Complex Analysis

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Chapter 1

The Complex Numbers

1.1 The Field of Complex Numbers

Definition. We define the set of **complex numbers** to be the collection of all ordered pairs of real numbers $\mathbb{C} = \{(a, b) : a, b \in \mathbb{R}\}$ together with the binary operations $+$ and $\cdot$ of **complex addition**, and **complex multiplication**, respectively defined by the rules

$$(a, b) + (c, d) = (a + c, b + d)$$
$$(a, b)(c, d) = (ac - bd, bc + ad)$$

Theorem 1.1.1. *The set of complex numbers $\mathbb{C}$ forms a field together with complex addition and complex multiplication.*

Corollary. *$\mathbb{C}$ is a field extension of the real numbers $\mathbb{R}$.*

Proof. The map $a \to (a, 0)$ from $\mathbb{R} \to \mathbb{C}$ defines an imbedding of $\mathbb{R}$ into $\mathbb{C}$. ■

Definition. We define the element $i = (0, 1)$ of $\mathbb{C}$ so that $i^2 = -1$, and the polynomial z^2+1 has as root i. We write $(a, b) = a + ib$. If $z = a + ib$, we call a the **real part** of z, and b the **imaginary part** of z and write $\operatorname{Re} z = a$ and $\operatorname{Im} z = z$.

Definition. Let $z = a+ib \in \mathbb{C}$. We define the **norm** (or **modulus**) of z to be $\|z\| = \sqrt{a^2 + b^2}$. We define the complex **conjugate** of z to be $\overline{z} = a - ib$.

Lemma 1.1.2. *For every $z \in \mathbb{C}$, $\|z\|^2 = z\overline{z}$.*

Proof. Let $z = a+ib$. Then $\overline{z} = a-ib$, and so $z\overline{z} = (a+ib)(a-ib) = a^2+b^2 = (\sqrt{a^2 + b^2})^2 = \|z\|^2$. ■

Corollary. *If $z \neq 0$, then $z^{-1} = \frac{1}{z} = \frac{\overline{z}}{\|z\|^2}$.*

Proof. The relation follows from above, and it remains to show that it is indeed the inverse element. Notice that if $z \in \mathbb{C}$ is nonzero, then $z\frac{\overline{z}}{\|z\|^2} = \frac{z\overline{z}}{\|z\|^2} = \frac{\|z\|^2}{\|z\|^2} = 1$. ■

Example 1.1. (1) Let $z = a + ib$. Then we get that $\frac{1}{z} = \frac{\overline{z}}{\|z\|}$ has real part $\operatorname{Re} \frac{1}{z} = \frac{a}{a^2+b^2}$ and imaginary part $\operatorname{Im} \frac{1}{z} = -\frac{b}{a^2+b^2}$.

(2) Let $z = a + ib$, and $c \in \mathbb{R}$. Then $\frac{z-c}{z+c} = \operatorname{Re} \frac{z-c}{z+c}$, so $\operatorname{Im} \frac{z-c}{z+c} = 0$.

(3) Let $z = a + ib$, then $z^3 = a^3 - 3ab^2 + i(3a^2b - b^3)$ So that $\operatorname{Re} z^3 = a^3 - 3ab^2$ and $\operatorname{Im} z = 3a^2b - b^3$.

(4) $\frac{3+i5}{1+i7} = \frac{19}{25} - i\frac{18}{25}$.

(5) $(-\frac{1}{2} + i\frac{\sqrt{3}}{2})^3 = 1$, and hence $(-\frac{1}{2} + i\frac{\sqrt{3}}{2})^6 = 1$.

(6) Notice that $i^n = 1, i, -1, -i$ whenever $n \equiv 0 \mod 4$, $n \equiv 1 \mod 4$, $n \equiv 2 \mod 4$, and $n \equiv 3 \mod 4$. respectively.

(7) $\| -2 + i\| = \sqrt{5}$, and $\|(2 + i)(4 + i3)\| = \|5 + i10\| = 5\sqrt{5}$.

Lemma 1.1.3. *The following are true for all $z, w \in \mathbb{C}$.*

(1) $\operatorname{Re} z = \frac{1}{2}(z + \overline{z})$ *and* $\operatorname{Im} z = \frac{1}{2i}(z - \overline{z})$.

(2) $\overline{(z + w)} = \overline{z} + \overline{w}$ *and* $\overline{zw} = \overline{z}\ \overline{w}$.

(3) $\|\overline{z}\| = \|z\|$.

Proof. Let $z = a + ib$ and $w = c + id$. Then notice that

$$\frac{(a + ib) + (a - ib)}{2} = \frac{2a + (ib - ib)}{2} = \frac{2a}{2} = a = \operatorname{Re} z$$

and

$$\frac{(a + ib) - (a - ib)}{2i} = \frac{(a - a) + 2ib}{2} = \frac{2ib}{2i} = b = \operatorname{Im} z$$

Moreover

$$\overline{(a + ib) + (c + id)} = \overline{(a + c) + i(b + d)} = (a + c) - i(b + d) = (a - ib) + (c - id)$$

And

$$\overline{(a + ib)(c + id)} = \overline{(ac - bd) + i(bc + ad)} = (ac - bd) - i(bc + ad) = (a - ib)(c - id)$$

so that $\overline{z + w} = \overline{z} + \overline{w}$ and $\overline{zw} = \overline{z}\ \overline{w}$.

Now, we have that $\|zw\|^2 = (zw)\overline{zw} = (zw)(\overline{z}\ \overline{w}) = (z\overline{z})(w\overline{w}) = \|z\|^2\|w\|^2$. Taking square roots, we get the result

$$\|zw\| = \|z\|\|w\|$$

Finally, notice that $\|z\|^2 = z\overline{z} = \overline{z}\ \overline{\overline{z}} = \|\overline{z}\|$. ■

Corollary. *The following are also true; provided $w \neq 0$.*

(1) $\overline{(\frac{z}{w})} = \frac{\overline{z}}{\overline{w}}$.

(2) $\|\frac{z}{w}\| = \frac{\|z\|}{\|w\|}$

Corollary. *If $z = z_1 + \cdots + z_n$, and $w = w_1 \dots w_n$, with $z_i, w_i \in \mathbb{C}$ for all $1 \leq i \leq n$, then*

(1) $\overline{z} = \overline{z_1} + \cdots + \overline{z_n}$.

(2) $\|w\| = \|w_1\| \ldots \|w_n\|$.

Proof. We prove both results by induction on n. For $n = 2$, we have already shown that $\overline{z} = \overline{z_1} + \overline{z_2}$ and $\|w\| = \|w_1\|\|w_2\|$. Now, for all $n \geq 2$, suppose that both

$$\begin{aligned} \overline{z} &= \overline{z_1} + \cdots + \overline{z_n} \\ \|w\| &= \|w_1\| \ldots \|w_n\| \end{aligned}$$

Then let $z' = z + z_{n+1}$ and $w' = ww_{n+1}$ for $z_{n+1}, w_{n+1} \in \mathbb{C}$. Then we have that

$$\begin{aligned} z' &= z + z_{n+1} = z_1 + \cdots + z_n + z_{n+1} \\ w' &= ww_{n+1} = w_1 \ldots w_n w_{n+1} \end{aligned}$$

so by the induction hypothesis, we have

$$\overline{z'} = \overline{(z + z_{n+1})} = \overline{z} + \overline{z_{n+1}} = \overline{z_1} + \cdots + \overline{z_n} + \overline{z_{n+1}}$$

and that

$$\|w'\| = \|ww_{n+1}\| = \|w\|\|w_{n+1}\| = \|w_1\| \ldots \|w_n\|\|w_{n+1}\|$$

which completes the proof. ∎

Lemma 1.1.4. *Let $z \in \mathbb{C}$. Then z is a real number if, and only if $z = \overline{z}$.*

Proof. If z is real, then $z = a + i0$, for some $a \in \mathbb{R}$, and hence $\overline{z} = a - i0 = z$. COnversely, suppose that $z = \overline{z}$. Then we have

$$\operatorname{Re} z = \frac{1}{2}(z + \overline{z}) = \frac{1}{2}(z + z) = z$$

so z has only a real part, and hence must be a real number. ∎

Lemma 1.1.5. *The following are true for all $z, w \in \mathbb{C}$.*

(1) $\|z + w\|^2 = \|z\|^2 + 2 \operatorname{Re} z\overline{w} + \|w\|^2$.

