Algebraic Topology

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Chapter 1

Categories.

1.1 Categories and Subcategories.

Definition. A **category** $\mathcal{C}$ is a collection of a class of **objects**, denoted $\text{obj}\,\mathcal{C}$ a collection of sets of **morphisms** $\text{Hom}\,(A, B)$ for each $A, B \in \text{obj}\,\mathcal{C}$ and a binary operation $\circ : \text{Hom}\,(A, B) \times \text{Hom}\,(B, C) \to \text{Hom}\,(A, C)$, defined by $(f, g) \to g \circ f$, called **composition** such that:

(1) Each $\text{Hom}\,(A, B)$ is pairwise disjoint for all $A, B \in \text{obj}\,\mathcal{C}$.

(2) $\circ$ is associative when defined; that is if either $(g \circ f) \circ h$ or $g \circ (f \circ h)$ are defined, then $(g \circ f) \circ h = g \circ (f \circ h)$, for morphisms f, g, h.

(3) For each $A \in \text{obj}\,\mathcal{C}$, there exists an **identity** morphism $1_A \in \text{Hom}\,(A, A)$ such that for each $B, C \in \text{obj}\,\mathcal{C}$, $1_A \circ f = f$ and $g \circ 1_A = g$ for each morphism $f \in \text{Hom}\,(B, A)$ and $g \in \text{Hom}\,(A, C)$.

We denote morphisms by $f : A \to B$ instead of $f \in (A, B)$.

Definition. Let $\mathcal{C}$ be a category and $f : A \to B$ a morphism in $\mathcal{C}$. We call A and B the **domain** and **codomain** of f, respectively, and we call the set $G_f = \{(a, f(a)) : a \in A\} \subseteq B$ the **graph** of f.

Example 1.1. (1) The category of all sets Set has as onjects the class of all sets. The morphisms in Set are all functions $f : A \to B$ where A and B are sets. The composition of Set is the usual composition of functions.

(2) The category of all topological spaces Top has as objects all topological spaces, and as morphisms all continuous maps $f : Y \to Y$ from a space X to a space Y. The composition is the usual composition.

(3) The category of all groups, Grp has as objects all groups and as morphisms all homomorphisms $f : G \to H$, under the usual composition.

(4) The category of rings with unit Rng has as objects all rings with unit, along with all ring homomorphisms $f : R \to K$ to be the morphisms under the usual composition.

Definition. We call a category a **subcategory** of a category $\mathcal{C}$ if $\operatorname{obj}\mathcal{A} \subseteq \operatorname{obj}\mathcal{C}$,$\operatorname{Hom}_{\mathcal{A}}(A,B) \subseteq \operatorname{Hom}_{\mathcal{C}}(A,B)$ for all $A, B \in \mathcal{C}$, and $\mathcal{A}$ inherits the composition of $\mathcal{C}$.

Example 1.2. (1) The category of all pairs of topological spaces (X, A) where A is a subspace of X, whose morphisms are pairs of continuous maps $f = (f_1, f_2)$ such that $f_1 i = j f_2$ where $i : A \to X$ and $j : B \to Y$ are inclusions, is a subcategory of Top. We denote this category Top^2.

$$\begin{tikzcd}
A \arrow[r, hook, "i"] \arrow[d, "f_2"'] & X \arrow[d, "f_1"] \\
B \arrow[r, hook, "j"'] & Y
\end{tikzcd}$$

(2) The category of all **pointed spaces**, Top^* is defined with the objects being all pairs $(X, \{x_0\})$, where $x_0 \in X$ with the morphisms of Top^2. Top^* is a subcategory of Top^2. We call x_0 the **base point**, and we call the morphisms of Top^* **pointed maps**.

(2) The category of all Abelian groups, Ab is a subcategory of Grp. Likewise, the category of all commutative rings with unit is a subcategory of Rng.

1.2 Commutative Diagrams and Congruences.

Definition. A **diagram** in a category is a directed graph with its vertex set a subset of objects, and whose edge set are morphisms between those objects. We call a diagram **commutative** if for any pair of objects, every pair of morphisms between those objects are equal. That is if A, A' and B, B' are pairs of objects with pairs of morphisms $f : A \to B$, $f : A \to A'$ and $f' : A' \to B'$, $g' : B \to B'$ we have that $g \circ f' = f \circ g'$

$$\begin{tikzcd}
A \arrow[r, "g"] \arrow[d, "f"'] & A' \arrow[d, "f'"] \\
B \arrow[r, "g"'] & B'
\end{tikzcd}$$

Definition. A **congruence** on a category $\mathcal{C}$ is an equivalence relation $\sim$ on morphisms in $\mathcal{C}$ such that:

(1) If $f \in \operatorname{Hom}(A, B)$, and $f \sim f'$, then $f' \in \operatorname{Hom}(A, B)$.

