\}$ a sequence in Y. Then for every $n \geq 1$, there is a sequence of points $\{x_n\}$ of X with $f(x_n) = y_n$, and $\{x_{n_k}\} \to x$. If $y = f(x)$, then by continuity, $\{y_{n_k}\} \to y$ so that Y is also compact.

Now, if X is connected, let $S \subseteq Y$ a nonempty set wich is both open and closed. Then $f^{-1}(S) \neq \emptyset$ and $f^{-1}(S)$ is also open and closed, so that $X = f^{-1}(S)$ by connectivity. This makes $S = Y$, and so Y must also be continuous. ■

Corollary. *If K is compact or connected in X, then $f(K)$ is compact or connected in Y.*

Corollary. *If $f : X \to \mathbb{R}$ is continuous, and X is connected, then $f(X)$ is an interval.*

Theorem 2.5.8 (The Intermediate Value Theorem)**.** *If $f[a,b] \to \mathbb{R}$ is continuous, with $f(a) \leq c \leq f(b)$, then there is an $x \in [a,b]$ with $f(x) = c$.*

Corollary. *If $K \subseteq X$ is compact, then there exist $x_0, y_0 \in K$ with $f(x_0) = \sup\{f(x) : x \in K\}$ and $f(y_0) = \inf\{f(y) : y \in K\}$.*

Corollary. *If $K \subseteq X$ is nonempty, and $x \in X$, there is a $y \in K$ for which $d(x,y) = d(x,K)$.*

Proof. Define $f : X \to \mathbb{R}$ by $f(y) = d(x,y)$. Then f is continuous, and by above, assumes a minimum value $yinK$. Then $f(y) \leq f(x)$ for all $x \in K$, so that $d(x,y) = d(x,K)$ by definition. ■

Theorem 2.5.9. *Let $f : X \to Y$ be continuous. If X is compact, then f is uniformly continuous.*

Proof. Let $\varepsilon > 0$ and suppose there is no such $\delta > 0$ for which the statement holds. Then each $\delta = \frac{1}{n}$ in particular fais. Then there exist $x_n, y_n \in X$ with $d(x_n, y_n) < \frac{1}{n}$, but where $\rho(f(x_n), f(y_n)) \geq \varepsilon$. Now, since X is compact, there is a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ converging to a point $x \in X$. Now, $d(x, y_{n_k}) \leq d(x, x_{n_k}) + \frac{1}{n_k}$ which goes to 0 as $k \to \infty$. SO $\{y_{n_k}\} \to x$. But if, $y = f(x)$, and $y = \lim f(x_{n_k}) = \lim f(y_{n_k})$, then we get

$$\varepsilon \leq \rho(f(x_{n_k}), f(y_{n_k})) \leq \rho(f(x_{n_k}), y) + \rho(y, f(y_{n_k})) = 0$$

which is a contradiction since $\varepsilon > 0$. ■

Definition. If $A, B \subseteq X$ are nonempty subsets of a metric space X, we define the **distance** between A and B to be

$$d(A,B) = \inf\{d(a,b) : a \in A, b \in B\}$$

Theorem 2.5.10. *let A and B be disjoint subsets of a metric space X; with B closed, and A compact. Then $d(A,B) > 0$.*

Proof. Define $f : X \to \mathbb{R}$ by $f(x) = d(x,B)$. Since A and B are disjoint, and B is closed, $f(a) > 0$ for all $a \in A$. Moreover, since A is compact, there is an $a \in A$ for which $0 < f(a) = \inf\{f(x) : x \in A\} = d(A,B)$. ■

Definition. Let X be a set, and (Y, ρ) a metric space; and let $\{f_n\}$ a sequence of functions from X to Y. We say that $\{f_n\}$ **converges uniformly** if for every $\varepsilon > 0$, there is an $N > 0$, dependent on ε such that

$$\rho(f(x), f_n(x)) < \varepsilon \text{ whenever } n \geq N$$

for all $x \in X$. We write $\{f_n\} \xrightarrow{\text{uniformly}} f$, or just $\{f_n\} \to f$.

Theorem 2.5.11. *If $f_n : X \to Y$ is continuous for each $n \geq 1$, and $\{f_n\} \xrightarrow{\textit{uniformly}} f$, then f is also continuous.*

Proof. Fix $x_0 \in X$ and let $\varepsilon > 0$. Since $\{f_n\} \to f$, there is a function f_n for which $\rho(f(x), f_n(x)) < \frac{\varepsilon}{3}$ for every $x \in X$. Since f_n is continuous, there is a $\delta > 0$ such that

$$\rho(f_n(x_0), f_n(x)) < \frac{\varepsilon}{3} \text{ whenever } d(x, x_0) < \delta$$

Therefore, if $d(x_0, x) < \delta$ we have

$$\rho(f(x_0), f(x)) \leq \rho(f(x_0), f_n(x_0)) + \rho(f_n(x_0), f_n(x)) + \rho(f_n(x), f(x)) < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon$$

so that f is continuous. ∎

Theorem 2.5.12 (The Weierstrass M-test)**.** *Let $u_n : X \to \mathbb{C}$ be a function such that $|u_n(x)| \leq M_n$, for all $x \in X$, and suppose that the sum $\sum M_n$ is finite. THen $\sum u_n$ is uniformly convergent.*

Proof. Let $f_n(x) = u_1(x) + \cdots + u_n(x)$. Then for $n > m$, $|f_n(x) - f_m(x)| = |u_{m+1}(x) + \cdots + u_n(x)| \leq \sum_{k=m+1}^{n} M_k$. Since $\sum M_k$ is finite, this sum converges, so that $\{f_n\}$ is Cauchy in $\mathbb{C}$. That is, there exists a $\xi \in \mathbb{C}$ for which $\{f_n(x)\} \to \xi$. Define then $f(x) = \xi$, then $f : X \to \mathbb{C}$ is a function with

$$|f(x) - f_n(x)| = |u_{m+1}(x) + \cdots + u_n(x)| \leq \sum_{k=m+1}^{n} |u_k(x)| \leq \sum_{k=m+1}^{n} M_k$$

Then for every $\varepsilon > 0$, there is an $N > 0$ such that $\sum M_k < \varepsilon$, whenever $n \geq N$. THus $|f(x) - f_n(x)| < \varepsilon$ for all $x \in X$. ∎

Chapter 3

Holomorphic Functions

3.1 Convergent Power Series

Definition. For a sequence $\{a_n\}$ of points of $\mathbb{C}$, the series $\sum_{n=0}^{\infty} a_n$ is said to **converge** to a point $z \in \mathbb{C}$ if for all $\varepsilon > 0$, there is an $N \in \mathbb{Z}^+$ such that $|s_m - z| < \varepsilon$, whenever $m \geq N$; where

$$s_m = \sum_{n=0}^{m} = a_n$$

is the n**-th partial sum**. Se say that the series $\sum a_n$ **converges absolutely** if the series $\sum |a_k|$ converges.

Lemma 3.1.1. *Let $\{a_n\}$ a sequence of points in $\mathbb{C}$. If the series $\sum a_n$ converges absolutely, then it converges.*

Proof. Let $\varepsilon > 0$ and put $z_n = a_0 + a_1 + \cdots + a_n$. Since the series $\sum |a_n|$ converges, there is an $N \in \mathbb{Z}^+$ such that $\sum_{n=N}^{\infty} |a_n| < \varepsilon$. Tus, if $m > k \geq N$, we have

$$|z_m - z_k| = \Big| \sum_{n=k+1}^{m} |a_n| \Big| \leq \sum_{n=k+1}^{m} |a_n| \leq \sum_{n=N}^{m} |a_n| < \varepsilon$$

This makes $\{z_n\}$ a Cauchy sequence in $\mathbb{C}$, si that $\{z_n\} \to z$. Therefore $\sum a_n = z$. ■

Definition. Let $\{a_n\}$ a sequence of points of $\mathbb{C}$. A **power series** about a point $z_0 \in \mathbb{C}$ is a series of the form

$$\sum a_n (z - z_0)^n$$

We say the power series is **convergent**, if the series converges.

Example 3.1. The **geometric series** $\sum z^n$ is a power series. Notice that

$$1 - z^{n+1} = (1 - z)(1 + z + \cdots + z^n)$$

so that

$$1 + z + \cdots + z^n = \frac{1 - z^{n+1}}{1 - z}$$

Now, when $|z| < 1$, $z^n \to 0$ and the series

$$\sum z^n = \frac{1}{1-z}$$

When $|z| > 1$, the series diverges.

Theorem 3.1.2. *Let $S = \sum a_n(z-z_0)^n$ be a power series, and define R such tht $0 \leq R \leq \infty$ by*

$$\frac{1}{R} = \limsup \sqrt[n]{|a_n|}$$

Then the following hold

(1) If $|z - z_0| < R$, then S converges absolutely.

(2) If $|z - a| > R$, then S diverges.

(3) If r is such that $0, r < R$, then S converges uniformly on the open ball $B(z_0, r)$.

Proof. Suppose without loss of generality, that $z_0 = 0$. If $|z| < R$, then there exists an r with $|z| < r < R$ and hence an $N \in \mathbb{Z}^+$ such that $\sqrt[n]{|a_n|} = \frac{1}{r}$ for all $n \geq N$; since $\frac{1}{r} < \frac{1}{R}$. Then we get

$$|a_n| < \frac{1}{r^n}$$

and so $|a_n z^n| < (\frac{|z|}{r})^n$. Hence, the tail, $\sum_{n=N}^{\infty} a_n z^n$ is dominated by the sum $\sum (\frac{|z|}{r})^n$, and since $\frac{|z|}{r} < 1$, we get that S converges absolutely for all $|z| < R$; i.e. the ball $B(0, R)$.

Now, suppose that $r < R$ and choose a $r < \rho < R$ as above. Take $N \in \mathbb{Z}^+$ such that $|a_n| < \frac{1}{\rho^n}$ for all $n \geq N$. Then if $|z| \leq r$, $|a^n_z| \leq (\frac{z}{\rho})^n$ and $\frac{r}{\rho} < 1$. By the Weierstrass M-test, we get that the series S converges uniformly on the ball $B(0, r)$.

Now, let $|z| > R$ and choose an r with $|z| > r > R$ so that $\frac{1}{r} < \frac{1}{R}$. Then $\sqrt[n]{|a_n|}$ gives infinitely many integers n with $\frac{1}{r} < \sqrt[n]{|a_n|}$. Hence

$$|a_n z^n| > \left(\frac{|z|}{r}\right)^n$$

and since $\frac{|z|}{r} > 1$, the terms become unbounded, making S diverge. ∎

Definition. We define the **radius of convergence** of a power series $\sum a_n(z_- z_0)^n$ to be a number R such that $0 \leq R \leq \infty$ and the following hold

(1) If $|z - z_0| < R$, then S converges absolutely.

(2) If $|z - a| > R$, then S diverges.

(3) If r is such that $0, r < R$, then S converges uniformly on the open ball $B(z_0, r)$.

Lemma 3.1.3. *If $\sum a_n(z-z_0)^n$ is a power series with radius of convergence $R > 0$, then*

$$R = \lim_{n \longrightarrow \infty} \left| \frac{a_n}{a_{n+1}} \right|$$

Proof. Without loss of generality, let $z_0 = 0$ and take $\alpha = \lim |\frac{a_n}{a_{n+1}}|$, and suppose that this limit does indeed exist. Suppose that $|z| < r < \alpha$ and take $N \in \mathbb{Z}^+$ such that $r < |\frac{a_n}{a_{n+1}}|$ for all $n \geq N$. Take $B = |a_N|r^N$. Then $|a_{N+1}r^{N+1} = |a_{N+1}|rr^N < |a_N|r^N = B$. That $|a_{N+2}|r^{N+2} = |a_{N+2}|rr^{N+2} < |a_{N+1}|r^{N+1} < B$. By induction we get $|a_n|r^N \leq B$ for all $n \geq N$. Then $|a_n z^n| = |a_n r^n|\frac{|z|^n}{r}$ for all $n \geq N$. Since $|z| < r$, we get that the series $\sum |a_n z^n|$ is dominated by a convergent series and hence is convergent itself.

Now, if $|z| > r > \alpha$. then $|a_n| < r|a_{n+1}|$ for all $n \geq N$, for some $N \in \mathbb{Z}^+$. We find that

$$|a_n r^n| \geq B = |a_N r^N|$$

so we get

$$|a_n z^n| \geq B\frac{|z|^n}{|r|^n}$$

and $B\frac{|z|^n}{|r|^n} \to \infty$ as $n \to \infty$. Therefore the series $\sum a_n z^n$ diverges so that $R \leq \alpha$. THis makes $R = \alpha$ and we are done. ■

Example 3.2. The **exponential series** defined by

$$\exp z = \sum \frac{z^n}{n!}$$

converges on all $\mathbb{C}$ and has radius of convergence $R = \infty$.

Lemma 3.1.4. *LEt $\sum a_n(z - z_0)^n$ and $\sum b_n(z - z_0)^n$ be convergent power series with radi of convergence greater than some $r > 0$. Let $c_n = \sum_{k=0}^n a_k b_{n-k}$. THen the series*

$$\sum (a_n + b_n)(z - z_0)^n \text{ and } \sum c_n(z - z_0)^n$$

are convergent power series with radi of convergent greater than r.

