

Field Theory and Galois Theory.

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February 13, 2023

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Chapter 1

Fields.

1.1 Field Extensions.

Definition. We define the **characteristic** of a field F to be the smallest positive integer p , such that $p \cdot 1 = 0$, where 1 is the identity of F . We write $\text{char } F = p$, and if no such p exists, then we write $\text{char } F = 0$.

Lemma 1.1.1. *Let F be a field, then $\text{char } F$ is either 0, or a prime integer.*

Proof. Let $\text{char } F = p$. If $p = 0$, then we are done. Now suppose that $p = mn$, with $m, n \in \mathbb{Z}^+$. Then $p \cdot 1 = (mn)1 = (n \cdot 1)(m \cdot 1) = mn = 0$, which makes m and n 0 divisors. Since F is a field, and hence an integral domain, this is impossible, and hence p must be prime. ■

Corollary. *If $\text{char } F = p$, then for all $a \in F$, $pa = \underbrace{a + \cdots + a}_{p \text{ times}}$.*

Proof. We have $pa = p(a \cdot 1) = (p \cdot 1)a$. ■

Example 1.1. (1) Both \mathbb{Q} and \mathbb{R} have $\text{char} = 0$. Similarly, $\text{char } \mathbb{Z} = 0$, even though \mathbb{Z} is just an integral domain.

(2) $\text{char } \mathbb{Z}/p\mathbb{Z} = p$ and $\text{char } \mathbb{Z}/p\mathbb{Z}[x] = p$ for any prime p .

Definition. We define the **prime subfield** of a field F to be the subfield of F generated by 1.

Example 1.2. (1) The prime subfields of \mathbb{Q} and \mathbb{R} is \mathbb{Q} .

(2) Let $\mathbb{Z}/p\mathbb{Z}(x)$ the field of rational functions over $\mathbb{Z}/p\mathbb{Z}$. Then the prime subfield of $\mathbb{Z}/p\mathbb{Z}(x)$ is $\mathbb{Z}/p\mathbb{Z}$. Similarly, the prime subfield for $\mathbb{Z}/p\mathbb{Z}[x]$ is also $\mathbb{Z}/p\mathbb{Z}$.

Definition. If K is a field containing a field F , then we call K **field extension** over F , and write K/F (not the quotient field!) or denote it by the diagram

$$\begin{array}{c} K \\ | \\ F \end{array}$$

Lemma 1.1.2. *Every field is a field extension of its prime subfield.*

Lemma 1.1.3. *Let K an extension over a field F . Then K is a vector space over F .*

Definition. Let K/F a field extension. We define the **degree** of K over F , $[K : F]$ to be the dimension of K/F as a vector space.

Definition. Let F be a field, and $f \in F[x]$ a polynomial. We call an element $\alpha \in R$ a **root** (or **zero**) of f if $f(\alpha) = 0$.

Lemma 1.1.4. *Let $\phi : F \rightarrow L$ a field homomorphism. Then either $\phi = 0$, or ϕ is 1-1.*

Lemma 1.1.5. *Let F be a field, and $p \in F[x]$ an irreducible polynomial. Then there exists a field K containing an embedding of F , such that p has a root in K .*

Proof. Consider $K = F[x]/(p)$. Since p is irreducible in a principle ideal domain, (p) is a maximal ideal, and hence K is a field. Now consider the canonical map $\pi : F[x] \rightarrow K$ taking $f \rightarrow f \bmod (p)$ and let $\phi = \pi|_F$. Then $\phi \neq 0$, since $\pi : 1 \rightarrow 1$. Then ϕ is 1-1. And so $\phi(F) \simeq F$.

Now, consider F as a subfield of K . Then $p(x \bmod (p)) \equiv p(x) \bmod (p) \equiv 0 \bmod (p)$, so that $x \bmod (p)$ is a root of p in K . ■

Corollary. *There exists a field extension of F containing a root of p .*

Theorem 1.1.6. *Let F be a field, and let $p \in F[x]$ an irreducible polynomial of degree n , and let $K = F[x]/(p)$, and $\theta = x \bmod (p)$. Then $\{1, \theta, \dots, \theta^{n-1}\}$ forms a basis for K as a vector space over F and $[K : F] = n$.*

Proof. Let $a \in F[x]$, since $F[x]$ is Euclidean domain, there exist $q, r \in F[x]$, $q \neq 0$ for which

$$a(x) = q(x)p(x) + r(x) \text{ where } \deg r < n$$

Now, since $pq \in (p)$, $a(x) \equiv r(x) \bmod (p)$, and every element of K is a polynomial of degree less than n . Then the elements $\{1, \theta, \dots, \theta^{n-1}\}$ span K .

Now, suppose that there are $b_0, \dots, b_{n-1} \in F$ not all 0 for which

$$b_0 + b_1\theta + \dots + b_{n-1}\theta^{n-1} = 0$$

Then

$$b_0 + b_1\theta + \dots + b_{n-1}\theta^{n-1} \equiv 0 \bmod (p)$$

so that $p|(b_0 + b_1\theta + \dots + b_{n-1}\theta^{n-1})$ in F . But $\deg p = n$ and p divides a polynomial of degree $n-1$, which is a contradiction. Therefore we are left with $b_0 = \dots = b_{n-1} = 0$. ■

Corollary. $K = \{\alpha_0 + a_1\theta + \dots + a_{n-1}\theta^{n-1} : a_i \in F \text{ for all } 1 \leq i \leq n-1\}$

Corollary. *If $a(\theta), b(\theta) \in K$, are elements of degree less than n , and the operations of polynomial addition, and polynomial multiplication mod (p) are defined, then K forms a field.*

Example 1.3. (1) Consider the polynomial $x^2 + 1$ over \mathbb{R} . Then one has the field

$$\mathbb{C} = \mathbb{R}[x]/(x^2 + 1)$$

an extension of \mathbb{R} of degree $[\mathbb{C} : \mathbb{R}] = 2$. Let i be a root of $x^2 + 1$ in this field, then $i^2 = -1$, and the elements of \mathbb{C} are of the form $a + ib$ where $a, b \in \mathbb{R}$. Then we have described the field of complex numbers, and the addition and multiplication (mod $x^2 + 1$) of these elements are the addition and multiplication of complex numbers.

One might also construct \mathbb{C} differently by defining the isomorphism

$$\mathbb{R}[x]/(x^2 + 1) \rightarrow \mathbb{C} \text{ taking } a + xb \rightarrow a + ib$$

(2) Consider again $x^2 + 1$ over \mathbb{Q} . Then we get the field

$$\mathbb{Q}(i) = \mathbb{Q}[x]/(x^2 + 1)$$

of degree $[\mathbb{Q}(i) : \mathbb{Q}] = 2$, and where i is a root of $x^2 + 1$, so that $i^2 = -1$. Then the elements of $\mathbb{Q}(i)$ are of the form $a + ib$ where $a, b \in \mathbb{Q}$, i.e. it is isomorphic to the set of all complex numbers with rational components.

(2) Consider $x^2 - 2$ over \mathbb{Q} . by Eisenstein's criterion for $p = 2$, $x^2 - 2$ is irreducible over \mathbb{Q} . Let α a root of $x^2 - 2$, so that $\alpha^2 = 2$. Then we have the field

$$\mathbb{Q}(\sqrt{2}) = \mathbb{Q}[x]/(x^2 - 2)$$

of degree $[\mathbb{Q}(\sqrt{2}) : \mathbb{Q}] = 2$, and whose elements are of the form $a + b\sqrt{2}$. One can define an isomorphism between $\mathbb{Q}(i)$ and $\mathbb{Q}(\sqrt{2})$ by taking $\sqrt{2} \rightarrow i$.

(3) The polynomial $x^3 - 2$ over \mathbb{Q} gives us the field

$$\mathbb{Q}(\sqrt[3]{2}) = \mathbb{Q}[x]/(x^3 - 2)$$

of degree $[\mathbb{Q}(\sqrt[3]{2}) : \mathbb{Q}] = 3$ over \mathbb{Q} . Here the elements are of the form $a + b\xi + c\xi^2$ where $\xi^3 = 2$.

(4) Denote \mathbb{F}_2 to be a finite field of 2 elements. Consider the polynomial $x^2 + x + 1$ over \mathbb{F}_2 which is irreducible. Then the field

$$\mathbb{F}_2(\alpha) = \mathbb{F}_2[x]/(x^2 + x + 1)$$

is a field of degree 2 over \mathbb{F}_2 , whose elements are of the form $a + b\alpha$, where $\alpha^2 = \alpha + 1$. In fact, one can generate this field using the fact that $\alpha^2 = \alpha + 1$.

(5) Let $F = K(t)$ the field of rational functions in t over a field K . Let $p(x) = x^2 - t \in F[x]$, then by Eisenstien's criterion with the ideal (t) , p is irreducible over $F[x]$. Let θ be a root for p , that is $\theta = \sqrt{t}$, then we get the field $K(t, \sqrt{t})$ of degree $[K(t, \sqrt{t}) : K] = 2$, whose elements are of the form $a(t) + b(t)\sqrt{t}$.

Lemma 1.1.7. *Let F be a subfield of a field K , and let $\alpha \in K$. Then there exists a unique minimal subfield of K containing F and α ; more precisely, it is the intersection of all subfields of K containing F and α .*

Definition. Let K be any extension of a field F , and let $\alpha, \beta, \dots \in K$. Then we define the subfield **generated** by α, β, \dots over F to be the unique minimal subfield containing all α, β, \dots and F and we denote it $F(\alpha, \beta, \dots)$. Moreover, we call K a **simple extension** of F if $K = F(\alpha, \beta, \dots)$. If $K = (F\alpha_1, \dots, \alpha_n)$ for $\alpha_1, \dots, \alpha_n \in K$, then it is a **finitely generated** simple extension.

Theorem 1.1.8. *Let F be a field, and $p \in F[x]$ irreducible, and let K an extension of F containing a root α of p . Then*

$$F(\alpha) \simeq F[x]_{(p)}$$

Proof. Consider the homomorphism $F[x] \rightarrow F(\alpha)$ taking $a(x) \rightarrow a(\alpha)$. Since $p(\alpha) = 0$, p is in the kernel of this homomorphism, and we get an induced homomorphism from $F[x]_{(p)} \rightarrow F(\alpha)$. Now, since p is irreducible, $F[x]_{(p)}$ is a field, and since the homomorphism takes $1 \rightarrow 1$, it is 1–1. Then by the first isomorphism theorem for ring homomorphisms these two fields are isomorphic. ■

Corollary. *If $\deg p = n$, then $F(\alpha) = \{a_0 + a_1\alpha + \dots + a_{n-1}\alpha^{n-1} : a_i \in F \text{ for all } 1 \leq i \leq n-1\}$ and $[F(\alpha) : F] = n$.*

Example 1.4. (1) The polynomial $x^2 - 2$ over \mathbb{Q} also has the root $-\sqrt{2}$ in \mathbb{R} , so that $\mathbb{Q}(-\sqrt{2})$ is of degree 2 over \mathbb{Q} with elements of the form $a - b\sqrt{2}$. Notice however that $\mathbb{Q}(-\sqrt{2}) \simeq \mathbb{Q}(\sqrt{2})$ by taking $a - b\sqrt{2} \rightarrow a + b\sqrt{2}$.

(2) The polynomial $x^3 - 2$ only has the solution $\xi = \sqrt[3]{2}$ in \mathbb{R} . However, in \mathbb{Q} it has the solutions given by

$$\sqrt[3]{2} \left(\frac{-1 \pm i\sqrt{3}}{2} \right)$$

So that the subfields generated by either of these three elements (over \mathbb{C}) are isomorphic.

Theorem 1.1.9. *Let $\phi : F \rightarrow L$ a field isomorphism and $p \in F[x]$, $q \in L[x]$ irreducible polynomials, where q is obtained by applying ϕ to the coefficients of p . Let α a root of p , and β a root of q . Then there exists an isomorphism $F(\alpha) \rightarrow L(\beta)$ taking $\alpha \rightarrow \beta$ and extending ϕ . That is, we have the following diagram*

$$\begin{array}{ccc} F(\alpha) & \longrightarrow & L(\beta) \\ \downarrow & & \downarrow \\ F & \xrightarrow{\phi} & L \end{array}$$

Proof. Notice that ϕ induces a ring homomorphism between $F[x]$ and $L[x]$, so that (p) is maximal. Since q is obtained from p , (q) is also maximal, so that $F[x]_{(p)}$ and $L[x]_{(q)}$ are fields. Then we have an isomorphism

$$F[x]_{(p)} \simeq L[x]_{(q)}$$

Then, if α is a root of p , and β a root of q , we obtain the isomorphism

$$F(\alpha) \simeq L(\beta)$$

moreover, this isomorphism takes $\alpha \rightarrow \beta$. ■

1.2 Algebraic Extensions.

Definition. Let K/F be a field extension. We say that an element $\alpha \in K$ is **algebraic** over F , provided there exists a polynomial over F having α as a root. Otherwise we call α **transcendental**. If every $\alpha \in K$ is algebraic, we call K **algebraic** and K/F an **algebraic extension**.

