

Algebraic Topology

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Chapter 1

Categories.

1.1 Categories and Subcategories.

Definition. A **category** \mathcal{C} is a collection of a class of **objects**, denoted $\text{obj } \mathcal{C}$ a collection of sets of **morphisms** $\text{Hom}(A, B)$ for each $A, B \in \text{obj } \mathcal{C}$ and a binary operation $\circ : \text{Hom}(A, B) \times \text{Hom}(B, C) \rightarrow \text{Hom}(A, C)$, defined by $(f, g) \rightarrow g \circ f$, called **composition** such that:

- (1) Each $\text{Hom}(A, B)$ is pairwise disjoint for all $A, B \in \text{obj } \mathcal{C}$.
- (2) \circ is associative when defined; that is if either $(g \circ f) \circ h$ or $g \circ (f \circ h)$ are defined, then $(g \circ f) \circ h = g \circ (f \circ h)$, for morphisms f, g, h .
- (3) For each $A \in \text{obj } \mathcal{C}$, there exists an **identity** morphism $1_A \in \text{Hom}(A, A)$ such that for each $B, C \in \text{obj } \mathcal{C}$, $1_A \circ f = f$ and $g \circ 1_A = g$ for each morphism $f \in \text{Hom}(B, A)$ and $g \in \text{Hom}(A, C)$.

We denote morphisms by $f : A \rightarrow B$ instead of $f \in (A, B)$.

Definition. Let \mathcal{C} be a category and $f : A \rightarrow B$ a morphism in \mathcal{C} . We call A and B the **domain** and **codomain** of f , respectively, and we call the set $G_f = \{(a, f(a)) : a \in A\} \subseteq B$ the **graph** of f .

Example 1.1. (1) The category of all sets Set has as objects the class of all sets. The morphisms in Set are all functions $f : A \rightarrow B$ where A and B are sets. The composition of Set is the usual composition of functions.

- (2) The category of all topological spaces Top has as objects all topological spaces, and as morphisms all continuous maps $f : X \rightarrow Y$ from a space X to a space Y . The composition is the usual composition.
- (3) The category of all groups, Grp has as objects all groups and as morphisms all homomorphisms $f : G \rightarrow H$, under the usual composition.
- (4) The category of rings with unit Rng has as objects all rings with unit, along with all ring homomorphisms $f : R \rightarrow K$ to be the morphisms under the usual composition.

Definition. We call a category a **subcategory** of a category \mathcal{C} if $\text{obj } \mathcal{A} \subseteq \text{obj } \mathcal{C}$, $\text{Hom}_{\mathcal{A}}(A, B) \subseteq \text{Hom}_{\mathcal{C}}(A, B)$ for all $A, B \in \mathcal{C}$, and \mathcal{A} inherits the composition of \mathcal{C} .

Example 1.2. (1) The category of all pairs of topological spaces (X, A) where A is a subspace of X , whose morphisms are pairs of continuous maps $f = (f_1, f_2)$ such that $f_1 i = j f_2$ where $i : A \rightarrow X$ and $j : B \rightarrow Y$ are inclusions, is a subcategory of Top . We denote this category Top^2 .

$$\begin{array}{ccc}
 A & \xhookrightarrow{i} & X \\
 f_2 \downarrow & & \downarrow f_1 \\
 B & \xhookrightarrow{j} & Y
 \end{array}$$

- (2) The category of all **pointed spaces**, Top^* is defined with the objects being all pairs $(X, \{x_0\})$, where $x_0 \in X$ with the morphisms of Top^2 . Top^* is a subcategory of Top^2 . We call x_0 the **base point**, and we call the morphisms of Top^* **pointed maps**.
- (2) The category of all Abelian groups, Ab is a subcategory of Grp . Likewise, the category of all commutative rings with unit is a subcategory of Rng .

1.2 Commutative Diagrams and Congruences.

Definition. A **diagram** in a category is a directed graph with its vertex set a subset of objects, and whose edge set are morphisms between those objects. We call a diagram **commutative** if for any pair of objects, every pair of morphisms between those objects are equal. That is if A, A' and B, B' are pairs of objects with pairs of morphisms $f : A \rightarrow B$, $f' : A' \rightarrow B'$ and $g : A \rightarrow A'$, $g' : B \rightarrow B'$ we have that $g' \circ f = f' \circ g$

$$\begin{array}{ccc}
 A & \xrightarrow{g} & A' \\
 f \downarrow & & \downarrow f' \\
 B & \xrightarrow{g'} & B'
 \end{array}$$

Definition. A **congruence** on a category \mathcal{C} is an equivalence relation \sim on morphisms in \mathcal{C} such that:

- (1) If $f \in \text{Hom}(A, B)$, and $f \sim f'$, then $f' \in \text{Hom}(A, B)$.
- (2) If $f \sim g$ and $f' \sim g'$, then $g \circ f \sim g' \circ f'$.

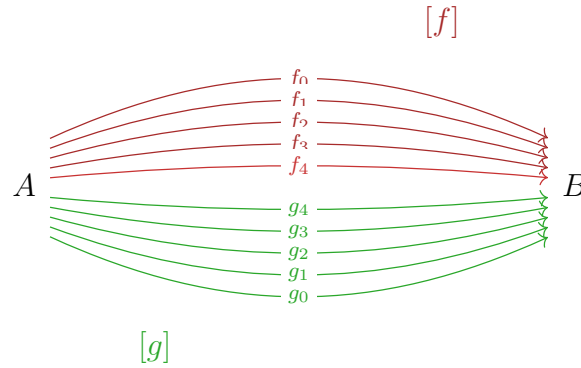


Figure 1.1: An equivalence relation between morphisms.

Theorem 1.2.1. Let \mathcal{C} be a category with congruence \sim . Define \mathcal{C}/\sim as follows:

- (1) $\text{obj } \mathcal{C}/\sim = \text{obj } \mathcal{C}$.
- (2) $\text{Hom}_{\mathcal{C}/\sim}(A, B) = \{[f] : f \in \text{Hom}_{\mathcal{C}}(A, B)\}$.
- (3) $[g] \circ [f] = [g \circ f]$

Then \mathcal{C}/\sim is a category.

Proof. We have by equivalence that $\text{obj } \mathcal{C}/\sim$ is a class. Moreover, since \sim partitions \mathcal{C} , it partitions all of the $\text{Hom}(A, B)$ for each A, B . So each $\text{Hom}(A, B)$ is a set, moreover, they are pairwise disjoint by definition of \sim . Now, notice that by hypothesis, composition in \mathcal{C}/\sim is well defined, so $[1_A] \circ [f] = [1_A \circ f] = [f]$ and $[g] \circ [1_A] = [g \circ 1_A] = [g]$. This makes \mathcal{C}/\sim a category. ■

Remark. One can think of the category \mathcal{C}/\sim as taking all morphisms with the same domain and codomain, and collapsing them into a single morphism.

Definition. Let \mathcal{C} be a category and \sim a congruence of \mathcal{C} . We call the category \mathcal{C}/\sim induced by \sim the **quotient category**.

1.3 Functors.

Definition. Let \mathcal{A} and \mathcal{C} be categories. We define a **covariant functor** to be a map $F : \mathcal{A} \rightarrow \mathcal{C}$ such that:

- (1) $A \in \text{obj } \mathcal{A}$ implies $F(A) \in \text{obj } \mathcal{C}$.
- (2) If $f : A \rightarrow B$ is a morphism in \mathcal{A} , then $F(f) : F(A) \rightarrow F(B)$ is a morphism in \mathcal{C} .

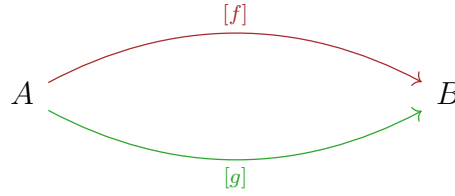


Figure 1.2: Morphisms in a category with the same domain and codomain get collapsed onto a single morphism in the corresponding quotient category.

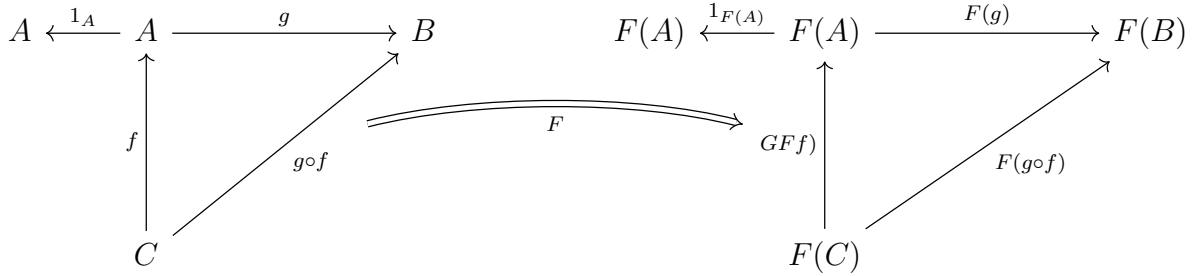


Figure 1.3: A covariant functor taking a diagram in one category to a diagram in the other.

- (3) For all morphisms f and g in \mathcal{A} , for which $g \circ f$ is defined, we have that $F(g \circ f) = F(g) \circ F(f)$, and $F(1_A) = 1_{F(A)}$.

Example 1.3. (1) We define the **forgetful functor** the map $F : \mathcal{C} \rightarrow \text{Set}$ that takes all objects in \mathcal{C} to their underlying sets, and morphisms in \mathcal{C} to themselves considered as functions under the usual composition. For example the forgetful functor $F : \text{Top} \rightarrow \text{Set}$ takes topological spaces X to their underlying sets, and continuous maps to themselves, considered just as functions.

- (2) The **identity functor** is the functor $I : \mathcal{C} \rightarrow \mathcal{C}$ that takes objects and morphisms in \mathcal{C} to themselves.
- (3) Let M be a topological space. Define $F_M : \text{Top} \rightarrow \text{Top}$ by $F_M : X \rightarrow X \times M$, and for each continuous map $f : X \rightarrow Y$, $F(f) : X \times M \rightarrow Y \times M$ is defined by $(x, m) \rightarrow (f(x), m)$. Then F_M is a functor.
- (4) Let $A \in \text{obj } \mathcal{C}$ and take the map $\text{Hom}(A, *) : \mathcal{C} \rightarrow \text{Set}$ that takes $A \rightarrow \text{Hom}(A, B)$ and for each morphism $f : B \rightarrow B'$, $\text{Hom}(A, f) : \text{Hom}(A, B) \rightarrow \text{Hom}(A, B')$ is given by $g \rightarrow f \circ g$. We call this functor the **covariant Hom functor**, and denote it f_* .

Definition. Let \mathcal{A} and \mathcal{C} be categories. We define a **contravariant functor** to be a map $G : \mathcal{A} \rightarrow \mathcal{C}$ such that:

- (1) $A \in \text{obj } \mathcal{A}$ implies $G(A) \in \text{obj } \mathcal{C}$.

- (2) If $f : A \rightarrow B$ is a morphism in \mathcal{A} , then $G(f) : G(B) \rightarrow G(A)$ is a morphism in \mathcal{C} .
- (3) For all morphisms f and g in \mathcal{A} , for which $g \circ f$ is defined, we have that $G(g \circ f) = G(f) \circ G(g)$, and $G(1_A) = 1_{G(A)}$.