(2) If $f \sim g$ and $f' \sim g'$, then $g \circ f \sim g' \circ f'$.

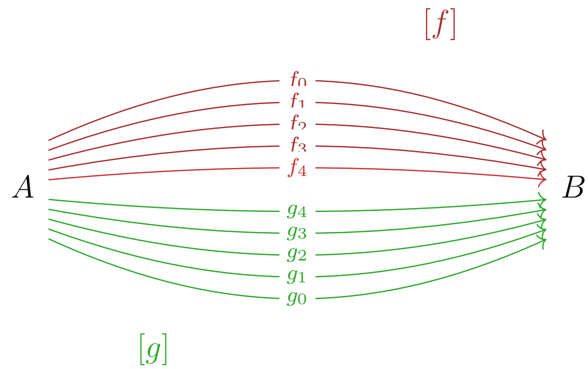

Figure 1.1: An equivalence relation between morphisms.

Theorem 1.2.1. *Let $\mathcal{C}$ be a category with congruence $\sim$. Define ${}^{\mathcal{C}}\!/\!_{\sim}$ as follows:*

(1) $\operatorname{obj} {}^{\mathcal{C}}\!/\!_{\sim} = \operatorname{obj} \mathcal{C}$.

(2) $\operatorname{Hom}_{{}^{\mathcal{C}}\!/\!_{\sim}}(A, B) = \{[f] : f \in \operatorname{Hom}_{\mathcal{C}}(A, B)\}$.

(3) $[g] \circ [f] = [g \circ f]$

Then ${}^{\mathcal{C}}\!/\!_{\sim}$ is a category.

Proof. We have by equivalence that $\operatorname{obj} {}^{\mathcal{C}}\!/\!_{\sim}$ is a class. Moreover, since $\sim$ partitions $\mathcal{C}$, it partions all of the $\operatorname{Hom}(A, B)$ for each A, B. So each $\operatorname{Hom}(A, B)$ is a set, moreover, they are pariwise disjoint by definition of $\sim$. Now, notice that by hypothesis, composition in ${}^{\mathcal{C}}\!/\!_{\sim}$ is well defined, so $[1_A] \circ [f] = [1_A \circ f] = [f]$ and $[g] \circ [1_A] = [g \circ 1_A] = [g]$. This makes ${}^{\mathcal{C}}\!/\!_{\sim}$ a category. ∎

Remark. On can think of the cateegory ${}^{\mathcal{C}}\!/\!_{\sim}$ as taking all morphisms with they same domain and codomain, and collapsing them into a single morphism.

Definition. Let $\mathcal{C}$ be a catogory and $\sim$ a congruence of $\mathcal{C}$. We call the category ${}^{\mathcal{C}}\!/\!_{\sim}$ induced by $\sim$ the **quotient category**.

1.3 Functors.

Definition. Let $\mathcal{A}$ and $\mathcal{C}$ be categories. We deine a **covariant functor** to be a map $F : \mathcal{A} \to \mathcal{C}$ such that:

(1) $A \in \operatorname{obj} \mathcal{A}$ implies $F(A) \in \operatorname{obj} \mathcal{C}$.

(2) If $f : A \to B$ is a morphism in $\mathcal{A}$, then $F(f) : F(A) \to F(B)$ is a morphism in $\mathcal{C}$.

$$
\begin{tikzcd}
A \arrow[r, bend left, "{[f]}"] \arrow[r, bend right, "{[g]}"'] & B
\end{tikzcd}
$$

Figure 1.2: Morphisms in a category with the same domain and codomain get collapsed onto a single morphism in the correspinding quotient category.

$$
\begin{tikzcd}
A & A \arrow[l, "1_A"'] \arrow[r, "g"] & B & & F(A) & F(A) \arrow[l, "1_{F(A)}"'] \arrow[r, "F(g)"] & F(B) \\
 & & {} \arrow[rr, Rightarrow, bend left, "F"'] & & {} & & \\
 & C \arrow[uu, "f"] \arrow[uur, "g\circ f"'] & & & & F(C) \arrow[uu, "GFf)"] \arrow[uur, "F(g\circ f)"'] &
\end{tikzcd}
$$

Figure 1.3: A covariant functor taking a diagram in on category to a diagram in the other.