3.2 Holomorphic Functions

Definition. Let U be an open set in $\mathbb{C}$, and $f : U \to \mathbb{C}$ a complex valued function. We call f **complex differentiable** at a point $z_0 \in U$ if

$$f'(z_0) = \lim_{h \longrightarrow 0} \frac{f(z_0 + h) - f(z_0)}{h} = \lim_{z \longrightarrow z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

exists, and we call $f'(z_0)$ the **complex derivative** of f at z_0. We call f **complex differentiable** on U if it is complex differentiable at every point $z_0 \in U$, and the value $f'(z_0)$ defines a function $f' : U \to \mathbb{C}$ called the **complex derivative** of f on U. If f is complex differentiable on U, and f' exists on U and is continuous, then we call f **continuously differentiable**, and say it is of class C^1.

Definition. Let U be an open set of $\mathbb{C}$, and let $f : U \to \mathbb{C}$ be a continuously differentiable complex valued function; i.e. of class C^1. We define the n**-th derivative** of f recuresively as

(1) $f^{(0)}(z) = f$ and $f^{(1)} = f'(z)$.

(2) $f^{(n+1)}(z) = (f^{(n)})'(z)$, provided $f^{(n)}(z)$ exists.

We say that f is **n-th differentiable** if $f^{(n)}$ exists. We say that f is **q-smooth** if $f^{(q)}(z)$ exists for some $q \geq 0$, and $f^{(q)}$ is continuous, and we call f of class C^q. We call f **smooth** if $f^{(n)}(z)$ exists and is continuous for any n, and we call f of class C^∞.

Lemma 3.2.1. *If $f : U \to \mathbb{C}$ is a complex valued function, complex differentiable at a point $z_0 \in U$, then f is continuous on U.*

Proof. We have

$$|f(z) - f(z_0)| = \left|\frac{f(z) - f(z_0)}{z - z_0}\right| |z - z_0|$$

Taking $z \to z_0$, we see that $|f(z) - f(z_0)| \to |f'(z_0)| \cdot 0 = 0$. ∎

Corollary. *If f is n-th differentiable, for all $n \in \mathbb{Z}^+$, then f is of class C^∞.*

Definition. We call a complex valued function $f : U \to \mathbb{C}$, on an open set U of $\mathbb{C}$, **holomorphic** on U if it is of class C^1 on U.

Theorem 3.2.2. *Let U be open in $\mathbb{C}$, and let $f : U \to \mathbb{C}$ and $g : U \to \mathbb{C}$ holomorphic on U. Then*

(1) $f + g$ is holomorphic on U with $(f + g)'(z) = f'(z) + g'(z)$.

(2) fg is holomorphic on U, with $(fg)'(z) = f'(z)g(z) + f(z)g'(z)$.

(3) For any $\alpha \in \mathbb{C}$, αf is holomorphic on U with $(\alpha f)'(z) = \alpha f'(z)$.

Corollary. *$\frac{f}{g}$ is holomorphic on U, provided that $g(z) \neq 0$ for all $z \in U$, and*

$$\left(\frac{f}{g}\right)'(z) = \frac{f'(z)g(z) - f(z)g'(z)}{g(z)^2}$$

Theorem 3.2.3 (The Chain Rule)**.** *Let U and V be open in $\mathbb{C}$. Let $f : U \to \mathbb{C}$ be holomorphic on U and let $g : V \to \mathbb{C}$ be holomorphic on V, with $f(U) \subseteq V$. Then $g \circ f$ is holomorphic on U, with*

$$(g \circ f)'(z) = g' \circ f(z) f'(z)$$

Proof. Fix $z_0 \in U$, and let $r > 0$ such that the open ball $B(z_0, r)$ is contained in U; i.e. that $B(z_0, r) \subseteq U$. It suffices to show that if $0 < |h_n| < r$, where the sequence $\{h_n\} \to 0$ as $n \to \infty$, then

$$\lim \frac{g \circ f(z_0 + h_n) - g \circ f(z_0)}{h_n}$$

exists and equals $g' \circ f(z_0) f'(z_0)$.

Suppose first that $f(z_0) \neq f(z_0 + h_n)$ for all $n \in \mathbb{Z}^+$. Then we have

$$\frac{g \circ f(z_0 + h_n) - g \circ f(z_0)}{h_n} = \frac{g \circ f(z_0 + h_n) - g \circ f(z_0)}{f(z_0 + h_n) - f(z_0)} \cdot \frac{f(z_0 + h_n) - f(z_0)}{h_n}$$

Since $\lim\left(f(z_0+h)-f(z_0)\right)=0$, we get

$$\lim \frac{g\circ f(z_0+h_n)-g\circ f(z_0)}{h_n}=g'\circ f(z_0)f'(z_0)$$

Now, suppose that $f(z_0)=f(z_0+h_n)$ for infinitely many n. Write $\{h_n\}=\{k_n\}\cup\{l_n\}$, where $f(z_0)\neq f(z_0+k_n)$ and $f(z_0)=f(z_0+l_n)$ for all n. Since f is holomorphic, and hence complex differentiable

$$f'(z_0)=\lim\frac{f(z_0+h_n)-f(z_0)}{l_n}=0$$

so that

$$\lim\frac{g\circ f(z_0+l_n)+g\circ f(z_0)}{l_n}=0$$

Now, by above we also get that

$$\lim\frac{g\circ f(z_0+k_n)+g\circ f(z_0)}{k_n}=g'\circ f(z_0)f'(z_0)$$

so that $g'\circ f(z_0)f'(z_0)=0$. ■

Definition. Let $A\subseteq\mathbb{C}$ an arbitrary set of $\mathbb{C}$. We call a complex valued function f **holomorphic** on A if it is analytic on some open set of $\mathbb{C}$ containing A.

Theorem 3.2.4. *Let $f(z)=\sum a_n(z-z_0)^n$ a convergent power series with radius of convergence $R>0$. Then the following are true.*

(1) The series $\sum\frac{n!}{(n-k)!}a_n(z-z_0)^{n-1}$ converges with radius of convergence R.

(2) f is smooth on the ball $B(z_0,R)$ with

$$f^{(n)}(z)=\sum\frac{n!}{(n-k)!}a_n(z-z_0)^{n-1}$$

(3) For all $n\geq 0$

$$a_n=\frac{f^{(n)}(z_0)}{n!}$$

Proof. Suppose without loss of generality that $z_0=0$. Then it is sufficient to show that f' exists and has the power series $\sum na_nz^{n-1}$. We have by definition that

$$\frac{1}{R}=\limsup\sqrt[n]{|a_n|}$$

Now, by L'ôpital's rule, we have

$$\lim\frac{\log n}{n-1}=0$$

so that $\lim\sqrt[n-1]{n}=1$. Let

$$\frac{1}{R'}=\limsup\sqrt[n-1]{|a_n|}$$

Then R' is the radius of convergence of the series

$$\sum na_n z^{n-1}$$

Notice that $\sum a_n z^{n-1} = \sum a_{n+1} z^n$, so that

$$z \sum a_{n+1} z^n + a_0 = \sum a_n z^n$$

Now, if $|z| < R$, and $z \neq 0$, then $\sum |a_n z^n| = \frac{1}{|z|}$. Moreover,

$$\sum |a_n z^n| + \frac{1}{|z|} a_0 < \infty$$

which makes $R \leq R'$. Therefore $R = R'$, and so the series $\sum na_n z^{n-1}$ converges and has radius of convergence R.

Now, for $|z| < R$, put $g(z) = \sum na_n z^{n-1}$ and $s_n(z) = \sum_{k=0}^{n} a_k z^n$ and $R(z) = \sum_{k=n+1}^{\infty} a_k z^k$. Let $w \in B$. Let $w \in B(0, R)$, the open ball of radius R about 0, and fix r such that $|w| < r < R$. Let $\delta > 0$ such that the closed ball $\overline{B}(w, \delta)$ of radius δ about w is contained in $B(0, R)$; that is, $\overline{B}(w, \delta) \subseteq B(0, R)$. Let $z \in B(w, \delta)$, then we see that

$$\frac{f(z) - f(w)}{z - w} - g(w) = \Big(\frac{s_n(z) - s_n(w)}{z - w} - s'(w)\Big) + s_n'(w) - g(w) + \frac{R_n(z) - R_n(w)}{z - w}$$

So that

$$\frac{f(z) - f(w)}{z - w} - g(w) = \frac{1}{z - w} \sum_{k=n+1}^{\infty} a_k (z^k - w^k)$$

However, notice that

$$\Big|\frac{z^k - w^k}{z - w}\Big| = |z^{k-1} + z^{k-2} w + \cdots + z w^{k-2} + w^{k-1}| \leq k r^{k-1}$$

Hence

$$\Big|\frac{R_n(z) - R_n(w)}{z - w}\Big| \leq \sum_{k=n+1}^{\infty} |a_k| k r^{k-1}$$

Since $r < R$, $\sum |a_k| k r^{k-1}$ converges so that for any $\varepsilon > 0$, there is an $N_1 \in \mathbb{Z}^+$ such that

$$\Big|\frac{R_n(z) - R_n(w)}{z - w}\Big| < \frac{\varepsilon}{3}$$

whenever $n \geq N_1$ and for all $z \in B(w, \delta)$. Also, notice that $\lim s_n'(w) = g(w)$, and that there exists an $N_2 \in \mathbb{Z}^+$ such that

$$|s_n'(w) - g(w)| < \frac{\varepsilon}{3}$$

whenever $n \geq N_2$. Now, let $N = \max\{N_1, N_2\}$ and choose $\delta > 0$ such that

$$\Big|\frac{s_n(z) - s_n(w)}{z - w}\Big| < \frac{\varepsilon}{3}$$

whenever $0 < |z - w| < \delta$. Then

$$\left|\frac{f(z) - f(w)}{z - w} - g(w)\right| < \varepsilon$$

whenever $0 < |z - w| < \delta$. Therefore $f'(z) = g(z)$.

Finally, observe that $f(0) = f^{(0)}(0) = a_0$. Using the power series

$$f^{(n)}(z) = \sum \frac{n!}{(n-k)!} a_n z^{n-k}$$

we find that

$$f^{(n)}(0) = n! a_n$$

and we are done. ∎

Corollary. *If $f(z) = \sum a_n (z - z_0)^n$ is a convergent power series with radius of convergence $R > 0$, then f is holomorphic on the open ball $B(z_0, R)$.*

Example 3.3. $\exp z = \sum \frac{z^n}{n!}$ is holomorphic on all of $\mathbb{C}$.

Lemma 3.2.5. *If U is open and connected in $\mathbb{C}$, and $f : U \to \mathbb{C}$ is complex differentiable on U with $f'(z) = 0$, then f is constant.*

Proof. Take $z_0 \in U$ and let $\omega_0 = f(z_0)$. Take $A = \{z \in U : f(z) = \omega_0\}$. And pick a point $z \in U$. Let $\{z_n\}$ a sequence of points in A converging to z; i.e. $\{z_n\} \to z$. Since $f(z_n) = \omega_0$ for all $n \geq 0$, and if f is continuous, we have $f(z) = \omega_0$ which makes $z \in A$, and A is closed in U.

Now, let $a \in A$ and take $e > A$ such that $B(a, \varepsilon) \subseteq U$. If $z \in B(a, \varepsilon)$, let

$$g(z) = f(tz + (1-t)a) \text{ where } 0 \leq t \leq 1$$

Then

$$\frac{g(t) - g(s)}{t - s} = \frac{g(t) - g(s)}{(t-s)z + (s-t)a} \frac{(t-s)z + (s-t)a}{t - a}$$

thus, if $t \to s$ we get

$$g'(s) = \lim \frac{g(t) - g(s)}{t - s} = f'(sz + (1-s)a)(z - a) = 0 \text{ for all } 0 \leq s \leq 1$$

This makes g constant, so that $f(z) = g(1) = g(0) = f(a) = \omega_0$ and hence $B(a, \varepsilon)A$ which makes A open. Therefore $A = U$ and this makes f constant on U. ∎

Example 3.4. (1) Differentiating $f'(z) = \exp z$ we get

$$f'(z) = \sum \frac{n}{n!} z^{n-1} = \sum \frac{z^{n-1}}{(n-1)!} = \sum \frac{z^n}{n!}$$

which makes $f'(z) = f(z)$. Taking $e^z = \exp z$, that is

$$\frac{d}{dz} e^z = e^z$$

Now, take $g(z) = e^z e^{a-z}$. Then $g'(z) = e^z e^{a-z} - e^z e^{a-z} = 0$ so that g is constant, and $g(z) = \omega$ for all $z \in \mathbb{C}$. Taking $e^0 = 1$, we get $\omega = g(0) = e^a$; moreover that $e^z e^{a-z} = a^a$. This shows that

$$e^{a+b} = e^a e^b \text{ and } e^z e^{-z} = 1 \text{ for all } a, b, z \in \mathbb{C}$$

and that

$$e^{-z} = \frac{1}{e^z}$$

since $e^z \neq 0$ for all $z \in \mathbb{C}$.

Moreover, notice that by the power series expansion, we have

$$\overline{\sum \frac{z^n}{n!}} = \sum \frac{\overline{z}^n}{n!}$$

so that

$$\overline{\exp z} = \exp \overline{z}$$

Now, if $\theta \in \mathbb{R}$, notice that $|e^{i\theta}| = e^{i\theta} e^{-i\theta} = e^0 = 1$ and $|e^z|^2 = e^z e^{\overline{z}} = e^{z+\overline{z}} = 2 \operatorname{Re} z$, and $|\exp z| = \exp 2 \operatorname{Re} z$.