Lemma 1.2.1. *Let α be algebraic over a field F . Then there exists a unique monic irreducible polynomial $m \in F[x]$ having α as a root. Moreover, if $f \in F[x]$ is a polynomial, then f has α as a root if, and only if $m|f$.*

Proof. Let m a polynomial of minimal degree having α as a root. Suppose, also that m is monic. Now, if m were reducible, then $m(x) = a(x)b(x)$ for some $a, b \in F[x]$ polynomials both of degree less than $\deg m$. Then we also have that $a(\alpha) = b(\alpha) = 0$, which contradicts that m is the polynomial of minimal degree satisfying that condition. Hence, m is irreducible.

Now, let $f \in F[x]$ have α as a root, then by the division theorem, there exist $q, r \in F[x]$, with $q \neq 0$ for which

$$f(x) = q(x)m(x) + r(x) \text{ where } \deg r < \deg m$$

Now, since $f(\alpha) = q(\alpha)m(\alpha) + r(\alpha) = 0$, then $r(\alpha) = 0$ for all x lest we contradict the minimality of m . Hence $m|f$. Conversely, if $m|f$, then f has α as a root.

Now, let g a polynomial of minimal degree for which $g(\alpha) = 0$. Then by above, we have that $\deg g = \deg m$, and that moreover, $m|g$ and $g|m$. therefore $g = m$ and uniqueness is established. ■

Corollary. *Let L/F be an extension, and α algebraic over F . Let $m_{\alpha,F}$ the unique monic irreducible polynomial over F having α as root, and $m_{\alpha,L}$ the unique monic irreducible polynomial over L having α as root. Then $m_{\alpha,L}|m_{\alpha,F}$ in $L[x]$.*

Definition. Let F be a field, and α algebraic over F . We define the **minimal polynomial** $m_{\alpha,F}$, to be the polynomial over F of minimal degree having α as a root. If the field is clear, we instead write m_α , or even just m when the root itself is also clear. We define the **degree** of α to be $\deg \alpha = \deg m_\alpha$.

Lemma 1.2.2. *Let α algebraic over F . Then*

$$F(\alpha) \simeq F[x]/(m_{\alpha,F})$$

Corollary. $[F(\alpha) : F] = \deg m_\alpha = \deg \alpha$.

Example 1.5.

- (1) The minimal polynomial for $\sqrt{2}$ over \mathbb{Q} is $x^2 - 2$.
- (3) The minimal polynomial for $\sqrt[3]{2}$ over \mathbb{Q} is $x^3 - 2$.
- (3) Let $n > 1$, then by the Eisenstein-Schömann criterion, $x^n - 2$ is irreducible over \mathbb{Q} . Moreover, $x^n - 2$ has as root in \mathbb{R} $\sqrt[n]{2}$. Then $\mathbb{Q}(\sqrt[n]{2})$ is a field of degree $[\mathbb{Q}(\sqrt[n]{2}) : \mathbb{Q}] = n$. Moreover $x^n - 2$ is the minimal polynomial of $\sqrt[n]{2}$. Notice, that over \mathbb{R} , $\deg [n]2 = 1$, and that $m_{\sqrt[n]{2}, \mathbb{R}}(x) = x - \sqrt[n]{2}$.
- (4) Consider $p(x) = x^3 - 3x - 1$ over \mathbb{Q} . Notice that p is irreducible over \mathbb{Q} and let α a root of p . Then $[\mathbb{Q}(\alpha) : \mathbb{Q}] = 3$.

Lemma 1.2.3. *An element α is algebraic over a field F if, and only if the simple extension $F(\alpha)/F$ is finite.*

Proof. If α is algebraic over F then $[F(\alpha) : F] = \deg \alpha \leq n$ if α satisfies a polynomial of degree n . Conversely, if α is an element of the finite extension K/F , of degree n , then the set $\{1, \alpha, \dots, \alpha^n\}$ is linearly dependent over F . Hence there exist $b_0, \dots, b_n \in F$ not all 0 for which

$$b_0 + b_1\alpha + \dots + a_n\alpha^n = 0$$

making α a root of a nonzero polynomial over F of degree $\deg \leq n$. ■

Corollary. *If an extension K/F is finite, then it is algebraic.*

Proof. If $\alpha \in K$ is algebraic, then K/F implies that $F(\alpha)/F$ is finite, since $F(\alpha) \subseteq K$. ■

Example 1.6. Let F a field of char $F \neq 2$, and let K an extension field of F of degree $[K : F] = 2$. Let $\alpha \in K$ not in F , then α satisfies an polynomial of at most degree 2 over F . Now, since $\alpha \notin F$, this polynomial must have degree greater than 1. Hence it satisfies a polynomial of degree 2. Then the minimal polynomial of α is a quadratic

$$m_\alpha(x) = x^2 + bx + c \text{ with } b, c \in F$$

Since $F \subseteq F(\alpha) \subseteq K$, and $F(\alpha)$ is a vector space over F of dimension 2, then we must have $K = F(\alpha)$; that is K/F is simple.

Now, the roots of m_α are

$$\alpha = \frac{-b \pm \sqrt{b^2 - 4c}}{2}$$

Since $\alpha \notin F$, $b^2 - 4c$ is not a square in F , and $\sqrt{b^2 - 4c}$ is a root of the equation $x^2 - (b^2 - 4c) = 0$ in K .

Conversely, $\sqrt{b^2 - 4c} = \pm(b + 2\alpha)$ which puts $\sqrt{b^2 - 4c} \in F(\alpha)$. That is $F(\sqrt{b^2 - 4c}) = F(\alpha)$. Moreover, $x^2 - (b^2 - 4c)$ does not have solutions in K .

We call field extensions K/F of degree 2 **quadratic field extension**, where $K = F(\sqrt{D})$, and D is a squarefree element of F .

Theorem 1.2.4. *Let $F \subseteq K \subseteq L$. Then $[L : F] = [L : K][K : F]$.*

Proof. Let $[L : K] = m$ and $[K : F] = n$. Let $\{\alpha_1, \dots, \alpha_m\}$ and $\{\beta_1, \dots, \beta_n\}$ be bases for the extensions L/K and K/F . Now, the elements of L over K are of the form

$$a_1\alpha_1 + \dots + a_m\alpha_m \text{ where } a_i \in K \text{ for all } 1 \leq i \leq m$$

Since each $a_i \in K$, which is an extension over F , they have the form

$$a_i = b_{i1}\beta_1 + \dots + b_{in}\beta_n \text{ where } b_{ij} \in F \text{ for all } 1 \leq j \leq n$$

That is, every element of L , as a vector space over F are of the form

$$\sum b_{ij}\alpha_i\beta_j$$

So the set $\{\alpha_1\beta_1, \dots, \alpha_m\beta_n\}$ spans L . It remains to show that this set is linearly independent.

Suppose that

$$\sum b_{ij}\alpha_i\beta_j = 0$$

for some $b_{ij} \in F$. Since $\{\alpha_1, \dots, \alpha_m\}$ are linearly independent in L over K , we have that the coefficients $a_1 = \dots = a_n = 0$ which makes

$$a_i = b_{i1}\beta_1 + \dots + b_{in}\beta_n = 0$$

Now, since $\{\beta_1, \dots, \beta_n\}$ is linearly independent in K over F , this implies that $b_{i1} = \dots = b_{in} = 0$ which makes the collection $\{\alpha_1\beta_1, \dots, \alpha_m\beta_n\}$ linearly independent, and hence, a basis. Moreover, notice that this basis has size mn . ■

Example 1.7. (1) The element $\sqrt{2} \notin \mathbb{Q}(\alpha)$, where α is the root of $x^3 - 3x - 1$; since $[\mathbb{Q}(\sqrt{2}) : \mathbb{Q}] = 2$, and $[\mathbb{Q}(\alpha) : \mathbb{Q}] = 3$.

(2) We have $[\mathbb{Q}(\sqrt[6]{2}) : \mathbb{Q}] = 6$, and since $(\sqrt[6]{2})^3 = \sqrt{2}$, we observe that $\mathbb{Q}(\sqrt{2}) \subseteq \mathbb{Q}(\sqrt[6]{2})$. Moreover, notice that by theorem 1.2.4 $[\mathbb{Q}(\sqrt[6]{2}) : \mathbb{Q}(\sqrt{2})] = 3$. Then we have the following tower of fields for

$$\begin{array}{c} \mathbb{Q} \subseteq \mathbb{Q}(\sqrt{2}) \subseteq \mathbb{Q}(\sqrt[6]{2}) \\ \mathbb{Q}(\sqrt[6]{2}) \\ \mid \\ \mathbb{Q}(\sqrt{2}) \\ \mid \\ \mathbb{Q} \end{array}$$

Lemma 1.2.5. *Let α, β be algebraic over a field F . Then $F(\alpha, \beta) = (F(\alpha))(\beta)$.*

Proof. By definition, $F(\alpha, \beta)$ contains F , and α , and hence contains $F(\alpha)$. It also contains β so that $(F(\alpha))(\beta) \subseteq F(\alpha, \beta)$. By the same argument, $(F(\alpha))(\beta)$ contains F , α and β so that $F(\alpha, \beta) \subseteq (F(\alpha))(\beta)$. ■

Corollary. The elements of $F(\alpha, \beta)$ are of the form $\sum a_{ij}\alpha^i\beta^j$, where $1 \leq i \leq \deg \alpha$ and $1 \leq j \leq \deg \beta$.

Example 1.8. Consider $\mathbb{Q}(\sqrt{2}, \sqrt{3})$ generated by $\sqrt{2}$ and $\sqrt{3}$. Notice that $\deg \sqrt{3} = 2$ over \mathbb{Q} so that $[\mathbb{Q}(\sqrt{2}, \sqrt{3}) : \mathbb{Q}(\sqrt{2})] \leq 2$. Now $[\mathbb{Q}(\sqrt{2}, \sqrt{3}) : \mathbb{Q}(\sqrt{2})] = 2$ if, and only if the polynomial $x^2 - 3$ is irreducible over $\mathbb{Q}(\sqrt{2})$. Then it is irreducible if, and only if $\sqrt{2} \in \mathbb{Q}(\sqrt{3})$. It can be shown that this is not the case by trying to find $a, b \in \mathbb{Q}$ for which $\sqrt{3} = a + b\sqrt{2}$. Moreover we have

$$[\mathbb{Q}(\sqrt{2}, \sqrt{3}) : \mathbb{Q}] = 4$$

Theorem 1.2.6. An extension field K/F is finite if, and only if it is generated by finitely many algebraic elements over F .

Proof. Let K/F finite of degree n , and $\{\alpha_1, \dots, \alpha_n\}$ a basis. Then by theorem 1.2.4, $[F(\alpha_i) : F] \mid [K : F]$ for all $1 \leq i \leq n$. So each α_i is algebraic over F . Then K is generated by finitely many algebraic elements.

Conversely, let $K = F(\alpha_1, \dots, \alpha_k) = (F(\alpha_1, \dots, \alpha_{k-1}))(\alpha_k)$. We obtain K by taking the extensions F_{i+1}/F_i iteratively, where $F_{i+1} = F_i(\alpha_{i+1})$, and obtain the sequence

$$F = F_0 \subseteq \dots \subseteq F_k = K$$

Now, if the elements $\alpha_1, \dots, \alpha_k$ are algebraic over F , each of $\deg \alpha_i = n_i$ for $1 \leq i \leq k$, then the extension F_{i+1}/F_i is a simple extension, and $[F_{i+1} : F_i] = \deg m_{\alpha_{i+1}} \leq \deg \alpha_{i+1} = n_{i+1}$. Then we have

$$[K : F] = [F_k : F_{k-1}] \dots [F_1, F] \leq n_1 \dots n_k$$

which makes K/F a finite extension. ■

Corollary. If α, β are algebraic over F , then so are $\alpha \pm \beta$, $\alpha\beta$, and $\alpha\beta^{-1}$ (for $\beta \neq 0$).

