Complex Analysis

Alec Zabel-Mena

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Contents

Chapter 1

The Complex Numbers

1.1 The Field of Complex Numbers and the Complex Plane

Definition. We define the set of **complex numbers** to be the collection of all ordered pairs of real numbers $\mathbb{C} = \{(a, b) : a, b \in \mathbb{R}\}$ together with the binary operations $+$ and $\cdot$ of **complex addition**, and **complex multiplication**, respectively defined by the rules

$$(a, b) + (c, d) = (a + c, b + d)$$
$$(a, b)(c, d) = (ac - bd, bc + ad)$$

Theorem 1.1.1. *The set of complex numbers $\mathbb{C}$ forms a field together with complex addition and complex multiplication.*

Corollary. *$\mathbb{C}$ is a field extension of the real numbers $\mathbb{R}$.*

Proof. The map $a \to (a, 0)$ from $\mathbb{R} \to \mathbb{C}$ defines an imbedding of $\mathbb{R}$ into $\mathbb{C}$. ∎

Definition. We define the element $i = (0, 1)$ of $\mathbb{C}$ so that $i^2 = -1$, and the polynomial z^2+1 has as root i. We write $(a, b) = a + ib$. If $z = a + ib$, we call a the **real part** of z, and b the **imaginary part** of z and write $\operatorname{Re} z = a$ and $\operatorname{Im} z = z$.

Definition. Let $z = a+ib \in \mathbb{C}$. We define the **norm** (or **modulus**) of z to be $\|z\| = \sqrt{a^2 + b^2}$. We define the complex **conjugate** of z to be $\overline{z} = a - ib$.

Lemma 1.1.2. *For every $z \in \mathbb{C}$, $\|z\|^2 = z\overline{z}$.*

Proof. Let $z = a+ib$. Then $\overline{z} = a-ib$, and so $z\overline{z} = (a+ib)(a-ib) = a^2+b^2 = (\sqrt{a^2 + b^2})^2 = \|z\|^2$. ∎

Corollary. *If $z \neq 0$, then $z^{-1} = \frac{1}{z} = \frac{\overline{z}}{\|z\|^2}$.*

Proof. The relation follows from above, and it remains to show that it is indeed the inverse element. Notice that if $z \in \mathbb{C}$ is nonzero, then $z\frac{\overline{z}}{\|z\|^2} = \frac{z\overline{z}}{\|z\|^2} = \frac{\|z\|^2}{\|z\|^2} = 1$. ∎

Example 1.1. (1) Let $z = a + ib$. Then we get that $\frac{1}{z} = \frac{\overline{z}}{\|z\|}$ has real part $\operatorname{Re}\frac{1}{z} = \frac{a}{a^2+b^2}$ and imaginary part $\operatorname{Im}\frac{1}{z} = -\frac{b}{a^2+b^2}$.

(2) Let $z = a + ib$, and $c \in \mathbb{R}$. Then $\frac{z-c}{z+c} = \operatorname{Re}\frac{z-c}{z+c}$, so $\operatorname{Im}\frac{z-c}{z+c} = 0$.

(3) Let $z = a + ib$, then $z^3 = a^3 - 3ab^2 + i(3a^2b - b^3)$ So that $\operatorname{Re} z^3 = a^3 - 3ab^2$ and $\operatorname{Im} z = 3a^2b - b^3$.

(4) $\frac{3+i5}{1+i7} = \frac{19}{25} - i\frac{18}{25}$.

(5) $(-\frac{1}{2} + i\frac{\sqrt{3}}{2})^3 = 1$, and hence $(-\frac{1}{2} + i\frac{\sqrt{3}}{2})^6 = 1$.

(6) Notice that $i^n = 1, i, -1, -i$ whenever $n \equiv 0 \mod 4$, $n \equiv 1 \mod 4$, $n \equiv 2 \mod 4$, and $n \equiv 3 \mod 4$. respectively.