(2) $\|z - w\|^2 = \|z\|^2 - 2 \operatorname{Re} z\overline{w} + \|w\|^2$.

(3) $\|z + w\|^2 + \|z - w\|^2 = 2(\|z\|^2 + \|w\|^2)$.

Proof. We first notice that $\|z+w\|^2 = (z+w)\overline{(z + w)} = (z+w)(\overline{z}+\overline{w}) = z\overline{z}+z\overline{w}+w\overline{z}+w\overline{w} = \|z\|^2 + z\overline{w} + w\overline{z} + \|w\|^2$. Now, let $z = a + ib$ and $w = c + id$. Then we have

$$\begin{aligned} (a + ib)(c - id) &= (ac + bd) - i(ad - bc) \\ (c + id)(a - ib) &= (ac + bd) + i(ad - bc) \end{aligned}$$

so that $z\overline{w} + w\overline{z} = 2(ac + bd) = 2\operatorname{Re} z\overline{w}$, and we are done. To get the identity for $\|z - w\|^2$, we simply replace w by $-w$, and use the above argument.

Now, we have that $\|z+w\|^2 = \|z^2\|+2\operatorname{Re} z\overline{w}+\|w\|^2$, and $\|z-w\|^2 = \|z^2\|-2\operatorname{Re} z\overline{w}+\|w\|^2$, so that adding them together, the terms $2\operatorname{Re} z\overline{w}$ cancel out and we are left with

$$\|z + w\|^2 + \|z - w\|^2 = 2(\|z\|^2 + \|w\|^2)$$

■

Lemma 1.1.6. *Let $R(z) \in \mathbb{C}(z)$ a rational function in z. Then if R has coefficients in $\mathbb{R}$, then $\overline{R(z)} = R(\overline{z})$.*

Proof. We first observe the polynomial $f \in \mathbb{C}[z]$, of finite degree $\deg f = n$, and of the form

$$f(z) = a_0 + a_1 z + \cdots + a_n z^n$$

Then if f has all coefficients in $\mathbb{R}$; i.e. $f \in \mathbb{R}[z]$, where $z \in \mathbb{C}$ is treated as indeterminant, then we have that since each $a_i \in \mathbb{R}$, then $\overline{a_i z^i} = \overline{a_i}\,\overline{z^i} = a_i \overline{z}^i$. So that

$$\overline{f(z)} = \overline{(a_0 + a_1 z + \cdots + a_n z^n)} = a_0 + a_1 \overline{z} + \cdots + a_n \overline{z}^n$$

which makes $\overline{f(z)} = f(\overline{z})$. Now, one can also extend f to a polynomial of infinite degree by taking $n \to \infty$, and the same holds.

Now, let $R(z) \in \mathbb{C}(z)$ a rational function. Recall that $R(z)$ is of the form

$$R(z) = \frac{f(z)}{g(z)} \text{ with } g \neq 0$$

for some polynomials $f, g \in \mathbb{C}[z]$. Then if R has all real coefficients, so do f and g, and hence we get

$$\overline{R(z)} = \frac{\overline{f(z)}}{\overline{g(z)}} = \frac{f(\overline{z})}{g(\overline{z})} = R(\overline{z})$$

which completes the proof. ■

1.2 The Complex Plane

Definition. We define the **complex plane** to be the space of points (x, y) of $\mathbb{R}^2$ for which $z = x + iy$.

Lemma 1.2.1. *For every $z, w \in \mathbb{C}$ $\|z + w\| \leq \|z\| + \|w\|$.*

Proof. Observe that $-\|z\| \leq \operatorname{Re} z \leq \|z\|$ for all $z \in \mathbb{C}$, so that $\operatorname{Re} z\overline{w} \leq \|z\overline{w}\| = \|z\|\|w\|$. So we get

$$\|z + w\|^2 = \|z\|^2 + \operatorname{Re} z\overline{w} + \|\overline{w}\| \leq \|z\|^2 + \|z\|\|w\| + \|\overline{w}\| = (\|z\| + \|w\|)^2$$

Taking square roots gives us the result. ■

Corollary. $\|z+w\| = \|z\| + \|w\|$ *if* $z = tw$ *for some* $t \geq 0$.

Corollary. *If* $z_1, \ldots, z_n \in \mathbb{C}$, *then* $\|z_1 + \cdots + z_n\| \leq \|z_1\| + \cdots + \|z_n\|$.

Proof. By induction on n. ∎

Corollary. *For all* $z, w \in \mathbb{C}$, $\big|\|z\| - \|w\|\big| \leq \|z - w\|$.

Proof. We have that $\|z\| \leq \|z-w\| + \|w\|$, and $\|w\| \leq \|z-w\| + \|z\|$. So we get $\|z\| - \|w\| \leq \|z - w\|$ and $-\|z - w\| \leq \|w\| - \|z\|$, so that $\big|\|z\| - \|w\|\big| \leq \|z - w\|$. ∎

Definition. We define the **polar form** of a complex number $z \in \mathbb{C}$ to be the polar coordinates (r, θ) where $r = \|z\|$ and θ is the angle between the line segment from 0 to z and the positive real axis. We call r the **modulus** of z, and θ the **argument** of z. We write $\theta = \arg z$.

Lemma 1.2.2. *Let* $z = r_1 \cos\theta_1 + r_1 i \sin\theta_1$ *and* $w = r_2 \cos\theta_2 + r_2 i \sin\theta_2$. *Then*

$$zw = r_1 r_2 \cos(\theta_1 + \theta_2) + r_1 r_2 i \sin(\theta_1 + \theta_2)$$

so that $\arg zw = \arg z + \arg w$.

Proof. We multiply the expanded forms of z and w together and use the trigonometric identities to get the result. ∎

Corollary. *If* $z_k = r_k \cos\theta_k + r_k i \sin\theta_k$, *then*

$$z_1 \ldots z_n = (r_1 \ldots r_n)\cos(\theta_1 + \cdots + \theta_n) + (r_1 \ldots r_n) i \sin(\theta_1 + \cdots + \theta_n)$$

Proof. By induction on n. ∎

Theorem 1.2.3 (DeMoivre's Theorem)**.** *For all integers* $n \geq 0$, *if* $z = \cos\theta + i\sin\theta$, *then*

$$z^n = \cos n\theta + i \sin n\theta$$

Proof. We use the corollory to lemma 1.2.2 recursively on z^n. ∎

Lemma 1.2.4. *FOr each nonzero* $a \in \mathbb{C}$, *and integer* $n \geq 2$, *the polynomial* $z^n - a$ *has has roots all* z *of the form*

$$z = \|a\|^{\frac{1}{n}}(\cos\frac{\alpha + 2k\pi}{n} + i\sin\frac{\alpha + 2k\pi}{n}) \textit{ for all } 0 \leq k \leq n-1$$

where $a = \|a\|\cos\alpha + \|a\| i \sin\alpha$

Proof. Let $a = \|a\|\cos\alpha + \|a\| i \sin\alpha$. Then we have $z^n - a = 0$ has as solution

$$z' = \|a\|^{\frac{1}{n}}(\cos\frac{\alpha + 2\pi}{n} + i\sin\frac{\alpha + 2\pi}{n})$$

The rest of the solutions are obtained by noting that $(z')^n - a = 0$. ∎

Definition. Let $a \in \mathbb{C}$ a nonzero complex number. We call the roots of the the polynomial $z^n - a \in \mathbb{C}[z]$ the **n-th roots** of a. We call the roots of $z^n - 1 \in \mathbb{C}[z]$ the **n-th roots of unity**.