Corollary. If L/F is an extension, then the collection of elements of L which are algebraic over F forms a subfield of L .

Example 1.9. (1) Consider the extension \mathbb{C}/\mathbb{Q} , and let $\text{cl } \mathbb{Q}$ the subfield of all elements of \mathbb{C} which are algebraic over \mathbb{Q} . Then $\sqrt[n]{2} \in \text{cl } \mathbb{Q}$ for all $n \geq 1$, so that $[\text{cl } \mathbb{Q} : \mathbb{Q}] \geq n$. This makes $\text{cl } \mathbb{Q}$ an infinite algebraic extension, and we call $\text{cl } \mathbb{Q}$ the **field of algebraic numbers**.

(2) Consider $\text{cl } \mathbb{Q} \cap \mathbb{R}$ as a subfield of \mathbb{R} (i.e. the subfield of all algebraic elements of \mathbb{Q}). Since \mathbb{Q} is countable, so is the field $\mathbb{Q}[x]$, and each polynomial in $\mathbb{Q}[x]$ has at most n roots in \mathbb{R} , hence the number of all algebraic elements of \mathbb{R} over \mathbb{Q} is also countable. This means that $\text{cl } \mathbb{Q}$ must also be countable. Now, since \mathbb{R} is uncountable, then there exist uncountably transcendental numbers of \mathbb{R} over \mathbb{Q} . Most notably the irrational numbers π and e are transcendental.

Theorem 1.2.7. If K is algebraic over F , and L algebraic over K , then L is algebraic over F .

Proof. Let $\alpha \in L$, since L is algebraic over K , there exists a $p \in K[x]$ having α as root. Let $p(x) = a_0 + a_1x + \cdots + a_nx^n$. Consider then $F(\alpha, a_0, \dots, a_n)$. Since K/F is algebraic, a_0, \dots, a_n are algebraic over F , and so $F(\alpha, a_0, \dots, a_n)$ is a finite extension over F . Then α generates an extension field of degree less than n , and we get

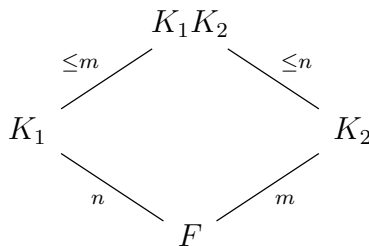
$$[F(\alpha, a_0, \dots, a_n) : F] = [F(\alpha, a_0, \dots, a_n) : F(a_0, \dots, a_n)][F(a_0, \dots, a_n) : F]$$

is finite, and $F(\alpha, a_0, \dots, a_n)$ is algebraic over F . That is, α is algebraic over F , and so L is algebraic over F . ■

Definition. Let K_1 and K_2 subfields of a field K . The **composite field** K_1K_2 is the smallest subfield of K containing both K_1 and K_2 .

Example 1.10. The composite field of $\mathbb{Q}(\sqrt[3]{2})$ and $\mathbb{Q}(\sqrt{2})$ is $\mathbb{Q}(\sqrt[6]{2})$.

Lemma 1.2.8. Let K_1 and K_2 be extensions of a field F contained in a field K . Then $[K_1K_2 : F] \leq [K_1 : F][K_2 : F]$ with equality holding if, and only if a basis of F in the other field is linearly independent. Moreover if $\{\alpha_1, \dots, \alpha_m\}$ and $\{\beta_1, \dots, \beta_n\}$ are bases for K_1 and K_2 , then $\{\alpha_1\beta_1, \dots, \alpha_m\beta_n\}$ span K_1K_2 .



Corollary. If $[K_1 : F] = m$, and $[K_2 : F] = n$ with m and n coprime, then $[K_1K_2 : F] = [K_1 : F][K_2 : F]$.

Proof. We have that $m, n | [K_1K_2 : F]$ and since $K_1, K_2 \subseteq K_1K_2$ are subfields of K_1K_2 , we get the least common multiple $[m, n] | [K_1K_2 : F]$. Now, since $(m, n) = 1$, we get $[m, n] = mn$ so that $mn \leq [K_1K_2 : F]$. ■

1.3 Splitting Fields

Definition. Let K be an extension of a field F . We say a polynomial f over F **splits completely** over K if f factors into linear factors over K . If f splits completely over K , and in no other proper subfield, then we say K is the **splitting field** of f over F .

Theorem 1.3.1. If f is a polynomial over a field F , then there exists a splitting field K of f over F .

Proof. Let E an extension of F with $[E : F] = n$. By induction on n , for $n = 1$, we take $E = F$ and we are done. Now, for $n \geq 1$, suppose the irreducible factors of f are of $\deg = 1$. Then f has all its roots in F , and hence splits completely over F . Then take $E = F$. On

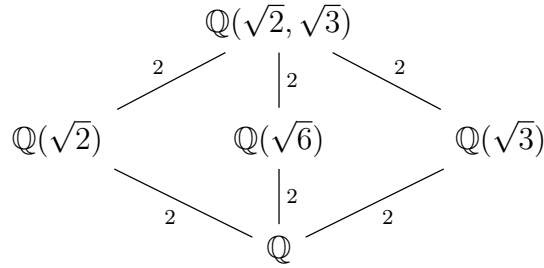
the other hand, if f has at least one irreducible factor of $\deg \geq 2$, then there is an extension E_1 of F for which f has the factor $(x - \alpha)$ for some root α . Then $f(x) = (x - \alpha)f_1(x)$ where $\deg f_1 = n - 1$. Therefore by the induction hypothesis, there is an extension E of E_1 containing all the roots of f_1 . Hence, it contains all the roots of f and f splits completely over E .

Now, let K be the intersection of all subfields of E for which f splits; i.e. all subfields containing the roots of f . Then by definition, K is the splitting field of f over F . ■

Definition. We call an extension K over a field F **normal**, if for any irreducible polynomial f over F with atleast one root in K , f splits completely in K over F . That is to say, K contains the splitting field of f over F .

Example 1.11. (1) The splitting field of $x^2 - 2$ over \mathbb{Q} is $\mathbb{Q}(\sqrt{2})$, since $x^2 - 2 = (x + \sqrt{2})(x - \sqrt{2})$ and $\pm\sqrt{2} \in \mathbb{Q}(\sqrt{2})$ and $[\mathbb{Q}(\sqrt{2}) : \mathbb{Q}] = 2$, so there is no other subfield in between.

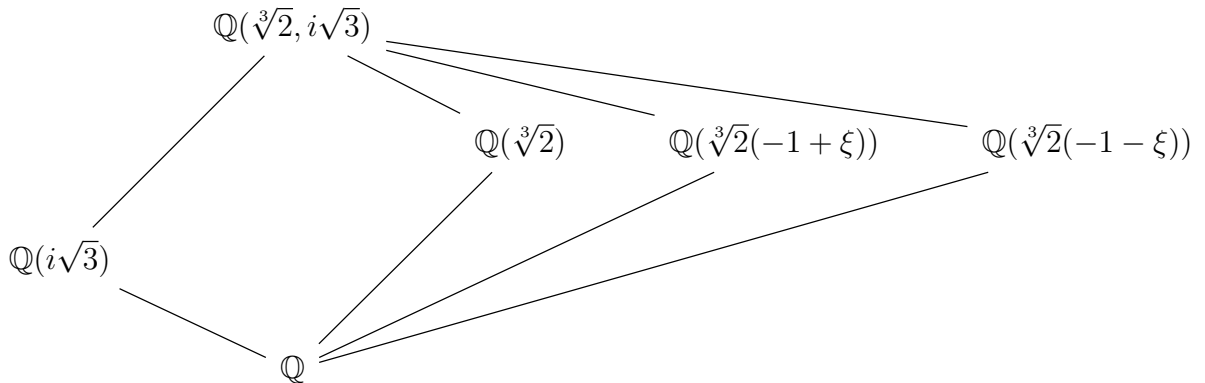
(2) The splitting field for $(x^2 - 2)(x^2 - 3) = (x + \sqrt{2})(x - \sqrt{2})(x + \sqrt{3})(x - \sqrt{3})$ is $\mathbb{Q}(\sqrt{2}, \sqrt{3})$. Now, $[\mathbb{Q}(\sqrt{2}, \sqrt{3}) : \mathbb{Q}] = 4$ and the lattice of fields is



(3) Let $\xi = i\sqrt[3]{2}$. Notice that $x^3 - 2$ factors into $x^3 - 2 = (x - \sqrt[3]{2})(x + \sqrt[3]{2}(-1 + \xi))(x + \sqrt[3]{2}(-1 - \xi))$. Now, $-1 + \xi, -1 - \xi \notin \mathbb{Q}(\sqrt[3]{2})$, so $\mathbb{Q}(\sqrt[3]{2})$ is not the splitting field for $x^3 - 2$. Let K be the splitting field of $x^3 - 2$. Then K contains $-1 \pm \xi$, so that $i\sqrt{3} \in K$. Thus

$$K = \mathbb{Q}(\sqrt[3]{2}, i\sqrt{3})$$

Moreover, $[\mathbb{Q}(\sqrt[3]{2}, i\sqrt{3}) : \mathbb{Q}(\sqrt[3]{2})] \geq 2$ and since $\mathbb{Q}(\sqrt[3]{2})$ is not the splitting field, $[\mathbb{Q}(\sqrt[3]{2}, i\sqrt{3}) : \mathbb{Q}(\sqrt[3]{2})] = 2$. Hence $[\mathbb{Q}(\sqrt[3]{2}, i\sqrt{3}) : \mathbb{Q}] = 6$. We have the following lattice.



- (4) Notice that $x^4 + 4 = (x^2 + 2x + 2)(x^2 - 2x + 2)$ over \mathbb{Q} which is irreducible by Eisenstein's criterion. Using the quadratic formula, we get ± 1 and $\pm i$ as the roots, moreover, notice that $\pm 1, \pm i \in \mathbb{Q}(i)$ and since $[\mathbb{Q}(i) : \mathbb{Q}] = 2$ there are no subfields between \mathbb{Q} and $\mathbb{Q}(i)$ so that $\mathbb{Q}(i)$ is the splitting field of $x^4 + 4$ over \mathbb{Q} .

Lemma 1.3.2. *A splitting field of a polynomial of degree n over a field F is of degree at most $n!$ over F .*

Proof. Let $f \in F[x]$ a polynomial of $\deg f = n$. Adjoining one root of f to F , we have an extension F_1/F of degree $[F_1 : F] = n$. Now, f over F_1 has at least one linear factor, and so any root of f satisfies a polynomial of degree $n - 1$. Hence proceeding inductively gives the result. ■

Example 1.12. Consider the polynomial $x^n - 1$ over \mathbb{Q} . Then the roots of $x^n - 1$ are of the form ξ where $\xi^n = 1$. Notice, that in \mathbb{C} , $\xi = e^{\frac{2i\pi}{n}}$, so that \mathbb{C} contains a splitting field of $x^n - 1$. Hence $\mathbb{Q}(\xi) \subseteq \mathbb{C}$ is a splitting field of $x^n - 1$ over \mathbb{Q} . Notice that the set of all roots ξ of $x^n - 1$ forms a cyclic group generated by ξ .

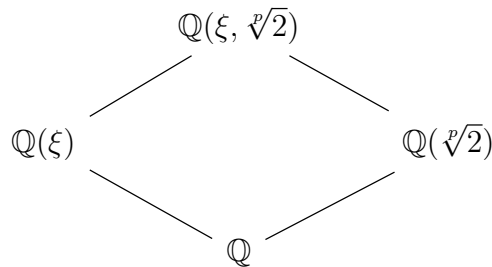
Definition. Consider a field F and the polynomial $x^n - 1$ over F . We call the roots ξ of $x^n - 1$, where $\xi^n = 1$ the **primitive n -th roots of unity** over F . We call $F(\xi)$ the **cyclotomic field** over F .

