

Algebraic Geometry.

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Chapter 1

Preliminaries

1.1 Affine Varieties

Definition. Let k be an algebraically closed field. We define **affine n -space** over k to be the set $\mathbb{A}^n(k)$ of n -tuples of elements of k . We write simply \mathbb{A}^n when k is understood. We call the elements of \mathbb{A}^n **points** and if $P = (a_1, \dots, a_n)$ is a point of \mathbb{A}^n , we call each a_i the **coordinates** of P .

Example 1.1. Let k be any algebraically closed field, and consider the multivariate polynomial ring $k[x_1, \dots, x_n]$. We can interpret the elements of $k[x_1, \dots, x_n]$ as functions from affine space $\mathbb{A}^n(k)$ to k by taking $f(P) = f(a_1, \dots, a_n)$, where $f \in k[x_1, \dots, x_n]$ and $P \in \mathbb{A}^n(k)$. This leads us to be able to talk about the set of zeros of a polynomial over k .

Definition. Let k be an algebraically closed field, and $f \in k[x_1, \dots, x_n]$ a multivariate polynomial over k . We define the **set of zeros** of f to be the set

$$Z(f) = \{P \in \mathbb{A}^n(k) : f(P) = 0\}$$

Let T be a subset of $k[x_1, \dots, x_n]$. Then we define the **set of zeros** of T to be

$$Z(T) = \bigcap_{f \in T} Z(f)$$

Now, if $\mathfrak{a} = (f_1, \dots, f_r)$ is an ideal of $k[x_1, \dots, x_n]$ generated by T , then we write $Z(T) = Z(\mathfrak{a}) = Z(f_1, \dots, f_r)$.

Definition. Let k be an algebraically closed field. We call a subset Y of \mathbb{A}^n an **algebraic set** if there exists some $T \subseteq k[x_1, \dots, x_n]$ for which Y is the set of zeros of T ; i.e. $Y = Z(T)$.

Lemma 1.1.1. *Let k be an algebraically closed field. Then algebraic sets of \mathbb{A}^n make \mathbb{A}^n into a topology under closed sets.*

Proof. Let $\mathbb{A}^n = Z(0)$ and $\emptyset = Z(1)$. Then \mathbb{A}^n and \emptyset are both algebraic. Now, let X and Y be algebraic, then there are S, T such that $X = Z(S)$ and $Y = Z(T)$. Now, let $P \in X \cup Y$, then P is a zero of any polynomial $f \in ST$, conversely, suppose that $P \in Z(ST)$ where

$P \notin Y$. There exists a polynomial $f \in S$ with $f(P) \neq 0$. Now, for any $g \in T$, we have that if $fg(P) = 0$, then $g(P) = 0$, so that $P \in S$. Therefore we have $X \cup Y = Z(ST)$, making $X \cup Y$ algebraic. So that the collection of algebraic sets is closed under finite intersection.

Lastly, consider a collection $\{Y_\alpha\}$ of algebraic sets, where $Y_\alpha = Z(T_\alpha)$ for some T_α . Let

$$Y = \bigcap Y_\alpha \text{ and } T = \bigcup T_\alpha$$

and let $P \in Y$. Then P is in every Y_α making it a zero of some $f_\alpha \in T_\alpha$, thus $P \in Z(T)$. Similarly, if $P \in Z(T)$, then $P \in Y$, making $Y = Z(T)$, and making the collection of algebraic sets closed under arbitrary intersections. ■

Definition. We define the **Zariski topology** on affine n -space \mathbb{A}^n to be the topology on \mathbb{A}^n whose closed sets are the algebraic sets of \mathbb{A}^n .

Example 1.2. Consider the Zariski topology on affine 1-space \mathbb{A}^1 . Now, since $k[x]$ is a PID, every algebraic set of \mathbb{A}^1 is the set of zeros of precisely one polynomial. Moreover, by the algebraic closure of k , for any nonzero polynomial f over k , we have

$$f(x) = c(x - a_1) \dots (x - a_n)$$

where $c, a_1, \dots, a_n \in k$. Then $Z(f) = \{a_1, \dots, a_n\}$, so that the algebraic sets of \mathbb{A}^1 are the emptyset, itself, and finite subsets. Thus the Zariski topology on \mathbb{A}^1 consists of finite sets, the emptyset, and \mathbb{A}^1 itself. Notice that this topology is not Hausdorff.

Definition. Let X be a topological space, and Y a subspace of X . We call Y **irreducible** if it cannot be written as the union $Y = Y_1 \cup Y_2$ of two sets Y_1 and Y_2 closed in Y . We make the convention that the emptyset is not irreducible.

Example 1.3. (1) Notice that the affine space \mathbb{A}^1 is irreducible. We have the only closed sets are finite sets, and since k is algebraically closed, and hence infinite, then \mathbb{A}^1 must be infinite.