Example 1.2. The n-th roots of unity are all complex numbers of the form

$$z = \cos\frac{2k\pi}{n} + i\sin\frac{2k\pi}{n} \text{ for all } 0 \leq k \leq n-1$$

Lemma 1.2.5. *Let $L \subseteq \mathbb{C}$ a straight line in $\mathbb{C}$. Then $L = \{z \in \mathbb{C} : \operatorname{Im}\frac{z-a}{b} = 0\}$, where $z = a + tb$ for some $t \in \mathbb{R}$.*

Proof. Let a be any point in L, and b the direction vector of L. Then if $z \in L$ $z = a + tb$ for some $t \in \mathbb{R}$. Since $b \neq 0$, $\operatorname{Im}\frac{z-a}{b} = 0$, since $t = \frac{z-a}{b}$, and $t \in \mathbb{R}$. ■

Corollary. *Let $H_a = \{z \in \mathbb{C} : \operatorname{Im}\frac{z-a}{b} > 0\}$ and $K_a = \{z \in \mathbb{C} : \operatorname{Im}\frac{z-a}{b} < 0\}$. Then $H_a = a + H_0$ and $K_a = a - K_0$.*

Proof. Suppose that $\|b\| = 1$, and let $a = 0$, then $H_0 = \{z \in \mathbb{C} : \operatorname{Im}\frac{z}{b} > 0\}$. Now, $b = \cos\beta + i\sin\beta$. If $z = r\cos\theta + ri\sin\theta$, then $\frac{z}{b} = r\cos(\theta - \beta) + ri\sin(\theta - \beta)$. So $z \in H_0$ if, and only if $\sin(\theta - \beta) > 0$; that is $\beta < \theta < \pi + \beta$, which makes H_0 the upper half plane about L.

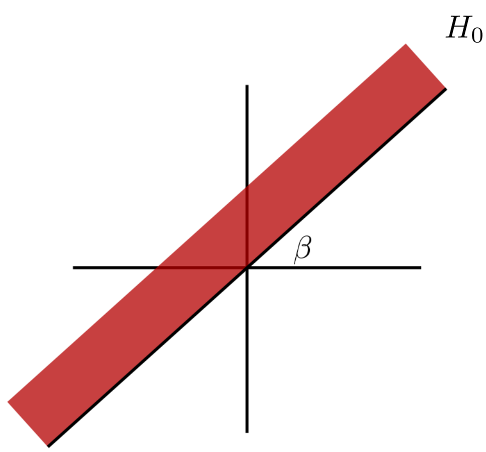

Figure 1.1:

Putting $H_a = \{z \in \mathbb{C} : \operatorname{Im}\frac{z-a}{b} > 0\}$, we get $H_a = a + H_0$. By similar reasoning, we get $K_a = a - K_0$, where $K_a = \{z \in \mathbb{C} : \operatorname{Im}\frac{z-a}{b} < 0\}$. ■

1.3 The Extended Complex Numbers

Definition. We define the **extended complex numbers** to be the set $\mathbb{C}_\infty = \mathbb{C} \cup \{\infty\}$.

Lemma 1.3.1. *$\mathbb{C}_\infty$ is homeomorphic to the unit sphere S^2 of $\mathbb{R}^3$.*

Proof. Identify $\mathbb{C}$ with the plane $\mathbb{R}^2$ as a subset of $\mathbb{R}^3$. Then $\mathbb{C}$ cuts the sphere S^2 along the equator. Now, let $N = (0, 0, 1)$ be the noth pole of S^2. For $z \in \mathbb{C}$, let L_z the line passing through z and N, and hence cuts S^3 at exactly one point $Z \neq N$. If $\|z\| > 1$, Z is in the northern hemisphere of S^2, and if $\|z\| < 1$, then Z is in the southern hemisphere. If $\|z\| = 1$, then $Z = z$. Then notice that as $\|z\| \to \infty$, then $Z \to N$; and so identify N with ∞ in $\mathbb{C}_\infty$.

Now, let $z = x+iy$ and $Z = (x_1, x_2, x_3)$ a point on S^2. Then $L_z = \{tN+(1-t)z : t \in \mathbb{R}\}$. Observe then that

$$L_z = \{((1-t)x, (1-t)y, t) : t \in \mathbb{R}\}$$

Then we get

$$1 = (1-t)^2\|z\|^2 + t^2$$

Taking $t \neq 1$ so that $z \neq \infty$

$$Z = \Big(\frac{2x}{\|z\|^2+1}, \frac{2y}{\|z\|^2+1}, \frac{\|z\|^2-1}{\|z\|^2+1}\Big)$$

additionally

$$Z = \Big(\frac{z+\overline{z}}{\|z\|^2+1}, -i\frac{z-\overline{z}}{\|z\|^2+1}, \frac{\|z\|^2-1}{\|z\|^2+1}\Big)$$

Taking $Z \neq N$ and $t = x_1$, we also get by definition of L_z, that

$$z = \frac{x_1 + ix_2}{1 - x_3}$$

Define now, the metric d on $\mathbb{C}_\infty$ by $d(z, w)$ is the distance between the points $Z = (x_1, x_2, x_3)$ and $W = (y_1, y_2, y_3)$ on S^2. Then we get

$$d(z, w) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2 + (x_3 - y_3)^2}$$

Squaring both sides we ovserve tha

$$d(z, w) = 2 - 2(x_1y_1 + x_2y_2 + x_3y_3)$$

Using the previous derivations of the components of Z, we finally obtain

$$d(z, w) = \frac{z\|z - w\|}{\sqrt{(\|z\|^2 + 1)(\|w\|^2 + 1)}} \text{ for all } z, w \in \mathbb{C}$$

When $w = \infty$, we have

$$d(z, \infty) = \frac{z}{\sqrt{\|z\|^2 + 1}}$$

Then d is the required homeomorphism. ■

Definition. We call the correspondence between S^2 and $\mathbb{C}_\infty$ the **stereographic projection** of S^2 onto $\mathbb{C}_\infty$.

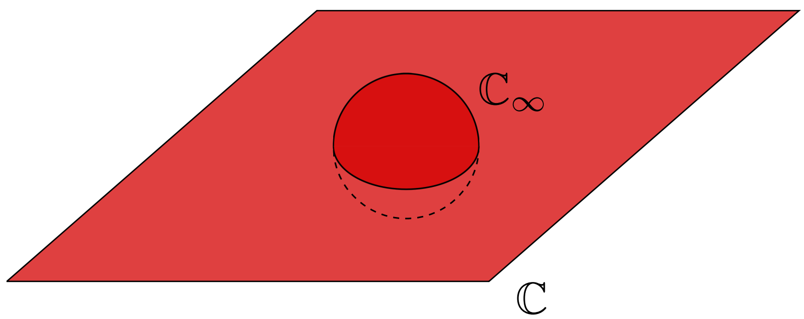

Figure 1.2: The Extended Complex Numbers.

Chapter 2

The Topology of $\mathbb{C}$.

2.1 Metric Spaces

Definition. A **metric space** is a set X together with a map $d : X \times X \to \mathbb{R}$ such that for all $x, y, z \in X$

(1) $d(x, y) \geq 0$ and $d(x, y) = 0$ if, and only if $x = y$.

(2) $d(x, y) = d(y, x)$.

(3) $d(x, y) \leq d(x, z) + d(z, y)$ (The Triangle Inequality).

We call d a **metric** on X. If $x \in X$, and $r > 0$, we define the **open ball** centered about x of **radius** r to be the set $B(x, r) = \{y \in X : d(x, y) < r\}$. We define the **closed ball** centered about x of radius r to be the set $\overline{B}(x, y) = \{y \in X : d(x, y) \leq r\}$.

Example 2.1. (1) The metric $d(x, y) = \|z - w\|$ makes $\mathbb{R}$ and $\mathbb{C}$ into metric spaces. In fact, d defines the norm on $\mathbb{C}$, i.e. $\|z\| = d(z, 0)$.

(2) If X is a metric space with metric d, and $Y X$, then d makes Y into a metric space.