(2) Define the follwing series $\sin z$ and $\cos z$ by

$$\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} + \cdots + (-1)^n \frac{z^{2n-1}}{(2n-1)!}$$
$$\cos z = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} + \cdots + (-1)^n \frac{z^{2n}}{(2n)!}$$

Then it can be shown that $\sin z$ and $\cos z$ are convergent power series with radius of convergence $R = \infty$, so that they are also holomorphic on all of $\mathbb{C}$. Differentiating the power series, we find that

$$\frac{d}{dz}\sin z = \cos z \text{ and } \frac{d}{dz}\cos z = -\sin z$$

We can also find that

$$\sin z = \frac{e^{iz} - e^{-iz}}{2} \text{ and } \cos z = \frac{e^{iz} + e^{-iz}}{2}$$

which shows that

$$\cos^2 z + \sin^2 z = 1$$

In partiuclar if $\theta \in \mathbb{R}$, we get $e^{i\theta} = \cos\theta + i \sin\theta$, and hence, for all $z \in \mathbb{Z}$, we get $z = |z| e^{i\theta}$ where $\theta = \arg z$. Since $\exp(a+b) = (\exp a)(\exp b)$, we find that $\arg \exp z = \operatorname{Im} z$.

Definition. We call a complex valued function f on $\mathbb{C}$ **periodic**, if there exists a $c \in \mathbb{C}$ such that $f(z+c) = f(z)$ for all $z \in \mathbb{C}$. We call c the **period** of f.

Example 3.5. If c is the period of $\exp z$, then $\exp z = \exp(z+c) = e^z e^c$, implying that $\exp c = 1$. Thus $\operatorname{Re} c = 0$, and so $c = i\theta$ for some $\theta \in \mathbb{R}$. Now, observe that

$$\exp c = \exp i\theta = \sin\theta + i\sin\theta = 1$$

which makes $\theta = 2\pi k$, and hence $c = 2i\pi k$, where $k \in \mathbb{Z}$.

Example 3.6 (The Complex Logarithm)**.** Define $\log w$ such that $w = \exp z$ whenever $z = \log w$. Since $\exp z \neq 0$ for all $z \in \mathbb{C}$, $\log 0$ is undefined. Now, let $\exp z = w$, where $w \neq 0$, if $z = x + iy$, then $|w| = e^x$ and $y = \arg w + 2\pi k$ for some $k \in \mathbb{Z}$. So the set of solutions to $\exp z$ is given by all $\log|w| + (\arg w + 2\pi k)$. Notice then that $\log|w|$ defines the natural logarithm of $|w|$ by definition of $\exp z$.

Definition. We define a **region** of $\mathbb{C}$ to be an open and connected set of $\mathbb{C}$.

Definition. Let U be a region of $\mathbb{C}$, and let $f : U \to \mathbb{C}$ a continuous complex valued function such that $z = \exp f(z)$. Then we call f a **branch of the logarithm**.

Lemma 3.2.6. *If U is a region in $\mathbb{C}$, and f is a branch of the logarithm in U, then then any other branch of logarithm is of the form*

$$f(z) + 2i\pi k \text{ where } k \in \mathbb{Z}^+$$

Proof. Let f be a branch of the logarithm in U and let $k \in \mathbb{Z}$. Take $g(z) = f(z) + 2i\pi k$. Then $\exp g = \exp f = z$ so that g is also a branch of the logarithm.

Conversely, suppose that f and g are both branches of the logarithm; then $g(z) = f(z) + 2i\pi k$, for some k (not necessarrily an integer). Now, define

$$h(z) = \frac{1}{2i\pi}(g(z) - f(z))$$

Then h is continuous on U, and $h(U) \subseteq \mathbb{Z}$. Now, since U is connected, then so is $h(U)$. Noticing that $h(z) = k$, this makes $k \in \mathbb{Z}$, and we are done. ∎

Definition. Let $U = \mathbb{C}\backslash\mathbb{R}_{\leq 0}$; that is, the complex numbers $\mathbb{C}$ slit about the negative real axis. Notice that U is connected, and that $z = re^{i\theta}$ where $r = |z|$ and $-\pi < \theta < \pi$. Define $f(z) = \log r + i\theta$. Since f is the sum of two continuous functions, i.e; $\log r$ for all $r \in \mathbb{R}\backslash\{0\}$ and the map $\theta \to i\theta$, we get that f is continuous on U. Therefore f defines a branch of the logarithm

Lemma 3.2.7. *Let U and V be open in $\mathbb{C}$. Let $f : U \to \mathbb{C}$ and $g : V \to \mathbb{C}$ be continuous complex valued functions with $f(U) \subseteq V$ and $g \circ f(z) = z$. Then if g is complex differentiable on V, f is complex differentiable on U, with*

$$f'(z) = \frac{1}{g' \circ f(z)}$$

Proof. Fix $z_0 \in U$ and let $h \in \mathbb{C}$ such that $h \neq 0$ and $h + z_0 \in U$. Then notice that $z_0 = g \circ f(z_0)$ and $z_0 + h = g \circ f(z_0 + h)$. This makes $f(z_0) \neq f(z_0 + h)$. Now, observe that

$$\frac{g \circ f(z + h_0) - g \circ f(z_0)}{h} = \frac{g \circ f(z + h_0) - g \circ f(z_0)}{f(z_0 + h) - f(z_0)} \frac{f(z_0 + h) - f(z_0)}{z - z_0} = 1$$

Now, taking the limit of both sides as $h \to 0$, we get $g' \circ f(z_0) f'(z_0) = 1$, and $f'(z_0)$ exists since $g' \circ f(z_0) \neq 0$. ∎

Corollary. *If g is holomorphic on V, then f is analyitic on U.*

Corollary. *If U is a connected; i.e. a region, and f is a branch of the logarithm on U, then f is holomorphic.*

Proof. Observe by definition that $z = \exp f(z)$. ∎

Definition. We define the **principle branch of the logarithm** on $\mathbb{C} \backslash \mathbb{R}_{\leq 0}$ to be the complex function $\log : \mathbb{C} \backslash \mathbb{R}_{\leq 0} \to \mathbb{C}$ defined by

$$\log z = \log r + i\theta \text{ where } z = r \exp i\theta$$

Definition. If f is a branch of the logarithm on some region U, and if $b \in \mathbb{Z}$. The **branch** of $bf(z)$ to be a complex function $g : U \to \mathbb{C}$ defined by $g(z) = expbf(z)$. We write $g(z) = z^b$, when the branch f is the principle branch of the logarithm.

Lemma 3.2.8. *Let U be a region, and f a branch of the logarithm and let $b \in \mathbb{Z}$. Then the branch of $bf(z)$, $g(z) = \exp bf(z)$ is holomorphic.*

Corollary. *The branch z^b of $\log z$ is holomorphic.*

Definition. Let $u : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ and $v : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be realvalued functions. We define the **Cauchy Riemann equations** to be the systems of equations

$$\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} = 0$$
$$\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = 0$$

Lemma 3.2.9. *Let $f : U \to \mathbb{C}$ a complex valued function with $f(z) = f(x + iy) = u(x, y) + iv(x, y)$. Then if f is holomorphic, we have*

$$f'(z) = \frac{\partial u}{\partial x}(x, y) + i\frac{\partial v}{\partial x}$$

Proof. We evaluate $f'(z)$ as $h \to 0$ along the real axis $\mathbb{R}$ of $\mathbb{C}$. ∎

Lemma 3.2.10. *Let $f : U \to \mathbb{C}$ a complex valued function with $f(z) = f(x + iy) = u(x, y) + iv(x, y)$. Then if f is holomorphic, we have*

$$f'(z) = \frac{\partial v}{\partial y}(x, y) - i\frac{\partial u}{\partial y}$$

Proof. We evaluate $f'(z)$ as $h \to 0$ along the imaginary axis $i\mathbb{R}$ of $\mathbb{C}$. ■

Theorem 3.2.11. *Let $u(z) = u(x, y)$ and $v(z) = v(x, y)$, where $z = x + iy$ be real valued functions on a region U with continuous partial derivarives. If $f : U \to \mathbb{C}$ is a complex valued function such that $f(z) = u(x, y) + iv(x, y)$, then f is holomorphic on U if, and only if u and v satisfy the Cauchy Riemann equations.*

Proof. Suppose that u and v satisfy the Cauchy Riemann equations, and let $B(z, r) \subseteq U$. If $h = s + itB(0, r)$, then

$$u(x + s, y + t) - u(x, y) = (u(x + s, y + t) - u(x, y + t)) + (u(x, y + t) - u(x, y))$$

By the mean value theorem for real valued functions, we have that there exist $s_1, t_1 \in B(0, r)$ with $|s_1| < |s|$ and $|t_1| < |t|$ for which

$$u(x + s, y + t) - u(x, y + t) = u_x(x + s_1, y + t)s$$

and

$$u(x, y + t) - u(x, y + t) = u_y(x, y + t)t$$

Letting

$$\phi(s, t) = (u(x + s, y + y) - u(x, y)) - (u_x(x, y)s - u_y(x, y)t)$$

Then

$$\frac{\phi(s, t)}{s + it} = \frac{s}{s + it}(u_x(x + s_1, y + t) - u_x(x, y)) + \frac{t}{s + it}(u_y(x, y + t_1) - u_y(x, y))$$

Now, since $|s| \leq |s + it|$ and $|t| \leq |s + it|$, and u_x and u_y are continuous, we get

$$\lim_{s+it \longrightarrow 0} \frac{\phi(s, t)}{s + it} = 0$$

So that

$$u(x + s, y + t) - u(x, y) = u_x(x, y)s + u_y(x, y)t + \phi(s, t)$$

Similarly, for v we get

$$v(x + s, y + t) - v(x, y) = v_x(x, y)s + v_y(x, y)t + \psi(s, t)$$

where

$$\lim_{s+it \longrightarrow 0} \frac{\psi(s, t)}{s + it} = 0$$

Notice then that

$$\frac{f(z + (s + it)) - f(z)}{s + it} = u_x(x, y) + iv(x, y) + \frac{\phi(s, t) + i\psi(s, t)}{s + it}$$

Taking $s + it \to 0$, makes f complex differentiable on U with $f'(z) = u_x(x, y) + iv(x, t)$. Since u_x and v_x are continuous, so is f, which makes f holomorphic.

Conversely, if we suppose that f is holomorphic, then by lemma 3.2.9 and lemma 3.2.10, we get

$$\frac{\partial u}{\partial x}(x, y) + i\frac{\partial v}{\partial x} = \frac{\partial v}{\partial y}(x, y) - i\frac{\partial u}{\partial y}$$

which shows that u and v satisfy the Cauch Riemann equations. ■

Definition. We call a real valued function $u : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ **harmonic** if

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

If $v : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a real valued function, and f is a complex valued function for which $f(z) = u(x, y) + iv(x, y)$, then we call v the **harmonic conjugate** of u.

Example 3.7. The function $u(x, y) = \log \sqrt{x^2 + y^2}$ is harmonic.

Theorem 3.2.12. *Let U be an open ball, or $\mathbb{C}$. If $u : U \to \mathbb{C}$ is harmonic, then u has a harmonic conjugate.*

Proof. Let $U = B(0, R)$ for $0 \leq R \leq \infty$, and let $u : U \to \mathbb{C}$ be harmonic. Define

$$v(x, y) = \int_0^y u_x(x, t)\ dt + \phi(x)$$

wheere ϕ is determined by taking $v_x + u_y = 0$. Then differentiating both sides, and by Leibniz's rule for differentiating under the integral, we have

$$\begin{aligned} v_x(x, y) &= \int_0^y u_{xx}(x, t)\ dt + \phi'(t) \\ &= -\int_0^y u_{xx}(x, t)\ dt + \phi'(x) \\ &= -u_y(x, y) + u_y(x, 0) + phi'(x) \end{aligned}$$

and $\phi'(x) = u_y(x, 0)$. Then u and $v = \int_0^y u_x(x, t)\ dt - \int_0^x u_y(s, 0)\ ds$ satisfy the Cauchy Riemann equations. This makes v a harmonic conjugate. ■

3.3 Möbius Transformations

Definition. We call a complex valued function S on C_∞, defined by

$$S(z) = \frac{az + b}{cx + d} \text{ where } a, b, c, d \in \mathbb{C}$$

a **Möbius transformation** if $ad - cd \neq 0$.

We have the following results for Möbius transformations.

Theorem 3.3.1. *The set of all Möbius transformations forms a group under function compositon.*

Proof. Since $\circ$ is associative, it suffices to show the identity, closure, and inverse laws. Indeed, notice that the function

$$I = \frac{1z + 0}{0z + 1} = z$$

is a Möbius transformation, and that for any Möbius transformation, S, $S = S \circ I = I \circ S$.

Now, Let S and T be Möbius transformations. Then

$$S(z) = \frac{az+b}{cz+d} \text{ and } T(z) = \frac{fz+g}{hz+l}$$

for $a, b, c, d, f, g, h, l \in \mathbb{C}$. Then we get

$$S \circ T(z) = \frac{aT(z)+b}{cT(z)+b}$$

and since $T(z) \in \mathbb{C}_\infty$, for all values of ζ, then $S \circ T$ is a Möbius transformation.

Finally, let

$$S^{-1}(z) = \frac{dz-b}{cz-a}$$

Then $S \circ S^{-1} = S^{-1} \circ S(z) = I(z)$, and we are done. ■

Definition. Let $a \in \mathbb{C}$. We call a Möbius trasnsformation of the form $T(z) = z + a$ a **translation** of z by a. We call a Möbius transformation $D(z) = az$ a **dilation** of z by a. Let $0 \leq t \leq 2\pi$. Then we call the Möbius transformation $R(z) = e^{it}z$ a **rotation** of z about t, and we call the Möbius transformation $S(z) = \frac{1}{z}$ an **inversion** of z.