(3) For all morphisms f and g in $\mathcal{A}$, for which $g \circ f$ is defined, we have that $F(g \circ f) = F(g) \circ F(f)$, and $F(1_A) = 1_{F(A)}$.

Example 1.3. (1) We define the **forgetful functor** the map $F : \mathcal{C} \to \mathrm{Set}$ that takes all objects in $\mathcal{C}$ to their underlying sets, and morphisms in $\mathcal{C}$ to themselves considered as functions under the usual composition. For example the forgetful functor $F : \mathrm{Top} \to \mathrm{Set}$ takes topological spaces X to their underlying sets, and continuous maps to themselves, considered just as functions.

(2) The **identity functor** is the functor $I : \mathcal{C} \to \mathcal{C}$ that takes objects and morphisms in $\mathcal{C}$ to themselves.

(3) Let M be a topological space. Define $F_M : \mathrm{Top} \to \mathrm{Top}$ by $F_M : X \to X \times M$, and for each continuous map $f : X \to Y$, $F(f) : X \times M \to Y \times M$ is defined by $(x, m) \to (f(x), m)$. Then F_M is a functor.

(4) Let $A \in \operatorname{obj} \mathcal{C}$ and take the map $\operatorname{Hom}(A, *) : \mathcal{C} \to \mathrm{Set}$ that takes $A \to \operatorname{Hom}(A, B)$ and for each morphism $f : B \to B'$, $\operatorname{Hom}(A, f) : \operatorname{Hom}(A, B) \to \operatorname{Hom}(A, B')$ is given by $g \to f \circ g$. With call this functor the **covariant Hom functor**, and denote it f_*.

Definition. Let $\mathcal{A}$ and $\mathcal{C}$ be categories. We deine a **contravariant functor** to be a map $G : \mathcal{A} \to \mathcal{C}$ such that:

(1) $A \in \operatorname{obj} \mathcal{A}$ implies $G(A) \in \operatorname{obj} \mathcal{C}$.

(2) If $f : A \to B$ is a morphism in $\mathcal{A}$, then $G(f) : G(B) \to G(A)$ is a morphism in $\mathcal{C}$.

(3) For all morphisms f and g in $\mathcal{A}$, for which $g \circ f$ is defined, we have that $G(g \circ f) = G(f) \circ G(g)$, and $G(1_A) = 1_{G(A)}$.

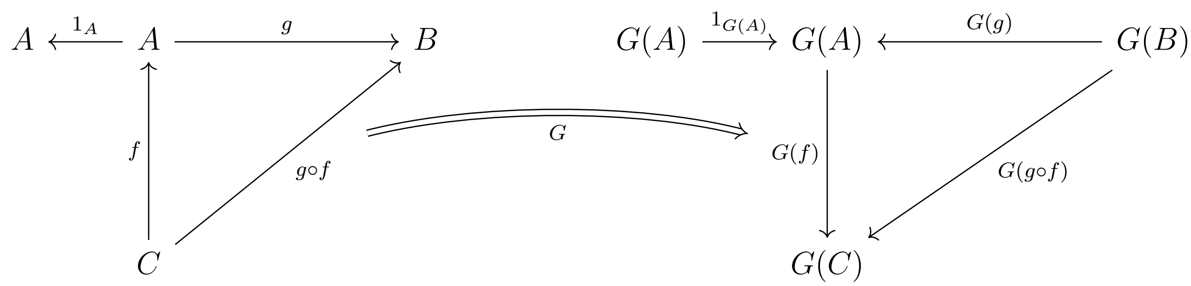

Figure 1.4: A contravariant functor taking a diagram in on category to a diagram in the other.

