Field Theory and Galois Theory.

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Contents

Chapter 1

Fields.

1.1 Field Extensions.

Definition. We define the **characteristic** of a field F to be the smallest positive integer p, such that $p \cdot 1 = 0$, where 1 is the identity of F. We write $\operatorname{char} F = p$, and if no such p exists, then we write $\operatorname{char} F = 0$.

Lemma 1.1.1. *Let F be a field, then $\operatorname{char} F$ is either 0, or a prime integer.*

Proof. Let $\Gamma F = p$. If $p = 0$, then we are done. Now suppose that $p = mn$, with $m, n \in \mathbb{Z}^+$. Then $p \cdot 1 = (mn)1 = (n \cdot 1)(m \cdot 1) = mn = 0$, which makes m and n 0 divisors. Since F is a field, and hence an integral domain, this is impossible, and hence p must be prime. ■

Corollary. *If $\operatorname{char} F = p$, then for all $a \in F$, $pa = \underbrace{a + \cdots + a}_{p \text{ times}}$.*

Proof. We have $pa = p(a \cdot 1) = (p \cdot 1)a$. ■

Example 1.1. (1) Both $\mathbb{Q}$ and $\mathbb{R}$ have char $= 0$. Similarly, $\operatorname{char} \mathbb{Z} = 0$, even though $\mathbb{Z}$ is just an integral domain.

(2) $\operatorname{char} \mathbb{Z}/p\mathbb{Z} = p$ and $\operatorname{char} \mathbb{Z}/p\mathbb{Z}[x] = p$ for any prime p.

Definition. We define the **prime subfield** of a field F to be the subfield of F generated by 1.

Example 1.2. (1) The prime subfields of $\mathbb{Q}$ and $\mathbb{R}$ is $\mathbb{Q}$.

(2) Let $\mathbb{Z}/p\mathbb{Z}(x)$ the field of rational functions over $\mathbb{Z}/p\mathbb{Z}$. Then the prime subfield of $\mathbb{Z}/p\mathbb{Z}(x)$ is $\mathbb{Z}/p\mathbb{Z}$. Similarly, the prime subfield for $\mathbb{Z}/p\mathbb{Z}[x]$ is also $\mathbb{Z}/p\mathbb{Z}$.

Definition. If K is a field containing a field F, then we call K **field extension** over F, and write K/F (not the quotient field!) or denote it by the diagram

$$\begin{array}{c} K \\ | \\ F \end{array}$$

Lemma 1.1.2. *Every field is a field extension of its prime subfield.*

Lemma 1.1.3. *Let K an extension over a field F. Then K is a vector space over F.*

Definition. Let ${}^{K}\!/_{F}$ a field extension. We define the **degree** of K over F, $[K : F]$ to be the dimension of ${}^{K}\!/_{F}$ as a vector space.

Definition. Let F be a field, and $f \in F[x]$ a polynomial. We call am element $\alpha \in R$ a **root** (or **zero**) of f if $f(\alpha) = 0$.

Lemma 1.1.4. *Let $\phi : F \to L$ a field homomorphism. Then either $\phi = 0$, or ϕ is 1–1.*

Lemma 1.1.5. *Let F be a field, and $p \in F[x]$ an irreducible polynomial. Then there exists a field K containing an embedding of F, such that p has a root in K.*

Proof. Consider $K = {}^{F[x]}\!/_{(p)}$. Siince p is irreducible in a principle ideal domain, (p) is a maximal idea, and hence K is a field. Now consider the canonical map $\pi : F[x] \to K$ taking $f \to f \mod (p)$ and let $\phi = \pi|_F$. Then $\phi \neq 0$, since $\pi : 1 \to 1$. Then ϕ is 1–1. And so $\phi(F) \simeq F$.

Now, consider F as a subfield of K. Then $p(x \mod (p)) \equiv p(x) \mod (p) \equiv 0 \mod (p)$, so that $x \mod (p)$ is a root of p in K. ■

Corollary. *There exists a field extension of F containing a root of p.*

Theorem 1.1.6. *Let F be a field, and let $p \in F[x]$ an irreducible polynomial of degree n, and let $K = {}^{F[x]}\!/_{(p)}$, and $\theta = x \mod (p)$. Then $\{1, \theta, \dots, \theta^{n-1}\}$ forms a basis for K as a vector space over F and $[K : F] = n$.*

Proof. Let $a \in F[x]$, since $F[x]$ is Euclidean domain, there exist $q, r \in F[x]$, $q \neq 0$ for which

$$a(x) = q(x)p(x) + r(x) \text{ where } \deg r < n$$

Now, since $pq \in (p)$, $a(x) \equiv r(x) \mod (p)$, and every element of K is a polynomial of degree less than n. Then the elements $\{1, \theta, \dots, \theta^{n-1}\}$ span K.

Now, suppose that there are $b_0, \dots, b_{n-1} \in F$ not all 0 for which

$$b_0 + b_1\theta + \cdots + b_{n-1}\theta^{n-1} = 0$$

Then

$$b_0 + b_1\theta + \cdots + b_{n-1}\theta^{n-1} \equiv 0 \mod (p)$$

so that $p|(b_0 + b_1\theta + \cdots + b_{n-1}\theta^{n-1})$ in F. But $\deg p = n$ and p divides a polynomial of degree $n - 1$, which is a contradiction. Therefore we are left with $b_0 = \cdots = b_{n-1} = 0$. ■

Corollary. $K = \{\alpha_0 + a_1\theta + \cdots + a_{n-1}\theta^{n-1} : a_i \in F \textit{ for all } 1 \leq i \leq n-1\}$

Corollary. *If $a(\theta), b(\theta) \in K$, are elements of degree less than n, and the operations of polynomial addition, and polynomial multiplication mod (p) are defined, then K forms a field.*

Example 1.3. (1) Consider the polynomial x^2+1 over $\mathbb{R}$. Then one has the field

$$\mathbb{C} = \mathbb{R}[x] \big/ (x^2+1)$$

an extension of $\mathbb{R}$ of degree $[\mathbb{C} : \mathbb{R}] = 2$. Let i be a root of x^2+1 in this field, then $i^2 = -1$, and the elements of $\mathbb{C}$ are of the form $a + ib$ where $a, b \in \mathbb{R}$. Then we have described the field of complex numbers, and the addition and multiplication (mod x^2+1) of these elements are the addition and multiplication of complex numbers.

One might also construct $\mathbb{C}$ differently by defining the isomorphism

$$\mathbb{R}[x] \big/ (x^2+1) \to \mathbb{C} \text{ taking } a + xb \to a + ib$$

(2) Consider again x^2+1 over $\mathbb{Q}$. Then we get the field

$$\mathbb{Q}(i) = \mathbb{Q}[x] \big/ (x^2+1)$$

of degree $[\mathbb{Q}(i) : \mathbb{Q}] = 2$, and where i is a root of x^2+1, so that $i^2 = -1$. Then the elements of $\mathbb{Q}(i)$ are of the form $a + ib$ where $a, b \in \mathbb{Q}$, i.e. it is isomorphic to the set of all complex numbers with rational components.

(2) Consider $x^2 - 2$ over $\mathbb{Q}$. by Eisenstein's criterion for $p = 2$, $x^2 - 2$ is irreducible over $\mathbb{Q}$. Let α a root of $x^2 - 2$, so that $\alpha^2 = 2$. Then we have the field

$$\mathbb{Q}(\sqrt{2}) = \mathbb{Q}[x] \big/ (x^2-2)$$

of degree $[Q(\sqrt{2}) : \mathbb{Q}] = 2$, and whose elements are of the form $a + b\sqrt{2}$. One can define an isomorphism between $\mathbb{Q}(i)$ and $\mathbb{Q}(\sqrt{2})$ by taking $\sqrt{2} \to i$.

(3) The polynomial $x^3 - 2$ over $\mathbb{Q}$ gives us the field

$$\mathbb{Q}(\sqrt[3]{2}) = \mathbb{Q}[x] \big/ (x^3-2)$$

of degree $[\mathbb{Q}(\sqrt[3]{2}) : \mathbb{Q}] = 3$ over 2. Here the elements are of the form $a + b\xi + c\xi^2$ where $\xi^3 = 2$.

(4) Denote $\mathbb{F}_2$ to be a finite field of 2 elements. Consider the polynomial $x^2 + x + 1$ over $\mathbb{F}_2$ which is irreducible. Then the field

$$\mathbb{F}_2(\alpha) = \mathbb{F}_2[x] \big/ (x^2+x+1)$$

is a field of degree 2 over $\mathbb{F}_2$, whose elements are of the form $a + b\alpha$, where $\alpha^2 = \alpha + 1$. In fact, one can generate this field using the fact that $\alpha^2 = \alpha + 1$.

(5) Let $F = K(t)$ the field of rational functions in t over a field K. Let $p(x) = x^2 - t \in F[x]$, then by Eisenstien's criterion with the ideal (t), p is irreducible over $F[x]$. Let θ be a root for p, that is $\theta = \sqrt{t}$, then we get the field $K(t, \sqrt{t})$ of degree $[K(t, \sqrt{t}) : K] = 2$, whose elements are of the form $a(t) + b(t)\sqrt{t}$.

Lemma 1.1.7. *Let F be a subfield of a field K, and let $\alpha \in K$. Then there exists a unique minimal subfield of K containing F and α; more preciesly, it is the intersection of all subfields of K containing F and α.*

Definition. Let K be any extension of a field F, and let $\alpha, \beta, \dots \in K$. Then we define the subfield **generated** by $\alpha, \beta, \dots$ over F to be the unique minimal subfield containing all $\alpha, \beta, \dots$ and F and we denote it $F(\alpha, \beta, \dots)$. Moreover, we call K a **simple extension** of F if $K = F(\alpha, \beta, \dots)$. If $K = (F\alpha_1, \dots, a_n)$ for $\alpha_1, \dots, a_n \in K$, then it is a **finitely generated** simple extension.

Theorem 1.1.8. *Let F be a field, and $p \in F[x]$ irreducible, and let K an extension of F containing a root α of p. Then*

$$F(\alpha) \simeq {}^{F[x]}\!\big/_{(p)}$$

Proof. Consider the homomorphism $F[x] \to F(\alpha)$ taking $a(x) \to a(\alpha)$. Since $p(\alpha) = 0$, p is in the kernel of this homomorphism, and we get an induced homomorphsim from ${}^{F[x]}\!\big/_{(p)} \to F(\alpha)$. Now, since p is irreducible, ${}^{F[x]}\!\big/_{(p)}$ is a field, and since the homomorphsim takes $1 \to 1$, it is 1–1. Then by the first isomorphsim theorem for ring homomorphsims these two fields are isomorphic. ∎

Corollary. *If $\deg p = n$, then $F(\alpha) = \{a_0 + a_1\alpha + \dots a_{n-1}\alpha^{n-1} : a_i \in F$ for all $1 \leq i \leq n-1\}$ and $[F(\alpha) : F] = n$.*

Example 1.4. (1) The polynomial $x^2 - 2$ over $\mathbb{Q}$ also has the root $-\sqrt{2}$ in $\mathbb{R}$, so that $\mathbb{Q}(-\sqrt{2})$ is of degree 2 over $\mathbb{Q}$ with elements of the form $a - b\sqrt{2}$. Notice however that $\mathbb{Q}(-\sqrt{2}) \simeq \mathbb{Q}(\sqrt{2})$ by taking $a - b\sqrt{2} \to a + b\sqrt{2}$.

(2) The polynomial $x^3 - 2$ only has the solution $\xi = \sqrt[3]{2}$ in $\mathbb{R}$. However, in $\mathbb{Q}$ it has the solutions given by

$$\sqrt[3]{2}\left(\frac{-1 \pm i\sqrt{3}}{2}\right)$$

So that the subfields generated by either of these three elements (over $\mathbb{C}$) are isomorphic.

Theorem 1.1.9. *Let $\phi : F \to L$ a field isomorphism and $p \in F[x]$, $q \in L[x]$ irreducible polynomials, where q is obtained by applying ϕ to the coefficients of p. Let α a root of p, and β a root of q. Then there exists an isomorphism $F(\alpha) \to L(\beta)$ taking $\alpha \to \beta$ and extending ϕ. That is, we have the following diagram*

$$\begin{tikzcd}
F(\alpha) \arrow[r] \arrow[d, dash] & L(\beta) \arrow[d, dash] \\
F \arrow[r, "\phi"'] & E
\end{tikzcd}$$

Proof. Notice that ϕ induces a ring homomorphism between $F[x]$ and $L[x]$, so that (p) is maximal. Since q is obtained from p, (q) is also maximal, so that ${F[x]}/{(p)}$ and ${L[x]}/{(q)}$ are fields. Then we have an isomorphsism

$$ {F[x]}/{(p)} \simeq {L[x]}/{(q)} $$

Then, if α is a root of p, and β a root of q, we obtain the isomorphism

$$ F(\alpha) \simeq L(\beta) $$

moreover, this isomorphism takes $\alpha \to \beta$. ∎

1.2 Algebraic Extensions.

Definition. Let ${K}/{F}$ be a field extension. We say that an element $\alpha \in K$ is **algebraic** over F, provided there exists a polynomial over F having α as a root. Otherwise we call α **transcendental**. If every $\alpha \in K$ is algebraic, we call K **algebraic** and ${K}/{F}$ an **algebraic extension**.