Lemma 3.3.2. *If S is a Möbius transformation, then S is the composition of translations, dilations, and inversions.*

Proof. Let

$$S(z) = \frac{az+b}{cz+d}$$

Suppose that $c = 0$, so that $S(z) = \frac{a}{d}z + b$. Then $S = S_2 \circ S_1$ where $S_1(z)$ is a translation by b and $S_2(z)$ is a dilation by $\frac{a}{d}$.

Now, if $c \neq 0$, then let $S_1(z) = z + \frac{d}{c}$, $S_2(z) = \frac{1}{z}$, $S_3(z) = \frac{bc-ad}{c^2}z$ and $S_4(z) = z + \frac{a}{c}$. Then $S = S_4 \circ S_3 \circ S_2 \circ S_1$. ■

Corollary. *Rotations are compositions of translations, dilations, and inversions.*

Definition. Let S be a Möbius transformation. We cal a point $z \in \mathbb{C}$ a **fixed point** of S if $S(z) = z$.

Lemma 3.3.3. *If*

$$S(z) = \frac{az+b}{cz+d}$$

is a Möbius transformation, and z is a fixed point, then $cz^2 + (d-a)z - b = 0$ and S has at most two fixed points; unless it is the identity transformation.

Proof. Suppose that $S \neq I$, and consider the equation

$$z = \frac{az+b}{cz+d}$$

to obtain a quadratic polynomial over $\mathbb{C}$, which has at most two roots in $\mathbb{C}$. ■

Lemma 3.3.4. *Let S be a Möbius transformation on $\mathbb{C}_\infty$, and let $a, b, c, d \in \mathbb{C}_\infty$ distinct points with $\alpha = S(a)$, $\beta = S(b)$ and $\gamma = S(c)$. If T is another Möbius transformation with this property, then $S = T$.*

Proof. Notice by hypothesis that the transformation $T^{-1} \circ S$ has a, b, and c as fixed points, which forces $T^{-1} \circ S = I$. ■

Definition. Let $z_2, z_3, z_4 \in \mathbb{C}_\infty$ and define the Möbius transformation $S : \mathbb{C}_\infty \to \mathbb{C}_\infty$ by

$$S(z) = \frac{\left(\frac{z-z_3}{z-z_4}\right)}{\left(\frac{z_2-z_3}{z_2-z_4}\right)} \text{ if } z_2, z_3, z_4 \in \mathbb{C}$$
$$S(z) = \frac{z - z_3}{z - z_4} \text{ if } z_2 = \infty$$
$$S(z) = \frac{z_2 - z_4}{z - z_4} \text{ if } z_3 = \infty$$
$$S(z) = \frac{z_2 - z_3}{z_2 - z_3} \text{ if } z_4 = \infty$$

and where $S(z_2) = 1$, $S(z_3) = 0$, and $S(z_4) = \infty$. Then if $z_1 \in \mathbb{C}_\infty$, we define the **cross ratio**, (z_1, z_2, z_3, z_4) of z_1 to be $S(z_1)$.

Example 3.8. $(z_2, z_2, z_3, z_4) = 1$, $(z_3, z_2, z_3, z_4) = 0$, and $(z_4, z_2, z_3, z_4) = 0$, by definition. Now, if M is aany Möbius transformation, and w_2, w_3, w_4 are points on M such that $M(w_1) = 1$, $M(w_3) = 0$, and $M(w_4) = \infty$, then $M(z) = (z, w_2, w_3, w_4)$.

Theorem 3.3.5. *IF $z_2, z_3, z_4 \in \mathbb{C}_\infty$ are distinct points, and T is a Möbius transformation, then $(z, z_2, z_3, z_4) = (T(z), T(z_2), T(z_3), T(z_4))$ for all $z \in \mathbb{C}_\infty$. That is, the cross ratio is invariant under transformations.*

Proof. Let $S = (z, z_2, z_3, z_4)$, then S is a Möbius transformation. Now, if $M = S \circ T^{-1}$, then $M(T(z_2)) = 1$, $M(T(z_3)) = 0$, and $M(T(z_4)) = \infty$, which makes $S \circ T^{-1} = (z, T(z_2), T(z_3), T(z_4))$. ■

Lemma 3.3.6. *If $z_2, z_3, z_4 \in \mathbb{C}_\infty$ and $w_2, w_3, w_4 \in \mathbb{C}_\infty$ are all distinct points, then there exists one, and only one Möbius transformation S for which $S(z_2) = w_2$, $S(z_3) = w_3$, and $S(z_4) = w_4$.*

Proof. Let $T(z) = (z, z_2, z, 3, z_4)$ and $M = (z, w, 2, w, 3, w_4)$. Put $S = M^{-1} \circ T$. Then $s(Z_2) = w_2$, $S(z_3) = z_3$,and $S(z_4) = w_4$. Now, if R is another Möbius transformation having this property, then $R^{-1} \circ S$ has 3 fixed points, which makes $R^{-1} \circ S = I$. ■

Lemma 3.3.7. *Let $z_1, z_2, 3, z_4 \in \mathbb{C}_\infty$ be distinct points. Then $(z, 1, z_2, z_3, z_4) \in \mathbb{R}$ if, and only if all the points lie on a circle.*

Proof. Define $S : \mathbb{C}_\infty \to \mathbb{C}_\infty$ by $S(z) = (z, z_2, z_3, z_4)$. Then it suffices to show that the preimage $S^{-1}(\mathbb{R}_\infty)$ is a circle.

Let
$$S(z) = \frac{az+b}{cz+d}$$
If $z \in \mathbb{R}$, and $w = S^{-1} = (z) \neq \infty$ then $x = S(w)$, so that $S(w) = \overline{S(w)}$. Thus
$$\frac{aw+b}{cw+b} = \frac{\overline{aw}+\overline{b}}{\overline{cw}+\overline{d}}$$
this gives
$$(a\overline{c} - \overline{a}c)|w|^2 + (a\overline{d} - \overline{b}c)w + (b\overline{c} - d\overline{a})\overline{w} + (b\overline{d} - \overline{b}d) = 0$$
If $a\overline{c} \in \mathbb{R}$, then $a\overline{c} - a\overline{c} = 0$. putting $\alpha = 2(a\overline{d} - \overline{b}c)$, and $\beta = i(b\overline{d} - \overline{b}d)$, and multuplying by the above equation by i gives
$$0 = \operatorname{Im}(\alpha w + \beta)$$
Since $\beta \in \mathbb{R}$, w lies on this line determined by α and β. Now, if $\alpha\overline{c} \notin \mathbb{R}$, then we get $|w|^2 + \gamma\overline{\gamma}w + w\overline{\gamma} - \delta = 0$, for $\gamma \in \mathbb{C}$ and $\delta \in \mathbb{R}$ constants. Hence $|w + \gamma| = \lambda$, where
$$\lambda = \sqrt{|\gamma^2 + \delta} = \left|\frac{ad - bc}{\overline{a}c - a\overline{c}}\right| > 0$$
Since γ and λ are independent of x, this defines a circle, and we are done. ∎

Theorem 3.3.8. *Möbius transformations take circles onto circles.*

Proof. Let Γ be a circle in $\mathbb{C}_\infty$, and S a Möbius transformation. Let $z_2, z_3, z_4 \in \Gamma$, diestinct, and put $w_2 = S(z_2)$, $w_3 = S(z_3)$, and $w_4 = S(z_4)$. then w_1 w_2 and w_3 determine a circle Γ'. Now, for every $z \in \mathbb{C}_\infty$, we have that $(z, z_2, z_3, z_4) = (S(z), w_2, w_3, w_4)$, so that if $z \in \Gamma$, $S(z) \in \Gamma'$; i.e. that $S(\Gamma) = \Gamma'$. ∎

Lemma 3.3.9. *For any two circle Γ and Γ' in $\mathbb{C}_\infty$, there is a Möbius transformation T, such that $T() = \Gamma'$, and T takes any three points on Γ to any three points on Γ'.*

Definition. Let Γ be a circle wthrough the discinct points $z_2, z_3, z_4 \in \mathbb{C}_\infty$. We call the points $z, z' \in \mathbb{C}_\infty$ **symmetric** with respect to $\mathbb{G}$ if $(z', z_2, z_3, z_4) = \overline{(z, z_2, z_3, z_4)}$.

Example 3.9. If Γ is a straight line, choosing $z_4 = \infty$, we get that
$$\frac{z' - z_3}{z_2 - z_4} = \frac{\overline{z} - \overline{z_3}}{\overline{z_2} - \overline{z_3}}$$
Giving $|z' - z_3| = |z - z_3|$. Which shows that z and z' are equidistant from each other. Moreover, we have that
$$\operatorname{Im}\frac{z' - z_3}{z_2 - z_4} = -\operatorname{Im}\frac{z - z_3}{z_2 - z_3}$$
hence, with the exception that $z \in \Gamma$, z and z' lie on different half planes determined by γ. That is, lhe line segmenent $[z, z'] \perp \Gamma$.

Now, let $\Gamma = \{z \in \mathbb{C} : |z - z_0| = R\}$ (the circle of radius R centered about z_0) for some $a \in \mathbb{C}$, and $0 < R < \infty$. Let $z_2, z_3, z_4 \in \Gamma$ distinct points on Γ. Then we have that

$$\begin{aligned}(z', z_2, z_3, z_4) &= \overline{(z, z_2, z_3, z_4)} \\ &= \overline{(z - z_0, z_2 - z_0, z_3 - z_0, z_4 - z_0)} \\ &= (\overline{z} - \overline{z_0}, \frac{R^2}{z_2 - z_0}, \frac{R^2}{z_3 - z_0}, \frac{R^2}{z_4 - z_0}) \\ &= (\frac{R^2}{\overline{z} - \overline{z_0}}, z_2 - z_0, z_3 - z_0, z_4 - z_0) \\ &= (\frac{R^2}{\overline{z} - \overline{z_0}} + z_0, z_2, z_3, z_4)\end{aligned}$$

so that $z' = \frac{R^2}{z - z_0}$ or $(z' - z_0)(\overline{z} - \overline{z_0}) = 0$, hence

$$\frac{z' - z_0}{z - a} = \frac{R^2}{|z - z_0|^2} > 0$$

and z' lies on the ray $[z, \infty) = \{z_0 + t(z - z_0) : 0 < t < \infty\}$ from z_0 to z. Now, suppose that z is inside of Γ, let L be the ray from z_0 trhough z, and construct the perpendicular $P \perp L$. Then P cuts Γ at a point z''. Construct the tangent through the point z'' and z'. Then z_0 and ∞ are symetrc to , and we obtain z' from z in this manner.

Theorem 3.3.10 (The Symmetric Principle). *If a Möbius transformation T takes a circle Γ_1 onto a circle Γ_2, then pairs of symmetric points with respect to Γ_1 are mapped onto pairs of symmetric points with respect to Γ_2.*

Proof. LEt $z_2, z_3, z_4 \in \Gamma$. If z and z' are symmetric with respect to Γ_1, then $(T(z'), T(z_2), T(z_3), T(z_4)) = (z', z_2, z_3, z_4) = \overline{(z, z_2, z_3, z_4)} = \overline{(T(z), T(z_2), T(z_3), T(z_4))}$. Which makes $T(z')$ symmetric to z with respect to Γ. ■

Definition. We define the **orientation** of a circle Γ to be an ordered triple (z_1, z_2, z_3) of points on Γ.

Example 3.10. Consider $\mathbb{R}$ as a circle, with $z_1, z_2, z_3 \in \mathbb{R}$. Then put $T(z) = (z, z_1, z_2, z_3) = \frac{az+b}{cz+d}$. Since $T(\mathbb{R}_\infty) = \mathbb{R}$, choose $a, b, c, d \in \mathbb{R}$ and we get

$$T(z) = \frac{az + b}{cz + d}(c\overline{z} + d) = \frac{ad - bc}{|cz + d|^2}\operatorname{Im} z$$

If $ad - bc > 0$, or $ad - bc < 0$, then the set $H_< = \{z \in \mathbb{C} : \operatorname{Im}(z, z_1, z_2, z_3) < 0\}$ is the upper, or lower half of the plane.

Definition. If (z_1, z_2, z_3) is an orientaton of a circle Γ, then we define the **right side** of Γ to be $\Gamma^> = \{z \in \mathbb{C} : \operatorname{Im}(z, z_1, z_2, z_3) > 0\}$. Similarly, we define the **left side** of Γ to be $\Gamma^> = \{z \in \mathbb{C} : \operatorname{Im}(z, z_1, z_2, z_3) < 0\}$.