(7) $\|-2 + i\| = \sqrt{5}$, and $\|(2 + i)(4 + i3)\| = \|5 + i10\| = 5\sqrt{5}$.

Lemma 1.1.3. *The following are true for all $z, w \in \mathbb{C}$.*

(1) $\operatorname{Re} z = \frac{1}{2}(z + \overline{z})$ *and* $\operatorname{Im} z = \frac{1}{2i}(z - \overline{z})$.

(2) $\overline{(z + w)} = \overline{z} + \overline{w}$ *and* $\overline{zw} = \overline{z}\,\overline{w}$.

(3) $\|\overline{z}\| = \|z\|$.

Proof. Let $z = a + ib$ and $w = c + id$. Then notice that

$$\frac{(a + ib) + (a - ib)}{2} = \frac{2a + (ib - ib)}{2} = \frac{2a}{2} = a = \operatorname{Re} z$$

and

$$\frac{(a + ib) - (a - ib)}{2i} = \frac{(a - a) + 2ib}{2} = \frac{2ib}{2i} = b = \operatorname{Im} z$$

Moreover

$$\overline{(a + ib) + (c + id)} = \overline{(a + c) + i(b + d)} = (a + c) - i(b + d) = (a - ib) + (c - id)$$

And

$$\overline{(a + ib)(c + id)} = \overline{(ac - bd) + i(bc + ad)} = (ac - bd) - i(bc + ad) = (a - ib)(c - id)$$

so that $\overline{z + w} = \overline{z} + \overline{w}$ and $\overline{zw} = \overline{z}\,\overline{w}$.

Now, we have that $\|zw\|^2 = (zw)\overline{zw} = (zw)(\overline{z}\,\overline{w}) = (z\overline{z})(w\overline{w}) = \|z\|^2\|w\|^2$. Taking square roots, we get the result

$$\|zw\| = \|z\|\|w\|$$

Finally, notice that $\|z\|^2 = z\overline{z} = \overline{z}\,\overline{\overline{z}} = \|\overline{z}\|$. ∎

Corollary. *The following are also true; provided $w \neq 0$.*

(1) $\overline{(\frac{z}{w})} = \frac{\overline{z}}{\overline{w}}$.

(2) $\|\frac{z}{w}\| = \frac{\|z\|}{\|w\|}$

Corollary. *If $z = z_1 + \cdots + z_n$, and $w = w_1 \ldots w_n$, with $z_i, w_i \in \mathbb{C}$ for all $1 \leq i \leq n$, then*

(1) $\overline{z} = \overline{z_1} + \cdots + \overline{z_n}$.

(2) $\|w\| = \|w_1\| \ldots \|w_n\|$.

Proof. We prove both results by induction on n. For $n = 2$, we have already shown that $\overline{z} = \overline{z_1} + \overline{z_2}$ and $\|w\| = \|w_1\|\|w_2\|$. Now, for all $n \geq 2$, suppose that both

$$\begin{aligned} \overline{z} &= \overline{z_1} + \cdots + \overline{z_n} \\ \|w\| &= \|w_1\| \ldots \|w_n\| \end{aligned}$$

Then let $z' = z + z_{n+1}$ and $w' = ww_{n+1}$ for $z_{n+1}, w_{n+1} \in \mathbb{C}$. Then we have that

$$\begin{aligned} z' &= z + z_{n+1} = z_1 + \cdots + z_n + z_{n+1} \\ w' &= ww_{n+1} = w_1 \ldots w_n w_{n+1} \end{aligned}$$

so by the induction hypothesis, we have

$$\overline{z'} = \overline{(z + z_{n+1})} = \overline{z} + \overline{z_{n+1}} = \overline{z_1} + \cdots + \overline{z_n} + \overline{z_{n+1}}$$

and that

$$\|w'\| = \|ww_{n+1}\| = \|w\|\|w_{n+1}\| = \|w_1\| \ldots \|w_n\|\|w_{n+1}\|$$

which completes the proof. ■

Lemma 1.1.4. *Let $z \in \mathbb{C}$. Then z is a real number if, and only if $z = \overline{z}$.*

