Topology

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Chapter 1

Topological Spaces and Continuous Functions.

1.1 Topological Spaces.

Definition. A **topology** on a set X is a collection $\mathcal{T}$ of subsets of X such that:

(1) $\emptyset \in \mathcal{T}$ and $X \in \mathcal{T}$

(2) For any collection $\{U_\alpha\}$ of subsets of X, $\bigcup_\alpha U_\alpha \in \mathcal{T}$.

(3) For any finite collection $\{U_i\}_{i=1}^n$ of subsets of X, $\bigcap_{i=1}^n U_i \in \mathcal{T}$.

We call the pair $(X, \mathcal{T})$ a **topological space**, and we call the elements of $\mathcal{T}$ **open sets**.

Example 1.1. (1) Let X be any set, the collection of all subsets of X, 2^X is a topology on X, which we call the **discrete topology**. We call the topology $\mathcal{T} = \{\emptyset, X\}$ the **indiscrete topology**.

(2) The set of three points $\{a, b, c\}$ has the 9 following topologies in figure 1.1.

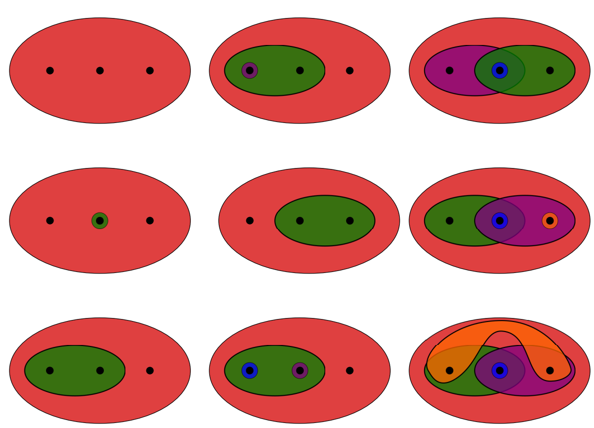

Figure 1.1: The Topologies on $\{a, b, c\}$.

(3) Let X be any set, and let $\mathcal{T}_f = \{U \subseteq X : X\backslash U$ is finite, or $X\backslash U = X\}$. Then $\mathcal{T}_f$ is a topology and called the **finite complement topology**.

(4) Let X be any set, and let $\mathcal{T}_c = \{U \subseteq X : X\backslash U$ is countable, or $X\backslash U = X\}$. Then $\mathcal{T}_c$ is a topology on X called the **countable complement topology**.

(5) Let X be any set and consider the collection $\mathcal{T}_\infty = \{U \subseteq X : X\backslash U = \emptyset, X\backslash U = X$, or $X\backslash U$ is infinite$\}$. Certainly, we have that $\emptyset, X \in \mathcal{T}_\infty$ as $X\backslash X = \emptyset$ and $X\backslash\emptyset = X$.

However, if $X = \mathbb{R}$, and we have the sets $(0,1)$, and $(-\infty,0)\cup(1,\infty)$, then $\mathbb{R}\backslash(0,1) = (-\infty,0]\cup[1,\infty)$, and $\mathbb{R}\backslash(-\infty,0)\cup(1,\infty) = [0,1]$, both of which are infinte, but, $[0,1]\cap((-\infty,0]\cup[1,\infty)) = \{0,1\}$, which is finite. So it is not true in general that the collection $\mathcal{T}_\infty$ is a topology.

Lemma 1.1.1. *Let X be a topological space. If $A \subseteq X$ is such that for each $x \in A$, there exists an open set U with $x \in U \subseteq A$, then A is also open in X.*

Proof. We have that for each $x \in A$, there is an open set U_a such that $x \in U_x \subseteq A$. Now, let $U = \bigcup_{x\in A} U_x$, which is open in X by definition. Then, we have $U \subseteq A$ by hypothesis; moreover, since $x \in A$ implies $x \in U_x$, then $x \in U$. This makes $A \subseteq U$, so $A = U$. ■

Definition. Let X be a set, and let $\mathcal{T}$ and $\mathcal{T}'$ be topologies on X. We say that $\mathcal{T}$ is **coarser** than $\mathcal{T}'$, and $\mathcal{T}'$ **finer** than $\mathcal{T}$ if $\mathcal{T} \subseteq \mathcal{T}'$. If two topologies are either coarser, or finer than each other, we call them **comparable**.

Example 1.2. The topologies $\mathcal{T}_f$ and $\mathcal{T}_c$ are comparable, and we see that $\mathcal{T}_f \subseteq \mathcal{T}_c$, so $\mathcal{T}_f$ is coarser than $\mathcal{T}_c$, and $\mathcal{T}_c$ is finer than $\mathcal{T}_f$.