Example 1.4. (1) Let F be a field, and Vec the category of all finite dimensional vector spaces over F, whose morphisms are linear transformations. Define the map $T : \text{Vec} \to \text{Vec}$ by taking $T : V \to V^{\perp}$, and $T : f \to f^T$. That is T takes vector spaces to their dual spaces, and linear transformation to their transpose. T is a contravariant functor called the **dual space functor.**

(2) Define $\text{Hom}\,(*, B) : \mathcal{C} \to \mathcal{C}$ by taking $\text{Hom}\,(*, B) : A \to \text{Hom}\,(A, B)$ and for each morphism $g : A \to A'$ in $\mathcal{C}$, $\text{Hom}\,(f, B) : \text{Hom}\,(A', B) \to \text{Hom}\,(A, B)$ is defined by taking $h \to h \circ g$. This is analogous to the covariant Hom functor, and we call it the **contravariant Hom functor.**

Definition. We call a morphism $f : A \to B$ an **equivalence** if there exists a morphism $g : B \to A$ such taht $f \circ g = 1_B$ and $g \circ f = 1_A$

Theorem 1.3.1. *Let $\mathcal{A}$ and $\mathcal{C}$ be categories, and $F : \mathcal{A} \to \mathcal{C}$ be a functor. If f is an equivalence in $\mathcal{A}$, then $F(f)$ is an equivalence in $\mathcal{C}$.*

Proof. Suppose that F is a covariant functor. Notice that if $f : A \to B$ is an equivalence, then there is a $g : B \to A$ with $f \circ g = 1_B$ and $g \circ f = 1_A$. Then $F(f \circ g) = F(f) \circ F(g) = F(1_B) = 1_{F(B)}$, and $F(g \circ f) = F(g) \circ F(f) = F(1_A) = 1_{F(A)}$.

Likewise, if F is contravariant, notice that $F(f) : B \to A$ and $F(g) : A \to B$. Then $F(f \circ g) = F(g) \circ F(f) = 1_{F(A)}$, and $F(g \circ f) = F(f) \circ F(g) = 1_{F(B)}$. In eithe case, we find that $F(f)$ is an equivalence in $\mathcal{C}$. ∎

Chapter 2

Homotopy, Convexity, and Connectedness.

2.1 Homotopy

Definition. If X and Y are topological spaces, and $f_0 : X \to Y$ and $f_1 : X \to Y$ are continuous maps, we say that f_0 is **homotopic** to f_1 if there exists a continuous map $F : X \times I \to Y$ with $F(x,0) = f_0(x)$ and $F(x,1) = f_1(x)$. We write $f_0 \simeq f_1$ and call F a **homotopy**. We also write $F : f_0 \simeq f_1$ to denote a homotopy between f_0 and f_1.

Lemma 2.1.1 (The Pasting Lemma)**.** *Let X is a topolgoical space that is covered by open sets $\{X_n\}$. If Y is some topological space for which there exist unique maps $f_n : X_n \to Y$ that coincide in the intersections of their domains, then there exists a unique map $f : X \to Y$ such that $f|_{X_n} = f_n$, for all n.*

Lemma 2.1.2. *Homotopy between continuous maps is an equivalence relation.*

Proof. Let $f : X \to Y$ be a continouos map. Define $F : X \times I \rangle Y$ by $(x,t) \to f(x)$ for all $(x,t) \in X \times I$. Then F is continuous by definition; moreover, $F(x,0) = F(x,1) = f(x)$, making $f \simeq f$.

Now suppose there exist a homotopy $F : f \simeq g$ for maps $f : X \to Y$ and $g : X \to Y$. Define the map $G : X \times I \to Y$ by $(x,t) \to F(x, 1-t)$. G is the composition of continuous maps, so G is continuous, moreover, $G(x,0) = F(x,1) = g(x)$ and $G(x,1) = F(x,0) = f(x)$, so that $g \simeq f$.

Lastly, suppose that $F : f \simeq g$ and $G : g \simeq h$ for maps f, g, h. Define the map $H : X \times I \to Y$ by:

$$H(x,t) = \begin{cases} F(x,2t), & \text{if } 0 \leq t \leq \frac{1}{2} \\ G(x, 2t-1), & \text{if } \frac{1}{2} \leq t \leq 1 \end{cases}$$

Notice that F and G conicide in their domains which cover X. Therefore, by the pasting lemma, H is continuous. Now notice also that $H(x,0) = F(x, 2 \cdot 0) = F(x,0) = f(x)$ and $H(x,1) = G(x, 2 \cdot 1 - 1) = G(x,1) = h(x)$. This makes $f \simeq h$. ■