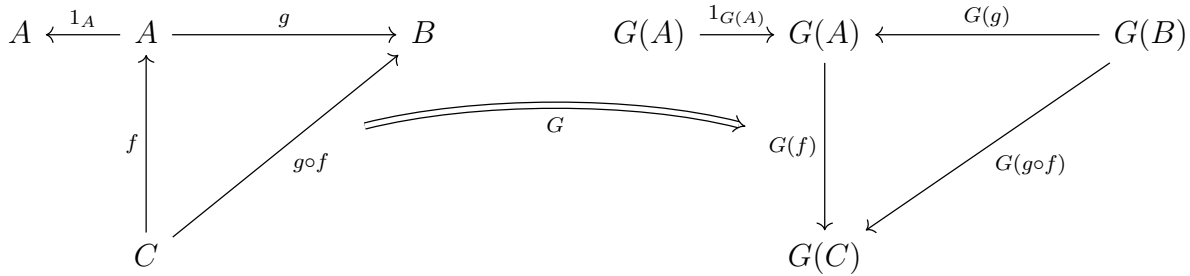


Figure 1.4: A contravariant functor taking a diagram in one category to a diagram in the other.

Example 1.4. (1) Let F be a field, and Vec the category of all finite dimensional vector spaces over F , whose morphisms are linear transformations. Define the map $T : \text{Vec} \rightarrow \text{Vec}$ by taking $T : V \rightarrow V^\perp$, and $T : f \rightarrow f^T$. That is T takes vector spaces to their dual spaces, and linear transformation to their transpose. T is a contravariant functor called the **dual space functor**.

- (2) Define $\text{Hom}(*, B) : \mathcal{C} \rightarrow \mathcal{C}$ by taking $\text{Hom}(*, B) : A \rightarrow \text{Hom}(A, B)$ and for each morphism $g : A \rightarrow A'$ in \mathcal{C} , $\text{Hom}(g, B) : \text{Hom}(A', B) \rightarrow \text{Hom}(A, B)$ is defined by taking $h \rightarrow h \circ g$. This is analogous to the covariant Hom functor, and we call it the **contravariant Hom functor**.

Definition. We call a morphism $f : A \rightarrow B$ an **equivalence** if there exists a morphism $g : B \rightarrow A$ such that $f \circ g = 1_B$ and $g \circ f = 1_A$.

Theorem 1.3.1. Let \mathcal{A} and \mathcal{C} be categories, and $F : \mathcal{A} \rightarrow \mathcal{C}$ be a functor. If f is an equivalence in \mathcal{A} , then $F(f)$ is an equivalence in \mathcal{C} .

Proof. Suppose that F is a covariant functor. Notice that if $f : A \rightarrow B$ is an equivalence, then there is a $g : B \rightarrow A$ with $f \circ g = 1_B$ and $g \circ f = 1_A$. Then $F(f \circ g) = F(f) \circ F(g) = F(1_B) = 1_{F(B)}$, and $F(g \circ f) = F(g) \circ F(f) = F(1_A) = 1_{F(A)}$.

Likewise, if F is contravariant, notice that $F(f) : B \rightarrow A$ and $F(g) : A \rightarrow B$. Then $F(f \circ g) = F(g) \circ F(f) = 1_{F(A)}$, and $F(g \circ f) = F(f) \circ F(g) = 1_{F(B)}$. In either case, we find that $F(f)$ is an equivalence in \mathcal{C} . ■

Chapter 2

Homotopy, Convexity, and Connectedness.

2.1 Homotopy

Definition. If X and Y are topological spaces, and $f_0 : X \rightarrow Y$ and $f_1 : X \rightarrow Y$ are continuous maps, we say that f_0 is **homotopic** to f_1 if there exists a continuous map $F : X \times I \rightarrow Y$ with $F(x, 0) = f_0(x)$ and $F(x, 1) = f_1(x)$. We write $f_0 \simeq f_1$ and call F a **homotopy**. We also write $F : f_0 \simeq f_1$ to denote a homotopy between f_0 and f_1 .

Lemma 2.1.1 (The Pasting Lemma). *Let X is a topological space that is covered by open sets $\{X_n\}$. If Y is some topological space for which there exist unique maps $f_n : X_n \rightarrow Y$ that coincide in the intersections of their domains, then there exists a unique map $f : X \rightarrow Y$ such that $f|_{X_n} = f_n$, for all n .*

Lemma 2.1.2. *Homotopy between continuous maps is an equivalence relation.*

Proof. Let $f : X \rightarrow Y$ be a continuous map. Define $F : X \times I \rightarrow Y$ by $(x, t) \rightarrow f(x)$ for all $(x, t) \in X \times I$. Then F is continuous by definition; moreover, $F(x, 0) = F(x, 1) = f(x)$, making $f \simeq f$.

Now suppose there exist a homotopy $F : f \simeq g$ for maps $f : X \rightarrow Y$ and $g : X \rightarrow Y$. Define the map $G : X \times I \rightarrow Y$ by $(x, t) \rightarrow F(x, 1 - t)$. G is the composition of continuous maps, so G is continuous, moreover, $G(x, 0) = F(x, 1) = g(x)$ and $G(x, 1) = F(x, 0) = f(x)$, so that $g \simeq f$.

Lastly, suppose that $F : f \simeq g$ and $G : g \simeq h$ for maps f, g, h . Define the map $H : X \times I \rightarrow Y$ by:

$$H(x, t) = \begin{cases} F(x, 2t), & \text{if } 0 \leq t \leq \frac{1}{2} \\ G(x, 2t - 1), & \text{if } \frac{1}{2} \leq t \leq 1 \end{cases}$$

Notice that F and G coincide in their domains which cover X . Therefore, by the pasting lemma, H is continuous. Now notice also that $H(x, 0) = F(x, 2 \cdot 0) = F(x, 0) = f(x)$ and $H(x, 1) = G(x, 2 \cdot 1 - 1) = G(x, 1) = h(x)$. This makes $f \simeq h$. ■

Definition. For any continuous map $f : X \rightarrow Y$ we define the **homotopy class** of f to be the equivalence class of all continuous maps homotopic to f . That is:

$$[f] = \{g : X \rightarrow Y : g \text{ is continuous and } g \simeq f\}$$

Lemma 2.1.3. Let $f_0 : X \rightarrow Y$, $f_1 : X \rightarrow Y$ and $g_0 : X \rightarrow Y$, $g_1 : X \rightarrow Y$ be continuous maps. If $f_0 \simeq f_1$ and $g_0 \simeq g_1$, then $g_0 \circ f_0 \simeq g_1 \circ f_1$. That is $[g_0 \circ f_0] = [g_1 \circ f_1]$.

Proof. Let $F : f_0 \simeq f_1$ and $G : g_0 \simeq g_1$ be the homotopies of f_0 into f_1 and g_0 into g_1 , respectively. Define the map $H : X \times I \rightarrow Y$ by taking $(x, t) \rightarrow G(f_0(x), t)$. Then we have that H is continuous by composition, and that $H(x, 0) = G(f_0(x), 0) = g_0(f_0(x))$, and $H(x, 1) = G(f_0(x), 1) = g_1(f_0(x))$. Thus we see that $g_0 \circ f_0 \simeq g_1 \circ f_0$.

Now define the map $K : X \times I \rightarrow Y$ by $K = g_1 \circ F$. We have that K is continuous by composition, and that $K(x, 0) = g_1 \circ f_0$ and $K(x, 1) = g_1 \circ f_1$, making $g_1 \circ f_0 \simeq g_1 \circ f_1$. Therefore, by transitivity of homotopy, $g_0 \circ f_0 \simeq g_1 \circ f_1$. ■

Theorem 2.1.4. Homotopy is a congruence on the category Top.

Proof. The proof follows by lemmas 2.1.2 and 2.1.3. ■

Definition. We call the quotient category of Top induced by homotopy the **homotopy category** and denote it hTop.

Definition. A continuous map $f : X \rightarrow Y$ is a **homotopy equivalence** if there exists a continuous map $g : Y \rightarrow X$ such that $f \circ g \simeq 1_Y$ and $g \circ f \simeq 1_X$. We say that the spaces X and Y have the same **homotopy type** if there exists a homotopy equivalence.

Definition. We call a continuous map **nullhomotopic** if it is homotopic to a constant map.

Example 2.1. The space of complex numbers \mathbb{C} and the unit circle S^1 have the same homotopy type.

Definition. Let Y and Z be topological spaces, and $X \subseteq Y$ a subspace of Y . If $f : X \rightarrow Z$ is a continuous map, then we call the map $g : Y \rightarrow Z$ defined by $g \circ i = f$ an **extension** of f , where $i : X \rightarrow Y$ is the inclusion map.

Theorem 2.1.5. Let $f : S^n \rightarrow Y$ be a continuous map into a topological space Y . The following are equivalent:

- (1) f is nullhomotopic.
- (2) f can be extended to a continuous map $B^{n+1} \rightarrow Y$.
- (3) There exists a constant map $k : S^n \rightarrow Y$, taking $x \rightarrow f(x_0)$, for all $x \in S^n$, such that $f \simeq k$, for $x_0 \in S^n$.

Proof. Notice that (3) implies (1) immediately. Now suppose that f is nullhomotopic. Then there exists a constant map $k : X \rightarrow Y$, such that for some $x_0 \in S^n$, $k : x \rightarrow x_0$ for all $x \in S^n$ implies that $f \simeq k$. Now, define the map $g : B^{n+1} \rightarrow Y$ by:

$$g(x) = \begin{cases} y_0, & \text{if } 0 \leq \|x\| \leq \frac{1}{2} \\ F(\frac{x}{\|x\|}, 2 - 2\|x\|), & \text{if } \frac{1}{2} \leq \|x\| \leq 1 \end{cases}$$

Notice, that if $\|x\| = \frac{1}{2}$, then $g(x) = F(2x, 1) = y_0$. Therefore, by the pasting lemma, g is continuous. Moreover, if $\|x\| = 1$, $g(x) = F(x, 0) = f$, which makes g an extension of f .

Now, suppose that there exists an extension $g : B^{n+1} \rightarrow Y$ of f . Since S^n is a subspace of B^{n+1} , we have that $g \circ i = g|_{S^n} = f$, where $i : Y \rightarrow S^n$ is an inclusion. Now, let $x_0 \in S^n$ and define the constant map $k : S^n \rightarrow Y$ by taking $x \rightarrow f(x_0)$ for all $x \in S^n$. Additionally, define the map $F : S^n \times I \rightarrow Y$ given by $F(x, t) = g((1-t)x + x_0t)$. We have that F is continuous by composition of continuous maps, and that $F(x, 0) = g(x) = f(x)$, since F has the domain $S^n \times I$, and that $F(x, 1) = g(x_0) = f(x_0)$, since F has the domain $S^n \times I$. This makes $f \simeq k$ with F as the associated homotopy. ■

2.2 Quotient Spaces

Definition. Let X be a topological space, and $X' = \{X_\alpha\}$ a partition of X . We define the **natural map** $q : X \rightarrow X'$ by taking $x \rightarrow X_\alpha$ where $x \in X_\alpha$. We define the **quotient topology** on X' to be the family:

$$\mathcal{T} = \{U' \subseteq X' : q^{-1}(U') \text{ is open in } X\}$$

We denote quotient spaces by X/q , X/X' , or X/\sim where \sim is an equivalence relation partitioning X into X' .

Example 2.2. (1) Consider the space $I = [0, 1]$ and let $A = \{0, 1\}$. The quotient space I/A identifies 0 to 1, and hence, under the quotient topology, is homeomorphic to S^1 .

(2) Consider the space $I \times I$ and define an equivalence relation $(x, 0) \sim (x, 1)$ for all $x \in I$. Then the quotient topology formed on $I \times I/\sim$ is homeomorphic to the cylinder $S^1 \times I$. Defining another equivalence $(0, y) \sim (1, y)$ for all $y \in I$, we get the quotient space on $S^1 \times I/\sim$ under this equivalence relations is homeomorphic to the torus $S^1 \times S^1$.

