# Field Theory and Galois Theory.

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# Chapter 1

# Fields.

## 1.1 Field Extensions.

**Definition.** We define the **characteristic** of a field $F$ to be the smallest positive integer $p$, such that $p \cdot 1 = 0$, where 1 is the identity of $F$. We write $\operatorname{char} F = p$, and if no such $p$ exists, then we write $\operatorname{char} F = 0$.

**Lemma 1.1.1.** *Let $F$ be a field, then $\operatorname{char} F$ is either $0$, or a prime integer.*

*Proof.* Let $\Gamma F = p$. If $p = 0$, then we are done. Now suppose that $p = mn$, with $m, n \in \mathbb{Z}^+$. Then $p \cdot 1 = (mn)1 = (n \cdot 1)(m \cdot 1) = mn = 0$, which makes $m$ and $n$ 0 divisors. Since $F$ is a field, and hence an integral domain, this is impossible, and hence $p$ must be prime. ■

**Corollary.** *If $\operatorname{char} F = p$, then for all $a \in F$, $pa = \underbrace{a + \cdots + a}_{p \text{ times}}$.*

*Proof.* We have $pa = p(a \cdot 1) = (p \cdot 1)a$. ■

**Example 1.1.**
(1) Both $\mathbb{Q}$ and $\mathbb{R}$ have char $= 0$. Similarly, $\operatorname{char} \mathbb{Z} = 0$, even though $\mathbb{Z}$ is just an integral domain.

(2) $\operatorname{char} \mathbb{Z}/p\mathbb{Z} = p$ and $\operatorname{char} \mathbb{Z}/p\mathbb{Z}[x] = p$ for any prime $p$.

**Definition.** We define the **prime subfield** of a field $F$ to be the subfield of $F$ generated by 1.

**Example 1.2.**
(1) The prime subfields of $\mathbb{Q}$ and $\mathbb{R}$ is $\mathbb{Q}$.

(2) Let $\mathbb{Z}/p\mathbb{Z}(x)$ the field of rational functions over $\mathbb{Z}/p\mathbb{Z}$. Then the prime subfield of $\mathbb{Z}/p\mathbb{Z}(x)$ is $\mathbb{Z}/p\mathbb{Z}$. Similarly, the prime subfield for $\mathbb{Z}/p\mathbb{Z}[x]$ is also $\mathbb{Z}/p\mathbb{Z}$.

**Definition.** If $K$ is a field containing a field $F$, then we call $K$ **field extension** over $F$, and write $K/F$ (not the quotient field!) or denote it by the diagram

$$\begin{array}{c} K \\ | \\ F \end{array}$$

**Lemma 1.1.2.** *Every field is a field extension of its prime subfield.*

**Lemma 1.1.3.** *Let $K$ an extension over a field $F$. Then $K$ is a vector space over $F$.*

**Definition.** Let ${}^{K}\!/_{F}$ a field extension. We define the **degree** of $K$ over $F$, $[K : F]$ to be the dimension of ${}^{K}\!/_{F}$ as a vector space.

**Definition.** Let $F$ be a field, and $f \in F[x]$ a polynomial. We call am element $\alpha \in R$ a **root** (or **zero**) of $f$ if $f(\alpha) = 0$.

**Lemma 1.1.4.** *Let $\phi : F \to L$ a field homomorphism. Then either $\phi = 0$, or $\phi$ is 1–1.*

**Lemma 1.1.5.** *Let $F$ be a field, and $p \in F[x]$ an irreducible polynomial. Then there exists a field $K$ containing an embedding of $F$, such that $p$ has a root in $K$.*

*Proof.* Consider $K = {}^{F[x]}\!/_{(p)}$. Siince $p$ is irreducible in a principle ideal domain, $(p)$ is a maximal idea, and hence $K$ is a field. Now consider the canonical map $\pi : F[x] \to K$ taking $f \to f \mod (p)$ and let $\phi = \pi|_F$. Then $\phi \neq 0$, since $\pi : 1 \to 1$. Then $\phi$ is 1–1. And so $\phi(F) \simeq F$.

Now, consider $F$ as a subfield of $K$. Then $p(x \mod (p)) \equiv p(x) \mod (p) \equiv 0 \mod (p)$, so that $x \mod (p)$ is a root of $p$ in $K$. ∎

**Corollary.** *There exists a field extension of $F$ containing a root of $p$.*

**Theorem 1.1.6.** *Let $F$ be a field, and let $p \in F[x]$ an irreducible polynomial of degree $n$, and let $K = {}^{F[x]}\!/_{(p)}$, and $\theta = x \mod (p)$. Then $\{1, \theta, \dots, \theta^{n-1}\}$ forms a basis for $K$ as a vector space over $F$ and $[K : F] = n$.*

*Proof.* Let $a \in F[x]$, since $F[x]$ is Euclidean domain, there exist $q, r \in F[x]$, $q \neq 0$ for which

$$a(x) = q(x)p(x) + r(x) \text{ where } \deg r < n$$

Now, since $pq \in (p)$, $a(x) \equiv r(x) \mod (p)$, and every element of $K$ is a polynomial of degree less than $n$. Then the elements $\{1, \theta, \dots, \theta^{n-1}\}$ span $K$.

Now, suppose that there are $b_0, \dots, b_{n-1} \in F$ not all 0 for which

$$b_0 + b_1\theta + \cdots + b_{n-1}\theta^{n-1} = 0$$

Then

$$b_0 + b_1\theta + \cdots + b_{n-1}\theta^{n-1} \equiv 0 \mod (p)$$

so that $p|(b_0 + b_1\theta + \cdots + b_{n-1}\theta^{n-1})$ in $F$. But $\deg p = n$ and $p$ divides a polynomial of degree $n - 1$, which is a contradiction. Therefore we are left with $b_0 = \cdots = b_{n-1} = 0$. ∎

**Corollary.** $K = \{\alpha_0 + a_1\theta + \cdots + a_{n-1}\theta^{n-1} : a_i \in F \textit{ for all } 1 \leq i \leq n - 1\}$

**Corollary.** *If $a(\theta), b(\theta) \in K$, are elements of degree less than $n$, and the operations of polynomial addition, and polynomial multiplication mod $(p)$ are defined, then $K$ forms a field.*

**Example 1.3.** (1) Consider the polynomial $x^2+1$ over $\mathbb{R}$. Then one has the field

$$\mathbb{C} = \mathbb{R}[x]\big/(x^2+1)$$

an extension of $\mathbb{R}$ of degree $[\mathbb{C} : \mathbb{R}] = 2$. Let $i$ be a root of $x^2+1$ in this field, then $i^2 = -1$, and the elements of $\mathbb{C}$ are of the form $a + ib$ where $a, b \in \mathbb{R}$. Then we have described the field of complex numbers, and the addition and multiplication ( mod $x^2+1$) of these elements are the addition and multiplication of complex numbers.

One might also construct $\mathbb{C}$ differently by defining the isomorphism

$$\mathbb{R}[x]\big/(x^2+1) \to \mathbb{C} \text{ taking } a + xb \to a + ib$$

(2) Consider again $x^2+1$ over $\mathbb{Q}$. Then we get the field

$$\mathbb{Q}(i) = \mathbb{Q}[x]\big/(x^2+1)$$

of degree $[\mathbb{Q}(i) : \mathbb{Q}] = 2$, and where $i$ is a root of $x^2+1$, so that $i^2 = -1$. Then the elements of $\mathbb{Q}(i)$ are of the form $a + ib$ where $a, b \in \mathbb{Q}$, i.e. it is isomorphic to the set of all complex numbers with rational components.

(2) Consider $x^2-2$ over $\mathbb{Q}$. by Eisenstein's criterion for $p = 2$, $x^2-2$ is irreducible over $\mathbb{Q}$. Let $\alpha$ a root of $x^2-2$, so that $\alpha^2 = 2$. Then we have the field

$$\mathbb{Q}(\sqrt{2}) = \mathbb{Q}[x]\big/(x^2-2)$$

of degree $[Q(\sqrt{2}) : \mathbb{Q}] = 2$, and whose elements are of the form $a + b\sqrt{2}$. One can define an isomorphism between $\mathbb{Q}(i)$ and $\mathbb{Q}(\sqrt{2})$ by taking $\sqrt{2} \to i$.

(3) The polynomial $x^3-2$ over $\mathbb{Q}$ gives us the field

$$\mathbb{Q}(\sqrt[3]{2}) = \mathbb{Q}[x]\big/(x^3-2)$$

of degree $[\mathbb{Q}(\sqrt[3]{2}) : \mathbb{Q}] = 3$ over 2. Here the elements are of the form $a + b\xi + c\xi^2$ where $\xi^3 = 2$.

(4) Denote $\mathbb{F}_2$ to be a finite field of 2 elements. Consider the polynomial $x^2+x+1$ over $\mathbb{F}_2$ which is irreducible. Then the field

$$\mathbb{F}_2(\alpha) = \mathbb{F}_2[x]\big/(x^2+x+1)$$

is a field of degree 2 over $\mathbb{F}_2$, whose elements are of the form $a + b\alpha$, where $\alpha^2 = \alpha + 1$. In fact, one can generate this field using the fact that $\alpha^2 = \alpha + 1$.

(5) Let $F = K(t)$ the field of rational functions in $t$ over a field $K$. Let $p(x) = x^2 - t \in F[x]$, then by Eisenstien's criterion with the ideal $(t)$, $p$ is irreducible over $F[x]$. Let $\theta$ be a root for $p$, that is $\theta = \sqrt{t}$, then we get the field $K(t, \sqrt{t})$ of degree $[K(t, \sqrt{t}) : K] = 2$, whose elements are of the form $a(t) + b(t)\sqrt{t}$.

**Lemma 1.1.7.** *Let $F$ be a subfield of a field $K$, and let $\alpha \in K$. Then there exists a unique minimal subfield of $K$ containing $F$ and $\alpha$; more preciesly, it is the intersection of all subfields of $K$ containing $F$ and $\alpha$.*

**Definition.** Let $K$ be any extension of a field $F$, and let $\alpha, \beta, \dots \in K$. Then we define the subfield **generated** by $\alpha, \beta, \dots$ over $F$ to be the unique minimal subfield containing all $\alpha, \beta, \dots$ and $F$ and we denote it $F(\alpha, \beta, \dots)$. Moreover, we call $K$ a **simple extension** of $F$ if $K = F(\alpha, \beta, \dots)$. If $K = (F\alpha_1, \dots, a_n)$ for $\alpha_1, \dots, a_n \in K$, then it is a **finitely generated** simple extension.

**Theorem 1.1.8.** *Let $F$ be a field, and $p \in F[x]$ irreducible, and let $K$ an extension of $F$ containing a root $\alpha$ of $p$. Then*

$$F(\alpha) \simeq {}^{F[x]}\!\diagup_{(p)}$$

*Proof.* Consider the homomorphism $F[x] \to F(\alpha)$ taking $a(x) \to a(\alpha)$. Since $p(\alpha) = 0$, $p$ is in the kernel of this homomorphism, and we get an induced homomorphsim from ${}^{F[x]}\!\diagup_{(p)} \to F(\alpha)$. Now, since $p$ is irreducible, ${}^{F[x]}\!\diagup_{(p)}$ is a field, and since the homomorphsim takes $1 \to 1$, it is 1–1. Then by the first isomorphsim theorem for ring homomorphsims these two fields are isomorphic. ∎

**Corollary.** *If $\deg p = n$, then $F(\alpha) = \{a_0 + a_1\alpha + \dots a_{n-1}\alpha^{n-1} : a_i \in F$ for all $1 \le i \le n-1\}$ and $[F(\alpha) : F] = n$.*

**Example 1.4.** (1) The polynomial $x^2 - 2$ over $\mathbb{Q}$ also has the root $-\sqrt{2}$ in $\mathbb{R}$, so that $\mathbb{Q}(-\sqrt{2})$ is of degree 2 over $\mathbb{Q}$ with elements of the form $a - b\sqrt{2}$. Notice however that $\mathbb{Q}(-\sqrt{2}) \simeq \mathbb{Q}(\sqrt{2})$ by taking $a - b\sqrt{2} \to a + b\sqrt{2}$.

(2) The polynomial $x^3 - 2$ only has the solution $\xi = \sqrt[3]{2}$ in $\mathbb{R}$. However, in $\mathbb{Q}$ it has the solutions given by

$$\sqrt[3]{2}\left(\frac{-1 \pm i\sqrt{3}}{2}\right)$$

So that the subfields generated by either of these three elements (over $\mathbb{C}$) are isomorphic.

**Theorem 1.1.9.** *Let $\phi : F \to L$ a field isomorphism and $p \in F[x]$, $q \in L[x]$ irreducible polynomials, where $q$ is obtained by applying $\phi$ to the coefficients of $p$. Let $\alpha$ a root of $p$, and $\beta$ a root of $q$. Then there exists an isomorphism $F(\alpha) \to L(\beta)$ taking $\alpha \to \beta$ and extending $\phi$. That is, we have the following diagram*

$$\begin{tikzcd} F(\alpha) \arrow[r] \arrow[d, dash] & L(\beta) \arrow[d, dash] \\ F \arrow[r, "\phi"'] & E \end{tikzcd}$$

*Proof.* Notice that $\phi$ induces a ring homomorphism between $F[x]$ and $L[x]$, so that $(p)$ is maximal. Since $q$ is obtained from $p$, $(q)$ is also maximal, so that $F[x]/(p)$ and $L[x]/(q)$ are fields. Then we have an isomorphsism

$$F[x]/(p) \simeq L[x]/(q)$$

Then, if $\alpha$ is a root of $p$, and $\beta$ a root of $q$, we obtain the isomorphism

$$F(\alpha) \simeq L(\beta)$$

moreover, this isomorphism takes $\alpha \to \beta$. ■

## 1.2 Algebraic Extensions.

**Definition.** Let $K/F$ be a field extension. We say that an element $\alpha \in K$ is **algebraic** over $F$, provided there exists a polynomial over $F$ having $\alpha$ as a root. Otherwise we call $\alpha$ **transcendental**. If every $\alpha \in K$ is algebraic, we call $K$ **algebraic** and $K/F$ an **algebraic extension**.

**Lemma 1.2.1.** *Let $\alpha$ be algebraic over a field $F$. Then there exista a unique monic irreducible polynomial $m \in F[x]$ having $\alpha$ as a root. Moreover, if $f \in F[x]$ is a polynomial, then $f$ has $\alpha$ as a root if, and only if $m|f$.*

*Proof.* Let $m$ a polynomial of minimal degree having $\alpha$ as a root. Suppose, also that , is monic. Now, if $m$ were reducible, then $m(x) = a(x)b(x)$ for some $a, b \in F[x]$ polynomials both of degree less than $\deg m$. Then we also have that $a(\alpha) = b(\alpha) = 0$, which contradicts that $m$ is the polynomial of minimal degree satisfying that condition. Hence, $m$ is irreducible.