Definition. For any continuous map $f : X \to Y$ we define the **homotopy class** of f to be the equivalence class of all continuous maps homotopic to f. That is:

$$[f] = \{g : X \to Y : g \text{ is continous and } g \simeq f\}$$

Lemma 2.1.3. *Let $f_0 : X \to Y$, $f_1 : X \to Y$ and $g_0 : X \to Y$, $g_1 : X \to Y$ be continuous maps. If $f_0 \simeq f_1$ and $g_0 \simeq g_1$, then $g_0 \circ f_0 \simeq g_1 \circ f_1$. That is $[g_0 \circ f_0] = [g_1 \circ f_1]$.*

Proof. Let $F : f_0 \simeq f_1$ and $G : g_0 \simeq g_1$ be the homotopies of f_0 into f_1 and g_0 into g_1, respectively. Define the map $H : X \times I \to Y$ by taking $(x, t) \to G(f_0(x), t)$. Then we have that H is continuous by composition, and that $H(x, 0) = G(f_0(x), 0) = g_0(f_0(x))$, and $H(x, 1) = G(f_0(x), 1) = g_1(f_0(x))$. Thus we see that $g_0 \circ f_0 \simeq g_1 \circ f_0$.

Now define the map $K : X \times I \to Y$ by $K = g_1 \circ F$. We have that K is continuous by composition, and that $K(x, 0) = g_1 \circ f_0$ and $K(x, 1) = g_1 \circ f_1$, making $g_1 \circ f_0 \simeq g_1 \circ f_1$. Therefore, by transitivity of homotopy, $g_0 \circ f_0 \simeq g_1 \circ f_1$. ■

Theorem 2.1.4. *Homotopy is a congruence on the category* Top.

Proof. The proof follows by lemmas 2.1.2 and 2.1.3. ■

Definition. We call the quotient category of Top induced by homotopy the **homotopy category** and denote it hTop.

Definition. A continuous map $f : X \to Y$ is a **homotopy equivalence** if there exists a continuous map $g : Y \to X$ such that $f \circ g \simeq 1_Y$ and $g \circ f \simeq 1_X$. We say that the spaces X and Y have the same **homotopy type** if there exists a homotopy equivalence.

Definition. We call a continuous map **nullhomotopic** if it is homotopic to a constant map.

Example 2.1. The space of complex numbers $\mathbb{C}$ and the unit circle S^1 have the same homnotopy type.

Definition. Let Y and Z be topological spaces, and $X \subseteq Y$ a subspace of Y. If $f : X \to Z$ is a continuous map, then we call the map $g : Y \to Z$ defined by $g \circ i = f$ an **extension** of f, where $i : X \to Y$ is the inclusion map.

Theorem 2.1.5. *Let $f : S^n \to Y$ be a continuous map into a topological space Y. The following are equivalent:*

(1) f is nullhomotopic.

(2) f can be extended to a continuous map $B^{n+1} \to Y$.

(3) There exists a constant map $k : S^n \to Y$, taking $x \to f(x_0)$, for all $x \in S^n$, such that $f \simeq k$, for $x_0 \in S^n$.

Proof. Notice that (3) implies (1) immediately. Now suppose that f is nullhomotopic. Then there exists a constant map $k : X \to Y$, such that for some $x_0 \in S^n$, $k : x \to x_0$ for all $x \in S^n$ implies that $f \simeq k$. Now, define the map $g : B^{n+1} \to Y$ by:

$$g(x) = \begin{cases} y_0, & \text{if } 0 \leq \|x\| \leq \frac{1}{2} \\ F(\frac{x}{\|x\|}, 2 - 2\|x\|), & \text{if } \frac{1}{2} \leq \|x\| \leq 1 \end{cases}$$

Notice, that if $\|x\| = \frac{1}{2}$, then $g(x) = F(2x, 1) = y_0$. Therefore, by the pasting lemma, g is continuous. Moreover, if $\|x\| = 1$, $g(x) = F(x, 0) = f$, which makes g an extension of f.