(3) Let $h : X \rightarrow Y$ be a map, and define $\ker h$ the equivalence relation on X such that $x \ker h x'$ if, and only if $h(x) = h(x')$. The quotient space $X/\ker h$ has the following relation to the natural map on X via the commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{h} & Y \\ q \downarrow & \searrow \phi & \\ X/\ker h & & \end{array}$$

Where $\phi : X/\ker h \rightarrow Y$ is a 1-1 map defined by $\phi([x]) = h(x)$.

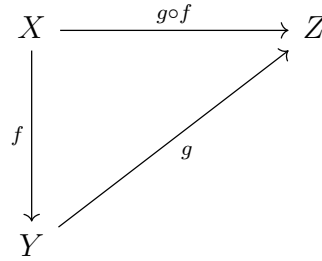
Definition. A continuous map $f : X \rightarrow Y$ of a topological space X onto a topological space Y is call an **identification** if a subset U of Y is open if, and only if $f^{-1}(U)$ is open in X . We denote the quotient space on X induced by f by X/f .

Example 2.3. (1) The natural map $q : X \rightarrow X/\sim$ is an identification, where \sim is an equivalence relation on X inducing the quotient topology.

(2) If $f : X \rightarrow Y$ takes spaces X onto Y , is open or closed, then f is an identification.

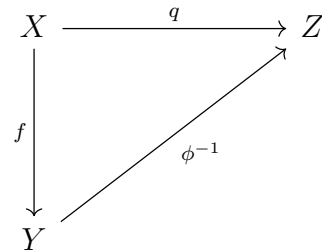
(3) If $f : X \rightarrow Y$ is a continuous map such that there exists a map $s : Y \rightarrow X$ such that $f \circ s = 1_Y$, then f is an identification. We call the map s a **section** of f .

Theorem 2.2.1. Let $f : X \rightarrow Y$ be a continuous map of a topological space X onto a topological space Y . f is an identification if, and only if for any topological space Z , and all maps $g : Y \rightarrow Z$, then g is continuous if, and only if $g \circ f$ is continuous.



Proof. Suppose that f is an identification. If g is continuous, then so is $g \circ f$, by continuity of f . On the other hand, if $g \circ f$ is continuous, letting V be open in Z we have $(g \circ f)^{-1}(V) = f^{-1}(g^{-1}(V))$ which is open in X . By hypothesis, $g^{-1}(V)$ is open in Y , which makes g continuous.

Now, suppose that g is continuous if, and only if $g \circ f$ is continuous. Let $Z = X/\ker f$, and $q : X \rightarrow X/\ker f$ the natural map. Additionally, define the 1-1 map $\phi : X/\ker f \rightarrow Y$ by $\phi([x]) = f(x)$. Since f is onto, we get that so is ϕ . Consider the following commutative diagram:

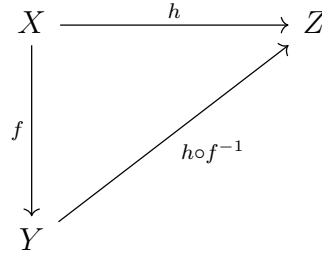


Then $\phi^{-1} \circ f = q$ is continuous which implies that ϕ^{-1} is continuous. ϕ is also continuous since q is an identification. Therefore ϕ is a homeomorphism between Y and Z . Notice now, that $f = \phi \circ q$. Then since q and ϕ are continuous, this makes f continuous by composition. Moreover, $f^{-1}(U) = q^{-1}(\phi^{-1}(U))$. Since q is an identification, $\phi^{-1}(U)$ is open in Z , which makes $f^{-1}(U)$ open in X . This makes f an identification. ■

f

Corollary. Let $f : X \rightarrow Y$ be an identification, and for some space Z , define $h : X \rightarrow Z$ to

be the continuous map constant on each fiber of f . Then $h \circ f^{-1} : Y \rightarrow Z$ is continuous.



Moreover $h \circ f^{-1}$ is open or closed if, and only if $h(U)$ is open or closed in Z whenever $U = f^{-1}(f(U))$ is open or closed in X .

Corollary. If $h : X \rightarrow Z$ is an identification, then the map $\phi : X/\ker h \rightarrow Z$ defined by $[x] \rightarrow h(x)$ is a homeomorphism.

2.3 Convexity and Contracibility

Definition. We call a subset X of \mathbb{R}^n **convex** if for every $x, y \in X$, the line segment joining x to y is convex. That is the line $tx + (1 - t)y \in X$ for all $t \in [0, 1]$.

Example 2.4. The sets \mathbb{R}^n , I^n , B^n and $\Delta(\mathbb{R}^n)$ are all convex. The sphere S^{n-1} is not convex.

Definition. We call a topological space X **contractible** if 1_X is nullhomotopic.

Example 2.5. (1) Let $X = \{x, y\}$ together with the topology $\mathcal{T} = \{\emptyset, \{x\}, X\}$. Then X is contractible under the topology \mathcal{T} . We call X together with \mathcal{T} the **Sierpinski space**.

(2) The space \mathbb{R}^n is contractible, but the sphere S^{n-1} is not contractible.

(3) Continuous images of contractible spaces need not be contractible.

Theorem 2.3.1. Every convex set is contractible.

Proof. Choose $x_0 \in X$ and consider the constant map $c : X \rightarrow X$ by $x \rightarrow x_0$ for all $x \in X$. Define $F : X \times I \rightarrow X$ by $F(x, t) = tx_0 + (1 - t)x$. This map is continuous, with $F(x, 0) = x = 1_X(x)$ and $F(x, 1) = x_0 = c(x)$. Therefore $1_X \simeq c$. ■

Lemma 2.3.2. If X is a contractible space, and homeomorphic to a space Y , then Y is also contractible.

Example 2.6. If X and Y are subspaces of \mathbb{R}^n , with X homeomorphic to Y , and X convex, then Y is contractible by lemma 2.3.2, however, Y may not be convex. This shows that not all contractible spaces are convex spaces.

Lemma 2.3.3. Contractible spaces are connected.

Corollary. *Convex sets are connected.*

Proof. This follows from theorem 2.3.1. ■

Definition. If X is a topological space, define the equivalence relation \sim on $X \times I$ by $(x, t) \sim (x', t')$ if, and only if $t = t' = 1$. Denote the equivalence classes of (x, t) as $[x, t]$. We call the quotient space $X \times I / \sim$ the **cone** over X , and denote it CX . We call the equivalence class $[x, 1]$ the **vertex** of CX .

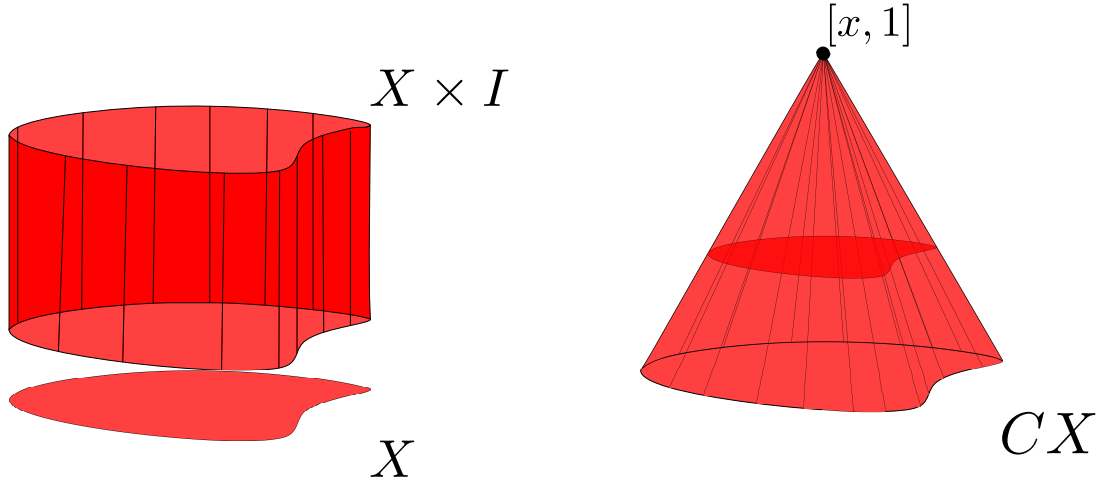


Figure 2.1: The space X and the cone CX formed by identifying all $t = 1$ of $X \times I$ to a point.

Example 2.7. (1) For topological spaces X and Y , every continuous map $f : X \times I \rightarrow Y$ with $f(x, 1) = y_0$ for some $y_0 \in Y$ induces a continuous map $Cf : CX \rightarrow Y$ by taking $[x, t] \rightarrow f(x, t)$.

(2) The cone over S^{n-1} is $CS^{n-1} = D^n$ and has the vertex 0.

Theorem 2.3.4. *For any topological space X , the cone over X is contractible.*

Proof. Define the map $F : CX \times I \rightarrow CX$ by taking $([x, t], s) \rightarrow [x, (1-s)t + s]$. This map is continuous by composition, moreover $F([x, t], 0) = [x, t]$ and $F([x, t], 1) = [x, 1]$ which makes $1_{CX} \simeq c$ where $c : CX \rightarrow CX$ is the constant map taking $[x, t] \rightarrow [x, 1]$ for all $x \in X$. ■

Theorem 2.3.5. *A topological space has the same homotopy type as a point if, and only if X is contractible.*

Proof. Let $\{a\}$ be a point space, and suppose that $X \simeq \{a\}$ have the same homotopy type. Then there are maps $f : X \rightarrow \{a\}$ and $g : \{a\} \rightarrow X$ with $a \xrightarrow{g} x_0$ such that $g \circ f \simeq 1_X$ and $f \circ g \simeq 1_{\{a\}}$. Notice that $g \circ f(x) = g(a) = x_0$, for all $x \in X$, so $g \circ f$ is constant. This makes 1_X (and 1_Y) nullhomotopic. Therefore X is contractible.

On the otherhand, supposing that X is contractible, let $1_X \simeq c$ where $c : X \rightarrow X$ is the constant map defined by $x \rightarrow x_0$ for all $x \in X$. Define the maps $f : X \rightarrow \{x_0\}$ and $g : \{x_0\} \rightarrow X$ by $x \xrightarrow{f} x_0$ and $x_0 \xrightarrow{g} x_0$. Observe that $g \circ f = 1_X$, and that $f \circ g \simeq 1_{\{x_0\}}$. Therefore X is of the same homotopy type as $\{x_0\}$. ■

Remark. This theorem shows that the simplest objects in \mathbf{hTop} are the contractible spaces.

Theorem 2.3.6. *If Y is a contractible space, then any two maps $X \rightarrow Y$ are homotopic.*

Proof. Suppose that $1_Y \simeq c$ where $c : Y \rightarrow Y$ takes $y \rightarrow y_0$ for all $y \in Y$. Define $g : X \rightarrow Y$ by taking $x \rightarrow y_0$ for all $x \in X$. If $f : X \rightarrow Y$ is any continuous map, then $f \simeq g$. Consider the diagram

$$X \longrightarrow Y \begin{array}{c} \xrightarrow{k} \\ \xleftarrow{1_Y} \end{array} Y$$

Since $1_Y \simeq k$, we get that $f = 1_Y \circ f \simeq k \circ f = g$. ■

Corollary. *Any two maps $X \rightarrow Y$ are nullhomotopic.*

2.4 Path Connectedness.

Definition. A **path** in a topological space X is a continuous map $f : [0, 1] \rightarrow X$ such that $f(0) = a$ and $f(1) = b$ for some $a, b \in X$. We call a and b the **endpoints** of f , we say f goes **from a to b** .

Definition. We call a topological space X **path connected** if there exists a path from a to b for all $a, b \in X$.

Example 2.8. The sphere S^n is path connected.