Lemma 1.2.1. *Let α be algebraic over a field F. Then there exista a unique monic irreducible polynomial $m \in F[x]$ having α as a root. Moreover, if $f \in F[x]$ is a polynomial, then f has α as a root if, and only if $m|f$.*

Proof. Let m a polynomial of minimal degree having α as a root. Suppose, also that , is monic. Now, if m were reducible, then $m(x) = a(x)b(x)$ for some $a, b \in F[x]$ polynomials both of degree less than $\deg m$. Then we also have that $a(\alpha) = b(\alpha) = 0$, which contradicts that m is the polynomial of minimal degree satisfying that condition. Hence, m is irreducible.

Now, let $f \in F[x]$ have α as a root, then by the divison theorem, there exist $q, r \in F[x]$, with $q \neq 0$ for which

$$ f(x) = q(x)m(x) + r(x) \text{ where } \deg r < \deg m $$

Now, since $f(\alpha) = q(\alpha)m(\alpha) + r(\alpha) = 0$, then $r(x) = 0$ for all x lest we contradict the minimality of m. Hence $m|f$. Conversely, if $m|f$, then f has α as a root.

Now, let g a polynomial of minimal degree for which $g(\alpha) = 0$. Then by above, we have that $\deg g = \deg m$, and that moreover, $m|g$ and $g|m$. therefore $g = m$ and uniqueness is established. ∎

Corollary. *Let ${L}/{F}$ be an extension, and α algebraic over F. Let $m_{\alpha,F}$ the unique monic irreducible polynomial over F having α as root, and $m_{\alpha,L}$ the unique monic irreducible polynomial over L having α as root. Then $m_{\alpha,L}|m_{\alpha,F}$ in $L[x]$.*

Definition. Let F be a field, and α algebraic over F. We define the **minimal polynomial** $m_{\alpha,F}$, to be the polynomial over F of minimal degree having α as a root. If the field is clear, we instead write m_α, or even just m when the root itself is also clear. We define the **degree** of α to be $\deg \alpha = \deg m_\alpha$.

Lemma 1.2.2. *Let α algebraic over F. Then*

$$F(\alpha) \simeq {}^{F[x]}\!/\!_{(m_{\alpha,F})}$$

Corollary. $[F(\alpha) : F] = \deg m_\alpha = \deg \alpha$.

Example 1.5.

(1) The minimal polynomial for $\sqrt{2}$ over $\mathbb{Q}$ is $x^2 - 2$.

(3) The minimal polynomial for $\sqrt[3]{2}$ over $\mathbb{Q}$ is $x^3 - 2$.

(3) Let $n > 1$, then by the Eisenstein-Schömann criterion, $x^n - 2$ is irreducible over $\mathbb{Q}$. Moreover, $x^n - 2$ has as root in $\mathbb{R}$ $\sqrt[n]{2}$. Then $\mathbb{Q}(\sqrt[n]{2})$ is a field of degree $[\mathbb{Q}(\sqrt[n]{2}) : \mathbb{Q}] = 2$. Moreover $x^n - 2$ is the minimal polynomial of $\sqrt[n]{2}$. Notice, that over $\mathbb{R}$, $\deg [n]2 = 1$, and that $m_{\sqrt[n]{2},\mathbb{R}}(x) = x - \sqrt[n]{2}$.

(4) Consider $p(x) = x^3 - 3x - 1$ over $\mathbb{Q}$. Notice that p is irreducible over $\mathbb{Q}$ and let α a root of p. Then $[\mathbb{Q}(\alpha) : \mathbb{Q}] = 3$.

Lemma 1.2.3. *An element α is algebraic over a field F if, and only if the simple extension ${}^{F(\alpha)}\!/\!_{F}$ is finite.*

Proof. If α is algebraic over F then $[F(\alpha) : F] = \deg \alpha \leq n$ if α satisfies a polynomial of degree n. Conversely, if α is an element of the finite extenson ${}^{K}\!/\!_{F}$, of degree n, then the set $\{1, \alpha, \dots, \alpha^n\}$ is linearly dependent over F. Hence there exist $b_0, \dots, b_n \in F$ not all 0 for which

$$b_0 + b_1\alpha + \cdots + a_n\alpha^n = 0$$

making α a root of a nonzero polynomial over F of degree $\deg \leq n$. ∎

Corollary. *If an extension ${}^{K}\!/\!_{F}$ is finite, then it is algebraic.*

Proof. If $\alpha \in K$ is algebraic, then ${}^{K}\!/\!_{F}$ implies that ${}^{F(\alpha)}\!/\!_{F}$ is finite, since $F(\alpha) \subseteq K$. ∎

Example 1.6. Let F a field of $\operatorname{char} F \neq 2$, and let K an extension field of F of degree $[K : F] = 2$. Let $\alpha \in K$ not in F, then α satisfies an polynomial of at most degree 2 over F. Now, since $\alpha \notin F$, this polynomial must have degree greater than 1. Hence it satisfies a polynomial of degree 2. Then the minimal polynomial of α is a quadratic

$$m_\alpha(x) = x^2 + bx + c \text{ with } b, c \in F$$

Since $F \subseteq F(\alpha) \subseteq K$, and $F(\alpha)$ is a vector space over F of dimension 2, then we must have $K = F(\alpha)$; that is ${}^{K}\!/\!_{F}$ is simple.

Now, the roots of m_α are

$$\alpha = \frac{-b \pm \sqrt{b^2 - 4c}}{2}$$

Since $\alpha \notin F$, $b^2 - 4c$ is not a square in F, and $\sqrt{b^2 - 4c}$ is a root of the equation $x^2 - (b^2 - 4c) = 0$ in K.

Conversely, $\sqrt{b^2 - 4c} = \pm(b + 2\alpha)$ which puts $\sqrt{b^2 - 4c} \in F(\alpha)$. That is $F(\sqrt{b^2 - 4c}) = \mathbb{F}(\alpha)$. Moreover, $x^2 - (b^2 - 4c)$ does not have solutions in K.

We call field extensions ${}^{K}\!/\!_{F}$ of degree 2 **quadratic field extension**, where $K = F(\sqrt{D})$, and D is a squarefree element of F.

Theorem 1.2.4. *Let $F \subseteq K \subseteq L$. Then $[L : F] = [L : K][K : F]$.*

Proof. Let $[L : K] = m$ and $[K : F] = n$. Let $\{\alpha_1, \ldots, \alpha_m\}$ and $\{\beta_1, \ldots, \beta_n\}$ be bases for the extensions ${}^{L}\!/_{K}$ and ${}^{K}\!/_{F}$. Now, the elements of L over K are of the form

$$a_1\alpha_1 + \cdots + a_m\alpha_m \text{ where } a_i \in K \text{ for all } 1 \leq i \leq m$$

Since each $a_i \in K$, which is an extension over F, they have the form

$$a_i = b_{i1}\beta_{i1} + \cdots + b_{in}\beta_{in} \text{ where } b_{ij} \in F \text{ for all } 1 \leq j \leq n$$

That is, every element of L, as a vector space over F are of the form

$$\sum b_{ij}\alpha_i\beta_j$$

So the set $\{\alpha_1\beta_1, \ldots \alpha_m\beta_n\}$ spans L. It remains to show that this set is linearly in dependent.

Suppose that

$$\sum b_{ij}\alpha_i\beta_j = 0$$

for some $b_{ij} \in F$. Since $\{\alpha_1, \ldots, \alpha_m\}$ are linearly indpendent in L over K, we have that the coefficients $a_1 = \cdots = a_n = 0$ which makes

$$a_i = b_{i1}\beta_{i1} + \cdots + b_{in}\beta_{in} = 0$$

Now, since $\{\beta_1, \ldots, \beta_n\}$ is linearly independent in K over F, this implies that $b_{i1} = \cdots = b_{in} = 0$ which makes the collection $\{\alpha_1\beta_1, \ldots \alpha_m\beta_n\}$ linearly independent, and hence, a basis. Moreover, notice that this basis has size mn. ■

Example 1.7. (1) The element $\sqrt{2} \notin \mathbb{Q}(\alpha)$, where α is the root of $x^3 - 3x - 1$; since $[\mathbb{Q}(\sqrt{2}) : \mathbb{Q}] = 2$, and $[\mathbb{Q}(\alpha) : \mathbb{Q}] = 3$.

(2) We have $[\mathbb{Q}(\sqrt[6]{2}) : \mathbb{Q}] = 6$, and since $(\sqrt[6]{2})^3 = \sqrt{2}$, we observe that $\mathbb{Q}(\sqrt{2}) \subseteq \mathbb{Q}(\sqrt[6]{2})$. Moreover, notice that by theorem 1.2.4 $[\mathbb{Q}(\sqrt[6]{2}) : Q(\sqrt{2})] = 3$. Then we have the following tower of fields for

$$\mathbb{Q} \subseteq \mathbb{Q}(\sqrt{2}) \subseteq \mathbb{Q}(\sqrt[6]{2})$$

$$\begin{array}{c} \mathbb{Q}(\sqrt[6]{2}) \\ \Big| \\ \mathbb{Q}(\sqrt{2}) \\ \Big| \\ \mathbb{Q} \end{array}$$

Lemma 1.2.5. *Let α, β be algebraic over a field F. Then $F(\alpha, \beta) = (F(\alpha))(\beta)$.*

Proof. By definition, $F(\alpha, \beta)$ contains F, and α, and hence contains $F(\alpha)$. It also contains β so that $(F(\alpha))(\beta) \subseteq F(\alpha, \beta)$. By the same argument, $(F(\alpha))(\beta)$ contains F, α and β so that $F(\alpha, \beta) \subseteq (F(\alpha))(b)$. ■

Corollary. *The elements of* $F(\alpha, \beta)$ *are of the form* $\sum a_{ij}\alpha^i b^j$, *where* $1 \leq i \leq \deg \alpha$ *and* $1 \leq j \leq \deg \beta$.

Example 1.8. Consider $\mathbb{Q}(\sqrt{2}, \sqrt{3})$ generated by $\sqrt{2}$ and $\sqrt{3}$. Notice that $\deg \sqrt{3} = 2$ over $\mathbb{Q}$ so that $[\mathbb{Q}(\sqrt{2}, \sqrt{3}) : \mathbb{Q}(\sqrt{2})] \leq 2$. Now $[\mathbb{Q}(\sqrt{2}, \sqrt{3}) : \mathbb{Q}(\sqrt{2})] = 2$ if, and only if the polynomial $x^2 - 3$ is irreducible over $\mathbb{Q}(\sqrt{2})$. Then it is irreducible if, and onyl if $\sqrt{2} \in \mathbb{Q}(\sqrt{3})$. It can be shown that this is not the case by trying to find $a, b \in \mathbb{Q}$ for which $\sqrt{3} = a + b\sqrt{2}$. Moreover we have

$$[\mathbb{Q}(\sqrt{2}, \sqrt{3}) : \mathbb{Q}] = 4$$

Theorem 1.2.6. *An extension field* ${}^{K}\!/_{F}$ *is finite if, and only if it is generated by finitely many algebraic elements over* F.

Proof. Let ${}^{K}\!/_{F}$ finite of degree n, and $\{\alpha_1, \dots, \alpha_n\}$ a basis. Then by theorem 1.2.4, $[F(\alpha_i) : F] | [K : F]$ for all $1 \leq i \leq n$. So each α_i is algebraic over F. Then K is generated by finitely many algebraic elements.

Conversely, let $K = F(\alpha_1, \dots, \alpha_k) = (F(\alpha_1, \dots a_{k-1}))(\alpha_k)$ We obtain K by taking the extensions ${}^{F_{i+1}}\!/_{F_i}$ iteratively, where $F_{i+1} = F_i(\alpha_{i+1})$, and obtain the sequence

$$F = F_0 \subseteq \dots \subseteq F_k = K$$

Now, if the elements $\alpha_1, \dots, \alpha_k$ are algebraic over F, each of $\deg \alpha_i = n_i$ for $1 \leq i \leq k$, then the extension ${}^{F_{i+1}}\!/_{F_i}$ is a simple extension, and $[F_{i+1} : F_i] = \deg m_{\alpha_{i+1}} \leq \deg \alpha_{i+1} = n_{i+1}$. Then we have

$$[K : F] = [F_k : F_{k-1}] \dots [F_1, F] \leq n_1 \dots n_k$$

which makes ${}^{K}\!/_{F}$ a finite extension. ■

Corollary. *If* α, β *are algebraic over* F, *then so are* $\alpha \pm \beta$, $\alpha\beta$, *and* $\alpha\beta^{-1}$ *(for* $\beta \neq 0$*).*

Corollary. *If* ${}^{L}\!/_{F}$ *is an extension, then the collection of elements of* L *which are algebraic over* F *forms a subfield of* L.

Example 1.9. (1) Consider the extension ${}^{\mathbb{C}}\!/_{\mathbb{Q}}$, and let $\operatorname{cl}\mathbb{Q}$ the subfield of all elements of $\mathbb{C}$ which are algebraic over $\mathbb{Q}$. Then $\sqrt[n]{2} \in \operatorname{cl} Q$ for all $n \geq 1$, so that $[\operatorname{cl}\mathbb{Q} : \mathbb{Q}] \geq n$. This makes $\operatorname{cl}\mathbb{Q}$ an infinite algebraic extension, and we call $\operatorname{cl}\mathbb{Q}$ the **field of algebraic numbers**.