Lemma 1.1.2. *If $\{Tc_\alpha\}$ is a collection of topologies on a set X, then the intersection of all $\mathcal{T}_\alpha$, $\bigcap T_\alpha$ is also a topology on X.*

Proof. Let $\mathcal{T} = \bigcap \mathcal{T}_\alpha$. We have that $\emptyset, X \in \mathcal{T}_\alpha$ for each α, so that $\emptyset, X \in \mathcal{T}$.

Now let $\{U_\alpha\}$ be a collection of open sets such that $U_\alpha \in \mathcal{T}_\alpha$ for each α. Then $U_\alpha \in \mathcal{T}$, for each α, so that $\bigcup U_\alpha \in \mathcal{T}$. Lastly, take a finite subcollection $\{U_i\}_{i=1}^n$ of $\{U_\alpha\}$, then $\bigcap_{i=1}^n U_i \in \mathcal{T}$ by similar reasoning. ■

Example 1.3. If X is any set, and $\{Tc_\alpha\}$ is a collection of topologies in X, it is not in general true that $\bigcup \mathcal{T}_\alpha$ is also a topology on X. Consider the 9 topologies on the set $X = \{a,b,c\}$ in the preceding examples. Let $\mathcal{T}_1 = \{\emptyset, \{a\}, \{a,b\}, X\}$, $\mathcal{T}_2 = \{\emptyset, \{a,b\}, \{b\}, \{b,c\}, \{c\}, X\}$, and let $\mathcal{T}_3 = \mathcal{T}_1 \cup \mathcal{T}_2$. The sets $\{a\}$ and $\{c\}$ have union $\{a.c\}$, however, $\{a,c\} \notin \mathcal{T}_3$.

1.2 The Basis and Subbasis for a Topology.

Definition. If X is a set, the **basis** for a topology on X is a collection $\mathcal{B}$ of subsets of X, called **basis elements**, such that:

(1) For every $x \in X$, there is a $B \in \mathcal{B}$ such that $x \in B$.

(2) For $B_1, B_2 \in \mathcal{B}$, if $x \in B_1 \cap B_2$, then there is a $B_3 \in \mathcal{B}$ such that $x \in B_3 \subseteq B_1 \cap B_2$

We define the topology $\mathcal{T}$ **generated** by $\mathcal{B}$ to be collection of open sets: $\mathcal{T} = \{U \subseteq X :$ for all $x \in U$, there exists a $B \in \mathcal{B}$ such that $x \in B\}$.

Theorem 1.2.1. *Let X be a set, and $\mathcal{B}$ a basis of X, then the collection of subsets of X, $\mathcal{T} = \{U \subseteq X :$ for all $x \in U$, there exists a $B \in \mathcal{B}$ such that $x \in B\}$ is a topology on X.*

Proof. Let $\mathcal{B}$ be a basis for a topology in X, and consider $\mathcal{T}$ as defined above. Cleary, $\emptyset \in X$ and so is X.

Now let $\{U_\alpha\}$ be a collection of subsets of X, and let $U = \bigcup U_\alpha$. Then if $x \in U$ for some α, there is a B_α such that $x \in B_\alpha \subseteq U_\alpha$, thus $x \in B_\alpha \subseteq U$.

Now let $x \in U_1 \cap U_2$, and choose $B_1, B_2 \in \mathcal{B}$ such that $x \in B_1 \subseteq U_1$ and $x \in B_2 \subseteq U_2$. Then by definition, there is a B_3 for which $x \in B_3 \subseteq B_1 \cap B_2$. Now suppose for arbitrary n, that $U = \bigcap_{i=1}^n U_i \in \mathcal{T}$, for some finite collection $\{U_i\}$ of subsets of X. Then by let $B_n, B_{n+1} \in \mathcal{B}$ such that $x \in B_n \subseteq U$ and $x \in B_{n+1} \subseteq U_{n+1}$. Then by our hypothesis, there is a B for which $x \in B \subseteq B_n \cap B_{n+1}$, thus $U \cap U_{n+1} = \bigcap_{i=1}^{n+1} U_i \in \mathcal{T}$. This make $\mathcal{T}$ a topology on X. ■

Example 1.4. (1) Let $\mathcal{B}$ be the set of all circular regions in the plane $\mathbb{R} \times \mathbb{R}$, then $\mathcal{B}$ satisfies the conditions needed for a basis.

(2) The collection $\mathcal{B}'$ in $\mathbb{R} \times \mathbb{R}$ of all rectangular region also forms a basis for a topology on $\mathbb{R} \times \mathbb{R}$.

(3) For any set X, the set of all 1-point subsets of X forms a basis for a topology on X.

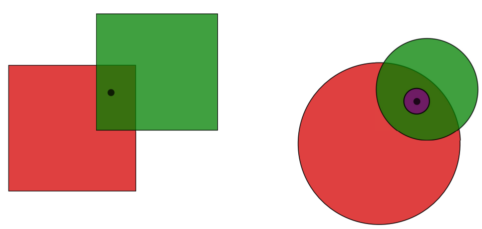

Figure 1.2: The basis for $\mathcal{B}$ and $\mathcal{B}'$ in $\mathbb{R} \times \mathbb{R}$ (see example (2)).

