Complex Analysis

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Text

Complex Analysis (4^{th} edition)

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Contents

Chapter 1

Complex Numbers and Funtions

1.1 Complex Numbers

1.2 Complex Valued Functions

Definition. We define a **complex valued function** to be a function $f : S \to \mathbb{C}$, where $S \subseteq \mathbb{C}$. Writing $f(z) = f(x + iy) = u(x, y) + iv(x, y)$, where $u : U_1 \times U_1 \to \mathbb{R}$ and $v : V_1 \times V_2 \to \mathbb{R}$ are real valued functions (with U_1, U_2, V_1, V_2 open in $\mathbb{R}$), we define the **real part** of f to be $\operatorname{Re} f = u(x, y)$, and the **imaginary part** of f to be $\operatorname{Im} f = v(x, y)$.

Remark. It should be noted that the domain of a complex valued function f depends on the domain of its real and imaginary parts, and vice versa.

Example 1.1. (1) The real and imaginary parts of the complex valued function $f(z) = x^3y + i\sin(x + y)$ to be $u(x, y) = x^3y$ and $v(x, y) = \sin(x + y)$, respectively.

(2) Consider the complex valued function $f(z) = z^n$, for $n \in \mathbb{Z}^+$. Writing $z = re^{i\theta}$, we get $f(z) = r^n \cos n\theta + ir^n \sin n\theta$. The real part of f is then $u(x, y) = r^n \cos n\theta$, and the imaginary part of f to be $v(x, y) = r^n \sin n\theta$.

Lettinh $\overline{B^1} = \{z \in \mathbb{C} : |z| \leq 1\}$ the closed unit ballm notice if $z \in \overline{B^1}$, then $|z^n| = |z|^n \leq 1^n = 1$, so that $z^n \in \overline{B^1}$, and hence $f(\overline{B^1}) = \overline{B^1}$.

Definition. We call the solutions to the polynomial $z^n - 1$ over $\mathbb{C}$ the complex n**-th roots of unity**.

Theorem 1.2.1. *Let ξ be a complex n-th root of unity. Then $\xi = e^{\frac{2i\pi}{n}}$.*

Corollory. *If ξ is an n-th root of unity, then so is ξ^k for all $k \in \mathbb{Z}/n\mathbb{Z}$.*

1.3 Complex Differentiation and Holomorphic Functions

Definition. Let U be an open set of $\mathbb{C}$, and let $w \in U$. We call a complex valued function $f : U \to \mathbb{C}$ **complex differentiable** at w if the limit

$$f'(w) = \lim_{h \longrightarrow 0} \frac{f(w+h) - f(w)}{h} = \lim_{z \longrightarrow w} \frac{f(z) - f(w)}{z - w}$$

exists. We call $f'(w)$ the **complex derivative** of f at w.

Theorem 1.3.1. *Let $f : U \to \mathbb{C}$ and $g : U \to \mathbb{C}$ be complex valued functions. If f and g are complex differentiable at a point $z \in U$, then following are true*

(1) $f + g$ is complex differentiable at z, with

$$(f+g)'(z) = f'(z) + g'(z)$$

(2) $(fg)'$ is complex differentiable at z, with

$$(fg)'(z) = f'(z)g(z) + f(z)g'(z)$$

Corollary. *The function $\frac{f}{g}$ is complex differentiable at z, provided $g(z) \neq 0$, with*

$$\left(\frac{f}{g}\right)'(z) = \frac{g(z)f'(z) - f(z) - g'(z)}{(g(z))^2}$$

Example 1.2. For all $n \in \mathbb{Z}^+$, the function $f(z) = z^n$ is complex differentiable on all of $\mathbb{C}$, with $f'(z) = nz^{n-1}$. In fact, z^n is what we call a "holomorphic" function.

Theorem 1.3.2 (The Chain Rule)**.** *Let U and V be open sets of $\mathbb{C}$, and let $f : U \to \mathbb{C}$, and $g : V \to \mathbb{C}$ be complex valued functions, with $f(U) \subseteq V$. If f is complex differentiable at a point $z \in Z$, and g is complex differentiable at the point $f(z) \in f(U)$, then $g \circ f$ is complex differentiable at z with*

$$(g \circ f)'(z) = (g' \circ f)(z)f'(z) = g'(f(z))f'(z)$$

Definition. We call a complex valued function $f : U \to \mathbb{C}$ **holomorphic** on U if it is complex differentiable at every point of U.

Remark. It is convention to simply say that f is "holomorphic" when it is holomorphic on all of $\mathbb{C}$.

Definition. Let $f : U \to \mathbb{C}$ a complex valued function with $f(z) = u(x, y) + iv(x, y)$. We define the **vector field** of f to be the map $F : U \to V \to \mathbb{R} \times \mathbb{R}$ defined by

$$F(x, y) = (u(x, y), v(x, y))$$

Where U and V are open in $\mathbb{R}$.