(2) Consider $\operatorname{cl}\mathbb{Q} \cap \mathbb{R}$ as a subfield of $\mathbb{R}$ (i.e. the subfield of all algebraic elements of $\mathbb{Q}$). Since $\mathbb{Q}$ is countable, so is the field $\mathbb{Q}[x]$, and each polynomial in $\mathbb{Q}[x]$ has at most n roots in $\mathbb{R}$, hence the number of all algebraic elements of $\mathbb{R}$ over $\mathbb{Q}$ is also countable. This means that $\operatorname{cl}\mathbb{Q}$ must also be countable. Now, since $\mathbb{R}$ is uncountable, then there exist uncountably transcendental numbers of $\mathbb{R}$ over $\mathbb{Q}$. Most notably the irrational numbers π and e are transcendental.

Theorem 1.2.7. *If* K *is algebraic over* F, *and* L *algebraic over* K, *then* L *is algebraic over* F.

Proof. Let $\alpha \in L$, since L is algebraic over K, there exists a $p \in K[x]$ having α as root. Let $p(x) = a_0 + a_1x + \cdots + a_nx^n$. Consider then $F(\alpha, a_0, \ldots, a_n)$. Since ${}^{K}\!/_{F}$ is algebraic, $a_0, \ldots, a_n$ are algebraic over F, and so $F(\alpha, a_0, \ldots, a_n)$ is a finite extension over F. Then α generates an extension field of degree less than n, and we get

$$[F(\alpha, a_0, \ldots, a_n) : F] = [F(\alpha, a_0, \ldots, a_n) : F(a_0, \ldots, a_n)][F(a_0, \ldots, a_n) : F]$$

is finite, and $F(\alpha, a_0, \ldots, a_n)$ is algebraic over F. That is, α is algebraic over F, and so L is algbraic over F. ■

Definition. Let K_1 and K_2 subfields of a field K. The **composite field** K_1K_2 is the smallest subfield of K containing both K_1 and K_2.

Example 1.10. The composite field of $\mathbb{Q}(\sqrt[3]{2})$ and $\mathbb{Q}(\sqrt{2})$ is $\mathbb{Q}(\sqrt[6]{2})$.

Lemma 1.2.8. *Let K_1 and K_2 be extensions of a field F contained in a field K. Then $[K_1K_2 : F] \leq [K_1 : F][K_2 : F]$ with equality holding if, and only if a basis of F in the other field is linearly independent. Moreover if $\{\alpha_1, \ldots, \alpha_m\}$ and $\{\beta_1, \ldots, \beta_n\}$ are bases for K_1 and K_2, then $\{\alpha_1, \beta_1, \ldots, \alpha_m\beta_n\}$ span K_1 and K_2.*

$$\begin{tikzcd}
 & K_1K_2 \arrow[dl, dash, "\leq m"'] \arrow[dr, dash, "\leq n"] & \\
K_1 \arrow[dr, dash, "n"'] & & K_2 \arrow[dl, dash, "m"] \\
 & F &
\end{tikzcd}$$

Corollary. *If $[K_1 : F] = m$, and $[K_2 : F] = n$ with m and n coprime, then $[K_1K_2 : F] = [K_1 : F][K_2 : F]$.*

Proof. We have that $m, n | [K_1K_2 : F]$ and since $K_1, K_2 \subseteq K_1K_2$ are subfields of K_1K_2, we get the least common multiple $[m, n] | [K_1K_2 : F]$. Now, since $(m, n) = 1$, we get $[m, n] = mn$ so that $mn \leq= [K_1K_2 : F]$. ■

1.3 Splitting Fields

Definition. Let K be an extension of a field F. We say a polynomial f over F **splits completely** over K if f factors into linear factors over K. If f splits completely over K, and in no other proper subfield, then we say K is the **splitting field** of f over F.

Theorem 1.3.1. *If f is a polynomial over a field F, then there exists a splitting field K of f over F.*

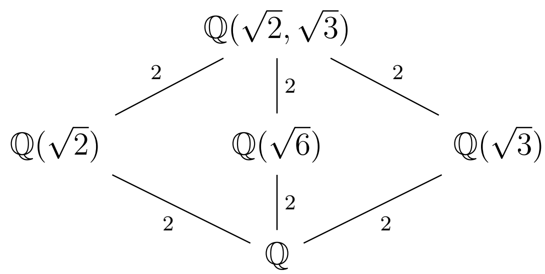
Proof. Let E an extension of F with $[E : F] = n$. By induction on n, for $n = 1$, we take $E = F$ and we are done. Now, for $n \geq 1$, suppose the irreducible factors of f are of $\deg = 1$. Then f has all its roots in F, and hence splits completely over F. Then take $E = F$. On the other hand, if f has at least one irreducible factor of deg ≥ 2, then there is an extension E_1 of F for which f has the factor $(x-\alpha)$ for some root α. Then $f(x) = (x-\alpha)f_1(x)$ where $\deg f_1 = n-1$. Therefore by the induction hypothesis, there is an extension E of E_1 containing all the roots of f_1. Hence, it contains all the roots of f and f splits completely over E.

Now, let K be the intersection of all subfields of E for which f splite; i.e. all subfields containing the roots of f. Then by definition, K is the splitting field of f over F. ■

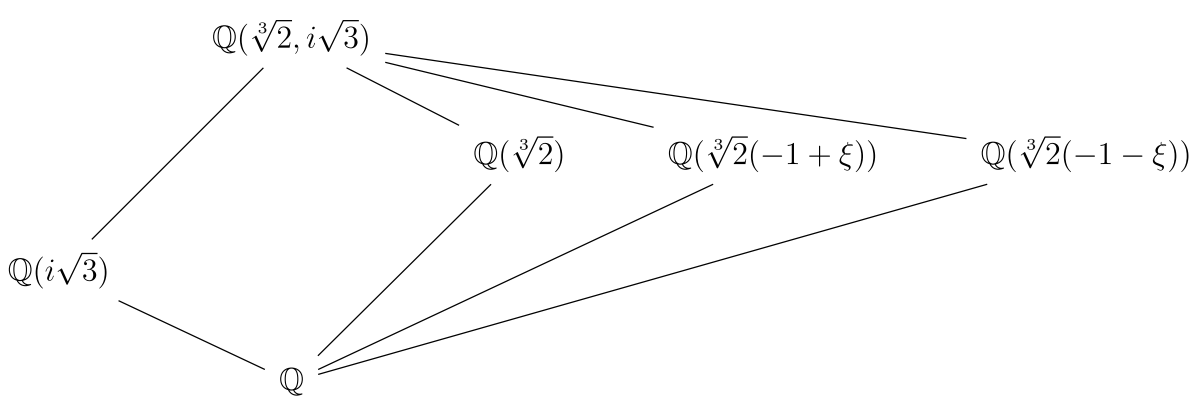
Definition. We call an extension K over a field F **normal**, if for any irreducible polynomial f over F with atleast one root in K, f splits completely in K over F. That is to say, K contains the splitting field of f over F.

Example 1.11. (1) The splitting field of x^2-2 over $\mathbb{Q}$ is $\mathbb{Q}(\sqrt{2})$, since $x^2-2 = (x+\sqrt{2})(x-\sqrt{2})$ and $\pm\sqrt{2} \in \mathbb{Q}(\sqrt{2})$ and $[\mathbb{Q}(\sqrt{2}) : \mathbb{Q}] = 2$, so there is no other subfield in between.

(2) The splitting field for $(x^2-2)(x^2-3) = (x+\sqrt{2})(x-\sqrt{2})(x+\sqrt{3})(x-\sqrt{3})$ is $\mathbb{Q}(\sqrt{2},\sqrt{3})$. Now, $[\mathbb{Q}(\sqrt{2},\sqrt{3}) : Q] = 4$ and the lattice of fields is



(3) Let $\xi = i\frac{\sqrt{3}}{2}$. Notice that x^3-2 factors into $x^3-2 = (x-\sqrt[3]{2})(x+\sqrt[3]{2}(-1+\xi))(x+\sqrt[3]{2}(-1-\xi))$. Now, $-1+\xi, -1-\xi \notin \mathbb{Q}(\sqrt[3]{2})$, so $\mathbb{Q}(\sqrt[3]{2})$ is not the splitting field for x^3-2. Let K be the splitting field of x^3-2. Then K conmtains $-1\pm\xi$, so that $i\sqrt{3} \in K$. Thus
$$K = \mathbb{Q}(\sqrt[3]{2}, i\sqrt{3})$$
Moreover, $[\mathbb{Q}(\sqrt[3]{2}, i\sqrt{3}) : \mathbb{Q}(\sqrt[3]{2})] \geq 2$ and since $\mathbb{Q}(\sqrt[3]{2})$ is not the splitting field, $[\mathbb{Q}(\sqrt[3]{2}, i\sqrt{3}) : \mathbb{Q}(\sqrt[3]{2})] = 2$. Hence $[\mathbb{Q}(\sqrt[3]{2}, i\sqrt{3}) : \mathbb{Q}] = 6$. We have the following lattice.



(4) Notice that $x^4+4 = (x^2+2x+2)(x^2-2x+2)$ over $\mathbb{Q}$ which is irreducible by Eisenstein's criterion. Using the quadratic formula, we get ± 1 and $\pm i$ as the roots, moreover, notice that $\pm 1, \pm i \in \mathbb{Q}(i)$ and since $[\mathbb{Q}(i) : \mathbb{Q}] = 2$ there are no subfields between $\mathbb{Q}$ and $\mathbb{Q}(i)$ so that $\mathbb{Q}(i)$ is the splitting field of $x^4 + 4$ over $\mathbb{Q}$.

Lemma 1.3.2. *A splitting field of a polynomial of degree n over a field F is of degree at most $n!$ over F.*

Proof. Let $f \in F[x]$ a polynomial of $\deg f = n$. Adjoining one root of f to F, we have an extension ${}^{F_1}\!/_{F}$ of degree $[F_1 : F] = n$. Now,f over F_1 has at leas one linear factor, and so any root of f satisfies a polynomial of degree $n-1$. Hence proceeding inductively gives the result. ■

Example 1.12. Consider the polynomial $x^n - 1$ over $\mathbb{Q}$. Then the roots of $x^n - 1$ are of the form ξ where $\xi^n = 1$. Notice, that in $\mathbb{C}$, $\xi = e^{\frac{2i\pi}{n}}$, so that $\mathbb{C}$ contains a splitting field of $x^n - 1$. Hence $\mathbb{Q}(\xi) \subseteq \mathbb{C}$ is a splitting field of $x^n - 1$ over $\mathbb{Q}$. Notice that the set of all roots ξ of $x^n - 1$ forms a cyclic group generated by ξ.

Definition. Consider a field F and the polynomial $x^n - 1$ over F. We call the roots ξ of $x^n - 1$, where $\xi^n = 1$ the **primitive n-th roots of unity** over F. We call $F(\xi)$ the **cyclotomic field** over F.

Example 1.13. Let p be a prime, and consider the splitting field $x^p - 2$ over $\mathbb{Q}$. If α is a root, then $\alpha^p = 2$ so that $(\xi\alpha)^p = 2$ where ξ is a primitive p-th root of unity over $\mathbb{Q}$. So the roots of $x^2 - 2$ are

$$\sqrt[p]{2} \text{ and } \xi\sqrt[p]{2}$$

Notice that $\frac{\xi\sqrt[p]{2}}{\sqrt[p]{2}} = \xi$ so the splitting field contains $\mathbb{Q}(\xi, \sqrt[p]{2})$, Moreover, $\mathbb{Q}(\xi, \sqrt[p]{2})$ contains all the roots of $x^p - 2$ so that $\mathbb{Q}(\xi, \sqrt[p]{2})$ is the splitting field of $x^p - 2$ over $\mathbb{Q}$.

Notice, that $\mathbb{Q}(\xi) \subseteq \mathbb{Q}(\xi, \sqrt[p]{2})$ so that $[\mathbb{Q}(\xi, \sqrt[p]{2}) : \mathbb{Q}(\xi)] \leq p$. not, since $\mathbb{Q}(\sqrt[p]{2})$ is also a subfield, we get $[\mathbb{Q}(\xi, \sqrt[p]{2}) : Q] \leq p(p-1)$. Since $(p, p-1) = 1$ (i.e. they are coprime), we have $p(p-1)|[\mathbb{Q}(\xi, \sqrt[p]{2}) : \mathbb{Q}]$ so that $[p]2) : \mathbb{Q}] = p(p-1)$. We have the follwing lattice.

$$\begin{tikzcd}
 & \mathbb{Q}(\xi, \sqrt[p]{2}) \arrow[dl, dash] \arrow[dr, dash] & \\
\mathbb{Q}(\xi) \arrow[dr, dash] & & \mathbb{Q}(\sqrt[p]{2}) \arrow[dl, dash] \\
 & \mathbb{Q} &
\end{tikzcd}$$

Theorem 1.3.3. *Let $\phi : F \to F'$ a field isomorphism. Let f and f' polynomials over F and F', where f' is obtained by applying ϕ to the coefficients of f. Let E and E' be splitting fields of f and f' over F and F', respectively. Then ϕ extends to an isomorphism betweenn E and E'; i.e. $E \simeq E'$.*

$$\begin{tikzcd}
E \arrow[r] \arrow[d, dash] & E' \arrow[d, dash] \\
F \arrow[r, "\phi"] & F'
\end{tikzcd}$$

Proof. Let $\deg f = n$. By induction on n. If f has all its roots in F, f splits completely over F, and f' over F'. Then take $E = F$ and $E' = F'$ and we are done for $n = 1$.