Proof. If z is real, then $z = a + i0$, for some $a \in \mathbb{R}$, and hence $\overline{z} = a - i0 = z$. COnversely, suppose that $z = \overline{z}$. Then we have

$$\operatorname{Re} z = \frac{1}{2}(z + \overline{z}) = \frac{1}{2}(z + z) = z$$

so z has only a real part, and hence must be a real number. ■

Lemma 1.1.5. *The following are true for all $z, w \in \mathbb{C}$.*

(1) $\|z + w\|^2 = \|z\|^2 + 2 \operatorname{Re} z\overline{w} + \|w\|^2$.

(2) $\|z - w\|^2 = \|z\|^2 - 2 \operatorname{Re} z\overline{w} + \|w\|^2$.

(3) $\|z + w\|^2 + \|z - w\|^2 = 2(\|z\|^2 + \|w\|^2)$.

Proof. We first notice that $\|z+w\|^2 = (z+w)\overline{(z+w)} = (z+w)(\overline{z}+\overline{w}) = z\overline{z}+z\overline{w}+w\overline{z}+w\overline{w} = \|z\|^2 + z\overline{w} + w\overline{z} + \|w\|^2$. Now, let $z = a+ib$ and $w = c+id$. Then we have

$$\begin{aligned}(a+ib)(c-id) &= (ac+bd) - i(ad-bc)\\ (c+id)(a-ib) &= (ac+bd) + i(ad-bc)\end{aligned}$$

so that $z\overline{w} + w\overline{z} = 2(ac+bd) = 2\operatorname{Re} z\overline{w}$, and we are done. To get the identity for $\|z-w\|^2$, we simply replace w by $-w$, and use the above argument.

Now, we have that $\|z+w\|^2 = \|z^2\| + 2\operatorname{Re} z\overline{w} + \|w\|^2$, and $\|z-w\|^2 = \|z^2\| - 2\operatorname{Re} z\overline{w} + \|w\|^2$, so that adding them together, the terms $2\operatorname{Re} z\overline{w}$ cancel out and we are left with

$$\|z+w\|^2 + \|z-w\|^2 = 2(\|z\|^2 + \|w\|^2)$$

■

Lemma 1.1.6. *Let $R(z) \in \mathbb{C}(z)$ a rational function in z. Then if R has coefficients in $\mathbb{R}$, then $\overline{R(z)} = R(\overline{z})$.*

Proof. We first observe the polynomial $f \in \mathbb{C}[z]$, of finite degree $\deg f = n$, and of the form

$$f(z) = a_0 + a_1 z + \cdots + a_n z^n$$

Then if f has all coefficients in $\mathbb{R}$; i.e. $f \in \mathbb{R}[z]$, where $z \in \mathbb{C}$ is treated as indeterminant, then we have that since each $a_i \in \mathbb{R}$, then $\overline{a_i z^i} = \overline{a_i}\,\overline{z^i} = a_i \overline{z}^i$. So that

$$\overline{f(z)} = \overline{(a_0 + a_1 z + \cdots + a_n z^n)} = a_0 + a_1\overline{z} + \cdots + a_n \overline{z}^n$$

which makes $\overline{f(z)} = f(\overline{z})$. Now, one can also extend f to a polynomial of infinite degree by taking $n \to \infty$, and the same holds.

Now, let $R(z) \in \mathbb{C}(z)$ a rational function. Recall that $R(z)$ is of the form

$$R(z) = \frac{f(z)}{g(z)} \text{ with } g \neq 0$$

for some polynomials $f, g \in \mathbb{C}[z]$. Then if R has all real coefficients, so do f and g, and hence we get

$$\overline{R(z)} = \frac{\overline{f(z)}}{\overline{g(z)}} = \frac{f(\overline{z})}{g(\overline{z})} = R(\overline{z})$$

which completes the proof. ■

Bibliography

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