Lemma 1.2.2. *Let X be a set, and $\mathcal{B}$ be a basis for a topology $\mathcal{T}$ on X. Then $\mathcal{T} = \{\bigcup B : B \in \mathcal{B}\}$.*

Proof. Given a collection $\{B_i\}_{i=1}^\infty$ of basis elements in $\mathcal{B}$, since they are all in $\mathcal{T}$, their unions are also in $\mathcal{T}$. Conversely, given $U \in \mathcal{T}$, then for every point $x \in U$, choose a $B_x \in \mathbb{B}_x$ such that $x \in B_x \subseteq U$, then $U = \bigcup_{x \in U} B_x$. ■

Lemma 1.2.3. *Let $(X, \mathcal{T})$ be a topological space, and let $\mathcal{C} \subseteq \mathcal{T}$ be a collection of open sets of X such that for every $x \in U$, there is a $C \in \mathcal{C}$ such that $x \in C \subseteq U$. Then $\mathcal{C}$ is the basis for a $\mathcal{T}$ on X.*

Proof. Take any $x \in X$, then there is a $C \in \mathcal{C}$ such that $x \in C \subseteq U$, thus the first condition for a basis is satisfied. Now let $x \in C_1 \cap C_2$ for $C_1, C_2 \in \mathcal{C}$, since $C_1 \cap C_2$ is open in X, there is a $C_3 \in \mathcal{C}$ such that $x \in C_3 \subseteq C_1 \cap C_2$. Therefore $\mathcal{C}$ is a basis for a topology on X.

Now let $\mathcal{T}_{\mathcal{C}}$ be the the topology generated by $\mathcal{C}$, now for $U \in \mathcal{T}$, we have by the hypothesis, that $U \in \mathcal{T}_{\mathcal{C}}$; and by lemma 1.2.2, $W \in \mathcal{T}_{\mathcal{C}}$ is the union of elements of $\mathcal{C}$, which is a subcollection of $\mathcal{T}$, thus $W \in \mathcal{T}$. Therefore $\mathcal{T}_{\mathcal{C}} = \mathcal{T}$. ■

Lemma 1.2.4. *Let $\mathcal{B}$ and $\mathcal{B}'$ be bases for topologies $\mathcal{T}$ and $\mathcal{T}'$ on X. Then the $\mathcal{T} \subseteq \mathcal{T}'$ if and only if for all $x \in X$, and all $B \in \mathcal{B}$, there is a $B' \in \mathcal{B}'$ such that $x \in B' \subseteq B$.*

Proof. Suppose first that $\mathcal{T} \subseteq \mathcal{T}'$, and let $x \in X$, and choose $B \in \mathcal{B}$ such that $x \in B$, then B is open in $\mathcal{T}$, thus it is open in $\mathcal{T}'$, thus there is a $B' \in \mathcal{B}'$ such that $x \in B' \subseteq B$. Conversely, suppose there is a $B' \in \mathcal{B}'$ for which $x \in B' \subseteq B$ for all $x \in X$, $B \in \mathcal{B}$. Take $x \in U \in \mathcal{T}$, since $\mathcal{B}$ generates $\mathcal{T}$, $x \in B \subseteq U$, since $B' \subseteq B$, this implies that $U \in \mathcal{T}'$ and $\mathcal{T} \subseteq \mathcal{T}'$. ■

Definition. If $\mathcal{B}$ is the collection of open intervals (a, b) in $\mathbb{R}$, we call the topology generated by $\mathcal{B}$ the **standard topology** on $\mathbb{R}$, and we denote it simply by $\mathbb{R}$.

Definition. If $\mathcal{B}$ is the collection of half open intervals $[a, b)$ in $\mathbb{R}$, we call the topology generated by $\mathcal{B}$ the **lower limit topology** on $\mathbb{R}$, and we denote it simply by $\mathbb{R}_l$. If $\mathcal{B}'$ is the collection of all half open intervals $(a, b]$ in $\mathbb{R}$, then we call the topology generated by $\mathcal{B}'$ the **upper limt topology** on $\mathbb{R}$, and denote it $\mathbb{R}_L$.

Definition. If $\mathcal{B}$ is the collection of all open intervals of the form $(a, b) \setminus \frac{1}{\mathbb{Z}^+}$, where $\frac{1}{\mathbb{Z}^+} = \{\frac{1}{n} : n \in \mathbb{Z}^+\}$, we call the topology generated by $\mathcal{B}$ the $\frac{1}{\mathbb{Z}^+}$**-topology** on $\mathbb{R}$, and we denote it $\mathbb{R}_{\frac{1}{\mathbb{Z}^+}}$.