Now, for $n \geq 1$, suppose the theorem is true. Let p an irreducible factor of f, and p' an irreducible factor of f'. If α and α' are roots of p and p', respectively, then extend ϕ to $F(\alpha)$ and $F'(\alpha')$. Then $f(x) = (x-\alpha)f_1(x)$ and $f'(x) = (x-\alpha')f_1'(x)$; with $\deg f_1 = \deg f_1' = n-1$. Then let E the splitting field of f_1 over $F(\alpha)$, and E' the splitting field of f_1' over $F'(\alpha')$

$$\begin{tikzcd}
E \arrow[r] \arrow[d, dash] & E' \arrow[d, dash] \\
F(\alpha') \arrow[r] \arrow[d, dash] & F'(\alpha') \arrow[d, dash] \\
F \arrow[r, "\phi"'] & F'
\end{tikzcd}$$

The the roots of f_1 and f_1' are in E and E', respectively, and hence so are the roots of f and f'. Then by the induction hypothesis, we can extend ϕ to E and E' so that $E \simeq E'$. ■

Corollary. *Any two splitting fields of a given polynomial over a field are isomorphic.*

Proof. Take ϕ to be the identity map. ■

1.4 Algebraic Closures.

Definition. We define the **algebraic closure** of a field F to be the algebraic extension, $\operatorname{cl} F$, over F for which every polynomial over F splits. We call a field K **algebraically closed** if every polynomial over K has at least one root in K.

Lemma 1.4.1. *A field K is algebraically closed if, and only if every polynomial over K has all of its roots in K.*

Proof. Certainly, if a polynomial f over K contains all of its roots in K, then K is algebraically closed, by definition.

Now, suppose that K is algebraically closed, and let f a polynomial over K. Then f contains at least one root in K. Hence $f(x) = (x - \alpha)f_1(x)$ for some root α of f, and where $f_1 \in K[x]$. But then by definition again, f_1 contains at least one root in K. Hence, we proceed until we exhaust all the roots of f, and obtain that every root of f lies in K. ■

Corollary. *K is algebraically closed if, and only if $\operatorname{cl} K = K$.*

Lemma 1.4.2. *Let F be a field, and $\operatorname{cl} F$ its algebraic closure. Then $\operatorname{cl} F$ is algebraically closed; i.e. $\operatorname{cl}(\operatorname{cl} F) = \operatorname{cl} F$.*

Proof. Let $f \in \operatorname{cl} F[x]$, and α a root of f. Then α generates all of $\operatorname{cl} F(\alpha)$, making $\operatorname{cl} F$ algebraic over F. Hence α is algebraic over F, but $\alpha \in \operatorname{cl} F$, so that $\operatorname{cl}(\operatorname{cl} F) = \operatorname{cl} F$. ■

Lemma 1.4.3. *For every field F, there exists an algebraically closed set containing F.*

Proof. Consider the polynomial ring $F[\dots, x_n, \dots]$ where $f(x_n)$ is a nonconstant polynomial over f. Consider the ideal (f). Then, if $(f) = (1)$, then

$$g_1 f_1(x_1) + \cdots + g_n f_n(x_n) = 1$$

where $g_i \in F[x_i]$. Then we get

$$g_1(x_1, \dots, x_m) f_1(x_1) + \cdots + g_n(x_1, \dots, x_m) f_n(x_n) = 1$$

Now, let F' an extension of F containing a root α_i of f_i. Then we observe that $0 = 1$ in the above equation which is a blatant contradiciton. S o (f) must be a proper ideal.

Now, by Zorn's lemma, there exists a maximal ideal M containing I. Then the quotient

$$K_1 = {}^{F[\dots, x_n, \dots]}\!/_{M}$$

is a field containing an imbedding of F. Moreover, f has a root in K_1, so that $f(x_n) \in (f) \subseteq M$. Then K_1 is a field in which every polynomial over F has a root. Proceeding as before with K_1, we obtain K_2 in which every polynomial over K_1 has a root. Hence, proceeding recursively, we obtain the sequence

$$F = K_0 \subseteq K_1 \subseteq K_n \subseteq \dots$$

in which everypolynomial over K_n has all its roots in K_{n+1}. Now, let

$$K = \bigcup K_n$$

Then $F \subseteq K$, and every polynomial over K has a root in K_N, for N large enough; but $K_N \subseteq K$, so that K is algebraically closed. ∎

Lemma 1.4.4. *Let K be algebraically closed, and let $F \subseteq K$. Then the collection of elements of the algbraic closure $\operatorname{cl} F$ of K that are algebraic over F is an algebraic closure of F.*

Proof. By definition, ${}^{\operatorname{cl} F}\!/_{F}$ is algebraic. Then every polynomial f over F splits over K into linear factors $(x - \alpha)$, where α is a root of f. So α is algebraic over F, and hence $\alpha \in \operatorname{cl} F$. then all linear factors have a coefficient in $\operatorname{cl} F$, so that f splits completely over $\operatorname{cl} F$. ∎

Corollary. *Algebraic closures are unique up to isomorphism.*

Theorem 1.4.5 (The Fundamental Theorem of Algebra)**.** *$\mathbb{C}$ is algebraically closed.*

Corollary. *$\mathbb{C}$ contains the an algebraic closuder of any of its subfields.*

1.5 Seperability.

Definition. Let f be a polynomial over a field F with factorization

$$f(x) = a_n (x - \alpha_1)^{n_1} \dots (x - \alpha_k)^{n_k}$$

where $\alpha_1, \dots, \alpha_k$ are roots of f, and a_n is the leading coefficient of f. If $n_i > 1$, we call α_i a **multiple root** of f, and if $n_i = 1$, we call α_i a **simple root**. We call n_i the **multiplicity** of α_i.

Definition. A polynomial over a field F is said to be **seperable** if it has only simple roots. Otherwise, we say it is **inseperable**.

Lemma 1.5.1. *Seperable polynomials have all their roots distinct.*

Definition. We say a field F is a **finite field** if it has a finite number of elements. If $|F| = n$, then we denote F as $\mathbb{F}_n$.

Lemma 1.5.2. *Every finite field has finite characteristic. Moreover, that characteristic is a prime integer.*

Proof. Recall that the characteristic is just the additive order of the element 1 in the field. Lemma 1.1.1 reiterates that any field of nonzero characteristic must have prime characteristic. ■

Example 1.14. (1) $x^2 - 2 = (x + \sqrt{2})(x - \sqrt{2})$ is seperable over $\mathbb{Q}$. However $(x^2 - 1)^n$ is inseperable.

(2) Consider $x^2 - t$ over the field $\mathbb{F}_2(t)$ of rational functions over t. $x^2 - t$ is irreducible, but inseperable. Let $\sqrt{t}$ a root, then $(x - \sqrt{t})^2 = x^2 - t$ since $\operatorname{char} \mathbb{F}_2 = 2$.

Definition. The **derivative** of a polynomial $f(x) = a_0 + a_1 x + \cdots + a_n x^n$ over a field F is the polynomial

$$Df(x) = a_1 + 2a_2 x + \cdots + n a_n x^{n-1}$$

over F.

Lemma 1.5.3. *For any two polynomials f and g over a field, the following are true.*

(1) $D(f + g) = Df + D(g)$.

(2) $D(fg) = fDg + gDf$.

Lemma 1.5.4. *A polynomial f has a multiple root α if, and only if α is a root of Df. Moreover, the minimal polynomial of α, m_α divides (f, Df).*

Proof. Let α a multiple root of f. Then $f(x) = (x - \alpha)^n g(x)$ for some polynomial g. Hence

$$Df(x) = n(x - \alpha)^{n-1} g(x) + (x - \alpha)^n Dg(x)$$

so that α is a root of Df.

Conversly, suppose that α is a root of both f and Df. Then $f(x) = (x - \alpha)g(x)$ for some polynomial g, and $Df(x) = g(x) + (x - \alpha)Dg(x)$. Now, since $Df(\alpha) = 0$, we get $h(\alpha) = 0$, so that h has a linear factor $(x - \alpha)$. This makes α a multiple root of f. ■

Corollary. *f is seperable if and only if $(f, Df) = 1$.*

Corollary. *Every irreducible polynomial ina field F of* $\operatorname{char} F = 0$ *is seperable. Moreover, a polynomial over such a field is irreducible if, and only if it is the product of distinct irreducible factors.*

Proof. Let p an irreducible polynomial over F of $\deg p = n$. Then $\deg Dp = n - 1$. Up to constant factors, the factors of p are 1 and itself, so that $(p, Dp) = 1$. This makes p seperable. Therefore every irreducible polynomial over F is seperable, and the rest follows. ■

Example 1.15. (1) Let p prime and $f(x) = x^{p^n} - x$ over the finite field $\mathbb{F}_p$, of $\operatorname{char} \mathbb{F}_p = p$. Then $Df(x) = p^n x^{p^n - 1} - 1 \equiv -1 \mod p$. Then Df has no roots, which makes f seperable.

(2) $D(x^n - 1) = nx^{n-1}$ for any field of char coprime to p. Then $D(x^n - 1)$ has a root 0 of multiplicity $n > 1$, but 0 is not a root of $x^n - 1$ so that $x^n - 1$ is seperable. That is, $x^n - 1$ has n distinct roots of unity ξ.

(3) Let F a field of $\operatorname{char} F = p$, where $p|n$. Then there are fewew than n distinct n-th roots of unity over F, since $n \equiv 0 \mod p$. Then $D(x^n - 1) = 0$, and every root of $x^n - 1$ is a multiple root.

Lemma 1.5.5. *If f is a polynomial over a field F whose derivative is* 0*, then there exist a polynomial g for which $f(x) = g(x^p)$ where* $\operatorname{char} F = p$.

Proof. Let $f(x) = a_0 + a_1 x + \cdots + a_n x^n$. Then $Df(x) = a_1 + \cdots + n a_n x^{n-1} = 0$, so that every exponent $i \equiv 0 \mod p$. That is, $f(x) = a_0 + a_1 x^p + \cdots + a_m x^{mp}$. Then let

$$g(x) = a_0 + a_1 x + \cdots + a_m x^m$$

then $f(x) = g(x^p)$. ■

Lemma 1.5.6. *Let F a field of* $\operatorname{char} F = p$. *The for every $a, b \in F$, $(a + b)^p = a^p + b^p$ and $(ab^p) = a^p b^p$.*

Proof. The binomial theorem gives

$$(a + b)^p = a^p + \binom{p}{1} a^{p-1} b + \cdots + \binom{p}{p-1} a b^{p-1} + b^p$$

Now, since $\binom{p}{i} \in \mathbb{Z}$ for any $1 \le i \le p - 1$, and p is prime (the charateristic of a field has to be prime), then $p|\binom{p}{i}$. Hence $\binom{p}{i} \equiv 0 \mod p$, so that the binomial exapnsion above reduces to

$$(a + b)^p \equiv a^p + b^p \mod p$$

Now, let $\phi : a \to a^p$, then ϕ is an automorphism of fields taking $(ab)^p = a^p b^p$. ■

Corollary. *Let F be a finite field of* $\operatorname{char} F = p$. *Then every element of F is a $p^{th} powerinF$.*

Definition. Let F be a field. We call the automorphism $F \to F$ defined by $a \to a^p$ where $p \in \mathbb{Z}$ the **Forbenius automorphism**.

Lemma 1.5.7. *Every irreducible polynomial in a finite field F is seperable.*

Proof. Suppose otherwise. Since F has finite characteristic, there is a polynomial q over F for which $p(x) = q(x^l)$, where p is the irreducible polynomial in question, and $\operatorname{char} F = l$. Let

$$q(x) = a_0 + a_1 x + \cdots + a_n x^n$$

then $a_i = b_i^p$ for some $b_i \in F$, and

$$\begin{aligned} p(x) &= q(x^l) \\ &= a_0 + a_1 x^p + \cdots + a_n x^{pn} \\ &= b_0^p + b_1^p x^p + \cdots + b_n^p x^{np} \\ &= (b_0 + b_1 x + \cdots + b_n x^n)^p \end{aligned}$$

which is a contradiction. ■

Definition. A field K of characteristic $\operatorname{char} K = p$ is called **perfect** if for every $a \in K$, there exists a $b \in K$ for which $a = b^p$, or $p = 0$.

Example 1.16. Let $n > 0$ and consider the splitting field of the polynomial $x^{p^n} - x$ over the finite field $\mathbb{F}_p$. Then $x^{p^n} - x$ has precisely p^n roots.

Let α, β be roots. Then $\alpha^{p^n} = \alpha$, and $\beta^{p^n} = \beta$. Then $(\alpha\beta)^{p^n} = \alpha\beta$ and $(\alpha^{-1})^{p^n} = \alpha^{-1}$. Moreover, $(\alpha + \beta)^{p^n} = \alpha + \beta$. So the set of p^n disctinct roots of $x^{p^n} - x$ is closed under addition, multiplication, and inverses in its splitting field. Let F be that splitting field. Notice that $F \subseteq E$, moreover, $[F : \mathbb{F}_p] = n$ so that $|F| = p^n$. We also have that $\mathcal{U}(F)$ is a cyclic group of order $p^n - 1$, so that $E \subseteq F$, since $\alpha^{p^n - 1} = 1$. Therefore E is the splitting field of $x^{p^n} - x$ over $\mathbb{F}_p$, and so contains all the roots of $x^{p^n} - x$. Hence finite fields of order p^n exist and are unique up to isomorphism.