Theorem 1.3.3. *If f is holomorphic on its domain, then F is real differentiable on its domain (resepctively to the domain of f) and has derivative*

$$\operatorname{Jac} F = \begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{pmatrix}$$

Where $\operatorname{Jac} F$ *is the Jacobian of* F.

Corollory. $f'(z) = \frac{\partial u}{\partial x} - i\frac{\partial v}{\partial y}$, *and the we have the following of equations*

$$\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} = 0$$
$$\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = 0$$

Theorem 1.3.4. *If $u : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, and $v : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are continuously real differentiable realvalued functions satisfying the equations*

$$\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} = 0$$
$$\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = 0$$

Then the function $f(z)u(x, y) + iv(x, y)$ is holomorphic on its domain.

Definition. Let $u : U_1 \times U_2 \to \mathbb{R}$ and $v : V_1 \times V_2 \to \mathbb{R}$ be continuously real differentiable real valued functions. We define the **Cauchy-Riemann equations** to be the set of equations

$$\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} = 0$$
$$\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = 0$$

Chapter 2

Power Series

2.1 Formal Power Series

Definition. Let F be a field, we define the set $F[[x]]$ of all series of the form

$$f(x) = \sum_{n=0}^{\infty} a_0 x^n, \text{ where } a_0, \dots, a_n, \dots \in F$$

the set of **formal power series** over F. We call the elements of $F[[x]]$ **formal power series**.

Definition. Let $f(x) = \sum_{n=0}^{\infty} a_0 x^n$ a formal power series over a field F. We define the **order** of f to be the smallest integer n for which $a_n \neq 0$, and write $\operatorname{ord} f = n$. We call the term a_0 of f the **constant term** of f.

Lemma 2.1.1. *Let F be a field, and define the operations $+$ and $\cdot$ on F by*

$$f(x) + g(x) = \sum_{n=0}^{\infty} c_n x^n \text{ where } c_n = a_n + b_n$$
$$f(x)g(x) = \sum_{n=0}^{\infty} d_n x^n \text{ where } c_n = \sum_{k=0}^{n} a_k b_{n-k}$$

Where $f(x) = \sum_{n=0}^{\infty} a_n x^n$ and $g(x) = \sum_{n=0}^{\infty} b_n x^n$ are formal power series over F. Then $F[[x]]$ forms a commutative ring under $+$ and $\cdot$.

Corollory. *Define the action $F \times F[[x]] \to F[[x]]$ by*

$$\alpha f(x) = \sum_{n=0}^{\infty} (\alpha a_n) x^n$$

Then $F[[x]]$ is an F-module under this action.

Lemma 2.1.2. *Let $f(x) = \sum_{n=0}^{\infty} a_n x^n$ and $g(x) = \sum_{n=0}^{\infty} b_n x^n$ be formal power series over a field F. Then $\operatorname{ord} fg = \operatorname{ord} f + \operatorname{ord} g$.*

Definition. Let $f \in F[[x]]$ be a formal power series over a field F. We say that a formal power series $g \in F[[x]]$ is an **inverse** of f if $fg = 1$.

Lemma 2.1.3. *If $f(x) = \sum_{n=0}^{\infty} a_n x^n$ is a formal power series over a field F, with nonzero constant term, then there exists an inverse of f.*

Proof. Consider the series $a_0^{-1} f(x)$ instead of f. Reacall also that the geometric series

$$\frac{1}{1-r} = 1 + r + r^2 + \dots$$

is a formal power series in r over F. Then $(1-r)(1+r+r^2+\dots) = 1$. Now, let $f(x) = 1-h(x)$, where $h(x) = -(a_1 x + a_2 x^2 + \dots)$ and consider $\phi(h) = 1 + h + h^2 + \dots$. Observbe that $\operatorname{ord} h^n \geq n$ sicne $h^n = (-1)a_1^n x^n + \dots$. Thus, if $m > n$, then h^m has all coefficients of order less than n equal to 0, and the n-th coefficient of ϕ is the n-th coefficient of the sum

$$1 + h + h^2 + \dots + h^n$$

Then, we get by the above geometric series that

$$(1-h(x))\phi(h) = (1-h(x))(1 + h + h(x)^2 + \dots) = 1 + \dots = 1$$

■

Example 2.1. Let $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots$. By lemma 2.1.3, since $\cos x$ has nonzero constant term, it has an invers $g(x) = \frac{1}{\cos x}$. Notice that

$$\begin{aligned}
\frac{1}{1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots} &= 1 + (\frac{x^2}{2!} - \frac{x^4}{4!} + \dots) + (\frac{x^2}{2!} - \frac{x^4}{4!} + \dots)^2 + \dots \\
&= 1 + \frac{x^2}{2!} - \frac{x^4}{4!} + \dots + \frac{x^4}{(2!)^2} \\
&= 1 + \frac{x^2}{2!} + (-\frac{1}{24} + \frac{1}{4})x^2 + \dots
\end{aligned}$$

Which gives coefficients of $g(x)$ up to order 4.