Lemma 1.2.5. *The topologies $\mathbb{R}_l$, $\mathbb{R}_L$, and $\mathbb{R}_{\frac{1}{\mathbb{Z}^+}}$ are all strictly finer than $\mathbb{R}$, but are not comparable with each other.*

Proof. Let (a, b) be a basis element for $\mathbb{R}$, and let $x \in (a, b)$, the basis element $[x, b) \in \mathbb{R}_l$ lies in (a, b) and contains x, however, there can be no interval (a, b) in $[x, b)$ as $x \leq a$, thus $\mathbb{R} \subset \mathbb{R}_l$; a similar argument holds for $\mathbb{R}_L$.

Similarly, for $(a, b) \in \mathbb{R}$, the basis element $(a, b)\setminus\frac{1}{\mathbb{Z}^+}$ of $\mathbb{R}_{\frac{1}{\mathbb{Z}^+}}$ lies in (a, b), however, choose the basis $B = (-1, 1)\setminus\frac{1}{\mathbb{Z}^+}$, and choose $0 \in B$, since $\mathbb{Z}^+$ is dense in $\mathbb{R}$, there is no interval (a, b) containing 0 and lying in B, thus $\mathbb{R} \subseteq \mathbb{R}_{\frac{1}{\mathbb{Z}^+}}$.

Now choose $[0, 1)$ in $\mathbb{R}_l$, and choose $\frac{1}{k} \in [0, 1)$ such that $k \in \mathbb{Z}^+$. Now $(0, 1) \subseteq [0, 1)]$, so we cannot say that $[0, 1)$ is a basis for $\mathbb{R}$, and moreover, $[0, 1)\setminus\frac{1}{\mathbb{Z}^+}$ cannot be said to be a basis in $\mathbb{R}_{\frac{1}{\mathbb{Z}^+}}$, thus $\mathbb{R}_l$ and $\mathbb{R}_{\frac{1}{\mathbb{Z}^+}}$ are incompararble, a similar argument holds for $\mathbb{R}_L$.

Lastly, let (a, b) be in $\mathbb{R}$ and choose $x \in (a, b)$. Then $(a, x]$ and $[x, b)$ are both in (a, b), however it is clear that $(a, x]$ and $[x, b)$ connot be contained in each other, thus $\mathbb{R}_l$ and $\mathbb{R}_L$ are incomparable. ■

Definition. A **subbasis**, $\mathcal{S}$, for a topology on X is a collection of subsets of X whose union equals X. We call the **topology generated by** $\mathcal{S}$ to be the collection of all unions of finite intersections of elements of $\mathcal{S}$, that is:

$$\mathcal{T} = \{\bigcup \bigcap_{i=1}^{n} S_i : S_i \in \mathcal{S} \text{ for } 1 \leq i \leq n\}$$

Theorem 1.2.6. *Let $\mathcal{S}$ be a subbasis for a topology on X. Then the collection $\mathcal{T} = \{\bigcup\bigcap_{i=1}^{n} S_i : S_i \in \mathcal{S}$ for $1 \leq i \leq n\}$ is a topology on X.*

Proof. It is sufficient to show that the collection $\mathcal{B}$ of all finite intersections of elements of $\mathcal{S}$ is a basis for a topology on X. By lemma 1.2.1, for $x \in X$, it belongs to an element S of $\mathcal{S}$, and therefore, to an element of $\mathcal{B}$. Now let $B_1 = \bigcap_{i=1}^{m} S_i$ and $B_2 = \bigcap_{j=1}^{n} S'_j$ be basis elements of $\mathcal{B}$. The intersection $\mathbb{B}_1 \cap B_2$ is a finite intersection of elements of $\mathcal{S}$, and hence also belongs in $\mathcal{B}$, and hence we can take another basis element B_3 such that $x \in B_3 \subseteq B_1 \cap B_2$. ■

1.3 The Order Topology.

Definition. Let X be a set with a simple order relation, and suppose that $|X| > 1$. Let $\mathcal{B}$ be the collection of sets of the following forms:

(1) All open intervals $(a, b) \in X$.

(2) All half open intervals $[a_0, b)$ where a_0 is the least element (if any) of X.

(3) All half open intervals of the form $(a, b_0]$ where b_0 is the greatest element (if any) of X.

Then $\mathcal{B}$ forms the basis for a topology on X called the **order topology**

Theorem 1.3.1. *The collection $\mathcal{B}$ forms a basis.*

Proof. Consider $x \in X$, if x is the least element of X, then it liess in all intervals of type (2), if it is the largest, then it lies in all intervals of type (3). If x is neither the least nor largest element, then $x \in (a_0, b_0)$ with a_0 and b_0 the least and largest elements (if any) of X. If no such elements exist, then $x \in (a, b)$, for some lowerbound a and upperbound b. Thus, in all three cases, there is a basis element containing x.