Now, suppose that there exists an extension $g : B^{n+1} \to Y$ of f. Since S^n is a subspace of B^{n+1}, we have that $g \circ i = g|_{S^n} = f$, where $i : Y \to S^n$ is an inclusion. Now, let $x_0 \in S^n$ and define the constant map $k : S^n \to Y$ by taking $x \to f(x_0)$ for all $x \in S^n$. Additionally, define the map $F : S^N \times I \to Y$ given by $F(x, t) = g((1 - t)x + x_0 t)$. We have that F is continuous by composition of continuous maps, and that $F(x, 0) = g(x) = f(x)$, since F has the domain $S^n \times I$, and that $F(x, 1) = g(x_0) = f(x_0)$, since F has the domain $S^n \times I$. This makes $f \simeq k$ with F as the associated homotopy. ■

2.2 Quotient Spaces

Definition. Let X be a topological space, and $X' = \{X_\alpha\}$ a partion of X. We define the **natrual map** $q : X \to X'$ by taking $x \to X_\alpha$ where $x \in X_\alpha$. We define the **quotient topology** on X' to be the family:

$$\mathcal{T} = \{U' \subseteq X' : q^{-1}(U') \text{ is open in } X\}$$

We denote quotient spaces by X/q, X/X', or $X/\sim$ where $\sim$ is an equivalence relation partitioning X into X'.

Example 2.2. (1) Consider the space $I = [0, 1]$ and let $A = \{0, 1\}$. The the quotient space I/A identifies 0 to 1, and hence, under the quotient topology, is homeomorphic to S^1.

(2) Consider the space $I \times I$ and define an equivalence relation $(x, 0) \sim (x, 1)$ for all $x \in I$. Then the quotient topology formed on $I \times I/\sim$ is homeomorphic to the cylinder $S^1 \times I$. Defining another equivalence $(0, y) \sim (1, y)$ for all $y \in I$, we get the quotient space on $S^1 \times I/\sim$ under this equivalence relations is homeomorphic to the torus $S^1 \times S^1$.

(3) Let $h : X \to Y$ be a map, and define $\ker h$ the equivalence relation on X such that $x \ker h x'$ if, and only if $h(x) = h(x')$. The quotient space $X/\ker h$ has the following relation to the natural map on X via the commutative diagram

$$\begin{tikzcd} X \arrow[r, "h"] \arrow[d, "q"'] & Y \\ X/\ker h \arrow[ur, "\phi"'] & \end{tikzcd}$$

Where $\phi : X/\ker h \to Y$ is a 1–1 map defined by $\phi([x]) = h(x)$.

Definition. A continuous map $f : X \to Y$ of a topological space X onto a topological space Y is call an **identification** if a subset U of Y is open if, and only if $f^{-1}(U)$ is open in X. We denote the quotient space on X induced by f by X/f.

Example 2.3. (1) The natural map $q : X \to {}^{X}\!/\!_{\sim}$ is an identification, where $\sim$ is an equivalence relation on X inducing the quotient topology.

(2) If $f : X \to Y$ takes spaces X onto Y, is open or closed, then f is an identification.

(3) If $f : X \to Y$ is a continuous map such that there exists a map $s : Y \to X$ such that $f \circ s = 1_Y$, then f is an identification. We call the map s a **section** of f.

Theorem 2.2.1. *Let $f : X \to Y$ be a continuous map of a topological space X onto a topological space Y. f is an identification if, and only if for any toopological space Z, and all maps $g : Y \to Z$, then g is continuous if, and only if $g \circ f$ is continuous.*

$$\begin{tikzcd} X \arrow[r, "g\circ f"] \arrow[d, "f"'] & Z \\ Y \arrow[ur, "g"'] & \end{tikzcd}$$

Proof. Suppos that f is an identification. If g is continuous, then so is $g \circ f$, by continuity of f. On the other hand, if $g \circ f$ is continuous, letting V be open in Z we have $(g \circ f)^{-1}(V) = f^{-1}(g^{-1}(V))$ which is open in X. By hypothesis, $g^{-1}(V)$ is open in Y, which makes g continuous.

Now, suppose that g is continuous if, and only if $g \circ f$ is continuous. Let $Z = {}^{X}\!/\!_{\ker f}$, and $q : X \to {}^{X}\!/\!_{\ker f}$ the natural map. Additionally, define the 1–1 map $\phi : {}^{X}\!/\!_{\ker f} \to Y$ by $\phi([x]) = f(x)$. Since f is onto, we get that so is ϕ. Consider the following commutative diagram:

$$\begin{tikzcd} X \arrow[r, "q"] \arrow[d, "f"'] & Z \\ Y \arrow[ur, "\phi^{-1}"'] & \end{tikzcd}$$