(2) Subspaces of irreducible spaces are irreducible and dense.

(3) If Y is an irreducible space of a topological space X , then the closure $\text{cl } Y$ is also irreducible in X .

Definition. We define an **algebraic affine variety** to be an irreducible closed subset of \mathbb{A}^1 under the Zariski topology. We define an open set of an affine variety to be a **quasi-affine variety**.

Definition. We define the **ideal** of a subset Y in \mathbb{A}^n , to be the set

$$I(Y) = \{f \in k[x_1, \dots, x_n] : f(P) = 0 \text{ for all } P \in Y\}$$

where k is an algebraically closed field.

Theorem 1.1.2 (Hilbert's Nullstellensatz). *Let k be an algebraically closed field and \mathfrak{a} an ideal of $k[x_1, \dots, x_n]$, and let $f \in k[x_1, \dots, x_n]$ be a polynomial vanishing at all points of $Z(\mathfrak{a})$. Then there exists an $r \in \mathbb{Z}^+$ for which $f^r \in \mathfrak{a}$.*

Lemma 1.1.3. *The following are true for any algebraically closed field k .*

- (1) *If $T_1, T_2 \subseteq k[x_1, \dots, x_n]$ with $T_1 \subseteq T_2$, then $Z(T_2) \subseteq Z(T_1)$.*
- (2) *If $Y_1, Y_2 \subseteq \mathbb{A}^n$ with $Y_1 \subseteq Y_2$, then $I(Y_2) \subseteq I(Y_1)$.*
- (3) *For any $Y_1, Y_2 \subseteq \mathbb{A}^n$, $I(Y_1 \cup Y_2) = I(Y_1) \cap I(Y_2)$.*
- (4) *For any ideal \mathfrak{a} of $k[x_1, \dots, x_n]$, $I(Z(\mathfrak{a})) = \sqrt{\mathfrak{a}}$, where*

$$\sqrt{\mathfrak{a}} = \{f \in k[x_1, \dots, x_n] : f^r \in \mathfrak{a} \text{ for some } r \in \mathbb{Z}^+\}$$

- (5) *For every $Y \subseteq \mathbb{A}^n$, $Z(I(Y)) = \text{cl } Y$ in the Zariski topology.*

Proof. Let $T_1 \subseteq T_2$ be subsets of $k[x_1, \dots, x_n]$, and choose a polynomial $f \in T_1$, and a point $P \in Z(T_2)$. We have by inclusion that $f \in T_2$, and that $f(P) = 0$. Now since $f \in T_1$, this puts $P \in Z(T_1)$ and we get the required inclusion. The proof for statement (2) is identical.

Now, let $P \in Y_1 \cup Y_2$, and choose a polynomial $f \in I(Y_1) \cap I(Y_2)$. Then we have that the point P is either contained in Y_1 or Y_2 (or both), so that $f(P) = 0$, which makes $I(Y_1 \cup Y_2) \subseteq I(Y_1) \cap I(Y_2)$. Conversely, if $f \in I(Y_1) \cap I(Y_2)$, then for any points $P \in Y_1 \cup Y_2$, $f(P) = 0$, which puts $f \in I(Y_1 \cup Y_2)$.

For part (4), notice this is a direct consequence of Hilbert's Nullstellensatz. Now, for part (5), notice that $Y \subseteq Z(I(Y))$, which is a closed set in the Zariski topology, so that $\text{cl } Y \subseteq Z(I(Y))$. Now, let W be a closed set in \mathbb{A}^n , containing Y . Then we have $W = Z(\mathfrak{a})$, for some ideal \mathfrak{a} of $k[x_1, \dots, x_n]$, so that $Y \subseteq Z(\mathfrak{a})$. Then by part (2), observe that $I(Y) \subseteq I(Z(\mathfrak{a}))$, but $\mathfrak{a} \subseteq I(Z(\mathfrak{a}))$, so by part (1), we have $Z(I(Y)) \subseteq Z(\mathfrak{a})$. This makes $Z(I(Y)) = \text{cl } Y$. ■

Corollary. *There exists a 1-1, inclusion reversing, correspondence of algebraic sets of \mathbb{A}^n onto radical ideals in $k[x_1, \dots, x_n]$; given by the maps*

$$\begin{aligned} Y &\rightarrow I(Y) \\ \mathfrak{a} &\rightarrow Z(\mathfrak{a}) \end{aligned}$$

Moreover, an algebraic set in \mathbb{A}^n is irreducible if, and only if its ideal in $k[x_1, \dots, x_n]$ is prime.

Proof. Notice that parts (1), (2), and (3) of the above lemma provide the required correspondence.