Definition. Let $f(x) = \sum_{n=0}^{\infty} a_n x^n$ a power series over a field F, and let $h(x) = c_1 x + \dots$ a power series of order greater than 1. We define the **substitute** of h in f to be the power series

$$f \circ h(x) = a_0 + a_1 h(x) + a_2 h(x)^2 + \dots$$

Definition. Let $f(x) = \sum_{n=0}^{\infty} a_n x^n$ and $g(x) = \sum_{n=0}^{\infty} b_n x^n$ be power series over a field F. We call f **congruent** to g **modulo** x^n if $a_k = b_k$ for all $k \in \mathbb{Z}/n\mathbb{Z}$. That is, f and g have the same coeficients of terms of order up to $n-1$. We write $f \equiv g \mod x^2$.

Lemma 2.1.4. *Congruence of power series modulo x^n defines an equivalence relation.*

Lemma 2.1.5. *If* $f_1 \equiv f_2 \mod x^n$ *and* $g_1 \equiv g_2 \mod x^n$*, then* $f_1 + g_1 \equiv f_2 g_2 \mod x^n$ *and* $f_1 g_1 \equiv f_2 g_2 \mod x^n$*. Moreover, if* h_1 *and* h_2 *are formal power series with zero constant term, and* $h_1 \equiv h_2 \mod x^n$*, then* $f_1 \circ h_1 \equiv f_1 \circ h_2 \mod x^n$*.*

Proof. We prove for substitutions of h_1 in f_1 only. Let p_1 and p_2 polynomials of degree $\deg = n-1$ such thta $f_1 \equiv p_1(x) \mod x^n$ and $f_2 \equiv p_2(x) \mod x^n$. By hypothesis, we get $p_1 \equiv p_2 \mod x^n$, and since $\deg p_1, \deg p_2 = n-1$, this makes $p_1 = p_2$. Then $f_1 \circ h \equiv p_1 \circ h = p_2 \circ h \equiv f_2 \circ h$. Now, let $q(x)$ the polynomial of degree $n-1$ such that $h_1 \equiv h_2 \equiv q(x) \mod x^n$Writing $p(x) = a_0 + a_1 x + \cdots + a_{n-1}x^{n-1}$. Then we get $p_1 \circ h_1 \equiv p_2 \circ h_2 \mod x^n$ and we are done. ■

Corollary. *Two power series* f *and* g *are equal if, and only if* $f \equiv g \mod x^n$ *for all* $n \in \mathbb{Z}^+$*.*

Corollary. $(f_1 + f_2) \circ h = (f_1 \circ h) + (f_2 \circ h)$*, and* $(f_1 f_2) \circ h = (f_1 \circ h)(f_2 \circ h)$*. That is, composition of power series distributes over the addition and multiplication of power series.*

Corollary. *Provided that* $\operatorname{ord} f_2 = 0$*, then*

$$\left(\frac{f_1}{f_2}\right) \circ h = \frac{f_1 \circ h}{f_2 \circ h}$$

Example 2.2. Considre the power series for $\frac{1}{\sin x}$. We have by definition that

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots = x(1 - \frac{x^2}{3!} + \frac{x^4}{5!})$$

so that

$$\frac{1}{\sin x} = \frac{1}{x(1 - \frac{x^2}{3!} + \frac{x^4}{5!})} = \frac{1}{x}(1 + \frac{x^2}{3!} - \frac{x^4}{5!} + \left(\frac{x^2}{3!}\right)^2 + \dots) = \frac{1}{x} + \frac{x}{3!} + \left(\frac{1}{(3!)^2} - \frac{1}{5!}\right)x^3 + \dots$$

2.2 Convergent Power Series

For the remainder of this chapter, we consider only formal power series if z over $\mathbb{C}$; i.e. all power series in $\mathbb{C}[[z]]$.

Definition. Let $\{z_n\}_{n \in \mathbb{Z}^+}$ a sequence of complex numbers, and consider the series $\sum_{n=0}^{\infty} z_n$. We define the n**-th partial sum** to be

$$s_n = \sum_{k=1}^{n} z_k$$

and we say that the series **converges** if there exists a $w \in \mathbb{C}$ for which $\lim \{s_n\} = w$ as $n \to \infty$. We call w the **sum** of the series.

Lemma 2.2.1. *Let* $A = \sum \alpha_n$ *and* $B = \sum\_n$ *be convergent series with* n*-th partial sums* s_n *and* t_n*. Then the sum and product of* A *and* B *converge, with*

$$A + B = \sum (\alpha_n + \_n) \text{ and } AB = \lim_{n \longrightarrow \infty} \{s_n t_n\}$$

Definition. Let $\sum \alpha_n$ a series of complex numbers. We say that $\sum \alpha_n$ **converges absolutely** if the series of real numbers $\sum |\alpha_n|$ converges.