Now suppose $B_1, B_2 \in \mathcal{B}$ such that $x \in B_1 \cap B_2$. If B_1 and B_2 are both of type (1), then let $B_1 = (a, b)$, $B_2 = (c, d)$, then $B_1 \cap B_2$ is an open interval of type (1), now fix B_1 to be of type one. If B_2 is of type (2), then letting $B_2 = [a_0, c)$, then $x \in [a_0, d)$ for some $d \in X$. Likewise, if $B_2 = (c, b_0]$, is of type (3), we get a similar result. Moreover, the results are analogous if we fix B_2 and let B_1 range between intervals of the three types. Thusm in all cases, there is a $B_3 \in \mathcal{B}$ such that $x \in B_3 \subseteq B_1 \cap B_2$. ■

Example 1.5. (1) The standard topology on $\mathbb{R}$ is the order topology on $\mathbb{R}$ induced by the usual order relation. We have that $\mathbb{R}$ under this topology has no intervals of type (2), nor (3), so all bases elements in the standard topology are open intervals in $\mathbb{R}$.

(2) Consider the dictionary order on $\mathbb{R} \times \mathbb{R}$. Since $\mathbb{R} \times \mathbb{R}$ has no intervals of type (2), nor (3), the bases of $\mathbb{R} \times \mathbb{R}$ under the dictionary order are the open intervals of the form $(a \times b, c \times d)$ Where $a \leq c$, and $b < d$.

(3) The positive integers $\mathbb{Z}^+$ with the least element 1 form an ordered set under the usual order. Taking $n > 1$, we see the bases of $\mathbb{Z}^+$ under the order topology are of the form $(n-1, n+1) = \{n\}$ and $[1, n) = \{1, \ldots, n-1\}$. Thus the order topology on $\mathbb{Z}^+$ is the discrete topology.

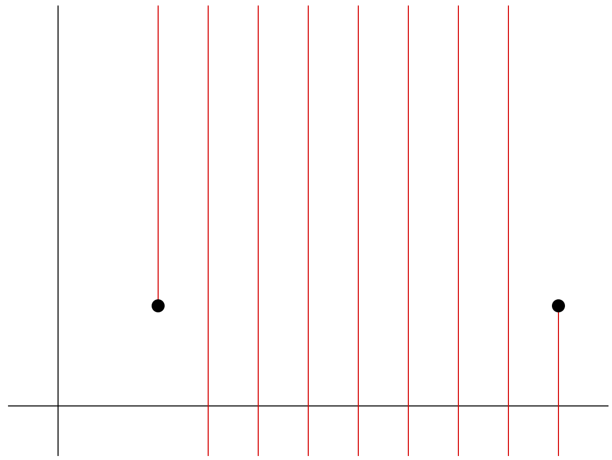

Figure 1.3: The order topology on $\mathbb{R} \times \mathbb{R}$.

(4) The set $X = \{1, 2\} \times \mathbb{Z}^+$ over the dictionary order is also an ordered set, with the least element 1×1. Denote $1 \times n$ as a_n and $2 \times n$ as b_n. Then X consist of the elements $a_1, a_2, \dots, b_1, b_2, \dots$.

Now take $\{b_1\}$, then any open set containing b_1 must have a basis about b_1, and also contains points a_i with $i \in \mathbb{Z}^+$; thus the order topology on X is not the discrete topology.

Definition. Let X be an ordered set, and let $a \in X$. There are two subsets in X, $(a, \infty) = \{x \in X : x > a\}$ and $(-\infty, a) = \{x \in X : x < a\}$ called **open rays** of X. There are also two sets $[a, \infty) = \{x \in X : x \geq a\}$ and $(-\infty, a] = \{x \in X : x \leq a\}$ called **closed rays** of X.

Theorem 1.3.2. *Let X be an ordered set. Then the collection of all open rays in X form a subbasis for the order topology on X.*

Proof. Let $\mathcal{S}$ be the collection of all open rays of X, let $a < b$ and (a, ∞), $(-\infty, b) \in \mathcal{S}$, then $(a, b) = (a, \infty) \cap (-\infty, b)$. Now take:

$$S = \bigcup_{a,b \in X} (a, b)$$

then $S \subseteq X$, likewise, since S runs through all intersections of open rays of X, it contains all open intervals in X, hence $X \subseteq S$, and so $X = S$ as required. ∎

1.4 The Product Topology.

Definition. Let X and Y be topological spaces. We define the **product topology** on $X \times Y$ to be the topology having as basis the collection

$$\mathcal{B} = \{U \times V \subseteq X \times Y : U \text{ is open in } X \text{ and } V \text{ is open in } Y\}$$

Theorem 1.4.1. *The collection $\mathcal{B} = \{U \times V \subseteq X \times Y : U$ is open in X and V is open in $Y\}$ forms a basis for the product topology on $X \times Y$.*

Proof. Clearly, we have that $X \times Y$ is a basis element of $\mathcal{B}$. Now take $U_1 \times V_1$ and $U_2 \times V_2$ in $\mathcal{B}$. Since $U_1 \times V_1 \cap U_2 \times V_2 = U_1 \cap U_2 \times V_1 \cap V_2$, since $U_1 \cap U_2$ and $V_1 \cap V_2$ are open in X and Y respectively, then we have that $U_1 \times V_1 \cap U_2 \times V_2$ is a basis element as well. ■

Theorem 1.4.2. *If $\mathcal{B}$ is the basis for a topology on X, and $\mathcal{C}$ is the basis for a topology on Y, then the collection:*

$$\mathcal{D} = \{B \times C : B \in \mathcal{B} \text{ and } C \in \mathcal{C}\}$$

Is a basis for the topology on $X \times Y$.