Now, suppose that Y is irreducible in \mathbb{A}^n , and take $fg \in I(Y)$. Then $Y \subseteq Z(fg) = Z(f) \cup Z(g)$, so that

$$Y = (Y \cap Z(f)) \cup (Y \cap Z(g))$$

which is the union of two closed sets in \mathbb{A}^n . Now, since Y is irreducible, we get that either $Y = Y \cap Z(f)$, or $Y = Y \cap Z(g)$; in either case, $Y \subseteq Z(f)$, or $Y \subseteq Z(g)$. This puts either $f \in I(Y)$, or $g \in I(Y)$, which makes the ideal $I(Y)$ prime.

Conversely if \mathfrak{p} is a prime ideal, let $Z(\mathfrak{p}) = Y_1 \cup Y_2$, then $\mathfrak{p} = I(Y_1) \cap I(Y_2)$, so that either $\mathfrak{p} = I(Y_1)$ or $\mathfrak{p} = I(Y_2)$, since \mathfrak{p} is prime. This makes $Z(\mathfrak{p}) = Y_1$ or $Z(\mathfrak{p}) = Y_2$, which makes $Z(\mathfrak{p})$ irreducible in \mathbb{A}^n . ■

Example 1.4. Consider k to be an algebraically closed field.

- (1) Notice that \mathbb{A}^n maps to the ideal (0) in $k[x_1, \dots, x_n]$, which is a prime ideal. This makes \mathbb{A}^n irreducible by the corollary to lemma 1.1.3.
- (2) Let $f \in k[x, y]$ be an irreducible polynomial. Then f generates the prime ideal (f) in $k[x, y]$, since $k[x, y]$ is a UFD. Thus the zero set $Y = Z(f)$ is irreducible, and closed in \mathbb{A}^n ; hence it is an affine variety. We call Y an **affine curve** in \mathbb{A}^n defined by the equation $f(x, y) = 0$ of **degree** $\deg f = d$. Now, moregenerally, if f is an irreducible polynomial in $k[x_1, \dots, x_n]$, then we call the affine variety $Y = Z(f)$ a **surface** in $n = 3$ and a **hypersurface** in $n > 3$.
- (3) A maximal ideal \mathfrak{m} of $k[x_1, \dots, x_n]$ corresponds to a minimal affine variety of \mathbb{A}^n , which are the point-sets $\{P\}$ of \mathbb{A}^n ; where $P = (a_1, \dots, a_n)$. Thus every maximal ideal of $k[x_1, \dots, x_n]$ is of the form $M = (x_1 - a_1, \dots, x_n - a_n)$ for some $a_1, \dots, a_n \in k$.
- (4) Consider the field \mathbb{R} which is not algebraically closed, and the curve defined by $x^2 + y^2 + 1 = 0$ in $\mathbb{A}^2(\mathbb{R})$. Notice that this curve is irreducible, but has no points in \mathbb{A}^2 (i.e. no roots in \mathbb{R}). This shows that if the field k is not algebraically closed, then the above results do not hold in general. Notice that the paraboloid $f(x, y) = x^2 + y^2 + 1$ (figure 1.1) does not intersect the real xy -plane

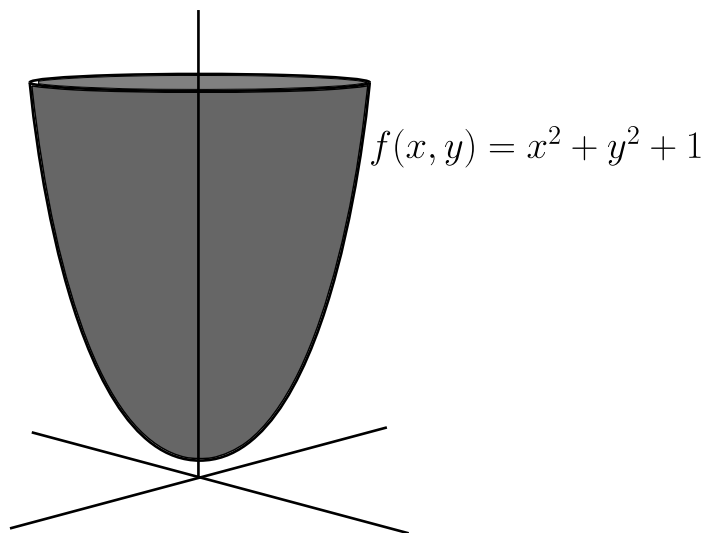


Figure 1.1: The curve $x^2 + y^2 + 1 = 0$ does not describe an affine variety.

Definition. Let k be an algebraically closed field, and Y an affine algebraic set of \mathbb{A}^n . We define the **affine coordinate ring** of Y to be the factor ring

$$A(Y) = k[x_1, \dots, x_n] / I(Y)$$

Lemma 1.1.4. *If Y is an affine variety, then $A(Y)$ is an integral domain. Moreover, there exists a 1–1 correspondence of finitely generated k -algebras onto affine coordinate rings of affine varieties.*

Bibliography

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