

# Topology

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# Chapter 1

## Topological Spaces and Continuous Functions.

### 1.1 Topological Spaces.

**Definition.** A **topology** on a set  $X$  is a collection  $\mathcal{T}$  of subsets of  $X$  such that:

- (1)  $\emptyset \in \mathcal{T}$  and  $X \in \mathcal{T}$
- (2) For any collection  $\{U_\alpha\}$  of subsets of  $X$ ,  $\bigcup_\alpha U_\alpha \in \mathcal{T}$ .
- (3) For any finite collection  $\{U_i\}_{i=1}^n$  of subsets of  $X$ ,  $\bigcap_{i=1}^n U_i \in \mathcal{T}$ .

We call the pair  $(X, \mathcal{T})$  a **topological space**, and we call the elements of  $\mathcal{T}$  **open sets**.

**Example 1.1.** (1) Let  $X$  be any set, the collection of all subsets of  $X$ ,  $2^X$  is a topology on  $X$ , which we call the **discrete topology**. We call the topology  $\mathcal{T} = \{\emptyset, X\}$  the **indiscrete topology**.

- (2) The set of three points  $\{a, b, c\}$  has the 9 following topologies in figure 1.1.

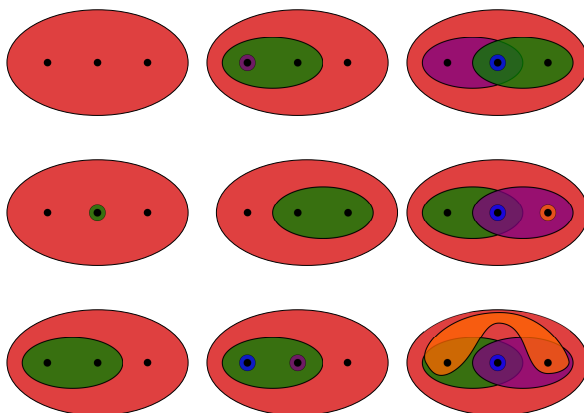


Figure 1.1: The Topologies on  $\{a, b, c\}$ .

- (3) Let  $X$  be any set, and let  $\mathcal{T}_f = \{U \subseteq X : X \setminus U \text{ is finite, or } X \setminus U = X\}$ . Then  $\mathcal{T}_f$  is a topology and called the **finite complement topology**.
- (4) Let  $X$  be any set, and let  $\mathcal{T}_c = \{U \subseteq X : X \setminus U \text{ is countable, or } X \setminus U = X\}$ . Then  $\mathcal{T}_c$  is a topology on  $X$  called the **countable complement topology**.
- (5) Let  $X$  be any set and consider the collection  $\mathcal{T}_\infty = \{U \subseteq X : X \setminus U = \emptyset, X \setminus U = X, \text{ or } X \setminus U \text{ is infinite}\}$ . Certainly, we have that  $\emptyset, X \in \mathcal{T}_\infty$  as  $X \setminus X = \emptyset$  and  $X \setminus \emptyset = X$ . However, if  $X = \mathbb{R}$ , and we have the sets  $(0, 1)$ , and  $(-\infty, 0) \cup (1, \infty)$ , then  $\mathbb{R} \setminus (0, 1) = (-\infty, 0] \cup [1, \infty)$ , and  $\mathbb{R} \setminus ((-\infty, 0) \cup (1, \infty)) = [0, 1]$ , both of which are infinite, but,  $[0, 1] \cap ((-\infty, 0] \cup [1, \infty)) = \{0, 1\}$ , which is finite. So it is not true in general that the collection  $\mathcal{T}_\infty$  is a topology.

**Lemma 1.1.1.** *Let  $X$  be a topological space. If  $A \subseteq X$  is such that for each  $x \in A$ , there exists an open set  $U$  with  $x \in U \subseteq A$ , then  $A$  is also open in  $X$ .*

*Proof.* We have that for each  $x \in A$ , there is an open set  $U_x$  such that  $x \in U_x \subseteq A$ . Now, let  $U = \bigcup_{x \in A} U_x$ , which is open in  $X$  by definition. Then, we have  $U \subseteq A$  by hypothesis; moreover, since  $x \in A$  implies  $x \in U_x$ , then  $x \in U$ . This makes  $A \subseteq U$ , so  $A = U$ . ■

**Definition.** Let  $X$  be a set, and let  $\mathcal{T}$  and  $\mathcal{T}'$  be topologies on  $X$ . We say that  $\mathcal{T}$  is **coarser** than  $\mathcal{T}'$ , and  $\mathcal{T}'$  **finer** than  $\mathcal{T}$  if  $\mathcal{T} \subseteq \mathcal{T}'$ . If two topologies are either coarser, or finer than each other, we call them **comparable**.

**Example 1.2.** The topologies  $\mathcal{T}_f$  and  $\mathcal{T}_c$  are comparable, and we see that  $\mathcal{T}_f \subseteq \mathcal{T}_c$ , so  $\mathcal{T}_f$  is coarser than  $\mathcal{T}_c$ , and  $\mathcal{T}_c$  is finer than  $\mathcal{T}_f$ .

**Lemma 1.1.2.** *If  $\{\mathcal{T}_\alpha\}$  is a collection of topologies on a set  $X$ , then the intersection of all  $\mathcal{T}_\alpha$ ,  $\bigcap \mathcal{T}_\alpha$  is also a topology on  $X$ .*

*Proof.* Let  $\mathcal{T} = \bigcap \mathcal{T}_\alpha$ . We have that  $\emptyset, X \in \mathcal{T}_\alpha$  for each  $\alpha$ , so that  $\emptyset, X \in \mathcal{T}$ .

Now let  $\{U_\alpha\}$  be a collection of open sets such that  $U_\alpha \in \mathcal{T}_\alpha$  for each  $\alpha$ . Then  $U_\alpha \in \mathcal{T}$ , for each  $\alpha$ , so that  $\bigcup U_\alpha \in \mathcal{T}$ . Lastly, take a finite subcollection  $\{U_i\}_{i=1}^n$  of  $\{U_\alpha\}$ , then  $\bigcap_{i=1}^n U_i \in \mathcal{T}$  by similar reasoning. ■

**Example 1.3.** If  $X$  is any set, and  $\{\mathcal{T}_\alpha\}$  is a collection of topologies in  $X$ , it is not in general true that  $\bigcup \mathcal{T}_\alpha$  is also a topology on  $X$ . Consider the 9 topologies on the set  $X = \{a, b, c\}$  in the preceding examples. Let  $\mathcal{T}_1 = \{\emptyset, \{a\}, \{a, b\}, X\}$ ,  $\mathcal{T}_2 = \{\emptyset, \{a, b\}, \{b\}, \{b, c\}, \{c\}, X\}$ , and let  $\mathcal{T}_3 = \mathcal{T}_1 \cup \mathcal{T}_2$ . The sets  $\{a\}$  and  $\{c\}$  have union  $\{a, c\}$ , however,  $\{a, c\} \notin \mathcal{T}_3$ .

## 1.2 The Basis and Subbasis for a Topology.

**Definition.** If  $X$  is a set, the **basis** for a topology on  $X$  is a collection  $\mathcal{B}$  of subsets of  $X$ , called **basis elements**, such that:

- (1) For every  $x \in X$ , there is a  $B \in \mathcal{B}$  such that  $x \in B$ .
- (2) For  $B_1, B_2 \in \mathcal{B}$ , if  $x \in B_1 \cap B_2$ , then there is a  $B_3 \in \mathcal{B}$  such that  $x \in B_3 \subseteq B_1 \cap B_2$

We define the topology  $\mathcal{T}$  **generated** by  $\mathcal{B}$  to be collection of open sets:  $\mathcal{T} = \{U \subseteq X : \text{for all } x \in U, \text{ there exists a } B \in \mathcal{B} \text{ such that } x \in B\}$ .

**Theorem 1.2.1.** *Let  $X$  be a set, and  $\mathcal{B}$  a basis of  $X$ , then the collection of subsets of  $X$ ,  $\mathcal{T} = \{U \subseteq X : \text{for all } x \in U, \text{ there exists a } B \in \mathcal{B} \text{ such that } x \in B\}$  is a topology on  $X$ .*

*Proof.* Let  $\mathcal{B}$  be a basis for a topology in  $X$ , and consider  $\mathcal{T}$  as defined above. Clearly,  $\emptyset \in \mathcal{T}$  and so is  $X$ .

Now let  $\{U_\alpha\}$  be a collection of subsets of  $X$ , and let  $U = \bigcup U_\alpha$ . Then if  $x \in U$  for some  $\alpha$ , there is a  $B_\alpha$  such that  $x \in B_\alpha \subseteq U_\alpha$ , thus  $x \in B_\alpha \subseteq U$ .

Now let  $x \in U_1 \cap U_2$ , and choose  $B_1, B_2 \in \mathcal{B}$  such that  $x \in B_1 \subseteq U_1$  and  $x \in B_2 \subseteq U_2$ . Then by definition, there is a  $B_3$  for which  $x \in B_3 \subseteq B_1 \cap B_2$ . Now suppose for arbitrary  $n$ , that  $U = \bigcap_{i=1}^n U_i \in \mathcal{T}$ , for some finite collection  $\{U_i\}$  of subsets of  $X$ . Then by let  $B_n, B_{n+1} \in \mathcal{B}$  such that  $x \in B_n \subseteq U$  and  $x \in B_{n+1} \subseteq U_{n+1}$ . Then by our hypothesis, there is a  $B$  for which  $x \in B \subseteq B_n \cap B_{n+1}$ , thus  $U \cap U_{n+1} = \bigcap_{i=1}^{n+1} U_i \in \mathcal{T}$ . This make  $\mathcal{T}$  a topology on  $X$ . ■

**Example 1.4.** (1) Let  $\mathcal{B}$  be the set of all circular regions in the plane  $\mathbb{R} \times \mathbb{R}$ , then  $\mathcal{B}$  satisfies the conditions needed for a basis.

(2) The collection  $\mathcal{B}'$  in  $\mathbb{R} \times \mathbb{R}$  of all rectangular region also forms a basis for a topology on  $\mathbb{R} \times \mathbb{R}$ .

(3) For any set  $X$ , the set of all 1-point subsets of  $X$  forms a basis for a topology on  $X$ .

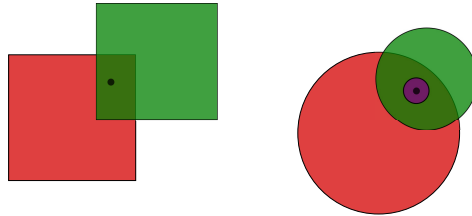


Figure 1.2: The basis for  $\mathcal{B}$  and  $\mathcal{B}'$  in  $\mathbb{R} \times \mathbb{R}$  (see example (2)).

**Lemma 1.2.2.** *Let  $X$  be a set, and  $\mathcal{B}$  be a basis for a topology  $\mathcal{T}$  on  $X$ . Then  $\mathcal{T} = \{\bigcup B : B \in \mathcal{B}\}$ .*

*Proof.* Given a collection  $\{B_i\}_{i=1}^\infty$  of basis elements in  $\mathcal{B}$ , since they are all in  $\mathcal{T}$ , their unions are also in  $\mathcal{T}$ . Conversely, given  $U \in \mathcal{T}$ , then for every point  $x \in U$ , choose a  $B_x \in \mathcal{B}$  such that  $x \in B_x \subseteq U$ , then  $U = \bigcup_{x \in U} B_x$ . ■

**Lemma 1.2.3.** *Let  $(X, \mathcal{T})$  be a topological space, and let  $\mathcal{C} \subseteq \mathcal{T}$  be a collection of open sets of  $X$  such that for every  $x \in U$ , there is a  $C \in \mathcal{C}$  such that  $x \in C \subseteq U$ . Then  $\mathcal{C}$  is the basis for a  $\mathcal{T}$  on  $X$ .*

*Proof.* Take any  $x \in X$ , then there is a  $C \in \mathcal{C}$  such that  $x \in C \subseteq U$ , thus the first condition for a basis is satisfied. Now let  $x \in C_1 \cap C_2$  for  $C_1, C_2 \in \mathcal{C}$ , since  $C_1 \cap C_2$  is open in  $X$ , there is a  $C_3 \in \mathcal{C}$  such that  $x \in C_3 \subseteq C_1 \cap C_2$ . Therefore  $\mathcal{C}$  is a basis for a topology on  $X$ .

Now let  $\mathcal{T}_{\mathcal{C}}$  be the topology generated by  $\mathcal{C}$ , now for  $U \in \mathcal{T}$ , we have by the hypothesis, that  $U \in \mathcal{T}_{\mathcal{C}}$ ; and by lemma 1.2.2,  $W \in \mathcal{T}_{\mathcal{C}}$  is the union of elements of  $\mathcal{C}$ , which is a subcollection of  $\mathcal{T}$ , thus  $W \in \mathcal{T}$ . Therefore  $\mathcal{T}_{\mathcal{C}} = \mathcal{T}$ . ■

**Lemma 1.2.4.** *Let  $\mathcal{B}$  and  $\mathcal{B}'$  be bases for topologies  $\mathcal{T}$  and  $\mathcal{T}'$  on  $X$ . Then the  $\mathcal{T} \subseteq \mathcal{T}'$  if and only if for all  $x \in X$ , and all  $B \in \mathcal{B}$ , there is a  $B' \in \mathcal{B}'$  such that  $x \in B' \subseteq B$ .*

*Proof.* Suppose first that  $\mathcal{T} \subseteq \mathcal{T}'$ , and let  $x \in X$ , and choose  $B \in \mathcal{B}$  such that  $x \in B$ , then  $B$  is open in  $\mathcal{T}$ , thus it is open in  $\mathcal{T}'$ , thus there is a  $B' \in \mathcal{B}'$  such that  $x \in B' \subseteq B$ . Conversely, suppose there is a  $B' \in \mathcal{B}'$  for which  $x \in B' \subseteq B$  for all  $x \in X$ ,  $B \in \mathcal{B}$ . Take  $x \in U \in \mathcal{T}$ , since  $\mathcal{B}$  generates  $\mathcal{T}$ ,  $x \in B \subseteq U$ , since  $B' \subseteq B$ , this implies that  $U \in \mathcal{T}'$  and  $\mathcal{T} \subseteq \mathcal{T}'$ . ■

**Definition.** If  $\mathcal{B}$  is the collection of open intervals  $(a, b)$  in  $\mathbb{R}$ , we call the topology generated by  $\mathcal{B}$  the **standard topology** on  $\mathbb{R}$ , and we denote it simply by  $\mathbb{R}$ .

**Definition.** If  $\mathcal{B}$  is the collection of half open intervals  $[a, b)$  in  $\mathbb{R}$ , we call the topology generated by  $\mathcal{B}$  the **lower limit topology** on  $\mathbb{R}$ , and we denote it simply by  $\mathbb{R}_l$ . If  $\mathcal{B}'$  is the collection of all half open intervals  $(a, b]$  in  $\mathbb{R}$ , then we call the topology generated by  $\mathcal{B}'$  the **upper limit topology** on  $\mathbb{R}$ , and denote it  $\mathbb{R}_L$ .

**Definition.** If  $\mathcal{B}$  is the collection of all open intervals of the form  $(a, b) \setminus \frac{1}{\mathbb{Z}^+}$ , where  $\frac{1}{\mathbb{Z}^+} = \{\frac{1}{n} : n \in \mathbb{Z}^+\}$ , we call the topology generated by  $\mathcal{B}$  the  **$\frac{1}{\mathbb{Z}^+}$ -topology** on  $\mathbb{R}$ , and we denote it  $\mathbb{R}_{\frac{1}{\mathbb{Z}^+}}$ .

**Lemma 1.2.5.** *The topologies  $\mathbb{R}_l$ ,  $\mathbb{R}_L$ , and  $\mathbb{R}_{\frac{1}{\mathbb{Z}^+}}$  are all strictly finer than  $\mathbb{R}$ , but are not comparable with each other.*

*Proof.* Let  $(a, b)$  be a basis element for  $\mathbb{R}$ , and let  $x \in (a, b)$ , the basis element  $[x, b) \in \mathbb{R}_l$  lies in  $(a, b)$  and contains  $x$ , however, there can be no interval  $(a, b)$  in  $[x, b)$  as  $x \leq a$ , thus  $\mathbb{R} \subset \mathbb{R}_l$ ; a similar argument holds for  $\mathbb{R}_L$ .

Similarly, for  $(a, b) \in \mathbb{R}$ , the basis element  $(a, b) \setminus \frac{1}{\mathbb{Z}^+}$  of  $\mathbb{R}_{\frac{1}{\mathbb{Z}^+}}$  lies in  $(a, b)$ , however, choose the basis  $B = (-1, 1) \setminus \frac{1}{\mathbb{Z}^+}$ , and choose  $0 \in B$ , since  $\mathbb{Z}^+$  is dense in  $\mathbb{R}$ , there is no interval  $(a, b)$  containing 0 and lying in  $B$ , thus  $\mathbb{R} \subseteq \mathbb{R}_{\frac{1}{\mathbb{Z}^+}}$ .

Now choose  $[0, 1)$  in  $\mathbb{R}_l$ , and choose  $\frac{1}{k} \in [0, 1)$  such that  $k \in \mathbb{Z}^+$ . Now  $(0, 1) \subseteq [0, 1)$ , so we cannot say that  $[0, 1)$  is a basis for  $\mathbb{R}$ , and moreover,  $[0, 1) \setminus \frac{1}{\mathbb{Z}^+}$  cannot be said to be a basis in  $\mathbb{R}_{\frac{1}{\mathbb{Z}^+}}$ , thus  $\mathbb{R}_l$  and  $\mathbb{R}_{\frac{1}{\mathbb{Z}^+}}$  are incomparable, a similar argument holds for  $\mathbb{R}_L$ .

Lastly, let  $(a, b)$  be in  $\mathbb{R}$  and choose  $x \in (a, b)$ . Then  $(a, x]$  and  $[x, b)$  are both in  $(a, b)$ , however it is clear that  $(a, x]$  and  $[x, b)$  cannot be contained in each other, thus  $\mathbb{R}_l$  and  $\mathbb{R}_L$  are incomparable. ■

**Definition.** A **subbasis**,  $\mathcal{S}$ , for a topology on  $X$  is a collection of subsets of  $X$  whose union equals  $X$ . We call the **topology generated by  $\mathcal{S}$**  to be the collection of all unions of finite intersections of elements of  $\mathcal{S}$ , that is:

$$\mathcal{T} = \left\{ \bigcup_{i=1}^n S_i : S_i \in \mathcal{S} \text{ for } 1 \leq i \leq n \right\}$$

**Theorem 1.2.6.** *Let  $\mathcal{S}$  be a subbasis for a topology on  $X$ . Then the collection  $\mathcal{T} = \{\bigcup \bigcap_{i=1}^n S_i : S_i \in \mathcal{S} \text{ for } 1 \leq i \leq n\}$  is a topology on  $X$ .*

*Proof.* It is sufficient to show that the collection  $\mathcal{B}$  of all finite intersections of elements of  $\mathcal{S}$  is a basis for a topology on  $X$ . By lemma 1.2.1, for  $x \in X$ , it belongs to an element  $S$  of  $\mathcal{S}$ , and therefore, to an element of  $\mathcal{B}$ . Now let  $B_1 = \bigcap_{i=1}^m S_i$  and  $B_2 = \bigcap_{j=1}^n S'_j$  be basis elements of  $\mathcal{B}$ . The intersection  $B_1 \cap B_2$  is a finite intersection of elements of  $\mathcal{S}$ , and hence also belongs in  $\mathcal{B}$ , and hence we can take another basis element  $B_3$  such that  $x \in B_3 \subseteq B_1 \cap B_2$ . ■

## 1.3 The Order Topology.

**Definition.** Let  $X$  be a set with a simple order relation, and suppose that  $|X| > 1$ . Let  $\mathcal{B}$  be the collection of sets of the following forms:

- (1) All open intervals  $(a, b) \in X$ .
- (2) All half open intervals  $[a_0, b)$  where  $a_0$  is the least element (if any) of  $X$ .
- (3) All half open intervals of the form  $(a, b_0]$  where  $b_0$  is the greatest element (if any) of  $X$ .

Then  $\mathcal{B}$  forms the basis for a topology on  $X$  called the **order topology**

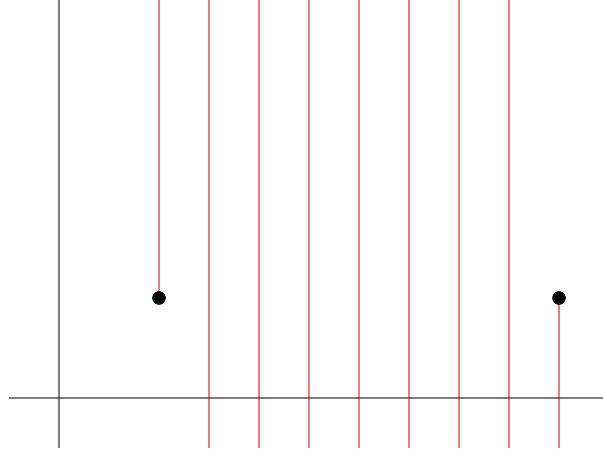
**Theorem 1.3.1.** *The collection  $\mathcal{B}$  forms a basis.*

*Proof.* Consider  $x \in X$ , if  $x$  is the least element of  $X$ , then it lies in all intervals of type (2), if it is the largest, then it lies in all intervals of type (3). If  $x$  is neither the least nor largest element, then  $x \in (a_0, b_0)$  with  $a_0$  and  $b_0$  the least and largest elements (if any) of  $X$ . If no such elements exist, then  $x \in (a, b)$ , for some lowerbound  $a$  and upperbound  $b$ . Thus, in all three cases, there is a basis element containing  $x$ .

Now suppose  $B_1, B_2 \in \mathcal{B}$  such that  $x \in B_1 \cap B_2$ . If  $B_1$  and  $B_2$  are both of type (1), then let  $B_1 = (a, b)$ ,  $B_2 = (c, d)$ , then  $B_1 \cap B_2$  is an open interval of type (1), now fix  $B_1$  to be of type one. If  $B_2$  is of type (2), then letting  $B_2 = [a_0, c)$ , then  $x \in [a_0, d)$  for some  $d \in X$ . Likewise, if  $B_2 = (c, b_0]$ , is of type (3), we get a similar result. Moreover, the results are analogous if we fix  $B_2$  and let  $B_1$  range between intervals of the three types. Thus in all cases, there is a  $B_3 \in \mathcal{B}$  such that  $x \in B_3 \subseteq B_1 \cap B_2$ . ■

**Example 1.5.** (1) The standard topology on  $\mathbb{R}$  is the order topology on  $\mathbb{R}$  induced by the usual order relation. We have that  $\mathbb{R}$  under this topology has no intervals of type (2), nor (3), so all bases elements in the standard topology are open intervals in  $\mathbb{R}$ .