Now, let $f \in F[x]$ have $\alpha$ as a root, then by the divison theorem, there exist $q, r \in F[x]$, with $q \neq 0$ for which

$$f(x) = q(x)m(x) + r(x) \text{ where } \deg r < \deg m$$

Now, since $f(\alpha) = q(\alpha)m(\alpha) + r(\alpha) = 0$, then $r(x) = 0$ for all $x$ lest we contradict the minimality of $m$. Hence $m|f$. Conversely, if $m|f$, then $f$ has $\alpha$ as a root.

Now, let $g$ a polynomial of minimal degree for which $g(\alpha) = 0$. Then by above, we have that $\deg g = \deg m$, and that moreover, $m|g$ and $g|m$. therefore $g = m$ and uniqueness is established. ■

**Corollary.** *Let $L/F$ be an extension, and $\alpha$ algebraic over $F$. Let $m_{\alpha,F}$ the unique monic irreducible polynomial over $F$ having $\alpha$ as root, and $m_{\alpha,L}$ the unique monic irreducible polynomial over $L$ having $\alpha$ as root. Then $m_{\alpha,L}|m_{\alpha,F}$ in $L[x]$.*

**Definition.** Let $F$ be a field, and $\alpha$ algebraic over $F$. We define the **minimal polynomial** $m_{\alpha,F}$, to be the polynomial over $F$ of minimal degree having $\alpha$ as a root. If the field is clear, we instead write $m_\alpha$, or even just $m$ when the root itself is also clear. We define the **degree** of $\alpha$ to be $\deg \alpha = \deg m_\alpha$.

**Lemma 1.2.2.** *Let $\alpha$ algebraic over $F$. Then*

$$F(\alpha) \simeq {}^{F[x]}\!/\!_{(m_{\alpha,F})}$$

**Corollary.** $[F(\alpha) : F] = \deg m_\alpha = \deg \alpha$.

**Example 1.5.**

(1) The minimal polynomial for $\sqrt{2}$ over $\mathbb{Q}$ is $x^2 - 2$.

(3) The minimal polynomial for $\sqrt[3]{2}$ over $\mathbb{Q}$ is $x^3 - 2$.

(3) Let $n > 1$, then by the Eisenstein-Schömann criterion, $x^n - 2$ is irreducible over $\mathbb{Q}$. Moreover, $x^n - 2$ has as root in $\mathbb{R}$ $\sqrt[n]{2}$. Then $\mathbb{Q}(\sqrt[n]{2})$ is a field of degree $[\mathbb{Q}(\sqrt[n]{2}) : \mathbb{Q}] = 2$. Moreover $x^n - 2$ is the minimal polynomial of $\sqrt[n]{2}$. Notice, that over $\mathbb{R}$, $\deg [n]2 = 1$, and that $m_{\sqrt[n]{2},\mathbb{R}}(x) = x - \sqrt[n]{2}$.

(4) Consider $p(x) = x^3 - 3x - 1$ over $\mathbb{Q}$. Notice that $p$ is irreducible over $\mathbb{Q}$ and let $\alpha$ a root of $p$. Then $[\mathbb{Q}(\alpha) : \mathbb{Q}] = 3$.

**Lemma 1.2.3.** *An element $\alpha$ is algebraic over a field $F$ if, and only if the simple extension ${}^{F(\alpha)}\!/\!_{F}$ is finite.*

*Proof.* If $\alpha$ is algebraic over $F$ then $[F(\alpha) : F] = \deg \alpha \leq n$ if $\alpha$ satisfies a polynomial of degree $n$. Conversely, if $\alpha$ is an element of the finite extenson ${}^{K}\!/\!_{F}$, of degree $n$, then the set $\{1, \alpha, \dots, \alpha^n\}$ is linearly dependent over $F$. Hence there exist $b_0, \dots, b_n \in F$ not all 0 for which

$$b_0 + b_1\alpha + \cdots + a_n\alpha^n = 0$$

making $\alpha$ a root of a nonzero polynomial over $F$ of degree $\deg \leq n$. ■

**Corollary.** *If an extension ${}^{K}\!/\!_{F}$ is finite, then it is algebraic.*

*Proof.* If $\alpha \in K$ is algebraic, then ${}^{K}\!/\!_{F}$ implies that ${}^{F(\alpha)}\!/\!_{F}$ is finite, since $F(\alpha) \subseteq K$. ■

**Example 1.6.** Let $F$ a field of $\operatorname{char} F \neq 2$, and let $K$ an extension field of $F$ of degree $[K : F] = 2$. Let $\alpha \in K$ not in $F$, then $\alpha$ satisfies an polynomial of at most degree 2 over $F$. Now, since $\alpha \notin F$, this polynomial must have degree greater than 1. Hence it satisfies a polynomial of degree 2. Then the minimal polynomial of $\alpha$ is a quadratic

$$m_\alpha(x) = x^2 + bx + c \text{ with } b, c \in F$$

Since $F \subseteq F(\alpha) \subseteq K$, and $F(\alpha)$ is a vector space over $F$ of dimension 2, then we must have $K = F(\alpha)$; that is ${}^{K}\!/\!_{F}$ is simple.

Now, the roots of $m_\alpha$ are

$$\alpha = \frac{-b \pm \sqrt{b^2 - 4c}}{2}$$

Since $\alpha \notin F$, $b^2 - 4c$ is not a square in $F$, and $\sqrt{b^2 - 4c}$ is a root of the equation $x^2 - (b^2 - 4c) = 0$ in $K$.

Conversely, $\sqrt{b^2 - 4c} = \pm(b + 2\alpha)$ which puts $\sqrt{b^2 - 4c} \in F(\alpha)$. That is $F(\sqrt{b^2 - 4c}) = \mathbb{F}(\alpha)$. Moreover, $x^2 - (b^2 - 4c)$ does not have solutions in $K$.

We call field extensions ${}^{K}\!/\!_{F}$ of degree 2 **quadratic field extension**, where $K = F(\sqrt{D})$, and $D$ is a squarefree element of $F$.

**Theorem 1.2.4.** *Let $F \subseteq K \subseteq L$. Then $[L : F] = [L : K][K : F]$.*

*Proof.* Let $[L : K] = m$ and $[K : F] = n$. Let $\{\alpha_1, \dots, \alpha_m\}$ and $\{\beta_1, \dots, \beta_n\}$ be bases for the extensions ${}^{L}\!/_{K}$ and ${}^{K}\!/_{F}$. Now, the elements of $L$ over $K$ are of the form

$$a_1\alpha_1 + \cdots + a_m\alpha_m \text{ where } a_i \in K \text{ for all } 1 \leq i \leq m$$

Since each $a_i \in K$, which is an extension over $F$, they have the form

$$a_i = b_{i1}\beta_{i1} + \cdots + b_{in}\beta_{in} \text{ where } b_{ij} \in F \text{ for all } 1 \leq j \leq n$$

That is, every element of $L$, as a vector space over $F$ are of the form

$$\sum b_{ij}\alpha_i\beta_j$$

So the set $\{\alpha_1\beta_1, \dots \alpha_m\beta_n\}$ spans $L$. It remains to show that this set is linearly in dependent.

Suppose that

$$\sum b_{ij}\alpha_i\beta_j = 0$$

for some $b_{ij} \in F$. Since $\{\alpha_1, \dots, \alpha_m\}$ are linearly indpendent in $L$ over $K$, we have that the coefficients $a_1 = \cdots = a_n = 0$ which makes

$$a_i = b_{i1}\beta_{i1} + \cdots + b_{in}\beta_{in} = 0$$

Now, since $\{\beta_1, \dots, \beta_n\}$ is linearly independent in $K$ over $F$, this implies that $b_{i1} = \cdots = b_{in} = 0$ which makes the collection $\{\alpha_1\beta_1, \dots \alpha_m\beta_n\}$ linearly independent, and hence, a basis. Moreover, notice that this basis has size $mn$. ■

**Example 1.7.** (1) The element $\sqrt{2} \notin \mathbb{Q}(\alpha)$, where $\alpha$ is the root of $x^3 - 3x - 1$; since $[\mathbb{Q}(\sqrt{2}) : \mathbb{Q}] = 2$, and $[\mathbb{Q}(\alpha) : \mathbb{Q}] = 3$.

(2) We have $[\mathbb{Q}(\sqrt[6]{2}) : \mathbb{Q}] = 6$, and since $(\sqrt[6]{2})^3 = \sqrt{2}$, we observe that $\mathbb{Q}(\sqrt{2}) \subseteq \mathbb{Q}(\sqrt[6]{2})$. Moreover, notice that by theorem 1.2.4 $[\mathbb{Q}(\sqrt[6]{2}) : Q(\sqrt{2})] = 3$. Then we have the following tower of fields for

$$\mathbb{Q} \subseteq \mathbb{Q}(\sqrt{2}) \subseteq \mathbb{Q}(\sqrt[6]{2})$$

$$\begin{array}{c} \mathbb{Q}(\sqrt[6]{2}) \\ | \\ \mathbb{Q}(\sqrt{2}) \\ | \\ \mathbb{Q} \end{array}$$

**Lemma 1.2.5.** *Let $\alpha, \beta$ be algebraic over a field $F$. Then $F(\alpha, \beta) = (F(\alpha))(\beta)$.*

*Proof.* By definition, $F(\alpha, \beta)$ contains $F$, and $\alpha$, and hence contains $F(\alpha)$. It also contains $\beta$ so that $(F(\alpha))(\beta) \subseteq F(\alpha, \beta)$. By the same argument, $(F(\alpha))(\beta)$ contains $F$, $\alpha$ and $\beta$ so that $F(\alpha, \beta) \subseteq (F(\alpha))(b)$. ■

**Corollary.** *The elements of $F(\alpha, \beta)$ are of the form $\sum a_{ij}\alpha^i b^j$, where $1 \leq i \leq \deg \alpha$ and $1 \leq j \leq \deg \beta$.*

**Example 1.8.** Consider $\mathbb{Q}(\sqrt{2}, \sqrt{3})$ generated by $\sqrt{2}$ and $\sqrt{3}$. Notice that $\deg \sqrt{3} = 2$ over $\mathbb{Q}$ so that $[\mathbb{Q}(\sqrt{2}, \sqrt{3}) : \mathbb{Q}(\sqrt{2})] \leq 2$. Now $[\mathbb{Q}(\sqrt{2}, \sqrt{3}) : \mathbb{Q}(\sqrt{2})] = 2$ if, and only if the polynomial $x^2 - 3$ is irreducible over $\mathbb{Q}(\sqrt{2})$. Then it is irreducible if, and onyl if $\sqrt{2} \in \mathbb{Q}(\sqrt{3})$. It can be shown that this is not the case by trying to find $a, b \in \mathbb{Q}$ for which $\sqrt{3} = a + b\sqrt{2}$. Moreover we have

$$[\mathbb{Q}(\sqrt{2}, \sqrt{3}) : \mathbb{Q}] = 4$$

**Theorem 1.2.6.** *An extension field ${}^{K}\!/_{F}$ is finite if, and only if it is generated by finitely many algebraic elements over $F$.*

*Proof.* Let ${}^{K}\!/_{F}$ finite of degree $n$, and $\{\alpha_1, \dots, \alpha_n\}$ a basis. Then by theorem 1.2.4, $[F(\alpha_i) : F] | [K : F]$ for all $1 \leq i \leq n$. So each $\alpha_i$ is algebraic over $F$. Then $K$ is generated by finitely many algebraic elements.

Conversely, let $K = F(\alpha_1, \dots, \alpha_k) = (F(\alpha_1, \dots a_{k-1}))(\alpha_k)$ We obtain $K$ by taking the extensions ${}^{F_{i+1}}\!/_{F_i}$ iteratively, where $F_{i+1} = F_i(\alpha_{i+1})$, and obtain the sequence

$$F = F_0 \subseteq \cdots \subseteq F_k = K$$

Now, if the elements $\alpha_1, \dots, \alpha_k$ are algebraic over $F$, each of $\deg \alpha_i = n_i$ for $1 \leq i \leq k$, then the extension ${}^{F_{i+1}}\!/_{F_i}$ is a simple extension, and $[F_{i+1} : F_i] = \deg m_{\alpha_{i+1}} \leq \deg \alpha_{i+1} = n_{i+1}$. Then we have

$$[K : F] = [F_k : F_{k-1}] \dots [F_1, F] \leq n_1 \dots n_k$$

which makes ${}^{K}\!/_{F}$ a finite extension. ∎

**Corollary.** *If $\alpha, \beta$ are algebraic over $F$, then so are $\alpha \pm \beta$, $\alpha\beta$, and $\alpha\beta^{-1}$ (for $\beta \neq 0$).*

**Corollary.** *If ${}^{L}\!/_{F}$ is an extension, then the collection of elements of $L$ which are algebraic over $F$ forms a subfield of $L$.*

**Example 1.9.** (1) Consider the extension ${}^{\mathbb{C}}\!/_{\mathbb{Q}}$, and let $\operatorname{cl}\mathbb{Q}$ the subfield of all elements of $\mathbb{C}$ which are algebraic over $\mathbb{Q}$. Then $\sqrt[n]{2} \in \operatorname{cl} Q$ for all $n \geq 1$, so that $[\operatorname{cl}\mathbb{Q} : \mathbb{Q}] \geq n$. This makes $\operatorname{cl}\mathbb{Q}$ an infinite algebraic extension, and we call $\operatorname{cl}\mathbb{Q}$ the **field of algebraic numbers**.

(2) Consider $\operatorname{cl}\mathbb{Q} \cap \mathbb{R}$ as a subfield of $\mathbb{R}$ (i.e. the subfield of all algebraic elements of $\mathbb{Q}$). Since $\mathbb{Q}$ is countable, so is the field $\mathbb{Q}[x]$, and each polynomial in $\mathbb{Q}[x]$ has at most $n$ roots in $\mathbb{R}$, hence the number of all algebraic elements of $\mathbb{R}$ over $\mathbb{Q}$ is also countable. This means that $\operatorname{cl}\mathbb{Q}$ must also be countable. Now, since $\mathbb{R}$ is uncountable, then there exist uncountably transcendental numbers of $\mathbb{R}$ over $\mathbb{Q}$. Most notably the irrational numbers $\pi$ and $e$ are transcendental.