Lemma 2.4.1. *If $f : X \rightarrow Y$ is a continuous map and X is a path connected space, then $f(X)$ is also path connected.*

Theorem 2.4.2. *If X is a path connected space, then X is a connected space.*

Proof. Suppose that X is disconnected. Then there exists a separation of X into disjoint open sets U and V . That is $X = U \cup V$. Suppose however that X is path connected. Then for points $a \in U$ and $b \in V$, there is a path $f : [0, 1] \rightarrow X$ from a to b . Since $[0, 1]$ is a connected space, so is $f([0, 1])$; however notice that $f([0, 1]) = (U \cap f([0, 1])) \cup (f([0, 1]) \cap V)$, which is a separation of $f([0, 1])$, since U and V form a separation. ■

Example 2.9. The converse of theorem 2.4.1 is not true in general. Consider the following two examples:

- (1) Consider the subspace $X = (0 \times [0, 1]) \cup G$ where G is the graph of $\sin \frac{1}{x}$ on the interval $(0, 2\pi]$. We have that X is connected, since the component containing G is closed, and $0 \times [0, 1] \subseteq \text{cl } G$. However, X is not path connected. We call the space X the **topologists sine curve**.
- (2) Another example of a connected space in \mathbb{R}^2 that is not path connected is the **topologist's whirlpool**.

Lemma 2.4.3. *Every contractible space is path connected.*

Lemma 2.4.4. *A topological space X is path connected if, and only if any two constant maps $X \rightarrow X$ are homotopic.*

Lemma 2.4.5. *If X is a contractible space and Y a path connected space, then any two continuous maps $X \rightarrow Y$ are homotopic.*

Corollary. *The continuous maps are nullhomotopic.*

Lemma 2.4.6. *If X and Y are path connected spaces, then so is $X \times Y$.*

Lemma 2.4.7. *If $f : X \rightarrow Y$ is a continuous map and X is a path connected space, then $f(X)$ is also path connected.*

Theorem 2.4.8. *If X is a topological space, then the relation \sim defined on X by $a \sim b$ if, and only if there is a path from a to b , is an equivalence relation.*

Proof. Consider the constant path $c : [0, 1] \rightarrow X$ where $c(x) = a$ for all $x \in A$. c is continuous, and $c(0) = c(1) = a$. So $a \sim a$.

Now suppose that for $a, b \in X$, that $a \sim b$. Then there is a path $f : [0, 1] \rightarrow X$ with $f(0) = a$ and $f(1) = b$. Consider the map $g : [0, 1] \rightarrow X$ defined by $g(t) = f(1 - t)$. g is continuous by composition, and $g(0) = f(1) = b$ and $g(1) = f(0) = a$, which makes $b \sim a$.

Lastly, suppose that $a \sim b$ and $b \sim c$ for some $a, b, c \in X$. Then there exist paths $f : [0, 1] \rightarrow X$ and $g : [0, 1] \rightarrow X$ with $f(0) = a$, $f(1) = b$, and $g(0) = b$, $g(1) = a$. Now, consider the map $h : [0, 1] \rightarrow X$ defined by:

$$h(t) = \begin{cases} f(2t), & \text{if } 0 \leq t \leq \frac{1}{2} \\ g(2t - 1), & \text{if } \frac{1}{2} \leq t \leq 1 \end{cases}$$

Notice that $f(\frac{1}{2}) = g(\frac{1}{2}) = f(1) = g(0) = b$, so the domains of f and g coincide. Therefore by the pasting lemma, h is continuous. Now, observe that $h(0) = f(0) = a$, and that $h(1) = g(1) = c$. This makes $a \sim c$. ■

Definition. We define the equivalence classes of X under path connectedness to be called **path components** of X .

Definition. We denote the collection of all path components of a topological space X to be $\pi_0(X)$; that is $\pi_0(X) = X/\sim$ (not necessarily as a quotient space). Moreover, we define the map $\pi_0(f) : \pi_0(X) \rightarrow \pi_0(Y)$ to be the map taking the path component C to the unique path component of Y containing $f(C)$.

Theorem 2.4.9. $\pi_0 : \text{Top} \rightarrow \text{Set}$ is a functor.

Proof. Consider $1_X : X \rightarrow X$ the identity on X . Let $\pi_0(X) = \{X_\alpha\}$ where X_α is a path component of X . We have that $\pi_0(1_X) : \pi_0(X) \rightarrow \pi_0(X)$ sends $X_\alpha \rightarrow X_\beta$ where X_β is the unique path component of X containing $1_X(X_\alpha) = X_\alpha$. However, since X_α and X_β are equivalence classes, we have $X_\alpha \subseteq X_\beta$ if and only if $\alpha = \beta$, i.e. $X_\alpha = X_\beta$. This makes $\pi_0(1_X) = 1_{\pi_0(X)}$.

Now let $f : X \rightarrow Y$ and $g : Y \rightarrow Z$ be continuous maps. Let $\pi_0(X) = \{X_\alpha\}$, $\pi_0(Y) = \{Y_\beta\}$, $\pi_0(Z) = \{Z_\gamma\}$ the collection of path components of X , Y , and Z , respectively. Now

consider X_α and Z_γ such that $\pi_0(g \circ f)(X_\alpha) = Z_\gamma$. Then Z_γ is the unique path component of Z containing $g(f(X_\alpha))$. Now, if Y_β is the unique path component of Y containing X_α , then $\pi_0(f)(X_\alpha) = Y_\beta$ and we see that $g(f(X_\alpha)) \subseteq g(Y_\beta)$. Moreover, if $Z_{\gamma'}$ is the unique path component of Z containing $g(Y_\beta)$, then $\pi_0(g)(Y_\beta) = Z_{\gamma'}$, and $g(Y_\beta) \subseteq Z_{\gamma'}$. But $g(f(X_\alpha)) \subseteq g(Y_\beta) \subseteq Z_{\gamma'}$; by above, and since path components partition their spaces, this makes $\gamma = \gamma'$. Thus $Z_\gamma = Z_{\gamma'}$ and we have that $g(f(X_\alpha)) \subseteq g(Y_\beta) \subseteq Z_\gamma$. Therefore Z_γ is the unique path component of Z containing both $g(f(X_\alpha))$ and $g(Y_\gamma)$; that is $\pi_0(g)(Y_\beta) = Z_\gamma$, where $\pi_0(f)(X_\alpha) = Y_\beta$. This implies that $\pi_0(g \circ f) = \pi_0(g) \circ \pi_0(f)$, which makes π_0 a functor. ■

Corollary. *If $f \simeq g$, then $\pi_0(f) = \pi_0(g)$.*

Proof. Suppose that $F : f \simeq g$ is a homotopy between the maps $f : X \rightarrow Y$ and $g : X \rightarrow Y$. Let C be a path component of X , then $C \times I$ is path connected by lemma 2.4.6. Thus by lemma 2.4.1, $F(C \times I)$ is also path connected. Notice then that:

$$f(C) = F(C \times 0) \subseteq F(C \times I)$$

and

$$g(C) = F(C \times 1) \subseteq F(C \times I)$$

So the unique path connected component of Y containing $F(C \times I)$ contains both $f(C)$ and $g(C)$. Therefore $\pi_0(f) = \pi_0(g)$. ■

Corollary. *If X and Y are topological spaces with the same homotopy type, then they have the same number of path components.*

Proof. Suppose that $f : X \rightarrow Y$ and $g : Y \rightarrow X$ are continuous maps with $g \circ f = 1_X$ and $f \circ g = 1_Y$. Since f is a homotopy equivalence, then $[f]$ is an equivalence in \mathbf{hTop} . Restricting π_0 to \mathbf{hTop} , this also gives use that $\pi_0([f])$ is an equivalence in \mathbf{Set} . That is f is 1-1 and onto. ■

Definition. A topological space X is **locally path connected** if, for each $x \in X$, and every open neighborhood U of x there is an open set V with $x \in V \subseteq U$ such that any two points in V can be joined by a path in U .

Example 2.10. Form the subspace X of \mathbb{R}^2 by adjoining a curve from $(0, 1)$ to $(\frac{1}{2\pi}, 0)$ on the topologist's sine curve. Then X is path connected, but not locally path connected.

Theorem 2.4.10. *A topological space is locally path connected if, and only if path components of open sets are open.*

Proof. Suppose that X is locally path connected, and let U be open in X . Let $x \in C$, where C is a path component of U . Then there is an open V with $x \in V \subseteq U$ such that every point of V can be joined to x by a path in U . Thus each point of V lies in the path component of x , which is C . Thus $V \subseteq C$, which makes C open.

Conversely, suppose that path components of open sets in X are open. Let U be an open set of X , and for some $x \in U$, let C be the path component of x in U . Then we have $x \in C \subseteq U$. Since C is open, this makes X locally path connected. ■

Corollary. *If X is locally path connected, then its path components are open.*

Corollary. *X is locally path connected if, and only if for every $x \in X$, and each open neighborhood U of x , there is an open path connected set V with $x \in V \subseteq U$.*

Corollary. *If X is locally path connected, then the connected components of every open set coincide with its path components. In particular the connected components of X coincide with the path components of X .*

Corollary. *If X is connected, and locally path connected, then X is connected.*

Definition. Let A be a subspace of a topological space X , and let $i : A \rightarrow X$ be the inclusion. Then A is a **deformation retract** of X if there is a continuous map $r : X \rightarrow A$ such that r is a retraction of X ; i.e. $r \circ i = 1_A$ and $i \circ r = 1_X$.

Lemma 2.4.11. *Every deformation retract is a retract.*

Theorem 2.4.12. *If A is a deformation retract of a topological space X , then X and A have the same homotopy type.*

Corollary. S^1 is a deformation retract of $\mathbb{C} \setminus 0$.

Proof. For every $z \in \mathbb{C} \setminus 0$, we can write z as $z = \rho e^{i\theta}$, where $\rho > 0$, and $0 \leq \theta \leq 2\pi$. Now, define $F : (\mathbb{C} \setminus 0) \times I \rightarrow \mathbb{C} \setminus 0$ by taking $(\rho e^{i\theta}, t) \rightarrow ((1-t)\rho + t)e^{i\theta}$. Notice that F is never 0, and that F is continuous, with $F(\rho e^{i\theta}, 0) = \rho e^{i\theta}$, $F(e^{i\theta}, 1) = e^{i\theta}$. Moreover $F(\rho e^{i\theta}, 1) = F(e^{i\theta}) = e^{i\theta}$. Writing S^1 as $S^1 = \{e^{i\theta} : 0 \leq \theta \leq 2\pi\}$. We see that F makes S^1 into a deformation retract of $\mathbb{C} \setminus 0$. ■

Corollary. S^1 has the same homotopy type as $\mathbb{C} \setminus 0$.

Definition. Let $f : X \rightarrow Y$ be a continuous map from a topological space X to a topological space Y . Define

$$M_f = (X \times I) \cup Y / \sim$$

Where $(X \times I) \cup Y$ is a disjoint union, and \sim is an equivalence relation defined by $(x, t) \sim y$ if $y = f(x)$ and $t = 1$. Denote the equivalence classes of (x, t) by $[x, t]$. We call the quotient space M_f the **mapping cylinder** of f .

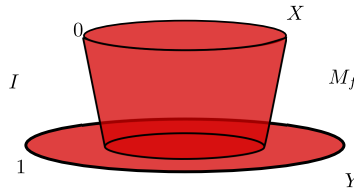


Figure 2.2: The mapping cylinder of a continuous map $f : X \rightarrow Y$.

Chapter 3

Simplexes.

3.1 Affine Spaces.

Definition. We call a subset $X \subseteq \mathbb{R}^n$ **affine** if for every $x, y \in X$, the line $l(x, y)$ passing through x and y is contained in X .