(3) Define $d(x + iy, a + ib) = \|x - a\| + \|y - b\|$. Then $(\mathbb{C}, d)$ is a metric space. We call d the **taxicab metric**.

(4) Define $d(x + iy, a + ib) = \max\{\|x - a\|, \|y - b\|\}$. Then $(\mathbb{C}, d)$ is a metric space. We call d the **square metric**.

(5) Let X be any set, and define $d : X \times X \to \mathbb{R}$ by

$$d(x, y) = \begin{cases} 1, & \text{if } x \neq y \\ 0, & \text{if } x = y \end{cases}$$

Then d is a metric on X. Notice also that for any $\varepsilon > 0$, that $B(x, \varepsilon) = \{x\}$ if $\varepsilon \leq 1$, and $B(x, \varepsilon) = X$ if $\varepsilon > 1$.

(6) Define d on $\mathbb{R}^n$ by

$$d(x,y) = \sqrt{\sum_{i=1}^{n}(x_i - y_i)^2}$$

Then d is a metric on $\mathbb{R}^n$ defining the general norm. That is $\|x\| = d(x,0)$.

(7) Let S and let $B(S)$ the set of all complex valued functions $f : S \to \mathbb{C}$ such that $\|f\|_\infty = \sup\{\|f(s\|) : s \in S\}$ is finite. That is, $B(S)$ is the set of all complex valued functions whose image is contained within a disk of finite radius. Define d on $B(S)$ by $d(f,g) = \|f-g\|_\infty$. Let $f,g,h \in B(S)$. Then

$$\|f(s) - g(s)\| = \|(f(s) - h(s)) - (h(s) - g(s))\| \leq \|f(s) - h(s)\| + \|h(s) - g(s)\|$$

taking least upper bounds, we get

$$\|f-g\|_\infty \leq \|f-h\|_\infty + \|h-g\|_\infty$$

Definition. Let X be a metric space together with metric d. We call a subset U of X **open** if there exists an $\varepsilon > 0$ for which $B(x,\varepsilon) \subseteq U$ for every $x \in U$.

Example 2.2. $U = \{z \in \mathbb{C} : a < \operatorname{Re} z < b\}$ is open in $\mathbb{C}$, but $U = \{z \in \mathbb{C} : \operatorname{Re} z < 0\}$ is not, as $B(0,\varepsilon) \notin U$ no matter how small we make ε.

Theorem 2.1.1. *Let X be a metric space with metric d. Then X is a topological space whose open sets are those subsets of X containing ε-balls for every element, and for $\varepsilon > 0$.*

Definition. We call a subset V of a metrix space (X,d) **closed** if $X \backslash V$ is open in X.

Lemma 2.1.2. *If (X,d) is a metric space, then it is a topology by closed sets.*

Definition. Let $A \subseteq X$ where X is a metric space. We define the **interior** of A to be the union of all open sets contained in A, and write $\operatorname{int} A$. We define the **closure** of A to be the intersection of all closed sets containing A and write $\operatorname{cl} A$. We define the **boundry** of A to be $\partial A = \operatorname{cl} A \cap \operatorname{cl}(X \backslash A)$.

Example 2.3. We have $\operatorname{int} \mathbb{Q}(i) = \emptyset$ and $\operatorname{cl} \mathbb{Q}(i) = \mathbb{C}$.

Lemma 2.1.3. *Let X be a metric space and A, BX. Then the following are true*

(1) A is open if, and only if $A = \operatorname{int} A$.

(2) A is closed if, and only if $A = \operatorname{cl} A$.

(3) $\operatorname{int} A = X \backslash \operatorname{cl}(X \backslash A)$, $\operatorname{cl} A = X \backslash \operatorname{int}(X \backslash A)$, and $\partial A = \operatorname{cl} A \backslash \operatorname{int} A$.

(4) $\operatorname{cl}(A \cup B) = \operatorname{cl} A \cup \operatorname{cl} B$.

(5) $x_0 \in \operatorname{int} A$ if, and only if there is an $\varepsilon > 0$ for which $B(x_0,\varepsilon) \subseteq A$.

(6) $x_0 \in \operatorname{cl} A$ if, and only if for every $\varepsilon > 0$, $B(x_0,\varepsilon) \cap A = \emptyset$.

Definition. A subset A of a metric space X is **dense** in X if $\operatorname{cl} A = X$.

Example 2.4. $\mathbb{Q}$ is dense in $\mathbb{R}$, notice that $\operatorname{cl} \mathbb{Q} = \mathbb{R}$. Moreover, $\mathbb{Q}(i)$ is dense in $\mathbb{C}$.

2.2 Connectedness in $\mathbb{C}$

Definition. We say a metric space X is connected provided there are no disjoint nonempty open sets $A, B \subseteq X$ for which $X = A \cup B$.

Lemma 2.2.1. *A metric space X is connected if its only closed and open sets are the emtpyset and itself.*

Example 2.5. Consider the space $X = \{z \in \mathbb{C} : \|z\| < 1\} \cup \{z \in \mathbb{C} : \|z - 3\| < 1\}$. Let $A = \{z \in \mathbb{C} : \|z\| < 1\}$ and $B = \{z \in \mathbb{C} : \|z - 3\| < 1\}$. Then then both A and B are open in X. Moreover, A is also closed in X as $B = X \backslash A$. So X is not connected.

Lemma 2.2.2. *A space $X \subseteq X$ is connected if, and only if it is an interval.*

Proof. Suppose that $X = [a, b]$, where $a, b \in \mathbb{R}$ and $a < b$. Let $A \subseteq X$ be open, with $a \in A$ and $b \in B$ and where $X \neq A$. Then there is an $\varepsilon > 0$ for which $[a, a + \varepsilon) \subseteq A$. Let $r = \sup\{\varepsilon : [a, a + \varepsilon) \subseteq A\}$. If $a \leq x < a + r$, putting $h = a + (r - x) > 0$ there is an $\varepsilon > 0$ for which $r - h < \varepsilon < r$ and $[a, a + \varepsilon) \subseteq A$. However, $a \leq a + (r - h) < a + \varepsilon$ putting $x \in A$. So that $[a, a + r) \subseteq A$. Now, if $a + r \in A$, then by the openness of A, there is a $\delta > 0$ with $[a + r, a + r + \delta] \subseteq A$, which puts $[a + r, a + r + \delta) \subseteq A$. But that contradicrts that r is a least upper boundl; so $a + r \notin A$.

Now, if A were closed, then $a + r \in B = X \backslash A$, which is open, so that there is a $\delta > 0$ such that $(a + r - \delta, a + r) \subseteq B$, which contradicts that $[a, a + r) \subseteq A$. ■

Remark. Note that the first part of this proof lacks the proof for the other types of intervals.

Definition. Let $z, w \in \mathbb{C}$. We define the **staight line segment** $[z, w]$ from z to w to be the set

$$[z, w] = \{tw + (1 - t)z : 0 \leq t \leq 1\}$$

A **polygon** from z to w is defined to be the set

$$P[z, w] = \bigcup_{k=1}^{n} [z_k, w_k]$$

where $z_1 = z$, $w_n = w$, and $z_{k+1} = w_k$ for all $1 \leq k \leq n - 1$. When the endpoints of the polygon are understood, we may simply just write P, or we enumerate the points of P as $P = [z, z_2, \ldots, z_n, w]$.

Theorem 2.2.3. *An open set U of $\mathbb{C}$ is connected if, and only if for all $z, w \in U$, there exists a polygon $P[z, w]$ from z to w contained in U.*

Proof. Let $P[z, w] \subseteq U$ be the given polygon. Suppose that U were not connected. Then there exist disjoint nonempty open sets Z and W of U (as a subspace of $\mathbb{C}$) for which $U = Z \cup W$. Let $z \in Z$ and $w \in W$. Consider the case for when $P[z, w] = [z, w]$. Define $S = \{s \in [0, 1] : sw + (1 - s)z \in A\}$ and $T = \{s \in [0, 1] : sw + (1 - s)z \in B\}$. Then notice that S and T are disjoint, and that $S \cup T = [0, 1]$. Moreover, they are open subsets of the interval $[0, 1] \subseteq \mathbb{R}$; but $[0, 1]$ is connected in $\mathbb{R}$, which is a contradiction. Therefore U must be connected.