**Theorem 1.2.7.** *If $K$ is algebraic over $F$, and $L$ algebraic over $K$, then $L$ is algebraic over $F$.*

*Proof.* Let $\alpha \in L$, since $L$ is algebraic over $K$, there exists a $p \in K[x]$ having $\alpha$ as root. Let $p(x) = a_0 + a_1x + \cdots + a_nx^n$. Consider then $F(\alpha, a_0, \ldots, a_n)$. Since ${}^{K}\!/_{F}$ is algebraic, $a_0, \ldots, a_n$ are algebraic over $F$, and so $F(\alpha, a_0, \ldots, a_n)$ is a finite extension over $F$. Then $\alpha$ generates an extension field of degree less than $n$, and we get

$$[F(\alpha, a_0, \ldots, a_n) : F] = [F(\alpha, a_0, \ldots, a_n) : F(a_0, \ldots, a_n)][F(a_0, \ldots, a_n) : F]$$

is finite, and $F(\alpha, a_0, \ldots, a_n)$ is algebraic over $F$. That is, $\alpha$ is algebraic over $F$, and so $L$ is algbraic over $F$. ■

**Definition.** Let $K_1$ and $K_2$ subfields of a field $K$. The **composite field** $K_1K_2$ is the smallest subfield of $K$ containing both $K_1$ and $K_2$.

**Example 1.10.** The composite field of $\mathbb{Q}(\sqrt[3]{2})$ and $\mathbb{Q}(\sqrt{2})$ is $\mathbb{Q}(\sqrt[6]{2})$.

**Lemma 1.2.8.** *Let $K_1$ and $K_2$ be extensions of a field $F$ contained in a field $K$. Then $[K_1K_2 : F] \leq [K_1 : F][K_2 : F]$ with equality holding if, and only if a basis of $F$ in the other field is linearly independent. Moreover if $\{\alpha_1, \ldots, \alpha_m\}$ and $\{\beta_1, \ldots, \beta_n\}$ are bases for $K_1$ and $K_2$, then $\{\alpha_1, \beta_1, \ldots, \alpha_m\beta_n\}$ span $K_1$ and $K_2$.*

$$\begin{tikzcd} & K_1K_2 & \\ K_1 \arrow[ur, dash, "\leq m"] & & K_2 \arrow[ul, dash, "\leq n"'] \\ & F \arrow[ul, dash, "n"] \arrow[ur, dash, "m"'] & \end{tikzcd}$$

**Corollary.** *If $[K_1 : F] = m$, and $[K_2 : F] = n$ with $m$ and $n$ coprime, then $[K_1K_2 : F] = [K_1 : F][K_2 : F]$.*

*Proof.* We have that $m, n | [K_1K_2 : F]$ and since $K_1, K_2 \subseteq K_1K_2$ are subfields of $K_1K_2$, we get the least common multiple $[m, n] | [K_1K_2 : F]$. Now, since $(m, n) = 1$, we get $[m, n] = mn$ so that $mn \leq= [K_1K_2 : F]$. ■

## 1.3 Splitting Fields

**Definition.** Let $K$ be an extension of a field $F$. We say a polynomial $f$ over $F$ **splits completely** over $K$ if $f$ factors into linear factors over $K$. If $f$ splits completely over $K$, and in no other proper subfield, then we say $K$ is the **splitting field** of $f$ over $F$.

**Theorem 1.3.1.** *If $f$ is a polynomial over a field $F$, then there exists a splitting field $K$ of $f$ over $F$.*

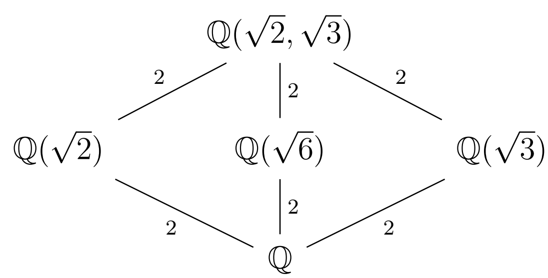
*Proof.* Let $E$ an extension of $F$ with $[E : F] = n$. By induction on $n$, for $n = 1$, we take $E = F$ and we are done. Now, for $n \geq 1$, suppose the irreducible factors of $f$ are of $\deg = 1$. Then $f$ has all its roots in $F$, and hence splits completely over $F$. Then take $E = F$. On the other hand, if $f$ has at least one irreducible factor of deg $\geq 2$, then there is an extension $E_1$ of $F$ for which $f$ has the factor $(x-\alpha)$ for some root $\alpha$. Then $f(x) = (x-\alpha)f_1(x)$ where $\deg f_1 = n-1$. Therefore by the induction hypothesis, there is an extension $E$ of $E_1$ containing all the roots of $f_1$. Hence, it contains all the roots of $f$ and $f$ splits completely over $E$.

Now, let $K$ be the intersection of all subfields of $E$ for which $f$ splite; i.e. all subfields containing the roots of $f$. Then by definition, $K$ is the splitting field of $f$ over $F$. ■

**Definition.** We call an extension $K$ over a field $F$ **normal**, if for any irreducible polynomial $f$ over $F$ with atleast one root in $K$, $f$ splits completely in $K$ over $F$. That is to say, $K$ contains the splitting field of $f$ over $F$.

**Example 1.11.** (1) The splitting field of $x^2-2$ over $\mathbb{Q}$ is $\mathbb{Q}(\sqrt{2})$, since $x^2-2 = (x+\sqrt{2})(x-\sqrt{2})$ and $\pm\sqrt{2} \in \mathbb{Q}(\sqrt{2})$ and $[\mathbb{Q}(\sqrt{2}) : \mathbb{Q}] = 2$, so there is no other subfield in between.

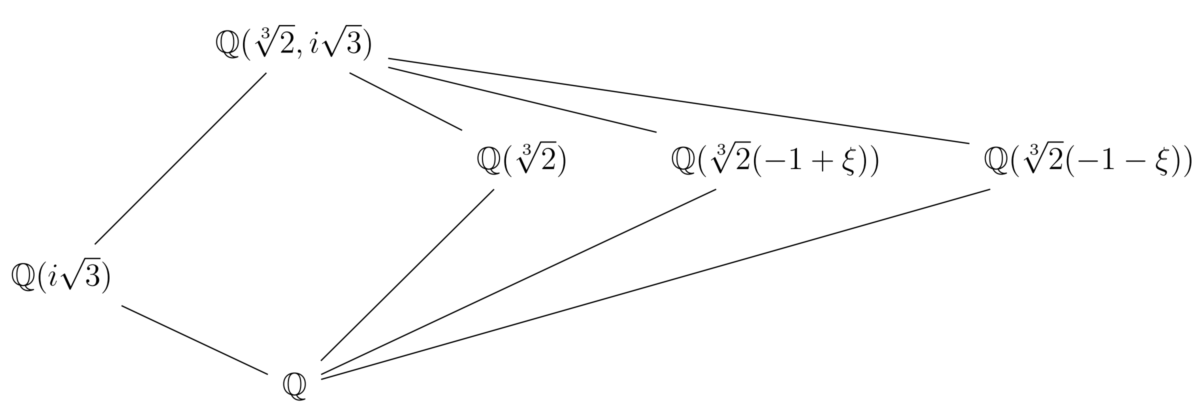
(2) The splitting field for $(x^2-2)(x^2-3) = (x+\sqrt{2})(x-\sqrt{2})(x+\sqrt{3})(x-\sqrt{3})$ is $\mathbb{Q}(\sqrt{2},\sqrt{3})$. Now, $[\mathbb{Q}(\sqrt{2},\sqrt{3}) : Q] = 4$ and the lattice of fields is




(3) Let $\xi = i\frac{\sqrt{3}}{2}$. Notice that $x^3-2$ factors into $x^3-2 = (x-\sqrt[3]{2})(x+\sqrt[3]{2}(-1+\xi))(x+\sqrt[3]{2}(-1-\xi))$. Now, $-1+\xi, -1-\xi \notin \mathbb{Q}(\sqrt[3]{2})$, so $\mathbb{Q}(\sqrt[3]{2})$ is not the splitting field for $x^3-2$. Let $K$ be the splitting field of $x^3-2$. Then $K$ conmtains $-1\pm\xi$, so that $i\sqrt{3} \in K$. Thus

$$K = \mathbb{Q}(\sqrt[3]{2}, i\sqrt{3})$$

Moreover, $[\mathbb{Q}(\sqrt[3]{2}, i\sqrt{3}) : \mathbb{Q}(\sqrt[3]{2})] \geq 2$ and since $\mathbb{Q}(\sqrt[3]{2})$ is not the splitting field, $[\mathbb{Q}(\sqrt[3]{2}, i\sqrt{3}) : \mathbb{Q}(\sqrt[3]{2})] = 2$. Hence $[\mathbb{Q}(\sqrt[3]{2}, i\sqrt{3}) : \mathbb{Q}] = 6$. We have the following lattice.



(4) Notice that $x^4+4=(x^2+2x+2)(x^2-2x+2)$ over $\mathbb{Q}$ which is irreducible by Eisenstein's criterion. Using the quadratic formula, we get $\pm 1$ and $\pm i$ as the roots, moreover, notice that $\pm 1, \pm i \in \mathbb{Q}(i)$ and since $[\mathbb{Q}(i) : \mathbb{Q}] = 2$ there are no subfields between $\mathbb{Q}$ and $\mathbb{Q}(i)$ so that $\mathbb{Q}(i)$ is the splitting field of $x^4+4$ over $\mathbb{Q}$.

**Lemma 1.3.2.** *A splitting field of a polynomial of degree $n$ over a field $F$ is of degree at most $n!$ over $F$.*

*Proof.* Let $f \in F[x]$ a polynomial of $\deg f = n$. Adjoining one root of $f$ to $F$, we have an extension ${}^{F_1}\!/_{F}$ of degree $[F_1 : F] = n$. Now,$f$ over $F_1$ has at leas one linear factor, and so any root of $f$ satisfies a polynomial of degree $n-1$. Hence proceeding inductively gives the result. ■

**Example 1.12.** Consider the polynomial $x^n - 1$ over $\mathbb{Q}$. Then the roots of $x^n-1$ are of the form $\xi$ where $\xi^n = 1$. Notice, that in $\mathbb{C}$, $\xi = e^{\frac{2i\pi}{n}}$, so that $\mathbb{C}$ contains a splitting field of $x^n-1$. Hence $\mathbb{Q}(\xi) \subseteq \mathbb{C}$ is a splitting field of $x^n-1$ over $\mathbb{Q}$. Notice that the set of all roots $\xi$ of $x^n-1$ forms a cyclic group generated by $\xi$.

**Definition.** Consider a field $F$ and the polynomial $x^n-1$ over $F$. We call the roots $\xi$ of $x^n-1$, where $\xi^n = 1$ the **primitive $n$-th roots of unity** over $F$. We call $F(\xi)$ the **cyclotomic field** over $F$.

**Example 1.13.** Let $p$ be a prime, and consider the splitting field $x^p-2$ over $\mathbb{Q}$. If $\alpha$ is a root, then $\alpha^p = 2$ so that $(\xi\alpha)^p = 2$ where $\xi$ is a primitive $p$-th root of unity over $\mathbb{Q}$. So the roots of $x^2-2$ are

$$\sqrt[p]{2} \text{ and } \xi\sqrt[p]{2}$$

Notice that $\frac{\xi\sqrt[p]{2}}{\sqrt[p]{2}} = \xi$ so the splitting field contains $\mathbb{Q}(\xi, \sqrt[p]{2})$, Moreover, $\mathbb{Q}(\xi, \sqrt[p]{2})$ contains all the roots of $x^p-2$ so that $\mathbb{Q}(\xi, \sqrt[p]{2})$ is the splitting field of $x^p-2$ over $\mathbb{Q}$.

Notice, that $\mathbb{Q}(\xi) \subseteq \mathbb{Q}(\xi, \sqrt[p]{2})$ so that $[\mathbb{Q}(\xi, \sqrt[p]{2}) : \mathbb{Q}(\xi)] \leq p$. not, since $\mathbb{Q}(\sqrt[p]{2})$ is also a subfield, we get $[\mathbb{Q}(\xi, \sqrt[p]{2}) : Q] \leq p(p-1)$. Since $(p, p-1) = 1$ (i.e. they are coprime), we have $p(p-1) | [\mathbb{Q}(\xi, \sqrt[p]{2}) : \mathbb{Q}]$ so that $[p]2) : \mathbb{Q}] = p(p-1)$. We have the follwing lattice.

$$\begin{tikzcd}
 & \mathbb{Q}(\xi, \sqrt[p]{2}) \arrow[dl, dash] \arrow[dr, dash] & \\
\mathbb{Q}(\xi) \arrow[dr, dash] & & \mathbb{Q}(\sqrt[p]{2}) \arrow[dl, dash] \\
 & \mathbb{Q} &
\end{tikzcd}$$

**Theorem 1.3.3.** *Let $\phi : F \to F'$ a field isomorphism. Let $f$ and $f'$ polynomials over $F$ and $F'$, where $f'$ is obtained by applying $\phi$ to the coefficients of $f$. Let $E$ and $E'$ be splitting fields of $f$ and $f'$ over $F$ and $F'$, respectively. Then $\phi$ extends to an isomorphism betweenn $E$ and $E'$; i.e. $E \simeq E'$.*

$$\begin{tikzcd}
E \arrow[r] \arrow[d, dash] & E' \arrow[d, dash] \\
F \arrow[r, "\phi"] & F'
\end{tikzcd}$$

*Proof.* Let $\deg f = n$. By induction on $n$. If $f$ has all its roots in $F$, $f$ splits completely over $F$, and $f'$ over $F'$. Then take $E = F$ and $E' = F'$ and we are done for $n = 1$.

Now, for $n \geq 1$, suppose the theorem is true. Let $p$ an irreducible factor of $f$, and $p'$ an irreducible factor of $f'$. If $\alpha$ and $\alpha'$ are roots of $p$ and $p'$, respectively, then extend $\phi$ to $F(\alpha)$ and $F'(\alpha')$. Then $f(x) = (x-\alpha)f_1(x)$ and $f'(x) = (x-\alpha')f_1'(x)$; with $\deg f_1 = \deg f_1' = n-1$. Then let $E$ the splitting field of $f_1$ over $F(\alpha)$, and $E'$ the splitting field of $f_1'$ over $F'(\alpha')$

$$\begin{tikzcd}
E \arrow[r] \arrow[d, no head] & E' \arrow[d, no head] \\
F(\alpha') \arrow[r] \arrow[d, no head] & F'(\alpha') \arrow[d, no head] \\
F \arrow[r, "\phi"'] & F'
\end{tikzcd}$$

The the roots of $f_1$ and $f_1'$ are in $E$ and $E'$, respectively, and hence so are the roots of $f$ and $f'$. Then by the induction hypothesis, we can extend $\phi$ to $E$ and $E'$ so that $E \simeq E'$. ■

**Corollary.** *Any two splitting fields of a given polynomial over a field are isomorphic.*

*Proof.* Take $\phi$ to be the identity map. ■

## 1.4 Algebraic Closures.

**Definition.** We define the **algebraic closure** of a field $F$ to be the algebraic extension, $\operatorname{cl} F$, over $F$ for which every polynomial over $F$ splits. We call a field $K$ **algebraically closed** if every polynomial over $K$ has at least one root in $K$.