Theorem 3.3.11 (The Orientation Principle)**.** *IF Γ_1 and Γ_2 are circles in $\mathbb{C}_\infty$, and T is a Möbius transformation, taking Γ_1 onto Γ_2, then T preserves orientation. That is, if (z_1, z_2, z_3) is an orientation of Γ_1, $(T(z_1), T(z_2), T(z_3))$ is an orientation of Γ_2. Moreover, T takes the right and left sides of Γ_1 onto the right and left sides of Γ_2, respectively.*

Example 3.11. (1) Consider the orientation $(1, 0, \infty)$ of $\mathbb{R}$. Since $(z, 1, 0, \infty) = z$, the right side of $\mathbb{R}$ with respect to $(1, 0, \infty)$ is the upper half plane. Now, to find an function $f : U \to \mathbb{C}$, with $U = \{z \in \mathbb{C} : \operatorname{Re} z > 0\}$ such that $f(U) = B(0, 1) = \{z \in \mathbb{C} : |z| < 1\}$ (i.e. the open unit ball), it is sufficient to find a Möbius transformaton mapping the imaginary axis $i\mathbb{R}$ onto the unit circle S^1. By the orientation principle, this transformation takes U onto $B(0, 1)$. Choose the orientation $(-i, 0, i)$ on $\mathbb{R}$, then the set $\{z : \operatorname{Re} z > 0\}$ is on the right side of $i\mathbb{R}$. Hence $i\mathbb{R}^> = \{z \in \mathbb{C} : \operatorname{Re} z > 0\}$.

Taking $S(z) = \frac{z_2}{z-i}$, and $R(z) = \frac{z-1}{z+1}$, then the function

$$g(z) = \frac{e^z - 1}{e^z + 1}$$

maps the strip $\{z \in \mathbb{C} : \Im z < \frac{\pi}{2}\}$ onto $B(0, 1)$.

(2) Let U_1 and U_2 be regions in $\mathbb{C}$, and we would like to find an functnion $f : U_1 \to U_2$ such that $f(U_1) = f(U_2)$. Now, map U_1 and U_2 onto the ball $B(0, 1)$ via a Möbius transformation. If this is possible, then we can obtain f by taking the composition of two functions $f_1 \circ f_2^{-1}$.

(3) Let U be the region insid the intersection of two circles Γ_1 and Γ_2; intersecting at ppints $a, b \in \mathbb{C}$, with $a \neq b$. Let L be the line segment $[a, b]$ with the orientation (∞, b, a) an dlet

$$T(z) = (z, \infty, b, a) = \frac{z - a}{b - a}$$

Then T maps L onto the closed interval $[0, 1]$, and since T maps circles onto circles, T maps Γ_1 onto Γ_2 through the points 0 and ∞. SO that $T(\Gamma_1)$ and $T(\Gamma_2)$ are straight lines in $\mathbb{C}_\infty$. By the orientation principle we get $T(U) = \{\omega \in \mathbb{C} : -\alpha < \arg \omega < \alpha\}$ for some $\alpha > 0$, or $T(U)$ is the complement of a closed sector.

Now, let $f_1(z) = e^{it} z^a$, with a an appropriate power and let $f_2(z) = \frac{z-1}{z+1}$. Then $f_1 \circ f_2$ takes U onto $B(0, 1)$.

Chapter 4

Complex Integration

4.1 Functions of Bounded Variation and The Riemann-Stieltjes Integral

Definition. Let $a, b \in \mathbb{R}$ with $a < b$. A complex valued function $\gamma : [a, b] \to \mathbb{C}$ is said to be of **bounded variation** if there exists an $M > 0$ such that for any partition $P = \{a = t_0 < \cdots < t_n =\}$ of $[a, b]$ the sum

$$v(\gamma, P) = \sum_{k=0}^{n-1} |\gamma(t_{k+1}) - \gamma(t_k)| \leq M$$

We call $v(\gamma, P)$ the **variation** of γ on P. We define the **total variation** of γ on $[a, b]$ to be

$$V(\gamma) = \sup \{v(\gamma, P)\}$$

for every partition P of $[a, b]$.

Lemma 4.1.1. *Let γ be a function of bounded variation. Then there exists an $M > 0$ for which $V(\gamma) \leq M$.*

Lemma 4.1.2. *Let $\gamma(t) = u(t) + iv(t)$ be a complex valued function. Then γ is of bounded variation if, and only if u and v are of bounded variation.*

Lemma 4.1.3. *If γ is a realvalued nondecreasing function, then γ is of bounded variation, and*

$$V(\gamma) = \gamma(b) - \gamma(a)$$

Lemma 4.1.4. *Let $\gamma : [a, b] \to \mathbb{C}$ be a function of bounded variation. THen the following are true.*

(1) If P is a partition of $[a, b]$, and Q is a refinement of P, then $v(\gamma, P) \leq v(\gamma, Q)$.

(2) If $\sigma : [a, b] \to \mathbb{C}$ is a function of bounded variation, and $\alpha, \beta \in \mathbb{C}$, then $\alpha\gamma + \beta\sigma$ is of bounded variation and $V(\alpha\gamma + \beta\sigma) = |\alpha|V(\gamma) + |\beta|V(\sigma)$.

Theorem 4.1.5. *Let $\gamma : [a,b] \to \mathbb{C}$ be a piecewise C^1 function. Then γ is of bounded variation and*

$$V(\gamma) = \int_a^b |\gamma'(t)|\ dt$$

Proof. Suppose without loss of generality that γ is of class C^1 on all $[a,b]$. Let $P = \{a = t_0 < \cdots < t_n = b\}$ a partition of $[a,b]$. Then by definition, we have

$$v(\gamma, P) = \sum_{k=0}^{n-1} |\gamma(y_{k+1}\gamma(y_k))| = \sum_{k=0}^{n-1} \Big| \int_{t_k}^{t_{k+1}} \gamma'(t)\ dt \Big| \leq \sum_{k=0}^{n-1} \int_{t_k}^{t_{k+1}} |\gamma'(t)|\ dt = \int_a^b |\gamma'(t)|\ dt$$

This gives $V(\gamma) \leq \int_a^b |\gamma'|$

Now, since γ' is continuous, it is uniformly continuous. Let $\varepsilon > 0$ and choose $\delta_1 > 0$ such that $|\gamma'(s) - \gamma'(t)| < \varepsilon$, whenever $0 < |s - t| < \delta_1$. Choose a $\delta_2 > 0$ such that if $\|P\| = \max\{t_{k+1} - t_k\} < \delta_2$, the

$$\Big| \int_a^b |\gamma(t)|\ dt - \sum_{k=0}^{n-1} \gamma'(\tau_k)|t_{k+1} - t_k| \Big| < \varepsilon$$

where $\tau_k \in [t_k, t_{k+1}]$. Then we get

$$\int_a^b |\gamma'(t)|\ dt \leq \varepsilon + \sum_{k=0}^{n-1} |\gamma'(\tau_k)|(t_{k+1} - t_k) = \varepsilon + \sum_{k=0}^{n-1} \Big| \int_a^b |\gamma(\tau_k)|\ dt \Big|$$

Then we get

$$\int_a^b |\gamma'(t)|\ dt \leq \varepsilon + \sum_{k=0}^{n-1} \Big| \int_{t_k}^{t_{k+1}} \gamma'(\tau_k) - \gamma'(t)\ dt \Big| + \sum_{k=0}^{n-1} \Big| \int_a^b \gamma'(t)\ dt \Big|$$

Now, take $\delta = \min\{\delta_1, \delta_2\}$. If $\|P\| < \delta$, then we get

$$|\gamma'(\tau_k) - \gamma'(t)| < \delta \text{ for all } \tau_k \in [t_k, t_{k+1}]$$

and

$$\int_a^b |\gamma'(t)|\ dt \leq \varepsilon + \varepsilon(b-a) + \sum_{k=0}^{n-1} |\gamma(t_{k+1} - \gamma(t_k))| = \varepsilon + \varepsilon(b-a) + v(\gamma, p) \leq \varepsilon + \varepsilon(b-a) + V(\gamma)$$

Then, as $\varepsilon \to 0$ from the positive side we observe that $\int_a^b |\gamma'| \leq V(\gamma)$ and equality is established. ■

Theorem 4.1.6. *Let $y : [a,b] \to \mathbb{C}$ be a function of bounded variation, and suppose that $f : [a,b] \to \mathbb{C}$ is a continuous complex valued function. Then there exists an $I \in \mathbb{C}$ such that for every $\varepsilon > 0$, there is a $\delta > 0$ such that whenever $P = \{a = t_0 < \cdots < t_n = b\}$ is a partition of $[a,b]$, with $\|P\| = \max\{t_{k+1} - t_k\} < \delta$, then*

$$\Big| I - \sum_{k=0}^{n} f(\tau_k)(\gamma(t_{k+1} - \gamma(t_k))) \Big| < \varepsilon$$

for any $\tau_k \in [t_k, t_{k+1}]$.

Proof. Since f is continuous on $[a,b]$ it is uniformly continuous on $[a,b]$. Then, by induction, choose $\delta_1 > \delta_2 > \dots$ such that if $|s-t| < \delta_n$, then $|f(s)-f(t)| < \frac{1}{m}$ for every $m \in \mathbb{Z}^+$. Now, let $\mathcal{P}_m$ be the collection of all partitions P of $[a,b]$ with $\|P\| < \delta_m$. Then $\mathcal{P}_{m+1} \subseteq \mathcal{P}_m$ for every $m \in \mathbb{Z}^+$. Define now

$$F_m = \text{cl}\left\{ \sum_{k=0}^{n-1} f(\tau_k)|\gamma(t_{k+1} - \gamma(t_k))| : P \in \mathcal{P}_m \right\}$$

then we claim that $F_{m+1} \subseteq F_m$ for every $m \in \mathbb{Z}^+$ and that $\text{diam}\, F_m \leq \frac{2}{m}V(\gamma)$.

Since $P_{m+1} \subseteq P_m$ then $F_{m+1} \subseteq F_m$. Now, let $P = \{a = t_0 < \dots < t_m = b\}$ a partition and take

$$S(P) = \sum_{k=0}^{n-1} f(\tau_k)|\gamma(t_{k+1}) - \gamma(t_k)| \text{ where } \tau_k \in [t_k, t_{k+1}]$$

Fix $m \geq 1$, then $P \in \mathcal{P}_m$. Letting Q be a refinement of P. Suppose then that $Q = P \cup \{t'\}$, and let $1 \leq p \leq m$ such that $t_p < t' < t_{p+1}$. If $t_p \leq \sigma \leq t' \leq \sigma'_{p+1}$, and

$$S(Q) = \sum_{k \neq p} (f(\sigma_k)(\gamma(t_{k+1}) - \gamma(t_k))) + f(\sigma)(\gamma(t') - \gamma(t_p)) + f(\sigma')(\gamma(t_{p+1}) - \gamma(t_p))$$

since $|f(\tau) - f(\sigma)| < \frac{1}{m}$ whenever $|\tau - \sigma| < \delta_m$, we find that

$$|S(P) - S(Q)| \leq \frac{1}{m}V(\gamma)$$

Letting $P, R \subseteq \mathcal{P}_m$ and taking $Q = P \cup R$, we get

$$|S(P) - S(Q)| \leq \frac{2}{m}V(\gamma)$$

and we get the diameter of F_m.

Now, by Cantor's theorem, there exists a unique $I \in F_m$ for every $m \in \mathbb{Z}^+$. If $\varepsilon > 0$ let $m > \frac{2}{\varepsilon}V(\gamma)$ so that $\varepsilon > \frac{2}{m}V(\gamma) \geq \text{diam}\, F_m$. Since $I \in F_m$, and $F_m \subseteq B(I, \varepsilon)$, taking $\delta = \delta_m$ gives us the final result. ■

Definition. Let $y : [a,b] \to \mathbb{C}$ be a function of bounded variation, and suppose that $f : [a,b] \to \mathbb{C}$ is a continuous complex valued function. We define the **Riemann-Stieltjes integral** of f with respect to γ over $[a,b]$ to be the unique $I \in \mathbb{C}$ such that for every $\varepsilon > 0$, there is a $\delta > 0$ and partition P of $[a,b]$ with $\|P\| < \delta$ for which

$$\left| I - \sum_{k=0}^{n} f(\tau_k)(\gamma(t_{k+1} - \gamma(t_k))) \right| < \varepsilon$$

for any $\tau_k \in [t_k, t_{k+1}]$. We write

$$I = \int_a^b f(t)\ d\gamma(t)$$

Theorem 4.1.7. *Let $f : [a,b] \to \mathbb{C}$ and $g : [a,b] \to \mathbb{C}$ continuous complex valued funtions, and let $\gamma : [a,b] \to \mathbb{C}$ and $\sigma : [a,b] \to \mathbb{C}$ of bounded variation on $[a,b]$. Then for every $\alpha, \beta \in \mathbb{C}$*

(1)

$$\int_a^b (\alpha f \beta g)\ d\gamma(t) = \alpha \int_a^b f\ d\gamma + \beta \int_a^b g\ d\gamma$$

(2)

$$\int_a^b f\ d(\alpha\gamma + \beta\sigma) = \alpha \int_a^b f\ d\gamma + \beta \int_a^b f\ d\sigma$$

Theorem 4.1.8. *Let $\gamma : [a,b] \to \mathbb{C}$ be of bounded variation and let $f : [a,b] \to \mathbb{C}$ a continuous complex valued function. If $a = t_0 < \cdots < t_n = b$ is a partition of $[a,b]$ then*

$$\int_a^b f\ d\gamma = \sum_{k=0}^{n-1} \int_{t_k}^{t_{k+1}} f\ d\gamma$$

Theorem 4.1.9. *If $\gamma : [a,b] \to \mathbb{C}$ is piecewise C^1 f bounded variation, and $f : [a,b] \to \mathbb{C}$ is a continuous complex valued function, then*

$$\int_a^b f\ d\gamma = \int_a^b f(t)\gamma'(t)\ dt$$

Proof. Suppose without loss of generality, that γ is of class C^1, and that $\gamma([a,b]) \subseteq \mathbb{R}$. Let $\varepsilon > 0$ and choose a $\delta > 0$ such that for any partition $P = \{a = t_0 < \cdots < t_n = b\}$ of $[a,b]$ with $\|P\| < \delta$ then

$$\Big| \int_a^b f\ d\gamma - \sum_{k=0}^{n-1} f(\tau_k)(\gamma(t_{k+1}) - \gamma(t_k)) \Big| < \frac{\varepsilon}{2}$$

and

$$\Big| \int_a^b f(t)\gamma'(t)\ dt - \sum_{k=0}^{n-1} f(\tau_k)\gamma'(\tau_k)(t_{k+1} - t_k) \Big| < \frac{\varepsilon}{2}$$

Then, by the mean value theorem for real valued derivatives, there exists a $\tau_k' \in [t_k, t_{k+1}]$ such that $\gamma(t_{k+1}) - \gamma(t_k) = \gamma'(\tau_k')(t_{k+1} - t_k)$, and we have

$$\sum_{k=0}^{n-1} f(\tau_k)(\gamma(t_{k+1}) - \gamma(t_k)) = \sum_{k=0}^{n-1} f(\tau_k')\gamma'(\tau_k')(t_{k+1} - t_k)$$

which gives us

$$| \int_a^b f(t)\ d\gamma - \int_a^b f(t)\gamma'(t)\ dt \Big|$$

∎

4.2 The Path Integral

Definition. We define a **path** on $\mathbb{C}$ to be a continuous complex valued functiom $\gamma : [a,b] \to \mathbb{C}$. We call $\gamma(a)$ the **initial point** of γ and $\gamma(b)$ the **end point** of γ. We define the **trace** of γ to be the image of γ (i.e. $\gamma([a,b])$) and denote it $\{\gamma\}$. We call the path γ **closed** if $\gamma(a) = \gamma(b)$.