Example 1.13. Let p be a prime, and consider the splitting field $x^p - 2$ over \mathbb{Q} . If α is a root, then $\alpha^p = 2$ so that $(\xi\alpha)^p = 2$ where ξ is a primitive p -th root of unity over \mathbb{Q} . So the roots of $x^p - 2$ are

$$\sqrt[p]{2} \text{ and } \xi \sqrt[p]{2}$$

Notice that $\frac{\xi \sqrt[p]{2}}{\sqrt[p]{2}} = \xi$ so the splitting field contains $\mathbb{Q}(\xi, \sqrt[p]{2})$. Moreover, $\mathbb{Q}(\xi, \sqrt[p]{2})$ contains all the roots of $x^p - 2$ so that $\mathbb{Q}(\xi, \sqrt[p]{2})$ is the splitting field of $x^p - 2$ over \mathbb{Q} .

Notice, that $\mathbb{Q}(\xi) \subseteq \mathbb{Q}(\xi, \sqrt[p]{2})$ so that $[\mathbb{Q}(\xi, \sqrt[p]{2}) : \mathbb{Q}(\xi)] \leq p$. not, since $\mathbb{Q}(\sqrt[p]{2})$ is also a subfield, we get $[\mathbb{Q}(\xi, \sqrt[p]{2}) : \mathbb{Q}] \leq p(p - 1)$. Since $(p, p - 1) = 1$ (i.e. they are coprime), we have $p(p - 1) | [\mathbb{Q}(\xi, \sqrt[p]{2}) : \mathbb{Q}]$ so that $[p]2 : \mathbb{Q}] = p(p - 1)$. We have the following lattice.



Theorem 1.3.3. *Let $\phi : F \rightarrow F'$ a field isomorphism. Let f and f' polynomials over F and F' , where f' is obtained by applying ϕ to the coefficients of f . Let E and E' be splitting fields of f and f' over F and F' , respectively. Then ϕ extends to an isomorphism between E and E' ; i.e. $E \simeq E'$.*

$$\begin{array}{ccc}
 E & \longrightarrow & E' \\
 \downarrow & & \downarrow \\
 F & \xrightarrow{\phi} & F'
 \end{array}$$

Proof. Let $\deg f = n$. By induction on n . If f has all its roots in F , f splits completely over F , and f' over F' . Then take $E = F$ and $E' = F'$ and we are done for $n = 1$.

Now, for $n \geq 1$, suppose the theorem is true. Let p an irreducible factor of f , and p' an irreducible factor of f' . If α and α' are roots of p and p' , respectively, then extend ϕ to $F(\alpha)$ and $F'(\alpha')$. Then $f(x) = (x - \alpha)f_1(x)$ and $f'(x) = (x - \alpha')f'_1(x)$; with $\deg f_1 = \deg f'_1 = n - 1$. Then let E the splitting field of f_1 over $F(\alpha)$, and E' the splitting field of f'_1 over $F'(\alpha')$

$$\begin{array}{ccc} E & \longrightarrow & E' \\ | & & | \\ F(\alpha) & \longrightarrow & F'(\alpha') \\ | & & | \\ F & \xrightarrow{\phi} & F' \end{array}$$

The roots of f_1 and f'_1 are in E and E' , respectively, and hence so are the roots of f and f' . Then by the induction hypothesis, we can extend ϕ to E and E' so that $E \simeq E'$. ■

Corollary. *Any two splitting fields of a given polynomial over a field are isomorphic.*

Proof. Take ϕ to be the identity map. ■

1.4 Algebraic Closures.

Definition. We define the **algebraic closure** of a field F to be the algebraic extension, $\text{cl } F$, over F for which every polynomial over F splits. We call a field K **algebraically closed** if every polynomial over K has at least one root in K .

Lemma 1.4.1. *A field K is algebraically closed if, and only if every polynomial over K has all of its roots in K .*

Proof. Certainly, if a polynomial f over K contains all of its roots in K , then K is algebraically closed, by definition.

Now, suppose that K is algebraically closed, and let f a polynomial over K . Then f contains at least one root in K . Hence $f(x) = (x - \alpha)f_1(x)$ for some root α of f , and where $f_1 \in K[x]$. But then by definition again, f_1 contains at least one root in K . Hence, we proceed until we exhaust all the roots of f , and obtain that every root of f lies in K . ■

Corollary. *K is algebraically closed if, and only if $\text{cl } K = K$.*

Lemma 1.4.2. *Let F be a field, and $\text{cl } F$ its algebraic closure. Then $\text{cl } F$ is algebraically closed; i.e. $\text{cl}(\text{cl } F) = \text{cl } F$.*

Proof. Let $f \in \text{cl } F[x]$, and α a root of f . Then α generates all of $\text{cl } F(\alpha)$, making $\text{cl } F$ algebraic over F . Hence α is algebraic over F , but $\alpha \in \text{cl } F$, so that $\text{cl}(\text{cl } F) = \text{cl } F$. ■

Lemma 1.4.3. *For every field F , there exists an algebraically closed set containing F .*

Proof. Consider the polynomial ring $F[\dots, x_n, \dots]$ where $f(x_n)$ is a nonconstant polynomial over F . Consider the ideal (f) . Then, if $(f) = (1)$, then

$$g_1 f_1(x_1) + \dots + g_n f_n(x_n) = 1$$

where $g_i \in F[x_i]$. Then we get

$$g_1(x_1, \dots, x_m) f_1(x_1) + \dots + g_n(x_1, \dots, x_m) f_n(x_n) = 1$$

Now, let F' an extension of F containing a root α_i of f_i . Then we observe that $0 = 1$ in the above equation which is a blatant contradiction. So (f) must be a proper ideal.

Now, by Zorn's lemma, there exists a maximal ideal M containing I . Then the quotient

$$K_1 = F[\dots, x_n, \dots] / M$$

is a field containing an imbedding of F . Moreover, f has a root in K_1 , so that $f(x_n) \in (f) \subseteq M$. Then K_1 is a field in which every polynomial over F has a root. Proceeding as before with K_1 , we obtain K_2 in which every polynomial over K_1 has a root. Hence, proceeding recursively, we obtain the sequence

$$F = K_0 \subseteq K_1 \subseteq K_2 \subseteq \dots$$

in which every polynomial over K_n has all its roots in K_{n+1} . Now, let

$$K = \bigcup K_n$$

Then $F \subseteq K$, and every polynomial over K has a root in K_N , for N large enough; but $K_N \subseteq K$, so that K is algebraically closed. ■

Lemma 1.4.4. *Let K be algebraically closed, and let $F \subseteq K$. Then the collection of elements of the algebraic closure $\text{cl } F$ of F that are algebraic over F is an algebraic closure of F .*

Proof. By definition, $\text{cl } F / F$ is algebraic. Then every polynomial f over F splits over K into linear factors $(x - \alpha)$, where α is a root of f . So α is algebraic over F , and hence $\alpha \in \text{cl } F$. then all linear factors have a coefficient in $\text{cl } F$, so that f splits completely over $\text{cl } F$. ■

Corollary. *Algebraic closures are unique up to isomorphism.*

Theorem 1.4.5 (The Fundamental Theorem of Algebra). *\mathbb{C} is algebraically closed.*

Corollary. *\mathbb{C} contains the an algebraic closure of any of its subfields.*

1.5 Seperability.

Definition. Let f be a polynomial over a field F with factorization

$$f(x) = a_n(x - \alpha_1)^{n_1} \dots (x - \alpha_k)^{n_k}$$

where $\alpha_1, \dots, \alpha_k$ are roots of f , and a_n is the leading coefficient of f . If $n_i > 1$, we call α_i a **multiple root** of f , and if $n_i = 1$, we call α_i a **simple root**. We call n_i the **multiplicity** of α_i .

Definition. A polynomial over a field F is said to be **seperable** if it has only simple roots. Otherwise, we say it is **inseperable**.

Lemma 1.5.1. *Seperable polynomials have all their roots distinct.*

Definition. We say a field F is a **finite field** if it has a finite number of elements. If $|F| = n$, then we denote F as \mathbb{F}_n .

Lemma 1.5.2. *Every finite field has finite characteristic. Moreover, that characteristic is a prime integer.*

Proof. Recall that the characteristic is just the additive order of the element 1 in the field. Lemma 1.1.1 reiterates that any field of nonzero characteristic must have prime characteristic. ■

Example 1.14. (1) $x^2 - 2 = (x + \sqrt{2})(x - \sqrt{2})$ is seperable over \mathbb{Q} . However $(x^2 - 1)^n$ is inseperable.

(2) Consider $x^2 - t$ over the field $\mathbb{F}_2(t)$ of rational functions over t . $x^2 - t$ is irreducible, but inseperable. Let \sqrt{t} a root, then $(x - \sqrt{t})^2 = x^2 - t$ since $\text{char } \mathbb{F}_2 = 2$.

Definition. The **derivative** of a polynomial $f(x) = a_0 + a_1x + \cdots + a_nx^n$ over a field F is the polynomial

$$Df(x) = a_1 + 2a_2x + \cdots + na_nx^{n-1}$$

over F .

Lemma 1.5.3. *For any two polynomials f and g over a field, the following are true.*

$$(1) D(f + g) = Df + D(g).$$

$$(2) D(fg) = fDg + gDf.$$

Lemma 1.5.4. *A polynomial f has a multiple root α if, and only if α is a root of Df . Moreover, the minimal polynomial of α , m_α divides (f, Df) .*

Proof. Let α a multiple root of f . Then $f(x) = (x - \alpha)^n g(x)$ for some polynomial g . Hence

$$Df(x) = n(x - \alpha)^{n-1}g(x) + (x - \alpha)^n Dg(x)$$

so that α is a root of Df .

Conversly, suppose that α is a root of both f and Df . Then $f(x) = (x - \alpha)g(x)$ for some polynomial g , and $Df(x) = g(x) + (x - \alpha)Dg(x)$. Now, since $Df(\alpha) = 0$, we get $h(\alpha) = 0$, so that h has a linear factor $(x - \alpha)$. This makes α a multiple root of f . ■

Corollary. *f is seperable if and only if $(f, Df) = 1$.*

Corollary. *Every irreducible polynomial in a field F of $\text{char } F = 0$ is seperable. Moreover, a polynomial over such a field is irreducible if, and only if it is the product of distinct irreducible factors.*

Proof. Let p an irreducible polynomial over F of $\deg p = n$. Then $\deg Dp = n - 1$. Up to constant factors, the factors of p are 1 and itself, so that $(p, Dp) = 1$. This makes p seperable. Therefore every irreducible polynomial over F is seperable, and the rest follows. ■

Example 1.15. (1) Let p prime and $f(x) = x^{p^n} - x$ over the finite field \mathbb{F}_p , of $\text{char } \mathbb{F}_p = p$. Then $Df(x) = p^n x^{p^n-1} - 1 \equiv -1 \pmod{p}$. Then Df has no roots, which makes f seperable.

(2) $D(x^n - 1) = nx^{n-1}$ for any field of char coprime to p . Then $D(x^n - 1)$ has a root 0 of multiplicity $n > 1$, but 0 is not a root of $x^n - 1$ so that $x^n - 1$ is seperable. That is, $x^n - 1$ has n distinct roots of unity ξ .

(3) Let F a field of char $F = p$, where $p|n$. Then there are fewew than n distinct n -th roots of unity over F , since $n \equiv 0 \pmod{p}$. Then $D(x^n - 1) = 0$, and every root of $x^n - 1$ is a multiple root.

Lemma 1.5.5. *If f is a polynomial over a field F whose derivative is 0, then there exist a polynomial g for which $f(x) = g(x^p)$ where $\text{char } F = p$.*

Proof. Let $f(x) = a_0 + a_1x + \cdots + a_nx^n$. Then $Df(x) = a_1 + \cdots + na_nx^{n-1} = 0$, so that every exponent $i \equiv 0 \pmod{p}$. That is, $f(x) = a_0 + a_1x^p + \cdots + a_mx^{mp}$. Then let

$$g(x) = a_0 + a_1x + \cdots + a_mx^m$$

then $f(x) = g(x^p)$. ■

Lemma 1.5.6. *Let F a field of char $F = p$. The for every $a, b \in F$, $(a + b)^p = a^p + b^p$ and $(ab)^p = a^p b^p$.*

Proof. The binomial theorem gives

$$(a + b)^p = a^p + \binom{p}{1}a^{p-1}b + \cdots + \binom{p}{p-1}ab^{p-1} + b^p$$

Now, since $\binom{p}{i} \in \mathbb{Z}$ for any $1 \leq i \leq p - 1$, and p is prime (the charateristic of a field has to be prime), then $p|\binom{p}{i}$. Hence $\binom{p}{i} \equiv 0 \pmod{p}$, so that the binomial exapnsion above reduces to

$$(a + b)^p \equiv a^p + b^p \pmod{p}$$

Now, let $\phi : a \rightarrow a^p$, then ϕ is an automorphism of fields taking $(ab)^p = a^p b^p$. ■

Corollary. *Let F be a finite field of char $F = p$. Then every element of F is a p^{th} power in F .*

Definition. Let F be a field. We call the automorphism $F \rightarrow F$ defined by $a \rightarrow a^p$ where $p \in \mathbb{Z}$ the **Forbenius automorphism**.