**Lemma 1.4.1.** *A field $K$ is algebraically closed if, and only if every polynomial over $K$ has all of its roots in $K$.*

*Proof.* Certainly, if a polynomial $f$ over $K$ contains all of its roots in $K$, then $K$ is algebraically closed, by definition.

Now, suppose that $K$ is algebraically closed, and let $f$ a polynomial over $K$. Then $f$ contains at least one root in $K$. Hence $f(x) = (x - \alpha)f_1(x)$ for some root $\alpha$ of $f$, and where $f_1 \in K[x]$. But then by definition again, $f_1$ contains at least one root in $K$. Hence, we proceed until we exhaust all the roots of $f$, and obtain that every root of $f$ lies in $K$. ■

**Corollary.** *$K$ is algebraically closed if, and only if $\operatorname{cl} K = K$.*

**Lemma 1.4.2.** *Let $F$ be a field, and $\operatorname{cl} F$ its algebraic closure. Then $\operatorname{cl} F$ is algebraically closed; i.e. $\operatorname{cl}(\operatorname{cl} F) = \operatorname{cl} F$.*

*Proof.* Let $f \in \operatorname{cl} F[x]$, and $\alpha$ a root of $f$. Then $\alpha$ generates all of $\operatorname{cl} F(\alpha)$, making $\operatorname{cl} F$ algebraic over $F$. Hence $\alpha$ is algebraic over $F$, but $\alpha \in \operatorname{cl} F$, so that $\operatorname{cl}(\operatorname{cl} F) = \operatorname{cl} F$. ■

**Lemma 1.4.3.** *For every field $F$, there exists an algebraically closed set containing $F$.*

*Proof.* Consider the polynomial ring $F[\dots, x_n, \dots]$ where $f(x_n)$ is a nonconstant polynomial over $f$. Consider the ideal $(f)$. Then, if $(f) = (1)$, then

$$g_1 f_1(x_1) + \cdots + g_n f_n(x_n) = 1$$

where $g_i \in F[x_i]$. Then we get

$$g_1(x_1, \dots, x_m) f_1(x_1) + \cdots + g_n(x_1, \dots, x_m) f_n(x_n) = 1$$

Now, let $F'$ an extension of $F$ containing a root $\alpha_i$ of $f_i$. Then we observe that $0 = 1$ in the above equation which is a blatant contradiciton. S o $(f)$ must be a proper ideal.

Now, by Zorn's lemma, there exists a maximal ideal $M$ containing $I$. Then the quotient

$$K_1 = {F[\dots, x_n, \dots]}\big/{M}$$

is a field containing an imbedding of $F$. Moreover, $f$ has a root in $K_1$, so that $f(x_n) \in (f) \subseteq M$. Then $K_1$ is a field in which every polynomial over $F$ has a root. Proceeding as before with $K_1$, we obtain $K_2$ in which every polynomial over $K_1$ has a root. Hence, proceeding recursively, we obtain the sequence

$$F = K_0 \subseteq K_1 \subseteq K_n \subseteq \dots$$

in which everypolynomial over $K_n$ has all its roots in $K_{n+1}$. Now, let

$$K = \bigcup K_n$$

Then $F \subseteq K$, and every polynomial over $K$ has a root in $K_N$, for $N$ large enough; but $K_N \subseteq K$, so that $K$ is algebraically closed. ∎

**Lemma 1.4.4.** *Let $K$ be algebraically closed, and let $F \subseteq K$. Then the collection of elements of the algbraic closure $\operatorname{cl} F$ of $K$ that are algebraic over $F$ is an algebraic closure of $F$.*

*Proof.* By definition, ${\operatorname{cl} F}\big/{F}$ is algebraic. Then every polynomial $f$ over $F$ splits over $K$ into linear factors $(x - \alpha)$, where $\alpha$ is a root of $f$. So $\alpha$ is algebraic over $F$, and hence $\alpha \in \operatorname{cl} F$. then all linear factors have a coefficient in $\operatorname{cl} F$, so that $f$ splits completely over $\operatorname{cl} F$. ∎

**Corollary.** *Algebraic closures are unique up to isomorphism.*

**Theorem 1.4.5** (The Fundamental Theorem of Algebra)**.** $\mathbb{C}$ *is algebraically closed.*

**Corollary.** $\mathbb{C}$ *contains the an algebraic closuder of any of its subfields.*

## 1.5 Seperability.

**Definition.** Let $f$ be a polynomial over a field $F$ with factorization

$$f(x) = a_n(x - \alpha_1)^{n_1} \dots (x - \alpha_k)^{n_k}$$

where $\alpha_1, \dots, \alpha_k$ are roots of $f$, and $a_n$ is the leading coefficient of $f$. If $n_i > 1$, we call $\alpha_i$ a **multiple root** of $f$, and if $n_i = 1$, we call $\alpha_i$ a **simple root**. We call $n_i$ the **multiplicity** of $\alpha_i$.

**Definition.** A polynomial over a field $F$ is said to be **seperable** if it has only simple roots. Otherwise, we say it is **inseperable**.

**Lemma 1.5.1.** *Seperable polynomials have all their roots distinct.*

**Definition.** We say a field $F$ is a **finite field** if it has a finite number of elements. If $|F| = n$, then we denote $F$ as $\mathbb{F}_n$.

**Lemma 1.5.2.** *Every finite field has finite characteristic. Moreover, that characteristic is a prime integer.*

*Proof.* Recall that the characteristic is just the additive order of the element 1 in the field. Lemma 1.1.1 reiterates that any field of nonzero characteristic must have prime characteristic. ■

**Example 1.14.** (1) $x^2 - 2 = (x + \sqrt{2})(x - \sqrt{2})$ is seperable over $\mathbb{Q}$. However $(x^2 - 1)^n$ is inseperable.

(2) Consider $x^2 - t$ over the field $\mathbb{F}_2(t)$ of rational functions over $t$. $x^2 - t$ is irreducible, but inseperable. Let $\sqrt{t}$ a root, then $(x - \sqrt{t})^2 = x^2 - t$ since $\operatorname{char} \mathbb{F}_2 = 2$.

**Definition.** The **derivative** of a polynomial $f(x) = a_0 + a_1 x + \cdots + a_n x^n$ over a field $F$ is the polynomial

$$Df(x) = a_1 + 2a_2 x + \cdots + n a_n x^{n-1}$$

over $F$.

**Lemma 1.5.3.** *For any two polynomials $f$ and $g$ over a field, the following are true.*

*(1)* $D(f + g) = Df + D(g)$.

*(2)* $D(fg) = fDg + gDf$.

**Lemma 1.5.4.** *A polynomial $f$ has a multiple root $\alpha$ if, and only if $\alpha$ is a root of $Df$. Moreover, the minimal polynomial of $\alpha$, $m_\alpha$ divides $(f, Df)$.*

*Proof.* Let $\alpha$ a multiple root of $f$. Then $f(x) = (x - \alpha)^n g(x)$ for some polynomial $g$. Hence

$$Df(x) = n(x - \alpha)^{n-1} g(x) + (x - \alpha)^n Dg(x)$$

so that $\alpha$ is a root of $Df$.

Conversly, suppose that $\alpha$ is a root of both $f$ and $Df$. Then $f(x) = (x - \alpha)g(x)$ for some polynomial $g$, and $Df(x) = g(x) + (x - \alpha)Dg(x)$. Now, since $Df(\alpha) = 0$, we get $h(\alpha) = 0$, so that $h$ has a linear factor $(x - \alpha)$. This makes $\alpha$ a multiple root of $f$. ■

**Corollary.** *$f$ is seperable if and only if $(f, Df) = 1$.*

**Corollary.** *Every irreducible polynomial ina field $F$ of* $\operatorname{char} F = 0$ *is seperable. Moreover, a polynomial over such a field is irreducible if, and only if it is the product of distinct irreducible factors.*

*Proof.* Let $p$ an irreducible polynomial over $F$ of $\deg p = n$. Then $\deg Dp = n-1$. Up to constant factors, the factors of $p$ are 1 and itself, so that $(p, Dp) = 1$. This makes $p$ seperable. Therefore every irreducible polynomial over $F$ is seperable, and the rest follows. ■

**Example 1.15.** (1) Let $p$ prime and $f(x) = x^{p^n} - x$ over the finite field $\mathbb{F}_p$, of $\operatorname{char} \mathbb{F}_p = p$. Then $Df(x) = p^n x^{p^n - 1} - 1 \equiv -1 \mod p$. Then $Df$ has no roots, which makes $f$ seperable.

(2) $D(x^n - 1) = nx^{n-1}$ for any field of char coprime to $p$. Then $D(x^n - 1)$ has a root 0 of multiplicity $n > 1$, but 0 is not a root of $x^n - 1$ so that $x^n - 1$ is seperable. That is, $x^n - 1$ has $n$ distinct roots of unity $\xi$.

(3) Let $F$ a field of $\operatorname{char} F = p$, where $p|n$. Then there are fewew than $n$ distinct $n$-th roots of unity over $F$, since $n \equiv 0 \mod p$. Then $D(x^n - 1) = 0$, and every root of $x^n - 1$ is a multiple root.

**Lemma 1.5.5.** *If $f$ is a polynomial over a field $F$ whose derivative is* 0, *then there exist a polynomial $g$ for which $f(x) = g(x^p)$ where* $\operatorname{char} F = p$.

*Proof.* Let $f(x) = a_0 + a_1 x + \cdots + a_n x^n$. Then $Df(x) = a_1 + \cdots + n a_n x^{n-1} = 0$, so that every exponent $i \equiv 0 \mod p$. That is, $f(x) = a_0 + a_1 x^p + \cdots + a_m x^{mp}$. Then let

$$g(x) = a_0 + a_1 x + \cdots + a_m x^m$$

then $f(x) = g(x^p)$. ■

**Lemma 1.5.6.** *Let $F$ a field of* $\operatorname{char} F = p$. *The for every $a, b \in F$, $(a+b)^p = a^p + b^p$ and $(ab^p) = a^p b^p$.*

*Proof.* The binomial theorem gives

$$(a+b)^p = a^p + \binom{p}{1} a^{p-1} b + \cdots + \binom{p}{p-1} a b^{p-1} + b^p$$

Now, since $\binom{p}{i} \in \mathbb{Z}$ for any $1 \leq i \leq p-1$, and $p$ is prime (the charateristic of a field has to be prime), then $p | \binom{p}{i}$. Hence $\binom{p}{i} \equiv 0 \mod p$, so that the binomial exapnsion above reduces to

$$(a+b)^p \equiv a^p + b^p \mod p$$

Now, let $\phi : a \to a^p$, then $\phi$ is an automorphism of fields taking $(ab)^p = a^p b^p$. ■

**Corollary.** *Let $F$ be a finite field of* $\operatorname{char} F = p$. *Then every element of $F$ is a $p^{th} powerin F$.*

**Definition.** Let $F$ be a field. We call the automorphism $F \to F$ defined by $a \to a^p$ where $p \in \mathbb{Z}$ the **Forbenius automorphism**.

**Lemma 1.5.7.** *Every irreducible polynomial in a finite field $F$ is seperable.*

*Proof.* Suppose otherwise. Since $F$ has finite characteristic, there is a polynomial $q$ over $F$ for which $p(x) = q(x^l)$, where $p$ is the irreducible polynomial in question, and $\operatorname{char} F = l$. Let

$$q(x) = a_0 + a_1 x + \cdots + a_n x^n$$

then $a_i = b_i^p$ for some $b_i \in F$, and

$$\begin{aligned} p(x) &= q(x^l) \\ &= a_0 + a_1 x^p + \cdots + a_n x^{pn} \\ &= b_0^p + b_1^p x^p + \cdots + b_n^p x^{np} \\ &= (b_0 + b_1 x + \cdots + b_n x^n)^p \end{aligned}$$

which is a contradiction. ■

**Definition.** A field $K$ of characteristic $\operatorname{char} K = p$ is called **perfect** if for every $a \in K$, there exists a $b \in K$ for which $a = b^p$, or $p = 0$.

**Example 1.16.** Let $n > 0$ and consider the splitting field of the polynomial $x^{p^n} - x$ over the finite field $\mathbb{F}_p$. Then $x^{p^n} - x$ has precisely $p^n$ roots.

Let $\alpha, \beta$ be roots. Then $\alpha^{p^n} = \alpha$, and $\beta^{p^n} = \beta$. Then $(\alpha\beta)^{p^n} = \alpha\beta$ and $(\alpha^{-1})^{p^n} = \alpha^{-1}$. Moreover, $(\alpha + \beta)^{p^n} = \alpha + \beta$. So the set of $p^n$ disctinct roots of $x^{p^n} - x$ is closed under addition, multiplication, and inverses in its splitting field. Let $F$ be that splitting field. Notice that $F \subseteq E$, moreover, $[F : \mathbb{F}_p] = n$ so that $|F| = p^n$. We also have that $\mathcal{U}(F)$ is a cyclic group of order $p^n - 1$, so that $E \subseteq F$, since $\alpha^{p^n - 1} = 1$. Therefore $E$ is the splitting field of $x^{p^n} - x$ over $\mathbb{F}_p$, and so contains all the roots of $x^{p^n} - x$. Hence finite fields of order $p^n$ exist and are unique up to isomorphism.

**Lemma 1.5.8.** *Let $f$ an irreducible polynomial over a field $F$ of* $\operatorname{char} F = p$. *Then there exists a unique integer $k \geq 0$ and a unique seperable polynomial $s$ such that $f(x) = s(x^{p^k})$.*

*Proof.* We have that since $\operatorname{char} F = p$, there exists a polynomial $f_1$ over $F$ for which $f(x) = f_1(x^p)$. Now, if $f_1$ is seperable, take $k = 1$ and we are done. Otherwise, there is a polynomial $f_2$ over $F$ for which $f_2(x) = f_2(x^p)$, so that $f(x) = f_1(x^p) = f_2(x^{p^2})$. Then proceeding in this fashion, we obtain a seperable polynomial $s$ for which $f(x) = s(x^{p^k})$ where $k \geq 0$. ■

**Definition.** Let $f$ an irreducible polynomial over a field of characteristic $p$, a prime. Let $f_s$ the polynomial for which $f(x) = f_s(x^{p^k})$ for some unique integer $k \geq 0$. Then we call the degree of $f_s$ the **seperable degree** of $f$ and write $\deg_s f = \deg f_s$. We call the integer $p^k$ the **inseperable degree** and write $\deg_i f = p^k$. We call $f_s$ the **seperable part** of $f$.