Lemma 2.2.2. *If $\sum \alpha_n$ is a series of complex numbers which converges absolutely, then it converges.*

Proof. Let $s_n = \sum_{k=1}^n \alpha_k$, then for $m \leq n$, notice that $s_n - s_m = \alpha + m_1 + \cdots + \alpha_n$, hence $|s_n - s_m| \leq \sum_{k=m+1}^n |\alpha_k|$. By absolute convergence, let $\varepsilon > 0$ then there exists an $N > 0$ such that $\sum |\alpha_k| < \varepsilon$ whenever $m, n \geq N$. Thus $|s_n - s_m| < \varepsilon$ which makes $\sum \alpha_n$ converge. ∎

Lemma 2.2.3. *Let $\sum c_n$ be a convergent series of real numbers greater than 0. If $\{\alpha_n\}$ is a sequence of complex numbers such that $|\alpha_n| < c_n$ for all $n \in \mathbb{Z}^+$, then $\sum \alpha_n$ converges absolutely.*

Proof. Notice that the partial sums $\sum_{k=1}^n c_n$ are bounded, hence $\sum |\alpha_n| \leq \sum c_k$. ∎

Lemma 2.2.4. *Let $\{\alpha_n\}$ a sequence of complex numbers. Then the following are true*

(1) If $\sum \alpha_n$ is absolutely convergent, then the series obtained by permuting terms is absolutely convergent, with the same limit.

(2) If $\sum_{n=1}^\infty (\sum_{m=1}^n \alpha_{mn})$ is absolutely convergent, then so is the series $\sum_{m=1}^n (\sum_{n=1}^\infty \alpha_{mn})$, and they converge to the same limit.

Definition. Let $S \subseteq \mathbb{C}$, and let f be a bounded complex valued function on S. We define the **sup norm** of f on S to be

$$\|f\|_S = \sup_{z \in S} \{|f(z)|\}$$

Lemma 2.2.5. *Let $S \subseteq \mathbb{C}$. The sup norm of a complex valued function on S defines a metric on $\mathbb{C}$.*

Definition. Let $\{f_n\}_{n \in \mathbb{Z}^+}$ a sequence of complex valued functions on a set $S \subseteq \mathbb{C}$. We say that the $\{f_n\}$ **converges uniformly** on S if there exists a bounded complex valued function f on S such that for all $\varepsilon > 0$, there is an $N > 0$ for which

$$\|f_n - f\|_S < \varepsilon \text{ whenever } n \geq N$$

We call $\{f_n\}$ **Cauchy** if for every $\varepsilon > 0$ there is an $N > 0$ for which

$$\|f_n - f_m\|_S < \varepsilon \text{ whenever } n, m \geq N$$

Theorem 2.2.6. *Let $\{f_n\}$ be a sequence of complex valued functions on a set $S \subseteq \mathbb{C}$. If $\{f_n\}$ is Cauchy, then it converges uniformly.*

Proof. We have for all $z \in S$, take $f(z) = \lim f_n(z)$ as $n \to \infty$. Then for $\varepsilon > 0$ there is an $N > 0$ for which $|f_n(z) - f_m(z)| < \varepsilon$ for al $z \in S$ and $m, n \geq N$. Now, for $n \geq N$, take $m(n) \geq N$ large enough so that $|f(z) - f_{m(n)}(z)| < \varepsilon$. Then we get that

$$|f(z) - f_n(z)| \leq |f(z) - f_{m(n)}(z)| + |f_{m(n)}(z) - f_n(z)| < \varepsilon + \|f_{m(n)} - f_n\| < 2\varepsilon$$

∎

Corollory. *If $\{f_n\}$ is bounded for all $n \in \mathbb{Z}^+$, then so is f.*

Definition. We say a series of complex valued functions on a domain $S \subseteq \mathbb{C}$, $\sum f_n$ **converges uniformly** if the sequence $\{s_n\}$ of n-th partial sums converges uniformly. We say that $\sum f_n$ **converges absolutely** if for all $z \in S$, $\sum |f_n(z)|$ converges.

Theorem 2.2.7 (The Comparison Test)**.** *Let $\{c_n\}$ be a sequence of real numbers greater than 0 such that $\sum c_n$ converges. Let $\{f_n\}$ a sequence of complex valued functions on a domain $S \subseteq \mathbb{C}$ such that $\|f_n\|_S \leq c_n$ for all $n \in \mathbb{Z}^+$. Then the series $\sum f_n$ converges uniformly, and converges absolutely.*

Proof. Let $m \leq n$. Then $\|s_n - s_m\| \leq \sum_{k=m+1}^{n} \|f_k\|_S \leq \sum c_k$. Since $\sum c_k$ converges, the uniform and absolute convergnce of $\sum f_n$ follows. ■