Lemma 1.5.8. *Let f an irreducible polynomial over a field F of $\operatorname{char} F = p$. Then there exists a unique integer $k \geq 0$ and a unique seperable polynomial s such that $f(x) = s(x^{p^k})$.*

Proof. We have that since $\operatorname{char} F = p$, there exists a polynomial f_1 over F for which $f(x) = f_1(x^p)$. Now, if f_1 is seperable, take $k = 1$ and we are done. Otherwise, there is a polynomial f_2 over F for which $f_2(x) = f_2(x^p)$, so that $f(x) = f_1(x^p) = f_2(x^{p^2})$. Then proceeding in this fashion, we obtain a seperable polynomial s for which $f(x) = s(x^{p^k})$ where $k \geq 0$. ■

Definition. Let f an irreducible polynomial over a field of characteristic p, a prime. Let f_s the polynomial for which $f(x) = f_s(x^{p^k})$ for some unique integer $k \geq 0$. Then we call the degree of f_s the **seperable degree** of f and write $\deg_s f = \deg f_s$. We call the integer p^k the **inseperable degree** and write $\deg_i f = p^k$. We call f_s the **seperable part** of f.

Lemma 1.5.9. *A polynomial f is seperable if, and only if $\deg_i f = 1$ and $\deg_s f = \deg f$. Moreover,*

$$\deg f = \deg_s f \cdot \deg_i f$$

Example 1.17. (1) $x^p - t$ over $\mathbb{F}_p(t)$ is irreducible with derivative $D = 0$. Hence $x^p - t$ is inseperable. We call $x^p - t$ a **purely inseperable polynomial**. Notice that $x^p - t$ has seperable part $(x - t)$.

(2) $x^{p^n} - t$ over $\mathbb{F}_p(t)$ is irreducible with seperable part $(x - t)$, and $\deg_i = p^n$.

(3) Let $f(x) = (x^{p^n} - t)(x^p - t)$ over $\mathbb{F}_p(t)$. Then p has two inseperable irreducible factors, and so is inseperable.

Definition. If K is an extension over a field F, we call an element $\alpha \in K$ **seperable** if its minimal polynomial is seperable, otherwise we call it **inseperable**. We call K/F **seperable** if every $\alpha \in K$ is seperable; otherwise we say that K/F is inseperable.

Lemma 1.5.10. *Every fnite extension of a perfect field is seperable.*

Corollary. *Finite extension fields of $\mathbb{Q}$ and $\mathbb{F}_p$ are seperable.*

1.6 Cyclotomic Polynomials.

Definition. We define **Euler's totient** to be the map $\phi : \mathbb{Z} \to \mathbb{Z}$ defined by the rule $\phi(n) = |\{a \in \mathbb{Z} : (a, n) = 1\}|$. That is, ϕ of n is the number of all integers less than n, coprime to n.

Definition. We define Ξ_n to be the **group of all primitive n-th roots of unity**, ξ for which $\xi^n = 1$.

Lemma 1.6.1. $\Xi_n \simeq \mathbb{Z}/n\mathbb{Z}$.

Proof. The map $a \to \xi^a$ defines the required isomorphism. ■

Corollary. $\operatorname{ord} \Xi_n = \phi(n)$ *where ϕ is Euler's totient.*

Proof. Since $\xi^n \equiv \xi^{0 \mod n} \equiv 1$, we have every non identity power of ξ has exponenct coprime to n. That is there are $\phi(n)$ such distinct powers ofξ. ■

Corollary. *If $d|n$, then $\Xi_d \leq \Xi_n$.*

Proof. Notice that if $d|n$, then $d = mn$ for some $m \in \mathbb{Z}^+$. Then $\xi^d = 1$ implies $(\xi^d) = \xi^{dm} = \xi^n = 1$. ■

Definition. We define the **n-th cyclotomic polynomial** to be the polynomial

$$\Phi_n(x) = \prod x - \xi$$

having as roots all n-primitive roots of unity.

Lemma 1.6.2. *The n-th cyclotomic polynomial Φ_n has degree* $\deg \Phi_n = \phi(n)$, *where ϕ is Euler's totient.*

Proof. Recall that $\operatorname{ord} \Xi_n = \phi(n)$, and since the elements of Ξ_n are the roots of Φ_n, there are $\phi(n)$ such roots. This puts $\deg \Phi_n = \phi(n)$. ■

Example 1.18 (Computing Cyclotomic Polynomials)**.** Recall that the polynomial $x^n - 1$ has as roots precisely all n-th roots of unity ξ, that is $\xi^n = 1$. If $x^n - 1 \in F[x]$, F a field, the the splitting field of $x^n - 1$ is $F(\xi)$. Then we have

$$x^n - 1 = \prod_{\xi \in \Xi_n} (x - \xi)$$

Now, grouping those factors where $\xi^d = 1$ for some $d|n$, then we have

$$x^n - 1 = \prod_{\xi \in \Xi_d} (x - \xi) \prod_{\xi \in \Xi_n} (x - \xi) = \prod d|n \prod_{\xi \in \Xi_n} (x - \xi) = \prod_{d|n} \Phi_n(x)$$

that is,

$$x^n - 1 = \prod_{d|n} \Phi_d(x)$$

which gives a method for computing Φ_n recursively.

We have $\Phi_1(x) = x - 1$ and $\Phi_2(x) = x + 1$. Now, $\Phi_3(x) = \Phi_1(x)\Phi_3(x) = (x - 1)\Phi_3(x)$, so that

$$Phi_n(x) = x^2 + x + 1$$

We have $\Phi_4(x) = \Phi_1(x)\Phi_2(x)\Phi_4(x) = (x - 1)(x + 1)\Phi_4(x) = (x^2 - 1)\Phi_4(x)$. So

$$\Phi_4(x) = x^2 + 1$$

Similarly,

$$\begin{aligned}
\Phi_5(x) &= x^4 + x^3 + x + 1 \\
\Phi_6(x) &= x^2 - x + 1 \\
\Phi_7(x) &= x^6 + x^5 + x^4 + x^3 + x^2 + x + 1 \\
\Phi_8(x) &= x^4 + 1 \\
\Phi_9(x) &= x^6 + x^x + 1 \\
\Phi_{10}(x) &= x^4 - x^3 + x^2 - x + 1 \\
\Phi_{11}(x) &= x^{10} + x^9 + x^8 + x^7 + x^6 + x^5 + x^4 + x^3 + x^2 + x + 1 \\
\Phi_{12}(x) &= x^4 - x^2 + 1
\end{aligned}$$

Also observe that if p is prime, then

$$\Phi_p(x) = \sum_{i=0}^{p-1} x^i = x^{p-1} + x^p + \cdots + x + 1$$

Lemma 1.6.3. *$\Phi_n(x)$ is monic over $\mathbb{Z}$.*

Proof. Notice that since $x^n - 1 = \prod \Phi_d(x)$, is monic, then each Φ_d must also be monic for all $d|n$.

Now, by induction on n, for $n = 1$, it is clear that $x - 1$ has coefficiencts in $\mathbb{Z}$ (if $x^n - 1 \in \mathbb{Z}[x]$ we are done, if not, just take $1_F \to 1_{\mathbb{Z}}$, whre F is the underlying field of $x^n - 1$). Now, suppose that $\Phi_d(x) \in \mathbb{Z}[x]$ for all $1 \leq d < n$, and $d|n$. Then $x^n - 1 = f(x)\Phi_n(x)$, where $f(x) = \prod \Phi_d(x)$ is monic over $\mathbb{Z}$. Moreover, $f|x^n - 1$, in the splitting field $\mathbb{Q}(\xi)$ (since we take $1_F \to F_{\mathbb{Z}}$), where $\xi^n = 1$. Then $f|x^n - 1$ over $\mathbb{Q}$ by the division theorem, and by Gauss' lemma, $f|x^n - 1$ in $\mathbb{Z}$. So $\Phi_n \in \mathbb{Z}[x]$. ■

Theorem 1.6.4. *Φ_n is irreducible over $\mathbb{Z}$.*

Proof. Again, if $x^n - 1 \in F[x]$ for some field F, take $1_F \to 1_{\mathbb{Z}}$ so that $x^n - 1 \in \mathbb{Z}[x]$. Suppose then that $\Phi_n(x) = f(x)g(x)$ where f and g are monic, and f is irreducible. Let $\xi^n = 1$, a primitive n-th root, so that ξ is a root of f. Then f is the minimal polynomial for ξ over $\mathbb{Q}$. Now, let p a prime such that $p \nmid n$. Then ξ^p is a n-th root, of f or g. If $f(\xi^p) = 0$, then for all a with $(a, n) = 1$, we have ξ^a is a root of f. Moreover, $a = p_1 \dots p_k$ where each $p_i \nmid n$ is prime. That means tht ξ^{p_1}, $(\xi^{p_1})^{p_2}, \dots, \xi^n$ are all roots of f making $f = \Phi_n$ and we are done.

Suppose then that $g(\xi^p) = 0$. Then ξ is root of $g(x^p)$, and since f is minimal, $f|g(x^p)$ in $\mathbb{Z}[x]$. Then we have $g(x^p) = f(x)h(x)$ for $f, h \in \mathbb{Z}[x]$. reducing mod p, we get $g(x^p) \equiv f(x)h(x) \mod p$ in $\mathbb{F}_p[x]$; but $g(x^p) \equiv (g(x))^p \mod p$. Since $\mathbb{F}_p[x]$ is a unique factorization domain, we get that $f \mod p$ and $g \mod p$ have a common factor. Then $\Phi_n(x) \equiv f(x)g(x) \mod p$ has a multiple root in $\mathbb{F}_p[x]$; implying that $x^n - 1$ has a multiple root, which is impossible; since $x^n - 1$ has n distinct roots. Therefore ξ^p is a root of f. ■

Corollary. *$[\mathbb{Q}(\xi) : \mathbb{Q}] = \phi(n)$.*

Proof. We have by above that Φ_n is the minimal polynomial for ξ over $\mathbb{Q}$. ■

Example 1.19. Let $\xi^8 = 1$ an 8-th root of unity. Then $[\mathbb{Q}(\xi) : \mathbb{Q}] = 4$ and $\mathbb{Q}(\xi)$ has minimal polynomial $\Phi_8(x) = x^4 + 1$. Moreover, $\mathbb{Q}(\xi)$ contains a primitive 4-th root of unity $i^4 = 1$ (over $\mathbb{C}$, $i^2 = -1$). So that $\mathbb{Q}(i) \subseteq \mathbb{Q}(\xi)$. We als get that $\xi + \xi^7 = \sqrt{2}$ (since $\xi = e^{\frac{2i\pi}{8}}$ over $\mathbb{C}$), and $\mathbb{Q}(\xi) = \mathbb{Q}(i, \sqrt{2})$.

Chapter 2

Galois Theory

2.1 Definitions and Examples.

Definition. An isomorphism of a field K onto itself is called an **automorphism**. We denote the set of all automorphisms of K $\operatorname{Aut} K$, and for $\sigma \in \operatorname{Aut} K$, we write $\sigma\alpha$ to mean $\sigma(\alpha)$. We say an automorphism σ of K **fixes** an element $\alpha \in K$ if $\sigma\alpha = \alpha$. We say σ **fixes** a subset $F \subseteq K$ if $\sigma\alpha = \alpha$ for all $\alpha \in F$. We denote $\operatorname{Aut} {}^{K}\!/_{F}$ to be the set of all automorphisms of K that fix F, where ${}^{K}\!/_{F}$ is a field extension.

Lemma 2.1.1. *Let K be a field. Then* $\operatorname{Aut} K$ *is a group. Moreover, if K is an extension of a field F, then* $\operatorname{Aut} {}^{K}\!/_{F} \leq \operatorname{Aut} K$.

Lemma 2.1.2. *Let K be an extension of F, and let $\alpha \in K$ algebraic over F. Then for every $\sigma \in \operatorname{Aut} {}^{K}\!/_{F}$, $\sigma\alpha$ is a root of the minimal polynomial of α over F; that is,* $\operatorname{Aut} {}^{K}\!/_{F}$ *permutes the roots of irreducible polynomials.*

Proof. Suppose that $a_0 + a_1\alpha + \cdots + a_{n-1}\alpha^{n-1} + \alpha^n = 0$, where $a_i \in F$ for all $1 \leq i \leq n$, and $a_n = 1$. Notice that if σ is an automorphism of K, then it is a homomorphism, moreover, since σ fixes F, and $a_i \in F$, we get $\sigma(a_i\alpha^i) = a_i\sigma\alpha^i$. Therefore,

$$\begin{aligned}
\sigma(a_0 + a_1\alpha + \cdots + a_{n-1}\alpha^{n-1} + \alpha^n) &= \sigma 0 = 0 \\
\sigma(a_0) + \sigma(a_1\alpha) + \cdots + \sigma(a_{n-1}\alpha^{n-1}) + \sigma(\alpha^n) &= 0 \\
a_0 + a_1\sigma\alpha + \cdots + a_{n-1}\sigma\alpha^{n-1} + \sigma\alpha^n &= 0
\end{aligned}$$

which makes $\sigma\alpha$ a root. ■

Example 2.1. (1) The identity map is an automorphism called the **trivial automorphsim**, and just maps elements of a field onto themselves. We denote this automorphism by ι. Notice additionally, that if σ is an automorphism of a field, then $\sigma : 1 \to 1$ and $\sigma : 0 \to 0$, so that $\sigma a = a$ for any element a in the prime subfield. That is, the automorphism group of a field fixes its prime subfield. In partucular, notice that $\mathbb{Q}$ and $\mathbb{F}_p$ have only the trivial automorphism, so that $\operatorname{Aut} \mathbb{Q} = \langle \iota \rangle$ and $\operatorname{Aut} \mathbb{F}_p = \langle \iota \rangle$.