**Lemma 1.5.9.** *A polynomial $f$ is seperable if, and only if* $\deg_i f = 1$ *and* $\deg_s f = \deg f$. *Moreover,*

$$\deg f = \deg_s f \cdot \deg_i f$$

**Example 1.17.** (1) $x^p - t$ over $\mathbb{F}_p(t)$ is irreducible with derivative $D = 0$. Hence $x^p - t$ is inseperable. We call $x^p - t$ a **purely inseperable polynomial**. Notice that $x^p - t$ has seperable part $(x - t)$.

(2) $x^{p^n} - t$ over $\mathbb{F}_p(t)$ is irreducible with seperable part $(x-t)$, and $\deg_i = p^n$.

(3) Let $f(x) = (x^{p^n} - t)(x^p - t)$ over $\mathbb{F}_p(t)$. Then $p$ has two inseperable irreducible factors, and so is inseperable.

**Definition.** If $K$ is an extension over a field $F$, we call an element $\alpha \in K$ **seperable** if its minimal polynomial is seperable, otherwise we call it **inseperable**. We call $K/F$ **seperable** if every $\alpha \in K$ is seperable; otherwise we say that $K/F$ is inseperable.

**Lemma 1.5.10.** *Every fnite extension of a perfect field is seperable.*

**Corollary.** *Finite extension fields of $\mathbb{Q}$ and $\mathbb{F}_p$ are seperable.*

## 1.6 Cyclotomic Polynomials.

**Definition.** We define **Euler's totient** to be the map $\phi : \mathbb{Z} \to \mathbb{Z}$ defined by the rule $\phi(n) = |\{a \in \mathbb{Z} : (a, n) = 1\}|$. That is, $\phi$ of $n$ is the number of all integers less than $n$, coprime to $n$.

**Definition.** We define $\Xi_n$ to be the **group of all primitive $n$-th roots of unity**, $\xi$ for which $\xi^n = 1$.

**Lemma 1.6.1.** $\Xi_n \simeq \mathbb{Z}/n\mathbb{Z}$.

*Proof.* The map $a \to \xi^a$ defines the required isomorphism. ∎

**Corollary.** $\operatorname{ord} \Xi_n = \phi(n)$ *where $\phi$ is Euler's totient.*

*Proof.* Since $\xi^n \equiv \xi^{0 \mod n} \equiv 1$, we have every non identity power of $\xi$ has exponenct coprime to $n$. That is there are $\phi(n)$ such distinct powers of$\xi$. ∎

**Corollary.** *If $d|n$, then $\Xi_d \leq \Xi_n$.*

*Proof.* Notice that if $d|n$, then $d = mn$ for some $m \in \mathbb{Z}^+$. Then $\xi^d = 1$ implies $(\xi^d) = \xi^{dm} = \xi^n = 1$. ∎

**Definition.** We define the **$n$-th cyclotomic polynomial** to be the polynomial

$$\Phi_n(x) = \prod x - \xi$$

having as roots all $n$-primitive roots of unity.

**Lemma 1.6.2.** *The $n$-th cyclotomic polynomial $\Phi_n$ has degree* $\deg \Phi_n = \phi(n)$*, where $\phi$ is Euler's totient.*

*Proof.* Recall that $\operatorname{ord} \Xi_n = \phi(n)$, and since the elements of $\Xi_n$ are the roots of $\Phi_n$, there are $\phi(n)$ such roots. This puts $\deg \Phi_n = \phi(n)$. ∎

**Example 1.18** (Computing Cyclotomic Polynomials)**.** Recall that the polynomial $x^n - 1$ has as roots precisely all $n$-th roots of unity $\xi$, that is $\xi^n = 1$. If $x^n - 1 \in F[x]$, $F$ a field, the the splitting field of $x^n - 1$ is $F(\xi)$. Then we have

$$x^n - 1 = \prod_{\xi \in \Xi_n} (x - \xi)$$

Now, grouping those factors where $\xi^d = 1$ for some $d|n$, then we have

$$x^n - 1 = \prod_{\xi \in \Xi_d} (x - \xi) \prod_{\xi \in \Xi_n} (x - \xi) = \prod d|n \prod_{\xi \in \Xi_n} (x - \xi) = \prod_{d|n} \Phi_n(x)$$

that is,

$$x^n - 1 = \prod_{d|n} \Phi_d(x)$$

which gives a method for computing $\Phi_n$ recursively.

We have $\Phi_1(x) = x - 1$ and $\Phi_2(x) = x + 1$. Now, $\Phi_3(x) = \Phi_1(x)\Phi_3(x) = (x-1)\Phi_3(x)$, so that

$$Phi_n(x) = x^2 + x + 1$$

We have $\Phi_4(x) = \Phi_1(x)\Phi_2(x)\Phi_4(x) = (x-1)(x+1)\Phi_4(x) = (x^2-1)\Phi_4(x)$. So

$$\Phi_4(x) = x^2 + 1$$

Similarly,

$$\begin{aligned}
\Phi_5(x) &= x^4 + x^3 + x + 1 \\
\Phi_6(x) &= x^2 - x + 1 \\
\Phi_7(x) &= x^6 + x^5 + x^4 + x^3 + x^2 + x + 1 \\
\Phi_8(x) &= x^4 + 1 \\
\Phi_9(x) &= x^6 + x^x + 1 \\
\Phi_{10}(x) &= x^4 - x^3 + x^2 - x + 1 \\
\Phi_{11}(x) &= x^{10} + x^9 + x^8 + x^7 + x^6 + x^5 + x^4 + x^3 + x^2 + x + 1 \\
\Phi_{12}(x) &= x^4 - x^2 + 1
\end{aligned}$$

Also observe that if $p$ is prime, then

$$\Phi_p(x) = \sum_{i=0}^{p-1} x^i = x^{p-1} + x^p + \cdots + x + 1$$

**Lemma 1.6.3.** *$\Phi_n(x)$ is monic over $\mathbb{Z}$.*

*Proof.* Notice that since $x^n - 1 = \prod \Phi_d(x)$, is monic, then each $\Phi_d$ must also be monic for all $d|n$.

Now, by induction on $n$, for $n = 1$, it is clear that $x - 1$ has coefficiencts in $\mathbb{Z}$ (if $x^n - 1 \in \mathbb{Z}[x]$ we are done, if not, just take $1_F \to 1_{\mathbb{Z}}$, whre $F$ is the underlying field of $x^n - 1$). Now, suppose that $\Phi_d(x) \in \mathbb{Z}[x]$ for all $1 \leq d < n$, and $d|n$. Then $x^n - 1 = f(x)\Phi_n(x)$, where $f(x) = \prod \Phi_d(x)$ is monic over $\mathbb{Z}$. Moreover, $f|x^n - 1$, in the splitting field $\mathbb{Q}(\xi)$ (since we take $1_F \to F_{\mathbb{Z}}$), where $\xi^n = 1$. Then $f|x^n - 1$ over $\mathbb{Q}$ by the division theorem, and by Gauss' lemma, $f|x^n - 1$ in $\mathbb{Z}$. So $\Phi_n \in \mathbb{Z}[x]$. ■

**Theorem 1.6.4.** *$\Phi_n$ is irreducible over $\mathbb{Z}$.*

*Proof.* Again, if $x^n - 1 \in F[x]$ for some field $F$, take $1_F \to 1_{\mathbb{Z}}$ so that $x^n - 1 \in \mathbb{Z}[x]$. Suppose then that $\Phi_n(x) = f(x)g(x)$ where $f$ and $g$ are monic, and $f$ is irreducible. Let $\xi^n = 1$, a primitive $n$-th root, so that $\xi$ is a root of $f$. Then $f$ is the minimal polynomial for $\xi$ over $\mathbb{Q}$. Now, let $p$ a prime such that $p \nmid n$. Then $\xi^p$ is a $n$-th root, of $f$ or $g$. If $f(\xi^p) = 0$, then for all $a$ with $(a, n) = 1$, we have $\xi^a$ is a root of $f$. Moreover, $a = p_1 \dots p_k$ where each $p_i \nmid n$ is prime. That means tht $\xi^{p_1}$, $(\xi^{p_1})^{p_2}, \dots, \xi^n$ are all roots of $f$ making $f = \Phi_n$ and we are done.

Suppose then that $g(\xi^p) = 0$. Then $\xi$ is root of $g(x^p)$, and since $f$ is minimal, $f|g(x^p)$ in $\mathbb{Z}[x]$. Then we have $g(x^p) = f(x)h(x)$ for $f, h \in \mathbb{Z}[x]$. reducing mod $p$, we get $g(x^p) \equiv f(x)h(x) \mod p$ in $\mathbb{F}_p[x]$; but $g(x^p) \equiv (g(x))^p \mod p$. Since $\mathbb{F}_p[x]$ is a unique factorization domain, we get that $f \mod p$ and $g \mod p$ have a common factor. Then $\Phi_n(x) \equiv f(x)g(x) \mod p$ has a multiple root in $\mathbb{F}_p[x]$ ; implying that $x^n - 1$ has a multiple root, which is impossible; since $x^n - 1$ has $n$ distinct roots. Therefore $\xi^p$ is a root of $f$. ■

**Corollary.** *$[\mathbb{Q}(\xi) : \mathbb{Q}] = \phi(n)$.*

*Proof.* We have by above that $\Phi_n$ is the minimal polynomial for $\xi$ over $\mathbb{Q}$. ■

**Example 1.19.** Let $\xi^8 = 1$ an 8-th root of unity. Then $[\mathbb{Q}(\xi) : \mathbb{Q}] = 4$ and $\mathbb{Q}(\xi)$ has minimal polynomial $\Phi_8(x) = x^4 + 1$. Moreover, $\mathbb{Q}(\xi)$ contains a primitive 4-th root of unity $i^4 = 1$ (over $\mathbb{C}$, $i^2 = -1$). So that $\mathbb{Q}(i) \subseteq \mathbb{Q}(\xi)$. We als get that $\xi + \xi^7 = \sqrt{2}$ (since $\xi = e^{\frac{2i\pi}{8}}$ over $\mathbb{C}$), and $\mathbb{Q}(\xi) = \mathbb{Q}(i, \sqrt{2})$.

# Chapter 2

# Galois Theory

## 2.1 Definitions and Examples.

**Definition.** An isomorphism of a field $K$ onto itself is called an **automorphism**. We denote the set of all automorphisms of $K$ $\operatorname{Aut} K$, and for $\sigma \in \operatorname{Aut} K$, we write $\sigma\alpha$ to mean $\sigma(\alpha)$. We say an automorphism $\sigma$ of $K$ **fixes** an element $\alpha \in K$ if $\sigma\alpha = \alpha$. We say $\sigma$ **fixes** a subset $F \subseteq K$ if $\sigma\alpha = \alpha$ for all $\alpha \in F$. We denote $\operatorname{Aut} {}^{K}\!/_{F}$ to be the set of all automorphisms of $K$ that fix $F$, where ${}^{K}\!/_{F}$ is a field extension.

**Lemma 2.1.1.** *Let $K$ be a field. Then $\operatorname{Aut} K$ is a group. Moreover, if $K$ is an extension of a field $F$, then $\operatorname{Aut} {}^{K}\!/_{F} \leq \operatorname{Aut} K$.*

**Lemma 2.1.2.** *Let $K$ be an extension of $F$, and let $\alpha \in K$ algebraic over $F$. Then for every $\sigma \in \operatorname{Aut} {}^{K}\!/_{F}$, $\sigma\alpha$ is a root of the minimal polynomial of $\alpha$ over $F$; that is, $\operatorname{Aut} {}^{K}\!/_{F}$ permutes the roots of irreducible polynomials.*

*Proof.* Suppose that $a_0 + a_1\alpha + \cdots + a_{n-1}\alpha^{n-1} + \alpha^n = 0$, where $a_i \in F$ for all $1 \leq i \leq n$, and $a_n = 1$. Notice that if $\sigma$ is an automorphism of $K$, then it is a homomorphism, moreover, since $\sigma$ fixes $F$, and $a_i \in F$, we get $\sigma(a_i\alpha^i) = a_i\sigma\alpha^i$. Therefore,

$$\begin{aligned}
\sigma(a_0 + a_1\alpha + \cdots + a_{n-1}\alpha^{n-1} + \alpha^n) &= \sigma 0 = 0 \\
\sigma(a_0) + \sigma(a_1\alpha) + \cdots + \sigma(a_{n-1}\alpha^{n-1}) + \sigma(\alpha^n) &= 0 \\
a_0 + a_1\sigma\alpha + \cdots + a_{n-1}\sigma\alpha^{n-1} + \sigma\alpha^n &= 0
\end{aligned}$$

which makes $\sigma\alpha$ a root. ∎

**Example 2.1.** (1) The identity map is an automorphism called the **trivial automorphsim**, and just maps elements of a field onto themselves. We denote this automorphism by $\iota$. Notice additionally, that if $\sigma$ is an automorphism of a field, then $\sigma : 1 \to 1$ and $\sigma : 0 \to 0$, so that $\sigma a = a$ for any element $a$ in the prime subfield. That is, the automorphism group of a field fixes its prime subfield. In partucular, notice that $\mathbb{Q}$ and $\mathbb{F}_p$ have only the trivial automorphism, so that $\operatorname{Aut} \mathbb{Q} = \langle \iota \rangle$ and $\operatorname{Aut} \mathbb{F}_p = \langle \iota \rangle$.

(2) If $\tau \in \operatorname{Aut}\mathbb{Q}(\sqrt{2}) = \operatorname{Aut} {}^{\mathbb{Q}(\sqrt{2})}\!/_{\mathbb{Q}}$, then $\tau\sqrt{2} = \pm\sqrt{2}$. Then $\tau$ fixes $\mathbb{Q}$, and we have that it sends elements $\tau : a + b\sqrt{2} \to a \pm b\sqrt{2}$. In the case of addition, we have that $\tau = \iota$ the identity. The latter case of subtraction gives $\tau = a + b\sqrt{2} \to a - b\sqrt{2}$, so that $\operatorname{Aut}\mathbb{Q}(\sqrt{2}) = \langle\tau\rangle$ a cyclic group of order 2 generated by $\tau$.