- (2) Consider the dictionary order on  $\mathbb{R} \times \mathbb{R}$ . Since  $\mathbb{R} \times \mathbb{R}$  has no intervals of type (2), nor (3), the bases of  $\mathbb{R} \times \mathbb{R}$  under the dictionary order are the open intervals of the form  $(a \times b, c \times d)$  Where  $a \leq c$ , and  $b < d$ .
- (3) The positive integers  $\mathbb{Z}^+$  with the least element 1 form an ordered set under the usual order. Taking  $n > 1$ , we see the bases of  $\mathbb{Z}^+$  under the order topology are of the form  $(n-1, n+1) = \{n\}$  and  $[1, n) = \{1, \dots, n-1\}$ . Thus the order topology on  $\mathbb{Z}^+$  is the discrete topology.

Figure 1.3: The order topology on  $\mathbb{R} \times \mathbb{R}$ .

- (4) The set  $X = \{1, 2\} \times \mathbb{Z}^+$  over the dictionary order is also an ordered set, with the least element  $1 \times 1$ . Denote  $1 \times n$  as  $a_n$  and  $2 \times n$  as  $b_n$ . Then  $X$  consist of the elements  $a_1, a_2, \dots, b_1, b_2, \dots$ .

Now take  $\{b_1\}$ , then any open set containing  $b_1$  must have a basis about  $b_1$ , and also contains points  $a_i$  with  $i \in \mathbb{Z}^+$ ; thus the order topology on  $X$  is not the discrete topology.

**Definition.** Let  $X$  be an ordered set, and let  $a \in X$ . There are two subsets in  $X$ ,  $(a, \infty) = \{x \in X : x > a\}$  and  $(-\infty, a) = \{x \in X : x < a\}$  called **open rays** of  $X$ . There are also two sets  $[a, \infty) = \{x \in X : x \geq a\}$  and  $(-\infty, a] = \{x \in X : x \leq a\}$  called **closed rays** of  $X$ .

**Theorem 1.3.2.** Let  $X$  be an ordered set. Then the collection of all open rays in  $X$  form a subbasis for the order topology on  $X$ .

*Proof.* Let  $\mathcal{S}$  be the collection of all open rays of  $X$ , let  $a < b$  and  $(a, \infty), (-\infty, b) \in \mathcal{S}$ , then  $(a, b) = (a, \infty) \cap (-\infty, b)$ . Now take:

$$S = \bigcup_{a, b \in X} (a, b)$$

then  $S \subseteq X$ , likewise, since  $S$  runs through all intersections of open rays of  $X$ , it contains all open intervals in  $X$ , hence  $X \subseteq S$ , and so  $X = S$  as required. ■

## 1.4 The Product Topology.

**Definition.** Let  $X$  and  $Y$  be topological spaces. We define the **product topology** on  $X \times Y$  to be the topology having as basis the collection

$$\mathcal{B} = \{U \times V \subseteq X \times Y : U \text{ is open in } X \text{ and } V \text{ is open in } Y\}$$

**Theorem 1.4.1.** The collection  $\mathcal{B} = \{U \times V \subseteq X \times Y : U \text{ is open in } X \text{ and } V \text{ is open in } Y\}$  forms a basis for the product topology on  $X \times Y$ .



*Proof.* Clearly, we have that  $X \times Y$  is a basis element of  $\mathcal{B}$ . Now take  $U_1 \times V_1$  and  $U_2 \times V_2$  in  $\mathcal{B}$ . Since  $U_1 \times V_1 \cap U_2 \times V_2 = U_1 \cap U_2 \times V_1 \cap V_2$ , since  $U_1 \cap U_2$  and  $V_1 \cap V_2$  are open in  $X$  and  $Y$  respectively, then we have that  $U_1 \times V_1 \cap U_2 \times V_2$  is a basis element as well. ■

**Theorem 1.4.2.** *If  $\mathcal{B}$  is the basis for a topology on  $X$ , and  $\mathcal{C}$  is the basis for a topology on  $Y$ , then the collection:*

$$\mathcal{D} = \{B \times C : B \in \mathcal{B} \text{ and } C \in \mathcal{C}\}$$

*Is a basis for the topology on  $X \times Y$ .*

*Proof.* By lemma 1.2.3, let  $W$  be an open set of  $X \times Y$ , and let  $x \times y \in W$ . Then there is a basis  $U \times V$  such that  $x \times y \in U \times V \subseteq W$ . Since  $\mathcal{B}$  and  $\mathcal{C}$  are bases of  $X$  and  $Y$  respectively, choosing  $B \in \mathcal{B}$  and  $C \in \mathcal{C}$ , we have that  $x \in B \subseteq U$ , and  $y \in C \subseteq V$ , thus  $x \times y \in B \times C \subseteq U \times V \subseteq W$ . Therefore,  $\mathcal{D}$  is the basis for a topology on  $X \times Y$ . ■

**Example 1.6.** The product of the standard topology on  $\mathbb{R}$  with itself is called the **standard topology on  $\mathbb{R} \times \mathbb{R}$** , and has as basis the collection of all products of open sets in  $\mathbb{R}$ . By theorem 1.4.2, if we take the collection of all open intervals  $(a, b) \times (c, d)$  in  $\mathbb{R} \times \mathbb{R}$ , we form a basis. Constructing this basis geometrically gives the interior of a rectangle, whose boundaries are the intervals  $(a, b)$  and  $(c, d)$ .

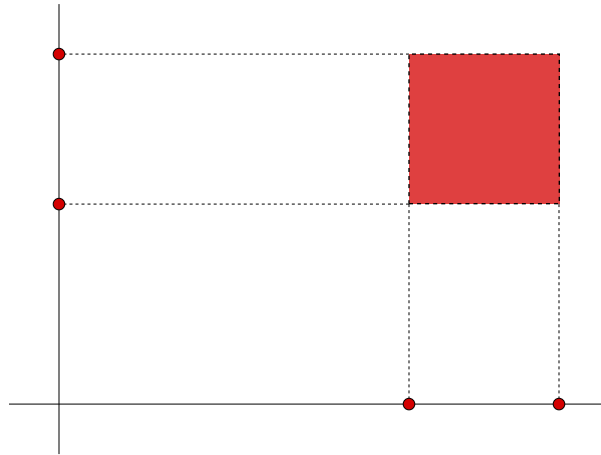


Figure 1.4: A basis element for  $\mathbb{R} \times \mathbb{R}$

**Definition.** Let  $\pi_1 : X \times Y \rightarrow X$  be defined such that  $\pi_1(x, y) = x$ , and define  $\pi_2 : X \times Y \rightarrow Y$  such that  $\pi_2(x, y) = y$ . We call  $\pi_1$  and  $\pi_2$  **projections** of  $X \times Y$  onto its first and second **factors**; that is onto  $X$  and  $Y$ , respectively.

Clearly,  $\pi_1$  and  $\pi_2$  are both onto. Now let  $U$  be open in  $X$ , then  $\pi_1^{-1}(U) = U \times Y$  is open in  $X \times Y$ ; similarly,  $\pi_2^{-1}(V) = X \times V$  is also open in  $X \times Y$ , for  $V$  open in  $Y$ .

**Example 1.7.** The maps  $\pi_1 : x \times y \rightarrow x$  and  $\pi_2 : x \times y \rightarrow y$  of  $X \times Y$  onto  $X$  and  $Y$ , respectively, are what we call “open maps”. Let  $U$  and  $V$  be open in  $X$  and  $Y$  respectively, then  $U \times V$  is open in  $X \times Y$ , and for every  $x \times y \in U \times V$ , we have  $\pi_1(x \times y) = x$  and  $\pi_2(x \times y) = y$ , so that  $\pi_1(U \times V) = U$  and  $\pi_2(U \times V) = V$  are open in  $X$  and  $Y$  respectively.

**Theorem 1.4.3.** *The collection  $\mathcal{S} = \{\pi_1^{-1}(U) : U \text{ is open in } X\} \cup \{\pi_2^{-1}(V) : V \text{ is open in } Y\}$  is a subbasis for the product topology on  $X \times Y$ .*

*Proof.* Let  $\mathcal{T}$  be the product topology on  $X \times Y$ , and let  $\mathcal{T}'$  be the topology generated by  $\mathcal{S}$ . Since every element of  $\mathcal{S}$  is open in  $\mathcal{T}$ ,  $\mathcal{T} \subseteq \mathcal{T}'$ . Conversely, consider the basis element  $U \times V$  of  $\mathcal{T}$ , then  $\pi_1^{-1}(U) \cap \pi_2^{-1}(V) = U \times Y \cap X \times V = U \times V$ , thus  $\mathcal{T} \subseteq \mathcal{T}'$ . Therefore,  $\mathcal{S}$  is a subbasis for the product topology. ■

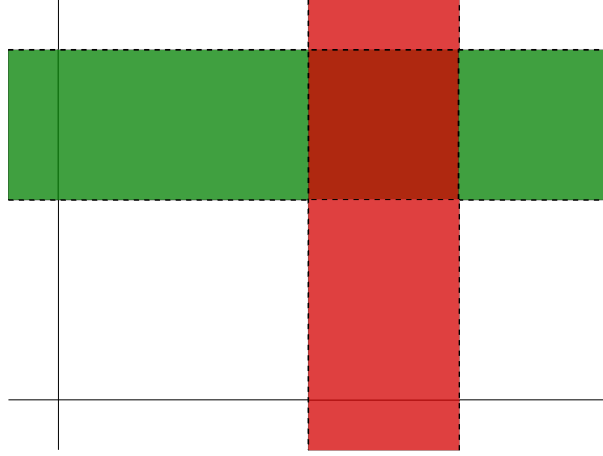


Figure 1.5: The inverse images,  $\pi_1^{-1}(U)$  and  $\pi_2^{-1}(V)$ , of the projections  $\pi_1$  and  $\pi_2$  onto the  $X \times Y$  plane.

## 1.5 The Subspace Topology.

**Theorem 1.5.1.** *Let  $X$  be a topological space with topology  $\mathcal{T}$ , and let  $Y \subseteq X$ . Then the collection:*

$$\mathcal{T}_Y = \{Y \cap U : U \in \mathcal{T}\}$$

*forms a topology on  $Y$ .*

*Proof.* Clearly,  $Y \cap \emptyset = \emptyset \in \mathcal{T}_Y$  and  $Y \cap X = Y \in \mathcal{T}_Y$ . Now consider the collection  $\{U_\alpha\}$ . Then  $\bigcup (Y \cap U_\alpha) = Y \cap \bigcup U_\alpha$ , similarly, for  $\{U_i\}_{i=1}^n$ ,  $\bigcap (Y \cap U_i) = Y \cap \bigcap U_i$ , hence  $\mathcal{T}$  is a topology on  $Y$ . ■

**Definition.** Let  $X$  be a topological space, and let  $Y \subseteq X$ . We call the  $\mathcal{T}$  defined in theorem 1.5.1 the **subspace topology** on  $Y$ . We say that  $U \subseteq Y$  is **open in  $Y$**  if  $U \in \mathcal{T}_Y$ .

**Lemma 1.5.2.** *Let  $\mathcal{B}$  be the basis for a topology on  $X$ . Then the collection  $\mathcal{B}_Y = \{B \cap Y : B \in \mathcal{B}\}$ , where  $Y \subseteq X$ , is a basis for the subspace topology on  $Y$ .*

*Proof.* Let  $U$  be open in  $X$ , and let  $y \in Y \cap U$ , and choose  $B \in \mathcal{B}$  such that  $y \in B \subseteq U$ , then  $y \in B \cap Y \subseteq U \cap Y$ , then by lemma 1.2.2,  $\mathcal{B}_Y$  is the basis for the subspace topology on  $Y$ . ■

**Lemma 1.5.3.** *Let  $Y$  be a subspace of  $X$ , If  $U \subseteq Y$  is open in  $Y$  and  $Y$  is open in  $X$ , then  $U$  is open in  $X$ .*

*Proof.* Let  $U \in \mathcal{T}_Y$ , then for some  $V \subseteq X$ ,  $U = Y \cap V$ . Now since  $Y$  is open in  $X$ , and so is  $V$ , then it follows that  $U$  is also open in  $X$ . ■

*Remark.* What this lemma says is that given a topological space  $X$ , and a subspace  $Y$  of  $X$ , then the subspace topology of  $Y$  is coarser than the topology on  $X$ , i.e.  $\mathcal{T}_Y \subseteq \mathcal{T}$ .

**Theorem 1.5.4.** *If  $A$  is a subspace of  $X$ , and  $B$  is a subspace of  $Y$ , then the product topology on  $A \times B$  is the topology that  $A \times B$  inherits as a subspace of  $X \times Y$ .*

*Proof.* We have that  $U \times V$  is the basis element for  $X \times Y$ , with  $U$  open in  $X$ , and  $V$  open in  $Y$ . Thus  $(U \times V) \cap (A \times B) = (U \cap A) \times (V \cap B)$  is a basis element for the subspace topology on  $X \times Y$ . Since  $U \cap A$  and  $V \cap B$  are open in the subspace topologies of  $A$  and  $B$  respectively, then  $(U \cap A) \times (V \cap B)$  is a basis for the product topology on  $A \times B$ . ■

**Example 1.8.** (1) Consider  $[0, 1] \subseteq \mathbb{R}$ . In the subspace topology of  $[0, 1]$ , we have as basis elements of the form  $(a, b) \cap [0, 1]$ , with  $(a, b) \subseteq \mathbb{R}$ . If we have that  $(a, b) \subseteq [0, 1]$ , then  $(a, b) \cap [0, 1] = (a, b)$ . On the other hand, if  $a \in [0, 1]$  or  $b \in [0, 1]$ , then we get  $(a, b) \cap [0, 1] = (a, 1]$  or  $(a, b) \cap [0, 1] = [0, b)$ , lastly if neither  $a$  nor  $b$  are in  $[0, 1]$ , then we have  $(a, b) \cap [0, 1] = [0, 1]$  only if  $[0, 1] \subseteq (a, b)$ , and  $(a, b) \cap [0, 1] = \emptyset$  otherwise.

The only one of these sets open in  $\mathbb{R}$  under the standard topology is  $(0, 1)$ .

(2) For  $[0, 1) \cup \{2\} \subseteq \mathbb{R}$ , the singleton  $\{2\}$  is open in the subspace topology on  $[0, 1) \cup \{2\}$ ; for observe, that  $(\frac{3}{5}, \frac{5}{2}) \cap ([0, 1) \cup \{2\}) = \{2\}$ , however, in the order topology, on that same set,  $\{2\}$  is not open. Any basis element on  $[0, 1) \cup \{2\}$  containing 2 is of the form  $(a, 2]$ , where  $a \in [0, 1) \cup \{2\}$ .

(3) The dictionary order on  $[0, 1] \times [0, 1]$  is a restriction of the dictionary order on  $\mathbb{R} \times \mathbb{R}$ . Now the set  $\{\frac{1}{2}\} \times (\frac{1}{2}, 1]$  is open in the subspace topology on  $[0, 1] \times [0, 1]$ , but it is not open in the dictionary order on the same set.

**Definition.** We call the set  $[0, 1] \times [0, 1]$  on the dictionary order the **ordered square**, and we denote it by  $I_0^2$ .

**Lemma 1.5.5.** *Let  $Y$  be a subspace of a topological space  $X$ , and let  $A \subseteq Y$ . Then the topology of  $A$  as a subspace of  $Y$  is the same as the topology of  $A$  as a subspace of  $X$ .*

*Proof.* Let  $V$  be open in  $Y$ , then  $V \cap A$  is open in  $A$  as a subspace of  $Y$ , however, since  $Y$  is a subspace of  $X$ ,  $V = U \cap Y$  for some  $U$  open in  $X$ , so  $V \cap A = (U \cap Y) \cap A = U \cap (A \cap Y) = U \cap A$ , making  $V \cap A$  open in  $A$  as a subspace of  $X$ .

Conversely, if  $U$  is open in  $X$ , then  $U \cap A$  is open in  $A$  as a subspace of  $X$ , additionally,  $U \cap Y$  is open in  $Y$  as a subspace of  $Y$ , so that  $(U \cap Y) \cap A = U \cap (A \cap Y) = U \cap A$ , which makes  $U \cap A$  open in  $A$  as a subspace of  $Y$ . ■

**Example 1.9.** Suppose that  $\mathcal{T}$  and  $\mathcal{T}'$  are topologies on a set  $X$ , with  $\mathcal{T} \subseteq \mathcal{T}'$ . Then if  $Y \subseteq X$ , then the topology of  $Y$  as a subspace of  $X$  under  $\mathcal{T}'$  is finer than the topology of  $Y$  as a subspace of  $X$  under  $\mathcal{T}$ ; for, if  $U$  is open in  $X$  under  $\mathcal{T}$ , then  $U \cap Y$  is open in  $Y$  under  $\mathcal{T}$ . But  $\mathcal{T} \subseteq \mathcal{T}'$ , which implies that  $U \cap Y$  is open in  $Y$  under  $\mathcal{T}'$ .

**Definition.** Let  $X$  be an ordered set. We say that a nonempty subset  $Y \subset X$  is **convex** in  $X$  if for each pair of points  $a, b \in Y$ , with  $a < b$ , then the open interval  $(a, b) \subseteq X$  is also contained in  $Y$ .

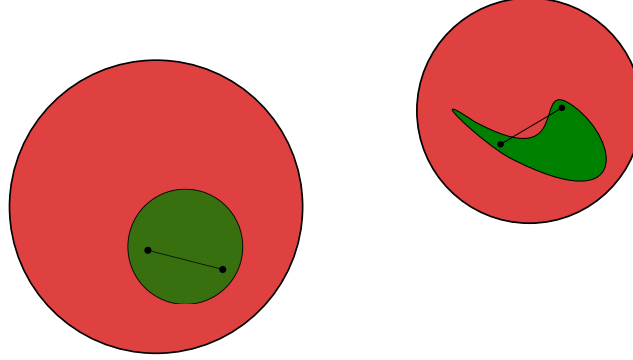


Figure 1.6: A convex set, and a nonconvex set.

**Example 1.10.** Let  $X$  be any ordered set. Then by definition, all open intervals and rays in  $X$  are convex in  $X$ .

**Theorem 1.5.6.** Let  $X$  be an ordered set on the order topology, and let  $Y \subseteq X$  be convex in  $X$ . Then the order topology on  $Y$  is the same as the subspace topology on  $Y$ .

*Proof.* Consider  $(a, \infty) \subseteq X$ . If  $a \in Y$ , then  $(a, \infty) \cap Y = \{x \in Y : x > a\}$ , which is by definition an open ray on  $Y$ . Now if  $a \notin Y$ , then  $a$  is either a lowerbound, or an upperbound. Then  $(a, \infty) \cap Y = \emptyset$  and  $(-\infty, a) \cap Y = Y$  if  $a$  is an upperbound, similarly, if  $a$  is a lowerbound we get  $(a, \infty) \cap Y = Y$  and  $(-\infty, a) \cap Y = \emptyset$ .

Since  $(a, \infty) \cap Y$  and  $(-\infty, a) \cap Y$  form a subbasis on the subspace topology on  $Y$ , and since they are also open in the order topology, then the order topology contains the subspace topology.

Now if  $(a, \infty)$  is an open ray in  $Y$ , then  $(a, \infty) = (b, \infty) \cap Y$ , with  $(b, \infty)$  some open ray in  $X$ , hence  $(a, \infty)$  is open in the subspace topology of  $Y$ , and since it also forms the subbasis for the order topology, we have that the order topology is contained within the subspace topology. Thus both topologies are equal. ■

## 1.6 Closed Sets and Limit Points.

**Definition.** A subset  $A$  of a topological space  $X$  is said to be **closed** if  $X \setminus A$  is open.

**Example 1.11.** (1) Consider  $[a, b] \subseteq \mathbb{R}$ , we have that  $\mathbb{R} \setminus [a, b] = (-\infty, a) \cup (b, \infty)$  which is open in  $\mathbb{R}$ . So  $[a, b]$  is closed.

(2) In  $\mathbb{R} \times \mathbb{R}$ , the set  $A = \{x \times y : x, y \geq 0\}$  (i.e the first quadrant of the plane) is closed, for  $\mathbb{R} \times \mathbb{R} \setminus A = ((-\infty, 0) \times \mathbb{R}) \cup (\mathbb{R} \times (-\infty, 0))$ , which is open in  $\mathbb{R} \times \mathbb{R}$ .

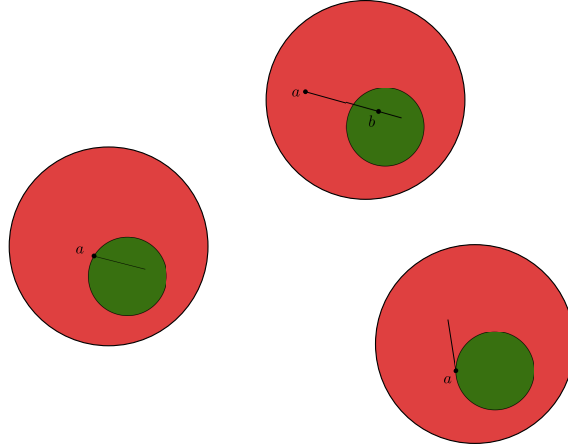


Figure 1.7: An illustration of theorem 1.5.6.

- (3) Consider the finite complement topology  $\mathcal{T}_f$  on a set  $X$ . We have that  $X \setminus X = \emptyset \in \mathcal{T}_f$ , so  $X$  is closed, similarly,  $\emptyset$  is also closed. Likewise, if  $A \subseteq X$  is a finite set, then  $X \setminus A$  is also finite, and hence  $A$  is also closed. Thus, we have that all the closed sets of  $\mathcal{T}_f$  are those finite subsets of  $X$ . As a consequence, this example also illustrates that sets can be both closed and open.
- (4) In the discrete topology  $2^X$ , every open set is closed. This is another example where open sets are also closed sets.
- (4) Consider  $[0, 1] \cup (2, 3)$  in the subspace topology on  $\mathbb{R}$ . We have that  $[0, 1]$  is open in the subspace topology on  $\mathbb{R}$ ;  $[0, 1] = [0, 1] \cup (2, 3) \cap (-\frac{2}{3}, \frac{3}{2})$ , similarly,  $(2, 3)$  is also open. Now taking  $[0, 1] \cup (2, 3) \setminus (2, 3) = [0, 1]$ , which is open, so  $[0, 1]$  is closed in the subspace topology on  $\mathbb{R}$ , by the same reasoning, so is  $(2, 3)$ .