Definition. We call a path $\gamma : [a,b] \to \mathbb{C}$ **rectifiable** if it is of bounded variation. Moreover, if γ is piecewise C^1, we define the **length** of γ to be the total variation of γ; that is

$$l(t) = \int_a^b |\gamma'(t)| \; dt$$

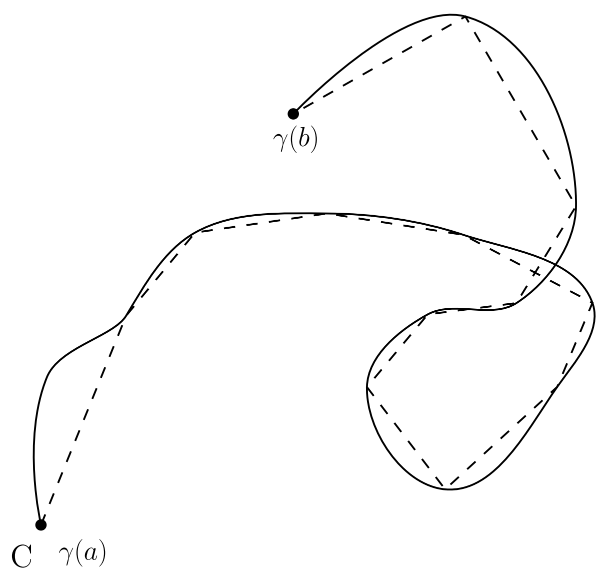

Figure 4.1: A rectifiable path in $\mathbb{C}$.

Definition. If $\gamma : [a,b] \to \mathbb{C}$ is a rectifiable path in $\mathbb{C}$, and f is a continuous complex valued function defined on $\{\gamma\}$, then the **path integral** of f **along** γ is defined to be

$$\int_\gamma f(z) \; dz = \int_a^b f \circ \gamma(t) \; d\gamma$$

Example 4.1. (1) Take $\gamma : [0, 2\pi] \to \mathbb{C}$ by $\gamma(t) = e^{it}$ and consider $f(z) = \frac{1}{z}$ defined on $\mathbb{C}\backslash\{0\}$. Then γ is differentiable, and a rectifiable path, so that

$$\int_\gamma \frac{dz}{z} = i \int_0^{2\pi} dt = 2i\pi$$

(2) Let $m \geq 0$, and $\gamma(t) = e^{it}$ on $0 \leq t \leq 2\pi$. Then

$$\int_\gamma z^m \; dz = i \int_0^{2\pi} e^{imt} e^{it} = 0$$

(3) Let $n \geq 0$ and $\gamma(t) = tb + (1-t)a$ on $0 \leq t \leq 1$, for $a, b \in \mathbb{C}$. Then

$$\int_\gamma z^n \, dz = (b-a) \int_0^1 (tb - (1-t)a)^t \, dt = \frac{b^{n+1-a^{n+1}}}{n+1}$$

Lemma 4.2.1. *If $\gamma : [a,b] \to \mathbb{C}$ is a recitifiable path and $\phi : [c,d] \to [a,b]$ is a nondecreasing function who's image is $[a,b]$, then $\gamma \circ \phi : [c,d] \to \mathbb{C}$ is a rectifiable path with trace $\{\gamma \circ \phi\} = \{\gamma\}$.*

Theorem 4.2.2. *If $\gamma : [a,b] \to \mathbb{C}$ is a rectifiable path, and $\phi : [c,d] \to [a,b]$ is a nondecreasing function who's image is $[a,b]$, the for every continuous complex valued function f defined on $\{\gamma\}$,*

$$\int_\gamma f = \int_{\gamma \circ \phi} f$$

Proof. Let $\varepsilon > 0$ and choose a $\delta_1 > 0$ and $S = \{c = s_0 < \cdots < s_n = d\}$ a partition of $[c,d]$ such that $\|S\| < \delta_1$ and $s_k \leq \sigma_k \leq \sigma_{k+1}$. We then get

$$\Big| \int_{\gamma \circ \phi} f - \sum_{k=0}^{n-1} f(\gamma \circ \phi(\sigma_k))(\gamma \circ \phi(s_{k+1}) - \gamma \circ \phi(s_k)) \Big| < \frac{\varepsilon}{2}$$

Choose $\delta_2 > 0$ and $T = \{a = t_0 < \cdots < t_m = b\}$ a partion of $[a,b]$ such that $\|T\| < \delta_2$ and $t_k \leq \tau_l \leq t_{l+1}$. Then

$$\Big| \int_\gamma f - \sum_{l=0}^{m-1} f(\tau_l)(\gamma(t_{l+1}) - \gamma(t_l)) \Big| < \frac{\varepsilon}{2}$$

Since ϕ is continuous on $[c,d]$, it is uniformly continuous, so that there exists a $\delta > 0$ chosen such that $\delta = \min\{\delta_1, \delta_2\}$ and for which $|\phi(s) - \phi(s')| < \varepsilon$ whenever $0 < |s - s'| < \delta$. Then if $\|S\| < \delta$, and $t_l = \phi(s_k)$, then $\|T\| < \delta$ and if $s_k \leq \sigma_k \leq s_{k+1}$ with $\tau_l = \phi(\sigma_k)$, then we get

$$\Big| \int_\gamma f - \int_{\gamma \circ \phi} f \Big| < \varepsilon$$

■

Definition. Let $\sigma : [a,b] \to \mathbb{C}$ and $\gamma : [a,b] \to \mathbb{C}$ be rectifiable paths. If $\phi : [a,b] \to [a,b]$ is a continuous strictly increasing functuion who's image is $[a,b]$, such that $\sigma = \gamma \circ \phi$, then we call ϕ a **change of parameter**. We call σ a **paramtrization** of γ about ϕ.

Definition. Let $\gamma : [a,b] \to \mathbb{C}$ be a path. We define the **reverse** of γ to be the path $\gamma^- : [-b, -a] \to \mathbb{C}$ defined by $\gamma^-(t) = \gamma(-t)$.

Lemma 4.2.3. *Let $\gamma : [a,b] \to \mathbb{C}$ be a rectifiable path, and suppos that f is a continuous complex valued function defined on $\{\gamma\}$. Then the following are true.*

(1)

$$\int_\gamma f = -\int_{\gamma^-} f$$

(2)

$$\Big|\int_\gamma f(z)\ dz\Big| \leq \int_\gamma |f(z)|\ |dz| \leq V(\gamma) \sup_{z\in\{\gamma\}} \{f(z)\}$$

(3) If $c \in \mathbb{C}$, then

$$\int_\gamma f(z)\ dz = \int_{\gamma+c} f(z-c)\ dz$$

Definition. Let f be a complex valued funtion defined on a domain U. We define a **primitive** for f on U to be a holomorphic function F on U such that $F' = f$.

Lemma 4.2.4. *If U is open in $\mathbb{C}$ and $\gamma : [a, b] \to \mathbb{C}$ a rectifiable path, with $\{\gamma\} \subseteq U$. If $f : U \to \mathbb{C}$ is a continuous complex valued function, then for every $\varepsilon > 0$, there exists a polygonal path Γ with $\{\Gamma\} \subseteq U$ such that $\Gamma(a) = \gamma(a)$, $\Gamma(b) = \gamma(b)$ and*

$$\Big|\int_\gamma f - \int_\Gamma f\Big| < \varepsilon$$

Proof. Suppose first that U is an open ball. Since $\{\gamma\}$ is compact, let d be the distance between $\{\gamma\}$ and ∂U. Then if $U = (c, r)$, we have $\{\gamma\} \subseteq B(c, \rho)$ where $\rho = r - \frac{d}{2}$. Moreover, f is uniformly continuous on $\operatorname{cl} B(c, \rho)$. Hence, without loss of generality, suppose that f is uniformly continuous on U. Choose then, a $\delta > 0$ such that $|f(z) - f(w)| < \varepsilon$ whenever $|z - w| < \delta$, for some $\varepsilon > 0$. If $\gamma : [a, b] \to \mathbb{C}$ is a rectifiable path, then γ is uniformly continuous, hence there exists a partion $P = \{a = t_0 < \cdots < t_n = b\}$ of $[a, b]$ for which $|\gamma(s) - \gamma(t)| < \frac{\delta}{2}$, for $s, t \in [t_k, t_{k+1}]$, and such that for $\tau_k \in [t_k, t_{k+1}]$, we have

$$\Big|\int_\gamma f - \sum_{k=0}^{n-1} f \circ \gamma(\tau_k)(\gamma(t_{k+1}) - \gamma(t_k))\Big| < \varepsilon$$

Now, define $\Gamma : [a, b] \to \mathbb{C}$ by

$$\Gamma(t) = \frac{(t_{k+1} - t)\gamma(t_k) + (t - t_k)\gamma(t_{k+1})}{t_{k+1} - t_k}$$

Then Γ is a polygonal path whose trace is contained in U, from $\gamma(t_k)$ to $\gamma(t_{k+1})$ on each $[t_k, t_{k+1}]$, and $|\Gamma(t) - \Gamma(\tau_k)| < \delta$. Now, by definition we have

$$\int_\Gamma f = \int_a^b f \circ \Gamma(t)\Gamma'(t)\ dt$$

which gives us

$$\int_\Gamma f = \sum_{k=0}^{n-1} \frac{\gamma(t_{k+1}) - \gamma(t_k)}{t_{k+1} - t_k} \int_{t_k}^{t_{k+1}} f \circ \Gamma(t)\ dt$$

Then

$$\Big|\int_\gamma f - \int_\Gamma f\Big| \leq \varepsilon + \Big|\sum_{k=0}^{n-1} f \circ \gamma(\tau_k)(\gamma(t_{k+1}) - \gamma(y_k)) - \int_\Gamma f\Big|$$

Hence

$$\int_\Gamma f \leq \varepsilon + \sum \frac{|\gamma(t_{k+1}) - \gamma(t_k)|}{t_{k+1} - t_k} \int_t^{t_{k+1}} |f \circ \Gamma(t) - f \circ \gamma(t)| \; dt \leq \varepsilon + \varepsilon \sum_{k=0}^{n-1} (\gamma(t_{k+1}) - \gamma(t_k)) \leq \varepsilon(1 + V(\gamma))$$

Now if U is an arbitrary open set of $\mathbb{C}$, sicne $\{\gamma\}$ is compact, there exists an $r > 0$ such that $r < d(\{\gamma\}, \partial U)$. Choose, then, a $\delta > 0$ such that $|\gamma(s) - \gamma(t)| < r$ whenever $|s - t| < \delta$. Let $P = \{a = t_0 < \cdots < t_n = b\}$ a partition of $[a, b]$ wit $\|P\| < \delta$. Then $|\gamma(t) - \gamma(t_k)| < r$ for all $t_k \leq t \leq t_{k+1}$. Defining then $\gamma_k : [t_k, t_{k+1}] \to U$ by $\gamma_k(t) = \gamma(t)$, then we get $\{\gamma\} \subseteq B(\gamma(t_k), r)$ for all $0 \leq k \leq n - 1$. Then by above, there is a polygonal path $\Gamma_k : [t_k, t_{k+1}] \to B(\gamma(t_k), r)$ such that $\Gamma_k(t_k) = \gamma_k(t_k)$ and $\Gamma_k(t_{k+1}) = \gamma(t_{k+1})$ and

$$\Big| \int_{\gamma_k} f - \int_{\Gamma_k} f \Big| < \frac{\varepsilon}{n}$$

Define then $\Gamma : [a, b] \to \mathbb{C}$ by $\Gamma(t)|_{[t_k, t_{k+1}]} = \Gamma_k(t)$. Then this Γ satisfies the properties above. ∎

Theorem 4.2.5 (The Fundamental Theorem of Calculus for Path Integrals)**.** *Let U be open in $\mathbb{C}$, and let $\gamma : [a, b] \to \mathbb{C}$ be a rectifiable path with $\{\gamma\} \subseteq U$. If $f : U \to \mathbb{C}$ is a continuous complex valued function with primitive $F : U \to \mathbb{C}$, then*

$$\int_\gamma f = F \circ \gamma(b) - F \circ \gamma(a)$$

Proof. Let $\gamma : [a, b] \to \mathbb{C}$ be peicewise C^1. Then we have by definition

$$\int_\gamma f = \int_a^b f \circ \gamma(t) \gamma'(t) \; dt = \int_a^b F' \circ \gamma(t) \gamma'(t) \; dt = F \circ \gamma(b) - F \circ \gamma(a)$$

by the fundamental theorem of calculus for real valued integrals.