**Lemma 2.1.3.** *Let $H \leq \operatorname{Aut} K$ for some field $K$. Then the collection $F$ of elements of $K$ fixed by $H$ is a subfield of $K$.*

*Proof.* LEt $h \in H$, and $a, b \in F$. Then $ha = a$, $hb = b$, so that $h(a \pm b) = a \pm b$, and $h(ab) = ab$, and $h(a^{-1}) = (ha)^{-1} = a^{-1}$. ∎

**Definition.** Let $K$ be a field. If $H \leq \operatorname{Aut} K$, we define the **fixed field** of $H$ to be the subfield of $K$ fixed by $H$, and we denote it $\mathcal{F}(H)$.

**Lemma 2.1.4.** *If $F_1 \subseteq F_2 \subseteq K$ are subfields of a field $K$, then $\operatorname{Aut} {}^{K}\!/_{F_2} \subseteq \operatorname{Aut} {}^{K}\!/_{F_1}$. Moreover, if $H_1 \leq H_2 \leq \operatorname{Aut} K$, then $\mathcal{F}(H_2) \subseteq \mathcal{F}(H_1)$.*

**Example 2.2.** (1) The fixed field of $\operatorname{Aut}\mathbb{Q}(\sqrt{2})$ is the field

$$F = \{a + b\sqrt{2} \in \mathbb{Q}(\sqrt{2}) : \sigma(a + b\sqrt{2}) = a + b\sqrt{2}\}$$

by definition. Then $a - b\sqrt{2} = a + b\sqrt{2}$ so that $b = 0$. Therefore $F = \mathbb{Q}$ and $\mathbb{Q}$ is the fixed field.

(2) Consider $\operatorname{Aut}\mathbb{Q}(\sqrt[3]{2}) = \langle\iota\rangle$. Then the fixed field of $\operatorname{Aut} {}^{\mathbb{Q}(\sqrt[3]{2})}\!/_{\mathbb{Q}}$ is $\mathbb{Q}(\sqrt[3]{2})$.

**Lemma 2.1.5.** *Let $E$ be the splitting field over a field $F$ of a polynomial $f(x)$ over $F$. Then*

$$\operatorname{ord}\operatorname{Aut} {}^{E}\!/_{F} \leq [E : F]$$

*Proof.* By induction on $[E : F]$. If $[E : F] = 1$, then $E = F$, and we are done. Now, for $[E : F] \geq 1$, $f(x)$ has atleast one irreducible factor $p(x)$ of degree $\deg p > 1$. Now, let $F'$ be the corresponding field to $F$ with splitting field $E'$, corresponding to $E$. Let $f'(x)$ be the polynomial over $F'$ the polynomial corresponding to $f$ over $F$, with irreducible factor $p'(x)$ corresponding to the irreducible factor $p$. Now, let $\alpha$ be a root of $p$. IF $\sigma$ is an extension of an isomorphism $\phi$ to $E$, then the restriction $\tau = \sigma|_{F(\alpha)}$ is an isomorphism of $F(\alpha)$ onto a subfield of $E'$. Since $\alpha$ generates $F(\alpha)$, $\tau$ is completely determined by its action on $\alpha$; i.e. $\tau\alpha$, so that $\tau\alpha$ is a root of $p'$. We then get the following diagram:

$$\begin{array}{ccc} E & \xrightarrow{\sigma} & E' \\ | & & | \\ F(\alpha) & \xrightarrow{\tau} & F'(\tau\alpha) \\ | & & | \\ F & \xrightarrow{\phi} & F' \end{array}$$

COnversly, let $\beta$ be a root of $p'$. Then there exist extensions $\tau$ and $\sigma$ of the isomorphism $\phi$ giving the above diagram (replace $\tau\alpha$ with $\beta$). Now, the number of extensions of $\phi$ to $\tau$ is equal to the number of distinct roots of $p'$. Since $\deg p = \deg p' = [F(\alpha) : F]$, the number of extensions to $\tau$ is at most $[F(\alpha) : F]$.

Now, notice that $[E : F(\alpha)] < [E : F]$. Therefore, by the induction hypothesis, the number of extensions of $\tau$ to $\sigma$ is at most $[E : F(\alpha)]$. Therefore, the number of extensions of $\phi$ to $\sigma$ is at most $[E : F(\alpha)][F(\alpha) : F] = [E : F]$.

Finally, if $F = F'$, we have $f = f'$ (and $p = p'$), and so $\phi$ is the identity map and $E = E'$. THis makes $\sigma$ an automorphism of $E$ which fixes $F$. The proof is complete. ■

**Corollary.** *If $K$ is the splitting field of a seperable polynomial $f(x)$ over a field $F$, then* $\operatorname{ord}\operatorname{Aut} {}^{K}\!/_{F} = [K : F]$.

**Definition.** We call a finite field extension ${}^{K}\!/_{F}$ a **Galois extension** if $\operatorname{ord}\operatorname{Aut} {}^{K}\!/_{F} = [K : F]$. We call $\operatorname{Aut} {}^{K}\!/_{F}$ the **Galois group** of ${}^{K}\!/_{F}$, and write $\operatorname{Gal} {}^{K}\!/_{F}$.

**Lemma 2.1.6.** *An extension $K$ over a field $F$ is Galois over if and only if it is normal and seperable.*

*Proof.* If $K$ is Galois over $F$, the result follws by definition. Now, let $K$ be normal and seperable. Let $\alpha \in K$. Then the minimal polynomial $m$ of $\alpha$ over $F$ is seperable. Moreover, $\alpha$ is a root of $K$, and since $K$ is normal, $m$ splits completely over $K$. Thus $K$ contains the splitting field of $m$, but since $m$ is minimal and irreducible, that makes $K$ the splitting field of some polynomial $f$ over $F$, having $m$ as a factor. By the above corollary, this makes $K$ Galois over $F$. ■

**Example 2.3.** (1) ${}^{\mathbb{Q}(\sqrt{2})}\!/_{\mathbb{Q}}$ is Galois, and $\operatorname{Gal} {}^{\mathbb{Q}(\sqrt{2})}\!/_{\mathbb{Q}} = \langle \sigma \rangle \simeq {}^{\mathbb{Z}}\!/_{2\mathbb{Z}}$, where $\sigma : a + b\sqrt{2} \to a - b\sqrt{2}$.

(2) Any quadratic extension field $K$ over $F$ is Galois over $F$, provided $\operatorname{char} F \neq 2$. Then any quadratic extension $K$ of $F$, of degree $[K : F] = 2$ is of the form $F(\sqrt{D})$, where $D \in \mathbb{Z}^+$ is squarefree. Hence $K = F(\sqrt{D})$ is the splitting field of the polynomial $x^2 - D$.

(3) $\mathbb{Q}(\sqrt[3]{2})$ is not Galois over $\mathbb{Q}$, since $\operatorname{ord}\operatorname{Aut} {}^{\mathbb{Q}(\sqrt[3]{2})}\!/_{\mathbb{Q}} = 1$, but $[\mathbb{Q}(\sqrt[3]{2}) : \mathbb{Q}] = 3$.

(4) $\mathbb{Q}(\sqrt{2}, \sqrt{3})$ is the splitting field of the seperable polynomial $(x^2 - 2)(x^2 - 3)$ over $\mathbb{Q}$. Hence $\mathbb{Q}(\sqrt{2}, \sqrt{3})$ is Galois over $\mathbb{Q}$, and has Galois group $\operatorname{Gal} {}^{\mathbb{Q}(\sqrt{2}, \sqrt{3})}\!/_{\mathbb{Q}}$ of order 6. Moreover, since the automorphisms of this group are completely determined by the roots $\sqrt{2}$ and $\sqrt{3}$, we get the possible automorphisms are given by the maps

$$\begin{array}{ll} \sqrt{2} \to \sqrt{2} & \sqrt{3} \to -\sqrt{3} \\ \sqrt{2} \to -\sqrt{2} & \sqrt{3} \to \sqrt{3} \\ \sqrt{2} \to \sqrt{2} & \sqrt{3} \to -\sqrt{3} \\ \sqrt{2} \to -\sqrt{2} & \sqrt{3} \to -\sqrt{3} \end{array}$$

Now, let $\sigma : \sqrt{2} \to -\sqrt{2}, \sqrt{3} \to \sqrt{3}$ and $\tau : \sqrt{2} \to \sqrt{2}, \sqrt{3} \to -\sqrt{3}$. Then $\sigma\tau : \sqrt{2} \to -\sqrt{2}, \sqrt{3} \to -\sqrt{3}$. Therefore we have

$$\operatorname{Gal} {}^{\mathbb{Q}(\sqrt{2}, \sqrt{3})}\!/_{\mathbb{Q}} = \langle \sigma, \tau \rangle \simeq V_4$$

where $V_4$ is the Klein 4-group.

We can also determine the fixed fields correspongidng to each subgroup of $\langle \sigma\tau \rangle$. That is, $\mathcal{F}(\langle \sigma\tau \rangle)$ is the set of all elements fixed by $\sigma\tau$ and has elements of the form $a+b\sqrt{6}$. So $\mathcal{F}(\langle \sigma\tau \rangle) = \mathbb{Q}(\sqrt{6})$. The table below lists the fixed fields of the Galois group considered.

| subgroup | fixed field |
|---|---|
| $\langle \iota \rangle$ | $\mathbb{Q}(\sqrt{2}, \sqrt{3})$ |
| $\langle \sigma \rangle$ | $\mathbb{Q}(\sqrt{3})$ |
| $\langle \sigma\tau \rangle$ | $\mathbb{Q}(\sqrt{6})$ |
| $\langle \tau \rangle$ | $\mathbb{Q}(\sqrt{2})$ |
| $\langle \sigma, \tau \rangle$ | $\mathbb{Q}$ |

(5) The roots of $x^3 - 2$ over $\mathbb{Q}$ are given by

$$\sqrt[3]{2} \qquad \xi\sqrt[3]{2} \qquad \xi^2\sqrt[3]{2}$$

where $\xi^3 = 1$ is the 3-rd root of unity. Additionally, the splitting field of $x^3 - 2$ over $\mathbb{Q}$ is $\mathbb{Q}(\sqrt[3]{2}, \xi\sqrt[3]{2})$ of degree 6. Now, $x^3 - 2$ is irreducible over $\mathbb{Q}$, and hence seperable over $\mathbb{Q}$. This makes $\mathbb{Q}(\sqrt[3]{2}, \xi\sqrt[3]{2})$ Galois over $\mathbb{Q}$, of order 6.

Consider now the set of generators $\sqrt[3]{2}$ and $\xi$. Then an automorphism $\sigma$ takes $\sqrt[3]{2} \to \sqrt[3]{2}, \xi\sqrt[3]{2}$, or $\xi^2\sqrt[3]{2}$, and takes $\xi \to \xi$ or $\xi^2$. Since these are the roots of the cyclotomic polynomial $\Phi_3(x) = x^2 + x + 12 + x + 1$, $\sigma$ is completely determined by the actions on $\sqrt[3]{2}$ and $\xi$. Hence there are 6 possible automorphisms.

Let

$$\begin{array}{ll} \sigma : \sqrt[3]{2} \to \xi\sqrt[3]{2} & \xi \to \xi \\ \tau : \sqrt[3]{2} \to \sqrt[3]{2} & \xi \to \xi^2 \end{array}$$

We obtain then the elements

$$\iota \qquad \sigma^2 \qquad \tau\sigma^2 = \sigma\tau$$

and we get the additional relations

$$\sigma^2 = \tau^2 = \iota$$

so that

$$\mathrm{Gal}\, {}^{\mathbb{Q}(\sqrt[3]{2}, \xi\sqrt[3]{2})}\!\big/_{\mathbb{Q}} = \langle \sigma, \tau \rangle \simeq S_3$$

The fixed field of $\langle \sigma^2 \rangle$ is $\mathbb{Q}(\xi)$.

(6) $\mathbb{Q}(\sqrt[4]{2})$ is not Galois over $\mathbb{Q}$. We have $[\mathbb{Q}(\sqrt[4]{2}) : \mathbb{Q}] = 4$ but that any automorphism takes $\sqrt[4]{2}$ onto $\pm\sqrt[4]{2}$, or $\pm i\sqrt[4]{2}$. But $\pm i\sqrt[4]{2} \notin \mathbb{Q}(\sqrt[4]{2})$. Notice however that $\mathbb{Q}(\sqrt[4]{2})$ is Galois over $\mathbb{Q}(\sqrt{2})$, and $\mathbb{Q}(\sqrt{2})$ is Galois over $\mathbb{Q}$.

(7) The extension field $\mathbb{F}_{p^n}$ is Galois over $\mathbb{F}_p$. Recall that $\mathbb{F}_{p^n}$ is the splitting field of the seperable polynomial $x^{p^n} - x$ over $\mathbb{F}_p$. Then $\operatorname{ord}\operatorname{Gal} {}^{\mathbb{F}_{p^n}}\!/_{\mathbb{F}_p} = n$ and the Frobenius automorphism given by

$$\sigma : \alpha \to \alpha^p$$

generates the Galois group, making it $\langle\sigma\rangle$, a cyclic group of order $n$.

(8) The extension $\mathbb{F}_2(x)$ is not Galois over $\mathbb{F}_2(t)$, since $x^2 - t$ is not seperable. Moreover, any automorphism of $\operatorname{Aut} {}^{\mathbb{F}_2(x)}\!/_{\mathbb{F}_2(t)}$ sends $x$ to the only root of $x^2 - t$, making it the trivial group.

## 2.2 The Fundamental Theorem of Galois Theory.

**Definition.** A **linear character** of a group $G$ with values in a field $L$ is a homomorphism $\chi : G \to \mathcal{U}(L)$. We say that distinct linear characters $\chi_1, \ldots, \chi_n$ of $G$ are **linearly independent** over $L$ if they are linearly independent, as functions, over $G$.