**Theorem 1.6.1.** *Let  $X$  be a topological space. Then:*

- (1)  $\emptyset$  and  $X$  are closed.
- (2) Arbitrary intersections of closed sets are closed.
- (3) Finite unions of closed sets are closed.

*Proof.* We have that  $X \setminus \emptyset = X$  and  $X \setminus X = \emptyset$ , both of which are open in  $X$ , so they are also closed in  $X$ . Now let  $\{U_\alpha\}$  be a collection of closed sets of  $X$ . We have that:

$$X \setminus \bigcap_{\alpha} U_{\alpha} = \bigcup_{\alpha} X \setminus U_{\alpha}.$$

Similarly, for  $\{U_i\}_{i=1}^n$ , we have

$$X \setminus \bigcup_{i=1}^n U_i = \bigcap_{i=1}^n X \setminus U_i.$$

Both of which are open in  $X$ . This completes the proof. ■

**Example 1.12.** For the following examples, let  $X$  and  $Y$  be topological spaces.

- (1) If  $Y$  is a subspace of  $X$ , and  $A$  is closed in  $Y$ , and  $Y$  is closed in  $X$ , then by theorem 1.6.1,  $A = U \cap Y$ , where  $U$  is closed in  $X$ . Then by theorem 1.6.1, we get that  $A$  is closed in  $X$ .
- (2) Suppose that  $A$  is closed in  $X$  and  $B$  is closed in  $Y$ . Then  $X \setminus A$  and  $Y \setminus B$  are open in  $X$  and  $Y$ , respectively, so  $(X \setminus A) \times (Y \setminus B) = (X \times Y) \setminus (A \times B)$  is open in  $X \times Y$ . Therefore  $A \times B$  is closed in  $X \times Y$ .

**Definition.** If  $Y$  is a subspace of  $X$ , we say that  $A$  is **closed in  $Y$**  if  $A \subseteq Y$  and  $A$  is closed in the subspace topology on  $Y$ .

**Theorem 1.6.2.** *Let  $Y$  be a subspace of  $X$ . Then  $A$  is closed in  $Y$  if and only if  $A$  equals the intersection of a closed set of  $X$  with  $Y$ .*

*Proof.* Suppose that  $A$  is closed in  $Y$ , then  $Y \setminus A$  is open in  $Y$ , hence we have that  $Y \setminus A = U \cap Y$  for some open set  $U$  of  $X$ . Now  $X \setminus U$  is closed in  $X$ , and with  $A \subseteq Y$ , we have that  $A = Y \cap (X \setminus U)$ .

Conversely, suppose that  $A = C \cap Y$ , with  $C$  closed in  $X$ . Then  $X \setminus C$  is open in  $X$ , hence  $(X \setminus C) \cap Y$  is open in  $Y$ , now since  $(X \setminus C) \cap Y = Y \setminus A$ , which is open, we have that  $A$  is closed in  $Y$ . ■

**Theorem 1.6.3.** *Let  $Y$  be a subspace of  $X$ . If  $A$  is closed in  $Y$ , and  $Y$  is closed in  $X$ , then  $A$  is closed in  $X$ ; that is, closure is transitive.*

*Proof.* By theorem 1.6.2, if  $A$  is closed in  $Y$ , then  $A = C \cap Y$  with  $C$  closed in  $X$ , now since  $Y$  is closed in  $X$ , then  $Y = D \cap X$  with  $D$  closed in  $X$ . Thus  $A = (C \cap D) \cap X$ , therefore,  $A$  is closed in  $X$ . ■

We now go over the concepts of the closure, and the interior of a set.

**Definition.** Let  $A \subseteq X$ , with  $X$  a topological space. The **interior** of  $A$  is defined to be the union of all open sets in  $A$ . The **closure** of  $A$  is defined to be the intersection of all closed sets containing  $A$ . We denote the interior and the closure of  $A$  as  $\text{Int } A$  and  $\text{cl } A$  respectively

We have by the very definitions that  $\text{Int } A \subseteq A \subseteq \text{cl } A$

**Lemma 1.6.4.**  *$\text{Int } A = A$  only when  $A$  is open, and  $\text{cl } A = A$  only when  $A$  is closed.*

*Proof.* Now, if  $A$  is open, then it is in the union of all open sets of  $A$ , hence  $A \subseteq \text{Int } A$ , likewise, if  $A$  is closed, then since  $\text{cl } A$  is the intersection of all closed sets containing  $A$ , we get  $\text{cl } A \subseteq A$ . ■

**Corollary.**  *$A$  is closed and open if and only if  $\text{Int } A = \text{cl } A$ .*

**Theorem 1.6.5.** *Let  $Y$  be a subspace of  $X$ , and let  $A \subseteq Y$ . Then  $\text{cl } A \cap Y$  is the closure of  $A$  in  $Y$ .*

*Proof.* Let  $\text{cl } A$  be the closure of  $A$  in  $X$ . Since  $\text{cl } A$  is closed in  $X$ , by theorem 1.6.2,  $\text{cl } A \cap Y$  is closed in  $Y$ , now we have that  $A \subseteq \text{cl } A \cap Y$ , and since  $\text{cl } A = \bigcap U$ , then  $\text{cl } A \subseteq \text{cl } A \cap Y$ .

Conversely, suppose that  $\text{cl } A$  is closed in  $Y$ , again by theorem 1.6.2, we have that  $\text{cl } A = C \cap Y$ , where  $C$  is closed in  $X$ , since  $A \subseteq \text{cl } A$ , then  $A \subseteq C$ , and since  $C$  is closed, then  $\text{cl } A \subseteq C$ , thus  $\text{cl } A \cap Y \subseteq \text{cl } A$ . ■

**Definition.** Let  $X$  be a topological space, and let  $x \in X$ . We call an open set  $U$  of  $X$  a **neighborhood** of  $x$  if  $x \in U$ .

**Theorem 1.6.6.** *If  $A \subseteq X$ , with  $X$  a topological space, then  $\text{cl } A$  is a neighborhood of  $x \in X$  if and only if for every neighborhood  $U$  of  $x$ ,  $A \cap U \neq \emptyset$ .*

*Proof.* We prove the contrapositive. If  $x \notin \text{cl } A$ , then  $U = X \setminus \text{cl } A$  is an open set containing  $x$ , disjoint from  $A$ . Conversely, suppose there is a neighborhood  $U$  of  $x$ , with  $U$  disjoint from  $A$ , then  $X \setminus U$  is closed, and therefore contains the closure of  $A$ , thus  $x \notin \text{cl } A$ . ■

**Corollary.**  *$\text{cl } A$  is a neighborhood of  $x$  if and only if for every basis element  $B$  of  $X$ , containing  $x$ , intersects  $A$ .*

*Proof.* This is a direct application of theorem 1.6.6, since basis elements are open sets. ■

**Example 1.13.** (1) We have the closure of  $(0, 1]$  in  $\mathbb{R}$  is the closed interval  $[0, 1]$ , since every neighborhood of 0 intersects  $(0, 1]$ . Now every point outside of  $[0, 1]$  has a neighborhood disjoint from  $[0, 1]$  (take the neighborhood  $(2, 3)$  of 2).

$$(2) \text{cl } \frac{1}{\mathbb{Z}^+} = \{0\} \cup \frac{1}{\mathbb{Z}^+} \text{ and } \text{cl } \{0\} \cup (1, 2) = \{0\} \cup [1, 2].$$

$$(3) \text{cl } \mathbb{Q} = \mathbb{R}, \text{cl } \mathbb{Z}^+ = \mathbb{Z}^+, \text{cl } \mathbb{R}^+ = \mathbb{R}^+ \cup \{0\}. \text{ This first follows from the density of } \mathbb{Q} \text{ in } \mathbb{R}. \text{ Every neighborhood } n \in \mathbb{Z}^+ \text{ intersects } \mathbb{Z}^+, \text{ so } \text{cl } \mathbb{Z}^+ \subseteq \mathbb{Z}^+, \text{ and we have that the neighborhood } (0, 1) \text{ of } 0 \text{ intersects } \mathbb{R}^+, \text{ so } \text{cl } \mathbb{R}^+ \subseteq \mathbb{R}^+ \cup \{0\}.$$

**Definition.** If  $A \subseteq X$ , with  $X$  a topological space, and if  $x \in X$ , we say that  $x$  is a **limit point** of  $A$  if every neighborhood of  $x$  intersects  $A$  at some distinct point. That is:  $x \in \text{cl } X \setminus \{x\}$ .

**Example 1.14.** (1) Consider  $(0, 1]$ , we have that  $0 \in [0, 1] = \text{cl } (0, 1] = \text{cl } [0, 1] \setminus \{0\}$ , so 0 is a limit point of  $(0, 1]$ , the same can be said for any  $x \in (0, 1]$ .

(2) For  $\frac{1}{\mathbb{Z}^+}$ , 0 is once again a limit point. Let  $x \in \mathbb{R}$  be nonzero, and let  $[x, b)$  be the neighborhood of  $x$  in the lower limit topology. Then  $[x, b) \cap \frac{1}{\mathbb{Z}^+} = \emptyset$  or  $\{x\}$ , hence, 0 is the only limit point of  $\frac{1}{\mathbb{Z}^+}$ .

(3)  $\text{cl } \{0\} \cup (1, 2) = \{0\} \cup [1, 2]$  has all of its limit points in  $[1, 2]$ . Likewise, every point in  $\mathbb{R}$  is a limit point of  $\mathbb{Q}$ .  $\mathbb{Z}^+$  has no limit points in  $\mathbb{R}$ , and the limit points of  $\mathbb{R}^+$  are all the points of  $\text{cl } \mathbb{R}^+$ .

**Theorem 1.6.7.** *Let  $A \subseteq X$ ,  $X$  a topological space, and let  $A'$  be the set of all limit points in  $A$ . Then  $\text{cl } A = A \cup A'$ .*

*Proof.* Let  $x \in A'$ , then every neighborhood of  $x$  intersects  $A$  at some distinct point  $x'$ , by definition, so by theorem 1.6.6,  $x \in \text{cl } A$ , hence  $A' \subseteq \text{cl } A$ , so  $A \cup A' \subseteq \text{cl } A$ . Now, let  $x \in \text{cl } A$ . If  $x \in A$ , we are done. Otherwise, since every neighborhood of  $x$  intersects  $A$ , we have that they intersect at distinct points, thus  $x \in A'$ , therefore  $\text{cl } A \subseteq A \cup A'$ . ■

**Corollary.**  $A \subseteq X$  is closed if and only if  $A' \subseteq A$ .

*Proof.* If  $A$  is closed, then  $\text{cl } A = A = A \cup A'$ , thus  $A' \subseteq A$ . The converse is obvious. ■

**Example 1.15.** Let  $X$  and  $Y$  be topological spaces, and let  $A \subseteq X$  and  $B \subseteq Y$ . Then  $\text{cl}(A \times B) = \text{cl } A \times \text{cl } B$ . It suffices to show that limit points of one closure is in the other and vice versa as  $A \times B \subseteq \text{cl}(A \times B)$  and  $A \times B \subseteq \text{cl } A \times \text{cl } B$ .

Suppose  $x' \times y' \in (A \times B)'$  is a limit point of  $A \times B$ , then for a neighborhood  $U \times V$  of  $x' \times y'$ , we have  $(U \times V) \cap (A \times B) = (U \cap A) \times (V \cap B) \neq \emptyset$ . Therefore  $U \cap A \neq \emptyset$  and  $V \cap B \neq \emptyset$ . This makes  $x' \in A'$  and  $y' \in B'$ . Therefore  $\text{cl}(A \times B) \subseteq \text{cl } A \times \text{cl } B$ . By similar reasoning, we have the reverse inclusion so that equality is established.

**Definition.** Let  $X$  be a topological space. A sequence of points of  $X$   $\{x_n\}$  is said to **converge** to a point  $x \in X$  if for every neighborhood  $U$  of  $x$ , there is an  $N \in \mathbb{Z}^+$  such that  $x_n \in U$  for all  $n \geq N$ .

**Example 1.16.** (1) Consider the following topological space on  $\{a, b, c\}$  in figure ??, and

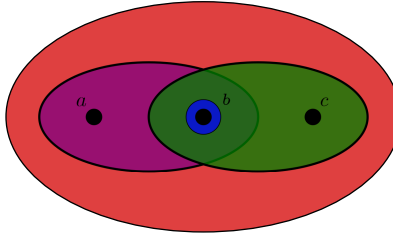


Figure 1.8: A topology on  $\{a, b, c\}$ , is not a Hausdorff space.

define the sequence  $\{x_n\}$  by  $x_n = b$  for all  $n \in \mathbb{Z}^+$ . The neighborhoods of  $a$ ,  $b$ , and  $c$  are  $U_a = \{a, b\}$ ,  $U_b = \{b\}$ , and  $U_c = \{b, c\}$ . Now let  $N > 0$ , then we see that for all  $n \geq N$ , that  $b \in U_a$ ,  $b \in U_b$ , and  $b \in U_c$  thus  $b$  converges to  $a$  and to  $c$ , and itself.

- (2) Consider the sequence of points  $\{\frac{1}{n}\}_{n \in \mathbb{Z}^+}$  of  $\mathbb{R}$  in the finite complement topology. Then for any neighborhood of  $x \in \mathbb{R}$ , which is of the form  $\mathbb{R} \setminus U$ , there are finitely many  $\frac{1}{n}$  not included. Thus we get that  $\lim_{n \rightarrow \infty} \frac{1}{n} = x$  for all  $x \in \mathbb{R}$ .

**Definition.** A topological space  $X$  is called a **Hausdorff space** if for each pair of distinct points  $x$  and  $y$ , there are neighborhoods  $U$  and  $V$  of  $x$  and  $y$  respectively such that  $U$  and  $V$  are disjoint.

**Example 1.17.** The topology of the previous example in figure 1.8 is not a Hausdorff space.

**Theorem 1.6.8.** Every finite point set in a Hausdorff space is closed.



*Proof.* Let  $X$  be a Hausdorff space, and let  $x_0 \in X$ . We have that  $\text{cl}\{x_0\} = \bigcap_{\{x_0\} \in U} U$ . Now let  $x \neq x_0 \in X$ . Since  $x \notin \{x_0\}$ , and  $X$  is Hausdorff, the intersections of the neighborhoods of  $x$  and  $x_0$  is empty, thus  $x \notin \text{cl}\{x_0\}$ , therefore  $\text{cl}\{x_0\} = \{x_0\}$ .

We can then extend this proof to finite point sets of size  $n$  by induction. ■

Now the condition that finite point sets be closed need not depend on whether or not  $X$  is a Hausdorff space. In fact, we can assume the following for some topological spaces.

**Axiom 1.6.1** (The  $T_1$  Axiom). *In any topological space, every finite point set of  $X$  is closed.*

**Theorem 1.6.9.** *Let  $X$  be a topological space satisfying the  $T_1$  axiom, and let  $A \subseteq X$ . Then a point  $x$  is a limit point of  $A$  if and only if every neighborhood of  $x$  contains infinitely many points of  $A$ .*

*Proof.* Let  $U_x$  be a neighborhood of  $x$ . If  $U_x$  intersects  $A$  at infinitely many points of  $A$ , then it intersects  $A$  at a point distinct from  $x$ , thus  $x$  is a limit point of  $A$ .

Conversely suppose that  $x$  is a limit point of  $A$ , and let  $U_x \cap A$  be finite, then  $U_x \cap (A \setminus \{x\})$  is also finite. Now let  $U_x \cap (A \setminus \{x\}) = \{x_1, \dots, x_m\}$ . By the  $T_1$  axiom,  $\{x_1, \dots, x_m\}$  is closed, so  $X \setminus \{x_1, \dots, x_m\}$  is open, thus  $U_x \cap (X \setminus \{x_1, \dots, x_m\})$  is a neighborhood of  $x$  that does not intersect  $A \setminus \{x\}$ , which contradicts that  $x$  is a limit point. ■

**Theorem 1.6.10.** *If  $X$  is a Hausdorff space, then a sequence of points of  $X$  converges to at most one point in  $X$ .*

*Proof.* Let  $\{x_n\}$  be a sequence of points converging to  $x$ , and let  $y \neq x$  and let  $U_x$  and  $U_y$  be neighborhoods of  $x$  and  $y$  respectively. Then  $U_x \cap U_y = \emptyset$ . Now since  $\{x_n\}$  converges to  $x$ , we have that for  $N > 0$ ,  $x_n \in U_x$  whenever  $n \geq N$ . Then  $x_n \notin U_y$ , and so  $\{x_n\}$  cannot converge to  $y$ . ■

**Definition.** Let  $\{x_n\}$  be a sequence in a Hausdorff space  $X$ . If  $\{x_n\}$  converges to a point  $x \in X$ , we call  $x$  the **limit** of  $\{x_n\}$  as  $n$  **approaches**  $\infty$  and we write  $\lim_{n \rightarrow \infty} x_n = x$  or  $\{x_n\} \rightarrow x$  as  $n \rightarrow \infty$ .

**Theorem 1.6.11.** *The following are true:*

- (1) *Every simply ordered set under the order topology is Hausdorff.*
- (2) *The product of two Hausdorff spaces is Hausdorff.*
- (3) *The subspace of a Hausdorff space is Hausdorff.*

*Proof.* (1) Let  $X$  be an ordered set under the order topology. Take  $x, y \in X$  distinct, and suppose without loss of generality that  $x < y$ . Then consider the neighborhoods  $(-\infty, x]$  and  $[y, \infty)$  of  $x$  and  $y$  respectively. Then  $(-\infty, x] \cap [y, \infty) = \emptyset$ .

- (2) Let  $X$  and  $Y$  be Hausdorff, and consider  $X \times Y$  in the product topology. Let  $x_1 \times y_1$  and  $x_2 \times y_2$  be distinct points, and let  $U_{x_1}, U_{x_2}, V_{y_1}$  and  $V_{y_2}$  be basis elements of  $x_1, x_2, y_1$ , and  $y_2$  respectively. Then they are neighborhoods of those elements respectively.

Now we have that  $U_{x_1} \times V_{y_1}$  and  $U_{x_2} \times V_{y_2}$  are basis elements of  $x_1 \times y_1$  and  $x_2 \times y_2$ , respectively, and hence neighborhoods of those elements respectively. Then we have  $(U_{x_1} \times V_{y_1}) \cap (U_{x_2} \times V_{y_2}) = (U_{x_1} \cap U_{x_2}) \times (V_{y_1} \cap V_{y_2}) = \emptyset \times \emptyset = \emptyset$ .

- (3) Let  $X$  be Hausdorff, and let  $Y$  be a subspace of  $X$ . Let  $x_1$  and  $x_2$  be distinct points, and let  $U_{x_1}$  and  $U_{x_2}$  be their neighborhoods. Since  $Y$  is open in  $X$ , then so are  $Y \cap U_{x_1}$  and  $Y \cap U_{x_2}$ , so they are also neighborhoods of  $x_1$  and  $x_2$  respectively. Then  $(Y \cap U_{x_1}) \cap (Y \cap U_{x_2}) = Y \cap (U_{x_1} \cap U_{x_2}) = \emptyset$ .

■

**Definition.** For any set  $X$ , we define the **diagonal** of  $X$  to be the set

$$\Delta(X) = \{x \times x : x \in X\} \quad (1.1)$$

**Theorem 1.6.12.** *A topological space  $X$  is Hausdorff if, and only if the diagonal  $\Delta(X)$  is closed.*

*Proof.* Suppose that  $X$  is Hausdorff, then for any  $x, y \in X$  distinct, there are neighborhoods  $U_x$  and  $U_y$  of  $x$  and  $y$  respectively with  $U_x$  and  $U_y$  disjoint. Notice then that  $X \setminus U_x$  and  $X \setminus U_y$  are closed in  $X$ , hence  $(X \times X) \setminus (U_x \times U_y)$  is also closed. Now, since  $x \neq y$  in the neighborhood of  $U_x \times U_y$  we get that the intersection:

$$\bigcap_{x, y \in X} (X \times X) \setminus (U_x \times U_y) = \Delta(X)$$

So  $\Delta(X)$  is closed.

Conversely, suppose that  $\Delta(X)$  is closed. Then  $(X \times X) \setminus \Delta(X) = \{x \times y : x \neq y\}$  is open. Now, let  $U \times V$  be a neighborhood for some  $x \times y \in (X \times X) \setminus \Delta(X)$  as a subspace of  $X \times X$ . Then  $U$  is a neighborhood for  $x$  and  $V$  is a neighborhood for  $y$ . Now, if  $U \cap V \neq \emptyset$ , there is some  $z \in U$  and  $z \in V$ , so that  $z \times z \in U \times V$ . But  $U \times V$  is a neighborhood contained in  $(X \times X) \setminus \Delta(X)$ , which contradicts  $z \times z \in U \times V$ . Therefore,  $U$  and  $V$  must be disjoint, which makes  $X$  into a Hausdorff space. ■

We conclude this section by looking at another characteristic of sets in topological spaces, closely related to the interior and closure of sets.

**Definition.** Let  $X$  be a topological space, and  $A \subseteq X$ . We define the **boundry** of  $A$

$$\partial A = \text{cl } A \cap \text{cl } (X \setminus A) \quad (1.2)$$



Figure 1.9: The boundry  $\partial C$  of a cylinder  $C$  whose interior is empty is the disjoint union of the two circles that enclose  $C$ .

**Example 1.18.** The following are true about the boundry:

- (1)  $\text{Int } A$  and  $\partial A$  are disjoint and  $\text{cl } A = \text{Int } A \cup \partial A$ .
- (2) If  $A$  is open and closed, then  $\partial A = \emptyset$ .
- (3) If  $A$  is open, then  $\partial A = \text{cl } A \setminus A$ .

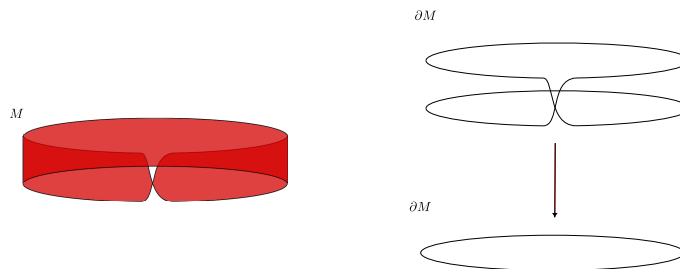


Figure 1.10: The Möbius strip is a well known example of an object with only one side. It can be recreated by taking a thin strip of tape, twisting it, and then gluing the ends together. It can be show that the boundry of a Möbius strip is homeomorphic to a circle.

## 1.7 Continuous Functions.

**Definition.** Let  $X$  and  $Y$  be topological spaces. We say that a mapping  $f : X \rightarrow Y$  is **continuous** if for each open set  $V$  in  $Y$ ,  $f^{-1}(V)$  is open in  $X$ .