On the otherhand, let $w \in Z$ and let $P = [z, z_2, \cdots z_n, w] \subseteq U$ SInce U is open, there is an $\varepsilon > 0$ such that $B(w, \varepsilon) \subseteq U$. Now, if $u \in B(w, \varepsilon)$, then $[w, u] \subseteq B(w, \varepsilon) \subseteq U$, so the polygon $Q = P \cup [w, u] \subseteq U$. Hence $B(w, \varepsilon) \subseteq Z$, which makes Z oepn. On the otherhand, consider $u \in U \backslash Z$, and let $\varepsilon > 0$ such that $B(u, \varepsilon) \subseteq U$. Then there is a $w \in Z \cap B(u, \varepsilon)$. Construct, then a polygon $P[z, u]$ so that $B(u, \varepsilon) \cap Z$ is empty. That is, $B(u, \varepsilon) \subseteq U \backslash Z$ making $U \backslash Z$ open, and hence Z closed. ■

Corollary. *If $U \subseteq \mathbb{C}$ is an open and connected set, then for all $z, w \in U$, there is a polygon $P[z, w]$ in U made up of straight line segments parallel to either the real axis, or the imaginary axis.*

Definition. Let X be a metric space. We call a subset $C \subset X$ a **connected component** if it is maximally connected in X.

Example 2.6. (1) A and B in example 2.5 are connected components.

(2) Let $X = \{\frac{1}{k} : k \in \mathbb{Z}^+\} \cup \{0\}$. Then every connected component is a point of x, and vise versa; with, the exception of 0.

Lemma 2.2.4. *Let X be a metric space with $x_0 \in X$. If $\{D_j\}$ is a collection of connected subsets of X, such that $x_0 \in D_j$, then the union $D = \bigcup D_j$ is connected.*

Proof. Let $A \subseteq D$, which is a metric space, for which A is both open and closed, and nonempty. Then $A \cap D_j$ is open and closed for all j. Now, since D_j is connected, either $A \cap D_j =$, or $A \cap D_j = D_j$. Since A is nonempty, we must have the latter case. Then there exists at least one index k for which $A \cap D_k = D_k$. Then if $x_0 \in A$, $x_0 \in A \cap D_k$ so that $x_0 \in D_k$ making $A \cap D_j = D_j$ for all j or $D_j \subseteq A$. In either case, we get $D = A$. ■

Theorem 2.2.5. *The connected components of a metric space partition the space.*

Proof. Let $\mathcal{D}$ the collection of all connected subsets of X containing a point $x_0 \in X$. Then $\mathcal{D}$ is nonempty by definition, and by hypothesis, we have that $C = \bigcup D_j$ is connected, and that $x_0 \in C$.

Now, suppose that $C \subseteq D$ for some connected st D. Then $x_0 \in D$ so that $D \in \mathcal{D}$, and hence $D \subseteq C$. This makes $C = D$, and hence C is a connected component of X. This then implies that $X = \bigcup C_j$ where $\{C_j\}$ is the collection of connected components of X.

Now, consider $\{C_j\}$, and suppose that for distinct components C_1 and C_2, that there is an $x_0 \in C_1 C_2$. Then $x_0 \in C_1$, and $x_0 \in C_2$ so that $C_1 = C_1 \cup C_2 = C_2$, which is a contradiction. Therefore the connected components are pairwise disjoint. ■

Lemma 2.2.6. *If X is a connected metric space with $A \subseteq X$, and $A \subseteq B \subseteq \operatorname{cl} A$, then B is also connected.*

Corollary. *Connected components of a metric space are closed.*

Theorem 2.2.7. *If U is open in $\mathbb{C}$, then U has countably many connected components; each of which is open.*

Proof. Let $C \subseteq U$ a connected component, with $x_0 \in C$. Since U is open, there is an $\varepsilon > 0$ for which $B(x_0, \varepsilon) \subseteq U$. Then $B(x_0, \varepsilon) \cup C$ is connected so that $B(x_0, \varepsilon) \cup C = C$, so that $B(x_0, \varepsilon) \subseteq C$. This makes each C open.

Now, let $S = \{a + ib \in \mathbb{Q}(i) : a + ib \in U\}$. Then S is countable by the density of $\mathbb{Q}(i)$ in $\mathbb{C}$, and each connected component of U contains a point of S. This implies there are countably many such components. ■

2.3 Completeness in $\mathbb{C}$

Definition. We say a sequence $\{x_n\}$ of points of a metric space X **converges** to a point $x \in X$ if for every $\varepsilon > 0$, there is anb $N \in \mathbb{Z}^+$ for which

$$d(x, x_n) < \varepsilon \text{ whenever } n \geq N$$

If $\{x_n\}$ converges to x, we write $\{x_n\} \to x$, or $\lim x_n = x$.

Lemma 2.3.1. *Let X be a metric space. A set $V \subseteq X$ is closed if, and only if for every sequece $\{x_n\}$ of points in V, $\{x_n\}$ converges to a point $x \in V$.*

Proof. If V is closed, and $\{x_n\} \to x$, then for every $\varepsilon > 0$ and $x_n \in B(x, \varepsilon)$, we get that $B(x, \varepsilon) \cap V \neq \emptyset$ so that $x \in \operatorname{cl} F = F$.

Conversly, suppose that V is not closed. Then there exists a point $x_0 \in \operatorname{cl} V \backslash V$. Then we get that for every $\varepsilon > 0$, the set $B(x_0, \varepsilon) \cap F \neq \emptyset$ so that for all $n \in \mathbb{Z}^+$, there is an $x_n \in B(x_0, \frac{1}{n}) \cap F$. This makes $d(x_0, x_n) < \frac{1}{n}$, so that $\{x_n\} \to x_0$. Since $x_0 F$, the condition fails. ■

Definition. We call a point $x \in X$ of a metric space X a **limit point** of a subset $A \subseteq X$ if there exists a sequence of points $\{x_n\}$ in A such that $\{x_n\} \to x$.

Example 2.7. Consider $\mathbb{C}$ and let $A = [0, 1] \cup \{i\}$. Then each point of $[0, 1]$ is a limit point of A, but i is not a limit point of A.

Lemma 2.3.2. *A subset of a metric space is closed if, and only if it contains all its limit points. Moreover, if A is a subset of a metric space X, then $\operatorname{cl} A = A \cup A'$, where A' is the collection of all limit points of A.*

Definition. We call a sequence $\{x_n\}$ of points of a metric space **Cauchy** if for every $\varepsilon > 0$ there is an $N \in \mathbb{Z}^+$ for which

$$d(x_m, x_m) < \varepsilon \text{ for all } m, n \geq N$$

If X is a metric space in which every Cauchy sequence converges in to a point in X, then we say X is **complete**.

Theorem 2.3.3. *The field $\mathbb{C}$ of complex numbers is complete.*

Proof. Let $\{z_n\}$ a Cauchy sequence of complex numbers with $z_n = x_n + iy_n$. Then the sequences $\{x_n\}$ and $\{y_n\}$ are Cauchy in $\mathbb{R}$. Since $\mathbb{R}$ is a complete metric space, we observe that there exist $x, y \in \mathbb{R}$ for which $\{x_n\} \to x$ and $\{y_n\} \to y$. This makes $\{z_n\} \to z$ with $z = x + iy \in \mathbb{C}$. ■

Definition. Let X be a metric space and $A \subseteq X$. We define the **diameter** of A to be the least upper bound:

$$\operatorname{diam} A = \sup \{d(x, y) : x, y \in A\}$$

of all distances of points in A.

Theorem 2.3.4 (Cantor's theorem)**.** *A metric space X is complete if, and only if for every decreasing sequence $\{F_n\}$ of nonempty closed sets, with $\operatorname{diam} F_n \to 0$ for all n, then the intersection*

$$F = \bigcap F_n$$

consists of a single point.