Theorem 2.2.8. *Let $S \subseteq \mathbb{C}$ and $\{f_n\}$ a sequence of continuous complex valued functions on S. If $\{f_n\}$ converges unifromly to a complex valued function f on S, then f is also continuous.*

Proof. let $\alpha \in S$ and n be large enough such that $\|f - f_n\|_S < \varepsilon$ for some $\varepsilon > 0$. By the continuity of f_n at α, choose ¿0 such that $|f_n(z) - f_n(\alpha)| < \varepsilon$ whenever $|z - \alpha| <$ ¿$Thenobservethat|f(z) - f(\alpha)| \leq |f(z) - f_n(z)| + |f_n(z) - f_n(\alpha)| + |f_n(\alpha) - f(\alpha)| < 2\|f - f_n\| + \varepsilon < 3\varepsilon$ ■

Theorem 2.2.9. *Let $\{a_n\}$ a sequence of complex numbers, and let $r > 0$ such that $\sum |a_n| r^n$ converges. Then the power series $\sum a_n z^n$ converges absolutely and converges uniformly whenever $|z| \leq r$.*

Example 2.3. (1) Let $r > 0$ and consider the series

$$\exp z = \sum_{n=0}^{\infty} \frac{z^n}{n!}$$

Then $\exp z$ converges absolutely and uniformly whenever $|z| \leq r$. Indeed, let $c_n = \frac{r^n}{n!}$, then

$$\frac{c_{n+1}}{c_n} = \frac{r}{n+1}$$

Taking $n \geq 2r$, notice that $\frac{c_{n+1}}{c_n} \leq \frac{1}{n}$ so that $c_{n+1} \leq \frac{1}{2} c_n$ for n large enough. Therefore there exists an n_0 such that

$$c_n \leq \frac{C}{2^{n-n_0}} \text{ for some constant } C$$

whenever $n \geq n_0$. Comparing this with the geometric series, we get absolute and uniform convergence as was required. Moreover, notice that the series $\exp z$ defines a continuous function on all of $\mathbb{C}$.

(2) Take the series $S(z) = \sum (-1)^n \frac{2^{2n+1}}{(2n+1)!}$ and $C(z) = \sum (-1)^n \frac{2^{2n}}{(2n)!}$. Both $S(z)$ and $C(z)$ converge absolutely and uniformly for all $|z| \leq r$. Moreover, they define continuous functions on al of $\mathbb{C}$.

Theorem 2.2.10. *Let $\sum a_n z^n$ a power series. If it does not converge absolutely for all $z \in \mathbb{C}$, then there exists a real number $r > 0$ such that $a_n z^n$ converges absolutely whenever $|z| \leq r$.*

Proof. Suppose that $\sum a_n z^n$ does not converge absolutely for all $z \in \mathbb{C}$. Let $r = \sup_{s \geq 0} \{s\}$ where $\sum a_n s^n$ converges. Then notice that $\sum |a_n||z|^n$ diverges wheneer $|z| > r$ and converges when $|z| < r$, by the comparison test. ■

Definition. The **radius of convergence** of a power series $\sum a_n z^n$ is a number $r > 0$ for which the series converges absolutely whenever $|z| < r$, and diverges whenever $|z| > r$. $\sum a_n z^n$ converges absolutely for all $z \in \mathbb{C}$, then we write $r = \infty$. We call $\sum a_n z^n$ a **convergent power series** if $r \neq 0$, and we say that it **converges** on an open ball $B(0, r)$.

Theorem 2.2.11. *Let $\sum a_n z^n$ a convergent power series with radius of convergence r. Then*

$$\frac{1}{r} = \limsup \sqrt[n]{|a_n|}$$

If $r = 0$, then the sequence of points $\{\sqrt[n]{a_n}\}$ is not bounded.

Proof. Let $t = \limsup \sqrt[n]{|a_n|}$, suppose first that $t \neq 0$ and that $t \neq \infty$. Given $\varepsilon > 0$, there is a finite number of points $n \in \mathbb{Z}^+$ for which $\sqrt[n]{|a_n|} \geq t + \varepsilon$. Thus, for all but finitely many n, we get $|a_n| < (t + \varepsilon)^n$, and $\sum a_n z^n$ converges if $|z| < \frac{1}{t+\varepsilon}$. By comparison with the geometric series, we conclude that $r \geq \frac{1}{t+\varepsilon}$ for all $\varepsilon > 0$; that is

$$r \geq \frac{1}{t}$$

Conversely, given $\varepsilon > 0$, there exist infinitely many $n \in \mathbb{Z}^+$ such that $\sqrt{|a_n|} \geq t - \varepsilon$, and hence $|a_n| \geq (t - \varepsilon)^n$. So we get that $\sum a_n z^n$ does not converge if $r = \frac{1}{t-\varepsilon}$, and its radius of convergence satisfies $r \leq \frac{1}{t-\varepsilon}$ for all $\varepsilon > 0$. That is

$$r \leq \frac{1}{t}$$

and equality is established. ■

Corollory. *If $\lim \sqrt[n]{|a_n|} = t$ exists, then $r = \frac{1}{t}$.*

Corollory. *If $\sum a_n z^n$ has radius of convergence $r > 0$, then there exists a $C > 0$ such that if $A > \frac{1}{r}$, then $|a_n| \leq CA^n$ for all n.*

Example 2.4.