Now if  $f : X \rightarrow Y$  is continuous, then for every open set  $V$  of  $Y$ ,  $f^{-1}(V)$  is open in  $X$ . Now suppose that  $\mathcal{B}$  is a basis of  $Y$ , then  $V = B_\alpha$ , hence  $f^{-1}(B_\alpha)$ , is open in  $X$ .

Similarly, if  $\mathcal{S}$  is a subbasis of  $Y$ , then for any basis element  $B$  of  $Y$ ,  $B = \bigcap_{i=1}^n S_i$ , which then implies that  $f^{-1}(B) = \bigcap_{i=1}^n f^{-1}(S_i)$ , thus  $f^{-1}(S_i)$  is also open in  $X$  for  $1 \leq i \leq n$ .

**Example 1.19.** (1) Let  $f : \mathbb{R} \rightarrow \mathbb{R}$  be a continuous realvalued function. Then for each open interval  $I \subseteq \mathbb{R}$ ,  $f^{-1}(I)$  is an open interval in  $\mathbb{R}$ , so take  $x_0 \in \mathbb{R}$  and  $\varepsilon > 0$ , and let  $I = (f(x_0) - \varepsilon, f(x_0) + \varepsilon)$ , then since  $x_0 \in f^{-1}(I)$ , there is a basis  $(a, b) \subseteq f^{-1}(I)$  about  $x_0$ . Then take  $\delta = \min\{x_0 - a, x_0 - b\}$ , then  $x \in (a, b)$  whenever  $0 < |x - x_0| < \delta$ , and we get that  $f(x) \in I$ , that is,  $|f(x) - f(x_0)| < \varepsilon$ . This is the definition of continuity defined in the real analysis. We can prove that the converse holds also.

If  $f : \mathbb{R} \rightarrow \mathbb{R}$  is continuous at a point  $x_0$ , then for every  $\varepsilon > 0$ , there is a  $\delta > 0$  such that  $|f(x) - f(x_0)| < \varepsilon$  whenever  $0 < |x - x_0| < \delta$ . Then we notice that  $x$  and  $x_0$  are distinct, furthermore,  $x_0 - \delta < x < x_0 + \delta$ , hence  $x \in (x_0 - \delta, x_0 + \delta)$  implies that  $f(x) \in (f(x_0) - \varepsilon, f(x_0) + \varepsilon)$ . Letting  $V_\delta(x_0) = (x_0 - \delta, x_0 + \delta)$  and  $V_\varepsilon(f(x_0)) = (f(x_0) - \varepsilon, f(x_0) + \varepsilon)$ , we have that whenever  $x \in V_\delta(x_0)$ , then  $f(x) \in V_\varepsilon(f(x_0)) \subseteq f^{-1}(V_\delta(x_0))$ . And so the topological definition of continuity is equivalent to the real analytic definition of continuity.

- (2) Let  $f : \mathbb{R} \rightarrow \mathbb{R}_l$  be defined such that  $f(x) = x$  for all  $x \in \mathbb{R}$ . Take  $[a, b) \subseteq \mathbb{R}_l$ , we have that  $f^{-1}([a, b)) = [a, b)$ , which is not open in  $\mathbb{R}$  (under the standard topology), hence  $f$  is not continuous. However, the map  $g : \mathbb{R}_l \rightarrow \mathbb{R}$  defined the same way is continuous since  $g^{-1}((a, b))$  is open in  $\mathbb{R}_l$ .
- (3) Consider the map  $f : \mathbb{R} \rightarrow \mathbb{R}$  defined by  $f(x) = x$  if  $x \in \mathbb{Q}$  and  $f(x) = 0$  if  $x \in \mathbb{R} \setminus \mathbb{Q}$ . Then  $f$  is continuous only at  $x = 0$ , for the limits

$$\lim_{x \rightarrow x_0^+} f(x)$$

and

$$\lim_{x \rightarrow x_0^-} f(x)$$

both fail to exist for  $x_0 \in \mathbb{Q}$  and  $x_0 \neq 0$ .

- (4) Let  $f : A \rightarrow B$  and  $g : C \rightarrow D$  be continuous and define  $f \times g : A \times C \rightarrow B \times D$  by  $f \times g : a \times c \rightarrow f(a) \times g(c)$ . Then  $f \times g$  is continuous. We have that  $U$  and  $V$  open in  $B$  and  $D$  make  $f^{-1}(U)$  and  $g^{-1}(V)$  open in  $A$  and  $C$ , respectively. Then it follows that  $U \times V$  is open in  $B \times D$ , and that  $(f \times g)^{-1}(U \times V) = f^{-1}(U) \times g^{-1}(V)$  is also open by the definition if  $f \times g$ . This makes  $f \times g$  continuous.

**Theorem 1.7.1.** *Let  $X$  and  $Y$  be topological spaces, and let  $f : X \rightarrow Y$  be a mapping of  $X$  into  $Y$ . Then the following are equivalent:*

- (1)  $f$  is continuous.
- (2) For every  $A \subseteq X$ ,  $f(\text{cl } A) \subseteq \text{cl } f(A)$ .
- (3) For every closed set  $B \subseteq Y$ ,  $f^{-1}(B)$  is closed in  $X$ .
- (4) For each  $x \in X$  and each neighborhood  $V$  of  $f(x)$ , there is a neighborhood  $U$  of  $x$  such that  $f(U) \subseteq V$ .

*Proof.* Let  $f$  be continuous and let  $A \subseteq X$ . Consider the neighborhood  $V$  of  $f(x)$ , then  $f^{-1}(V)$  is open in  $X$ , and intersects  $A$  at a point  $y$ . Then  $V \cap f(A) = f(y)$ , thus  $f(x) \in \text{cl } f(A)$ .

Now let  $B$  be closed in  $Y$ , and let  $A = f^{-1}(B)$ . Then we have that  $f(A) = f(f^{-1}(B)) \subseteq B$ , thus  $x \in \text{cl } A$ .

Now let  $V$  be open in  $Y$ , so that  $B = Y \setminus V$  is closed in  $Y$ , and  $f^{-1}(B) = f^{-1}(Y) \setminus f^{-1}(V) = X \setminus f^{-1}(V)$  which is closed in  $X$ , hence  $f^{-1}(V)$  is open in  $X$ .

Now let  $x \in X$ , and let  $V$  be a neighborhood of  $f(x)$ . Then  $U = f^{-1}(V)$  is a neighborhood of  $x$  for which  $f(U) \subseteq V$ . Finally let  $V$  be open in  $Y$ , and let  $x \in f^{-1}(V)$ , then  $f(x) \in V$ , so there is a neighborhood  $U_x$  of  $x$  for which  $f(U_x) \subseteq V$ , then  $U_x \subseteq f^{-1}(V)$ , then  $f^{-1}(V)$  is a union of open sets, and hence open in  $X$ . ■

**Example 1.20.** Suppose  $f : X \rightarrow Y$  is continuous, and that  $AX$ . Let  $x \in A'$  be a limit point of  $A$ . Certainly, we have  $x \in \text{cl } A$  so then  $f(x) \in f(\text{cl } A) \subseteq \text{cl } f(A)$ . It remains to see if  $x \in (f(A))'$  then. Let  $V$  be open in  $Y$  such that  $x \in U = f^{-1}(V)$ . Then  $U \cap (A \setminus x) \neq \emptyset$ . So then we get  $V \cap (f(A) \setminus f(x)) \neq \emptyset$ . If  $z \in U \cap A$ , then  $f(z) = f(x)$ , so that  $f(U \cap A) = \{f(x)\}$ , which means that  $x \notin (f(A))'$ , in general. It is the case that  $f(x)$  is a limit point of  $f(A)$  whenever  $f$  is 1-1.

**Definition.** Let  $X$  and  $Y$  be topological spaces, and  $f : X \rightarrow Y$  be a 1-1 mapping of  $X$  onto  $Y$ . We call  $f$  a **homeomorphism** if both  $f$  and  $f^{-1}$  are continuous.

**Lemma 1.7.2.** *Let  $X$  and  $Y$  be topological spaces and let  $f : X \rightarrow Y$  be a homeomorphism. Then  $f(U)$  is open if and only if  $U$  is open.*

*Proof.* We have that both  $f : X \rightarrow Y$  and  $f^{-1} : Y \rightarrow X$  are continuous 1 – 1 of  $X$  and  $Y$  onto each other (respectively). Now let  $U$  be open in  $X$ , then  $U = f^{-1}(V)$ , for some set  $V$  open in  $Y$ . Notice then, that  $f(U) = f(f^{-1}(V)) = V$ , thus  $f(U)$  is open in  $Y$ . Conversely, let  $V = f(U)$  be open in  $Y$  for some open set  $U$  in  $X$ , then  $U = f^{-1}(V)$ , so by definition of continuity,  $U$  is open in  $X$ . ■

We can also characterize now when the topology of one set is finer or coarser than the topology of another given another given a continuous mapping.

**Lemma 1.7.3.** *If  $X$  and  $X'$  are two sets with topologies  $\mathcal{T}$  and  $\mathcal{T}'$ , respectively, then if  $f : X' \rightarrow Y$  is the identity mapping, then  $f$  is continuous if, and only if  $\mathcal{T} \subseteq \mathcal{T}'$*

*Proof.* Suppose that  $f$  is continuous, then for every  $V$  open in  $X$  under  $\mathcal{T}$ ,  $f^{-1}(V)$  is open in  $X'$  under  $\mathcal{T}'$ . Since  $f$  is the identity, we get that  $V = f^{-1}(V)$ , so that  $V$  is open in  $X'$ .

Conversely, if  $\mathcal{T} \subseteq \mathcal{T}'$ , then for every  $V$  open in  $X$ ,  $V$  is open in  $X'$ . Notice then  $V = f^{-1}(V) \subseteq X'$ . Therefore,  $f$  is continuous. ■

**Corollary.**  *$f : X' \rightarrow X$  is a homomorphism if, and only if  $\mathcal{T} = \mathcal{T}'$ .*

**Definition.** Let  $X$  and  $Y$  be topological spaces and let  $f : X \rightarrow Y$  be a continuous 1 – 1 mapping of  $X$  into  $Y$ , and consider  $f(X)$  as a subspace of  $Y$ . We call  $f : X \rightarrow f(X)$  a **topological imbedding** if  $f$  is a homeomorphism of  $X$  onto  $f(X)$ .

**Example 1.21.** (1) The map  $f : \mathbb{R} \rightarrow \mathbb{R}$  defined by  $f(x) = 3x + 1$  is a homeomorphism whose inverse is  $f^{-1}(y) = \frac{1}{3}(y - 1)$ , both  $f$  and  $f^{-1}$  are continuous.

(2) The map  $f : (-1, 1) \rightarrow \mathbb{R}$  defined by  $f(x) = \frac{x^2}{1-x^2}$  has as its inverse the map  $f : \mathbb{R} \rightarrow (-1, 1)$  defined by  $f^{-1}(y) = \frac{2y}{1+\sqrt{1+4y^2}}$ . Both  $f$  and  $f^{-1}$  are continuous, so  $f$  is a homeomorphism.

(3) The map  $g : \mathbb{R}_l \rightarrow \mathbb{R}$  defined by  $g(x) = x$  is not a homeomorphism, despite being continuous, as  $g^{-1}(1)$  is undefined.

(4) Let  $S^1$  be the unit circle in  $\mathbb{R}^2$ , which is a subspace of  $\mathbb{R}^2$ , and define  $f : [0, 1) \rightarrow S^1$  by  $f(t) = (\cos(2\pi t), \sin(2\pi t))$ . Clearly  $f$  is 1 – 1 onto  $S^1$ , and continuous, however  $f^{-1}$  is not continuous as  $f([0, \frac{1}{4}))$  is not open in  $S^1$  as  $f(0)$  is in no open set of  $\mathbb{R}^2$  such that  $U \cap S^1 = f([0, 1))$ .

(5) Consider the mappings  $g : [0, 1) \rightarrow \mathbb{R}^2$  by  $g(t) = (\cos(2t\pi), \sin(2t\pi))$ . Now  $g$  is 1 – 1 and continuous, and we have that  $g([0, 1)) \subseteq S^1$ , however since  $g$  is not a homeomorphism,  $g$  fails to be a topological embedding.

(6) Let  $X$  and  $Y$  be topological spaces, and consider the maps  $f : x \rightarrow x \times y_0$  and  $g : y \rightarrow y_0 \times x$  for some  $x_0 \in X$  and  $y_0 \in Y$ . We see that  $f$  and  $g$  are both onto  $f(X)$  and  $g(Y)$ , respectively, and that  $x \times y_0 = x' \times y_0$  and  $x_0 \times y = x_0 \times y'$  both imply that  $x = x'$  and  $y = y'$ , so  $f$  and  $g$  are also 1 – 1.

Lastly, we see that if  $U \times y_0$  and  $x_0 \times V$  are open in  $X \times Y$ , then  $f^{-1}(U \times y_0)$  and  $g^{-1}(x_0 \times V)$  are open in  $X$  and  $Y$ , respectively. This makes  $f$  and  $g$  continuous, and

by the same argument, so are  $f^{-1}$  and  $g^{-1}$ . Thus  $f$  and  $g$  are imbeddings of  $X$  onto  $f(X) = X \times y_0$  and  $g(X) = x_0 \times Y$ .

**Theorem 1.7.4** (Constructions for continuous functions.). *Let  $X$  and  $Y$  be topological spaces, then:*

- (1) (Constant construction) *If  $f : X \rightarrow Y$  maps  $x \rightarrow y_0$  for all  $x \in X$ , then  $f$  is continuous.*
- (2) (Inclusion) *If  $A \subseteq X$  is a subspace, then the inclusion mapping  $\iota : A \rightarrow Y$  is continuous.*
- (3) (Construction by composition) *If  $f : X \rightarrow Y$  and  $g : Y \rightarrow Z$  are continuous, then  $f \circ g : X \rightarrow Z$  is also continuous.*
- (4) (Domain restriction) *If  $f : X \rightarrow Y$  is continuous and  $A \subseteq X$ , then  $f : A \rightarrow Y$  is continuous.*
- (5) (Range restriction) *if  $f : X \rightarrow Y$ , and  $Z \subseteq Y$  such that  $f(X) \subseteq Z$ , then  $f : X \rightarrow Z$  is continuous.*
- (6) (Range expansion) *If  $f : X \rightarrow Y$  is continuous, and  $Y \subseteq Z$  is a subspace of  $Z$ , then  $f : X \rightarrow Z$  is continuous.*
- (7) (Local Formulation) *The map  $f : X \rightarrow Y$  is continuous if  $X$  can be written as the union of open sets  $U_\alpha$  such that  $f : U_\alpha \rightarrow Y$  is continuous for all  $\alpha$ .*

*Proof.* (1) Let  $f(x) = y_0$  for all  $x \in X$ , and let  $V$  be open in  $Y$ , then  $f^{-1}(V) = X$  or  $\emptyset$  depending on if  $y_0 \in V$  or not. In either case,  $f^{-1}(V)$  is open.

(2) If  $U$  is open in  $X$ , then  $f^{-1}(U) = U \cap A$  which is open in the subspace topology of  $X$ .

(3) If  $U$  is open in  $Z$ ,  $g^{-1}(U)$  is open in  $Y$ , hence  $f^{-1}g^{-1}(U)$  is open in  $X$ .

(4) Notice that  $f_A = \iota \circ f = f : A \rightarrow Y$  which is continuous by (2) and (3).

(5) Let  $f : A \rightarrow Y$  be continuous and let  $f(X) \subseteq Z \subseteq Y$ . Let  $B$  be open in  $Z$ , so  $B = Z \cap U$  for some  $U$  open in  $Y$ . Now by hypothesis, we have that  $f^{-1}(U) \subseteq f^{-1}(B)$ , hence  $f^{-1}(B)$  is open in  $X$ , thus  $f : X \rightarrow Z$  is continuous.

(6) Let  $f$  be as in (5), and let  $Y \subseteq Z$  be a subspace of  $Z$ . Then the mapping  $h : X \rightarrow Z$  defined by  $h = \iota \circ f$  is continuous.

(7) Let  $X = \bigcup U_\alpha$  where  $U_\alpha$  is open in  $X$ , and  $f : U_\alpha \rightarrow Y$  is continuous for all  $\alpha$ . Let  $V$  be open in  $Y$ , then  $f^{-1}(V) \cap U_\alpha = f_{U_\alpha}^{-1}(V)$ , and since  $f$  is continuous on  $U_\alpha$ , then  $f^{-1}(V) = \bigcup f_{U_\alpha}^{-1}(V)$  is open in  $X$ . ■

**Theorem 1.7.5** (The pasting lemma). *Let  $X = A \cup B$  with  $A$  and  $B$  closed in  $X$ , and let  $f : A \rightarrow Y$  and  $g : B \rightarrow Y$  be continuous. If  $f(x) = g(x)$  for all  $x \in A \cap B$ , then we can construct a mapping  $h : X \rightarrow Y$  defined by  $h(x) = \begin{cases} f(x), & x \in A \\ g(x), & x \in B \end{cases}$ . Then  $h$  is continuous.*

*Proof.* Let  $C$  be closed in  $Y$ , then  $h^{-1}(C) = f^{-1}(C) \cup g^{-1}(C)$ . Since  $f$  and  $g$  are continuous, then  $f^{-1}(C)$  and  $g^{-1}(C)$  are closed in  $A$  and  $B$ , respectively. Thus  $h^{-1}(C)$  is closed in  $X$ . ■

**Example 1.22.** Define  $h : \mathbb{R} \rightarrow \mathbb{R}$  by  $h(x) = \begin{cases} x, & x \leq 0 \\ \frac{x}{2}, & x \geq 0 \end{cases}$ . We have that  $x$  and  $\frac{x}{2}$  are continuous on their respective domains, intersecting at 0, i.e.  $x : (-\infty, 0] \rightarrow \mathbb{R}$ ,  $\frac{x}{2} : [0, \infty) \rightarrow \mathbb{R}$ , and  $\{0\} = (-\infty, 0] \cap [0, \infty)$ . Thus  $h$  is continuous on  $\mathbb{R}$ .

However,  $k, l : \mathbb{R} \rightarrow \mathbb{R}$  defined by  $k(x) = \begin{cases} x - 2, & x \leq 0 \\ x + 2, & x \geq 0 \end{cases}$  and  $l(x) = \begin{cases} x - 2, & x < 0 \\ x + 2, & x \geq 0 \end{cases}$  are not continuous. We have that their domains intersect at 0, but that  $k(0) = \pm 2$ , (so  $k$  isn't even a function). Likewise,  $(-\infty, 0) \cap [0, \infty) = \emptyset$ , which is open in  $\mathbb{R}$  see 1.11.

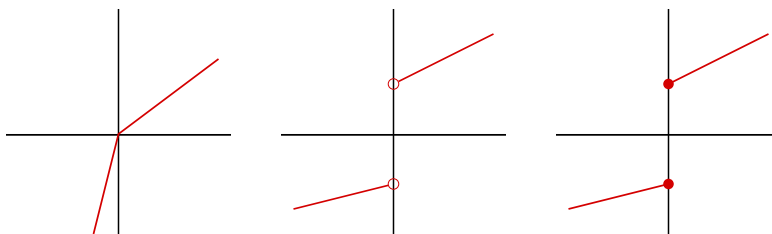


Figure 1.11: The mappings  $h$ ,  $k$ , and  $l$ .

**Theorem 1.7.6.** Let  $f : A \rightarrow X \times Y$  be defined by  $f(a) = (f_1(a), f_2(a))$ , where  $f_1 : A \rightarrow X$  and  $f_2 : A \rightarrow Y$ . Then  $f$  is continuous if and only if  $f_1$  and  $f_2$  are continuous.

*Proof.* Let  $\pi_1 : X \times Y \rightarrow X$  and  $\pi_2 : X \times Y \rightarrow Y$  be projections onto  $X$  and  $Y$  respectively. Since  $\pi_1^{-1}(U) = U \times Y$  and  $\pi_2^{-1}(V) = X \times V$  are both open in  $X \times Y$ ,  $\pi_1$  and  $\pi_2$  are continuous. Then notice that  $f_1(a) = \pi_1 \circ f(a)$  and  $f_2(a) = \pi_2 \circ f(a)$ , both of which are continuous.

Now suppose that  $f_1$  and  $f_2$  are continuous. We have that  $a \in f^{-1}(U \times V)$  if and only if  $f_1(a) \in U$  and  $f_2(a) \in V$ , then  $f_1^{-1}(U)$  and  $f_2^{-1}(V)$  are open in  $A$ , hence so is  $f^{-1}(U \times V) = f_1^{-1}(U) \cap f_2^{-1}(V)$ . ■

We now conclude the section with some more definitions and examples.

**Definition.** We define the **parametrized curve** of the plane  $\mathbb{R}^2$  to be the continuous function  $f : [a, b] \rightarrow \mathbb{R}^2$  defined by  $f(t) = (x(t), y(t))$ . If  $f$  is in a vector field, then we define  $f(t) = x(t)i + y(t)j$  where  $i = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$  and  $j = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ .

**Example 1.23.** The function  $f(t) = ((\cos(t)), \sin(t))$  is a parametrization of the curve  $x^2 + y^2 = 1$ , i.e. the unit circle  $S^1$ .

**Example 1.24.** Suppose  $\{A_\alpha\}$  is a finite collection of subsets of a topological space  $X$ , and that  $X = \bigcup_\alpha A_\alpha$ . Suppose  $f : X \rightarrow Y$  is defined such that the maps  $f_\alpha = f|_{A_\alpha} : A_\alpha \rightarrow Y$  are continuous for each  $\alpha$ . Then  $f$  is also continuous provided that each  $A_\alpha$  is closed. For, if each  $A_\alpha$  is closed, then  $X$  is the finite union of closed sets. Moreover, for any  $V \subseteq Y$

closed, we get  $f_\alpha^{-1}(V)$  is closed in  $A_\alpha$  as a subspace of  $X$ . So  $f^{-1}(V) = \bigcup f_\alpha^{-1}(V)$  is closed in  $X = \bigcup A_\alpha$ , via local formulation, Making  $f$  continuous.

This fact is not true however if  $\{A_\alpha\}$  is countable, but not finite. Consider the collection  $\{[\frac{1}{n}, 1]\}_{n \in \mathbb{Z}^+}$ , of the space  $[0, 1]$ . We have  $[0, 1] = \bigcup_{n \in \mathbb{Z}^+} [\frac{1}{n}, 1]$ , and each  $[\frac{1}{n}, 1]$  is closed in  $[0, 1]$  as a subspace of  $\mathbb{R}$ . However, the function  $f(x) = \frac{1}{n}$  if  $x \in [\frac{1}{n}, 1]$  and  $f(x) = 1$  otherwise is continuous on each  $[\frac{1}{n}, 1]$ , but its local formulation is not continuous on  $[0, 1]$ .