Lemma 3.1.1. *Affine sets are convex.*

Proof. Note that the line $l(x, y)$ contains the segment $l[x, y]$ which is in X for every $x, y \in X$. ■

Theorem 3.1.2. *If $\{X_\alpha\}$ is a collection of affine (or convex) sets in \mathbb{R}^n , then the intersection of all X_α is affine (or convex) in \mathbb{R}^n .*

Proof. Let $X = \bigcap X_\alpha$ and let $x, y \in X$. let $l(x, y)$ be the line passing through x and y , then $l(x, y) \in X_\alpha$ for every α , since $x, y \in X_\alpha$ which is affine. This makes $l(x, y) \in X$, which makes X affine in \mathbb{R}^n . The proof for convexity of X is the same except using the line segment $l[x, y]$. ■

Definition. An **affine combination** of points $x_0, \dots, x_m \in \mathbb{R}^n$ is a point $x \in \mathbb{R}^n$ such that

$$x = t_0x_1 + \dots + t_mx_m$$

Where $\sum t_i = 1$. A **convex combination** is an affine combination in which each $t_i \geq 0$ for $0 \leq i \leq m$.

Example 3.1. The line $tx + (1 - t)y$ is a convex combination in \mathbb{R}^n .

Definition. We say a subset $X \subseteq \mathbb{R}^n$ **spans** an affine set $[X]$ if $[X]$ is the intersection of all affine subsets containing X . Similarly, we say X **spans** a convex set $[X]$ if $[X]$ is the intersection of all convex subsets containing X . We call these the affine and convex **hulls**, respectively.

Theorem 3.1.3. *If $x_0, \dots, x_m \in \mathbb{R}^n$, then the convex hull $[x_0, \dots, x_m]$ is the set of all convex combinations of x_0, \dots, x_m .*

Proof. Let S be the set of all convex combinations of x_0, \dots, x_m , then $[x_0, \dots, x_m] \subseteq S$. Now, let $t_j = 1$ and $t_i = 0$, then $x_i \in S$ for all j . Moreover, let $\alpha = \sum a_i x_i$ and $\beta = \sum b_i x_i$ where $\sum a_i = \sum b_i = 1$. Then for $t \in [0, 1]$ we have

$$t\alpha + (1-t)\beta = t \sum a_i x_i + (1-t) \sum b_i x_i = \sum (t(a_i x_i) + (1-t)b_i x_i)$$

moreover, $t \sum a_i + (1-t) \sum b_i = 1$ and $ta_i + (1-t)b_i \geq 0$ for all $0 \leq i \leq m$, so $t\alpha + (1-t)\beta$ is a convex combination in S .

Now, let X be any convex set containing $\{x_0, \dots, x_m\}$. By induction on m , for $m = 0$, $S = \{x_0\}$. Now let $m \geq 0$ and $t_i \geq 0$ with $\sum t_i = 1$. Assume without loss of generality that $t_0 \neq 1$. Then

$$y = \left(\frac{t_1}{1-t_0}\right)x_0 + \dots + \left(\frac{t_m}{1-t_0}\right)x_m \in X$$

which makes $x = t_0 x_0 + (1-t_0)y \in X$. This makes $S \subseteq [x_0, \dots, x_m]$. ■

Definition. We call points $x_0, \dots, x_m \in \mathbb{R}^n$ **affinely independent** if $\{x_1 - x_0, \dots, x_m - x_0\}$ is linearly independent in \mathbb{R}^n as a vector space. We say the points x_0, \dots, x_m are **affinely dependent** if they are not affinely independent.

Theorem 3.1.4. *For any points $x_0, \dots, x_m \in \mathbb{R}^n$, the following are equivalent:*

- (1) x_0, \dots, x_m are affinely independent.
- (2) If $s_0, \dots, s_m \in \mathbb{R}$ such that $\sum s_i x_i = 0$ and $\sum s_i = 0$, then $s_0 = \dots = s_m = 0$.
- (3) If A is an affine set spanned by x_0, \dots, x_m , then every $x \in A$ can be written as a unique affine combination of x_0, \dots, x_m .

Proof. Suppose, that x_0, \dots, x_m are affinely independent. Let $a_0, \dots, a_m \in \mathbb{R}$ such that $\sum a_i = 0$ and $\sum a_i x_i = 0$. We see that

$$\sum a_i x_i = \sum a_i x_i - 0 \cdot x_0 = \sum a_i x_i - x_0 \sum a_i = \sum a_i (x_i - x_0) = 0$$

Now, since x_0, \dots, x_m are affinely independent, $x_1 - x_0, \dots, x_m - x_0$ are linearly independent which implies that $a_0 = \dots = a_m = 0$.

Now, suppose that if $\sum a_i = 0$ and $\sum a_i x_i = 0$, then $a_0 = \dots = a_m = 0$. Let A be an affine set spanned by x_0, \dots, x_m and suppose there is an $x \in A$ for which $x = \sum a_i x_i$ and $x = \sum b_i x_i$. Then

$$\sum a_i x_i = \sum b_i x_i$$

so that $\sum (a_i - b_i)x_i = 0$. Notice also that $\sum a_i - \sum b_i = \sum a_i - \sum b_i = 1 - 1 = 0$, so that by hypothesis, $a_i - b_i = 0$ for each i . That is $a_i = b_i$.

Finally, suppose that A is an affine set spanned by x_0, \dots, x_m for which every $x \in A$ can be written uniquely as an affine combination of these points. That is $x = \sum a_i x_i$ where $\sum a_i = 1$. Now, if $m = 0$, we get each $x = a_0 x_0$ which is trivially affinely independent. Now, suppose $m \geq 0$ and by induction on m , suppose that $x_1 - x_0, \dots, x_m - x_0$ are linearly

dependent. Then there exists $a_0, \dots, a_m \in \mathbb{R}$ not all 0 such that $\sum a_i(x_i - x_0) = 0$. Choose $r_j \neq 0$ then

$$\sum \frac{r_i}{r_j}(x_i - x_0) = 0$$

Suppose then, without loss of generality that $r_j = 1$. Then $x_j \in \{x_0, \dots, x_m\}$ which gives x_j two affine combinations:

$$\begin{aligned} x_j &= 1 \cdot x_j \\ x_j &= -\sum a_i x_i + (1 + \sum a_i)x_0 \end{aligned}$$

This contradicts that each $x \in A$ has a unique representation as an affine combination, hence $x_1 - x_0, \dots, x_m - x_0$ have to be linearly independent, making x_0, \dots, x_m affinely independent. ■

Corollary. *Given points $x_0, \dots, x_m \in \mathbb{R}^n$, affine independence on the points is independent of their ordering.*

Corollary. *If $A \subseteq \mathbb{R}^n$ is an affine set spanned by affinely independent points x_0, \dots, x_m , then it is the translation of an m -dimensional subspace V of \mathbb{R}^n as a vector space.*

Proof. Let $p_0 = x_0$ and V subspace of \mathbb{R}^n as a vector space with basis $\{x_1 - x_0, \dots, x_m - x_0\}$. If $z \in A$, then $z = \sum a_i x_i$ where $\sum a_i = 1$. Then $z = \sum a_i x_i + a_0 x_0 = \sum a_i x_i - \sum a_i x_0 + (a_0 + \sum a_i)x_0 = \sum a_i(x_i - x_0) + x_0 \in V + x_0$. By similar reasoning, if we have $z \in V + p_0$, then $z \in A$. ■

Definition. We say a set of points $a_1, \dots, a_k \in \mathbb{R}^n$ are in **general position** if every $n + 1$ of its points are affinely independent.

Theorem 3.1.5. *Given $k \geq 0$, \mathbb{R}^n contains k points in general position.*

Proof. For $0 \leq k \leq n + 1$, take the origin 0 together with any $k - 1$ elements of a basis of \mathbb{R}^n . These points are in general position.

Now, suppose that $k > n + 1$, and choose $r_1, \dots, r_k \in \mathbb{R}$ and define

$$a_i = (r_i, r_i^2, \dots, r_i^n) \text{ for } 1 \leq i \leq k$$

Suppose additionally that the points a_1, \dots, a_k are not in general position. Then $n + 1$ of the points a_{i_0}, \dots, a_{i_n} which are affinely dependent. Then $a_{i_1} - a_{i_0}, \dots, a_{i_n} - a_{i_0}$ are linearly dependent. Then there exist s_0, \dots, s_n , not all 0 such that

$$\sum s_j(a_{i_j} - a_{i_0}) = 0$$

Consider now, the $n \times n$ south east block, V^* of the $(n + 1) \times (n + 1)$ Vandermonde matrix obtained from r_{i_0}, \dots, r_{i_n} . Let $\sigma = (s_0, \dots, s_n)$, then the equation above give the matrix equation

$$V^* \sigma^T = 0 \tag{3.1}$$

Now, since V^* is nonsingular, and each of the r_{i_j} is distinct, we get that $\sigma = 0$, which contradicts our assumption that a_{i_0}, \dots, a_{i_n} are affinely independent. ■

Definition. Let $x_0, \dots, x_m \in \mathbb{R}^n$ be affinely independent and let $x \in \mathbb{R}^n$ be such that $x = \sum t_i x_i$. We call the $(m+1)$ -tuple $(t_0, x_0, \dots, t_m, x_m)$ the **barycentric coordinates** for x .

Definition. Let $x_0, \dots, x_m \in \mathbb{R}^n$ be affinely independent. We call the convex set spanned by each of these points, $[x_0, \dots, x_m]$ an m -simplex. We call the n -simplex $[e_0, \dots, e_n]$ of \mathbb{R}^{n+1} , where $\{e_0, \dots, e_n\}$ is the standard basis of \mathbb{R}^{n+1} the **standard** n -simplex, and we denote it Δ^n .

Theorem 3.1.6. *If $x_0, \dots, x_m \in \mathbb{R}^n$ are affinely independent then each $x \in [x_0, \dots, x_m]$ is the unique convex combination of barycentric coordinates.*

Proof. Note that barycentric coordinates are unique by theorem 3.1.4. ■

Definition. If $x_0, \dots, x_m \in \mathbb{R}^n$ are affinely independent, the **barycenter** of $[x_0, \dots, x_m]$ is the point $\frac{\sum x_i}{m+1}$ of $[x_0, \dots, x_m]$.

Example 3.2. (1) $[x_0]$ is a 0-simplex with x_0 as its barycenter.

(2) The 1-simplex $[x_0, x_1]$ has as its barycenter the point $\frac{x_0+x_1}{2}$, which is the midpoint of a closed line segment between x_0 and x_1 .

(3) The 2-simplex $[x_0, x_1, x_2]$ has barycenter $\frac{x_0+x_1+x_2}{3}$ which is the geometric barycenter of a triangle.

(4) Let Δ^n the standard n -simplex. Every point $x \in \Delta^n$ has the form $\sum t_i e_i$, which is represented as (t_0, \dots, t_n) in \mathbb{R}^{n+1} as a vector space. Therefore the barycentric coordinates of any point in Δ^n are precisely its cartesian coordinates.

Definition. Let $[x_0, \dots, x_m]$ be an m -simplex. We define the **face opposite** of x_i to be the set

$$[x_0, \dots, \hat{x}_i, \dots, x_m] = \left\{ \sum t_j x_j : t_j \geq 0, \sum t_j = 1, \text{ and } t_i = 0 \right\}$$

We define a **k -face** of $[x_0, \dots, x_m]$ to be a k -simplex spanned by $k+1$ vertices of $[x_0, \dots, x_m]$. We define the **boundary** of $[x_0, \dots, x_m]$ to be the union of all faces opposite x_i for all $0 \leq i \leq m$, and we write $\partial[x_0, \dots, x_m]$.