**Theorem 2.2.1.** *If $\chi_1, \ldots, \chi_n$ are distinct linear characters of a group $G$ with values in a field $L$, then they are linearly independent over $L$.*

*Proof.* Suppose that $\chi_1, \ldots, \chi_n$ are linearly dependent, and choose a dependence relation with minimum of $m$ nonzero coefficients $a_1, \ldots, a_m \in L$, so that

$$a_1\chi_1 + \cdots + a_n\chi_m = 0$$

Then for any $g \in G$, we have

$$a_1\chi_1(g) + \cdots + a_n\chi_m(g) = 0$$

Now, let $g_0 \in G$, with $\chi_1(g_0) \neq \chi_m(g_0)$. Then

$$a_1\chi_1(g_0g) + \cdots + a_n\chi_n(g_0g) = a_1\chi(g_0)\chi(g) + \cdots + a_m\chi_m(g_0)\chi_m(g) = 0$$

multiplying the preceeding equation with the above by $\chi_m(g_0)$ and subtracting from the above equation, we get

$$a_1(\chi_1(g_0) - \chi_m(g_0))\chi_1(g) + \cdots + a_m(\chi_1(g_0) - \chi_m(g_0))\chi_m(g) = 0$$

which gives a linear dependence relation with fewer than $m$ nonzero coefficients; which contradicts our choice of $m$. Therefore $\chi_1, \ldots, \chi_n$ must be linearly independent. ∎

**Corollary.** *If $\sigma_1, \ldots, \sigma_n$ are distinct embeddings of a field $K$ into a field $L$, then they are linearlyy independent as functions.*

**Theorem 2.2.2.** *Let $G = \{\sigma_1, \ldots, s_n\}$ where $\sigma_1 = \iota$ a subgroup of automorphisms of a field $K$, and let $F$ be the corresponding fixed field. THen*

$$[K : F] = \operatorname{ord} G = n$$

*Proof.* Suppose that $n > [K : F]$, and consider the basis $\{\omega_1, \dots, \omega_m\}$ of ${}^{K}\!/_{F}$ as a vector space so that $[K : F] = m$. Then the matrix equation

$$\begin{pmatrix} \sigma_1\omega_1 & \dots & \sigma_n\omega_m \\ \vdots & \ddots & \vdots \\ \sigma_n\omega_1 & \dots & \sigma_n\omega_m \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_m \end{pmatrix} = 0 \tag{2.1}$$

has nontrivial solution $\begin{pmatrix} \beta_1 \\ \vdots \\ \beta_n \end{pmatrix}$. Let $a_1, \dots, a_m \in F$, so that $\sigma_i\alpha_j = \alpha_j$ for each $1 \leq i \leq n$ and $1 \leq j \leq m$. Multyplying by $\begin{pmatrix} \sigma_1 a_1 \\ \vdots \\ \sigma_m a_1 \end{pmatrix}$, we obtain

$$\begin{pmatrix} a_1\sigma_1\omega_1 & \dots & a_1\sigma_1\omega_1 \\ \vdots & \ddots & \vdots \\ a_m\sigma_m\omega_1 & \dots & a_m\sigma_n\omega_m \end{pmatrix} \begin{pmatrix} \beta_1 \\ \vdots \\ b_n \end{pmatrix} = 0$$

so that we can obtain the equation

$$\sigma_1(a_1\omega_1 + \dots + a_m\omega_m)\beta_1 + \dots + \sigma_n(a_1\omega_1 + \dots + a_m\omega_m)\beta_n = 0$$

Where $\beta_1, \dots, \beta_n$ are not all 0. Now, since $\{\omega_1, \dots, \omega_m\}$ is an $F$-basis for $K$, for all $\alpha \in K$, we get that $\alpha = a_1\omega_1 + \dots + a_m\omega_m$. So we have from the above equation

$$(\sigma_1\alpha)\beta_1 + \dots (\sigma_n\alpha)\beta_n = 0$$

so that $\{\sigma_1, \dots, \sigma_n\}$ are linearly dependent over $K$; which contradicts the above corollary. No $n \leq [K : F]$.

Now, suppose that $n < [K : F]$, and thte tere are more than $n$ $F$-linearly independent elements $\alpha_1, \dots, \alpha_{n+1} \in K$. Then

$$\begin{pmatrix} \sigma_1\alpha_1 & \dots & \sigma_1\alpha_{n+1} \\ \vdots & \ddots & \vdots \\ \sigma_n\alpha_1 & \dots & \sigma_n\alpha_{n+1} \end{pmatrix} \begin{pmatrix} x_1 \\ \dots \\ x_{n+1} \end{pmatrix} = 0$$

has nontrivial solution with entries $\beta_1, \dots, \beta_{n+1} \in K$. Now, if $\beta_i \in F$ for all $1 \leq i \leq n+1$, we get an immediate contradiction of the linear independence of $\{\alpha_1, \dots, \alpha_{n+1}\}$ over $F$. So at least one $\beta_i \notin F$.

Now, choose a nontrivial solution with minimum of $r$ nonzero entries $\beta_i$. Suppose also that $\beta_r = 1$, then at least one $\beta_i \notin F$, for $1 \leq i \leq r-1$, and so $r > 1$. SUppose then that $\beta_1 \notin F$. Then the matrix equation

$$\begin{pmatrix} \sigma_1\alpha_1 & \dots & \sigma_1\alpha_{r-1} & \sigma_1\alpha_r \\ \vdots & \ddots & \vdots & \\ \sigma_n\alpha_n & \dots & \sigma_n\alpha_{r-1} & \sigma_n\alpha_r \end{pmatrix} \begin{pmatrix} \beta_1 \\ \vdots \\ \beta_{r-1} \\ 1 \end{pmatrix} = 0$$

Now, since $\beta_1 \notin F$, there exists an automorphsim $\sigma_{k_0}$ of $K$ with $\sigma_{k_0}\beta_1 \neq \beta_1$ for $1 \leq k_0 \leq n$. Applying $\sigma_{k_0}$ to each row of the above equation yields a row of the form

$$\sigma_{k_0}(\sigma_j\alpha_1)(\sigma_{k_0}\beta_1) + \cdots + \sigma_{k_0}(\sigma_j\alpha_{r-1})(\sigma_{k_0}\beta_{r-1}) + \sigma_j\alpha_r = 0$$

However, since $G$ is a group, $\sigma_{k_0}\sigma_j = \sigma_i$ for $1 \leq i, j \leq n$, so we get

$$(\sigma_i\alpha_1)(\sigma_{k_0}\beta_1) + \cdots + (\sigma_i\alpha_{r-1})(\sigma_{k_0}\beta_{r-1}) + \sigma_i\alpha_r = 0$$

Subtracting this equation from the one preceeding it, we obtain

$$(\sigma_i\alpha_1)(\beta_1 - \sigma_{k_0}\beta_1) + \cdots + (\sigma_i\alpha_{r-1})(\beta_{r-1} - \sigma_{k_0}\beta_{r-1}) = 0$$

with $x_1 = \beta_1 - \sigma_{k_0}\beta_1 \neq 0$. This choice of $k_0$ gives fewer than $r$ nonzero coefficients of a nontrivial solutions, which contradicts the choice of $r$. Therefore $n = [K : F]$. ■

**Corollary.** *If $K$ is a finite extension over a field $F$, then*

$$\operatorname{ord} \operatorname{Aut} {}^{K}\!/\!_{F} \leq [K : F]$$

*with equality holding if, and only if $F$ is the fixed field of* $\operatorname{Aut} {}^{K}\!/\!_{F}$.

*Proof.* Let $F_1$ be the fixed field of $\operatorname{Aut} {}^{K}\!/\!_{F}$, so that $F \subseteq F_1 \subseteq K$. TYhen $[K : F_1] = \operatorname{ord} \operatorname{Aut} {}^{K}\!/\!_{F}$, hence

$$[K : F] = (\operatorname{ord} \operatorname{Aut} {}^{K}\!/\!_{F})[F_1 : F]$$

■

**Corollary.** *If $G$ is a finite subgroup of automorphisms of a field $K$, and $F$ is its fixed field, then* $\operatorname{Aut} {}^{K}\!/\!_{F} = G$ *so that $K$ is Galois over $F$ with Galois group $G$.*

*Proof.* By definition, we have that since $G$ fixes the elements of $F$, then $G \leq \operatorname{Aut} {}^{K}\!/\!_{F}$. THen $\operatorname{ord} G = [K : F]$ and by the above corollary, we get

$$\operatorname{ord} \operatorname{Aut} {}^{K}\!/\!_{F} \leq [K : F]$$

so that

$$[K : F] = \operatorname{ord} G \leq \operatorname{ord} \operatorname{Aut} {}^{K}\!/\!_{F} \leq [K : F]$$

and equality holds. ■

**Corollary.** *If $G$ and $H$ are distinct finite subgroups of* $\operatorname{Aut} K$*, then their fixed fields are also distinct.*

*Proof.* Let $F_G$ the fixed field of $G$, and $F_H$ the fixed field of $H$. I f $F_G = F_H$, then we have that $H$ fixes $F_G$, and since any automorphism fixing $F_G$ is in $G$, we have $H \leq G$. By similar reasoning, we get $G \leq H$ so that $G = H$. ■

**Theorem 2.2.3.** *The extension $K$ over a field $F$ is Galois if, and only if $K$ is the splitting field of some seperable polynomial in $F$. Moreover, every irreducible polynomial over $F$ having at least one root in $K$ splits over $K$.*

*Proof.* By lemma 2.1.5, the splitting field of a seperable polynomial over a field is Galois.

Now, suppose that $K$ is Galois over $F$, and let $p(x) \in F[x]$ an irreducible polynomial with a root $\alpha \in K$. Consider, for each $\sigma_i \in \operatorname{Gal} {}^{K}\!/_{F}$ the elements

$$\alpha \qquad \sigma_2\alpha \qquad \dots \qquad \sigma_n\alpha$$

where $\sigma_1 = \iota$, and let

$$\alpha \qquad \alpha_2 \qquad \dots \qquad \alpha_n$$

be the distinct elements taken on by these permutations (in no particular order). If $\tau \in \operatorname{Gal} {}^{K}\!/_{F}$, by the group law, we get $\tau\sigma_i = \sigma_j$ for all $1 \leq i, j \leq n$. APplying $\tau$ to $\alpha_i$ we het permutations of the elements $\alpha, \alpha_2, \dots, \alpha_n$. Then the polynomial $f(x) = (x - \alpha)(x - \alpha_2)\dots(x - \alpha_n)$ has coefficients fixed by the elements of $\operatorname{Gal} {}^{K}\!/_{F}$. That is, the coefficients lie in the fixed field $F$. Hence $f \in F[x]$.

Now, since $p$ is irreducible with root $\alpha$, it is the minimal polynomial for $\alpha$ over $F$, and hence $p|f$. Moreover, we can pbserve that $f|p$, so that $p(x) = f(x)$, which makes $p(x)$ seperable with all its roots in $K$.

Now, let $\{\omega_1, \dots, \omega_n\}$ be a basis for ${}^{K}\!/_{F}$ as a vector space, and let $p_i(x)$ the minimal polynomial for $\omega_i$ over $F$ for all $1 \leq i \leq n$. Then $p_i$ is seperable, with roots in $K$. Let $g(x) = p_1(x)\dots p_n(x)$ (where this product is squarefree). Then if $E$ is the splitting field of $g$ over $F$, then $\omega_i \in E$ for all $1 \leq i \leq n$, so that $K \subseteq E$. On the otherhand, since $g$ splits over $K$, we get $E \subseteq K$, and so $K = E$ is the splitting field of $g$ over $F$. ■

**Definition.** Let $K$ be an extension of a field $F$. If $\alpha \in K$, and $\sigma \in \operatorname{Gal} {}^{K}\!/_{F}$, we call the permutations $\sigma\alpha$ **Galois conjugates** (or simply **conjugates**) of $\alpha$ over $F$. If $E$ is a sbufield of $K$ containing $F$, then we call $\sigma E$ the **conjugate field** of $E$over $F$.

**Theorem 2.2.4** (The Fundamental Theorem of Galois Theory)**.** *Let $K$ be Galois over a field $F$ with Galois group $G$, and let $E$ be an intermidiate field of $K$ over $F$ which is fixed by some subgroup $H \leq G$. Then the following are true.*

*(1) There is a 1–1 correspondence between the subgroups of $G$ onto the fixed fields of ${}^{K}\!/_{F}$; that is, $\mathcal{F}$, treated as a map, is 1–1 and onto.*

*(2) If $\sigma \in G$, then $\sigma E$ is fixed by $\sigma H \sigma^{-1}$; that is, $\sigma E = \mathcal{F}(\sigma H \sigma^{-1})$.*

*(3) $K$ is Galois over $E$, and $E$ is normal over $F$ if, and only if $H$ is normal in $G$.*

*(4) If $H$ is normal in $G$, then*

$$\operatorname{Gal} {}^{E}\!/_{F} \simeq {}^{G}\!/_{H}$$

*(5) Independntly of whether or not $E$ is normal over $F$, we have that*

$$[E : F] = [G : H]$$

*Proof.* Let $\mathcal{G}(E) = \operatorname{Aut} {}^{K}\!/_{E}$, that is, as a map, $\mathcal{G}$ sends a intermidiate field of ${}^{K}\!/_{F}$ to that group of $E$-automorphisms of $K$; i.e. all automorphisms that fix the elements of $E$. Now, consider the mapping

$$H \to \mathcal{F}(H) \to \mathcal{G}\mathcal{F}(H)$$

and take $\sigma \in H$. Then, by definition, $\sigma$ fixes the field $\mathcal{F}(H)$, so that $\sigma \in \mathcal{G}(\mathcal{F}(H)) = \mathcal{G}\mathcal{F}(H)$. Then $H \leq \mathcal{G}\mathcal{F}(H)$. Now, we have that the fixed field $\mathcal{F}(H)$ contains the fixed field of $\mathcal{G}\mathcal{F}(H)$, i.e. $\mathcal{F}(\mathcal{G}\mathcal{F}(H))$, which is $H$. That is, $\mathcal{G}\mathcal{F}(H) \leq H$. Therefore, we have $\mathcal{G}\mathcal{F}(H) = H$. Conversely, consider the mapping

$$E \to \mathcal{G}(E) \to \mathcal{F}\mathcal{G}(E)$$

Observe that $\mathcal{F}$) is the fixed field of $\mathcal{G}(E) = \operatorname{Aut} {}^{K}\!/_{E}$, by definition, $\mathcal{F}\mathcal{G}(K) = K$. This establishes the 1–1 correspondence of the subgroups of $G$ onto the fixed fields of ${}^{K}\!/_{F}$.

Now, since $E = \mathcal{F}(H)$, observe that $\mathcal{F}(\sigma H s^{-1})$ consists of all elements of $K$ which are fixed by $\sigma\tau\sigma^{-1}$ for all $\tau \in H$; that is, all $\alpha \in K$ for which $\sigma\tau\sigma^{-1}(\alpha) = \alpha$. Observe, then that $\tau\sigma^{-1}(\alpha) = \tau(\sigma^{-1}\alpha) = \sigma^{-1}\alpha$, so that $\tau$ fixes $\sigma^{-1}\alpha$. Then $\sigma^{-1}\alpha \in \mathcal{F}(H)$. That is, $\alpha \in \sigma\mathcal{F}(H) = \sigma E$, therefore $\mathcal{F}(\sigma H \sigma^{-1}) = \sigma E$.