*Remark.* Theorem 1.7.4 only states that the local formulation of a mapping is continuous only if its domain is the union of open sets. Thus a local formulation where the domain is a union of closed sets need not be continuous.

**Definition.** Let  $X$  be a topological space, and  $\{A_\alpha\}$  a collection of subsets of  $X$ . We call  $\{A_\alpha\}$  **locally finite** if for each  $x \in X$ , there is a neighborhood  $U$  of  $x$  such that  $U \cap A_\alpha \neq \emptyset$  only for a finite number of  $\alpha$ .

**Lemma 1.7.7.** Let  $X$  be a topological space, and  $\{A_\alpha\}$  a locally finite collection of subsets of  $X$ . If  $f : X \rightarrow Y$  is defined such that  $f_\alpha = f|_{A_\alpha} : A_\alpha \rightarrow Y$  is continuous for each  $\alpha$ , then  $f$  is continuous.

*Proof.* If  $\{A_\alpha\}$  is locally finite, then for each  $x \in X$ , there is a neighborhood  $U_x$  of  $x$  such that  $U_x \cap A_\alpha \neq \emptyset$  for only a finite number of  $\alpha$ . Then  $U_x \cap A_\alpha$  is open in  $A_\alpha$  as a subspace of  $X$ . Then  $V_x = A_\alpha \setminus (U_x \cap A_\alpha)$  is closed in  $A_\alpha$ , and  $V_x = f_\alpha^{-1}(W)$  for some  $W$  closed in  $Y$ . Thus  $\bigcup_{x \in X} V_x = \bigcup_\alpha f_\alpha^{-1}(W) = f^{-1}(W)$  (which is a finite union) is closed in  $X$ . This makes  $f$  continuous. ■

**Definition.** Let  $X$ ,  $Y$ , and  $Z$  be topological space. We say a map  $F : X \times Y \rightarrow Z$  is **continuous in each variable** if the maps  $F_X : X \rightarrow Z$  and  $F_Y : Y \rightarrow Z$  defined by  $F_X(x) = F(x \times y_0)$  and  $F_Y(y) = F(x_0 \times y)$  are continuous.

**Lemma 1.7.8.** If  $X$ ,  $Y$ , and  $Z$  are topological spaces, and  $F : X \times Y \rightarrow Z$  is continuous, then  $F$  is continuous in each variable separately.

*Proof.* If  $F$  is continuous, then  $W$  open in  $Z$  implies that  $F^{-1}(W)$  is open in  $X \times Y$ , and that  $U \times V = F^{-1}(W)$  for  $U$  and  $V$  open in  $X$  and  $Y$  respectively. Then  $F$  is continuous at  $U \times y_0$ , for  $y_0 \in V$ ; that is  $F^{-1}(U \times y_0)$  is open in  $X \times Y$ . So, by definition,  $F_X(U)$  is open in  $Z$ , and  $F_X$  is continuous. By similar reasoning, we can also conclude that  $F_Y$  is continuous, and that  $F$  is continuous in each variable. ■

**Example 1.25.** Define  $F : \mathbb{R} \rightarrow \mathbb{R}$  by

$$F(x \times y) = \begin{cases} \frac{xy}{x^2+y^2}, & \text{if } x \times y \neq 0 \times 0 \\ 0, & \text{otherwise} \end{cases}$$

Define  $F_X(x) = F(x \times y_0)$ , then  $F_X(x) = \frac{xy_0}{x^2+y_0^2}$  if  $x \neq 0$  and  $y_0 \neq 0$ , and 0 otherwise. Then when  $y_0 = 0$  we see that

$$\lim_{x \rightarrow 0} F(x \times y_0) = \lim_{x \rightarrow 0} F_X(x) = F_X(0) = F(0 \times 0) = 0$$



so that  $F_X$  is continuous at  $x = 0$ , and it is also continuous everywhere else. So  $F_X$  is continuous. Similarly, the mapping  $F_Y(y) = F(x_0 \times y)$  is also continuous, and  $F$  is continuous in each variable.

$F$  however, is not continuous in general. Define  $g(x) = F(x \times x)$  so we get

$$g(x) = \begin{cases} \frac{1}{2}, & \text{if } x \neq 0 \\ 0, & \text{otherwise} \end{cases}$$

which is not continuous. So  $F$  cannot be continuous at the points  $x \times x$ , for it degenerates into a noncontinuous function. In fact, if we have  $\frac{xy}{x^2+y^2} = \frac{1}{2}$ , then we get  $x^2+y^2-2xy = (x-y)^2 = 0$  and  $x \times y \neq 0 \times 0$ . This means  $F^{-1}(\{\frac{1}{2}\}) = \{x \times x : x \neq 0\}$  is not closed.



# Chapter 2

## More on Topological Spaces

### 2.1 The Product Space and Box Topologies.

We now explore more about the product topology.

**Definition.** Let  $J$  be an indexed set, and let  $X$  be a set. We define a  **$J$ -tuple** of elements of  $X$  to be a map  $x : J \rightarrow X$ , where if  $\alpha \in J$ , then  $x(\alpha) = x_\alpha$ , and we call it the  **$\alpha$ -th coordinate** of  $x$ . We write  $(x_\alpha)_\alpha \in J$ , or just simply  $(x_\alpha)$

**Definition.** Let  $\{A_\alpha\}$  be an indexed family, and let  $X = \bigcup_{\alpha \in J} A_\alpha$ . We define the **cartesian product** of  $\{A_\alpha\}$ ,  $\prod_{\alpha \in J} A_\alpha$  to be the set of all  $J$ -tuples  $(x_\alpha)$  of elements of  $X$ , where  $x_\alpha \in A_\alpha$

**Theorem 2.1.1.** *Let  $\{X_\alpha\}$  be a family of topological spaces, and consider the cartesian product  $\prod X_\alpha$ . Then the collection of all cartesian products  $\prod U_\alpha$ , where  $U_\alpha$  is open in  $X_\alpha$ , for all  $\alpha$ , forms a basis for the topology on  $\prod X_\alpha$ .*

*Proof.* Clearly  $\prod X_\alpha$  itself is a basis element by the first condition. Now consider  $\prod U_\alpha$  and  $\prod V_\alpha$ , then  $\prod U_\alpha \cap \prod V_\alpha = \prod (U_\alpha \cap V_\alpha)$ , which is also a basis element. ■

**Definition.** Let  $\{X_\alpha\}$  be a family of topological spaces, and take as basis the collection of all products  $\prod U_\alpha$  where  $U_\alpha$  is open in  $X_\alpha$ . We call the topology generated by this basis the **box topology** on  $\prod X_\alpha$ .

**Definition.** Let  $\pi_\beta : \prod X_\alpha \rightarrow X_\beta$  be defined by  $\pi_\beta((x_\alpha)) = x_\beta$ . We call this map the **projection mapping** of  $\prod X_\alpha$  onto  $X_\beta$

**Theorem 2.1.2.** *Let  $\mathcal{S}_\beta = \{\pi_\beta^{-1}(U_\beta) : U_\beta \text{ is open in } X_\beta\}$ , and let  $\mathcal{S} = \bigcup \mathcal{S}_\beta$ . Then  $\mathcal{S}$  forms the basis for a topology on  $\prod X_\alpha$ .*

*Proof.* Since  $U_\beta$  is open in  $X_\beta$ ,  $\pi_\beta^{-1}(U_\beta) \subseteq \prod X_\alpha$ . Taking  $\bigcup \mathcal{S}$ , we get that  $\bigcup \pi_\beta^{-1}(U_\beta) = \prod X_\beta$  for all  $\beta$ . Thus  $\mathcal{S}$  is a subbasis. ■

**Definition.** Let  $\pi_\beta$  be a projection mapping of  $\prod X_\alpha$  onto  $X_\beta$ , and take as subbasis the collection of all  $\pi_\beta^{-1}(U_\beta)$  where  $U_\beta$  is open in  $X_\beta$ . We call the topology generated by this subbasis the **product space topology**, or more generally the **product topology** on  $\prod X_\alpha$ .

**Theorem 2.1.3.** *The box topology on  $\prod X_\alpha$  has as basis all sets of the form  $\prod U_\alpha$ , where  $U_\alpha$  is open in  $X_\alpha$ , and the product space topology has as basis all sets of the form  $\prod U_\alpha$ , where  $U_\alpha$  is open in  $X_\alpha$ , and  $U_\alpha = X_\alpha$  except only for finitely many  $\alpha$ .*

*Proof.* That the box topology has as a basis all sets of the form  $\prod U_\alpha$  is clear. Now consider the basis  $\mathcal{B}$  that  $\mathcal{S}$  generates, and let  $\beta_1, \dots, \beta_n$  be a finite set of distinct indices and let  $U_{\beta_i}$  be open in  $X_{\beta_i}$ , and  $U_\alpha = X_\alpha$  for all other  $\alpha$ . Since  $B \in \mathcal{B}$  is a finite intersection of elements of  $\mathcal{S}$ , we have that  $B = \bigcap_{i=1}^n \beta_i^{-1}(U_{\beta_i})$ .

Now a point  $x = (x_\alpha) \in B$  if and only if the  $\beta_i$ -th coordinate is in  $U_{\beta_i}$ , for  $1 \leq i \leq n$ , hence membership depends only on a finite number of  $\alpha$ , thus  $B = \prod U_\alpha$  where  $U_\alpha = X_\alpha$  for  $\alpha \neq \beta_i$  for  $1 \leq i \leq n$ . ■

**Corollary.** *The box topology on  $\prod X_\alpha$  is finer than the product topology on  $\prod X_\alpha$ ; moreover, if  $\{X_i\}_{i=1}^n$  is a finite family of topologies, then the box topology, and the product topology on  $\prod_{i=1}^n X_i$  are equal.*

For convention, from now on when we consider the product  $\prod X_\alpha$ , we assume that it is under the product space topology.

**Theorem 2.1.4.** *Suppose the topology on  $X_\alpha$  is given by a basis  $\mathcal{B}_\alpha$ . The collection of all sets  $\prod B_\alpha$  where  $B_\alpha \in \mathcal{B}_\alpha$  for each  $\alpha$  is a basis for the box topology on  $\prod X_\alpha$ .*

*The same collection for a finite number of  $\alpha$ , and where  $B_\alpha = X_\alpha$  for all other  $\alpha$  forms a basis for the product space topology on  $\prod X_\alpha$ .*

*Proof.* Let  $\mathcal{B}$  be the collection of all  $\prod B_\alpha$ , where  $B_\alpha \in \mathcal{B}_\alpha$ . Now each  $X_\alpha$  is already its own basis, hence so is  $\prod X_\alpha$ . Now let  $\prod U_\alpha$  and  $\prod V_\alpha$  be basis elements. Since  $\prod U_\alpha \cap \prod V_\alpha = \prod U_\alpha \cap V_\alpha$ , for finite alpha, and since  $\prod U_\alpha \cap \prod V_\alpha = \prod X_\alpha$  for all other  $\alpha$  (in the case of the product space topology), we get another basis element. Hence  $\mathcal{B}$  is a basis for the box topology, and, provided the necessary condition, is also a basis for the product topology. ■

**Theorem 2.1.5.** *Let  $A_\alpha$  be a subspace of  $X_\alpha$ . Then  $\prod A_\alpha$  is a subspace of  $\prod X_\alpha$  under both the box and product space topologies.*

*Proof.* Since  $\prod A_\alpha \cap \prod U_\alpha = \prod (A_\alpha \cap U_\alpha)$ , and  $A_\alpha \cap U_\alpha$  is a basis element for  $X_\alpha$  under the subspace topology, then it follows that  $\prod (A_\alpha \cap U_\alpha)$  is a basis element for the same topology on  $\prod X_\alpha$ , thus  $\prod A_\alpha$  is a subspace. ■

**Theorem 2.1.6.** *If  $X_\alpha$  is a Hausdorff space, then so is  $\prod X_\alpha$  under both the box and product space topologies.*

*Proof.* Since  $X_\alpha$  is a Hausdorff space, a sequence of points of  $X_\alpha$ ,  $\{x_{\alpha_n}\}$  converges to at most one point. Now construct a sequence  $\{x_n\}$  where  $x_i = x_{\alpha_i}$  and  $x_{\alpha_i}$  is the  $i$ -th term of  $(x_\alpha)$ , we see that  $\{x_{\alpha_n}\}$  is a subsequence of  $\{x_n\}$ , by definition, and hence  $\{x_n\}$  must also converge at at most one point. ■

**Example 2.1.** For Euclidean space  $\mathbb{R}^n$ , a basis consists of all products of the form  $(a_1, b_1) \times \dots \times (a_n, b_n)$  where  $(a_i, b_i)$  is an open interval for all  $1 \leq i \leq n$ . Since  $\mathbb{R}^n$  is a finite product space, both the box and product topologies on  $\mathbb{R}^n$  are the same.

**Theorem 2.1.7.** *If  $A_\alpha \subseteq X_\alpha$ , then  $\prod \text{cl } A_\alpha = \text{cl } \prod A_\alpha$*

*Proof.* Let  $x = (x_\alpha) \in \prod \text{cl } A_\alpha$  and let  $U = \prod U_\alpha$  be a basis element (for either topology). Choosing  $y_\alpha \in U_\alpha \cap A_\alpha$ , for each  $\alpha$ , let  $y = (y_\alpha)$ . Then  $y \in U$ , and  $y \in \prod A_\alpha$ , hence  $x \in \text{cl } \prod A_\alpha$ .

Now suppose that  $x \in \prod A_\alpha$  (in either topology). Let  $V_\beta$  be an open set of  $X_\beta$  containing  $x_\beta$ . Since  $\pi_\beta^{-1}(V_\beta)$  is open in  $\prod X_\alpha$  (in either topology), it contains a point  $y = (y_\alpha)$  of  $\prod A_\alpha$ . Then  $y_\beta \in V_\beta \cap A_\beta$ , hence  $x \in \text{cl } A_\beta$ . ■

**Theorem 2.1.8.** *Let  $f : A \rightarrow \prod X_\alpha$  be defined by  $f(a) = (f_\alpha(a))$ , where  $f_\alpha : A \rightarrow X_\alpha$ . Letting  $\prod X_\alpha$  have the product space topology,  $f$  is continuous if and only if  $f_\alpha$  is continuous for each  $\alpha$ .*

*Proof.* We know that the projection mapping  $\pi_\beta$  is continuous. Now suppose that  $f$  is continuous, and notice that  $f_\beta = \pi_\beta \circ f$ , which makes  $f_\beta$  continuous for each  $\beta$ .

On the other hand, suppose that  $f_\beta$  is continuous for each  $\beta$ . Notice that  $f_\beta^{-1} = f^{-1} \circ \pi_\beta^{-1}$ , since  $\pi_\beta^{-1}(U_\beta)$  is open in  $\prod X_\alpha$ , then so is  $f^{-1} \circ \pi_\beta^{-1}(U_\beta) = f_\beta^{-1}(U_\beta)$ . This makes  $f$  continuous. ■

**Example 2.2.** Theorem 2.1.8 holds only for the product space topology and fails in general for the box topology. Consider  $\mathbb{R}^\omega$  and define the map  $f : \mathbb{R} \rightarrow \mathbb{R}^\omega$  by  $f(t) = (t, t, t, \dots)$ . We have that  $f_n(t) = t$  is continuous, which makes  $f$  continuous under the product topology. Now consider the box topology: let  $B = (-1, 1) \times (-\frac{1}{2}, \frac{1}{2}) \times (-\frac{1}{3}, \frac{1}{3}) \times \dots$ , and suppose that  $f^{-1}(B)$  were open. Then it contains some interval  $(-\delta, \delta)$ , about 0, thus  $\pi_\beta \circ f((-\delta, \delta)) = f_\beta((-\delta, \delta)) = (-\delta, \delta) \subseteq (-\frac{1}{n}, \frac{1}{n})$ , which is absurd. Thus the only implication of the theorem that holds for the box topology is that  $f_\alpha$  is continuous only when  $f$  is continuous.

## 2.2 The Metric Topology

**Definition.** A **metric** (or **distance function**) on a set  $X$  is a map  $d : X \times X \rightarrow \mathbb{R}$  satisfying the following for all  $x, y, z \in X$ :

- (1)  $d(x, y) \geq 0$  and  $d(x, y) = 0$  if and only if  $x = y$ .
- (2)  $d(x, y) = d(y, x)$ .
- (3)  $d(x, y) \leq d(x, z) + d(z, y)$  (The Triangle Inequality).

We call  $d(x, y)$  the **distance** between  $x$  and  $y$ , and given  $\varepsilon > 0$ , we define the  **$\varepsilon$ -ball centered about  $x$**  to be the set  $B_d(x, \varepsilon) = \{y \in X : d(x, y) < \varepsilon\}$ .

**Lemma 2.2.1.** *Let  $d$  be a metric on  $X$ . For  $x, y \in X$ , and  $B_d(x, \varepsilon)$  an  $\varepsilon$ -ball centered about  $x$ , there is a  $\delta$ -ball centered about  $y$ ,  $B_d(y, \delta)$  such that  $B_d(y, \delta) \subseteq B_d(x, \varepsilon)$ .*

*Proof.* Let  $y \in B_d(x, \varepsilon)$  and let  $\delta = \varepsilon - d(x, y)$ , and take  $z \in B_d(y, \delta)$ , then we have that  $d(y, z) < \varepsilon - d(x, y)$ , thus  $d(x, z) \leq d(x, y) + d(y, z) < \varepsilon$  which completes the proof. ■

**Theorem 2.2.2.** *Let  $d$  be a metric on  $X$ . Then the collection of all  $\varepsilon$ -balls about  $x$ , for some  $x \in X$  forms the basis for a topology on  $X$ .*

*Proof.* Clearly  $x \in B_d(x, \varepsilon)$ , by definition, so it remains to show that the intersection of two  $\varepsilon$ -balls contains an  $\varepsilon$ -ball. Let  $B_1$  and  $B_2$  be  $\varepsilon$ -balls about  $x$ , and let  $y \in B_1 \cap B_2$ . By lemma 2.2.1, there are  $\delta_1, \delta_2 > 0$  such that  $B_d(y, \delta_1) \subseteq B_1$  and  $B_d(y, \delta_2) \subseteq B_2$ . Now take  $\delta = \min\{\delta_1, \delta_2\}$ , then we see that  $B_d(y, \delta) \subseteq B_1 \cap B_2$ . ■

**Definition.** If  $d$  is a metric on  $X$ , we call the topology having as basis the collection of all  $\varepsilon$ -balls centered about  $x$ , for some  $x \in X$  and  $\varepsilon > 0$ , the **metric topology** induced by  $d$ .

**Corollary.** *A set  $U$  is open in the metric topology induced by  $d$  if and only if for each  $y \in U$ , and  $\delta > 0$ , there is a  $\delta$ -ball centered about  $y$  contained in  $U$ .*

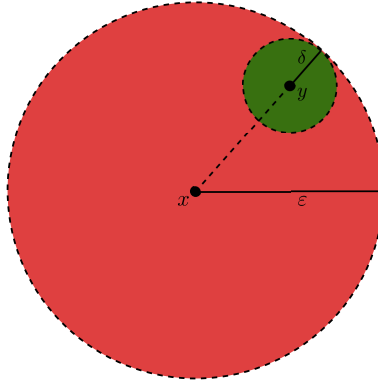


Figure 2.1: All  $\varepsilon$ -balls centered about  $x$  are open in the metric topology by lemma 2.2.1.

**Example 2.3.** (1) Define  $d(x, y) = 1$  if  $x \neq y$  and  $d(x, y) = 0$  if  $x = y$ . Clearly  $d$  is a metric on  $X$ , and induces the discrete topology on  $X$ . The basis  $B_d(x, 1) = \{x\}$

(2) The standard metric on  $\mathbb{R}$  is defined to be  $d(x, y) = |x - y|$  and is a metric on  $\mathbb{R}$  (that is, the absolute  $|\cdot|$  is a metric on  $\mathbb{R}$ ). This metric induces the standard topology on  $\mathbb{R}$  as we see that it has basis  $B_d(x, \varepsilon) = \{y \in \mathbb{R} : |x - y| < \varepsilon\} = \{y \in \mathbb{R} : y - \varepsilon < x < y + \varepsilon\} = (y - \varepsilon, y + \varepsilon)$ .

(3) Define  $d : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}$  by

$$d(x, y) = \sum_{i=1}^n |x_i - y_i|$$

We have that  $d$  is a metric, for  $d(x, y) \geq 0$  and  $d(x, y) = 0$  if and only if  $x = y$ . Likewise,  $d(x, y) = d(y, x)$ , since  $|x_i - y_i| = |y_i - x_i|$ .

Lastly, we see for  $x, y, z \in \mathbb{R}^n$ ,  $|x_i - y_i| \leq |x_i - z_i| + |z_i - y_i|$ , so that  $\sum |x_i - y_i| \leq \sum |x_i - z_i| + \sum |z_i - y_i|$ , so  $d(x, y) \leq d(x, z) + d(z, y)$ .

Now, consider the open balls under  $d$ , and let  $y \in B(x, \varepsilon)$ . Then  $d(x, y) = \sum |x_i - y_i| < \varepsilon$ , which makes  $|x_i - y_i| < \varepsilon$ , and hence  $y_i - \varepsilon < x_i < y_i + \varepsilon$ , for all  $1 \leq i \leq n$ , so  $y_i \in (x_i - \varepsilon, x_i + \varepsilon)$ . This puts  $y \in (x_1 - \varepsilon, x_1 + \varepsilon) \times \cdots \times (x_n - \varepsilon, x_n + \varepsilon)$ . So  $d$  induces the usual topology on  $\mathbb{R}^n$ . We call this metric the **taxicab metric**.

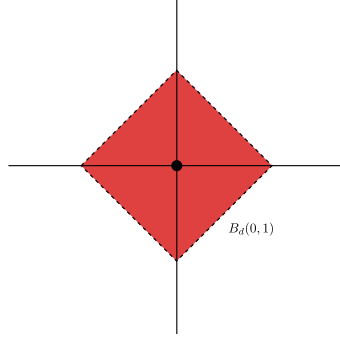


Figure 2.2: The open ball of radius 1 centered about the point  $0 \times 0$  in the taxicab metric

(4) For the metric  $d' : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}$  defined by

$$d'(x, y) = \sqrt[p]{\sum_{i=1}^n |x_i - y_i|^p}$$

for some  $p$  also induces the usual topology on  $\mathbb{R}^n$ . For if  $y \in B(x, \varepsilon)$ , then  $d'(x, y) = \sqrt[p]{\sum_{i=1}^n |x_i - y_i|^p} < \varepsilon^p$ , which makes  $|x_i - y_i|^p < \varepsilon^p$ , so that  $|x_i - y_i| < \varepsilon$ , which is the same as the above example.