Then $\phi^{-1} \circ f = q$ is continuous which implies that ϕ^{-1} is continuous. ϕ is also continuou since q is an identification. Therefore ϕ is a homeomorphism between Y and Z. Notice now, that $f = \phi \circ q$. Then since q and ϕ are continuous, this makes f continuous by composition. Moreover, $f^{-1}(U) = q^{-1}(\phi^{-1}(U))$. Since q is an identification, $\phi^{-1}(U)$ is open in Z, which makes $f^{-1}(U)$ open in X. This makes f an identification. ■

f

Corollary. *Let $f : X \to Y$ be an identification, and for some space Z, define $h : X \to Z$ to* *be the continuous map constant on each fiber of f. Then $h \circ f^{-1} : Y \to Z$ is continuous.*

$$\begin{tikzcd} X \arrow[r, "h"] \arrow[d, "f"'] & Z \\ Y \arrow[ur, "h\circ f^{-1}"'] & \end{tikzcd}$$

Moreover $h \circ f^{-1}$ is open or closed if, and only if $h(U)$ is open or closed in Z whenever $U = f^{-1}(f(U))$ is open or closed in X.

Corollary. *If $h : X \to Z$ is an identification, then the map $\phi : X/_{\ker h} \to Z$ defined by $[x] \to h(x)$ is a homeomorphism.*

2.3 Convexity and Contracibilty

Definition. We call a subset X of $\mathbb{R}^n$ **convex** if for every $x, y \in X$, the line segment joining x to y is convex. That is the line $tx + (1-t)y \in X$ for all $t \in [0, 1]$.

Example 2.4. The sets $\mathbb{R}^n$, I^n, B^n and $\Delta(\mathbb{R}^n)$ are all convex. The sphere S^{n-1} is not convex.

Definition. We call a topological space X **contracible** if 1_X is nullhomotopic.

Example 2.5. (1) Let $X = \{x, y\}$ together with the topolgoy $\mathcal{T} = \{\emptyset, \{x\}, X\}$. Then X is contractible under the topology $\mathcal{T}$. We call X together with $\mathcal{T}$ the **Sierpinski space**.

(2) The space $\mathbb{R}^n$ is contractible, but the sphere S^{n-1} is not contractible.

(3) Continuous images of contractible spaces need not be contractible.

Theorem 2.3.1. *Every convex set is contractible.*

Proof. Choose $x_0 \in X$ and consider the constant map $c : X \to X$ by $x \to x_0$ for all $x \in X$. Define $F : X \times I \to X$ by $F(x, t) = tx_0 + (1-t)x$. This map is continuous, with $F(x, 0) = x = 1_X(x)$ and $F(x, 1) = x_0 = c(x)$. Therefore $1_X \simeq c$. ■

Lemma 2.3.2. *If X is a contractible space, and homeomorphic to a space Y, then Y is also contractible.*

Example 2.6. If X and Y are subspaces of $\mathbb{R}^n$, with X homeomorphic to Y, and X convex, then Y is contractible by lemma 2.3.2, however, Y may not be convex. This shows that not all contractible spaces are convex spaces.

Lemma 2.3.3. *Contractible spaces are connected.*

Corollary. *Convex sets are connected.*

Proof. This follows from theorem 2.3.1. ∎

Definition. If X is a topological space, define the equivalence relation $\sim$ on $X \times I$ by $(x,t) \sim (x',t')$ if, and only if $t = t' = 1$. Denote the equivalence classes of (x,t) as $[x,t]$. We call the quotient space ${}^{X \times I}\!/\!{}_{\sim}$ the **cone** over X, and denote it CX. We call the equivalence class $[x,1]$ the **vertex** of CX.