(2) If $\tau \in \operatorname{Aut}\mathbb{Q}(\sqrt{2}) = \operatorname{Aut} {}^{\mathbb{Q}(\sqrt{2})}\!/\!_{\mathbb{Q}}$, then $\tau\sqrt{2} = \pm\sqrt{2}$. Then τ fixes $\mathbb{Q}$, and we have that it sends elements $\tau : a + b\sqrt{2} \to a \pm b\sqrt{2}$. In the case of addition, we have that $\tau = \iota$ the identity. The latter case of subtraction gives $\tau = a + b\sqrt{2} \to a - b\sqrt{2}$, so that $\operatorname{Aut}\mathbb{Q}(\sqrt{2}) = \langle\tau\rangle$ a cyclic group of order 2 generated by τ.

Lemma 2.1.3. *Let $H \leq \operatorname{Aut} K$ for some field K. Then the collection F of elements of K fixed by H is a subfield of K.*

Proof. LEt $h \in H$, and $a, b \in F$. Then $ha = a$, $hb = b$, so that $h(a \pm b) = a \pm b$, and $h(ab) = ab$, and $h(a^{-1}) = (ha)^{-1} = a^{-1}$. ■

Definition. Let K be a field. If $H \leq \operatorname{Aut} K$, we define the **fixed field** of H to be the subfield of K fixed by H, and we denote it $\mathcal{F}(H)$.

Lemma 2.1.4. *If $F_1 \subseteq F_2 \subseteq K$ are subfields of a field K, then $\operatorname{Aut} {}^{K}\!/\!_{F_2} \subseteq \operatorname{Aut} {}^{K}\!/\!_{F_1}$. Moreover, if $H_1 \leq H_2 \leq \operatorname{Aut} K$, then $\mathcal{F}(H_2) \subseteq \mathcal{F}(H_1)$.*

Example 2.2. (1) The fixed field of $\operatorname{Aut}\mathbb{Q}(\sqrt{2})$ is the field

$$F = \{a + b\sqrt{2} \in \mathbb{Q}(\sqrt{2}) : \sigma(a + b\sqrt{2}) = a + b\sqrt{2}\}$$

by definition. Then $a - b\sqrt{2} = a + b\sqrt{2}$ so that $b = 0$. Therefore $F = \mathbb{Q}$ and $\mathbb{Q}$ is the fixed field.

(2) Consider $\operatorname{Aut}\mathbb{Q}(\sqrt[3]{2}) = \langle\iota\rangle$. Then the fixed field of $\operatorname{Aut} {}^{\mathbb{Q}(\sqrt[3]{2})}\!/\!_{\mathbb{Q}}$ is $\mathbb{Q}(\sqrt[3]{2})$.

Lemma 2.1.5. *Let E be the splitting field over a field F of a polynomial $f(x)$ over F. Then*

$$\operatorname{ord}\operatorname{Aut} {}^{E}\!/\!_{F} \leq [E : F]$$

Proof. By induction on $[E : F]$. If $[E : F] = 1$, then $E = F$, and we are done. Now, for $[E : F] \geq 1$, $f(x)$ has atleast one irreducible factor $p(x)$ of degree $\deg p > 1$. Now, let F' be the corresponding field to F with splitting field E', corresponding to E. Let $f'(x)$ be the polynomial over F' the polynomial corresponding to f over F, with irreducible factor $p'(x)$ corresponding to the irreducible factor p. Now, let α be a root of p. IF σ is an extension of an isomorphism ϕ to E, then the restriction $\tau = \sigma|_{F(\alpha)}$ is an isomorphism of $F(\alpha)$ onto a subfield of E'. Since α generates $F(\alpha)$, τ is completely determined by its action on α; i.e. $\tau\alpha$, so that $\tau\alpha$ is a root of p'. We then get the following diagram:

$$\begin{tikzcd}
E \arrow[r, "\sigma"] \arrow[d, dash] & E' \arrow[d, dash] \\
F(\alpha) \arrow[r, "\tau"] \arrow[d, dash] & F'(\tau\alpha) \arrow[d, dash] \\
F \arrow[r, "\phi"] & F'
\end{tikzcd}$$

COnversly, let β be a root of p'. Then there exist extensions τ and σ of the isomorphism ϕ giving the above diagram (replace $\tau\alpha$ with β). Now, the number of extensions of ϕ to τ is equal to the number of distinct roots of p'. Since $\deg p = \deg p' = [F(\alpha) : F]$, the number of extensions to τ is at most $[F(\alpha) : F]$.

Now, notice that $[E : F(\alpha)] < [E : F]$. Therefore, by the induction hypothesis, the number of extensions of τ to σ is at most $[E : F(\alpha)]$. Therefore, the number of extensions of ϕ to σ is at most $[E : F(\alpha)][F(\alpha) : F] = [E : F]$.

Finally, if $F = F'$, we have $f = f'$ (and $p = p'$), and so ϕ is the identity map and $E = E'$. THis makes σ an automorphism of E which fixes F. The proof is complete. ■

Corollary. *If K is the splitting field of a seperable polynomial $f(x)$ over a field F, then* $\operatorname{ord} \operatorname{Aut} {}^{K}\!/_{F} = [K : F]$.

Definition. We call a finite field extension ${}^{K}\!/_{F}$ a **Galois extension** if $\operatorname{ord} \operatorname{Aut} {}^{K}\!/_{F} = [K : F]$. We call $\operatorname{Aut} {}^{K}\!/_{F}$ the **Galois group** of ${}^{K}\!/_{F}$, and write $\operatorname{Gal} {}^{K}\!/_{F}$.

Lemma 2.1.6. *An extension K over a field F is Galois over if and only if it is normal and seperable.*

Proof. If K is Galois over F, the result follws by definition. Now, let K be normal and seperable. Let $\alpha \in K$. Then the minimal polynomial m of α over F is seperable. Moreover, α is a root of K, and since K is normal, m splits completely over K. Thus K contains the splitting field of m, but since m is minimal and irreducible, that makes K the splitting field of some polynomial f over F, having m as a factor. By the above corollary, this makes K Galois over F. ■

Example 2.3. (1) ${}^{\mathbb{Q}(\sqrt{2})}\!/_{\mathbb{Q}}$ is Galois, and $\operatorname{Gal} {}^{\mathbb{Q}(\sqrt{2})}\!/_{\mathbb{Q}} = \langle \sigma \rangle \simeq {}^{\mathbb{Z}}\!/_{2\mathbb{Z}}$, where $\sigma : a + b\sqrt{2} \to a - b\sqrt{2}$.

(2) Any quadratic extension field K over F is Galois over F, provided $\operatorname{char} F \neq 2$. Then any quadratic extension K of F, of degree $[K : F] = 2$ is of the form $F(\sqrt{D})$, where $D \in \mathbb{Z}^+$ is squarefree. Hence $K = F(\sqrt{D})$ is the splitting field of the polynomial $x^2 - D$.

(3) $\mathbb{Q}(\sqrt[3]{2})$ is not Galois over $\mathbb{Q}$, since $\operatorname{ord} \operatorname{Aut} {}^{\mathbb{Q}(\sqrt[3]{2})}\!/_{\mathbb{Q}} = 1$, but $[\mathbb{Q}(\sqrt[3]{2}) : \mathbb{Q}] = 3$.

(4) $\mathbb{Q}(\sqrt{2}, \sqrt{3})$ is the splitting field of the seperable polynomial $(x^2 - 2)(x^2 - 3)$ over $\mathbb{Q}$. Hence $\mathbb{Q}(\sqrt{2}, \sqrt{3})$ is Galois over $\mathbb{Q}$, and has Galois group $\operatorname{Gal} {}^{\mathbb{Q}(\sqrt{2}, \sqrt{3})}\!/_{\mathbb{Q}}$ of order 6. Moreover, since the automorphisms of this group are completely determined by the roots $\sqrt{2}$ and $\sqrt{3}$, we get the possible automorphisms are given by the maps

$$\begin{array}{ll} \sqrt{2} \to \sqrt{2} & \sqrt{3} \to -\sqrt{3} \\ \sqrt{2} \to -\sqrt{2} & \sqrt{3} \to \sqrt{3} \\ \sqrt{2} \to \sqrt{2} & \sqrt{3} \to -\sqrt{3} \\ \sqrt{2} \to -\sqrt{2} & \sqrt{3} \to -\sqrt{3} \end{array}$$

Now, let $\sigma : \sqrt{2} \to -\sqrt{2}, \sqrt{3} \to \sqrt{3}$ and $\tau : \sqrt{2} \to \sqrt{2}, \sqrt{3} \to -\sqrt{3}$. Then $\sigma\tau : \sqrt{2} \to -\sqrt{2}, \sqrt{3} \to -\sqrt{3}$. Therefore we have

$$\operatorname{Gal} {}^{\mathbb{Q}(\sqrt{2}, \sqrt{3})}\!/_{\mathbb{Q}} = \langle \sigma, \tau \rangle \simeq V_4$$

where V_4 is the Klein 4-group.

We can also determine the fixed fields correspongidng to each subgroup of $\langle \sigma\tau \rangle$. That is, $\mathcal{F}(\langle \sigma\tau \rangle)$ is the set of all elements fixed by $\sigma\tau$ and has elements of the form $a+b\sqrt{6}$. So $\mathcal{F}(\langle \sigma\tau \rangle) = \mathbb{Q}(\sqrt{6})$. The table below lists the fixed fields of the Galois group considered.

subgroup	fixed field
$\langle \iota \rangle$	$\mathbb{Q}(\sqrt{2}, \sqrt{3})$
$\langle \sigma \rangle$	$\mathbb{Q}(\sqrt{3})$
$\langle \sigma\tau \rangle$	$\mathbb{Q}(\sqrt{6})$
$\langle \tau \rangle$	$\mathbb{Q}(\sqrt{2})$
$\langle \sigma, \tau \rangle$	$\mathbb{Q}$

(5) The roots of $x^3 - 2$ over $\mathbb{Q}$ are given by

$$\sqrt[3]{2} \qquad \xi\sqrt[3]{2} \qquad \xi^2\sqrt[3]{2}$$

where $\xi^3 = 1$ is the 3-rd root of unity. Additionally, the splitting field of $x^3 - 2$ over $\mathbb{Q}$ is $\mathbb{Q}(\sqrt[3]{2}, \xi\sqrt[3]{2})$ of degree 6. Now, $x^3 - 2$ is irreducible over $\mathbb{Q}$, and hence seperable over $\mathbb{Q}$. This makes $\mathbb{Q}(\sqrt[3]{2}, \xi\sqrt[3]{2})$ Galois over $\mathbb{Q}$, of order 6.

Consider now the set of generators $\sqrt[3]{2}$ and ξ. Then an automorphism σ takes $\sqrt[3]{2} \to \sqrt[3]{2}, \xi\sqrt[3]{2}$, or $\xi^2\sqrt[3]{2}$, and takes $\xi \to \xi$ or ξ^2. Since these are the roots of the cyclotomic polynomial $\Phi_3(x) = x^2 + x + 12 + x + 1$, σ is completely determined by the actions on $\sqrt[3]{2}$ and ξ. Hence there are 6 possible automorphisms.

Let

$$\begin{aligned} \sigma : \sqrt[3]{2} \to \xi\sqrt[3]{2} \qquad & \xi \to \xi \\ \tau : \sqrt[3]{2} \to \sqrt[3]{2} \qquad & \xi \to \xi^2 \end{aligned}$$

We obtain then the elements

$$\iota \qquad \sigma^2 \qquad \tau\sigma^2 = \sigma\tau$$

and we get the additional relations

$$\sigma^2 = \tau^2 = \iota$$

so that

$$\mathrm{Gal}\, {}^{\mathbb{Q}(\sqrt[3]{2}, \xi\sqrt[3]{2})}\!\big/_{\mathbb{Q}} = \langle \sigma, \tau \rangle \simeq S_3$$

The fixed field of $\langle \sigma^2 \rangle$ is $\mathbb{Q}(\xi)$.

(6) $\mathbb{Q}(\sqrt[4]{2})$ is not Galois over $\mathbb{Q}$. We have $[\mathbb{Q}(\sqrt[4]{2}) : \mathbb{Q}] = 4$ but that any automorphism takes $\sqrt[4]{2}$ onto $\pm\sqrt[4]{2}$, or $\pm i\sqrt[4]{2}$. But $\pm i\sqrt[4]{2} \notin \mathbb{Q}(\sqrt[4]{2})$. Notice however that $\mathbb{Q}(\sqrt[4]{2})$ is Galois over $\mathbb{Q}(\sqrt{2})$, and $\mathbb{Q}(\sqrt{2})$ is Galois over $\mathbb{Q}$.