Proof. Suppose that X is complete. Let $\{F_n\}$ a sequence of closed sets such that

(1) $F_{n+1} \subseteq F_n$; i.e. $\{F_n\}$ is a decreasing sequence.

(2) $\lim \operatorname{diam} F_n \to 0$.

Let $x_n \in F_n$. If $n, m \geq N$ then $x_m, x_n \in F_N$ so that $d(x_m, x_n) \leq \operatorname{diam} F_n$ by definition. By hypothesis, choose an N large enough such that $\operatorname{diam} F_N < \varepsilon$ for some $\varepsilon > 0$. This makes the sequence $\{x_n\}$ Cauchy. Then by the completeness of X $\{x_n\} \to x$ for some $x \in X$. Since $x_n \in F_n$ for all $n \geq N$, we get that $F_n \subseteq F_N$ and hence $x \in F_N$ which puts

$$x \in F = \bigcap F_n$$

Now, if $y \in F$, then $x, y \in F_n$ for all n which gives $d(x, y) \leq \operatorname{diam} F_n \to 0$. So $d(x, y) = 0$ which makes $x = y$ and so $F = \{x\}$.

Conversely, let $\{x_n\}$ be Cauchy in X, and take $F_n = \operatorname{cl}\{x_n, x_{n+1}, \dots\}$. Then $F_{n+1} \subseteq F_n$, making $\{F_n\}$ decreasing sequence. If $\varepsilon > 0$, choose an $N > 0$ such that $d(x_m, x_n) < \varepsilon$ for any $m, n \geq N$. Then $\operatorname{diam} F_n \leq \varepsilon$. By hypothesis, there is an $x_0 \in X$ such that $F = \bigcap F_n = \{x_0\}$. Moreover, $x_0 \in F_n$ so that $d(x_0, x_m) \leq \operatorname{diam} F_n \to 0$, which puts $\{x_n\} \to x \in X$ which makes X complete. ■

Lemma 2.3.5. *If X is a complete metric space, and $Y \subseteq X$, then Y is complete if, and only if Y is closed in X.*

Proof. Suppose that Y is complete and let y a limit point of Y. Then there exists a sequence $\{y_n\}$ of points of Y for which $\{y_n\} \to y$. This makes $\{y_n\}$ Cauchy, and so $\{y_n\} \to x_0 \in Y$. It follows that $y = x_0$, so that $Y' \subseteq Y$ and hence Y is closed. ■

2.4 Compactness in $\mathbb{C}$

Definition. Let X be a metric space. We say an collection $\{U_n\}$ of open sets of X **covers** a subset K of X if $K \subseteq \bigcup U_n$. We call $\{U_n\}$ an **open cover** of K. We call K **compact** if every open cover of K has a finite open subcover.

Lemma 2.4.1. *If K is compact in a metric space X, then K is closed. Moreover, if $F \subseteq K$ is closed, then F is also compact.*

Proof. Certainly, we have $K \subseteq \operatorname{cl} K$. Now, let $x_0 \in \operatorname{cl} K$, then $B(x_0, \varepsilon) \cap K$ is nonempty for every $\varepsilon > 0$. Let $G_n = X \backslash \overline{B}(x_0, \frac{1}{n})$, and suppose that $x_0 \notin K$. Then each G_n is open in X, and $K \subseteq \bigcup G_n$. Since K is compact, then ther is an $m \in \mathbb{Z}^+$ for whcih $K \subseteq \bigcup_{n=1}^{m} G_n$. Notice, however that $G_1 \subseteq G_2 \subseteq \cdots \subseteq G_m \subseteq \ldots$ so that $K \subseteq G_m = X \backslash \overline{B}(x_0, \frac{1}{m})$, so that $B(x_0, \frac{1}{n}) \cap K = \emptyset$; a contradiction! Therefore $x_0 \in K$ and $K = \operatorname{cl} K$. ■

Definition. Let X be a set. We say a collection $\{F_n\}$ of subsets of X has the **finite intersection property** (**FIP**) if the intersection of any finite subcollection of $\{F_n\}$ is nonempty.

Lemma 2.4.2. *A set K of a metric space X is compact if, and only if for every collection of closed sets $\{F_n\}$ satisfying the finite intersection property, the intersection*

$$F = \bigcap F_n$$

is nonempty.

Proof. Let K be compact in X, and $\{F_n\}$ a collection of closed sets of X with the FIP. Suppose that $F = \bigcap F_n = \emptyset$. Now, take $\mathcal{G} = \{X \backslash F_n\}$ the collecton of open sets. Then observe that

$$\bigcup X \backslash F_n = X \backslash \bigcap F_n = X \backslash F = X$$

by hypothesis. SInce $K \subseteq K$, $\mathcal{G}$ covers K, and since K is compact, there is a finite subcover $\{X \backslash F_i\}_{i=1}^{n}$ of K. That is

$$K \subseteq \bigcup_{i=1}^{n} X \backslash F_i = X \backslash \bigcap_{i=1}^{n} F_i \subseteq X$$

since $\bigcap_{i=1}^{n} F_n \neq \emptyset$. But then $\bigcap_{i=1}^{n} F_i \subseteq X \backslash K$, and since $F_i \subseteq K$ for all $1 \leq i \leq n$, this makes $\bigcap_{i=1}^{n} F_i =$; a contradiction! ■

Corollary. *Compact metric spaces are complete.*

Corollary. *If X is compact, then every infinite set in X has a limit point in X.*

Proof. Let $S \subseteq X$ infinite, and suppose the set of all limit points of S in X, S', is empty. Consider the sequence $\{a_n\}$ of distinct points of S, and take $F_n = \{a_n, a_{n+1}, \ldots\}$. Then F_n has no limit points in X so that $F_n' = \emptyset$. Then $F_n' \subseteq F_n$ so that F_n is closed. Thus $\{F_n\}$ has the finite intersection property. But since $a_1 \neq \ldots \neq a_n \neq$, we get $\bigcap F_n = \emptyset$; which contradicts the above. Therefore S' is nonempty. ■

Definition. We call a metric space **sequentially compact** if every sequence of point in the space has a convergent subsequence.

Lemma 2.4.3 (Lebesgue's Covering Lemma)**.** *If X is a sequentially compact metric space, and $\mathcal{G}$ is an open cover of X, then there is an $\varepsilon > 0$ such that if $x \in X$ there is a $G \in \mathcal{G}$ with $B(x, \varepsilon) \subseteq G$.*

Proof. Suppose by contradiction that for every open cover $\mathcal{G}$ of X there is no ε for which the statement holds. Then for every $n \in \mathbb{Z}^+$, there is an $x_n \in X$ for which $B(x_n, \frac{1}{n}) \nsubseteq G$. Now, since X is sequentially compact, there is a point $x_0 \in X$ and s subsequence $\{x_{n_k}\}$ of a sequence $\{x_n\}$ for which $\{x_{n_k}\} \to x_0$. Let $G_0 \in \mathcal{G}$ such that $x_0 \in G_0$. Choose $\varepsilon > 0$ such that $n_k \geq N$ and $n_k > \frac{1}{\varepsilon}$. Let $y \in B(x_{n_k}, \frac{1}{n_k})$. THen $d(x_0, y) \leq d(x_0, x_{n_k}) + d(x_{n_k}, y) < \frac{\varepsilon}{2} + \frac{1}{n_k} < \varepsilon$. So that $B(x_{n_k}, \frac{1}{n_k}) \subseteq B(x_0, \varepsilon)$. But that contradicts our choice of $\{x_{n_k}\}$. ■

Definition. We say a subset K of a metric space X is **totally bounded** if for any $\varepsilon > 0$ there exist a sequence $\{x_n\}$ of points of X for which $K = \bigcup_{k=1}^{n} B(x_k, \varepsilon)$.

Theorem 2.4.4. *The following are equivalent in every metric space X.*

(1) X is compact.

(2) Every infinite set of X has a limit point in X.

(3) X is sequentially compact.

(4) X is complete, and totally bounded.