**Lemma 2.2.3.** *If  $X$  is a metric space, then the metric of  $X$  is a continuous function.*

*Proof.* Let  $d : X \times X \rightarrow \mathbb{R}$  be the metric of  $X$ , and let  $U \in \mathbb{R}$ . Then if  $x \times y \in d(U)^{-1}$ , then  $a = d(x, y) \in U$ . We also have then in the usual metric of  $\mathbb{R}$ , that  $(a - \varepsilon, a + \varepsilon) \subseteq U$ , for some  $\varepsilon > 0$ . Now, take  $x' \times y' \in B(x, \frac{\varepsilon}{2}) \times B(y, \frac{\varepsilon}{2})$ . By the triangle inequality, we have that  $|d(x', y') - d(x, y)| \leq d(x, x') + d(y, y') < \varepsilon$ , so that  $d(x', y') \in (a - \varepsilon, a + \varepsilon) \subseteq U$ . This makes  $B(x, \frac{\varepsilon}{2}) \times B(y, \frac{\varepsilon}{2}) \subseteq d^{-1}(U)$  open in  $X \times X$ , which makes  $d^{-1}(U)$  open in  $X$  ■

**Definition.** If  $X$  is a topological space, we call  $X$  **metrizable** if there is a metric  $d$  which induces the topology on  $X$ . A **metric space** is a metrizable space  $X$  together with the metric inducing the topology of  $X$ .

**Definition.** Let  $X$  be a metric space with metric  $d$ . A subset  $A \subseteq X$  is said to be **bounded** if there is an  $M > 0$  such that  $d(a_1, a_2) \leq M$  for all  $a_1, a_2 \in A$ . We define the **diameter** of a bounded set  $A$  to be  $\text{diam } A = \sup\{d(a_1, a_2) : a_1, a_2 \in A\}$ .

It is easy to see that boundedness of a set does not depend on the topology of  $X$ , but on the metric.

**Theorem 2.2.4.** *Let  $X$  be a metric space with metric  $d$  and define  $\bar{d} : X \times X \rightarrow \mathbb{R}$  by  $\bar{d} = \min\{d(x, y), 1\}$  for all  $x, y \in X$ . Then  $\bar{d}$  is a metric on  $X$  that induces the same topology as  $d$ .*

*Proof.* Clearly we have that  $0 \leq \bar{d}(x, y) \leq 1$ , and that  $\bar{d}(x, y) = \min\{d(x, y), 1\} = \min\{d(y, x), 1\} = \bar{d}(y, x)$ . It remains to show the triangle inequality.

Now if  $d(x, z) \leq 1$  and  $d(z, y) \leq 1$ , then by the triangle inequality  $d(x, y) \leq 1$  and  $\bar{d}(x, y) \leq \bar{d}(x, z) + \bar{d}(z, y)$ . Now if  $d(x, z) < 1$  and  $d(z, y) < 1$ , we get the same conclusion. Thus we see that  $\bar{d}$  is a metric on  $X$ .

Now take as basis the collection of all  $\varepsilon$ -balls with  $0 < \varepsilon < 1$ , and any basis element of  $x$  contains such an  $\varepsilon$ -ball, thus  $\bar{d}$  induces the same topology as  $d$ . ■

**Definition.** Let  $X$  be a metric space with metric  $d$ . We define the **standard bounded metric** to be the metric  $\bar{d} : X \times X \rightarrow \mathbb{R}$  defined by the rule  $\bar{d}(x, y) = \min\{d(x, y), 1\}$  for any  $x, y \in X$ .

**Definition.** Let  $x \in \mathbb{R}^n$ . We define the **norm** of  $x$ ,  $\|x\|$ , to be  $\|x\| = \sqrt{x_1^2 + \cdots + x_n^2}$ . We define the **square metric**  $\rho$  on  $\mathbb{R}^n$  to be  $\rho(x, y) = \{|x_1 - y_1|, \dots, |x_n - y_n|\}$ .

Before we show that  $\|\cdot\|$  and  $\rho$  are metrics, we introduce the following:

**Definition.** Let  $x, y \in \mathbb{R}^n$ . We define the **inner product** of  $x$  and  $y$  to be:

$$\langle x, y \rangle = x_1 y_1 + \cdots + x_n y_n \quad (2.1)$$

**Lemma 2.2.5.** For  $x, y \in \mathbb{R}^n$ ,  $\langle x, y + z \rangle = \langle x, y \rangle + \langle x, z \rangle$  and  $|\langle x, y \rangle| \leq \|x\| \|y\|$ .

*Proof.* We have  $\langle x, y + z \rangle = x_1(y_1 + z_1) + \cdots + x_n(y_n + z_n) = (x_1 y_1 + \cdots + x_n y_n) + (x_1 z_1 + \cdots + x_n z_n) = \langle x, y \rangle + \langle x, z \rangle$ .

Now if  $x = 0$  and  $y = 0$ , then  $|\langle x, y \rangle| = \|x\| \|y\| = 0$ , so suppose that both  $x, y \neq 0$ , and let  $a = \frac{1}{\|x\|}$  and  $b = \frac{1}{\|y\|}$ . Notice that  $\|ax + by\| \geq 0$  and  $\|ax - by\| \geq 0$ , then  $\|ax + by\|^2 \|ax - by\|^2 = \|x\|^2 \|y\|^2 - |\langle x, y \rangle|^2 \geq 0$ , hence  $|\langle x, y \rangle| \leq \|x\| \|y\|$ . ■

*Remark.* We call the last relation in the lemma the **Cauchy-Schwarz inequality**.

**Theorem 2.2.6.** Both the norm and square metrics make  $\mathbb{R}^n$  into a metric space.

*Proof.* We start with the norm. Now clearly, since  $\sqrt{x} \geq 0$  (for real numbers),  $\|x - y\| \geq 0$ , and if  $\|x - y\| = 0$  then  $(x_1 - y_1)^2 + \cdots + (x_n - y_n)^2 = 0$  hence  $x_i = y_i$  for all  $1 \leq i \leq n$ , and if  $x_i = y_i$ , then clearly  $\|x - y\| = 0$ . We also see that  $\|x - y\| = \|y - x\|$ .

Now consider  $z \in \mathbb{R}^n$ , notice that  $\|x + y\|^2 = \langle x + y, x + y \rangle = \langle x + y, x \rangle + \langle x + y, y \rangle = \langle x, x \rangle + \langle y, y \rangle \leq \|x\|^2 + \|y\|^2$ . Using this we have  $\|x - z\| + \|z - y\| \geq \|x - y\|$  (square the left hand side and evaluate), so  $\|\cdot\|$  is a metric on  $\mathbb{R}^n$ .

Now consider the square metric. Clearly we have that  $\rho(x, y) \geq 0$  and that  $\rho(x, y) = 0$  if and only if  $x = y$  (since  $|\cdot|$  is also a metric), we also see that  $\rho(x, y) = \rho(y, x)$ .

Now let  $x \in \mathbb{R}^n$ , and we have for all  $1 \leq i \leq n$  that  $|x_i - y_i| \leq |x_i - z_i| + |z_i - y_i|$ , hence by definition  $|x_i - y_i| \leq \rho(x, z) + \rho(z, y)$ , thus  $\rho(x, y) \leq \rho(x, z) + \rho(z, y)$  and  $\rho$  is a metric. ■

**Lemma 2.2.7.** Let  $d$  and  $d'$  be metrics on  $X$  and let  $\mathcal{T}$  and  $\mathcal{T}'$  be the topologies induced by  $d$  and  $d'$  respectively.  $\mathcal{T} \subseteq \mathcal{T}'$  if and only if for each  $x \in X$ , and  $\varepsilon > 0$ , there is a  $\delta > 0$  such that  $B_{d'}(x, \delta) \subseteq B_d(x, \varepsilon)$ .

*Proof.* Suppose that  $\mathcal{T} \subseteq \mathcal{T}'$ , and take  $B_d(x, \varepsilon)$  in  $\mathcal{T}$ , then by lemma 2.2.1, there is a  $B'$  in  $\mathcal{T}'$  or which  $B' \subseteq B_d(x, \varepsilon)$ , hence there is a  $\delta$ -ball about  $x$  for which  $B_{d'}(x, \delta) \subseteq B'$ .

Conversly, suppose for  $x \in X$  and  $\varepsilon > 0$ , that there is a  $\delta > 0$  for which  $B_{d'}(x, \delta) \subseteq B_d(x, \varepsilon)$ . Given a basis  $B$  of  $\mathcal{T}$ , there is an  $\varepsilon$ -ball about  $x$  contained in  $B$ , hence  $B_d(x, \varepsilon)$  is also in  $B$ , thus we have that  $\mathcal{T} \subseteq \mathcal{T}'$ . ■



**Theorem 2.2.8.** *The norm and the square metric both induce the product topology on  $\mathbb{R}^n$ .*

*Proof.* Notice that  $\rho(x, y) \leq \|x - y\| \leq \rho(x, y)\sqrt{n}$ . This first inequality shows that  $B_{\|\cdot\|}(x, \varepsilon) \subseteq B_\rho(x, \varepsilon)$ , and the second shows that  $B_\rho(x, \frac{\varepsilon}{\sqrt{n}}) \subseteq B_{\|\cdot\|}(x, \varepsilon)$ , thus both  $\|\cdot\|$  and  $\rho$  induce the same topology.

Now let  $B = (a_1, b_1) \times \cdots \times (a_n, b_n)$  be a basis for the product topology on  $\mathbb{R}^n$ . Since, for each  $i$ , there is an  $\varepsilon_i > 0$  such that  $(x_i - \varepsilon_i, x_i + \varepsilon_i) \subseteq (a_i, b_i)$ , choosing  $\varepsilon = \min\{\varepsilon_1, \dots, \varepsilon_n\}$ , we see that  $B_\rho(x, \varepsilon) \subseteq B$ . Conversely, given  $y \in B_\rho(x, \varepsilon)$ , notice that  $B_\rho(x, \varepsilon) = (x_1 - \varepsilon, x_1 + \varepsilon) \times \cdots \times (x_n - \varepsilon, x_n + \varepsilon)$ , thus  $B \subseteq B_\rho(x, \varepsilon)$ . Thus the topologies are the same. ■

**Definition.** Given an index set  $J$  and given points  $x = (x_\alpha)$  and  $y = (y_\alpha)$  of  $\mathbb{R}^J$ , we define the **uniform metric**  $\bar{\rho}$  on  $\mathbb{R}^J$  by  $\bar{\rho}(x, y) = \sup\{\bar{d}(x_\alpha, y_\alpha) : \alpha \in J\}$ , where  $\bar{d}$  is the standard bounded metric on  $\mathbb{R}^J$ . We call the topology induced by  $\bar{\rho}$  the **uniform topology**.

**Theorem 2.2.9.** *The uniform topology on  $\mathbb{R}^J$  is finer than the product topology on  $\mathbb{R}^J$  and coarser than the box topology on  $\mathbb{R}^J$ , and all three topologies are different if  $J$  is infinite.*

*Proof.* Let  $x = (x_\alpha)$  and  $\prod U_\alpha$  be a basis element for the product topology, and let  $\alpha_1, \dots, \alpha_n$  be the indices for which  $U_\alpha \neq \mathbb{R}$ , and for each  $i$ , choose  $\varepsilon_i > 0$  such that the  $\varepsilon_i$ -ball about  $x_{\alpha_i}$ ,  $B_{\bar{d}}(x_{\alpha_i}, \varepsilon_i) \subseteq U_{\alpha_i}$ . Let  $\varepsilon = \min\{\varepsilon_1, \dots, \varepsilon_n\}$ , and let  $z = (z_\alpha) \in \mathbb{R}^J$  be such that  $\bar{\rho}(x, z) < \varepsilon$ . Then  $\bar{d}(x_\alpha, z_\alpha) < \varepsilon$  for all  $\alpha$ . Hence  $B_{\bar{\rho}}(x, \varepsilon) \subseteq \prod U_\alpha$  for all  $\alpha$ ; so the uniform topology is finer.

Likewise, consider  $B$  the  $\varepsilon$ -ball about  $x$  in the  $\bar{\rho}$  metric. Then the box  $U = \prod (x_\alpha - \frac{\varepsilon}{2}, x_\alpha + \frac{\varepsilon}{2})$  and  $y \in U$  if  $\bar{d}(x_\alpha, y_\alpha) < \frac{\varepsilon}{2}$ , then  $\bar{\rho}(x, y) \leq \frac{\varepsilon}{2}$ , so  $U \subseteq B$ , and the uniform topology is coarser.

Now in the case where  $J$  is infinite, if  $J$  is uncountable, we are done, since there is no way to map the indices of  $J$  onto  $\mathbb{Z}^+$ . So consider the case where  $J$  is countable, and map  $J \rightarrow \mathbb{Z}^+$  by  $\alpha_i \rightarrow i$ . Let  $U = \prod (x_i - \varepsilon, x_i + \varepsilon)$  and consider a base  $B_{\bar{\rho}}(x, \varepsilon)$ . We have that for  $y \in B_{\bar{\rho}}(x, \varepsilon)$  that  $\bar{d}(x_\alpha, y_\alpha) = \min\{\rho(x_i, y_i), 1\}$ , we have that  $\bar{d}(x_i, y_i) = \bar{\rho}(x_i, y_i)$  or 1, and if we choose  $0 < \varepsilon < 1$ , then  $\bar{\rho}$  fails to put  $B_{\bar{\rho}}(x, \varepsilon)$  inside of  $U$ . Likewise, the basis  $\prod U_\alpha$  (in the product topology) fails to be contained in  $B_{\bar{\rho}}(x, \varepsilon)$  by the same argument. Thus the uniform topology is not necessarily finer than the box topology, nor coarser than the product topology in  $\mathbb{R}^J$ , when  $J$  is infinite. ■

*Remark.* Clearly the box, product and uniform topologies on  $\mathbb{R}^J$  are the same when  $J$  is finite, as the box and product topologies are the same for finite product spaces.

**Theorem 2.2.10.** *Let  $\bar{d}(a, b) = \min\{|a - b|, 1\}$  be the standard bounded metric on  $\mathbb{R}$ , and for  $x, y \in \mathbb{R}^\omega$ , define:*

$$D(x, y) = \sup_i \left\{ \frac{\bar{d}(x_i, y_i)}{i} \right\} \quad (2.2)$$

*Then  $D$  is a metric that induces the product space topology on  $\mathbb{R}^\omega$*

*Proof.* Since  $\bar{d}$  is a metric,  $D$  satisfies the conditions for a metric, it is worth looking into the case for the triangle inequality. Notice that for all  $i$ ,  $D(x, y) \leq \frac{\bar{d}(x_i, y_i)}{i} \leq \frac{\bar{d}(x_i, z_i)}{i} + \frac{\bar{d}(z_i, y_i)}{i} \leq D(x, z) + D(z, y)$ . Therefore  $D(x, y) \leq D(x, z) + D(z, y)$ .

Now let  $U$  be open in the metric topology induced by  $D$ , and let  $x \in U$ . Choose an  $\varepsilon$ -ball  $B_D(x, \varepsilon) \subseteq U$  and choose  $N > 0$  arbitrarily large such that  $\frac{1}{N} < \varepsilon$  and let  $V = (x_1 - \varepsilon, x_1 + \varepsilon) \times \cdots \times (x_N - \varepsilon, x_N + \varepsilon) \times \mathbb{R} \times \cdots \times \mathbb{R} \dots$ . Given  $y \in \mathbb{R}^\omega$ , we have  $\frac{\bar{d}(x_i, y_i)}{i} \leq \frac{1}{N}$  for

all  $i \geq N$ , thus  $D(x, y) \leq \max\{\frac{\bar{d}(x_1, y_1)}{1}, \dots, \frac{\bar{d}(x_N, y_N)}{N}, \frac{1}{N}\}$ . Now if  $y \in V$ , then the expression is less than  $\varepsilon$ , so  $V \subseteq B_D(x, \varepsilon)$ .

Conversely let  $U = \prod U_i$  where  $U_i$  is open in  $\mathbb{R}$  for  $i = \alpha_1, \dots, \alpha_n$ , and  $U_i = \mathbb{R}$  for all other indices. Given  $x \in U$ , for  $i = \alpha_1, \dots, \alpha_n$ , and letting  $\varepsilon = \min\{\frac{\varepsilon_i}{i} : i = \alpha_1, \dots, \alpha_n\}$  where  $\frac{1}{N} < \varepsilon_i$  for each  $i$ . Letting  $B_d(x, \varepsilon)$  be the corresponding  $\varepsilon$ -ball under the metric  $d$ , we see that  $\frac{\bar{d}(x_i, y_i)}{i} \leq D(x, y) < \varepsilon$ . Now if  $i = \alpha_1, \dots, \alpha_n$ , then  $\bar{d}(x_i, y_i) < \varepsilon_i \leq 1$ , hence  $|x_i - y_i| < \varepsilon_i$ , putting  $y \in U$ . ■

## 2.3 More on Metric Spaces

We go more in depth on metric spaces here.

**Theorem 2.3.1.** *If  $A$  is a subspace of a metric space  $X$ , with metric  $d$ , then  $d$  restricted to  $A \times A$  makes  $A$  into a metric space.*

*Proof.* Clearly  $d : A \times A \rightarrow \mathbb{R}$  is a metric. So consider the  $\varepsilon$ -ball about  $x$ ,  $B_d(x, \varepsilon)$  in  $X$ ; restricting  $d$  to  $A \times A$ , consider  $A \cap B_d(x, \varepsilon)$ . For  $y \in A$ , there is a  $\delta$ -ball about  $y$  such that  $B_d(y, \delta) \subseteq B_d(x, \varepsilon)$ ; then  $B_d(y, \delta) \subseteq B_d(x, \varepsilon)$ . This makes  $A$  as a subspace, into a metric space. ■

**Corollary.** *The metric topology on  $A$  is the same as the subspace topology on  $A$ .*

**Theorem 2.3.2.** *The Hausdorff axiom is satisfied in every metric space.*

*Proof.* If  $x, y \in X$  are distinct, let  $\varepsilon = \frac{1}{2}d(x, y)$ , by the triangle inequality, we have that  $B_d(x, \varepsilon)$  and  $B_d(y, \varepsilon)$  are disjoint. ■

**Theorem 2.3.3.** *Countable products of metrizable spaces are metrizable.*

*Proof.* Let  $X$  be a metric space with metric  $d$ . Define  $\bar{d}(x, y) = \min\{d(x, y), 1\}$  on  $X$  and define  $D(x, y) = \sup\{\frac{\bar{d}(x_i, y_i)}{i}\}$  on  $X^\omega$ . It is clear that both  $\bar{d}$  and  $D$  are metrics on  $X$  and  $X^\omega$  respectively. We would like to show that  $D$  induces the product topology on  $X^\omega$ .

Let  $U$  be open and let  $x \in U$ . Choose  $B_D(x, \varepsilon) \subseteq U$  and choose  $N$  large enough such that  $\frac{1}{N} < \varepsilon$ . Now let  $V = (x_1 - \varepsilon, x_1 + \varepsilon) \times \dots \times (x_N - \varepsilon, x_N + \varepsilon) \times X \times \dots$  be a basis in the product topology on  $X^\omega$ . Given  $y \in X^\omega$ , such that  $\frac{\bar{d}(x_i, y_i)}{i} \leq \frac{1}{N}$ , we have by definition that  $D(x, y) \leq \max\{\bar{d}(x_1, y_1), \frac{\bar{d}(x_2, y_2)}{2}, \dots, \frac{\bar{d}(x_N, y_N)}{N}, \frac{1}{N}\}$ . If  $y \in V$ , we get that  $V \subseteq B_D(x, \varepsilon)$  and we are done.

Conversely let  $U = \prod U_i$  be a basis of the product topology where  $U_i$  is open in  $X$  for  $i = 1, \dots, n$  and  $U_i = X$  for all other indices. Now let  $x \in U$  and choose an interval about  $x_i$ ,  $(x_i - \varepsilon_i, x_i + \varepsilon_i)$  lying in  $U_i$  with  $0 < \varepsilon_i \leq 1$  for all  $i$ . Choose  $\varepsilon = \min\{\varepsilon_1, \frac{\varepsilon_2}{2}, \dots, \frac{\varepsilon_n}{n}\}$ . Then  $x \in B_D(x, \varepsilon) \subseteq U$ , for if  $y \in B_D(x, \varepsilon)$ , we have that  $\frac{\bar{d}(x_i, y_i)}{i} \leq D(x, y) < \varepsilon$ , hence  $\varepsilon \leq \frac{\varepsilon_i}{i}$  and  $d(x_i, y_i) < \varepsilon_i$ , and so  $y \in U_i$ . Therefore  $D$  induces the product space topology. ■

*Remark.* This theorem generalizes theorem 2.2.9 for any countable product space of a metric space  $X$ . Hence we can take theorem 2.2.9 as a corollary to this theorem.

We would now like to study continuous functions in metric spaces, which brings us into the realm of analysis. We show that the “ $\varepsilon$ - $\delta$ ” definition, and the sequence definition of continuity carry over.

**Theorem 2.3.4.** *Let  $f : X \rightarrow Y$  with  $X$  and  $Y$  metric spaces with metric  $d_X$  and  $d_Y$  respectively. Then  $f$  is continuous if and only if for  $x \in X$ , and  $\varepsilon > 0$ , there is a  $\delta > 0$  such that  $d_Y(f(x), f(y)) < \varepsilon$  whenever  $d_X(x, y) < \delta$ .*

*Proof.* Suppose that  $f$  is continuous and consider  $f^{-1}(B(f(x), \varepsilon))$  open in  $X$ . Then it contains a  $\delta$ -ball  $B(x, \delta)$  about  $x$ . If  $y \in B(x, \delta)$ , then  $f(y) \in B(f(x), \varepsilon)$ , as is required.