Lemma 1.5.7. *Every irreducible polynomial in a finite field F is seperable.*

Proof. Suppose otherwise. Since F has finite characteristic, there is a polynomial q over F for which $p(x) = q(x^l)$, where p is the irreducible polynomial in question, and $\text{char } F = l$. Let

$$q(x) = a_0 + a_1x + \cdots + a_nx^n$$

then $a_i = b_i^p$ for some $b_i \in F$, and

$$\begin{aligned} p(x) &= q(x^l) \\ &= a_0 + a_1x^p + \cdots + a_nx^{pn} \\ &= b_0^p + b_1^p x^p + \cdots + b_n^p x^{np} \\ &= (b_0 + b_1x + \cdots + b_nx^n)^p \end{aligned}$$

which is a contradiction. ■

Definition. A field K of characteristic $\text{char } K = p$ is called **perfect** if for every $a \in K$, there exists a $b \in K$ for which $a = b^p$, or $p = 0$.

Example 1.16. Let $n > 0$ and consider the splitting field of the polynomial $x^{p^n} - x$ over the finite field \mathbb{F}_p . Then $x^{p^n} - x$ has precisely p^n roots.

Let α, β be roots. Then $\alpha^{p^n} = \alpha$, and $\beta^{p^n} = \beta$. Then $(\alpha\beta)^{p^n} = \alpha\beta$ and $(\alpha^{-1})^{p^n} = \alpha^{-1}$. Moreover, $(\alpha + \beta)^{p^n} = \alpha + \beta$. So the set of p^n distinct roots of $x^{p^n} - x$ is closed under addition, multiplication, and inverses in its splitting field. Let F be that splitting field. Notice that $F \subseteq \mathbb{F}_{p^n}$, moreover, $[F : \mathbb{F}_p] = n$ so that $|F| = p^n$. We also have that $\mathcal{U}(F)$ is a cyclic group of order $p^n - 1$, so that $F_{p^n} \subseteq F$, since $\alpha^{p^n-1} = 1$. Therefore \mathbb{F}_{p^n} is the splitting field of $x^{p^n} - x$ over \mathbb{F}_p , and so contains all the roots of $x^{p^n} - x$. Hence finite fields of order p^n exist and are unique up to isomorphism.

Lemma 1.5.8. *Let f an irreducible polynomial over a field F of $\text{char } F = p$. Then there exists a unique integer $k \geq 0$ and a unique separable polynomial s such that $f(x) = s(x^{p^k})$.*

Proof. We have that since $\text{char } F = p$, there exists a polynomial f_1 over F for which $f(x) = f_1(x^p)$. Now, if f_1 is separable, take $k = 1$ and we are done. Otherwise, there is a polynomial f_2 over F for which $f_2(x) = f_2(x^p)$, so that $f(x) = f_1(x^p) = f_2(x^{p^2})$. Then proceeding in this fashion, we obtain a separable polynomial s for which $f(x) = s(x^{p^k})$ where $k \geq 0$. ■

Definition. Let f an irreducible polynomial over a field of characteristic p , a prime. Let f_s the polynomial for which $f(x) = f_s(x^{p^k})$ for some unique integer $k \geq 0$. Then we call the degree of f_s the **separable degree** of f and write $\deg_s f = \deg f_s$. We call the integer p^k the **inseparable degree** and write $\deg_i f = p^k$. We call f_s the **separable part** of f .

Lemma 1.5.9. *A polynomial f is separable if, and only if $\deg_i f = 1$ and $\deg_s f = \deg f$. Moreover,*

$$\deg f = \deg_s f \cdot \deg_i f$$

Example 1.17. (1) $x^p - t$ over $\mathbb{F}_p(t)$ is irreducible with derivative $D = 0$. Hence $x^p - t$ is inseparable. We call $x^p - t$ a **purely inseparable polynomial**. Notice that $x^p - t$ has separable part $(x - t)$.

- (2) $x^{p^n} - t$ over $\mathbb{F}_p(t)$ is irreducible with separable part $(x - t)$, and $\deg_i = p^n$.
- (3) Let $f(x) = (x^{p^n} - t)(x^p - t)$ over $\mathbb{F}_p(t)$. Then p has two inseparable irreducible factors, and so is inseparable.

Definition. If K is an extension over a field F , we call an element $\alpha \in K$ **separable** if its minimal polynomial is separable, otherwise we call it **inseparable**. We call K/F **separable** if every $\alpha \in K$ is separable; otherwise we say that K/F is inseparable.

Lemma 1.5.10. *Every finite extension of a perfect field is separable.*

Corollary. *Finite extension fields of \mathbb{Q} and \mathbb{F}_p are separable.*

1.6 Cyclotomic Polynomials.

Definition. We define **Euler's totient** to be the map $\phi : \mathbb{Z} \rightarrow \mathbb{Z}$ defined by the rule $\phi(n) = |\{a \in \mathbb{Z} : (a, n) = 1\}|$. That is, ϕ of n is the number of all integers less than n , coprime to n .

Definition. We define Ξ_n to be the **group of all primitive n -th roots of unity**, ξ for which $\xi^n = 1$.

Lemma 1.6.1. $\Xi_n \simeq \mathbb{Z}/n\mathbb{Z}$.

Proof. The map $a \rightarrow \xi^a$ defines the required isomorphism. ■

Corollary. $\text{ord } \Xi_n = \phi(n)$ where ϕ is Euler's totient.

Proof. Since $\xi^n \equiv \xi^{0 \bmod n} \equiv 1$, we have every non identity power of ξ has exponent coprime to n . That is there are $\phi(n)$ such distinct powers of ξ . ■

Corollary. If $d|n$, then $\Xi_d \leq \Xi_n$.

Proof. Notice that if $d|n$, then $d = mn$ for some $m \in \mathbb{Z}^+$. Then $\xi^d = 1$ implies $(\xi^d)^m = \xi^{dm} = \xi^n = 1$. ■

Definition. We define the **n -th cyclotomic polynomial** to be the polynomial

$$\Phi_n(x) = \prod (x - \xi)$$

having as roots all n -primitive roots of unity.

Lemma 1.6.2. *The n -th cyclotomic polynomial Φ_n has degree $\deg \Phi_n = \phi(n)$, where ϕ is Euler's totient.*

Proof. Recall that $\text{ord } \Xi_n = \phi(n)$, and since the elements of Ξ_n are the roots of Φ_n , there are $\phi(n)$ such roots. This puts $\deg \Phi_n = \phi(n)$. ■

Example 1.18 (Computing Cyclotomic Polynomials). Recall that the polynomial $x^n - 1$ has as roots precisely all n -th roots of unity ξ , that is $\xi^n = 1$. If $x^n - 1 \in F[x]$, F a field, the the splitting field of $x^n - 1$ is $F(\xi)$. Then we have

$$x^n - 1 = \prod_{\xi \in \Xi_n} (x - \xi)$$

Now, grouping those factors where $\xi^d = 1$ for some $d|n$, then we have

$$x^n - 1 = \prod_{\xi \in \Xi_d} (x - \xi) \prod_{\xi \in \Xi_n} (x - \xi) = \prod_{d|n} d \prod_{\xi \in \Xi_n} (x - \xi) = \prod_{d|n} \Phi_n(x)$$

that is,

$$x^n - 1 = \prod_{d|n} \Phi_d(x)$$

which gives a method for computing Φ_n recursively.

We have $\Phi_1(x) = x - 1$ and $\Phi_2(x) = x + 1$. Now, $\Phi_3(x) = \Phi_1(x)\Phi_3(x) = (x - 1)\Phi_3(x)$, so that

$$\Phi_3(x) = x^2 + x + 1$$

We have $\Phi_4(x) = \Phi_1(x)\Phi_2(x)\Phi_4(x) = (x - 1)(x + 1)\Phi_4(x) = (x^2 - 1)\Phi_4(x)$. So

$$\Phi_4(x) = x^2 + 1$$

Similarly,

$$\begin{aligned} \Phi_5(x) &= x^4 + x^3 + x^2 + x + 1 \\ \Phi_6(x) &= x^2 - x + 1 \\ \Phi_7(x) &= x^6 + x^5 + x^4 + x^3 + x^2 + x + 1 \\ \Phi_8(x) &= x^4 + 1 \\ \Phi_9(x) &= x^6 + x^3 + 1 \\ \Phi_{10}(x) &= x^4 - x^3 + x^2 - x + 1 \\ \Phi_{11}(x) &= x^{10} + x^9 + x^8 + x^7 + x^6 + x^5 + x^4 + x^3 + x^2 + x + 1 \\ \Phi_{12}(x) &= x^4 - x^2 + 1 \end{aligned}$$

Also observe that if p is prime, then

$$\Phi_p(x) = \sum_{i=0}^{p-1} x^i = x^{p-1} + x^{p-2} + \cdots + x + 1$$

Lemma 1.6.3. $\Phi_n(x)$ is monic over \mathbb{Z} .

Proof. Notice that since $x^n - 1 = \prod_{d|n} \Phi_d(x)$, is monic, then each Φ_d must also be monic for all $d|n$.

Now, by induction on n , for $n = 1$, it is clear that $x - 1$ has coefficients in \mathbb{Z} (if $x^n - 1 \in \mathbb{Z}[x]$ we are done, if not, just take $1_F \rightarrow 1_{\mathbb{Z}}$, where F is the underlying field of $x^n - 1$). Now, suppose that $\Phi_d(x) \in \mathbb{Z}[x]$ for all $1 \leq d < n$, and $d|n$. Then $x^n - 1 = f(x)\Phi_n(x)$, where $f(x) = \prod \Phi_d(x)$ is monic over \mathbb{Z} . Moreover, $f|x^n - 1$, in the splitting field $\mathbb{Q}(\xi)$ (since we take $1_F \rightarrow F_{\mathbb{Z}}$, where $\xi^n = 1$). Then $f|x^n - 1$ over \mathbb{Q} by the division theorem, and by Gauss' lemma, $f|x^n - 1$ in \mathbb{Z} . So $\Phi_n \in \mathbb{Z}[x]$. ■

Theorem 1.6.4. Φ_n is irreducible over \mathbb{Z} .

Proof. Again, if $x^n - 1 \in F[x]$ for some field F , take $1_F \rightarrow 1_{\mathbb{Z}}$ so that $x^n - 1 \in \mathbb{Z}[x]$. Suppose then that $\Phi_n(x) = f(x)g(x)$ where f and g are monic, and f is irreducible. Let $\xi^n = 1$, a primitive n -th root, so that ξ is a root of f . Then f is the minimal polynomial for ξ over \mathbb{Q} . Now, let p a prime such that $p \nmid n$. Then ξ^p is a n -th root, of f or g . If $f(\xi^p) = 0$, then for all a with $(a, n) = 1$, we have ξ^a is a root of f . Moreover, $a = p_1 \dots p_k$ where each $p_i \nmid n$ is prime. That means that $\xi^{p_1}, (\xi^{p_1})^{p_2}, \dots, \xi^n$ are all roots of f making $f = \Phi_n$ and we are done.

Suppose then that $g(\xi^p) = 0$. Then ξ is root of $g(x^p)$, and since f is minimal, $f|g(x^p)$ in $\mathbb{Z}[x]$. Then we have $g(x^p) = f(x)h(x)$ for $f, h \in \mathbb{Z}[x]$. reducing mod p , we get $g(x^p) \equiv f(x)h(x) \pmod{p}$ in $\mathbb{F}_p[x]$; but $g(x^p) \equiv (g(x))^p \pmod{p}$. Since $\mathbb{F}_p[x]$ is a unique factorization domain, we get that $f \pmod{p}$ and $g \pmod{p}$ have a common factor. Then $\Phi_n(x) \equiv f(x)g(x) \pmod{p}$ has a multiple root in $\mathbb{F}_p[x]$; implying that $x^n - 1$ has a multiple root, which is impossible; since $x^n - 1$ has n distinct roots. Therefore ξ^p is a root of f . ■

Corollary. $[\mathbb{Q}(\xi) : \mathbb{Q}] = \phi(n)$.

