# Algebraic Geometry.

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# Contents

# Chapter 1

# Preliminaries

## 1.1 Affine Varieties

**Definition.** Let $k$ be an algebraically closed field. We define **affine $n$-space** over $k$ to be the set $\mathbb{A}^n(k)$ of $n$-tuples of elements of $k$. We write simply $\mathbb{A}^n$ when $k$ is understood. We call the elements of $\mathbb{A}^n$ **points** and if $P = (a_1, \ldots, a_n)$ is a point of $\mathbb{A}^n$, we call each $a_i$ the **coordinates** of $P$.

**Example 1.1.** Let $k$ be any algebraically closed field, and consider the multivariate polynomial ring $k[x_1, \ldots, x_n]$. We can interperate the elements of $k[x_1, \ldots, x_n]$ as functions from affine space $\mathbb{A}^n(k)$ to $k$ by taking $f(P) = f(a_1, \ldots, a_n)$, where $f \in k[x_1, \ldots, x_n]$ and $P \in \mathbb{A}^n(k)$. This leads us to be able to talk about the set of zeros of a polynomial over $k$.

**Definition.** Let $k$ be an algebraically closed field, and $f \in k[x_1, \ldots, x_n]$ a multivariate polynomial over $k$. We define the **set of zeros** of $f$ to be the set

$$Z(f) = \{P \in \mathbb{A}^n(k) : f(P) = 0\}$$

Let $T$ be a subset of $k[x_1, \ldots, x_n]$. Then we define the **set of zeros** of $T$ to be

$$Z(T) = \bigcap_{f \in T} Z(f)$$

Now, if $I = (f_1, \ldots, f_r)$ is an ideal of $k[x_1, \ldots, x_n]$ generated by $T$, then we write $Z(T) = Z(I) = Z(f_1, \ldots, f_r)$.

**Definition.** Let $k$ be an algebraically closed field. We call a subset $Y$ of $\mathbb{A}^n$ an **algebraic set** if there exists some $T \subseteq k[x_1, \ldots, x_n]$ for which $Y$ is the set of zeros of $T$; i.e. $Y = Z(T)$.

**Lemma 1.1.1.** *Let $k$ be an algebraically closed field. Then algebraic sets of $\mathbb{A}^n$ make $\mathbb{A}^n$ into a topology under closed sets.*

*Proof.* Let $\mathbb{A}^n = Z(0)$ and $\emptyset = Z(1)$. Then $\mathbb{A}^n$ and $\emptyset$ are both algebraic. Now, let $X$ and $Y$ be algebraic, then there are $S, T$ such that $X = Z(S)$ and $Y = Z(T)$. Now, let $P \in X \cup Y$, then $P$ is a zero of any polynomial $f \in ST$, conversly, suppose that $P \in Z(ST)$ where

$P \notin Y$. There exists a polynomial $f \in S$ with $f(P) \neq 0$. Now, for any $g \in T$, we have that if $fg(P) = 0$, then $g(P) = 0$, so that $P \in S$. Therefore we have $X \cup Y = Z(ST)$, making $X \cup Y$ algebraic. So that the collection of algebraic sets is closed under finite intersection.

Lastly, consider a collection $\{Y_\alpha\}$ of algebraic sets, where $Y_\alpha = Z(T_\alpha)$ for some $T_\alpha$. Let

$$Y = \bigcap Y_\alpha \text{ and } T = \bigcup T_\alpha$$

and let $P \in Y$. Then $P$ is in every $Y_\alpha$ making it a zero of some $f_\alpha \in T_\alpha$, thus $P \in Z(T)$. Similarly, if $P \in Z(T)$, then $P \in Y$, making $Y = Z(T)$, and making the collection of algebraic sets closed under arbitrary intersections. ■

**Definition.** We define the **Zariski topology** on affine $n$-space $\mathbb{A}^n$ to be the topology on $\mathbb{A}^n$ whose open sets are complements of algebraic sets.

**Example 1.2.** Consider the Zariski topology on affine 1-space $\mathbb{A}^1$. Now, since $k[x]$ is a PID, every algebraic set of $\mathbb{A}^1$ is the set of zeros of preciesly one polynomial. Moreover, by the algebraic closure of $k$, for any nonzero polynomial $f$ over $k$, we have

$$f(x) = c(x - a_1) \dots (x - a_n)$$

where $c, a_1, \dots, a_n \in k$. Then $Z(f) = \{a_1, \dots, a_n\}$, so that the algebraic sets of $\mathbb{A}^1$ are the emptyset, itself, and finite subsets. Thus the Zariski topology on $\mathbb{A}^1$ consists of complements of finite sets, the emptyset, and $\mathbb{A}^1$ itself. Notice that this topology is not Hausdorff.

**Definition.** Let $X$ be a topological space, and $Y$ a subspace of $X$. We call $Y$ **irreducible** if it cannot be written as the union $Y = Y_1 \cup Y_2$ of two sets $Y_1$ and $Y_2$ closed in $Y$. We make the convention that the emptyset is not irreducible.

**Example 1.3.** (1) Notice that the affine space $\mathbb{A}^1$ is irreducible. We have the only closed sets are finite sets, and since $k$ is algebraically closed, and hence infinite, then $\mathbb{A}^1$ must be infinite.

(2) Subspaces of irreducible spaces are irreducible and dense.

(3) If $Y$ is an irreducible space of a topological space $X$, then the closure $\operatorname{cl} Y$ is also irreducible in $X$.

**Definition.** We define an **algebraic affine variety** to be an irreducible closed subset of $\mathbb{A}^1$ under the Zariski topology. We define an open set of an affine variety to be a **quasi-affine variety**.

# Bibliography

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[2] I. N. Herstein, *Topics in algebra.* New York: Wiley, 1975.