(7) The extension field $\mathbb{F}_{p^n}$ is Galois over $\mathbb{F}_p$. Recall that $\mathbb{F}_{p^n}$ is the splitting field of the seperable polynomial $x^{p^n} - x$ over $\mathbb{F}_p$. Then $\operatorname{ord} \operatorname{Gal} {}^{\mathbb{F}_{p^n}}\!\diagup_{\mathbb{F}_p} = n$ and the Frobenius automorphism given by

$$\sigma : \alpha \to \alpha^p$$

generates the Galois group, making it $\langle \sigma \rangle$, a cyclic group of order n.

(8) The extension $\mathbb{F}_2(x)$ is not Galois over $\mathbb{F}_2(t)$, since $x^2 - t$ is not seperable. Moreover, any automorphism of $\operatorname{Aut} {}^{\mathbb{F}_2(x)}\!\diagup_{\mathbb{F}_2(t)}$ sends x to the only root of $x^2 - t$, making it the trivial group.

2.2 The Fundamental Theorem of Galois Theory.

Definition. A **linear character** of a group G with values in a field L is a homomorphism $\chi : G \to \mathcal{U}(L)$. We say that distinct linear characters $\chi_1, \ldots, \chi_n$ of G are **linearly independent** over L if they are linearly independent, as functions, over G.

Theorem 2.2.1. *If $\chi_1, \ldots, \chi_n$ are distinct linear characters of a group G with values in a field L, then they are linearly independent over L.*

Proof. Suppose that $\chi_1, \ldots, \chi_n$ are linearly dependent, and choose a dependence relation with minimum of m nonzero coefficients $a_1, \ldots, a_m \in L$, so that

$$a_1\chi_1 + \cdots + a_n\chi_m = 0$$

Then for any $g \in G$, we have

$$a_1\chi_1(g) + \cdots + a_n\chi_m(g) = 0$$

Now, let $g_0 \in G$, with $\chi_1(g_0) \neq \chi_m(g_0)$. Then

$$a_1\chi_1(g_0g) + \cdots + a_n\chi_n(g_0g) = a_1\chi(g_0)\chi(g) + \cdots + a_m\chi_m(g_0)\chi_m(g) = 0$$

multiplying the preceeding equation with the above by $\chi_m(g_0)$ and subtracting from the above equation, we get

$$a_1(\chi_1(g_0) - \chi_m(g_0))\chi_1(g) + \cdots + a_m(\chi_1(g_0) - \chi_m(g_0))\chi_m(g) = 0$$

which gives a linear dependence relation with fewer than m nonzero coefficients; which contradicts our choice of m. Therefore $\chi_1, \ldots, \chi_n$ must be linearly independent. ∎

Corollary. *If $\sigma_1, \ldots, \sigma_n$ are distinct embeddings of a field K into a field L, then they are linearlyy independent as functions.*

Theorem 2.2.2. *Let $G = \{\sigma_1, \ldots, s_n\}$ where $\sigma_1 = \iota$ a subgroup of automorphisms of a field K, and let F be the corresponding fixed field. THen*

$$[K : F] = \operatorname{ord} G = n$$

Proof. Suppose that $n > [K : F]$, and consider the basis $\{\omega_1, \ldots, \omega_m\}$ of ${}^{K}\!/_{F}$ as a vector space so that $[K : F] = m$. Then the matrix equation

$$\begin{pmatrix} \sigma_1\omega_1 & \ldots & \sigma_n\omega_m \\ \vdots & \ddots & \vdots \\ \sigma_n\omega_1 & \ldots & \sigma_n\omega_m \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_m \end{pmatrix} = 0 \tag{2.1}$$

has nontrivial solution $\begin{pmatrix} \beta_1 \\ \vdots \\ \beta_n \end{pmatrix}$. Let $a_1, \ldots, a_m \in F$, so that $\sigma_i\alpha_j = \alpha_j$ for each $1 \leq i \leq n$ and $1 \leq j \leq m$. Multyplying by $\begin{pmatrix} \sigma_1 a_1 \\ \vdots \\ \sigma_m a_1 \end{pmatrix}$, we obtain

$$\begin{pmatrix} a_1\sigma_1\omega_1 & \ldots & a_1\sigma_1\omega_1 \\ \vdots & \ddots & \vdots \\ a_m\sigma_m\omega_1 & \ldots & a_m\sigma_n\omega_m \end{pmatrix} \begin{pmatrix} \beta_1 \\ \vdots \\ b_n \end{pmatrix} = 0$$

so that we can obtain the equation

$$\sigma_1(a_1\omega_1 + \cdots + a_m\omega_m)\beta_1 + \cdots + \sigma_n(a_1\omega_1 + \cdots + a_m\omega_m)\beta_n = 0$$

Where $\beta_1, \ldots, \beta_n$ are not all 0. Now, since $\{\omega_1, \ldots, \omega_m\}$ is an F-basis for K, for all $\alpha \in K$, we get that $\alpha = a_1\omega_1 + \cdots + a_m\omega_m$. So we have from the above equation

$$(\sigma_1\alpha)\beta_1 + \ldots (\sigma_n\alpha)\beta_n = 0$$

so that $\{\sigma_1, \ldots, \sigma_n\}$ are linearly dependent over K; which contradicts the above corollary. No $n \leq [K : F]$.

Now, suppose that $n < [K : F]$, and thte tere are more than n F-linearly independent elements $\alpha_1, \ldots, \alpha_{n+1} \in K$. Then

$$\begin{pmatrix} \sigma_1\alpha_1 & \ldots & \sigma_1\alpha_{n+1} \\ \vdots & \ddots & \vdots \\ \sigma_n\alpha_1 & \ldots & \sigma_n\alpha_{n+1} \end{pmatrix} \begin{pmatrix} x_1 \\ \ldots \\ x_{n+1} \end{pmatrix} = 0$$

has nontrivial solution with entries $\beta_1, \ldots, \beta_{n+1} \in K$. Now, if $\beta_i \in F$ for all $1 \leq i \leq n+1$, we get an immediate contradiction of the linear independence of $\{\alpha_1, \ldots, \alpha_{n+1}\}$ over F. So at least one $\beta_i \notin F$.

Now, choose a nontrivial solution with minimum of r nonzero entries β_i. Suppose also that $\beta_r = 1$, then at least one $\beta_i \notin F$, for $1 \leq i \leq r-1$, and so $r > 1$. SUppose then that $\beta_1 \notin F$. Then the matrix equation

$$\begin{pmatrix} \sigma_1\alpha_1 & \ldots & \sigma_1\alpha_{r-1} & \sigma_1\alpha_r \\ \vdots & \ddots & \vdots & \\ \sigma_n\alpha_n & \ldots & \sigma_n\alpha_{r-1} & \sigma_n\alpha_r \end{pmatrix} \begin{pmatrix} \beta_1 \\ \vdots \\ \beta_{r-1} \\ 1 \end{pmatrix} = 0$$

Now, since $\beta_1 \notin F$, there exists an automorphsim σ_{k_0} of K with $\sigma_{k_0}\beta_1 \neq \beta_1$ for $1 \leq k_0 \leq n$. Applying σ_{k_0} to each row of the above equation yields a row of the form

$$\sigma_{k_0}(\sigma_j\alpha_1)(\sigma_{k_0}\beta_1) + \cdots + \sigma_{k_0}(\sigma_j\alpha_{r-1})(\sigma_{k_0}\beta_{r-1}) + \sigma_j\alpha_r = 0$$

However, since G is a group, $\sigma_{k_0}\sigma_j = \sigma_i$ for $1 \leq i, j \leq n$, so we get

$$(\sigma_i\alpha_1)(\sigma_{k_0}\beta_1) + \cdots + (\sigma_i\alpha_{r-1})(\sigma_{k_0}\beta_{r-1}) + \sigma_i\alpha_r = 0$$

Subtracting this equation from the one preceeding it, we obtain

$$(\sigma_i\alpha_1)(\beta_1 - \sigma_{k_0}\beta_1) + \cdots + (\sigma_i\alpha_{r-1})(\beta_{r-1} - \sigma_{k_0}\beta_{r-1}) = 0$$

with $x_1 = \beta_1 - \sigma_{k_0}\beta_1 \neq 0$. This choice of k_0 gives fewer than r nonzero coefficients of a nontrivial solutions, which contradicts the choice of r. Therefore $n = [K : F]$. ■

Corollary. *If K is a finite extension over a field F, then*

$$\operatorname{ord}\operatorname{Aut} {}^{K}\!/_{F} \leq [K : F]$$

with equality holding if, and only if F is the fixed field of $\operatorname{Aut} {}^{K}\!/_{F}$.

Proof. Let F_1 be the fixed field of $\operatorname{Aut} {}^{K}\!/_{F}$, so that $F \subseteq F_1 \subseteq K$. TYhen $[K : F_1] = \operatorname{ord}\operatorname{Aut} {}^{K}\!/_{F}$, hence

$$[K : F] = (\operatorname{ord}\operatorname{Aut} {}^{K}\!/_{F})[F_1 : F]$$

■

Corollary. *If G is a finite subgroup of automorphisms of a field K, and F is its fixed field, then* $\operatorname{Aut} {}^{K}\!/_{F} = G$ *so that K is Galois over F with Galois group G.*

Proof. By definition, we have that since G fixes the elements of F, then $G \leq \operatorname{Aut} {}^{K}\!/_{F}$. THen $\operatorname{ord} G = [K : F]$ and by the above corollary, we get

$$\operatorname{ord}\operatorname{Aut} {}^{K}\!/_{F} \leq [K : F]$$

so that

$$[K : F] = \operatorname{ord} G \leq \operatorname{ord}\operatorname{Aut} {}^{K}\!/_{F} \leq [K : F]$$

and equality holds. ■

Corollary. *If G and H are distinct finite subgroups of* $\operatorname{Aut} K$, *then their fixed fields are also distinct.*

Proof. Let F_G the fixed field of G, and F_H the fixed field of H. I f $F_G = F_H$, then we have that H fixes F_G, and since any automorphism fixing F_G is in G, we have $H \leq G$. By similar reasoning, we get $G \leq H$ so that $G = H$. ■

Theorem 2.2.3. *The extension K over a field F is Galois if, and only if K is the splitting field of some seperable polynomial in F. Moreover, every irreducible polynomial over F having at least one root in K splits over K.*

Proof. By lemma 2.1.5, the splitting field of a seperable polynomial over a field is Galois.

Now, suppose that K is Galois over F, and let $p(x) \in F[x]$ an irreducible polynomial with a root $\alpha \in K$. Consider, for each $\sigma_i \in \operatorname{Gal} {}^{K}\!/_{F}$ the elements

$$\alpha \qquad \sigma_2\alpha \qquad \dots \qquad \sigma_n\alpha$$

where $\sigma_1 = \iota$, and let

$$\alpha \qquad \alpha_2 \qquad \dots \qquad \alpha_n$$

be the distinct elements taken on by these permutations (in no particular order). If $\tau \in \operatorname{Gal} {}^{K}\!/_{F}$, by the group law, we get $\tau\sigma_i = \sigma_j$ for all $1 \leq i, j \leq n$. APplying τ to α_i we het permutations of the elements $\alpha, \alpha_2, \dots, \alpha_n$. Then the polynomial $f(x) = (x - \alpha)(x - \alpha_2)\dots(x - \alpha_n)$ has coefficients fixed by the elements of $\operatorname{Gal} {}^{K}\!/_{F}$. That is, the coefficients lie in the fixed field F. Hence $f \in F[x]$.

Now, since p is irreducible with root α, it is the minimal polynomial for α over F, and hence $p|f$. Moreover, we can pbserve that $f|p$, so that $p(x) = f(x)$, which makes $p(x)$ seperable with all its roots in K.

Now, let $\{\omega_1, \dots, \omega_n\}$ be a basis for ${}^{K}\!/_{F}$ as a vector space, and let $p_i(x)$ the minimal polynomial for ω_i over F for all $1 \leq i \leq n$. Then p_i is seperable, with roots in K. Let $g(x) = p_1(x) \dots p_n(x)$ (where this product is squarefree). Then if E is the splitting field of g over F, then $\omega_i \in E$ for all $1 \leq i \leq n$, so that $K \subseteq E$. On the otherhand, since g splits over K, we get $E \subseteq K$, and so $K = E$ is the splitting field of g over F. ■

Definition. Let K be an extension of a field F. If $\alpha \in K$, and $\sigma \in \operatorname{Gal} {}^{K}\!/_{F}$, we call the permutations $\sigma\alpha$ **Galois conjugates** (or simply **conjugates**) of α over F. If E is a sbufield of K containing F, then we call σE the **conjugate field** of Eover F.

Theorem 2.2.4 (The Fundamental Theorem of Galois Theory)**.** *Let K be Galois over a field F with Galois group G, and let E be an intermidiate field of K over F which is fixed by some subgroup $H \leq G$. Then the following are true.*

(1) There is a 1–1 correspondence between the subgroups of G onto the fixed fields of ${}^{K}\!/_{F}$; that is, $\mathcal{F}$, treated as a map, is 1–1 and onto.