Proof. By lemma 1.2.3, let W be an open set of $X \times Y$, and let $x \times y \in W$. Then there is a basis $U \times V$ such that $x \times y \in U \times V \subseteq W$. Since $\mathcal{B}$ and $\mathcal{C}$ are bases of X and Y respectively, choosing $B \in \mathcal{B}$ and $C \in \mathcal{C}$, we have that $x \in B \subseteq U$, and $y \in C \subseteq Y$, thus $x \times y \in B \times C \subseteq U \times V \subseteq W$. Therefore, $\mathcal{D}$ is the basis for a topology on $X \times Y$. ■

Example 1.6. The product of the standard topology on $\mathbb{R}$ with itself is called the **standard topology on** $\mathbb{R} \times \mathbb{R}$, and has as basis the collection of all products of open sets in $\mathbb{R}$. By theorem 1.4.2, if we take the collection of all open intervals $(a, b) \times (c, d)$ in $\mathbb{R} \times \mathbb{R}$, we form a basis. Constructing this basis geometrically gives the interior of a rectangle, whose boundaries are the intervals (a, b) and (c, d).

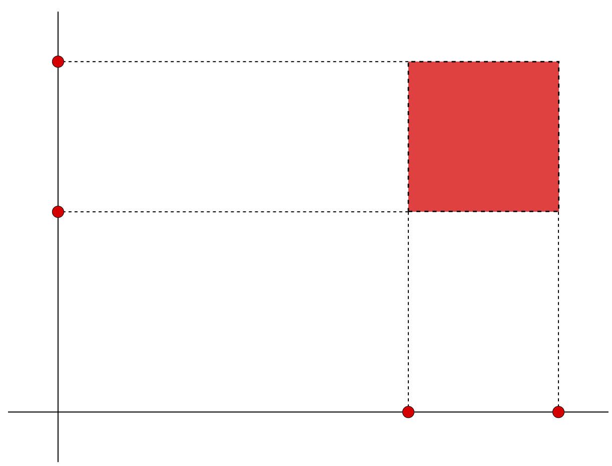

Figure 1.4: A basis element for $\mathbb{R} \times \mathbb{R}$

Definition. Let $\pi_1 : X \times Y \to X$ be defined such that $\pi_1(x, y) = x$, and define $\pi_2 : X \times Y \to Y$ such that $\pi_2(x, y) = y$. We call π_1 and π_2 **projections** of $X \times Y$ onto its first and second **factors**; that is onto X and Y, respectively.

Clearly, π_1 and π_2 are both onto. Now let U be open in X, then $\pi_1^{-1}(U) = U \times Y$ is open in $X \times Y$; similarly, $\pi_2^{-1}(V) = X \times V$ is also open in $X \times Y$, for V open in Y.

Example 1.7. The maps : $\pi_1 : x \times y \to x$ and $\pi_2 : x \times y \to y$ of $X \times Y$ onto X and Y, repsectively, are what we call "open maps". Let U and V be open in X and Y respectively, then $U \times V$ is open in $X \times Y$, and for every $x \times y \in U \times V$, we have $\pi_1(x \times y) = x$ and $\pi_2(x \times y) = y$, so that $\pi_1(U \times V) = U$ and $\pi_2(U \times V) = V$ are open in X and Y respectively.

Theorem 1.4.3. *The collection* $\mathcal{S} = \{\pi_1^{-1}(U) : U \text{ is open in } X\} \cup o\{\pi_2^{-1}(V) : V \text{ is open in } Y\}$ *is a subbasis for the product topology on* X*.*

Proof. Let $\mathcal{T}$ be the product topology on $X \times Y$, and let $\mathcal{T}'$ be the topology generated by $\mathcal{S}$. Since every element of $\mathcal{S}$ is open in $\mathcal{T}$, $\mathcal{T} \subseteq \mathcal{T}'$. Conversely, consider the basis element $U \times V$ of $\mathcal{T}$, then $\pi_1^{-1}(U) \cap \pi_2^{-1}(V) = U \times Y \cap X \times V = U \times V$, thus $\mathcal{T} \subseteq \mathcal{T}'$. Therefore, $\mathcal{S}$ is a subbasis for the product topology. ■