More generally, if $\varepsilon > 0$, there exists a polygonal path $\Gamma : [a, b] \to \mathbb{C}$ with $\Gamma(a) = \gamma(a)$ and $\Gamma(b) = \gamma(b)$ such that

$$\Big| \int_\gamma f - \int_\Gamma f \Big| < \varepsilon$$

But Γ is piecewise C^1, so that

$$\Big| \int_\gamma f - (F \circ \gamma(b) - F \circ \gamma(a)) \Big| < \varepsilon$$

and we are done. ∎

Corollary. *If γ is a close rectifiable curve, and f is a continuous complex valued function defined on $\{\gamma\}$ and having a primitive, then*

$$\int_\gamma f = 0$$

4.3 Analytic Functions

Definition. Let $f : U \to \mathbb{C}$ be a complex valued function on an open set U of $\mathbb{C}$. We call f **analytic** at z_0 if f has a power series representation converging on the open ball $B(z_0, R)$, with $R > 0$; that is,

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n \text{ for all } |z - z_0| < R$$

and R is the radius of convergence of the series. We call f **analytic** on U if it is analytic at ever point of U. We call f **entire** if it is analytic on all $\mathbb{C}$.

Theorem 4.3.1 (Leibniz's Rule). *Let $\phi : [a, b] \times [c, d] \to \mathbb{C}$ be a continuous complex valued function, and define $g : [c, d] \to \mathbb{C}$ by*

$$g(t) = \int_a^b \phi(s, t)\ ds$$

Then g is continous, moreover if $D_t\phi$ exists and is continuous on $[a, b] \times [c, d]$, then g is of class C^1 and

$$g'(t) = \int_a^b D_t\phi(s, t)\ ds$$

Proof. That ϕ is continuous implies that g is continuous. Now, fix $t_0 \in [c, d]$ and let $\varepsilon > 0$, and let $\phi_t = D_t\phi$. Since ϕ_t is continuous it is uniformly continuous on $[a, b] \times [c, d]$, thus, there exists a $\delta > 0$ for which $|\phi_t(s', t') - \phi_t(s, t)| < \varepsilon$ whenever $(s - s')^2 + (t - t')^2 < \delta^2$. In particular, we have $|\phi_t(s, t) - \phi(s, t_0)| < \varepsilon$ whenever $|t - t_0| < \delta$ and $a \leq sb$. Then we get

$$\Big| \int_{t_0}^{t} \phi_t(s, \tau) - \phi_t(s, t_0)\ dt \Big| < \varepsilon|t - t_0|$$

whenver $|t - t_0| < \delta$, for s fixed. Now, let

$$\Phi(t) = \phi(s, t) - \phi_t(s, t_0)$$

then Φ is a primitive of $\phi_t(s, t) - \phi_t(s, t_0)$. Thus, by the fundamental theorem of calculus, it follows that

$$|\phi(s, t) - \phi(s, t_0) - (t - t_0)\phi_t(s, t_0)| < \varepsilon|t - t_0|$$

for every $s \in [a, b]$, when $|t - t_0| < \varepsilon$. This gives

$$\Big| \frac{g(t) - g(t_0)}{t - t_0} - \int_a^b \phi_t(s, t_0)\ ds \Big| \leq \varepsilon(b - a)$$

whenever $0 < |t - t_0| < \delta$. Since g is differentiable and $D_t(\phi)$ is continuous, we get that g' is continuous. ■

Example 4.2. Let $\phi(s, t) = \frac{e^{is}}{e^{is} - tz}\ ds$ for $0 \leq t \leq 1$ and $0 \leq s \leq 2\pi$. Let $|z| < 1$. Then ϕ is of class C^1. Then

$$g(t) = \int_0^{2\pi} \frac{e^{is}}{e^{is} - tz}\ ds$$

Then $g(0) = 2\pi$, and now

$$g'(t) = \int_0^{2\pi} \frac{ze^{is}}{(e^{is} - tz)^2}\ ds$$

with t fixed. Then

$$\Phi(s) = \frac{iz}{e^{is} - tz}$$

has as derivative

$$\Phi'(s) = \frac{ze^{is}}{(e^{is} - tz)^2}$$

hence $g'(z) = \Phi(2\pi) - \Phi(0) = 0$ making g constant. Hence $g(1) = 2\pi$ and we get

$$\int_0^{2\pi} \frac{e^{is}}{e^{is} - tz}\ ds = 2\pi$$

for all $|z| < 1$.

Lemma 4.3.2. *Let $f : U \to \mathbb{C}$ be holomorphic on an open set U of $\mathbb{C}$, and suppose that $\overline{B}(z_0, r) \subseteq U$ with $r > 0$. If $\gamma(t) = z_0 + re^{it}$ on $0 \le t \le 2\pi$, then*

$$f(z) = \frac{1}{2i\pi} \int_\gamma \frac{f(w)}{w - z}\ dw$$

for every $|z - z_0| < r$

Proof. Consider $U_1 = \{\frac{z - z_0}{r} : z \in U\}$ and let $g(z) = f(z_0 + rz)$. Then without loss of generality, suppose that $z_0 = 0$, and $r = 1$, so that we work with the unit ball $B(0,1)$.

Fix $|z| < 1$ and let

$$\phi(s,t) = \frac{f((1-t)z + te^{is})e^{is}}{e^{is} - z} - f(z)$$

for all $0 \le t \le 1$, and $0 \le s \le 2\pi$. Notice that $|(1-t)z + te^{is}| \le 1$ so that g is of class C^1.

Now, we have

$$g(0) = \int_0^{2\pi} \phi(s,0)\ ds = f(z) \int_0^{2\pi} \frac{e^{is}}{e^{is} - z}\ ds - 2\pi f(z) = 0$$

so that we get

$$\int_0^{2\pi} \frac{e^{is}}{e^{is} - z}\ ds = 2\pi$$

now, by leibniz's rule, we get

$$g'(t) = \int_0^{2\pi} D_t\phi(s,t)\ ds$$

however, for $0 \le t \le 1$, we have

$$\Phi(s) = -i\frac{f(z + t(e^{is} - z))}{t}$$

is a primitive of $D_t\phi$, and that $g'(t) = \Phi(2\pi) - \Phi(0) = 0$ for all $0 \leq t \leq 1$, which makes g constant. So we get

$$f(z) = \int_0^{2\pi} \frac{f(e^{is})e^{is}}{e^{is} - z}\, ds = \frac{1}{2i\pi} \int_\gamma \frac{f(w)}{w - z}\, dw \tag{4.1}$$

■

Lemma 4.3.3. *Let γ be a rectifiable path in $\mathbb{C}$. Then, if $\{f_n\}$ is a sequence of continuous complex valued functions, defined on $\{\gamma\}$, converging uniformly to a continuous complex valued function f defined on $\{\gamma\}$, then we have*

$$\int_\gamma f_n = \int_\gamma f$$

Proof. Let $\varepsilon > 0$. Then there exists an $N \in \mathbb{Z}^+$ such that

$$|f_n(z) - f(z)| < \frac{\varepsilon}{V(\gamma)}$$

for ever $z \in \{\gamma\}$, and $n \geq N$. Then we get

$$\Big| \int_\gamma f_n - \int_y f \Big| = \Big| \int_\gamma (f_n - f) \Big| \leq \int_\gamma |f_n(z) - f(z)|\, |dz| \leq \frac{\varepsilon}{V(\gamma)} V(\gamma) \leq \varepsilon$$

■

Theorem 4.3.4. *Let $f : U \to \mathbb{C}$ be a complex valued function on an open set U of $\mathbb{C}$. Then f is holomorphic if, and only if it is analyitic.*

Proof. It was shown by theorem 3.2.4 that functions which are analytic on their domain are holomorphic on their domain. It remains to show the converse.

Suppose first that f is holomorphic. Let $0 < r < R$ so that $\overline{B}(z_0, r) \subseteq B(z_0, R)$ and take $y(t) = z_0 + re^{it}$ on $0 \leq t \leq 2\pi$. Then we have that

$$f(z) = \frac{1}{2i\pi} \int_\gamma \frac{f(w)}{w - z}\, dw \text{ for all } |z - z_0| < r$$

since $w \in \{\gamma\}$, and $|z - z_0| < r$, we have

$$\frac{|f(w)||z - z_0|^n}{|w - z_0|^{n+1}} \leq \frac{M}{r} \Big(\frac{z - z_0}{r} \Big)^n$$

with $M = \max \{|f(w)|\}$ for all $|w - z_0| < r$. Then we get

$$\frac{|z - z_0|}{r} < 1$$

so by the Weierstrass M-test, the series

$$\sum_{n=0}^{\infty} f(z) \frac{(z - z_0)^n}{(w - z_0)^{n+1}}$$

converges uniformly on $\{\gamma\}$. Using a geometric series, we get

$$\frac{1}{w-z_0} = \frac{1}{1-\frac{z-z_0}{w-z_0}} = \frac{1}{w-z_0}\sum_{n=0}^{\infty}\left(\frac{z-z_0}{w-z_0}\right)^n$$

Since $|z-z_0| < r$. Now, multiplying both sides by $(\frac{f(z)}{2i\pi})$ and integrating along the circle defined by $|w-z_0| < r$ gives us $f(z)$. Now, by lemma 4.3.3, and integrating under the sum, we get

$$f(z) = \sum_{n=0}^{\infty} a_n(z-z_0)^n$$

where

$$a_n = \frac{1}{2i\pi}\int_\gamma \frac{f(w)}{(w-z_0)^{n+1}}\,dw$$

Therefore, f converges for $|z-z_0| < r$, and moreover each a_n is independent of the path γ, so that we get the required power series converging on $|z-z_0| < R$. ■

Corollary. *If f is analytic on U, then for every $z_0 \in U$ with $|z-z_0| < R$,*

$$f(z) = \sum_{n=0}^{\infty} a_n(z-z_0)^n$$

where the radius of convergence $R > 0$ is gven by the distance between z_0 and ∂U.

Corollary. *Analytic functions are of class C^∞.*

Corollary. *If f is analytic on U and $\overline{B}(z_0, r) \subseteq U$, then*

$$f^{(n)}(z_0) = \frac{n!}{2i\pi}\int_\gamma \frac{f(w)}{(w-z)^{n+1}}\,dw$$

where $\gamma(t) = z_0 + re^{it}$ on $0 \le t \le 2\pi$.

Corollary. *A function is entire if and only if it is holomorphic on all $\mathbb{C}$.*

Remark. This theorem shows that the definitions for analytic functions and holomorphic functions are equivalent. Therefore, we make the convention of calling functions holomorphic when we want to refer to their differential properties, and analytic when we want to refer to them as power series expansions.

Theorem 4.3.5 (Cauchy's Estimate)**.** *Let $f : U \to \mathbb{C}$ be analytic on an open set U of $\mathbb{C}$, whose power series expansion converges on the ball $B(z_0, R)$. If there is an $M > 0$ such that $|f(z)| \le M$ for every $z \in B(z_0, R)$, then*

$$|g^{(n)}(z_0)| \le \frac{n!M}{R^n}$$

Proof. By the above corollory, we have

$$|f^{(n)}(z_0)| \leq \frac{n!}{2\pi}\int_\gamma \left|\frac{f(w)}{(w-z)^N n+1}\right| dw \leq \frac{n!}{2\pi}\frac{M}{R^{n+1}}2\pi r = \frac{n!M}{R^n}$$

■

Lemma 4.3.6. *Let $f : U \to \mathbb{C}$ be analytic on an open set U of $\mathbb{C}$, and suppose that γ is a closed rectifiable path on the ball $B(z_0, R)$, with $R > 0$ the radius of convergence of f at z_0. Then f has a primitive and*

$$\int_\gamma f = 0$$

Proof. We have that

$$f(z) = \sum a_n(z - z_0)^n \text{ for all } |z - z_0| < R$$

Let

$$F(z) = \sum \frac{a_n}{n+1}(z - z_0)^{n+1}$$

Since $\sqrt{n+1} \to 1$ as $n \to \infty$, F has the same radius of convergenc of F, and hence converges on the ball $B(z_0, R)$. Moreover, notice that $F' = f$ for every $|z - z_0| < R$. ■

4.4 Roots of Analytic Functions

Definition. If $f : U \to \mathbb{C}$ is an analytic function, and $z_0 \in U$, satisfies $f(z_0) = 0$, then we call z_0 a **root** (or **zero**) of f, of **multiplicity** $m \geq 1$. provided that $f(z) = (z - z_0)^m g(z)$ where $g : U \to \mathbb{C}$ is an analytic function such that $g(z_0) \neq 0$.