Now suppose that for  $x \in X$  and  $\varepsilon > 0$ , that there is a  $\delta > 0$  such that  $d_Y(f(x), f(y)) < \varepsilon$  whenever  $d_X(x, y) < \delta$ , for  $x \in X$ . Let  $V$  be open in  $Y$  and let  $x \in f^{-1}(V)$ . Then  $f(x) \in V$ , hence there is an  $\varepsilon$ -ball  $B(f(x), \varepsilon) \subseteq V$ . By hypothesis, there is a  $\delta > 0$  such that  $f(B(x, \delta)) \subseteq B(f(x), \varepsilon)$ , hence  $B(x, \delta) \subseteq f^{-1}(V)$  which makes  $f^{-1}(V)$  open. ■

**Lemma 2.3.5** (The Sequence Lemma). *Let  $X$  be a topological space and let  $A \subseteq X$ . If there is a sequence of points of  $A$  converging to  $x \in X$ , then  $x \in \text{cl } A$ . The converse holds if  $X$  is metrizable.*

*Proof.* Suppose for some sequence  $\{x_n\} \subseteq A$  that  $x_n \rightarrow x$ . By theorem 1.6.6, we have every neighborhood of  $x$  contains points of  $A$ , hence  $x \in \text{cl } A$ . Conversely, suppose that  $X$  is metrizable with metric  $d$ , and let  $x \in \text{cl } A$ . For  $n \in \mathbb{Z}^+$ , take  $B_d(x, \frac{1}{n})$  and take  $\{x_n\} = B_d(x, \frac{1}{n}) \cap A$ . Then  $x_n \rightarrow x$ , for: any open set  $U$  of  $x$  contains an  $\varepsilon$ -ball about  $x$ ,  $B_d(x, \varepsilon)$ , so choose  $N$  large enough so that  $\frac{1}{N} < \varepsilon$ , hence  $U$  contains  $x_i$  for all  $i \geq N$ . ■

**Theorem 2.3.6** (The Sequential Criterion). *Let  $f : X \rightarrow Y$  be continuous, then for every convergent sequence  $\{x_n\}$  converging to  $x \in X$ , the sequence  $\{f(x_n)\}$  converges to  $f(x)$ . the converse holds if  $X$  is metrizable.*

*Proof.* Let  $f$  be continuous and suppose that  $x_n \rightarrow x$ . Let  $V$  be a neighborhood of  $f(x)$ , then  $f^{-1}(V)$  is a neighborhood of  $x$ ; hence there is an  $N > 0$  such that  $x_n \in f^{-1}(V)$  whenever  $n \geq N$ , thus  $f(x_n) \in V$  whenever  $n \geq N$ .

Conversely suppose that for every  $\{x_n\}$  converging to  $x$ , that  $\{f(x_n)\}$  converges to  $f(x)$ , and let  $A \subseteq X$ . if  $x \in \text{cl } A$ , by the sequence lemma, there is a sequence  $\{x_n\} \subseteq A$  converging to  $x$ . Now since  $f(x_n) \rightarrow f(x)$ . and  $f(x_n) \in f(A)$ , by the sequence lemma again,  $f(x) \in \text{cl } f(A)$ . Thus  $f(\text{cl } A) \subseteq \text{cl } f(A)$  and we are done. ■

We now consider methods for constructing continuous functions on metric spaces.

**Lemma 2.3.7.** *The additions, subtraction, and multiplication operations are continuous from  $\mathbb{R} \times \mathbb{R}$  to  $\mathbb{R}$ . The quotient operation is continuous from  $\mathbb{R} \times \mathbb{R} \setminus \{0\}$  to  $\mathbb{R}$ .*

**Theorem 2.3.8.** *If  $X$  is a topological space and if  $f, g : X \rightarrow \mathbb{R}$  are continuous, then  $f + g$ ,  $f - g$ , and  $fg$  are continuous; moreover if  $g(x) \neq 0$  for all  $x \in X$ , then  $\frac{f}{g}$  is also continuous.*

*Proof.* The map  $h : X \rightarrow \mathbb{R} \times \mathbb{R}$  defined by  $h(x) = (f(x), g(x))$  is continuous. Then notice that  $f + g(x) = +(f(x), g(x)) = + \circ h(x)$ , so by the above lemma, we get that  $f + g$  is continuous. We also have that  $f - g$  is continuous for  $f - g(x) = +(f(x), -g(x))$ . The same argument holds for  $fg$  and  $\frac{f}{g}$ . ■

**Definition.** Let  $f_n : X \rightarrow Y$  be a sequence of functions from  $X$  to the metric space  $Y$ , with metric  $d$ . We say that the sequence  $\{f_n\}$  **converges uniformly** to the function  $f : X \rightarrow Y$  if for  $\varepsilon > 0$ , there is an integer  $N > 0$  such that  $d(f_n(x), f(x)) < \varepsilon$  whenever  $n \geq N$ , for all  $x \in X$ .

**Theorem 2.3.9.** Let  $f_n : X \rightarrow Y$  be a sequence of continuous functions from the topological space  $X$  to the metric space  $Y$ . If  $\{f_n\}$  converges uniformly to  $f$ , then  $f$  is continuous.

*Proof.* Let  $V$  be open in  $Y$  and let  $x_0 \in f^{-1}(V)$ . Let  $y_0 = f(x_0)$  and choose  $\varepsilon > 0$  such that  $B(y_0, \varepsilon) \subseteq V$ . By uniform convergence, choosing  $N > 0$  so that whenever  $n \geq N$ ,  $d(f_n(x), f(x)) < \frac{\varepsilon}{3}$  for all  $x \in X$ . By the continuity of  $f_N$ , choose a neighborhood  $U$  of  $x_0$  such that  $f_N(U) \subseteq B(f_N(x_0), \frac{\varepsilon}{3})$ . Then if  $x \in U$ , we have  $d(f(x), f_N(x)) < \frac{\varepsilon}{3}$ ,  $d(f_N(x), f_N(x_0)) < \frac{\varepsilon}{3}$ , and  $d(f_N(x_0), f(x_0)) < \frac{\varepsilon}{3}$  by the triangle inequality we get  $d(f(x), f(x_0)) < \varepsilon$  which completes the proof. ■

**Example 2.4.** (1)  $\mathbb{R}^\omega$  is not metrizable in the box topology. Let  $A = \{(x_1, x_2, \dots) \in \mathbb{R}^\omega : x_i > 0\}$  and consider  $0 = (0, 0, \dots) \in \mathbb{R}^\omega$ .  $0 \in \text{cl } A$  if for any basis element  $B = (a_1, b_1) \times (a_2, b_2) \times \dots$ ,  $0 \in B$ ; then  $B \cap A \neq \emptyset$  (take the point  $\frac{1}{2}b \in \mathbb{R}^\omega$ ). Now let  $\{a_n\}$  be a sequence of points of  $A$  with  $a_n = (x_{1n}, x_{2n}, \dots, x_{in}, \dots)$ , since  $x_{in} > 0$ , construct a basis element  $B' = (-x_{11}, x_{11}) \times (-x_{22}, x_{22}) \times \dots$ . Then  $0 \in B'$ , but  $\{a_n\} \not\subseteq B'$  for the point  $x_{nn} \notin (-x_{nn}, x_{nn})$ . Thus  $a_n \not\rightarrow 0$ .

(2) An uncountable product of  $\mathbb{R}$  with itself is not metrizable. Let  $J$  be uncountable, and let  $A = \{(x_\alpha) \in \mathbb{R}^J : x_\alpha = 1 \text{ for all but finitely many } \alpha\}$ . Consider  $0 \in \mathbb{R}^J$  and let  $U$  be a basis for containing  $0$ . Now  $U_\alpha \neq \mathbb{R}$  for  $\alpha_1, \dots, \alpha_n$ , so let  $(x_\alpha) \in A$  be defined by letting  $x_\alpha = 0$  for  $\alpha_1, \dots, \alpha_n$  and  $x_\alpha = 1$  for all other indices. Then  $(x_\alpha) \in A \cap U_\alpha$ .

Now let  $\{a_n\} \subseteq A$  and for  $n \in \mathbb{Z}^+$  let  $J_n = \{\alpha \in J : a_{\alpha n} \neq 1\}$ . Then we see that  $\bigcup J_n$  is a countable union of finite sets, and hence countable itself. Now since  $J$  is uncountable, there is a  $\beta \in J$  for which  $\beta \notin \bigcup J_n$ , so  $a_{\beta n} = 1$ . Letting  $U_\beta = (-1, 1)$  in  $\mathbb{R}$  let  $U = \pi_\beta^{-1}(U_\beta)$  in  $\mathbb{R}^J$ . Then  $U$  is a neighborhood of  $0$  not containing any points of  $\{a_n\}$ , so  $a_n \not\rightarrow 0$ .

**Lemma 2.3.10.** Let  $X$  and  $Y$  be metric spaces with metrics  $d_X$  and  $d_Y$ , and suppose that  $f : X \rightarrow Y$  is a function such that for all  $x, y \in X$ ,

$$d_Y(f(x), f(y)) = d_X(x, y)$$

then  $f$  is an imbedding.

*Proof.* First, we have that  $f$  is 1-1, for if  $f(x) = f(y)$ , then  $d_Y(f(x), f(y)) = d_X(x, y) = 0$ , making  $x = y$ . So  $f$  is 1-1 of  $X$  onto  $f(X)$ .

We also have by theorem 2.3.4, that  $f$  is continuous if, and only if for every  $\varepsilon > 0$ , there is a  $\delta > 0$  such that  $d_Y(f(x), f(y)) < \varepsilon$  whenever  $d_X(x, y) < \delta$ . When  $x = y$ , we can take  $\varepsilon$  and  $\delta$  to be any real number. Now, if  $x \neq y$ , just take  $\varepsilon = d_X(x, y) + 1$  and  $\delta = d_Y(f(x), f(y)) + 1$ , which makes  $f$  continuous.

On the other hand, if  $B_{d_X}(x, \varepsilon)$  is an open ball in  $X$ , then by definition of  $f$ , we get that  $f(B_{d_X}(x, \varepsilon)) = B_{d_Y}(f(x), \varepsilon)$ , which is an open ball in  $f(X)$ . Therefore,  $f^{-1}$  is continuous, and we have that  $f$  is an imbedding of  $X$  onto  $f(X)$ . ■

**Definition.** Let  $X$  and  $Y$  be metric spaces. We call a function  $f : X \rightarrow Y$  an **isometric imbedding** if for every  $x, y \in X$  we have

$$d_Y(f(x), f(y)) = d_X(x, y) \quad (2.3)$$

## 2.4 The Quotient Topology

**Definition.** Let  $X$  and  $Y$  be topological spaces, and let  $p : X \rightarrow Y$  be onto. We say that  $p$  is a **quotient map** if a subset  $U \subseteq Y$  is open if and only if  $p^{-1}(U) \subseteq X$  is open. We say that a subset  $U$  is **saturated** with respect to  $p$  if  $U = p^{-1}(V)$  for some  $V \subseteq Y$ . That is a saturated subset is the preimage of some subset of  $Y$ .

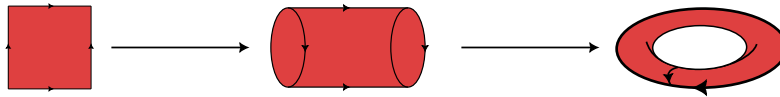


Figure 2.3: Gluing the edges of the unit square to form a cylinder, and then a torus.

**Definition.** Let  $f : X \rightarrow Y$  be a map, with  $X$  and  $Y$  topological spaces. We say that  $f$  is an **open map** if for each open subset  $U \subseteq X$ ,  $f(U) \subseteq Y$  is also open. We say  $f$  is a **closed map** if for each closed subset  $U \subseteq X$ ,  $f(U) \subseteq Y$  is closed.

**Lemma 2.4.1.** *If  $p : X \rightarrow Y$  is a continuous map of  $X$  onto  $Y$ , for topological spaces  $X$  and  $Y$ , that is either open or closed, then  $p$  is a quotient map.*

**Lemma 2.4.2.** *A continuous map  $p : X \rightarrow Y$  of  $X$  onto  $Y$  is a quotient map if, and only if it takes saturated open sets to open sets, or saturated closed sets to closed sets.*

*Proof.* Suppose that  $p$  is a quotient map, then by definition,  $U$  is open in  $Y$  if, and only if  $p^{-1}(U)$  is open in  $X$ . Now, let  $V$  be a saturated open set of  $X$  such that  $V = p^{-1}(U)$ , then we get that  $U$  is open since  $V$  is open. Thus  $p$  takes saturated open sets to open sets. Alternatively, if  $U$  is closed, we simply consider the open set  $Y \setminus U$  and repeat the proof above.

Now, suppose that  $p$  is onto and continuous, and takes saturated open sets to open sets. That is, if  $V = p^{-1}(U)$  is open in  $X$ , then  $U$  is open in  $Y$ . Certainly, if  $U$  is open in  $Y$ , then  $V$  is open in  $X$ . Conversely if  $V$  is open in  $X$ , then by hypothesis,  $p(p^{-1}(U)) = U$  is open in  $Y$  and this makes  $p$  into a quotient map. Moreover, if  $p$  takes saturated closed subsets to closed subsets, then just consider the open sets  $X \setminus V = p^{-1}(Y \setminus U)$  and repeat the proof above. ■

*Remark.* One consequence of this is that open or closed maps taking one space onto another are quotient maps.

**Example 2.5.** (1) A quotient map need not be open nor closed. Consider the projection  $\pi_1$  on  $\mathbb{R} \times \mathbb{R}$ , which is a quotient map. Consider  $A = \{x \times y \in \mathbb{R} \times \mathbb{R} : x \geq 0 \text{ or } y = 0\}$  of  $\mathbb{R} \times \mathbb{R}$ , and take  $q = \pi_1|_A$ . Then  $q$  is also a quotient map by inheritance, but it is not

open in  $\mathbb{R}$ . We have  $A$  is open in its subspace topology, so that  $q(A) = [x, \infty)$ , which is not open in  $\mathbb{R}$ . Likewise,  $q$  is not closed, for take  $U = \{x \times y : 0 < x < 1 \text{ or } y \geq 0\}$ , which is closed in  $A$  as a subspace, however,  $q(U) = (0, 1)$ , which is not closed in  $\mathbb{R}$ .

- (2) Let  $X$  be the subspace  $[0, 1] \cup [2, 3]$  and  $Y$  be the subspace  $[0, 2]$  in  $\mathbb{R}$ , and defined  $p : X \rightarrow Y$  by  $p(x) = \begin{cases} x, & x \in [0, 1] \\ x - 1, & x \in [2, 3] \end{cases}$ . We have that  $p$  is continuous onto, by the pasting lemma, furthermore,  $p$  is also closed; hence  $p$  is a quotient map.  $p$  is not open however, as  $p([0, 1]) = [0, 1]$  is closed in  $Y$ .

Now if  $A$  is the subspace  $[0, 1) \cup [2, 3]$  of  $X$ , then  $p : A \rightarrow Y$  is continuous onto, but fails to be closed. So  $p|_A$  is not a quotient map, despite the fact that  $[2, 3]$  is open in  $A$  and saturated with respect to  $p|_A$ .

- (3) Let  $\pi_1 : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$  be the projection map that takes  $x \times y \rightarrow x$ . Clearly  $\pi_1$  is continuous onto, and  $\pi_1$  is also open as  $\pi_1(U \times V) = U$  which is open in  $Y$ ; hence  $\pi_1$  is a quotient map. Now consider the closed set  $C = \{x \times y : xy = 1\}$  in  $\mathbb{R} \times \mathbb{R}$ .  $\pi_1(C) = \mathbb{R} \setminus \{0\}$  which is not closed.
- (4) Consider the map  $p : \mathbb{R}^n \setminus \{0\} \rightarrow S^{n-1}$  taking punctured Euclidean space onto the unit circle via the rule  $x \rightarrow \frac{x}{\|x\|}$  then the saturated open sets of  $\mathbb{R}^n \setminus \{0\}$  are unions of rays originating from the origin 0. We notice that  $p$  takes saturated open sets onto open sets, which makes  $p$  a quotient map.

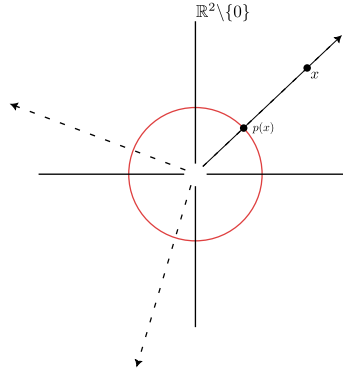


Figure 2.4: Saturated open sets of  $\mathbb{R}^2 \setminus \{0\}$  onto  $S^1$  via the map  $p : x \rightarrow \frac{x}{\|x\|}$ .

**Lemma 2.4.3.** *If  $p : X \rightarrow Y$  is an open quotient map, then  $Y$  is Hausdorff if, and only if  $p^{-1}(\Delta(Y)) = \{x \times y \in X \times X : p(x) = p(y)\}$  is closed.*

*Proof.* Suppose that  $Y$  is Hausdorff, and let  $x \times y \notin p^{-1}(\Delta(Y))$ . Then  $p(x) \neq p(y)$  and there exist disjoint neighborhoods  $U$  of  $p(x)$  and  $V$  of  $p(y)$ . Now, we have that  $p^{-1}(U)$  and  $p^{-1}(V)$  are open in  $X$ , hence  $p^{-1}(U) \times p^{-1}(V)$  is open in  $X \times X$  and is a neighborhood of  $x \times y$ . Now, if  $z \times z' \in p^{-1}(\Delta(Y)) \cap (p^{-1}(U) \times p^{-1}(V))$ , then  $p(z) \in U$  and  $p(z') \in V$ , and  $p(z) = p(z')$ ; but  $U \cap V = \emptyset$  making  $p(z) \neq p(z')$ , a contradiction, thus  $p^{-1}(\Delta(Y))$  and  $p^{-1}(U) \times p^{-1}(V)$  are disjoint. We have that  $X \times X \setminus p^{-1}(\Delta(Y))$  is open, making  $p^{-1}(\Delta(Y))$  closed.

On the otherhand, suppose that  $\Delta(Y)^{-1}$  is closed. Now, take  $x, y \in Y$  distinct, then let  $x', y' \in X$  such that  $p(x') = x$  and  $p(y') = y$ , then  $x' \times y' \notin p^{-1}(\Delta(Y))$ . Then there exists a neighborhood  $U \times V$  of  $x' \times y'$  disjoint from  $p^{-1}(\Delta(Y))$ . Now consider  $p(U)$  and  $p(V)$ ; since  $p$  is an open map, then these sets are open. Moreover, we have that  $x \in p(U)$  and  $y \in p(V)$ , so  $p(U)$  and  $p(V)$  are neighborhoods of  $x$  and  $y$ . Now, since  $U \times V$  is disjoint from  $p^{-1}(\Delta(Y))$ , we get that  $p(U)$  and  $p(V)$  are disjoint. Therefore,  $Y$  must be Hausdorff. ■

*Remark.* This gives us a useful check for determining when spaces arising from open quotient maps are Hausdorff. A more general check is presented later.

**Theorem 2.4.4.** *Let  $X$  be a topological space, and  $A$  a set, and let  $p : X \rightarrow Y$  be onto. Define  $\mathcal{T}$  to be the collection of subsets  $U$  of  $A$  for which  $p^{-1}(U)$  is open in  $X$ . Then  $\mathcal{T}$  is a unique topology for which  $p$  is a quotient map.*

*Proof.* Clearly  $\emptyset, A \in \mathcal{T}$ , for  $p^{-1}(\emptyset) = \emptyset$  and  $p^{-1}(A) = X$ . Now let  $\{U_\alpha\}$  and  $\{U_i\}_{i=1}^n$  be collections of subsets of  $A$ . Then

$$p^{-1}\left(\bigcup U_\alpha\right) = \bigcup p^{-1}(U_\alpha)$$

and

$$p^{-1}\left(\bigcap_{i=1}^n U_i\right) = \bigcap_{i=1}^n p^{-1}(U_i).$$

Thus  $\mathcal{T}$  is a topology on  $A$ . Now notice that this also makes  $p$  into a quotient map.

Now suppose that there is another topology  $\mathcal{T}'$  for which  $p$  is a quotient map. Clearly  $\mathcal{T} \subseteq \mathcal{T}'$ , now if  $p$  is open, then for  $U$  open in  $X$ ,  $p(U)$  is open in  $A$ , hence so are their preimages, and since  $p$  is continuous and onto this makes  $\mathcal{T}' \subseteq \mathcal{T}$ , thus  $\mathcal{T}$  is unique. Likewise, by similar reasoning with closed sets, if  $p$  is closed,  $\mathcal{T}$  is still unique. ■

**Definition.** If  $X$  is a topological space, and  $A$  a set, and  $p : X \rightarrow A$  is onto, then there is exactly one topology  $\mathcal{T}$  on  $A$  for which  $p$  is a quotient map. We call this topology the **quotient topology** on  $A$  induced by  $p$ .

**Example 2.6.** Let  $p : \mathbb{R} \rightarrow A$ , with  $A = \{a, b, c\}$  be defined by

$$p(x) = \begin{cases} a, & x > 0 \\ b, & x < 0 \\ c, & x = 0 \end{cases}$$

Then the quotient topology on  $A$  is the topology  $\mathcal{T} = \{\emptyset, \{a\}, \{b\}, \{a, b\}, A\}$ .

**Definition.** Let  $X$  be a topological space and  $X/p$  be a partition of  $X$  into disjoint subsets whose disjoint union is  $X$ . Let  $p : X \rightarrow X/p$  be onto such that  $p : x \rightarrow U$  if  $x \in U$ . We call  $X/p$  in the quotient topology induced by  $p$  the **quotient space** of  $X$ , or the **decomposition space** of  $x$ .

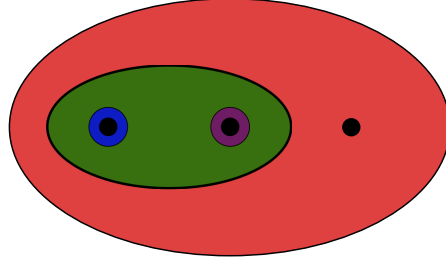


Figure 2.5: The quotient topology on  $A = \{a, b, c\}$  induced by  $p$ .

We can take an equivalence relation  $\sim$  on  $X$  by taking  $x \sim y$  if  $x, y \in U$  for  $U \in X/p$ . Then the quotient space is the collection of equivalence classes of  $X$ ; that is we can think of obtaining  $X^*$  by “identifying” those pairs of equivalent points. Similarly, we can also describe the quotient space  $X/p$  by noting a subset  $U$  of equivalence classes where  $p^{-1}(U) = \bigcup_{V \in U} V$  is the union of all equivalence classes in  $U$ . We will denote the quotient space by  $X/p$  or  $X/\sim$ .

We have the following lemma for determining when a map is a quotient map.