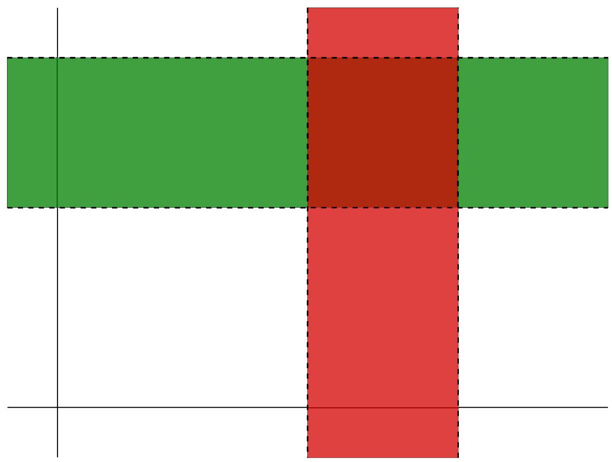

Figure 1.5: The inverse images, $\pi_1^{-1}(U)$ and $\pi_2^{-1}(V)$, of the projections π_1 and π_2 onto the $X \times Y$ plane.

1.5 The Subspace Topology.

Theorem 1.5.1. *Let* X *be a topological space with topology* $\mathcal{T}$*, and let* $Y \subseteq X$*. Then the collection:*

$$\mathcal{T}_y = \{Y \cap U : U \in \mathcal{T}\}$$

forms a topology on Y*.*

Proof. Cleary, $Y \cap \emptyset = \emptyset \in \mathcal{T}_Y$ and $Y \cap X = Y \in \mathcal{T}_Y$. Now consider the collection $\{U_\alpha\}$. Then $\bigcup(Y \cap U_\alpha) = Y \cap \bigcup U_\alpha$, similarly, for $\{U_i\}_{i=1}^n$, $\bigcap(Y \cap U_i) = Y \cap \bigcap U_i$, hence $\mathcal{T}$ is a topology on Y. ■

Definition. Let X be a topological space, and let $Y \subseteq X$. We call the $\mathcal{T}$ defined in theorem 1.5.1 the **subspace topology** on Y. We say that $U \subseteq Y$ is **open in** Y if $U \in \mathcal{T}_Y$.

Lemma 1.5.2. *Let* $\mathcal{B}$ *be the basis for a topology on* X*. Then the collection* $\mathcal{B}_Y = \{B \cap Y : B \in \mathcal{B}\}$*, where* $Y \subseteq X$*, is a basis for the subspace topology on* Y*.*

Proof. Let U be open in X, and let $y \in Y \cap U$, and choose $B \in \mathcal{B}$ such that $y \in B \subseteq U$, then $y \in B \cap Y \subseteq U \cap Y$, then by lemma 1.2.2, $\mathcal{B}_y$ is the basis fpr the subspace topology on Y. ■

Lemma 1.5.3. *Let Y be a subspace of X, If $U \subseteq Y$ is open in Y and Y is open in X, then U is open in X.*

Proof. Let $U \in \mathcal{T}_Y$, then for some $V \subseteq X$, $U = Y \cap V$. Now since Y is open in X, and so is V, then it follows that U is also open in X. ■

Remark. What this lemma says is that given a topological space X, and a subspace Y of X, then the subspace topology of Y is courser than the topology on X, i.e. $\mathcal{T}_Y \subseteq \mathcal{T}$.

Theorem 1.5.4. *If A is a subspace of X, and B is a subspace of Y, then the product topology on $A \times B$ is the topology that $A \times B$ inherits as a subspace of $X \times Y$.*

Proof. We have that $U \times V$ is the basis element for $X \times Y$, with U open in X, and V open in Y. Thus $(U \times V) \cap (A \times B) = (U \cap A) \times (V \cap B)$ is a basis element for the subspace topology on $X \times Y$. Since $U \cap A$ and $V \cap B$ are open in the subspace topologies of A and B respectively, then $(U \cap A) \times (V \cap B)$ is a basis for the product topology on $A \times B$. ■

Example 1.8. (1) Consider $[0, 1] \subseteq \mathbb{R}$. In the subspace topology of $[0, 1]$, we have as basis elements of the form $(a, b) \cap [0, 1]$, with $(a, b) \subseteq \mathbb{R}$. If we have that $(a, b) \subseteq [0, 1]$, then $(a, b) \cap [0, 1] = (a, b)$. On the other hand, if $a \in [0, 1]$ or $b \in [0, 1]$, then we get $(a, b) \cap [0, 1] = (a, 1]$ or $(a, b) \cap [0, 1] = [0, b)$, lastly if neither a nor b are in $[0, 1]$, then we have $(a, b) \cap [0, 1] = [0, 1]$ only if $[0, 1] \subseteq (a, b)$, and $(a, b) \cap [0, 1] = \emptyset$ otherwise.

The only one of these sets open in $\mathbb{R}$ under the standard topology is $(0, 1)$.