(1) The radius of convergence of the series $\sum n! z^n$ is $r = 0$, since $\sqrt[n]{n!}$ is unbounded as $n \to \infty$.

(2) The radius of convergence for the series $\exp z = \sum \frac{z^n}{n!}$ is $r = \infty$, as $\sqrt[n]{\frac{1}{n!}} \to 0$ as $n \to \infty$. That is, the series $\exp z$ converges on all of $\mathbb{C}$.

(3) The radius of convergence of $\sum \frac{n!}{n^n} z^n$ is $r = e$, where e is Euler's constant. Observe that $\lim \frac{n!}{n^n} = \frac{1}{e}$.

Theorem 2.2.12 (The Ratio Test)**.** *If* $\{a_n\}$ *is a sequence of positive real numbers, for which* $\lim \frac{a_{n+1}}{a_n} = A$ *exists, then* $\lim \sqrt[n]{a_n} = A$*.*

Proof. Suppose that $A > 0$, given $\varepsilon > 0$, take n_0 such that $A - \varepsilon \leq \frac{a_{n+1}}{a_n} \leq A + \varepsilon$, for all $n \geq n_0$. Without loss of generality, suppose that $\varepsilon < A$, so that $A - \varepsilon > 0$. THen

$$a_n = a_1 \prod_{k=1}^{n_0-1} \frac{a_k + 1}{a_k} \prod_{k=n_0}^{n} \frac{a_k + 1}{a_k}$$

By induction, there exists constants $C_1(\varepsilon)$ and $C_2(\varepsilon)$ suc that

$$C_1(\varepsilon)(A - \varepsilon)^{n-n_0} \leq a_n \leq C_2(\varepsilon)(A + \varepsilon)^{n-n_0}$$

Put $C_1'(\varepsilon) = C_1(\varepsilon)(A - \varepsilon)^{-n_0}$ nad $C_2'(\varepsilon) = C_2(\varepsilon)(A + \varepsilon)^{-n_0}$, then

$$(A - \varepsilon)\sqrt[n]{C_1'(\varepsilon)} \leq \sqrt[n]{a_n} \leq (A + \varepsilon)\sqrt[n]{C_2'(\varepsilon)}$$

Then, there exists $N \geq n_0$ such that $\sqrt[n]{C_1'(\varepsilon)} = 1 + 1(n)$, with $|1(n)| \leq \frac{\varepsilon}{A-\varepsilon}$ and $\sqrt[n]{C_2'(\varepsilon)} = 1 + 2(n)$ and $|2(n)| \leq \frac{\varepsilon}{A+\varepsilon}$ for all $n \geq N$. Then

$$A - \varepsilon + 1(n)(A - \varepsilon) \leq \sqrt[n]{a_n} \leq A + \varepsilon + 2(n)(A + \varepsilon)$$

which shows that

$$|\sqrt[n]{a_n} - A| < 2\varepsilon$$

For the case that $A = 0$, it is easy. ■

Example 2.5. Let $a \neq 0$ a complex number. We define the **binomial coefficient** of α **choose** n, where $n \in \mathbb{Z}^+$ to be

$$\binom{\alpha}{n} = \frac{\alpha(\alpha - 1)\ldots(\alpha - n + 1)}{n!}$$

and we define the **binomial sereies**

$$(1 + z)^{\alpha} = \sum \binom{\alpha}{n} z \text{ where } \binom{\alpha}{0} = 1$$

By the ratio test, we get that $r = 1$ if α is not an integer greater than 0.

2.3 Properties on Power Series

Theorem 2.3.1. *Let* $f(z)$ *and* $g(z)$ *be formal power series which converge absolutely on the open ball* $B(0, r)$, $r > 0$*. Then* $f + g$, fg, *and* αf, *where* $\alpha \in \mathbb{C}$, *also converge on* $B(0, r)$*. Moreover, we have*

(1) $(f + g)(z) = f(z) + g(z)$

(2) $(fg)(z) = f(z)g(z)$

(3) $(\alpha f)(z) = \alpha f(z)$

Proof. Let $f(z) = \sum a_n z^n$ and let $g(z) = \sum b_n z^n$. Then $fg(z) = \sum c_n z^n$, where $c_n = \sum_{k=0}^{n} a_k b_{n-k}$. Now, let $0 < r < s$, then there exists a $C > 0$ such that for al $n \in \mathbb{Z}^+$, $|a_n| \leq \frac{C}{s^n}$, and $|b_n| \leq \frac{C}{s^n}$. So we have

$$|c_n| \leq \sum |a_k b_{n-k}| \leq (n+1)\frac{C^2}{n}$$

notice that

$$\lim \sqrt[n]{(n+1)C} - 1$$

so that $\limsup \sqrt[n]{|c_n|} = \frac{1}{s}$ for all $s < r$. Thus we have

$$\limsup \sqrt[n]{|c_n|} \leq \frac{1}{r}$$

and so fg converges absolutely on $B(0, r)$. Notice also that $\sum |a_k||b_{n-k}||z|^n$ also converges as well.