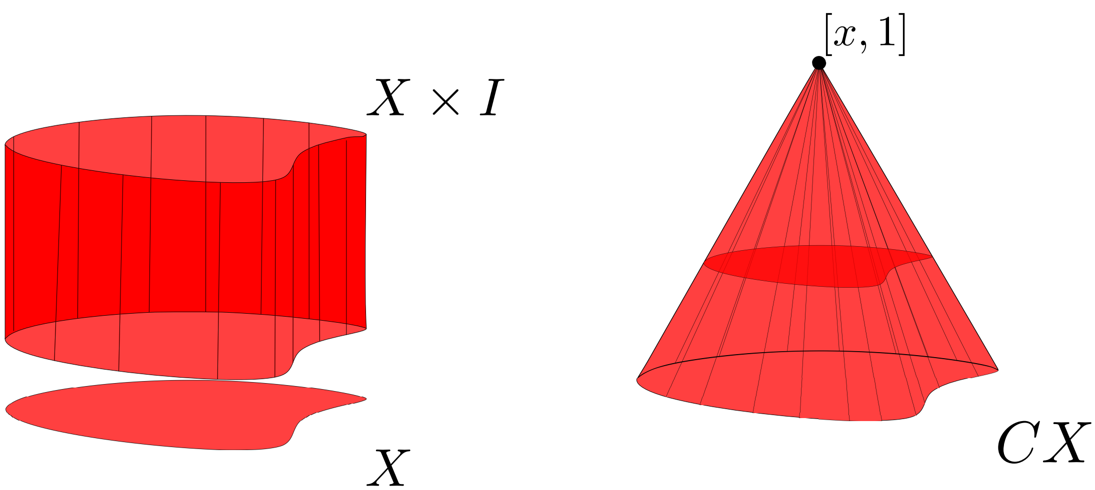

Figure 2.1: The space X and the cone CX formed by identifying all $t = 1$ of $X \times I$ to a point.

Example 2.7. (1) For topological spaces X and Y, every continuous map $f : X \times I \to Y$ with $f(x,1) = y_0$ for some $y_0 \in Y$ induces a continuous map $Cf : CX \to Y$ by taking $[x,t] \to f(x,t)$.

(2) The cone over S^{n-1} is $CS^{n-1} = D^n$ and has the vertex 0.

Theorem 2.3.4. *For any topological space X, the cone over X is contractible.*

Proof. Define the map $F : CX \times I \to CX$ by taking $([x,t],s) \to [x,(1-s)t+s]$. This map is continuous by composition, moreover $F([x,t],0) = [x,t]$ and $F([x,t],1) = [x,1]$ which makes $1_{CX} \simeq c$ where $c : CX \to CX$ is the constant map taking $[x,t] \to [x,1]$ for all $x \in X$. ∎

Theorem 2.3.5. *A topological space has the same homotopy type as a point if, and only if X is contractible.*

Proof. Let $\{a\}$ be a point space, and suppose that $X \simeq \{a\}$ have the same homotopy type. Then there are maps $f : X \to \{a\}$ and $g : \{a\} \to X$ with $a \xrightarrow{g} x_0$ such that $g \circ f \simeq 1_X$ and $f \circ g \simeq 1_{\{a\}}$. Notice that $g \circ f(x) = g(a) = x_0$, for all $x \in X$, so $g \circ f$ is constant. This makes 1_X (and 1_Y) nullhomotopic. Therefore X is contractible.

On the otherhand, supposing that X is contractible, let $1_X \simeq c$ where $c : X \to X$ is the constant map defined by $x \to x_0$ for all $x \in X$. Define the maps $f : X \to \{x_0\}$ and $g : \{x_0\} \to X$ by $x \xrightarrow{f} x_0$ and $x_0 \xrightarrow{g} x_0$. Observe that $g \circ f = 1_X$, and that $f \circ g \simeq 1_{\{x_0\}}$ Therefore X is of the same homotopy type as $\{x_0\}$. ∎

Remark. This theorem shows that the simplest objects in hTop are the contractible spaces.

Theorem 2.3.6. *If Y is a contractible space, then any two maps $X \to Y$ are homotopic.*

Proof. Suppose that $1_Y \simeq c$ where $c : Y \to Y$ takes $y \to y_0$ for all $y \in Y$. Defie $g : X \to Y$ by taking $x \to y_0$ for all $x \in X$. If $f : X \to Y$ is any continuous map, then $f \simeq g$. Consider the diagram

$$\begin{tikzcd} X \arrow[r] & Y \arrow[r, shift left, "k"] \arrow[r, shift right, "1_Y"'] & Y \end{tikzcd}$$

Since $1_Y \simeq k$, we get that $f = 1_Y \circ f \simeq k \circ f = g$. ■

Corollary. *Any two maps $X \to Y$ are nullhomotopic.*

2.4 Path Connectedness.

Definition. A **path** in a topological space X is a continuous map $f : [0,1] \to X$ such that $f(0) = a$ and $f(1) = b$ for some $a, b \in X$. We call a and b the **endpoints** of f, we say f goes **from** a **to** b.

Bibliography

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