(2) If $\sigma \in G$, then σE is fixed by $\sigma H \sigma^{-1}$; that is, $\sigma E = \mathcal{F}(\sigma H \sigma^{-1})$.

(3) K is Galois over E, and E is normal over F if, and only if H is normal in G.

(4) If H is normal in G, then

$$\operatorname{Gal} {}^{E}\!/_{F} \simeq {}^{G}\!/_{H}$$

(5) Independntly of whether or not E is normal over F, we have that

$$[E : F] = [G : H]$$

Proof. Let $\mathcal{G}(E) = \operatorname{Aut} {}^{K}\!/_{E}$, that is, as a map, $\mathcal{G}$ sends a intermidiate field of ${}^{K}\!/_{F}$ to that group of E-automorphisms of K; i.e. all automorphisms that fix the elements of E. Now, consider the mapping

$$H \to \mathcal{F}(H) \to \mathcal{GF}(H)$$

and take $\sigma \in H$. Then, by definition, σ fixes the field $\mathcal{F}(H)$, so that $\sigma \in \mathcal{G}(\mathcal{F}(H)) = \mathcal{GF}(H)$. Then $H \leq \mathcal{GF}(H)$. Now, we have that the fixed field $\mathcal{F}(H)$ contains the fixed field of $\mathcal{GF}(H)$, i.e. $\mathcal{F}(\mathcal{GF}(H))$, which is H. That is, $\mathcal{GF}(H) \leq H$. Therefore, we have $\mathcal{GF}(H) = H$. Conversely, consider the mapping

$$E \to \mathcal{G}(E) \to \mathcal{FG}(E)$$

Observe that $\mathcal{F})$ is the fixed field of $\mathcal{G}(E) = \operatorname{Aut} {}^{K}\!/_{E}$, by definition, $\mathcal{FG}(K) = K$. This establishes the 1–1 correspondence of the subgroups of G onto the fixed fields of ${}^{K}\!/_{F}$.

Now, since $E = \mathcal{F}(H)$, observe that $\mathcal{F}(\sigma H s^{-1})$ consists of all elements of K which are fixed by $\sigma\tau\sigma^{-1}$ for all $\tau \in H$; that is, all $\alpha \in K$ for which $\sigma\tau\sigma^{-1}(\alpha) = \alpha$. Observe, then that $\tau\sigma^{-1}(\alpha) = \tau(\sigma^{-1}\alpha) = \sigma^{-1}\alpha$, so that τ fixes $\sigma^{-1}\alpha$. Then $\sigma^{-1}\alpha \in \mathcal{F}(H)$. That is, $\alpha \in \sigma\mathcal{F}(H) = \sigma E$, therefore $\mathcal{F}(\sigma H \sigma^{-1}) = \sigma E$.

For the third statement, notice that since K is Galois over F, then it is normal and seperable. This makes E normal and seperable over F, so that by lemma 2.1.6, ${}^{E}\!/_{F}$ is Galois. Now, let σ be a 1–1 F-homomorphism which is from $K \to E$. Then σ can be extended to a 1–1 F-homomorphism from $K \to K$, by lemma 1.2.5. If ${}^{E}\!/_{F}$ is normal, then for every $\sigma \in G$, σ fixes E, and E is fixed by a normal subgroup of G by the previous statement.

Consider now, the homomorphism from $G \to \operatorname{Gal} {}^{E}\!/_{F}$ given by $\sigma \to \sigma|_E$. This map is onto, with kernel consisting of all automorphisms which fix K. Then $\operatorname{Gal} {}^{E}\!/_{F} = H$, moreover, since ${}^{K}\!/_{F}$ is normal, we get $H \trianglelefteq G$. Therefore by the first isomorphism theorem (for groups), we get

$$\operatorname{Gal} {}^{E}\!/_{F} \simeq {}^{G}\!/_{H}$$

where ${}^{G}\!/_{H}$ is understood to be the quotient group. Finally, we also have that $[K : F] = [K : E][E : F]$. Since both ${}^{K}\!/_{F}$ and ${}^{E}\!/_{F}$ are Galois, this makes

$$\operatorname{ord} G = [E : F] \operatorname{ord} H$$

which makes $[E : F] = [G : H]$ by the definition of the index of a subgroup. ■

Example 2.4. (1) The lattices of $\mathbb{Q}(\sqrt{2}, \sqrt{3})$ and $\mathbb{Q}(\sqrt[3]{2}, \xi)$ (where $\xi^3 = 1$) indicate all of the subfields of these fields. We have that the lattice of $\mathbb{Q}(\sqrt{2}, \sqrt{3})$ is isomorphic to the lattice of the Klein 4-group V_4, which has all its subgroups normal. Thus we get every subfield of $\mathbb{Q}(\sqrt{2}, \sqrt{3})$ is Galoix over $\mathbb{Q}$.

On the other hand, the lattice for $\mathbb{Q}(\sqrt[3]{2}, \xi)$ is isomorphic to the lattice of S^3 where the only normal subgroup is the nontrivial subgroup of order 3; moreover, only $\mathbb{Q}(\xi)$ is Galois over $\mathbb{Q}$ with $\operatorname{Gal} {}^{\mathbb{Q}(\xi)}\!/_{\mathbb{Q}} \simeq {}^{S_3}\!/_{\langle\sigma\rangle}$, where $\langle\sigma\rangle$ is the cyclic subgroup of order 2.

$$
\begin{tikzcd}[column sep=small]
 & \mathbb{Q}(\sqrt{2},\sqrt{3}) \arrow[dl, dash] \arrow[d, dash] \arrow[dr, dash] & \\
\mathbb{Q}(\sqrt{2}) \arrow[dr, dash] & \mathbb{Q}(\sqrt{6}) \arrow[d, dash] & \mathbb{Q}(\sqrt{3}) \arrow[dl, dash] \\
 & \mathbb{Q} &
\end{tikzcd}
\qquad
\begin{tikzcd}[column sep=small]
 & \langle \sigma, \tau, \eta \rangle \arrow[dl, dash] \arrow[d, dash] \arrow[dr, dash] & \\
\langle \sigma \rangle \arrow[dr, dash] & \langle \tau \rangle \arrow[d, dash] & \langle \eta \rangle \arrow[dl, dash] \\
 & \langle \iota \rangle &
\end{tikzcd}
$$

Figure 2.1: The lattice of subfields of $\mathbb{Q}(\sqrt{2,\sqrt{3}})$ and the lattice of subgroups of $\operatorname{Gal} {}^{\mathbb{Q}(\sqrt{2,\sqrt{3}})}\!/_{\mathbb{Q}}$.

(2) Consider $\mathbb{Q}(\sqrt{2}+\sqrt{3}) \subseteq \mathbb{Q}(\sqrt{2},\sqrt{3})$. If $m(x)$ is the minimal polynomial of $\sqrt{2}+\sqrt{3}$, then observe that is has as roots the distinct conjugates

$$\pm\sqrt{2} \pm \sqrt{3}$$

so that

$$m(x) = (x+(\sqrt{2}+\sqrt{3}))(x-(\sqrt{2}+\sqrt{3}))(x+(\sqrt{2}-\sqrt{3}))(x-(\sqrt{2}-\sqrt{3})) = x^4+10x+1$$

Moreover, $x^4+10x+1$ is irreducible. Then only the automorphism ι of $\{\iota, \sigma, \tau, \sigma\tau\}$ fixes $\sqrt{2}+\sqrt{3}$ so that the fixing group of $\mathbb{Q}(\sqrt{2}+\sqrt{3})$ is preciesly that of $\mathbb{Q}(\sqrt{2},\sqrt{3})$. So $\mathbb{Q}(\sqrt{\ }+\sqrt{3}) = \mathbb{Q}(\sqrt{2},\sqrt{3})$.

(3) Consider the splitting field of x^8+1 over $\mathbb{Q}$, generated by the elements $\sqrt[8]{2}$ and ξ, where $\xi^8 = 1$ (i.e. a primitive 8-th root of unity). Let $\zeta = \sqrt[8]{2}$, and notice that $\zeta^4 = \sqrt{2}$, and the splitting field of x^8-2 over $\mathbb{Q}$ is $\mathbb{Q}(\sqrt[8]{2}, i)$ of degree $[\mathbb{Q}(\sqrt[8]{2}, i) : \mathbb{Q}] = 16 = 4^2$. Consider then all possible maps on ζ and i given by $\zeta \to \xi^a\zeta, i \to \pm i$. Define then the automorphisms

$$\sigma : \zeta \to \xi\zeta, i \to i \text{ and } \tau : \zeta \to \zeta, i \to -i$$

Since $\xi = \frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2} = \frac{1+i}{2}\zeta^4$, we compute that $\sigma : \xi \to \xi^5$, and $\tau : \xi \to \xi^7$. We can then compute the Galois group by noting that $\sigma^8 = \tau^2 = \iota$, and $\sigma\tau = \tau\sigma^3$, so that

$$\operatorname{Gal} {}^{\mathbb{Q}(\sqrt[8]{2}, i)}\!/_{\mathbb{Q}} = \langle \sigma, \tau : \sigma^8 = \tau^2 = \iota \text{ and } \sigma\tau = \tau\sigma^3 \rangle$$

which describes the quasidihedral group of order 16.

2.3 Finite Fields

We reiterate some previous results about finite fields.

Lemma 2.3.1. *Let E be a finite field over $\mathbb{F}_p$. Then E is an extension of finite degree $[E : \mathbb{F}_p] = n$. Moreover, if $|E| = p^n$, and E is the splitting field of the polynomial $x^{p^n} - x$ over $\mathbb{F}_p$.*

Proof. Suppose that E is a finite field, but that the extension ${}^{E}\!/_{\mathbb{F}_p}$ is infinite. The E, as a vector space over $\mathbb{F}_p$, has an infinite basis $\{\alpha_1, \alpha_2, \dots\}$. Moreover, since every element of E is a linear comination of this basis, we obtain a contradiction as there are infinite such combinations, but E is finite. Therefore $[E : \mathbb{F}_p] = n$, for some $n \in \mathbb{Z}^+$.

Let α, β be roots. Then $\alpha^{p^n} = \alpha$, and $\beta^{p^n} = \beta$. Then $(\alpha\beta)^{p^n} = \alpha\beta$ and $(\alpha^{-1})^{p^n} = \alpha^{-1}$. Moreover, $(\alpha + \beta)^{p^n} = \alpha + \beta$. So the set of p^n disctinct roots of $x^{p^n} - x$ is closed under addition, multiplication, and inverses in its splitting field. Let F be that splitting field. Notice that $F \subseteq E$, moreover, $[F : \mathbb{F}_p] = n$ so that $|F| = p^n$. We also have that $\mathcal{U}(F)$ is a cyclic group of order $p^n - 1$, so that $E \subseteq F$, since $\alpha^{p^n - 1} = 1$. Therefore E is the splitting field of $x^{p^n} - x$ over $\mathbb{F}_p$, and so contains all the roots of $x^{p^n} - x$. Notice that since E is a splitting field, it is unique up to isomorphism. ■

Remark. Since the splitting fields of $x^{p^n} - x$ over $\mathbb{F}_p$ are unique up to isomorphism, we denote them by $\mathbb{F}_p$ from now on.

Corollary. *$\mathbb{F}_{p^n}$ is Galois over $\mathbb{F}_p$ with Galois group isomorphic to ${}^{\mathbb{Z}}\!/_{n\mathbb{Z}}$.*

Proof. Notice that $\mathbb{F}_{p^n}$ is normal and seperable over $\mathbb{F}_p$. Morever, that the Frobenius automorphism generates the Galois group of order n. ■

Corollary. *All subfields of $\mathbb{F}_{p^n}$ are Galois over $\mathbb{F}_p$, and in 1–1 with the divisors of n. Moreover, they are of the form $\mathbb{F}_{p^d}$ for all $d|n$.*

Proof. We have that

$$\operatorname{Gal} {}^{\mathbb{F}_{p^n}}\!/_{\mathbb{F}_p} = \langle \sigma \rangle \simeq {}^{\mathbb{Z}}\!/_{n\mathbb{Z}}$$

where $\sigma : \alpha \to \alpha^p$ is the Frobenius automorphism. By the fundamental theorem of Galois theory, each subfield of $\mathbb{F}_{p^n}$ corresponds to a subgroup of ${}^{\mathbb{Z}}\!/_{n\mathbb{Z}}$, which are defined by the divisors of n. Hence, there is precisely one field $\mathbb{F}_{p^d}$ for each $d|n$, with $[\mathbb{F}_{p^d} : \mathbb{F}_p] = d$. Now, since ${}^{\mathbb{Z}}\!/_{n\mathbb{Z}}$ is Abelian, every subgroup is normal, and so each $\mathbb{F}_{p^d}$ is normal over $\mathbb{F}_p$. Since they are also seperable, the are Galois over $\mathbb{F}_p$. ■

Corollary. *The fields $\mathbb{F}_{p^d}$ are precisely those fixed by σ^d; that is, $\mathcal{F}(\langle \sigma^d \rangle) = \mathbb{F}_{p^d}$ for all $d|n$.*

Bibliography

[1] D. Dummit, *Abstract algebra.* Hoboken, NJ: John Wiley & Sons, Inc, 2004.

[2] I. N. Herstein, *Topics in algebra.* New York: Wiley, 1975.