Example 3.3. (1) Note that $\partial[e_0, \dots, e_n] = \partial\Delta^n$.

(2) Given any m -simplex, it has $\binom{m+1}{k+1}$ k -faces.

Theorem 3.1.7. *Let $S = [x_0, \dots, x_n]$ be an n -simplex. The following are true*

(1) *If $u, v \in S$, then $\|u - v\| \leq \sup_i \|u - x_i\|$*

(2) $\text{diam } S = \sup_{i,j} \|x_i - x_j\|$

(3) *If b is the barycenter of S , then $\|b - x_i\| \leq \frac{n}{n+1} \text{diam } S$.*

Proof. Let $u, v \in S$, and $v = \sum t_i x_i$ where $t_i \geq 0$ and $\sum t_i = 1$. Then $\|u - v\| = \|u - \sum t_i x_i\| = \|u \sum t_i - \sum t_i x_i\| \leq \sum t_i \|u - x_i\| \leq \sum t_i \sup \|u - x_i\| = \sup \|u - x_i\|$. It also follow that the second statement is true using the properties of least upperbounds.

Now, let $b = \frac{x_0 + \dots + x_n}{n+1}$ the barycenter of S . Then

$$\begin{aligned} \|b - x_i\| &= \left\| \frac{1}{n+1} \sum x_j - x_i \right\| \\ &= \left\| \frac{1}{n+1} \sum x_j - \frac{1}{n+1} \sum x_i \right\| \\ &= \left\| \frac{1}{n+1} \sum x_j - x_i \right\| \\ &\leq \frac{1}{n+1} \sum \|x_j - x_i\| \\ &\leq \frac{n}{n+1} \sup \|x_j - x_i\| \\ &= \frac{n}{n+1} \text{diam } S \end{aligned}$$

■

3.2 Affine Maps.

Definition. Let $x_0, \dots, x_m \in \mathbb{R}^n$ be affinely independent points and let A be the affine set spanned by these points. An **affine map** is a map $T : A \rightarrow \mathbb{R}^k$, with $k \leq n$ such that

$$T\left(\sum a_i x_i\right) = \sum a_i T(x_i)$$

whenever $\sum a_i = 1$ for $a_1, \dots, a_m \in \mathbb{R}$.

Theorem 3.2.1. *If $[x_0, \dots, x_m]$ and $[y_0, \dots, y_n]$ are m and n -simplexes, and $f : \{x_0, \dots, x_m\} \Rightarrow [y_0, \dots, y_n]$ is a map, then there exists a unique affine map $T : [x_0, \dots, x_m] \rightarrow [y_0, \dots, y_n]$ such that $T(x_i) = f(x_i)$ for all $1 \leq i \leq m$.*

Chapter 4

The Fundamental Group.

4.1 The Fundamental Groupoid

Definition. Let $f : \rightarrow_X$ and $g : I \rightarrow X$ be paths with $f(1) = g(0)$. We define **path multiplication** to be the operation

$$f * g = \begin{cases} f(2t), & \text{if } 0 \leq t \leq \frac{1}{2} \\ g(2t - 1), & \text{if } \frac{1}{2} \leq t \leq 1 \end{cases}$$

We call $f * g$ the **path product** of f and g .

Lemma 4.1.1. *The path product of two paths is a continuous map.*

Proof. This follows from the pasting lemma. ■

Corollary. *The path product is a path.*

Proof. Notice that if f and g are paths with $f(1) = g(0)$, then $f * g(0) = f(0)$ and $f * g(1) = g(1)$. ■

Definition. Let X be a topological space and A a subspace of X . Let $f_0 : X \rightarrow Y$ and $f_1 : X \rightarrow Y$, are continuous maps with $f_0|_A = f_1|_A$. we say that f_0 is **homotopic** to f_1 **relative** A , and write $f_0 \simeq f_1 \text{ rel } A$ or $f_0 \simeq_A f_1$, if there is a continuous map $F : X \times I \rightarrow Y$ that defines a homotpy between f_0 and f_1 , and $F(a, t) = f_0(a) = f_1(a)$ for all $a \in A$. We call F the **relative homotopy**.

Lemma 4.1.2. *Relative homotopy is an equivalence relation.*

Definition. Let ∂I be the boundry of $I = [0, 1]$ in \mathbb{R} . The equivalence class of $f : I \rightarrow X \text{ rel } \partial I$ is called the **path class** of f , and denoted $[f]$.

Theorem 4.1.3. *Let f_0, f_1 and g_0, g_1 be paths on X with $f_0 \simeq f_1 \text{ rel } \partial I$ and $g_0 \simeq g_1 \text{ rel } \partial I$. If $f_0(1) = g_0(0)$ and $f_1(1) = g_1(0)$ then $f_0 * g_0 \simeq f_1 * g_1 \text{ rel } \partial I$.*

Proof. Let $F : f_0 \sim_{\partial I} f_1$ and $G : g_0 \sim_{\partial I} g_1$ be the relative homotopies between f_0 and f_1 , and g_0 and g_1 , respectively. Define the map $H : I \times I \rightarrow Y$ by

$$H(s, t) = \begin{cases} F(2t, s), & \text{if } 0 \leq t \leq \frac{1}{2} \\ G(2t - 1, s), & \text{if } \frac{1}{2} \leq t \leq 1 \end{cases}$$

H is continuous by the pasting lemma, moreover, $H(0, s) = F(0, s) = f_0 * g_0(s)$, and $H(1, s) = G(1, s) = f_1 * g_1(s)$. Additionally, ∂I is fixed by H . This makes H a relative homotopy. ■

Definition. Let $f : I \rightarrow X$ be a path from x_0 to x_1 on X . The **origin** of f is x_0 and we write $x_0 = \alpha(f)$, and the **end** of f is x_1 and we write $x_1 = \omega(f)$. We call f a **closed** path if $\alpha(f) = \omega(f)$. We define the maps $i_p : I \rightarrow X$ and $i_q : I \rightarrow X$ given by $i_p(t) = \alpha(f)$, and $i_q(t) = \omega(f)$ to be the **constant paths**. We define the **inverse path** of f to be the path $f(1 - t)$ and denote it f^{-1} .

Definition. A set G together with an operation $*$ is called a **groupoid** if for all $a, b \in G$ we have:

- (1) $a * (b * c) = (a * b) * c$ (Associativity)
- (2) There exist elements $e_1, e_2 \in G$ such that $a * e_1 = a$ and $e_1 * a = a$.
- (3) For ever $a \in G$, there is an element $a^{-1} \in G$ such that $a * a^{-1} = e_1$ and $a^{-1} * a = e_2$.

Theorem 4.1.4. *For any topological space, the collection of all path classes forms a groupoid under path multiplication. More precisely, if $p = \alpha[f]$ and $q = \omega[f]$ then:*

- (1) *Path multiplication is associative whenever defined.*
- (2) *$i_p * f \simeq f \text{ rel } \partial I$ and $f * i_q \simeq f \text{ rel } \partial I$.*
- (3) *$f * f^{-1} \simeq i_p \text{ rel } \partial I$ and $f^{-1} * f \simeq i_q \text{ rel } \partial I$.*

Proof. Let $[f]$ be a path with $p = \alpha[f]$ and $q = \omega[f]$. Consider the following space (see figure 4.1) of $I \times I$ together with the line joining the points $(0, 1)$ and $(\frac{1}{2}, 0)$. This line has equation $2s = t - 1s$. Now, define $\theta_t : [\frac{1-t}{2}, 1] \rightarrow [0, 1]$ to be the affine map:

$$\theta_t(s) = \frac{s - \frac{1-t}{2}}{1 - \frac{1-t}{2}}$$

Define the map $H : I \times I \rightarrow I$ by:

$$H(s, t) = \begin{cases} p, & \text{if } 2s \leq 1 - t \\ f(\theta_t(s)), & \text{if } 2s \geq 1 - t \end{cases}$$

By the pasting lemma, H is continuous. Moreover, $H(0, t) = p = \alpha(f) = i_p * f$, and $H(1, t) = f(\theta_t(1)) = f(t)$. Lastly, ∂I is fixed on H , so we have $i_p * f \simeq f \text{ rel } \partial I$. By similar reasoning, we also get $f * i_q \simeq f \text{ rel } \partial I$.

Now, consider the space of figure ?? on $I \times I$ together with slanted lines. Construct a

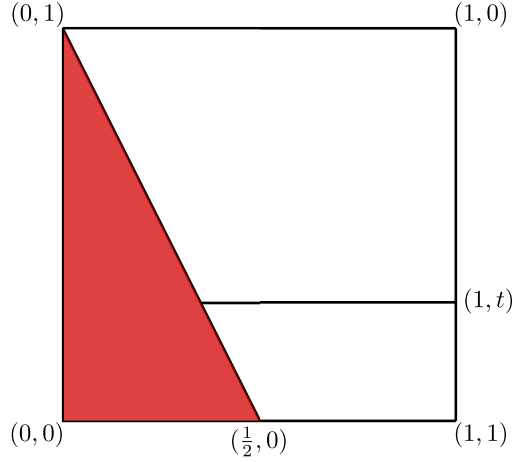


Figure 4.1:

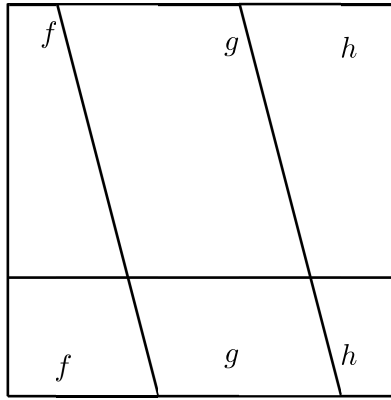


Figure 4.2:

continuous map defined by the affine map from $[0, \frac{1}{2}]$ to $[0, \frac{2-t}{4}]$. It follows that $f * (g * h) \simeq (f * g) * h \text{ rel } \partial I$.

Finally, subdivided $I \times I$, again, as in figure ?? and define the map $H : I \times I \rightarrow X$ by

$$H(s, t) = \begin{cases} f(2(s(1-t))), & \text{if } 0 \leq s \leq \frac{1}{2} \\ f(2(1-s)(1-t)), & \text{if } \frac{1}{2} \leq s \leq 1 \end{cases}$$

This map is continuous by the pasting lemma with $H(0, t) = f * f^{-1}$, and $H(1, t) = i_q(t)$. Therefore $f * f^{-1} \simeq i_p \text{ rel } \partial I$. Again by similar reasoning, we get $f^{-1} * f \simeq i_q \text{ rel } \partial I$. ■

Definition. Let X be a topological space and $x_0 \in X$. The **fundamental group** of X with **basepoint** x_0 is the collection of all path classes on X which are closed at x_0 . We denote it $\pi_1(X, x_0)$.

Theorem 4.1.5. *The fundamental group of a topological space is a group under path multiplication for every basepoint in the space.*

4.2 The π_1 Functor

Theorem 4.2.1. *The map $\pi_1 : \text{Top} \rightarrow \text{Grp}$ is a covariant functor. Moreover, if $h : (X, x_0) \rightarrow (Y, y_0)$ and $k : (X, x_0) \rightarrow (Y, y_0)$ are continuous maps such that $h \simeq k \text{ rel } x_0$, then $\pi_1 h = \pi_1 k$.*

Proof. Let $[f] \in \pi_1(X, x_0)$ and define $\pi_1 h : [f] \rightarrow [h \circ f]$. Notice that the map $h \circ f : I \rightarrow Y$ is continuous by composition and well defined. Moreover $h \circ f$ is a closed path in Y at y_0 . This shows that $\pi_1 h([f]) = [h \circ f] \in \pi_1(Y, y_0)$. Now, if $f \simeq f' \text{ rel } \partial I$, then $h \circ f \simeq h \circ f' \text{ rel } \partial I$, so that if f and g are closed paths in X , at x_0 , then $h \circ (f * g) = (h \circ f) * (h \circ g)$. This makes $\pi_1 h$ into a homomorphism. Moreover, $\pi_1 1_{(X, x_0)} = 1_{\pi_1(X, x_0)}$. This shows that π_1 is a functor, and that it is covariant.