Proof. We have by above that Φ_n is the minimal polynomial for ξ over \mathbb{Q} . ■

Example 1.19. Let $\xi^8 = 1$ an 8-th root of unity. Then $[\mathbb{Q}(\xi) : \mathbb{Q}] = 4$ and $\mathbb{Q}(\xi)$ has minimal polynomial $\Phi_8(x) = x^4 + 1$. Moreover, $\mathbb{Q}(\xi)$ contains a primitive 4-th root of unity $i^4 = 1$ (over \mathbb{C} , $i^2 = -1$). So that $\mathbb{Q}(i) \subseteq \mathbb{Q}(\xi)$. We also get that $\xi + \xi^7 = \sqrt{2}$ (since $\xi = e^{\frac{2i\pi}{8}}$ over \mathbb{C}), and $\mathbb{Q}(\xi) = \mathbb{Q}(i, \sqrt{2})$.

Chapter 2

Galois Theory

2.1 Definitions and Examples.

Definition. An isomorphism of a field K onto itself is called an **automorphism**. We denote the set of all automorphisms of K $\text{Aut } K$, and for $\sigma \in \text{Aut } K$, we write $\sigma\alpha$ to mean $\sigma(\alpha)$. We say an automorphism σ of K **fixes** an element $\alpha \in K$ if $\sigma\alpha = \alpha$. We say σ **fixes** a subset $F \subseteq K$ if $\sigma\alpha = \alpha$ for all $\alpha \in F$. We denote $\text{Aut } K/F$ to be the set of all automorphisms of K that fix F , where K/F is a field extension.

Lemma 2.1.1. *Let K be a field. Then $\text{Aut } K$ is a group. Moreover, if K is an extension of a field F , then $\text{Aut } K/F \leq \text{Aut } K$.*

Lemma 2.1.2. *Let K be an extension of F , and let $\alpha \in K$ algebraic over F . Then for every $\sigma \in \text{Aut } K/F$, $\sigma\alpha$ is a root of the minimal polynomial of α over F ; that is, $\text{Aut } K/F$ permutes the roots of irreducible polynomials.*

Proof. Suppose that $a_0 + a_1\alpha + \cdots + a_{n-1}\alpha^{n-1} + \alpha^n = 0$, where $a_i \in F$ for all $1 \leq i \leq n$, and $a_n = 1$. Notice that if σ is an automorphism of K , then it is a homomorphism, moreover, since σ fixes F , and $a_i \in F$, we get $\sigma(a_i\alpha^i) = a_i\sigma\alpha^i$. Therefore,

$$\begin{aligned}\sigma(a_0 + a_1\alpha + \cdots + a_{n-1}\alpha^{n-1} + \alpha^n) &= \sigma 0 = 0 \\ \sigma(a_0) + \sigma(a_1\alpha) + \cdots + \sigma(a_{n-1}\alpha^{n-1}) + \sigma(\alpha^n) &= 0 \\ a_0 + a_1\sigma\alpha + \cdots + a_{n-1}\sigma\alpha^{n-1} + \sigma\alpha^n &= 0\end{aligned}$$

which makes $\sigma\alpha$ a root. ■

Example 2.1. (1) The identity map is an automorphism called the **trivial automorphism**, and just maps elements of a field onto themselves. We denote this automorphism by ι . Notice additionally, that if σ is an automorphism of a field, then $\sigma : 1 \rightarrow 1$ and $\sigma : 0 \rightarrow 0$, so that $\sigma a = a$ for any element a in the prime subfield. That is, the automorphism group of a field fixes its prime subfield. In particular, notice that \mathbb{Q} and \mathbb{F}_p have only the trivial automorphism, so that $\text{Aut } \mathbb{Q} = \langle \iota \rangle$ and $\text{Aut } \mathbb{F}_p = \langle \iota \rangle$.

- (2) If $\tau \in \text{Aut } \mathbb{Q}(\sqrt{2}) = \text{Aut } \mathbb{Q}(\sqrt{2})/\mathbb{Q}$, then $\tau\sqrt{2} = \pm\sqrt{2}$. Then τ fixes \mathbb{Q} , and we have that it sends elements $\tau : a + b\sqrt{2} \rightarrow a \pm b\sqrt{2}$. In the case of addition, we have that $\tau = \iota$ the identity. The latter case of subtraction gives $\tau : a + b\sqrt{2} \rightarrow a - b\sqrt{2}$, so that $\text{Aut } \mathbb{Q}(\sqrt{2}) = \langle \tau \rangle$ a cyclic group of order 2 generated by τ .

Lemma 2.1.3. *Let $H \leq \text{Aut } K$ for some field K . Then the collection F of elements of K fixed by H is a subfield of K .*

Proof. Let $h \in H$, and $a, b \in F$. Then $ha = a$, $hb = b$, so that $h(a \pm b) = a \pm b$, and $h(ab) = ab$, and $h(a^{-1}) = (ha)^{-1} = a^{-1}$. ■

Definition. Let K be a field. If $H \leq \text{Aut } K$, we define the **fixed field** of H to be the subfield of K fixed by H , and we denote it $\mathcal{F}(H)$.

Lemma 2.1.4. *If $F_1 \subseteq F_2 \subseteq K$ are subfields of a field K , then $\text{Aut } K/F_2 \subseteq \text{Aut } K/F_1$. Moreover, if $H_1 \leq H_2 \leq \text{Aut } K$, then $\mathcal{F}(H_2) \subseteq \mathcal{F}(H_1)$.*

Example 2.2. (1) The fixed field of $\text{Aut } \mathbb{Q}(\sqrt{2})$ is the field

$$F = \{a + b\sqrt{2} \in \mathbb{Q}(\sqrt{2}) : \sigma(a + b\sqrt{2}) = a + b\sqrt{2}\}$$

by definition. Then $a - b\sqrt{2} = a + b\sqrt{2}$ so that $b = 0$. Therefore $F = \mathbb{Q}$ and \mathbb{Q} is the fixed field.

- (2) Consider $\text{Aut } \mathbb{Q}(\sqrt[3]{2}) = \langle \iota \rangle$. Then the fixed field of $\text{Aut } \mathbb{Q}(\sqrt[3]{2})/\mathbb{Q}$ is $\mathbb{Q}(\sqrt[3]{2})$.

Lemma 2.1.5. *Let E be the splitting field over a field F of a polynomial $f(x)$ over F . Then*

$$|\text{Aut } E/F| \leq [E : F]$$

Proof. By induction on $[E : F]$. If $[E : F] = 1$, then $E = F$, and we are done. Now, for $[E : F] \geq 1$, $f(x)$ has at least one irreducible factor $p(x)$ of degree $\deg p > 1$. Now, let F' be the corresponding field to F with splitting field E' , corresponding to E . Let $f'(x)$ be the polynomial over F' the polynomial corresponding to f over F , with irreducible factor $p'(x)$ corresponding to the irreducible factor p . Now, let α be a root of p . If σ is an extension of an isomorphism ϕ to E , then the restriction $\tau = \sigma|_{F(\alpha)}$ is an isomorphism of $F(\alpha)$ onto a subfield of E' . Since α generates $F(\alpha)$, τ is completely determined by its action on α ; i.e. $\tau\alpha$, so that $\tau\alpha$ is a root of p' . We then get the following diagram:

$$\begin{array}{ccc} E & \xrightarrow{\sigma} & E' \\ | & & | \\ F(\alpha) & \xrightarrow{\tau} & F'(\tau\alpha) \\ | & & | \\ F & \xrightarrow{\phi} & F' \end{array}$$

Conversely, let β be a root of p' . Then there exist extensions τ and σ of the isomorphism ϕ giving the above diagram (replace $\tau\alpha$ with β). Now, the number of extensions of ϕ to τ is

equal to the number of distinct roots of p' . Since $\deg p = \deg p' = [F(\alpha) : F]$, the number of extensions to τ is at most $[F(\alpha) : F]$.

Now, notice that $[E : F(\alpha)] < [E : F]$. Therefore, by the induction hypothesis, the number of extensions of τ to σ is at most $[E : F(\alpha)]$. Therefore, the number of extensions of ϕ to σ is at most $[E : F(\alpha)][F(\alpha) : F] = [E : F]$.

Finally, if $F = F'$, we have $f = f'$ (and $p = p'$), and so ϕ is the identity map and $E = E'$. This makes σ an automorphism of E which fixes F . The proof is complete. ■

Corollary. *If K is the splitting field of a separable polynomial $f(x)$ over a field F , then $\text{ord Aut } K/F = [K : F]$.*

Definition. We call a finite field extension K/F a **Galois extension** if $\text{ord Aut } K/F = [K : F]$. We call $\text{Aut } K/F$ the **Galois group** of K/F , and write $\text{Gal } K/F$.

Lemma 2.1.6. *An extension K over a field F is Galois over F if and only if it is normal and separable.*

Proof. If K is Galois over F , the result follows by definition. Now, let K be normal and separable. Let $\alpha \in K$. Then the minimal polynomial m of α over F is separable. Moreover, α is a root of m , and since K is normal, m splits completely over K . Thus K contains the splitting field of m , but since m is minimal and irreducible, that makes K the splitting field of some polynomial f over F , having m as a factor. By the above corollary, this makes K Galois over F . ■

Example 2.3. (1) $\mathbb{Q}(\sqrt{2})/\mathbb{Q}$ is Galois, and $\text{Gal } \mathbb{Q}(\sqrt{2})/\mathbb{Q} = \langle \sigma \rangle \simeq \mathbb{Z}/2\mathbb{Z}$, where $\sigma : a + b\sqrt{2} \rightarrow a - b\sqrt{2}$.

(2) Any quadratic extension field K over F is Galois over F , provided $\text{char } F \neq 2$. Then any quadratic extension K of F , of degree $[K : F] = 2$ is of the form $F(\sqrt{D})$, where $D \in \mathbb{Z}^+$ is squarefree. Hence $K = F(\sqrt{D})$ is the splitting field of the polynomial $x^2 - D$.

(3) $\mathbb{Q}(\sqrt[3]{2})$ is not Galois over \mathbb{Q} , since $\text{ord Aut } \mathbb{Q}(\sqrt[3]{2})/\mathbb{Q} = 1$, but $[\mathbb{Q}(\sqrt[3]{2}) : \mathbb{Q}] = 3$.

(4) $\mathbb{Q}(\sqrt{2}, \sqrt{3})$ is the splitting field of the separable polynomial $(x^2 - 2)(x^2 - 3)$ over \mathbb{Q} . Hence $\mathbb{Q}(\sqrt{2}, \sqrt{3})$ is Galois over \mathbb{Q} , and has Galois group $\text{Gal } \mathbb{Q}(\sqrt{2}, \sqrt{3})/\mathbb{Q}$ of order 6. Moreover, since the automorphisms of this group are completely determined by the roots $\sqrt{2}$ and $\sqrt{3}$, we get the possible automorphisms are given by the maps

$$\begin{array}{ll} \sqrt{2} \rightarrow \sqrt{2} & \sqrt{3} \rightarrow -\sqrt{3} \\ \sqrt{2} \rightarrow -\sqrt{2} & \sqrt{3} \rightarrow \sqrt{3} \\ \sqrt{2} \rightarrow \sqrt{2} & \sqrt{3} \rightarrow -\sqrt{3} \\ \sqrt{2} \rightarrow -\sqrt{2} & \sqrt{3} \rightarrow -\sqrt{3} \end{array}$$

Now, let $\sigma : \sqrt{2} \rightarrow -\sqrt{2}, \sqrt{3} \rightarrow \sqrt{3}$ and $\tau : \sqrt{2} \rightarrow \sqrt{2}, \sqrt{3} \rightarrow -\sqrt{3}$. Then $\sigma\tau : \sqrt{2} \rightarrow -\sqrt{2}, \sqrt{3} \rightarrow -\sqrt{3}$. Therefore we have

$$\text{Gal } \mathbb{Q}(\sqrt{2}, \sqrt{3})/\mathbb{Q} = \langle \sigma, \tau \rangle \simeq V_4$$

where V_4 is the Klein 4-group.