Lemma 4.4.1. *If f is an entire function, then*

$$f(z) = \sum_{n=0}^{\infty} a_n z^n$$

Proof. Since f is entire, it is analytic on all $\mathbb{C}$ by definition, and hence has a power series expansion about the point $0 \in \mathbb{C}$. ■

Theorem 4.4.2 (Liouville's Theorem)**.** *Bounded entire functions are constant.*

Proof. Let f be an entire function, and $M > 0$ such that $|f(z)| \leq M$ for all $z \in \mathbb{C}$. Now, since f is entire, it is analytic about any open ball $B(z_0, R)$, and by Cauchy's estimate, we have

$$|f'(z)| \leq \frac{M}{R}$$

Then as $R \to \infty$, $|f'(z)| \to 0$ making $f'(z) = 0$ for all $z \in \mathbb{C}$, and hence making f constant. ■

Theorem 4.4.3 (The Fundamental Theorem of Algebra)**.** *Any nonconstant polynomial over $\mathbb{C}$ has atleast one root in $\mathbb{C}$.*

Proof. Let $p(z)$ be a nonconstant polynomial over $\mathbb{C}$, and suppose there exists no roots of p in $\mathbb{C}$; that is, $p(z) \neq 0$ for all $z \in \mathbb{C}$. Now, let $q(z) = \frac{1}{p(z)}$. Since p is a polynomial, it is entire, and our hypothesis on p makes q entire as well. Moreover, $\lim |p(z)| \to \infty$ as $|z| \to \infty$, so that $\lim |q(z)| \to 0$. Since q is continuous on the closed ball $\overline{B}(0, R)$, there is an $M > 0$ for which $|q(z)| \leq M$ for all $z \in \overline{B}(0, R)$, so that q is constant by Liouville's theorem. Therefore, p must also be constant, which contradicts our assumption on p. Therefore, p must have atleast one root in $\mathbb{C}$. ■

Corollary. *If $\alpha_1, \dots, \alpha_n$ are roots of $p(z)$, of multiplicities $k_1, \dots, k_n$, then $p(z) = (z - \alpha_1)^{k_1} \dots (z - \alpha_n)^{k_n}$, where* $\deg p = k_1 + \dots, k_n$.

Example 4.3. Let $f(z) = \cos \frac{1+z}{1-z}$ on the ball $B(0, 1)$. Then $\frac{1+z}{1-z}$ maps $B(0, 1)$ onto the half plane $H^+ = \{z \in : \operatorname{Re} z > 0\}$. Then the roots of f are given by

$$\frac{n\pi - 1}{n\pi + 1} 1 \mod 2$$

for all $n \in \mathbb{Z}^+$ and n odd. So f has infinitely many roots in $\mathbb{C}$, however, notice that as $n \to \infty$, $\frac{n\pi - 1}{n\pi + 1} \to 1 \notin B(0, 1)$.

Theorem 4.4.4. *Let U be a region of $\mathbb{C}$, and let $f : U \to \mathbb{C}$ be analytic on U. Then the following are equivalent.*

(1) $f(z) = 0$ for all $z \in U$.

(2) There exists a $z_0 \in U$ for which $f^{(n)}(z_0) = 0$ for all $n \geq 0$.

(3) The set of roots of f has a limit point in U.

Proof. First suppose that $f(z) = 0$ on its domain. Then $f^{(n)}(z) = 0$ for every $z \in U$, and f admits all of its roots as limit points of the set $Z = \{z \in \mathbb{C} : f(z) = 0\}$.

Now, suppose that the set Z of roots of of f admits a limit point $z_0 \in U$. Let $R > 0$ such that $B(z_0, R)U$. Now, since f is continuous, we get $f(z_0) = 0$. Now, let $n \in \mathbb{Z}^+$ such that $f^{(i)}(z_0) = 0$ for all $1 \leq i \leq n-1$, and $f^{(n)} \neq 0$. Expanding f as a power series about z_0 gives

$$f(z) = \sum a_n (z - z_0)^n$$

for all $|z - z_0| < R$,. Now, if

$$g(z) = \sum_{k \neq m} a_n (z - z_0)^{k-n}$$

then g is analytic on the ball $B(z_0, R)$, and $f(z) = (z - z_0)^n g(z)$, with $g(z_0) = an_n \neq 0$. Then fnd $0 < r < R$ such that $g(z) \neq 0$ for all $|z - z_0| < r$. Since z_0 is a limit point of the set Z, there is a $w \in Z$ for which $f(w) = n$ and $0 < |w - z_0| < r$, so that $(w - z_0)^n g(w) = 0$, which makes $g(w) = 0$, which is a contradiction. Hence $f^{(n)}(z_0) = 0$.

Conversely, let $Z^{(n)} = \{z_0 \in U : f^{(n)}(z) = 0 \text{ for all } n \geq 0\}$, and suppose that there exists a point $z_0 \in U$ for which $f^{(n)}(z_0) = 0$ for all such n; that is, that $Z^{(n)}$ is nonempty. Since U is connected, $Z^{(n)}$ is open in U. Now, let $z \in \operatorname{cl} Z^{(n)}$, and choose a sequence $\{z_k\}_{k \in \mathbb{Z}^+}$ of points of $Z^{(n)}$, such that $\{z_k\} \to z$ as $k \to \infty$; that is, z is a limit point of $Z^{(n)}$. Since $f^{(n)}$ is continuous, by the sequential criterion, we get

$$f^{(n)}(z) = \lim f^{(n)}(z_k) = 0$$

so that $z \in Z^{(n)}$, and $Z^{(n)}$ contains a limit point. This makes $Z^{(n)}$ closed, and hence $Z^{(n)} = U$ by the connectedness of U. So that $f(z) = 0$ for all $z \in U$. ■

Corollary. *If f and g are analytic on a region U, then $f(z) = g(z)$ for every $z \in U$ if, and only if the set $Z = \{z \in U : f(z) = g(z)\}$ contains a limit point in U.*

Proof. Consider the analytic function $f - g$. ■

Corollary. *If f is analytic on a region U such that f is not identically 0, then for every $z_0 \in U$ with $f(z_0) = 0$, there exists an $n \in \mathbb{Z}^+$ and an analytic function $g : U \to \mathbb{C}$ such that $g(z_0) \neq 0$ and $f(z) = (z - z_0)^n g(z)$*

Proof. Let $n \geq 1$, the largest such integer for which $f^{(k)}(z_0) = 0$ for all $0 \leq k \leq n$. Define

$$g(z) = \frac{f(z)}{(z - z_0)^n}$$

for all $z \neq z_0$, and

$$g(z_0) = \frac{f^{(n)}(z_0)}{n!}$$

when $z = z_0$. Then g is analytic on $U \backslash \{z_0\}$, then by the above theorem, g is analytic in all of U. ■

Theorem 4.4.5 (The Maximum Modulus Theorem)**.** *Let U be a region, and $f : U \to \mathbb{C}$ analytic on U such that there exists a $z_0 \in U$ with $|f(z)| \leq |f(z_0)|$ for all $z \in U$. Then f is constant.*

Proof. Let $\overline{B}(z_0, r)$ a closed ball in U for $0 \leq r$, and let $\gamma(t) = z_0 + re^{it}$ on $0 \leq t \leq 2\pi$. Then we have

$$|f(z_0)| = \frac{1}{2i\pi} \int_\gamma \frac{f(w)}{w - z_0}\, dw = \frac{1}{2i\pi} \int_0^{2\pi} f(z_0 + re^{it})\, dt$$

Then we get

$$|f(z_0)| \leq \left| \frac{1}{2i\pi} \int_0^{2\pi} f(z_0 + re^{it})\, dt \right| \leq \int_0^{2\pi} |f(z_0 + re^{it})|\, dt \leq |f(z_0)|$$

since $|f(z_0 + re^{it})| \leq |f(z_0)|$, we get

$$\int_0^{2\pi} |f(z_0)| - |f(z_0 + re^{it})|\, dt = 0$$

which makes $|f(z_0)| = f(z_0 + reit)$, for all $0 \leq t \leq 2\pi$. Then f maps any ball $B(z_0, R)$ of U into the circle described by $|z| = |\alpha|$ for some $\alpha \in \mathbb{C}$, with $f(z_0) = \alpha$. Thus f is constant on $B(z_0, R)$. In particularm $f(z) = \alpha$ for every $|z - \alpha| < R$. ■

4.5 Winding Numbers and Cauchy's Integral Formula

Lemma 4.5.1. *Let $\gamma : [0,1] \to \mathbb{C}$ be a closed rectifiable path. Define, for $z_0 \notin \{\gamma\}$ the path integral*

$$W = \frac{1}{2i\pi} \int_\gamma \frac{dz}{z - z_0}$$

Then $W \in \mathbb{Z}$.

Proof. Suppose, without loss of generality, that γ is of class C^1. Define $g : [0,1] \to \mathbb{C}$ by

$$g(t) = \int_0^t \frac{\gamma'(s)}{\gamma(s) - z_0}\, ds$$

hence $g(0) = 0$ and $g(1) = \int_\gamma \frac{dz}{z - z_0}$. Moreover, notice that

$$g'(t) = \frac{\gamma'(s)}{\gamma(s) - z_0} \text{ on } 0 \leq t \leq 2\pi$$

Then we get $D(\exp(-g)(\gamma - z_0)) = 0$ so that $\exp(-g)(\gamma - z_0)$ is the constant function, and $\exp(-g)(\gamma(0) - z_0) = \exp(-g)(\gamma(1) - z_0) = 2i\pi k$, and $k \in \mathbb{Z}$. Notice that k is precisely W. ■

Definition. Let γ be a rectifiable closed path in $\mathbb{C}$. We define the **winding number** of γ **around** some point $z_0 \notin \{\gamma\}$ to be the path integral given by

$$W(\gamma, z_0) = \frac{1}{2i\pi} \int_\gamma \frac{dz}{z - z_0}$$

Lemma 4.5.2. *If γ and σ be closed rectifiable paths having the same initial points. Then the following are true*

(1) $W(\gamma, z_0) = -W(\gamma^-, z_0)$ for all $z_0 \notin \{\gamma\}$.

(2) $W(\gamma + \sigma, z_0) = W(\gamma, z_0) + W(\sigma, z_0)$ for all $z_0 \notin \{y\} \cup \{\sigma\}$.

Definition. Let γ be a closed path. We define the **interior** of γ to be the open set U of $\mathbb{C}$ having as boundry the trace of γ, i.e. $\partial U = \{\gamma\}$. We denote the interior of γ by $\operatorname{int} \gamma$. We call an open set U of the set $\mathbb{C} \backslash \{\gamma\}$ a component **bounded** by γ if $U \subseteq \operatorname{int} \gamma$, and **unbounded** otherwise.

Theorem 4.5.3. *Let γ be a closed rectifiable path in $\mathbb{C}$. Then $W(\gamma, z_0)$ is constant for any z_0 in a component U of $\mathbb{C} \backslash \{\gamma\}$. Moreover, if z_0 is in an unbounded component, then $W(\gamma, z_0) = 0$.*

Proof. Let $f : U \to \mathbb{C}$ be defined by $f(z) = W(\gamma, z)$. Now, fix a $z_0 \in U$, and let r be the distance between z_0 and $\{\gamma\}$. If $|a - b| < \delta < \frac{r}{2}$, then we get

$$|f(a) - f(b)| = \frac{1}{2i\pi} \Big| \int_\gamma \frac{a - b}{(z - a)(z - b)}\, dz \Big| \leq \frac{|a - b|}{2i\pi} \int_\gamma \frac{|dz|}{|z - a||z - b|}$$

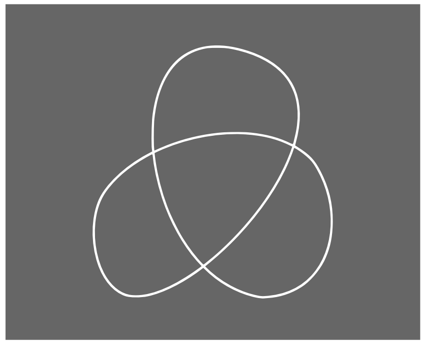

Figure 4.2: A closed rectifiable path γ (removed from $\mathbb{C}$) with its bounded components. Notice that the component lying outside of $\{\gamma\}$ is an unbounded component.

now, for $|a-b| < \frac{r}{2}$, and $z \in \{\gamma\}$, we get $|z-a| \leq r > \frac{r}{2}$, thus $|f(a)-f(b)| < \frac{2\delta}{\pi r^2}V(\gamma)$ so if $\varepsilon > 0$, choosing $\delta < \min\{\frac{r}{2}, \frac{\pi r^2}{2V(\gamma)}\}$, and we get that f is continuous.

Now, if U is a component of $\mathbb{C}\backslash\{\gamma\}$, then $f(U)$ is connected. However, notice that $f(U) \subseteq \mathbb{Z}$, so that $f(U)$ must be pointset. Therefore f is constant on U.

Now, let V be an unbounded component of $\mathbb{C}\backslash\{\gamma\}$; i.e. $V \int \gamma$; then there exists an $R > 0$ such that the set $\{z \in V : |z| > R\} \subseteq V$. If $\varepsilon > 0$, choose a point z_0 with $|z_0| > R$, and $|z - z_0| > \frac{V(\gamma)}{2\pi\varepsilon}$ uniformly for $z \in \{\gamma\}$. Then $|W(\gamma, z_0)| < \varepsilon$ so that $W(\gamma, z_0) \to 0$. Since $W(\gamma, z_0)$ is constant, this makes $W(\gamma, z_0) = 0$ on V. ■

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