Proof. We have that if X is compact, then every infinite set of X has their limit points in X, by the above corollory.

Suppopse every infinite set of X has a limit point in X. Let $\{x_n\}$ a sequence, and suppose without loss of generality, that all the points are distinct. Then $\{x_n\}$ has a limit point x_0. Then there exist an $x_{n_1} \in B(x_0, 1)$. Similarly, there is an $n_2 > n_1$ with $x_{n_2} \in B(x_0, \frac{1}{2})$. Continuiing in this manner, we get for some $n_k > n_{k-1}$, that $x_{n_k} \in B(x_0, \frac{1}{k})$, so that $\{x_{n_k}\} \to x_0$; and so X is sequentially compact.

Suppose now that X is sequentially compact, and let $\{x_n\}$ be a Cauchy sequence. By the sequential compactness of $\{x_n\}$, it has a convergent subsequence, which makes X complete. Now, let $\varepsilon > 0$ and fix $x_1 \in X$. If $X = B(x_1, \varepsilon)$, we are done. Otherwise, choose an $x_2 \in X \backslash B(x_1, \varepsilon)$. If $X = B(x_1, \varepsilon) \cup B(x_2, \varepsilon)$ we are done. Otherwise, continuing in this manner, we find a sequence $\{x_n\}$ of points with $x_{n+1} \in X \backslash \bigcup_{k=1}^{n} B(x_k, \varepsilon)$. Which implies for $m \neq n$, that $d(x_m, x_n) \geq \varepsilon > 0$. Contradictiong that X is sequentially compact. So we have that X must be totally bounded.

Conversely, suppose that X is complete and totally bounded. Let $\{x_n\}$ a sequence of distint points. Then there is a $y_1 \in X$ and a subsequence $\{x_n^{(1)}\}$ of $\{x_n\}$ for which $\{x_n^{(1)}\} \subseteq B(y_1, 1)$. There also exists a $y_2 \in X$ and s a subsequence $\{x_n^{(2)}\}$ of $\{x_n^{(1)}\}$ such that $\{x_n^{(2)}\} \subseteq B(y_2, \frac{1}{2})$. Continuing in this manner, for all $k \geq 2$, there is a $y_k \in X$ and a subsequence $\{x_n^{(k)}\}$ of $\{x_n^{(k-1)}\}$ for which $\{x_n^{(k)}\} \subseteq B(y_k, \frac{1}{k})$. Take $K_k \operatorname{cl}\{x_n^{(k)}\}$. Then

$$\operatorname{diam} F_k \leq \frac{1}{k}$$

and $\{F_k\}$ is a decreasing collection of closed sets. Thus the intersection $F = \{x_0\}$ is a single point. So $x_0 \in F_k$, so that

$d(x_0, x_n^{(k)}) \leq F_k \leq \frac{1}{k}$ so that $\{x_n^{(k)}\} \to x_0$, making X sequentially compact.

Finally, if X is sequentially compact, and $\mathcal{G}$ is an open cover of X, then there exists an $\varepsilon > 0$ such that for every $x \in X$, there is a $G \in \mathcal{G}$, with $B(x, \varepsilon) \subseteq G$. Hence there is a sequence $\{x_n\}$ of points of X for which $X = \bigcup B(x_n, \varepsilon)$ (i.e. X is totally bounded). Then there is a $G_n \in \mathcal{G}$ for all $1 \leq k \leq n$ for which $B(x_k, \varepsilon) \subseteq G_k$. So tghat $X = \bigcup G_k$ which makes X compact. ∎

Theorem 2.4.5 (Heine-Borel)**.** *A subset K of $\mathbb{R}^n$ is compact if, and only if it is closed and bounded.*

Proof. Suppose that K is compact, then K is closed by lemma 2.4.1, and K is also totally bounded, which makes K bounded. So K is closed and bounded in $\mathbb{R}^n$.

Conversely, suppose that K is closed and bounded. Then there are sequences $\{a_k\}_{k=1}$ and $\{b_k\}_{k=1}^n$ for which $K \subseteq [a_1, b_1] \times [a_n, b_n]$. Now, since $\mathbb{R}^n$ is complete, and K is closed, K is also complete. Hence it remains to show that K is totally bounded. Let $\varepsilon > 0$, and write K as the union of n-dimensional rectangles of diameters less than ε. Then $K \subseteq \bigcup_{k=1}^m B(x_k, \varepsilon)$ where x_k is contained in one of the rectangles, for all $1 \leq k \leq m$. This makes K totally bounded, and therefore, compact. ■

2.5 Continuity and Uniform Convergence in $\mathbb{C}$

Definition. Let (X, d) and (Y, ρ) be metric spaces, and $f : X \to Y$ a function. We say that f is **continuous** at a point $a \in X$ if for every $\varepsilon > 0$, there is a $\delta > 0$ for which

$$\rho(f(x), y) < \varepsilon \text{ whenever } 0 < d(x, a) < \delta$$

for some $y \in Y$ and we write $\lim_{x \longrightarrow a} f(x) = y$, or simply $f \to y$. If f is continuous at every point in X, we say that f is **continuous** on X (or simply that f is **continuous**).

Lemma 2.5.1. *Let X and Y be metric spaces. If $f : X \to Y$ is a function, then the following statements are equivalent for any $a \in X$ with $y = f(a)$.*

(1) f is continuous at a.

(2) For any $\varepsilon > 0$ $f^{-1}(B(y, \varepsilon))$ contains a ball centered about a.

(3) If $\{x_n\}$ is a sequence of points of X converging to a, then the sequence $\{f(x_n)\}$ converges to y.

Lemma 2.5.2. *Let X and Y be metric spaces, and $f : X \to Y$ a function. The following statements are equivalent.*

(1) f is continuous on X.

(2) For any open set U of Y, $f^{-1}(U)$ is open in X.

(3) For any closed set V of Y, $f^{-1}(V)$ is closed in X.

Lemma 2.5.3. *Let $f : X \to \mathbb{C}$ and $g : X \to \mathbb{C}$ be complex-valued functions. If f and g are continuous, then for every $\alpha, \beta \in \mathbb{C}$, we have*

(1) $\alpha f + \beta g$ is continuous.

(2) fg is continuous, and $\frac{f}{g}$ is continuous provided $g(z) \neq 0$ for all $z \in X$.

Lemma 2.5.4. *If $f : X \to Y$ and $g : Y \to Z$ are continuous, then $g \circ f : X \to Z$ is continuous.*

Definition. We call a function $f : X \to Y$ **uniformly continuous** if for every $\varepsilon > 0$, there is a $\delta > 0$, depending on ε, such that

$$\rho(f(x), f(y)) < \varepsilon \text{ whenever } d(x, y) < \delta$$

We call f **Lipschitz continuous** if there exists an $M > 0$ such that

$$\rho(f(x), f(y)) = Md(x, y) \text{ for all } x, y \in X$$

Lemma 2.5.5. *Lipschitz continuous functions are uniformly continuous, and uniformly contiuous functions are continuous.*

Definition. Let X be a metric space, and $A \subseteq X$ a nonempty subset. We define the **distance** from a point $x \in X$ to A to be

$$d(x, A) = \inf \{d(x, a) : a \in A\}$$

Lemma 2.5.6. *Let X a metric space, and $A \subseteq X$ nonempty. The following are true.*

(1) $d(x, A) = d(x, \operatorname{cl} A)$.

(2) $d(x, A) = 0$ if, and only if $x \in \operatorname{cl} A$.

(3) $|d(x, A) - d(y, A)| \leq d(x, y)$ for all $x, y \in X$.

Proof. Let $A \subseteq B$. Then by definition, $d(x, B) \leq d(x, A)$, so that $d(x, \operatorname{cl} A) \leq d(x, A)$. Now, if $\varepsilon > 0$, there is a $y \in \operatorname{cl} A$ for which $d(x, y) \leq d(x, \operatorname{cl} A) + \frac{\varepsilon}{2}$, and there exists an $a \in A$ with $d(y, a) < \frac{\varepsilon}{2}$. Then

$$|d(x, y) - d(x, a)| < d(y, a) < \frac{\varepsilon}{2}$$

by the triangle inequality. Then $d(x, a) < d(x, y) + \frac{\varepsilon}{2}$ so that $d(x, A) < d(x, \operatorname{cl} A) + \frac{\varepsilon}{2}$. That is $d(x, A) \leq d(x, \operatorname{cl} A)$.