Now, let $h \simeq k \text{ rel } \partial I$. Then $h \circ f \simeq k \circ f \text{ rel } \partial I$, where f is some closed path in X at x_0 . That is $\pi_1 h = \pi_1 k$. \blacksquare

Definition. We define the **pointed homotopy category** hTop to be the quotient category of Top^* arising from the congruence of relative homotopy.

Theorem 4.2.2. *Let $x_0 \in X$, and X_0 the path component of X containing x_0 . Then $\pi_1(X_0, x_0) \simeq \pi_1(X, x_0)$.*

Proof. Consider the inclusion $j : (X_0, x_0) \rightarrow (X, x_0)$ and let $[f] \in \ker \pi_1 j$. Then $j \circ f \simeq c \text{ rel } \partial I$ where $c : I \rightarrow X$ is the constant path from t to x_0 . Let $F : j \circ f \simeq c$ a homotopy, then $F(0, 0) = x_0$, the space $F(I \times I)$ is path connected, and $F(I \times I) \subseteq X$. It remains to show that f is nullhomotopic. Notice that $F(I \times I)$ makes $\pi_1 j$ 1-1. Now, if f is a closed path in X_0 at x_0 , then $f(I) \subseteq X_0$. Define $f' : I \rightarrow X_0$ by $f'(t) = f(t)$ for all $t \in I$. Then notice that $j \circ f' = f$, this makes $\pi_1 j$ onto. \blacksquare

Theorem 4.2.3. *If X is a path connected space, then the fundamental group of X is independent of the basepoint chosen; i.e. if $x_0, x_1 \in X$, then $\pi_1(X, x_0) \simeq \pi_1(X, x_1)$.*

Proof. Let $\gamma : I \rightarrow X$ be a path from x_0 to x_1 . Define the map $\phi : \pi_1(X, x_0) \rightarrow \pi_1(X, x_1)$ by $\phi : [f] \rightarrow [\gamma][f][\gamma^{-1}]$. By theorem 4.1.4, ϕ is an isomorphism. \blacksquare

Definition. Let H and K be sets. We define the **projections** $p : H \times K \rightarrow H$ and $q : H \times K \rightarrow K$ defined by $p : h \times k \rightarrow h$ and $q : h \times k \rightarrow k$. Additionally, for some set L , there exist maps $\alpha : L \rightarrow H$ and $\beta : L \rightarrow K$ such that the map $\alpha \times \beta : L \rightarrow H \times K$ satisfies $p \circ (\alpha \times \beta) = \alpha$ and $q \circ (\alpha \times \beta) = \beta$.

$$\begin{array}{ccccc}
 & & H \times K & & \\
 & \swarrow p & \uparrow \alpha \times \beta & \searrow q & \\
 H & & & & K \\
 & \nwarrow \alpha & \downarrow & \nearrow \beta & \\
 & & L & &
 \end{array}$$

Theorem 4.2.4. *For pointed spaces (X, x_0) and (Y, y_0) , we have that $\pi_1(X \times Y, x_0 \times y_0) \simeq \pi_1(X, x_0) \times \pi_1(Y, y_0)$.*

Proof. Let p and q the projections of $(X \times Y, x_0 \times y_0)$ onto (X, x_0) and (Y, y_0) , respectively. Then $\pi_1 p \times \pi_1 q : \pi_1(X \times Y, x_0 \times y_0) \rightarrow \pi_1(X, x_0) \times \pi_1(Y, y_0)$ is a homomorphism. If f is a closed path, in $X \times Y$ at $x_0 \times y_0$, then $\pi_1 p \times \pi_1 q([f]) = [pf] \times [qf]$. Now, let g be a closed path in X at x_0 and h a closed path in Y at y_0 and define $\theta : [g] \times [h] \rightarrow [g \times h]$. Then θ is well defined and is the incerse of $\pi_1 p \times \pi_1 q$. ■

Lemma 4.2.5. *Suppose that $F : \phi_0 \simeq \phi_1$ is a free homotopy, where $\phi_0 : X \rightarrow Y$ and $\phi_1 : X \rightarrow Y$ are continuous. Choose an $x_0 \in X$ and let $\lambda = F(x_0, t)$ the path in Y from $\phi_0(x_0)$ to $\phi_1(x_0)$. Then there is a commutative diagram*

$$\begin{array}{ccc} \pi_1(X, x_0) & \xrightarrow{\pi_1 \phi_1} & \pi_1(Y, \phi_1(x_0)) \\ & \searrow \pi_1 \phi_0 & \downarrow \psi \\ & & \pi_1(Y, \phi_0(x_0)) \end{array}$$

where $\psi : [g] \rightarrow [\lambda * g * \lambda^{-1}]$.

Proof. Let $f : I \rightarrow X$ a closed path at X and define $G : I \times I \rightarrow Y$ by $t \times s \rightarrow F(f(s), t)$. Then $G : \phi_0 \circ f \simeq \phi_1 \circ f$. Consider the triangulations of the box $I \times I$ and define a continuous

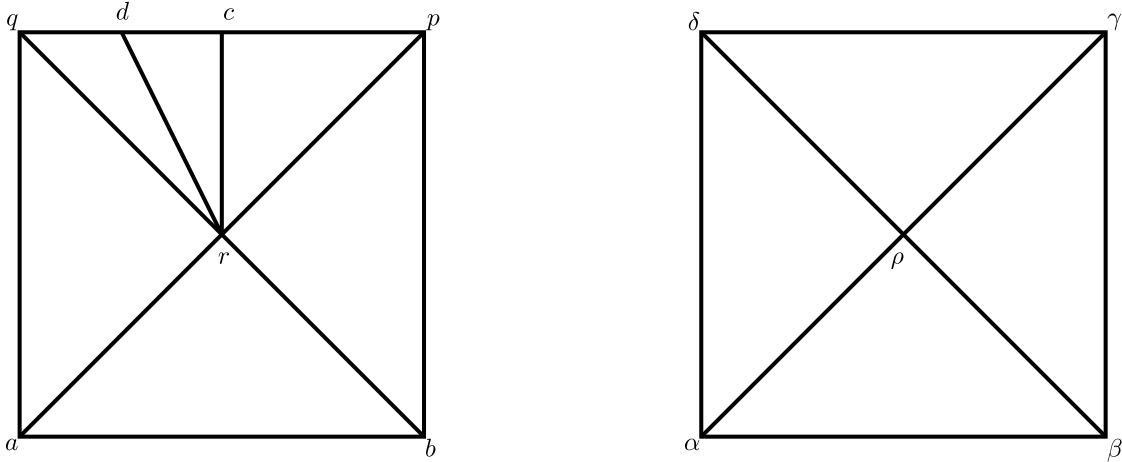


Figure 4.3:

map $H : I \times I \rightarrow I \times I$ by defining it in each triangle and applying the pasting lemma. Let

$$\begin{aligned} H(a) &= H(q) = \alpha \\ H(b) &= H(p) = \beta \\ H(c) &= \gamma \\ H(d) &= \delta \\ H(r) &= \rho \end{aligned}$$

Then the map $J = G \circ H$ is a relative homotopy with $J : \phi_0 \circ f \simeq \lambda * (\phi_1 \circ f) * \lambda^{-1} \text{ rel } \partial I$. So $\pi_1 \phi_0([f]) = [\phi_0 \circ f] = [\lambda * (\phi_1 \circ f) * \lambda^{-1}]$ and $\psi \circ \pi_1 \phi_1([f]) = \psi([\phi_1 \circ f]) = [\lambda * (\phi_1 \circ f) * \lambda^{-1}]$ as desired. ■

Corollary. *If ϕ_0 and ϕ_1 are homotopic, then*

(1) $\pi_1 \phi_0$ and $\pi_1 \phi_1$ are conjugate.

(2) If $\pi_1(X, y_0)$ is Abelian, then $\pi_1 \phi_0 = \pi_1 \phi_1$.

Theorem 4.2.6. *If $\beta : X \rightarrow Y$ is a homotopy equivalence, then it induces homomorphism $\pi_1 \beta$ is an isomorphism for every $x_0 \in X$.*

Proof. Let $\alpha : Y \rightarrow X$ a continuous map with $\alpha \circ \beta = 1_X$ and $\beta \circ \alpha = 1_Y$. Then the lower triangle of the diagram

$$\begin{array}{ccccc}
 & & \pi_1(Y, \beta(x_0)) & & \\
 & \nearrow \pi_1 \beta & & \searrow \pi_1 \alpha & \\
 \pi_1(X, x_0) & \xrightarrow{\pi_1(\alpha \circ \beta)} & & \xrightarrow{\quad} & \pi_1(X, \alpha \circ \beta(x_0)) \\
 & \searrow 1 & & \swarrow \psi & \\
 & & \pi_1(X, x_0) & &
 \end{array}$$

commutes. Since ψ , defined in lemma 4.2.5 is an isomorphism, then so is $\pi_1(\alpha \circ \beta)$. Moreover, the top triangle also commutes since π_1 is a functor, and $\pi_1(\alpha \circ \beta) = \pi_1 \alpha \circ \pi_1 \beta$, so that $\pi_1 \beta$ is 1-1 and $\pi_1 \alpha$ is onto. A similar diagram arising from $\beta \circ \alpha = 1_Y$ shows that $\pi_1 \beta$ is onto. ■

Corollary. *If X and Y are path connected with the same homotopy type, then for any $x_0 \in X$, $y_0 \in Y$, $\pi_1(X, x_0) \simeq \pi_1(Y, y_0)$.*

Corollary. *If X is a contractible space, then for any $x_0 \in X$, $\pi_1(X, x_0) \simeq \langle 1 \rangle$, the trivial group.*

Corollary. *If $\beta : (X, x_0) \rightarrow (Y, y_0)$ is freely nullhomotopic, then the induced isomorphism $\pi_1 \beta$ is trivial.*

Proof. Let $k : X \rightarrow Y$ the constant map at y . Then $\pi_1 k$ is trivial since $\pi_1 k([f]) = [k \circ f]$ and $k \circ f$ is constant. Then if $\beta \simeq k$ by lemma 4.2.5, there is an isomorphism ψ with $\psi \circ \pi_1 \beta = \pi_1 k$, so that $\pi_1 \beta = \psi^{-1} \pi_1 k$, which makes $\pi_1 \beta$ trivial. ■

Definition. We call a topological space **simply connected** if it is path connected with trivial fundamental group.