We can also determine the fixed fields corresponding to each subgroup of $\langle \sigma\tau \rangle$. That is, $\mathcal{F}(\langle \sigma\tau \rangle)$ is the set of all elements fixed by $\sigma\tau$ and has elements of the form $a+b\sqrt{6}$. So $\mathcal{F}(\langle \sigma\tau \rangle) = \mathbb{Q}(\sqrt{6})$. The table below lists the fixed fields of the Galois group considered.

subgroup	fixed field
$\langle \iota \rangle$	$\mathbb{Q}(\sqrt{2}, \sqrt{3})$
$\langle \sigma \rangle$	$\mathbb{Q}(\sqrt{3})$
$\langle \sigma\tau \rangle$	$\mathbb{Q}(\sqrt{6})$
$\langle \tau \rangle$	$\mathbb{Q}(\sqrt{2})$
$\langle \sigma, \tau \rangle$	\mathbb{Q}

(5) The roots of $x^3 - 2$ over \mathbb{Q} are given by

$$\sqrt[3]{2} \qquad \qquad \xi \sqrt[3]{2} \qquad \qquad \xi^2 \sqrt[3]{2}$$

where $\xi^3 = 1$ is the 3-rd root of unity. Additionally, the splitting field of $x^3 - 2$ over \mathbb{Q} is $\mathbb{Q}(\sqrt[3]{2}, \xi \sqrt[3]{2})$ of degree 6. Now, $x^3 - 2$ is irreducible over \mathbb{Q} , and hence separable over \mathbb{Q} . This makes $\mathbb{Q}(\sqrt[3]{2}, \xi \sqrt[3]{2})$ Galois over \mathbb{Q} , of order 6.

Consider now the set of generators $\sqrt[3]{2}$ and ξ . Then an automorphism σ takes $\sqrt[3]{2} \rightarrow \sqrt[3]{2}, \xi \sqrt[3]{2}$, or $\xi^2 \sqrt[3]{2}$, and takes $\xi \rightarrow \xi$ or ξ^2 . Since these are the roots of the cyclotomic polynomial $\Phi_3(x) = x^2 + x + 1$, σ is completely determined by the actions on $\sqrt[3]{2}$ and ξ . Hence there are 6 possible automorphisms.

Let

$$\begin{aligned} \sigma : \sqrt[3]{2} &\rightarrow \xi \sqrt[3]{2} & \xi &\rightarrow \xi \\ \tau : \sqrt[3]{2} &\rightarrow \sqrt[3]{2} & \xi &\rightarrow \xi^2 \end{aligned}$$

We obtain then the elements

$$\iota \qquad \qquad \sigma^2 \qquad \qquad \tau\sigma^2 = \sigma\tau$$

and we get the additional relations

$$\sigma^2 = \tau^2 = \iota$$

so that

$$\text{Gal } \mathbb{Q}(\sqrt[3]{2}, \xi \sqrt[3]{2}) / \mathbb{Q} = \langle \sigma, \tau \rangle \simeq S_3$$

The fixed field of $\langle \sigma^2 \rangle$ is $\mathbb{Q}(\xi)$.

(6) $\mathbb{Q}(\sqrt[4]{2})$ is not Galois over \mathbb{Q} . We have $[\mathbb{Q}(\sqrt[4]{2}) : \mathbb{Q}] = 4$ but that any automorphism takes $\sqrt[4]{2}$ onto $\pm \sqrt[4]{2}$, or $\pm i \sqrt[4]{2}$. But $\pm i \sqrt[4]{2} \notin \mathbb{Q}(\sqrt[4]{2})$. Notice however that $\mathbb{Q}(\sqrt[4]{2})$ is Galois over $\mathbb{Q}(\sqrt{2})$, and $\mathbb{Q}(\sqrt{2})$ is Galois over \mathbb{Q} .

- (7) The extension field \mathbb{F}_{p^n} is Galois over \mathbb{F}_p . Recall that \mathbb{F}_{p^n} is the splitting field of the separable polynomial $x^{p^n} - x$ over \mathbb{F}_p . Then $\text{ord Gal } \mathbb{F}_{p^n}/\mathbb{F}_p = n$ and the Frobenius automorphism given by

$$\sigma : \alpha \rightarrow \alpha^p$$

generates the Galois group, making it $\langle \sigma \rangle$, a cyclic group of order n .

- (8) The extension $\mathbb{F}_2(x)$ is not Galois over $\mathbb{F}_2(t)$, since $x^2 - t$ is not separable. Moreover, any automorphism of $\text{Aut } \mathbb{F}_2(x)/\mathbb{F}_2(t)$ sends x to the only root of $x^2 - t$, making it the trivial group.

2.2 The Fundamental Theorem of Galois Theory.

Definition. A **linear character** of a group G with values in a field L is a homomorphism $\chi : G \rightarrow \mathcal{U}(L)$. We say that distinct linear characters χ_1, \dots, χ_n of G are **linearly independent** over L if they are linearly independent, as functions, over G .

Theorem 2.2.1. *If χ_1, \dots, χ_n are distinct linear characters of a group G with values in a field L , then they are linearly independent over L .*

Proof. Suppose that χ_1, \dots, χ_n are linearly dependent, and choose a dependence relation with minimum of m nonzero coefficients $a_1, \dots, a_m \in L$, so that

$$a_1\chi_1 + \dots + a_m\chi_m = 0$$

Then for any $g \in G$, we have

$$a_1\chi_1(g) + \dots + a_m\chi_m(g) = 0$$

Now, let $g_0 \in G$, with $\chi_1(g_0) \neq \chi_m(g_0)$. Then

$$a_1\chi_1(g_0g) + \dots + a_m\chi_m(g_0g) = a_1\chi_1(g_0)\chi_1(g) + \dots + a_m\chi_m(g_0)\chi_m(g) = 0$$

multiplying the preceding equation with the above by $\chi_m(g_0)$ and subtracting from the above equation, we get

$$a_1(\chi_1(g_0) - \chi_m(g_0))\chi_1(g) + \dots + a_m(\chi_1(g_0) - \chi_m(g_0))\chi_m(g) = 0$$

which gives a linear dependence relation with fewer than m nonzero coefficients; which contradicts our choice of m . Therefore χ_1, \dots, χ_n must be linearly independent. ■

Corollary. *If $\sigma_1, \dots, \sigma_n$ are distinct embeddings of a field K into a field L , then they are linearly independent as functions.*

Theorem 2.2.2. *Let $G = \{\sigma_1, \dots, \sigma_n\}$ where $\sigma_1 = \iota$ a subgroup of automorphisms of a field K , and let F be the corresponding fixed field. Then*

$$[K : F] = \text{ord } G = n$$

Proof. Suppose that $n > [K : F]$, and consider the basis $\{\omega_1, \dots, \omega_m\}$ of K/F as a vector space so that $[K : F] = m$. Then the matrix equation

$$\begin{pmatrix} \sigma_1\omega_1 & \dots & \sigma_n\omega_m \\ \vdots & \ddots & \vdots \\ \sigma_n\omega_1 & \dots & \sigma_n\omega_m \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_m \end{pmatrix} = 0 \quad (2.1)$$

has nontrivial solution $\begin{pmatrix} \beta_1 \\ \vdots \\ \beta_n \end{pmatrix}$. Let $a_1, \dots, a_m \in F$, so that $\sigma_i\alpha_j = \alpha_j$ for each $1 \leq i \leq n$ and

$1 \leq j \leq m$. Multiplying by $\begin{pmatrix} \sigma_1 a_1 \\ \vdots \\ \sigma_m a_1 \end{pmatrix}$, we obtain

$$\begin{pmatrix} a_1\sigma_1\omega_1 & \dots & a_1\sigma_1\omega_m \\ \vdots & \ddots & \vdots \\ a_m\sigma_m\omega_1 & \dots & a_m\sigma_m\omega_m \end{pmatrix} \begin{pmatrix} \beta_1 \\ \vdots \\ \beta_n \end{pmatrix} = 0$$

so that we can obtain the equation

$$\sigma_1(a_1\omega_1 + \dots + a_m\omega_m)\beta_1 + \dots + \sigma_n(a_1\omega_1 + \dots + a_m\omega_m)\beta_n = 0$$

Where β_1, \dots, β_n are not all 0. Now, since $\{\omega_1, \dots, \omega_m\}$ is an F -basis for K , for all $\alpha \in K$, we get that $\alpha = a_1\omega_1 + \dots + a_m\omega_m$. So we have from the above equation

$$(\sigma_1\alpha)\beta_1 + \dots + (\sigma_n\alpha)\beta_n = 0$$

so that $\{\sigma_1, \dots, \sigma_n\}$ are linearly dependent over K ; which contradicts the above corollary. No $n \leq [K : F]$.

Now, suppose that $n < [K : F]$, and tht there are more than n F -linearly independent elements $\alpha_1, \dots, \alpha_{n+1} \in K$. Then

$$\begin{pmatrix} \sigma_1\alpha_1 & \dots & \sigma_1\alpha_{n+1} \\ \vdots & \ddots & \vdots \\ \sigma_n\alpha_1 & \dots & \sigma_n\alpha_{n+1} \end{pmatrix} \begin{pmatrix} x_1 \\ \dots \\ x_{n+1} \end{pmatrix} = 0$$

has nontrivial solution with entries $\beta_1, \dots, \beta_{n+1} \in K$. Now, if $\beta_i \in F$ for all $1 \leq i \leq n+1$, we get an immediate contradiction of the linear independence of $\{\alpha_1, \dots, \alpha_{n+1}\}$ over F . So at least one $\beta_i \notin F$.

Now, choose a nontrivial solution with minimum of r nonzero entries β_i . Suppose also that $\beta_r = 1$, then at least one $\beta_i \notin F$, for $1 \leq i \leq r-1$, and so $r > 1$. Suppose then that $\beta_1 \notin F$. Then the matrix equation

$$\begin{pmatrix} \sigma_1\alpha_1 & \dots & \sigma_1\alpha_{r-1} & \sigma_1\alpha_r \\ \vdots & \ddots & \vdots & \vdots \\ \sigma_n\alpha_1 & \dots & \sigma_n\alpha_{r-1} & \sigma_n\alpha_r \end{pmatrix} \begin{pmatrix} \beta_1 \\ \vdots \\ \beta_{r-1} \\ 1 \end{pmatrix} = 0$$

Now, since $\beta_1 \notin F$, there exists an automorphism σ_{k_0} of K with $\sigma_{k_0}\beta_1 \neq \beta_1$ for $1 \leq k_0 \leq n$. Applying σ_{k_0} to each row of the above equation yields a row of the form

$$\sigma_{k_0}(\sigma_j\alpha_1)(\sigma_{k_0}\beta_1) + \cdots + \sigma_{k_0}(\sigma_j\alpha_{r-1})(\sigma_{k_0}\beta_{r-1}) + \sigma_j\alpha_r = 0$$

However, since G is a group, $\sigma_{k_0}\sigma_j = \sigma_i$ for $1 \leq i, j \leq n$, so we get

$$(\sigma_i\alpha_1)(\sigma_{k_0}\beta_1) + \cdots + (\sigma_i\alpha_{r-1})(\sigma_{k_0}\beta_{r-1}) + \sigma_i\alpha_r = 0$$

Subtracting this equation from the one preceeding it, we obtain

$$(\sigma_i\alpha_1)(\beta_1 - \sigma_{k_0}\beta_1) + \cdots + (\sigma_i\alpha_{r-1})(\beta_{r-1} - \sigma_{k_0}\beta_{r-1}) = 0$$

with $x_1 = \beta_1 - \sigma_{k_0}\beta_1 \neq 0$. This choice of k_0 gives fewer than r nonzero coefficients of a nontrivial solutions, which contradicts the choice of r . Therefore $n = [K : F]$. ■

Corollary. *If K is a finite extension over a field F , then*

$$\text{ord Aut } K/F \leq [K : F]$$

with equality holding if, and only if F is the fixed field of $\text{Aut } K/F$.