**Lemma 2.4.5.** *If  $p : X \rightarrow Y$  is a continuous map, and there exists a continuous map  $f : Y \rightarrow X$  such that  $p \circ f = i$  the identity map on  $Y$ , then  $p$  is a quotient map.*

*Proof.* By lemma 2.4.1,  $p$  is a quotient map if  $p$  is open or closed. Let  $U$  be open in  $X$ , since  $f$  is continuous,  $f^{-1}(U)$  is open in  $Y$ , and since  $p$  is continuous,  $p^{-1}(f^{-1}(U))$  is open in  $X$ . Since  $i$  is the identity on  $Y$ , we have  $p^{-1}(f^{-1}(U)) = U$ , so that  $f^{-1}(U) = p(U)$ , which is open. Therefore  $p$  is open and hence a quotient map. ■

**Example 2.7.** (1) Let  $X = \{x \times y : x^2 + y^2 \leq 1\}$  be the closed unit ball in  $\mathbb{R}^2$  and let  $X/\sim$  be the quotient space of  $X$  by taking its equivalence classes to be all one point sets  $\{x \times y\}$  for which  $x^2 + y^2 < 1$ , and the unit circle  $S^1$ . Take  $p : X \rightarrow X/\sim$  by the rule

$$p : x \times y \rightarrow \begin{cases} \{x \times y\}, & x^2 + y^2 < 1 \\ S^1 & x^2 + y^2 = 1 \end{cases}$$

Clearly  $p$  is onto by definition, and we have that for any one point sets and  $S_1$  in  $X/\sim$  that  $p^{-1}(\{x \times y\}) = B_{\|\cdot\|}(x, 1)$ , and  $p^{-1}(S^1) = S_1$ , thus open sets in  $X$  are open in  $X/\sim$  (likewise closed), so  $p$  is open which makes it into a quotient map. Now let  $S^2 = \{x \times y \times z : x^2 + y^2 + z^2 = 1\}$  be the **unit sphere** in  $\mathbb{R}^3$ , then we can map  $X/\sim \rightarrow S^2$  by taking  $\{x \times y\} \rightarrow \{x \times y \times z : x^2 + y^2 + z^2 < 1\}$  and  $S^1 \rightarrow S^1$ . Then we have that  $X/\sim$  is homeomorphic to the unit sphere.

- (2) Let  $X$  be the unit square  $[0, 1] \times [0, 1]$  and define the quotient space  $X/\sim$  of  $X$  to be the collection of all point sets  $\{x \times y\}$  for which  $x \times y \in (0, 1) \times (0, 1)$ , together with the sets  $\{x \times 0, x \times 1\}$ ,  $\{0 \times y, 1 \times y\}$  with  $x \times y \in (0, 1) \times (0, 1)$ , and the set  $\{0 \times 0, 0 \times 1, 1 \times 0, 1 \times 1\}$ , then we can show that  $X/\sim$  is homeomorphic to a torus.
- (3) Consider  $\mathbb{R} \times \mathbb{R}$  under the equivalence relation  $\sim$ , where  $x_0 \times y_0 \sim x \times y$  if  $x_0 + y_0^2 = x + y^2$ , and consider  $\mathbb{R} \times \mathbb{R}/\sim$  as a quotient space. Then let  $g(x \times y) = x + y^2$ . We have that



$y^2 \geq 0$ , but  $x \in \mathbb{R}$  is arbitrary, so  $g$  is onto  $\mathbb{R}$ . Likewise,  $\lim g(x \times y) = g(x_0 \times y_0) = x_0 + y_0^2$  as  $x \times y \rightarrow x_0 \times y_0$ , which makes  $g$  continuous. Lastly, let  $f(x) = x \times 0$ , then  $g \circ f = i$  the identity map on  $\mathbb{R}$ . This makes  $g$  into a quotient map. Then the space  $\mathbb{R} \times \mathbb{R} /_g = \mathbb{R} \times \mathbb{R} /_{\sim}$  is homeomorphic to a cylindrical paraboloid in  $\mathbb{R}^3$ . We can imagine this by taking a piece of paper (representing  $\mathbb{R} \times \mathbb{R}$ ), and bending it into a parabola shape.

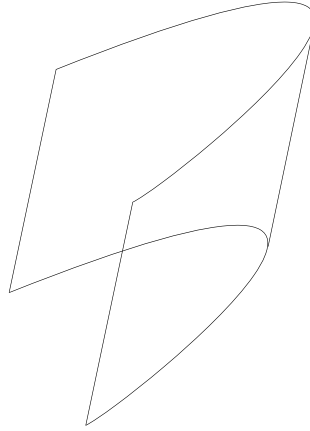


Figure 2.6: The space  $\mathbb{R} \times \mathbb{R} /_g$ , where  $g(x \times y) = x + y^2$

- (4) Taking  $\mathbb{R} \times \mathbb{R}$  under the equivalence relation  $\sim$  where  $x_0 \times y_0 \sim x \times y$  if  $x_0^2 + y_0^2 = x^2 + y^2$ . Let  $h(x \times y) = x^2 + y^2$ , then  $h$  is onto  $\mathbb{R}^+$  and is also continuous. Moreover, it is a quotient map, for let  $f(x) = \sqrt{x} \times 0$ , which makes  $h \circ f = i$  the identity map on  $\mathbb{R}$ . Then  $\mathbb{R} \times \mathbb{R} /_h = \mathbb{R} \times \mathbb{R} /_{\sim}$  is homeomorphic to an elliptic paraboloid. This can be visualized by taking a sheet of rubber and making a dimple in the middle and pushing it deeper.

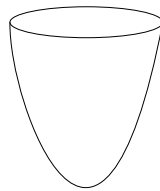


Figure 2.7: The space  $\mathbb{R} \times \mathbb{R} /_h$ , where  $h(x \times y) = x^2 + y^2$

**Theorem 2.4.6.** *Let  $p : X \rightarrow Y$  be a quotient map and let  $A$  be a subspace of  $X$  saturated with respect to  $p$ . Let  $q : A \rightarrow p(A)$ , then:*

- (1) *If  $A$  is either open or closed, then  $q$  is a quotient map.*
- (2) *If  $p$  is either open or closed, then  $q$  is a quotient map.*

*Proof.* We first have that  $q^{-1}(V) = p^{-1}(V)$  since  $V \subseteq A$  and  $A$  is saturated, well  $p^{-1}(V) \subseteq A$ . We also have that  $p(U \cap A) = U \cap p(A)$ . Now suppose that  $y = p(u) = p(a)$  for  $u \in U$

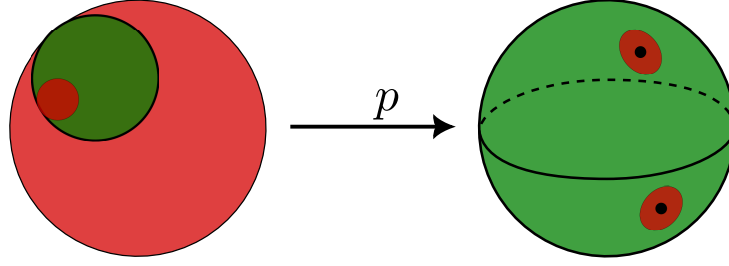


Figure 2.8: Transforming the unit disk into the unit sphere

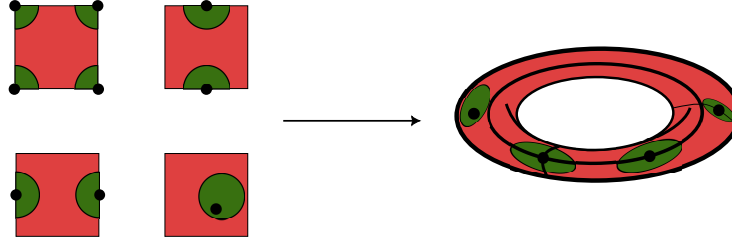


Figure 2.9: Transforming the unit square into torus

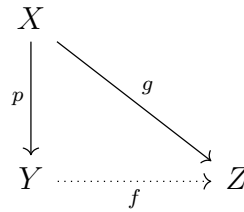
and  $a \in A$ . Since  $A$  is saturated, it contains  $p^{-1}(p(a)) = p^{-1}(p(u))$ , hence  $y = p(u)$  with  $u \in U \cap A$ , so  $P(U) \cap p(A) \subseteq P(U \cap A)$ .

Now suppose that  $A$ , or that  $p$  is open. Given  $V \subseteq p(A)$ , suppose  $q^{-1}(V) \subseteq A$  is open. If  $A$  is open in  $X$ , then  $q^{-1}(V)$  is also open in  $X$ , and since  $q^{-1}(V) = p^{-1}(V)$ , then  $V$  is open in  $Y$ , and since  $p$  is a quotient map, then  $V$  is open in  $p(A)$ . Now if  $p$  is open, since  $q^{-1}(V) = p^{-1}(V)$  and  $q^{-1}(V)$  is open in  $A$ , then  $p^{-1}(V) = U \cap A$ , thus  $V = p(p^{-1}(V)) = p(U \cap A) = p(U) \cap p(A)$ . Now  $p(U)$  is open in  $Y$ , so  $V$  is open in  $p(A)$ . In either case we have that  $q$  is a quotient map. ■

**Lemma 2.4.7.** *The composition of two quotient maps is a quotient map,*

*Proof.* Let  $p : X \rightarrow Y$  and  $q : Y \rightarrow Z$  be quotient maps and consider  $q \circ p : X \rightarrow Z$ . Let  $V \subseteq Z$  be open, then  $q^{-1}(V) = U$  is open in  $Y$  which implies that  $p^{-1}(U)$  is open in  $X$ . Hence  $p^{-1}(q^{-1}(V))$  is open in  $X$ . ■

**Theorem 2.4.8.** *Let  $p : X \rightarrow Y$  be a quotient map and let  $Z$  be a topological space with  $g : X \rightarrow Z$  a map constant on all  $p^{-1}(\{y\})$  for each  $y \in Y$ . Then  $g$  induces a map  $f : Y \rightarrow Z$  such that  $f \circ p = g$ , where  $f$  is continuous if and only if  $g$  is continuous, and where  $f$  is a quotient map if and only if  $g$  is a quotient map.*



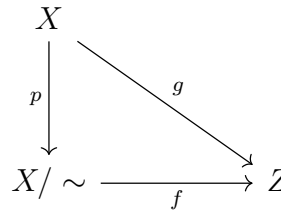
*Proof.* For each  $y \in Y$ , since  $g$  is constant on all  $p^{-1}(\{y\})$ ,  $g(p^{-1}(\{y\}))$  is a one point set in

$Z$ ; let it be  $f(y)$ . Then we have a map  $f : Y \rightarrow Z$  such that  $f(p(x)) = g(x)$  for each  $x$ . Now if  $f$  is continuous, then so is  $g$ ; likewise if  $f$  is a quotient map.

Conversely, suppose that  $g$  is continuous, given  $V \subseteq Z$  open,  $g^{-1}(V)$  is open in  $X$ . Now  $g^{-1}(V) = p^{-1}(f^{-1}(V))$ , and since  $p$  is a quotient map,  $f^{-1}(V)$  is open in  $Y$ , hence  $f$  is continuous. Now suppose that  $g$  is a quotient map, for  $V \subseteq Z$ , if  $f^{-1}(V)$  is open in  $Y$ , then  $p^{-1}(f^{-1}(V))$  is open in  $X$  by continuity of  $p$ , thus  $g^{-1}(V)$  is open in  $X$ , then  $V$  is open in  $Z$ ; this makes  $f$  a quotient map. ■

**Corollary.** Let  $g : X \rightarrow Z$  be continuous and let  $X/\sim = \{g^{-1}(\{z\}) : z \in Z\}$  be the following quotient space for  $X$ . Then

- (1) The map  $g$  induces a continuous 1-1 map  $f : X/\sim \rightarrow Z$  of  $X/\sim$  onto  $Z$  which is a homeomorphism if and only if  $g$  is a quotient map.



- (2) If  $Z$  is Hausdorff, then so is  $X/\sim$ .

*Proof.* By the theorem 2.4.6,  $g$  induces a continuous map  $f : X/\sim \rightarrow Z$  which is 1-1 and (clearly) onto. Suppose that  $f$  is a homeomorphism. Then both  $f$  and  $p : X \rightarrow X/\sim$  are quotient maps, which makes  $g$  a quotient map. On the other hand, if  $g$  is a quotient map, then so is  $f$ , which makes  $f$  1-1 continuous, and hence a homeomorphism.

Now suppose that  $Z$  is Hausdorff, given points of  $X/\sim$ , their images under  $f$  are distinct, and hence they possess disjoint neighborhoods  $U$  and  $V$ . Then  $f^{-1}(U) \cap f^{-1}(V) = \emptyset$ , which makes  $X/\sim$  Hausdorff. ■

**Corollary.** Suppose that  $q : X \rightarrow Y$  and  $q' : X \rightarrow Y'$  are quotient maps such that  $q(x) = q(x')$  if, and only if  $q'(x) = q'(x')$ , then  $Y$  is homeomorphic to  $Y'$ .

*Proof.* By the above theorem, there exists maps  $\tilde{q}$  and  $\tilde{q}'$  such that  $\tilde{q} \circ q' = q$  and  $\tilde{q}' \circ q = q'$ . It then suffices to show that  $\tilde{q}$  and  $\tilde{q}'$  are inverses of each other. That is

to say, that  $q \circ q' = i$ , where  $i$  is the identity on  $Y'$ . Notice then that  $i \circ q' = \tilde{q}' \circ q = \tilde{q}' \circ \tilde{q} \circ q'$ . Now, since both  $q$  and  $q'$  are onto, they are right invertible, so taking right inverse of  $q'$ , we get  $i = \tilde{q}' \circ \tilde{q}$ . ■

We conclude with some examples and extra results.

**Example 2.8.** (1) Let  $X = \{[0, 1] \times n : n \in \mathbb{Z}^+\}$  be the subspace of  $\mathbb{R}^2$  and let  $Z = \{x \times \frac{x}{n} : x \in [0, 1] n \in \mathbb{Z}^+\}$  be another subspace of  $\mathbb{R}^2$ . We have that  $X$  is the union of countably many disjoint line segments in  $\mathbb{R}^2$  and  $Z$  is the union of countably many line segments in  $\mathbb{R}^2$  having a common endpoint.

Now define  $g : X \rightarrow Z$  by  $g : x \times n \rightarrow x \times \frac{x}{n}$ . We have that  $g$  is continuous onto. Now consider the quotient space  $X/g$  whose elements are  $g^{-1}(\{z\})$  to be the space obtained

from  $X$  by identifying the equivalence classes of  $\{0\} \times \mathbb{Z}^+$  to be points. Now we have that  $g$  induces a 1-1 continuous map  $f : X/g \rightarrow Z$  of  $X/g$  onto  $Z$ ; but  $f$  is not a homeomorphism. It is sufficient to show that  $g$  is not a quotient map consider the sequence  $\{\frac{1}{n} \times n\}$  of points of  $X$ . This sequence is closed in  $X$  since it has no limit points, it is also saturated with respect to  $g$ . However  $g(A)$  is not closed in  $Z$  for it consists of the sequence of points  $\{\frac{1}{n} \times \frac{1}{n^2}\}$  whose limit point is the origin (see 2.10).

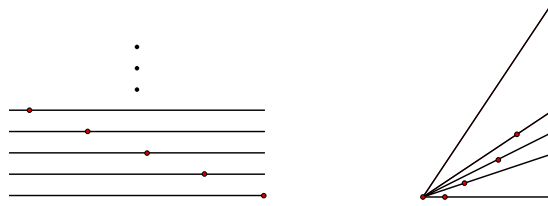


Figure 2.10:

- (2) Not it is not in general true that the product of two quotient maps is itself a quotient map. Consider  $\mathbb{R}$  and let  $\mathbb{R}/\sim$  be the quotient space obtained from  $\mathbb{R}$  by taking  $\mathbb{Z}^+ \sim b$  for some point  $b$ . Now let  $p : \mathbb{R} \rightarrow \mathbb{R}/\sim$  be the quotient map and consider  $\mathbb{Q}$  as a subspace of  $\mathbb{R}$ . Let  $i : \mathbb{Q} \rightarrow \mathbb{Q}$  be the identity map, then  $p \times i : \mathbb{R} \times \mathbb{Q} \rightarrow \mathbb{R}/\sim \times \mathbb{Q}$  is not a quotient map. For each  $n$ , take the sequence  $\{\frac{\sqrt{2}}{n}\}$  and consider the straight lines of slope  $\pm 1$  in  $\mathbb{R}^2$  through the point  $n \times \frac{\sqrt{2}}{n}$ . Let  $U_n$  be the set of all  $\mathbb{R} \times \mathbb{Q}$  points lying above or beneath both these line, and between the lines  $x = n \pm \frac{1}{4}$ . We have that  $U_n$  is open in  $\mathbb{R} \times \mathbb{Q}$  as it contains the set  $\{n\} \times \mathbb{Q}$ .

Now let  $U = \bigcup U_n$ , then  $U$  is open in  $\mathbb{R} \times \mathbb{Q}$ , and it is saturated with respect to  $p \times i$ , as  $\mathbb{Z}^+ \times q \in \mathbb{R} \times \mathbb{Q}$  for all  $q \in \mathbb{Q}$ , so assume that  $U' = p \times i(U)$  is open in  $\mathbb{R}/\sim \times \mathbb{Q}$ . Now since  $\mathbb{Z}^+ \times 0 \subseteq U$ , the point  $b \times 0 \in U'$ , so  $U'$  contains an open set of the form  $W \times I_\delta$  where  $W$  is a neighborhood of  $b$  in  $\mathbb{R}/\sim$ , and  $I_\delta = \{y \in \mathbb{Q} : |y| < \delta\}$ . Then  $p^{-1}(W) \times I_\delta \subseteq U$ . Choosing  $n$  large enough so that  $\frac{\sqrt{2}}{n} < \delta$ , since  $p^{-1}(W)$  is open in  $\mathbb{R}$  and contains  $\mathbb{Z}^+$ , choose  $0 < \varepsilon < \frac{1}{4}$  so that  $(n - \varepsilon, n + \varepsilon) \subseteq p^{-1}(W)$ . Then  $U$  contains the subset  $(n - \varepsilon, n + \varepsilon) \times I_\delta \subseteq \mathbb{R} \times \mathbb{Q}$ , however there are points that need not lie in  $U$ , which contradicts that  $U'$  is open in  $\mathbb{R}/\sim$ . So  $p \times i$  cannot be a quotient map.

- (3) Likewise, the product of two quotient maps need not be a quotient map. Consider the space  $\mathbb{R}$  and the quotient space  $\mathbb{R}/p$  Where  $p : \mathbb{R} \rightarrow \mathbb{R}/p$  is the quotient map taking  $\mathbb{Z}^+$  to a point  $b$ . Consider also  $\mathbb{Q}$  as a subspace of  $\mathbb{R}$  and take the identity map  $i : \mathbb{Q} \rightarrow \mathbb{Q}$ . Then  $i$  is also a quotient map, but the product  $p \times i$  is not a quotient map.

For each  $n \in \mathbb{Z}^+$ , let  $c_n = \frac{\sqrt{2}}{n}$  and consider the lines in  $\mathbb{R}$  with slopes  $\pm 1$ , through the point  $n \times c_n$ . Now, let  $U_n$  be all the points of  $\mathbb{R} \times \mathbb{Q}$  that lie above these lines, or below these lines, and between the lines  $x = \frac{n \pm 1}{4}$ .  $U_n$  is open in  $\mathbb{R} \times \mathbb{Q}$ , as it contains the set  $\{n\} \times \mathbb{Q}$ . Now let

$$U = \bigcup_{n \in \mathbb{Z}^+} U_n$$

Then  $U$  is also open in  $\mathbb{R} \times \mathbb{Q}$ , and it is saturated with respect to  $p \times i$ , as it contains  $\mathbb{Z}^+ \times \{q\}$  for each  $q \in \mathbb{Q}$ . Then suppose that  $p \times i(U)$  is open in  $\mathbb{R}/p \times \mathbb{Q}$ .

We have that  $U$  contains the set  $\mathbb{Z}^+ \times 0$ , so  $p \times i(U)$  contains the point  $b \times 0$ . So  $p \times i(U)$  contains a neighborhood of the form  $W \times I_\delta$ ,  $W$  a neighborhood of  $b$ , and  $I_\delta = \{y \in \mathbb{Q} : |y| < \delta\}$ . Then we have

$$p^{-1}(W) \times I_\delta \subseteq U$$

Now, choose  $n$  sufficiently large so that  $c_n < \delta$ . Since  $p^{-1}(W)$  is open in  $\mathbb{R}$ , and contains  $\mathbb{Z}^+$ , choose  $\varepsilon < \frac{1}{4}$ , so that  $(n - \varepsilon, n + \varepsilon) \subseteq p^{-1}(W)$ . Then  $U$  also contains a subset of the form  $(n - \varepsilon, n + \varepsilon) \times I_\delta$ . But figure 2.11 shows that there are points of  $(n - \varepsilon, n + \varepsilon) \times I_\delta$  not lying in  $U$ , taking  $x \times y = n = \frac{1}{2}\varepsilon \times y$  with  $|y - c_n| < \frac{1}{2}\varepsilon$ . Thus  $U$  cannot be open and we have a contradiction. Therefore  $p \times i$  is not a quotient map.

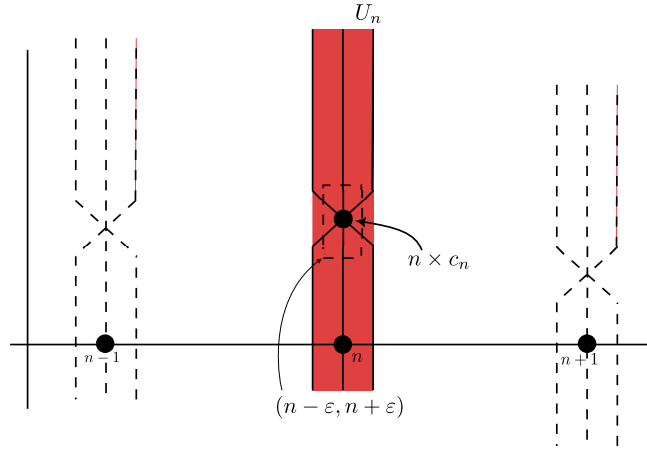


Figure 2.11: The product of two quotient maps need not be a quotient map.

**Definition.** Let  $X$  be a topological space, and  $A \subseteq X$ . We call a continuous map  $r : X \rightarrow A$  a **retraction** of  $X$  onto  $A$  if  $r : a \rightarrow a$  for all  $a \in A$ .

**Lemma 2.4.9.** *For any topological space  $X$  and  $A \subseteq X$ , a retraction of  $X$  onto  $A$  is a quotient map.*

*Proof.* Let  $r$  be a retraction of  $X$  onto  $A$ , and let  $f : A \rightarrow X$  be such that  $r \circ f = i$  the identity map in  $A$ . Let  $U$  be open in  $A$ , then  $U = r^{-1}(U)$  is open in  $X$ , and since  $r \circ f = i$ , we get  $f^{-1}(r^{-1}(U)) = f^{-1}(U)$  must be open in  $A$ . this makes  $f$  continuous. Therefore by lemma 2.4.4,  $r$  is a quotient map. ■



# Bibliography

- [1] J. Munkres, *Topology*. New York, NY: Pearson, 2018.