Now, let $f_N(z) = a_0 + a_1 z + \cdots + a_N z^N$ and $g_N(z) = b_0 + b_1 z + \cdots + b_N z^N$ be polynomials in z over $\mathbb{C}$ of degree N (i.e. the terms of f and g of order less than N). Then we get $f(z) = \lim f_N(z)$ and $g(z) = \lim g_N(z)$ as $N \to \infty$, moreover

$$|(fg)_N(z) - f_N(z)g_N(z)| \leq \sum_{n=N+1}^{\infty} \sum_{k=0}^{n} |a_k||b_{n-k}||z|^n$$

which converges, that is, $(fg)(z) = \lim f_N g_n = f(z)g(z)$. ■

Theorem 2.3.2. *Let $f(z) = \sum a_n z^n$ and $g(z) = b_n z^n$. Then the following are true*

(1) If f is nonconstant and convergent with radius of convergence $r > 0$ and $f(0) = 0$, then there exists an $s > 0$ for which $f(z) \neq 0$ whenever $|z| \leq s$, provided $z \neq 0$.

(2) If f and g converge, with $f(x) = g(x)$ for all x in an infinite set having 0 as a limit point, then $f(z) = g(z)$ for all z; i.e. $a_n = b_n$ for all $n \in \mathbb{Z}^+$.

Proof. Write $f(z) = amz^m + A = a_m z^m(1 + b_1 z + b_2 z^2 + \dots) = a_m z^m(1 + h(z))$ where $a_m \neq 0$, and $h(z) = b_1 z + b_2 z^2 + \dots$ a power series having radius of convergence $r > 0$ and 0 constant term. Then for $|z|$ small, $|h(z)|$ is small, and hnece $1 + h(z) \neq 0$. Now, if $z \neq 0$, then $a_m z^m \neq 0$ and we are done with the first assertion.

Now, let $h(t) = f(t) - g(t) = \sum (a_n - b_n)t^n$. Let S have an infinite set having 0 as a limit point. Then for every $x \in S$, $h(x) = 0$, by above, we get that $h(z) = 0(z) = 0 + 0z + 0z^2 + \dots$; i.e. $a_n - b_n = 0$ for all $n \in \mathbb{Z}^+$, and we are done with the second assertion. ■

Example 2.6. (1) There exists at most one convergent power series $f(z) = \sum a_n z^n$ for which $f(x) = e^x$ for all $x \in [-\varepsilon, \varepsilon]$, given some $\varepsilon > 0$. Then any extension of e^x to $\mathbb{C}$ is unique, morever, the series $\exp z = \sum \frac{z^n}{n!}$ coincides with that extension, i.e. $\exp z = e^z$.

Moreover, we have that

$$\exp iz = \sum \frac{(iz)^n}{n!}$$

so that $\exp iz = C(z) + iS(z)$, where $S(z)$ and $C(z)$ were defined in example 2.3. It can also be shown that $C(z)$ and $S(z)$ coincide with expanding cos and sin to $\mathbb{C}$; i.e. $C(z) = \cos z$ and $S(z) = \sin z$.

In fact, if $f(z)$ and $g(z)$ are power series, with constant term 0, then $(\exp f(z))(\exp g(z)) = \exp(f(z) + g(z))$. Indded, by defition, we have that

$$\exp(f(z) + g(z)) = \sum \frac{(f(z) + g(z))^n}{n!}$$

On the other hand, we get

$$(\exp f(z))(\exp g(z)) = \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{f(z)^n g(z)^{n-k}}{k!(n-k)!} = \sum \frac{(f(z) + g(z))^n}{n!}$$

Taking $f(z) = z$ and $g(w) = w$ (i.e. the constant series $0 + 1z + 0z^2 + \dots$ and $0 + 1w + w^2 + \dots$). We get $(\exp z)(\exp w) = \exp(z + w)$; and we get the familiar properties for e^x extended to the complex function $\exp z = e^z$.

(2) Let $C(z)$ and $S(z)$ the power series for which $\exp z = C(z) + iS(z)$. Notice that the series $S(z)^2 + C(z)^2$ has radius of convergence 1, indeed, $S(z)^2 + C(z)^z = 1$, since there exists at most one series with this property, the series $1 + 0z + 0z^2 + \dots$. Thus $\sin z$ and $\cos z$ have the property that $\sin^2 z + \cos^2 z = 1$.