Proof. Let F_1 be the fixed field of $\text{Aut } K/F$, so that $F \subseteq F_1 \subseteq K$. Then $[K : F_1] = \text{ord Aut } K/F$, hence

$$[K : F] = (\text{ord Aut } K/F)[F_1 : F]$$

■

Corollary. *If G is a finite subgroup of automorphisms of a field K , and F is its fixed field, then $\text{Aut } K/F = G$ so that K is Galois over F with Galois group G .*

Proof. By definition, we have that since G fixes the elements of F , then $G \leq \text{Aut } K/F$. Then $\text{ord } G = [K : F]$ and by the above corollary, we get

$$\text{ord Aut } K/F \leq [K : F]$$

so that

$$[K : F] = \text{ord } G \leq \text{ord Aut } K/F \leq [K : F]$$

and equality holds. ■

Corollary. *If G and H are distinct finite subgroups of $\text{Aut } K$, then their fixed fields are also distinct.*

Proof. Let F_G the fixed field of G , and F_H the fixed field of H . If $F_G = F_H$, then we have that H fixes F_G , and since any automorphism fixing F_G is in G , we have $H \leq G$. By similar reasoning, we get $G \leq H$ so that $G = H$. ■

Theorem 2.2.3. *The extension K over a field F is Galois if, and only if K is the splitting field of some separable polynomial in F . Moreover, every irreducible polynomial over F having at least one root in K splits over K .*

Proof. By lemma 2.1.5, the splitting field of a separable polynomial over a field is Galois.

Now, suppose that K is Galois over F , and let $p(x) \in F[x]$ an irreducible polynomial with a root $\alpha \in K$. Consider, for each $\sigma_i \in \text{Gal } K/F$ the elements

$$\alpha \qquad \qquad \qquad \sigma_2 \alpha \qquad \qquad \qquad \dots \qquad \qquad \qquad \sigma_n \alpha$$

where $\sigma_1 = \text{id}$, and let

$$\alpha \qquad \qquad \qquad \alpha_2 \qquad \qquad \qquad \dots \qquad \qquad \qquad \alpha_n$$

be the distinct elements taken on by these permutations (in no particular order). If $\tau \in \text{Gal } K/F$, by the group law, we get $\tau \sigma_i = \sigma_j$ for all $1 \leq i, j \leq n$. Applying τ to α_i we get permutations of the elements $\alpha, \alpha_2, \dots, \alpha_n$. Then the polynomial $f(x) = (x - \alpha)(x - \alpha_2) \dots (x - \alpha_n)$ has coefficients fixed by the elements of $\text{Gal } K/F$. That is, the coefficients lie in the fixed field F . Hence $f \in F[x]$.

Now, since p is irreducible with root α , it is the minimal polynomial for α over F , and hence $p|f$. Moreover, we can observe that $f|p$, so that $p(x) = f(x)$, which makes $p(x)$ separable with all its roots in K .

Now, let $\{\omega_1, \dots, \omega_n\}$ be a basis for K/F as a vector space, and let $p_i(x)$ the minimal polynomial for ω_i over F for all $1 \leq i \leq n$. Then p_i is separable, with roots in K . Let $g(x) = p_1(x) \dots p_n(x)$ (where this product is squarefree). Then if E is the splitting field of g over F , then $\omega_i \in E$ for all $1 \leq i \leq n$, so that $K \subseteq E$. On the otherhand, since g splits over K , we get $E \subseteq K$, and so $K = E$ is the splitting field of g over F . ■

Definition. Let K be an extension of a field F . If $\alpha \in K$, and $\sigma \in \text{Gal } K/F$, we call the permutations $\sigma\alpha$ **Galois conjugates** (or simply **conjugates**) of α over F . If E is a subfield of K containing F , then we call σE the **conjugate field** of E over F .

Theorem 2.2.4 (The Fundamental Theorem of Galois Theory). *Let K be Galois over a field F with Galois group G , and let E be an intermediate field of K over F which is fixed by some subgroup $H \leq G$. Then the following are true.*

- (1) *There is a 1-1 correspondence between the subgroups of G onto the fixed fields of K/F ; that is, \mathcal{F} , treated as a map, is 1-1 and onto.*
- (2) *If $\sigma \in G$, then σE is fixed by $\sigma H \sigma^{-1}$; that is, $\sigma E = \mathcal{F}(\sigma H \sigma^{-1})$.*
- (3) *K is Galois over E , and E is normal over F if, and only if H is normal in G .*
- (4) *If H is normal in G , then*

$$\text{Gal } E/F \simeq G/H$$

- (5) *Independently of whether or not E is normal over F , we have that*

$$[E : F] = [G : H]$$

Proof. Let $\mathcal{G}(E) = \text{Aut } K/E$, that is, as a map, \mathcal{G} sends a intermediate field of K/F to that group of E -automorphisms of K ; i.e. all automorphisms that fix the elements of E . Now, consider the mapping

$$H \rightarrow \mathcal{F}(H) \rightarrow \mathcal{G}\mathcal{F}(H)$$

and take $\sigma \in H$. Then, by definition, σ fixes the field $\mathcal{F}(H)$, so that $\sigma \in \mathcal{G}(\mathcal{F}(H)) = \mathcal{G}\mathcal{F}(H)$. Then $H \leq \mathcal{G}\mathcal{F}(H)$. Now, we have that the fixed field $\mathcal{F}(H)$ contains the fixed field of $\mathcal{G}\mathcal{F}(H)$, i.e. $\mathcal{F}(\mathcal{G}\mathcal{F}(H))$, which is H . That is, $\mathcal{G}\mathcal{F}(H) \leq H$. Therefore, we have $\mathcal{G}\mathcal{F}(H) = H$. Conversely, consider the mapping

$$E \rightarrow \mathcal{G}(E) \rightarrow \mathcal{F}\mathcal{G}(E)$$

Observe that \mathcal{F} is the fixed field of $\mathcal{G}(E) = \text{Aut } K/E$, by definition, $\mathcal{F}\mathcal{G}(K) = K$. This establishes the 1–1 correspondence of the subgroups of G onto the fixed fields of K/F .

Now, since $E = \mathcal{F}(H)$, observe that $\mathcal{F}(\sigma H \sigma^{-1})$ consists of all elements of K which are fixed by $\sigma \tau \sigma^{-1}$ for all $\tau \in H$; that is, all $\alpha \in K$ for which $\sigma \tau \sigma^{-1}(\alpha) = \alpha$. Observe, then that $\tau \sigma^{-1}(\alpha) = \tau(\sigma^{-1}\alpha) = \sigma^{-1}\alpha$, so that τ fixes $\sigma^{-1}\alpha$. Then $\sigma^{-1}\alpha \in \mathcal{F}(H)$. That is, $\alpha \in \sigma \mathcal{F}(H) = \sigma E$, therefore $\mathcal{F}(\sigma H \sigma^{-1}) = \sigma E$.

For the third statement, notice that since K is Galois over F , then it is normal and separable. This makes E normal and separable over F , so that by lemma 2.1.6, E/F is Galois. Now, let σ be a 1–1 F -homomorphism which is from $K \rightarrow E$. Then σ can be extended to a 1–1 F -homomorphism from $K \rightarrow K$, by lemma 1.2.5. If E/F is normal, then for every $\sigma \in G$, σ fixes E , and E is fixed by a normal subgroup of G by the previous statement.

Consider now, the homomorphism from $G \rightarrow \text{Gal } E/F$ given by $\sigma \rightarrow \sigma|_E$. This map is onto, with kernel consisting of all automorphisms which fix E . Then $\text{Gal } E/F = H$, moreover, since K/F is normal, we get $H \trianglelefteq G$. Therefore by the first isomorphism theorem (for groups), we get

$$\text{Gal } E/F \simeq G/H$$

where G/H is understood to be the quotient group. Finally, we also have that $[K : F] = [K : E][E : F]$. Since both K/F and E/F are Galois, this makes

$$\text{ord } G = [E : F] \text{ord } H$$

which makes $[E : F] = [G : H]$ by the definition of the index of a subgroup. ■

Example 2.4. (1) The lattices of $\mathbb{Q}(\sqrt{2}, \sqrt{3})$ and $\mathbb{Q}(\sqrt[3]{2}, \xi)$ (where $\xi^3 = 1$) indicate all of the subfields of these fields. We have that the lattice of $\mathbb{Q}(\sqrt{2}, \sqrt{3})$ is isomorphic to the lattice of the Klein 4-group V_4 , which has all its subgroups normal. Thus we get every subfield of $\mathbb{Q}(\sqrt{2}, \sqrt{3})$ is Galois over \mathbb{Q} .

On the other hand, the lattice for $\mathbb{Q}(\sqrt[3]{2}, \xi)$ is isomorphic to the lattice of S^3 where the only normal subgroup is the nontrivial subgroup of order 3; moreover, only $\mathbb{Q}(\xi)$ is Galois over \mathbb{Q} with $\text{Gal } \mathbb{Q}(\xi)/\mathbb{Q} \simeq S_3/\langle \sigma \rangle$, where $\langle \sigma \rangle$ is the cyclic subgroup of order 2.

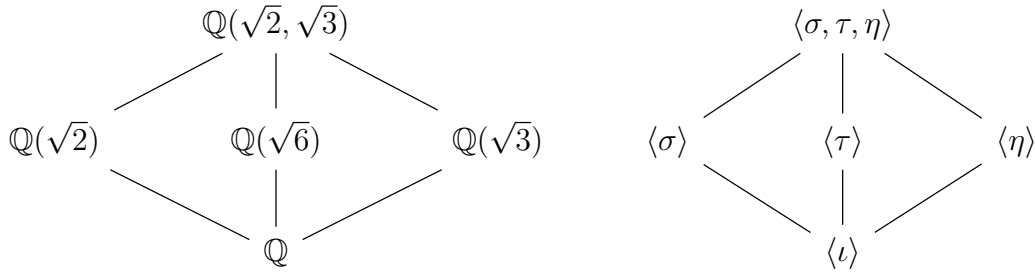


Figure 2.1: The lattice of subfields of $\mathbb{Q}(\sqrt{2}, \sqrt{3})$ and the lattice of subgroups of $\text{Gal } \mathbb{Q}(\sqrt{2}, \sqrt{3})/\mathbb{Q}$.

- (2) Consider $\mathbb{Q}(\sqrt{2} + \sqrt{3}) \subseteq \mathbb{Q}(\sqrt{2}, \sqrt{3})$. If $m(x)$ is the minimal polynomial of $\sqrt{2} + \sqrt{3}$, then observe that it has as roots the distinct conjugates

$$\pm\sqrt{2} \pm \sqrt{3}$$

so that

$$m(x) = (x + (\sqrt{2} + \sqrt{3}))(x - (\sqrt{2} + \sqrt{3}))(x + (\sqrt{2} - \sqrt{3}))(x - (\sqrt{2} - \sqrt{3})) = x^4 + 10x + 1$$

Moreover, $x^4 + 10x + 1$ is irreducible. Then only the automorphism ι of $\{\iota, \sigma, \tau, \sigma\tau\}$ fixes $\sqrt{2} + \sqrt{3}$ so that the fixing group of $\mathbb{Q}(\sqrt{2} + \sqrt{3})$ is precisely that of $\mathbb{Q}(\sqrt{2}, \sqrt{3})$. So $\mathbb{Q}(\sqrt{2} + \sqrt{3}) = \mathbb{Q}(\sqrt{2}, \sqrt{3})$.

- (3) Consider the splitting field of $x^8 + 1$ over \mathbb{Q} , generated by the elements $\sqrt[8]{2}$ and ξ , where $\xi^8 = 1$ (i.e. a primitive 8-th root of unity). Let $\zeta = \sqrt[8]{2}$, and notice that $\zeta^4 = \sqrt{2}$, and the splitting field of $x^8 - 2$ over \mathbb{Q} is $\mathbb{Q}(\sqrt[8]{2}, i)$ of degree $[\mathbb{Q}(\sqrt[8]{2}, i) : \mathbb{Q}] = 16 = 4^2$. Consider then all possible maps on ζ and i given by $\zeta \rightarrow \xi^a \zeta, i \rightarrow \pm i$. Define then the automorphisms

$$\sigma : \zeta \rightarrow \xi \zeta, i \rightarrow i \text{ and } \tau : \zeta \rightarrow \zeta, i \rightarrow -i$$

Since $\xi = \frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2} = \frac{1+i}{2}\zeta^4$, we compute that $\sigma : \xi \rightarrow \xi^5$, and $\tau : \xi \rightarrow \xi^7$. We can then compute the Galois group by noting that $\sigma^8 = \tau^2 = \iota$, and $\sigma\tau = \tau\sigma^3$, so that

$$\text{Gal } \mathbb{Q}(\sqrt[8]{2}, i)/\mathbb{Q} = \langle \sigma, \tau : \sigma^8 = \tau^2 = \iota \text{ and } \sigma\tau = \tau\sigma^3 \rangle$$

which describes the quasidihedral group of order 16.

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