For the third statement, notice that since $K$ is Galois over $F$, then it is normal and seperable. This makes $E$ normal and seperable over $F$, so that by lemma 2.1.6, ${}^{E}\!/_{F}$ is Galois. Now, let $\sigma$ be a 1–1 $F$-homomorphism which is from $K \to E$. Then $\sigma$ can be extended to a 1–1 $F$-homomorphism from $K \to K$, by lemma 1.2.5. If ${}^{E}\!/_{F}$ is normal, then for every $\sigma \in G$, $\sigma$ fixes $E$, and $E$ is fixed by a normal subgroup of $G$ by the previous statement.

Consider now, the homomorphism from $G \to \operatorname{Gal} {}^{E}\!/_{F}$ given by $\sigma \to \sigma|_E$. This map is onto, with kernel consisting of all automorphisms which fix $K$. Then $\operatorname{Gal} {}^{E}\!/_{F} = H$, moreover, since ${}^{K}\!/_{F}$ is normal, we get $H \trianglelefteq G$. Therefore by the first isomorphism theorem (for groups), we get

$$\operatorname{Gal} {}^{E}\!/_{F} \simeq {}^{G}\!/_{H}$$

where ${}^{G}\!/_{H}$ is understood to be the quotient group. Finally, we also have that $[K : F] = [K : E][E : F]$. Since both ${}^{K}\!/_{F}$ and ${}^{E}\!/_{F}$ are Galois, this makes

$$\operatorname{ord} G = [E : F] \operatorname{ord} H$$

which makes $[E : F] = [G : H]$ by the definition of the index of a subgroup. ■

**Example 2.4.** (1) The lattices of $\mathbb{Q}(\sqrt{2}, \sqrt{3})$ and $\mathbb{Q}(\sqrt[3]{2}, \xi)$ (where $\xi^3 = 1$) indicate all of the subfields of these fields. We have that the lattice of $\mathbb{Q}(\sqrt{2}, \sqrt{3})$ is isomorphic to the lattice of the Klein 4-group $V_4$, which has all its subgroups normal. Thus we get every subfield of $\mathbb{Q}(\sqrt{2}, \sqrt{3})$ is Galoix over $\mathbb{Q}$.

On the other hand, the lattice for $\mathbb{Q}(\sqrt[3]{2}, \xi)$ is isomorphic to the lattice of $S^3$ where the only normal subgroup is the nontrivial subgroup of order 3; moreover, only $\mathbb{Q}(\xi)$ is Galois over $\mathbb{Q}$ with $\operatorname{Gal} {}^{\mathbb{Q}(\xi)}\!/_{\mathbb{Q}} \simeq {}^{S_3}\!/_{\langle\sigma\rangle}$, where $\langle\sigma\rangle$ is the cyclic subgroup of order 2.

$$
\begin{tikzcd}[arrows=dash]
 & \mathbb{Q}(\sqrt{2},\sqrt{3}) \arrow[dl] \arrow[d] \arrow[dr] & \\
\mathbb{Q}(\sqrt{2}) \arrow[dr] & \mathbb{Q}(\sqrt{6}) \arrow[d] & \mathbb{Q}(\sqrt{3}) \arrow[dl] \\
 & \mathbb{Q} &
\end{tikzcd}
\qquad
\begin{tikzcd}[arrows=dash]
 & \langle \sigma,\tau,\eta \rangle \arrow[dl] \arrow[d] \arrow[dr] & \\
\langle \sigma \rangle \arrow[dr] & \langle \tau \rangle \arrow[d] & \langle \eta \rangle \arrow[dl] \\
 & \langle \iota \rangle &
\end{tikzcd}
$$

Figure 2.1: The lattice of subfields of $\mathbb{Q}(\sqrt{2,\sqrt{3}})$ and the lattice of subgroups of $\operatorname{Gal}{}^{\mathbb{Q}(\sqrt{2,\sqrt{3}})}\!\!\diagup_{\mathbb{Q}}$.

(2) Consider $\mathbb{Q}(\sqrt{2}+\sqrt{3}) \subseteq \mathbb{Q}(\sqrt{2},\sqrt{3})$. If $m(x)$ is the minimal polynomial of $\sqrt{2}+\sqrt{3}$, then observe that is has as roots the distinct conjugates

$$\pm\sqrt{2}\pm\sqrt{3}$$

so that

$$m(x) = (x+(\sqrt{2}+\sqrt{3}))(x-(\sqrt{2}+\sqrt{3}))(x+(\sqrt{2}-\sqrt{3}))(x-(\sqrt{2}-\sqrt{3})) = x^4+10x+1$$

Moreover, $x^4+10x+1$ is irreducible. Then only the automorphism $\iota$ of $\{\iota,\sigma,\tau,\sigma\tau\}$ fixes $\sqrt{2}+\sqrt{3}$ so that the fixing group of $\mathbb{Q}(\sqrt{2}+\sqrt{3})$ is preciesly that of $\mathbb{Q}(\sqrt{2},\sqrt{3})$. So $\mathbb{Q}(\sqrt{\ }+\sqrt{3}) = \mathbb{Q}(\sqrt{2},\sqrt{3})$.

(3) Consider the splitting field of $x^8+1$ over $\mathbb{Q}$, generated by the elements $\sqrt[8]{2}$ and $\xi$, where $\xi^8 = 1$ (i.e. a primitive 8-th root of unity). Let $\zeta = \sqrt[8]{2}$, and notice that $\zeta^4 = \sqrt{2}$, and the splitting field of $x^8-2$ over $\mathbb{Q}$ is $\mathbb{Q}(\sqrt[8]{2},i)$ of degree $[\mathbb{Q}(\sqrt[8]{2},i) : \mathbb{Q}] = 16 = 4^2$. Consider then all possible maps on $\zeta$ and $i$ given by $\zeta \to \xi^a\zeta, i \to \pm i$. Define then the automorphisms

$$\sigma : \zeta \to \xi\zeta, i \to i \text{ and } \tau : \zeta \to \zeta, i \to -i$$

Since $\xi = \frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2} = \frac{1+i}{2}\zeta^4$, we compute that $\sigma : \xi \to \xi^5$, and $\tau : \xi \to \xi^7$. We can then compute the Galois group by noting that $\sigma^8 = \tau^2 = \iota$, and $\sigma\tau = \tau\sigma^3$, so that

$$\operatorname{Gal}{}^{\mathbb{Q}(\sqrt[8]{2},i)}\!\!\diagup_{\mathbb{Q}} = \langle \sigma,\tau : \sigma^8 = \tau^2 = \iota \text{ and } \sigma\tau = \tau\sigma^3 \rangle$$

which describes the quasidihedral group of order 16.

## 2.3 Finite Fields

We reiterate some previous results about finite fields.

**Lemma 2.3.1.** *Let $E$ be a finite field over $\mathbb{F}_p$. Then $E$ is an extension of finite degree $[E : \mathbb{F}_p] = n$. Moreover, if $|E| = p^n$, and $E$ is the splitting field of the polynomial $x^{p^n} - x$ over $\mathbb{F}_p$.*

*Proof.* Suppose that $E$ is a finite field, but that the extension ${E}/{\mathbb{F}_p}$ is infinite. The $E$, as a vector space over $\mathbb{F}_p$, has an infinite basis $\{\alpha_1, \alpha_2, \dots\}$. Moreover, since every element of $E$ is a linear comination of this basis, we obtain a contradiction as there are infinite such combinations, but $E$ is finite. Therefore $[E : \mathbb{F}_p] = n$, for some $n \in \mathbb{Z}^+$.

Let $\alpha, \beta$ be roots. Then $\alpha^{p^n} = \alpha$, and $\beta^{p^n} = \beta$. Then $(\alpha\beta)^{p^n} = \alpha\beta$ and $(\alpha^{-1})^{p^n} = \alpha^{-1}$. Moreover, $(\alpha+\beta)^{p^n} = \alpha + \beta$. So the set of $p^n$ disctinct roots of $x^{p^n} - x$ is closed under addition, multiplication, and inverses in its splitting field. Let $F$ be that splitting field. Notice that $F \subseteq E$, moreover, $[F : \mathbb{F}_p] = n$ so that $|F| = p^n$. We also have that $\mathcal{U}(F)$ is a cyclic group of order $p^n - 1$, so that $E \subseteq F$, since $\alpha^{p^n-1} = 1$. Therefore $E$ is the splitting field of $x^{p^n} - x$ over $\mathbb{F}_p$, and so contains all the roots of $x^{p^n} - x$. Notice that since $E$ is a splitting field, it is unique up to isomorphism. ■

*Remark.* Since the splitting fields of $x^{p^n} - x$ over $\mathbb{F}_p$ are unique up to isomorphism, we denote them by $\mathbb{F}_p$ from now on.

**Corollary.** *$\mathbb{F}_{p^n}$ is Galois over $\mathbb{F}_p$ with Galois group isomorphic to ${\mathbb{Z}}/{n\mathbb{Z}}$.*

*Proof.* Notice that $\mathbb{F}_{p^n}$ is normal and seperable over $\mathbb{F}_p$. Morever, that the Frobenius automorphism generates the Galois group of order $n$. ■

**Corollary.** *All subfields of $\mathbb{F}_{p^n}$ are Galois over $\mathbb{F}_p$, and in 1–1 with the divisors of $n$. Moreover, they are of the form $\mathbb{F}_{p^d}$ for all $d|n$.*

*Proof.* We have that

$$\operatorname{Gal} {\mathbb{F}_{p^n}}/{\mathbb{F}_p} = \langle \sigma \rangle \simeq {\mathbb{Z}}/{n\mathbb{Z}}$$

where $\sigma : \alpha \to \alpha^p$ is the Frobenius automorphism. By the fundamental theorem of Galois theory, each subfield of $\mathbb{F}_{p^n}$ corresponds to a subgroup of ${\mathbb{Z}}/{n\mathbb{Z}}$, which are defined by the divisors of $n$. Hence, there is precisely one field $\mathbb{F}_{p^d}$ for each $d|n$, with $[\mathbb{F}_{p^d} : \mathbb{F}_p] = d$. Now, since ${\mathbb{Z}}/{n\mathbb{Z}}$ is Abelian, every subgroup is normal, and so each $\mathbb{F}_{p^d}$ is normal over $\mathbb{F}_p$. Since they are also seperable, the are Galois over $\mathbb{F}_p$. ■

**Corollary.** *The fields $\mathbb{F}_{p^d}$ are precisely those fixed by $\sigma^d$; that is, $\mathcal{F}(\langle\sigma^d\rangle) = \mathbb{F}_{p^d}$ for all $d|n$.*

**Example 2.5.** The irreducible polynomial $x^4 + 1$ over $\mathbb{Z}$ is reducible $\mod p$, by $p$ a prime integer. Consider $x^4 + 1 \in \mathbb{F}_p[x]$, if $p = 2$, then $x^4 + 1 = (x+1)^4$ and we are done, since $\Gamma\mathbb{F}_2 = 2$. Now, if $p$ is an odd prime, notice that $p \equiv 1, 3, 5, 7 \mod 8$ so that $p^2 \equiv 1 \mod 8$. That is, $8|p^2 - 1$. Then the polynomial $x^8 - 1|x^{p^2-1} - 1$. We have then that $x^4 + 1|x^8 - 1|x^{p^2-1} - 1|x^{p^2} - x$; so that the roots of $x^4 + 1 \mod p$ are also the roots of $x^{p^2} - x \mod p$, and equivalently, are fixed by the automorphism $\sigma^2$, $\sigma$ being the Frobenius automorphism. Since $\mathbb{F}_{p^2}$ is the splitting field of $x^{p^2} - x$, and hence consists of exactly the roots of $\mathbb{F}_{p^2}$, we get that if $\alpha$ is a root of $x^{p^2} - x$, then $[\mathbb{F}_p(\alpha) : \mathbb{F}_p] \leq 2$ so that $x^4 + 1$ is not irreducible over $\mathbb{F}_p$.

**Lemma 2.3.2.** *The finite field $\mathbb{F}_{p^n}$ is simple.*

*Proof.* We have that $\mathcal{U}(\mathbb{F}_p)$ is a multiplicative group of finite order; moreover, it is the finite subgroup of the multiplicative group of a field. Therefore $\mathcal{U}(\mathbb{F}_p)$ is cyclic. Then if $\alpha$ is a generator of $\mathcal{U}(\mathbb{F}_p)$, then $\mathbb{F}_{p^n} = \mathbb{F}_p(\alpha)$ and there exists an irreducible polynomial of degree $n$ over $\mathbb{F}_p$. ■

**Lemma 2.3.3.** *$x^{p^n} - x$ is precisely the product of all irreducible polynomials of degree* $\deg =$ *$d$, $d|n$ over $\mathbb{F}_p$.*

*Proof.* Consider $\mathbb{F}_{p^r}$ as the quotient ring ${}^{\mathbb{F}_p[x]}\!/\!_{(m)}$, where $m(x)$ os the minimal polynomial of a root $\alpha$ of $x^{p^n} - x$. Then it follows that $m(x)|x^{p^n} - x$ and $\deg m = n$.

COnversely, if $p(x)$ is an irreducible polynomial of degree $\deg p = d$, with $p|x^{p^n} - x$, if $\alpha$ is a root of $p$, then $\mathbb{F}_p(\alpha) \subseteq \mathbb{F}_{p^n}$; and $[\mathbb{F}_p(\alpha) : \mathbb{F}_p] = d$. That makes $d|n$, and $\mathbb{F}_p(\alpha)$ is Galois over $\mathbb{F}_p$. Now, since $\mathbb{F}_{p^n}$ consists preecisely of the roots of $x^{p^n} - x$, grouping together the factores $x - \alpha$ according to the degre $d$, we obtain the result. ■

**Definition.** We define the **Möbius function** of an integer $n \in \mathbb{Z}^+$ to be

$$\mu(n) = \begin{cases} 1, & \text{if } n = 1 \\ 0, & \text{if } n \text{ has a square factor} \\ (-1)^r, & \text{if } n \text{ has } r \text{ distinct prime factors} \end{cases}$$

**Theorem 2.3.4.** *The number of irreducible polynomials of degree $n$ over $\mathbb{F}_p$ is*

$$\psi(n) = \frac{1}{n}\sum_{d|n} \mu(d) p^{\frac{n}{d}}$$

*Proof.* Let $\psi(n)$ the number of irreducible polynomials of degree $\deg = n$ over $\mathbb{F}_p$. By the above lemma, we get that $p^n = \sum_{d|n} d\psi(d)$. Using the Möbius inversion formula for $n\psi(n)$, we get that $n\psi(n) = \sum_{d|n} \mu(d) p^{\frac{n}{d}}$. ■

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