(3) Consider the binomial series $B(z) = \sum \binom{\alpha}{n} z^n$, for $\alpha = \frac{1}{m}$, $m \in \mathbb{Z}^+$. Then $B(z)$ has radius of convergence $r = 1$. Moreover, by some elementary calculus, it can be shown that

$$B(z)^m = z + x \text{ for all } x \in \mathbb{R} \text{ small enough}$$

Thus $B(z)^m = 1 + z$, and so the series $(1 + z)^{\frac{1}{m}}$ converges whenever $|z| < 1$.

Definition. Let $f(z) = \sum a_n z^n$ be a formal power series, and let $\phi(z) = \sum c_n z^n$ a formal power series with nonnegative real coefficients. We say that f is **dominated** by ϕ if $|a_n| \leq c_n$ for al $n \in \mathbb{Z}^+$. We write $f = O(\phi)$, or $f \preceq \phi$.

Lemma 2.3.3. *If ϕ and ψ are power series with nonnegative real coefficients, and let $f(z)$ and $g(z)$ be formal power series. Then if $f \preceq \phi$ an $g \preceq \psi$, then*

$$f + g \preceq \phi + \psi \textit{ and } fg \preceq \phi\psi$$

Theorem 2.3.4. *Let $f(z)$ be a convergent power series with radius of convergence $r > 0$ and nonzero constant term. Let g be the inverse of f. Then g is also convergent with nonzero radius of convergence.*

Proof. Without loss of generality, suppose that the constant term of f is 1. That is

$$f(z) = 1 + a_1 z + a_2 z^2 + \cdots = 1 - h(z)$$

where $h(z)$ is a power series with constant term 0. Then there exists an $A > 0$ such that $|a_n| \leq A$ for all $n \geq 1$. Choosing A large enough, choose $C = 1$. Then

$$\frac{1}{f(z)} = \frac{1}{1-h(z)} = 1 + h(z) + h(z)^2 + \dots$$

But $h(z)$ is dominated by the series $\sum A^n z^n = \frac{Az}{1-Az}$. SO that $\frac{1}{f(z)} = g(z)$ satisfies

$$g(z) \preceq 1 + \frac{Az}{1-Az} + (\frac{Az}{1-Az})^2 + \dots = \frac{1}{1-\frac{Az}{1-Az}} = (1-Az)(1+2Az+(2Az)^2+\dots)$$

and

$$\frac{1}{1-\frac{Az}{1-Az}} = (1-Az)(1+2Az+(2Az)^2+\dots) \preceq (1+Az)(1+2Az+(2Az)^2+\dots)$$

That is, $g(z)$ is dominated by a convergent power series; hence g converges and has nonzero radius of convergence. ■

Theorem 2.3.5. *Let $f(z) = \sum a_n z^n$ and $h(z) = \sum b_n z^n$ be convergent power series where the constant term of h is 0. If f is absolutely convergent whenever $|z| \leq r$, given $r > 0$, and there is an $s > 0$ for which*

$$\sum |b_n| s^n \leq r$$

then the formal power series $f \circ h(z) = \sum a_n (\sum b_k z^k)^n$ converges absolutely whenever $|z| \leq s$.

Proof. Let $g(z) = \sum c_n z^n$. Then

$$g(z) \preceq \sum |a_n| (\sum |b_k|)^n$$

by hypothesis, we have that $\sum |a_n|(\sum |b_k|)^n$ converges absolutely whenever $|z| \leq s$, so that g does as well.

Now, let $f_N(z) = a_0 + a_1 z + \dots + a_{N-1} z^{N-1}$ a polynomial of degree $N-1$. Observe then that

$$f \circ h(z) - f_N \circ h(z) \preceq \sum |a_n| (\sum |b_k|)^n$$

so that $f \circ h(z) = g(z)$. By absolute convergence, given $\varepsilon > 0$, there is an $N_0 > 0$ such that

$$|g(z) - f_N \circ h(z)| < \varepsilon \text{ whenever } N \geq N_0 \text{ and } |z| \leq s$$

Since $f_N \to f$ as $N \to \infty$ on the open ball $B(0, r)$, choose N_0 large enough so that $|f_N \circ h(z) - f \circ h(z)| < \varepsilon$ for all $N \geq N$; i.e. $|g(z) - f \circ h(z)| < 2\varepsilon$. ■

Example 2.7. (1) Let $m \in \mathbb{Z}^+$ and $h(z)$ a convergent power series with constant term 0. We take the m-th root $\sqrt[m]{1+h(z)}$ using the binomial series with $\alpha = \frac{1}{m}$. Thus $B \circ h(z) = B(h(z))$ converges.

(2) Define $f(w) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{w^n}{n}$. Define $\log z$ for all $|z-1| < 1$ by $\log z = f(z-1)$. It can be shown that $\exp(\log z) = z$.

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