Now, if $x \in \operatorname{cl} A$, then $d(x, \operatorname{cl} A) = d(x, A) = 0$. Conversly, if $d(x, A) = 0$, then consider the decreasing sequence $\{a_n\}$ of A such that $\lim d(x, a_n) = d(x, A)$. Then $\lim d(x, a_n) = 0$ so that $\lim a_n = x$, so that $x \in \operatorname{cl} A$.

Finally, we have for $a \in A$ that $d(x, a) \leq d(x, y) + d(y, a)$, so that $d(x, A) \leq \inf \{d(x, y) + d(y, a) : a \in A\}$ $d(x, y) + d(y, A)$. This gives $d(x, A) - d(y, A) \leq d(x, y)$. Similar reasoning also gives $d(y, A) - d(x, A) \leq d(x, y)$ so that

$$|d(x, A) - d(y, A)| \leq d(x, y) \text{ for all } x, y \in X$$

■

Corollary. *The function $f : X \to \mathbb{R}$ defined by $f(x) = d(x, A)$ is Lipschitz continuous.*

Theorem 2.5.7. *Let $f : X \to Y$ be continuous. Then following are true.*

(1) If X is compact, then so is $f(X)$.

(2) If X is connected, so is $f(X)$.

Proof. Without loss of generality, suppose $f(X) = Y$. If X is compact, et $\{y_n\}$ a sequence in Y. Then for every $n \geq 1$, there is a sequence of points $\{x_n\}$ of X with $f(x_n) = y_n$, and $\{x_{n_k}\} \to x$. If $y = f(x)$, then by continuity, $\{y_{n_k}\} \to y$ so that Y is also compact.

Now, if X is connected, let $S \subseteq Y$ a nonempty set wich is both open and closed. Then $f^{-1}(S) \neq \emptyset$ and $f^{-1}(S)$ is also open and closed, so that $X = f^{-1}(S)$ by connectivity. This makes $S = Y$, and so Y must also be continuous. ■

Corollary. *If K is compact or connected in X, then $f(K)$ is compact or connected in Y.*

Corollary. *If $f : X \to \mathbb{R}$ is continuous, and X is connected, then $f(X)$ is an interval.*

Theorem 2.5.8 (The Intermediate Value Theorem)**.** *If $f[a, b] \to \mathbb{R}$ is continuous, with $f(a) \leq c \leq f(b)$, then there is an $x \in [a, b]$ with $f(x) = c$.*

Corollary. *If $K \subseteq X$ is compact, then there exist $x_0, y_0 \in K$ with $f(x_0) = \sup\{f(x) : x \in K\}$ and $f(y_0) = \inf\{f(y) : y \in K\}$.*

Corollary. *If $K \subseteq X$ is nonempty, and $x \in X$, there is a $y \in K$ for which $d(x, y) = d(x, K)$.*

Proof. Define $f : X \to \mathbb{R}$ by $f(y) = d(x, y)$. Then f is continuous, and by above, assumes a minimum value $yinK$. Then $f(y) \leq f(x)$ for all $x \in K$, so that $d(x, y) = d(x, K)$ by definition. ■

Theorem 2.5.9. *Let $f : X \to Y$ be continuous. If X is compact, then f is uniformly continuous.*

Proof. Let $\varepsilon > 0$ and suppose there is no such $\delta > 0$ for which the statement holds. Then each $\delta = \frac{1}{n}$ in particular fais. Then there exist $x_n, y_n \in X$ with $d(x_n, y_n) < \frac{1}{n}$, but where $\rho(f(x_n), f(y_n)) \geq \varepsilon$. Now, since X is compact, there is a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ converging to a point $x \in X$. Now, $d(x, y_{n_k}) \leq d(x, x_{n_k}) + \frac{1}{n_k}$ which goes to 0 as $k \to \infty$. SO $\{y_{n_k}\} \to x$. But if, $y = f(x)$, and $y = \lim f(x_{n_k}) = \lim f(y_{n_k})$, then we get

$$\varepsilon \leq \rho(f(x_{n_k}), f(y_{n_k})) \leq \rho(f(x_{n_k}), y) + \rho(y, f(y_{n_k})) = 0$$

which is a contradiction since $\varepsilon > 0$. ■

Definition. If $A, B \subseteq X$ are nonempty subsets of a metric space X, we define the **distance** between A and B to be

$$d(A, B) = \inf\{d(a, b) : a \in A, b \in B\}$$

Theorem 2.5.10. *let A and B be disjoint subsets of a metric space X; with B closed, and A compact. Then $d(A, B) > 0$.*

Proof. Define $f : X \to \mathbb{R}$ by $f(x) = d(x, B)$. Since A and B are disjoint, and B is closed, $f(a) > 0$ for all $a \in A$. Moreover, since A is compact, there is an $a \in A$ for which $0 < f(a) = \inf\{f(x) : x \in A\} = d(A, B)$. ■

Definition. Let X be a set, and (Y, ρ) a metric space; and let $\{f_n\}$ a sequence of functions from X to Y. We say that $\{f_n\}$ **converges uniformly** if for every $\varepsilon > 0$, there is an $N > 0$, dependent on ε such that

$$\rho(f(x), f_n(x)) < \varepsilon \text{ whenever } n \geq N$$

for all $x \in X$. We write $\{f_n\} \xrightarrow{\text{uniformly}} f$, or just $\{f_n\} \to f$.

Theorem 2.5.11. *If $f_n : X \to Y$ is continuous for each $n \geq 1$, and $\{f_n\} \xrightarrow{\textit{uniformly}} f$, then f is also continuous.*

Proof. Fix $x_0 \in X$ and let $\varepsilon > 0$. Since $\{f_n\} \to f$, there is a function f_n for which $\rho(f(x), f_n(x)) < \frac{\varepsilon}{3}$ for every $x \in X$. Since f_n is continuous, there is a $\delta > 0$ such that

$$\rho(f_n(x_0), f_n(x)) < \frac{\varepsilon}{3} \text{ whenever } d(x, x_0) < \delta$$

Therefore, if $d(x_0, x) < \delta$ we have

$$\rho(f(x_0), f(x)) \leq \rho(f(x_0), f_n(x_0)) + \rho(f_n(x_0), f_n(x)) + \rho(f_n(x), f(x)) < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon$$

so that f is continuous. ∎

Theorem 2.5.12 (The Weierstrass M-test)**.** *Let $u_n : X \to \mathbb{C}$ be a function such that $\|u_n(x)\| \leq M_n$, for all $x \in X$, and suppose that the sum $\sum M_n$ is finite. THen $\sum u_n$ is uniformly convergent.*

Proof. Let $f_n(x) = u_1(x) + \cdots + u_n(x)$. Then for $n > m$, $\|f_n(x) - f_m(x)\| = \|u_{m+1}(x) + \cdots + u_n(x)\| \leq \sum_{k=m+1}^{n} M_k$. Since $\sum M_k$ is finite, this sum converges, so that $\{f_n\}$ is Cauchy in $\mathbb{C}$. That is, there exists a $\xi \in \mathbb{C}$ for which $\{f_n(x)\} \to \xi$. Define then $f(x) = \xi$, then $f : X \to \mathbb{C}$ is a function with

$$\|f(x) - f_n(x)\| = \|u_{m+1}(x) + \cdots + u_n(x)\| \leq \sum_{k=m+1}^{n} \|u_k(x)\| \leq \sum_{k=m+1}^{n} M_k$$

Then for every $\varepsilon > 0$, there is an $N > 0$ such that $\sum M_k < \varepsilon$, whenever $n \geq N$. THus $\|f(x) - f_n(x)\| < \varepsilon$ for all $x \in X$. ∎

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