4.3 The Fundamental Group of S^1 .

Lemma 4.3.1. *If $\pi_1(X, y_0) \neq \langle 1 \rangle$ for some pointed space (Y, y_0) , then $\pi_1(S^1, 1) \neq \langle 1 \rangle$.*

Proof. We defer the proof. ■

Lemma 4.3.2. *Let X be a compact convex subset of \mathbb{R}^k , and $f : (X, x_0) \rightarrow (S^1, 1)$ a continuous map. For $t_0 \in \mathbb{Z}$, there exists a unique continuous map $\tilde{f} : (X, x_0) \rightarrow (\mathbb{R}, t_0)$ with $\exp \tilde{f} = f$, where $\exp t = e^{2i\pi t}$*

$$\begin{array}{ccc}
 (\mathbb{R}, t_0) & & \\
 \uparrow \tilde{f} & \searrow \exp & \\
 (X, x_0) & \xrightarrow{f} & (S^1, 1)
 \end{array}$$

Proof. Since X is compact and metrizable under the norm $\|\cdot\|$, we have that f is uniformly continuous. That is, there is an $\varepsilon > 0$ for which $\|f(x) - f(y)\| < 2$ whenever $\|x - y\| < \varepsilon$, and where $2 = \text{diam } S^1$ so that $f(x)$ and $f(y)$ are not antipodal. Now, since X is compact and metrizable, it is also bounded. So there exists an $n > 0$ with

$$\frac{\|x - x_0\|}{n} < \varepsilon \text{ for all } x_0 \in X$$

Now, partition the line segment $l(x, x_0) \in X$ with endpoints x and x_0 into intervals of equal length according to the partition $\{x_0, x_1, \dots, x_n = x\}$. Then $\|x_j - x_{j+1}\| = \frac{\|x - x_0\|}{n} < \varepsilon$, hence $\frac{f(x_{j+1})}{f(x_j)} \neq -1$, i.e. $f(x_j)$ and $f(x_{j+1})$ are not antipodal for all j . Now, define $g_j : X \rightarrow S^1 \setminus \{-1\}$ for all $0 \leq j \leq n-1$ by the rule

$$g_j(x) = \frac{f(x_{j+1})}{f(x_j)}$$

Then g_j is continuous, and $g_j(x_0) = 1$ for all j . Now, considering S^1 as a subspace of \mathbb{C} , since it is a multiplicative group, there exists a product

$$f(x) = f(x_0) \frac{f(x_1)}{f(x_0)} \frac{f(x_2)}{f(x_1)} \cdots \frac{f(x_n)}{f(x_{n-1})} = f(x_0) g_0(x) \cdots g_{n-1}(x)$$

Consider now the restriction of \exp to $(-\frac{1}{2}, \frac{1}{2})$, for which it is a homeomorphism. Let $\lambda : S^1 \setminus \{-1\} \rightarrow (-\frac{1}{2}, \frac{1}{2})$ the inverse map of \exp . Then $\lambda(1) = 0$, moreover, $g_j(X) \subseteq S^1 \setminus \{-1\}$ and $\lambda \circ g_j$ is defined and continuous. Thus, define the map $\tilde{f} : X \rightarrow \mathbb{R}$ by

$$\tilde{f} = t_0 + \lambda(g_0(x)) + \cdots + \lambda(g_{n-1}(x))$$

Then \tilde{f} is continuous with $\tilde{f}(x_0) = t_0$. Moreover, since \exp is a homomorphism of groups

$$\exp \tilde{f} = (\exp t_0) g_0(x) \cdots g_{n-1}(x) = f$$

Now, let $\tilde{g} : X \rightarrow \mathbb{R}$ a continuous map with $\exp g = f$, and $g(x_0) = t_0$. Define $h : X \rightarrow \mathbb{R}$ by $h = \tilde{f} - \tilde{g}$. Then h is continuous, and $\exp h = \exp \tilde{f} - \tilde{g} = 1$, that is

$$\frac{\exp \tilde{f}}{\exp \tilde{g}} = 1$$

so that $\exp \tilde{f} = \exp \tilde{g}$. Taking λ on both sides we get $\tilde{f} = \tilde{g}$ which establishes the uniqueness. ■

Corollary. *Let $f : (I, \partial I) \rightarrow (S^1, 1)$ be a continuous map. Then there exists a unique $\tilde{f} : I \rightarrow \mathbb{R}$ with $\exp \tilde{f} = f$ and $\tilde{f}(0) = 0$. Moreover, if $g : (I, \partial I) \rightarrow (S^1, 1)$ is a map for which $f \simeq g \text{ rel } \partial I$, then $\tilde{f} \simeq \tilde{g} \text{ rel } \partial I$.*

Proof. Notice that $I \times I$ is compact and convex. Choose 0×0 as the basepoint, and let $F : I \times I \rightarrow S^1$ be the relative homotopy $F : f \simeq g \text{ rel } \partial I$. By the above theorem there exists a map $\tilde{F} : I \times I \rightarrow \mathbb{R}$ with $\exp \tilde{F} = F$ and $\tilde{F}(0, 0) = 0$. Then it can be shown that $\tilde{F} : \tilde{f} \simeq \tilde{g} \text{ rel } \partial I$ (hint let $\phi_i(s) = \tilde{F}(s, i)$ and $\theta_i(t) = \tilde{F}(i, t)$ for $i \in \mathbb{Z}/2\mathbb{Z}$). ■

Definition. Let X be a compact convex subset of \mathbb{R}^k , and $f : (X, x_0) \rightarrow (S^1, 1)$ a continuous map. For $t_0 \in \mathbb{Z}$, we call the unique map $\tilde{f} : X \rightarrow \mathbb{R}$ such that $\exp \tilde{f} = f$ and $\tilde{f}(x_0) = t_0$ the **lifting** of f into \mathbb{R} .

Example 4.1. One example of a lifting is the winding number. Consider $S^1 = \{z \in \mathbb{C} : \|z\| = 1\}$ as a subspace of \mathbb{C} . We define the **winding number** of a parametrization $f : (I, \partial I) \rightarrow (S^1, 1)$ of S^1 to be

$$W(f) = \frac{1}{2i\pi} \oint_f \frac{dz}{z}$$

One can lift f to the map \tilde{f} and compute $W(f)$ as a Riemann (or Lebesgue) integral. Take $z = f(t) = \exp \tilde{f}(t)$ and $dz = z 2i\pi \tilde{f}'(t) dt$. Then

$$W(f) = \frac{1}{2i\pi} \oint_f \frac{dz}{z} = \int_0^1 \tilde{f}'(t) dt = \tilde{f}(1) - \tilde{f}(0)$$

Definition. If $f : (I, \partial I) \rightarrow (S^1, 1)$ is continuous, we define the **degree** of f to be $\deg f = \tilde{f}(1)$, where \tilde{f} is the unique lifting of f , and $\tilde{f}(0) = 0$.

Theorem 4.3.3. $\pi_1(S^1, 1) \simeq \mathbb{Z}$, in particular, $\deg f * g = \deg f + \deg g$.

Proof. Define $d : \pi_1(S^1, 1) \rightarrow \mathbb{Z}$ by $[f] \rightarrow \deg f$. By lemma 4.3.2, d is well defined. Moreover, notice that the map $f : z \rightarrow z^m$ has degree $\deg f = m$, for all $m \in \mathbb{Z}$, so that d is onto.

Now, suppose that $\deg f = 0$, where f is a closed path in S^1 at 1, then $\tilde{f}(1) = 0$ so that \tilde{f} is a closed path in \mathbb{R} at 0. Notice that \exp induces a homomorphism $\pi_1(\mathbb{R}, 0) \rightarrow \pi_1(S^1, 1)$ by $[\tilde{f}] \rightarrow [\exp \tilde{f}] = [f]$. Since \mathbb{R} is contractible, then $\pi_1(\mathbb{R}, 0) = \langle 1 \rangle$ so that $[\tilde{f}] = [f] = 1$ is the identity map.

Now, suppose that f and g are closed paths in S^1 at 1, with degrees $\deg f = m$ and $\deg g = n$. Let \tilde{g} be the lifting of g with $\tilde{g}(0) = 0$, and define $\tilde{\gamma}(t) = m + \tilde{g}(t)$. Then γ is a

path in \mathbb{R} from m to $m+n$. Let \tilde{f} the lifting of f with $\tilde{f}(0) = 0$. Then $\tilde{f} * \tilde{\gamma}$ is a path in \mathbb{R} from 0 to $m+n$ where

$$\exp(\tilde{f} * \tilde{\gamma}) = \begin{cases} \exp \tilde{f}(2t) & \text{if } 0 \leq t \leq \frac{1}{2} \\ \exp \tilde{g}(2t-1) & \text{if } \frac{1}{2} \leq t \leq 1 \end{cases}$$

since \tilde{f} is a lifting of f , $\exp \tilde{f} = f$, and $\exp \tilde{\gamma} = e^{2i\pi m} \exp \tilde{g} = g$. Since $m \in \mathbb{Z}$ and \tilde{g} is a lifting of g we get $\exp \tilde{f} * \tilde{\gamma} = f * g$, thus $\deg f * g = \deg f + \deg g = m + n$. Moreover notice that $\ker d = \langle 1 \rangle$, and so we are done. ■

Corollary. S^1 is not simply connected.

Corollary. $\pi_1(S^1, t) = \pi_1 S^1$.

Proof. Notice that since S^1 is path connected, its fundamental group is independent of any base point. ■

Corollary. Two closed paths in S^1 are homotopic relative ∂I if, and only if they have the same degree.

Theorem 4.3.4 (The Fundamental Theorem of Algebra). *Every nonconstant polynomial with coefficients in \mathbb{C} as at least one root in \mathbb{C} .*

Proof. Define $\Sigma_\rho = \{z \in \mathbb{C} : \|z\| = \rho\}$ the circle of radius ρ centered at the point $0 + i_0$ in \mathbb{C} . For $n \geq 0$ define $f_\rho^n : \Sigma_\rho \rightarrow \mathbb{C} \setminus \{0\}$ the restriction to Σ_ρ of the map $z \rightarrow z^n$. Consider then the composition $h : S^1 \xrightarrow{\rho z} \Sigma_\rho \xrightarrow{z^n} \mathbb{C} \setminus \{0\} \xrightarrow{\|z\|} S^1$. Then $h(z) = z^n$. Suppose that f_ρ^n is freely nullhomotopic, then so is h , and hence $\pi_1 h : \pi_1 S^1 \rightarrow \pi_1 S^1$ is trivial. In particular, $\pi_1 h([\exp]) = [h \circ \exp] = [\exp^n]$ is trivial. That is \exp^n is nullhomotopic relative ∂I , so $\deg \exp^n = 0$, but $\deg \exp^n = n > 0$, which is a contradiction. Thus f_ρ^n is not nullhomotopic.

Now, consider the polynomial

$$g(z) = z^n + a_{n-1}z^{n-1} + \cdots + a_1z + a_0 \in \mathbb{C}[z]$$

and choose $\rho > \max\{\sum_{i=0}^{n-1} \|a_i\|\}$ and define the map $F : \Sigma_\rho \rightarrow \mathbb{C}$ by

$$F(z, t) = z^n + \sum_{i=0}^{n-1} (1-t)a_i z^i$$

Then $F : g_{\Sigma_\rho} \text{ rel } f_\rho^n$. Suppose that $F(I \times I) \not\subseteq \mathbb{C} \setminus \{0\}$, then $F(z, t) = 0$ for some $t \in I$, and $z \in \Sigma_\rho$. Then

$$z^n = - \sum_{i=0}^{n-1} (1-t)a_i z^i$$

by the triangle inequality, we have

$$\rho^n \leq \sum_{i=0}^{n-1} (1-t)\|a_i\|\rho^i \leq \rho^{n-1} \left(\sum_{i=0}^{n-1} \|a_i\| \right)$$

For $\rho > 1$, we get $\rho^i \leq \rho^{n-1}$, and we get $\rho \leq \sum_{i=0}^{n-1} \|a_i\|$, which contradicts our choice of ρ .

Suppose now that g has no complex roots. Define $G : \Sigma_\rho \times I \rightarrow \mathbb{C} \setminus \{0\}$ by $G(z, t) = g((-1t)z)$. Notice that $G : g|_{\Sigma_\rho} \simeq k$, where k is the constant map at a_0 . Thus $g|_{\Sigma_\rho}$ is nullhomotopic, which makes f_ρ^n null homotopic. A contradiction. Therefore g must have at least one root in \mathbb{C} ■

Chapter 5

Singular Homology

Bibliography

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