(2) For $[0, 1) \cup \{2\} \subseteq \mathbb{R}$, the singletoun $\{2\}$ is open in the subspace topology on $[0, 1) \cup \{2\}$; for observe, that $(\frac{3}{5}, \frac{5}{2}) \cap ([0, 1) \cup \{2\}) = \{2\}$, however, in the order topology, on that same set, $\{2\}$ is not open. Any basis element on $[0, 1) \cup \{2\}$ containing 2 is of the form $(a, 2]$, where $a \in [0, 1) \cup \{2\}$.

(3) The dictionary order on $[0, 1] \times [0, 1]$ is a restriction of the dictionary order on $\mathbb{R} \times \mathbb{R}$. Now the set $\{\frac{1}{2}\} \times (\frac{1}{2}, 1]$ is open in the subspace topology on $[0, 1] \times [0, 1]$, but it is not open in the dictionary order on the same set.

Definition. We call the set $[0, 1] \times [0, 1]$ on the dictionary odere the **ordered square**, and we denote it by I_0^2.

Lemma 1.5.5. *Let Y be a subspace of a topological space X, and let $A \subseteq Y$. Then the topology of A as a subspace of Y is the same as the topolgy of A as a subspace of X.*

Proof. Let V be open in Y, then $V \cap A$ is open in A as a subspace of Y, however, since Y is a subspace of X, $V = U \cap Y$ for some U open in X, so $V \cap A = (U \cap Y) \cap A = U \cap (A \cap Y) = U \cap A$, making $V \cap A$ open in A as a subspace of X.

Conversely, if U is open in X, then $U \cap A$ is open in A as a subspace of X, additionally, $U \cap Y$ is open in Y as a subspace of Y, so that $(U \cap Y) \cap A = U \cap (A \cap Y) = U \cap A$, which makes $U \cap A$ open in A as a subspace of Y. ■

Example 1.9. Suppose that $\mathcal{T}$ and $\mathcal{T}'$ are topologies on a set X, with $\mathcal{T} \subseteq \mathcal{T}'$. Then if $Y \subseteq X$, then the topology of Y as a subspace of X under $\mathcal{T}'$ is finer than the topology of Y as a subspace of X under $\mathcal{T}$; for, if U is open in X under $\mathcal{T}$, then $U \cap Y$ is open in Y under $\mathcal{T}$. But $\mathcal{T} \subseteq \mathcal{T}'$, which implies that $U \cap Y$ is open in Y under $\mathcal{T}'$

Definition. Let X be an ordered set. We say that a nonempty subset $Y \subset X$ is **convex** in X if for each pair of points $a, b \in Y$, with $a < b$, then the open interval $(a, b) \subseteq X$ is also contained in Y.

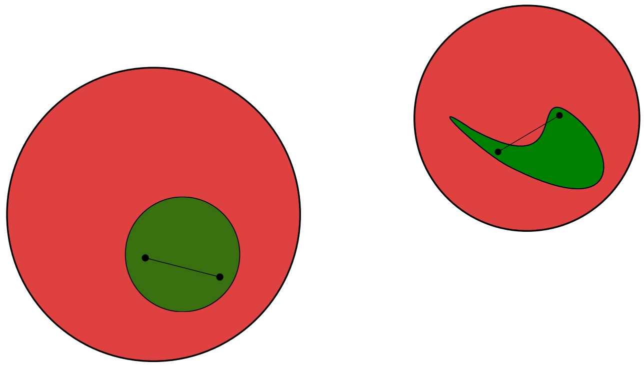

Figure 1.6: A convex set, and a nonconvex set.

Example 1.10. Let X be any ordered set. Then by definition, all open intervals and rays in X are convex in X.

Theorem 1.5.6. *Let X be an ordered set on the order toplogy, and let $Y \subseteq X$ be convex in X. Then the order topology on Y is the same as the subspace topology on Y.*

Proof. Consider $(a, \infty) \subseteq X$. If $a \in Y$, then $(a, \infty) \cap Y = \{x \in Y : x > a\}$, which is by definition an open ray on Y. Now if $a \notin Y$, then a is either a lowerbound, or an upperbound. Then $(a, \infty) \cap Y = \emptyset$ and $(-\infty, a) \cap Y = Y$ if a is an upperbound, similarly, if a is a lowerbound we get $(a, \infty) \cap Y = Y$ and $(-\infty, a) \cap Y = \emptyset$.

Since $(a, \infty) \cap Y$ and $(-\infty, a) \cap Y$ form a subbasis on the subspace topology on Y, and since they are also open in the order topology, then the order topology contains the subspace topology.

Now if (a, ∞) is an open ray in Y, then $(a, \infty) = (b, \infty) \cap Y$, with (b, ∞) some open ray in X, hence (a, ∞) is open in the subspace topology of Y, and since it also forms the subbasis for the order topology, we have that the order topology is contained within the subspace topology. Thus both topologies are equal. ■

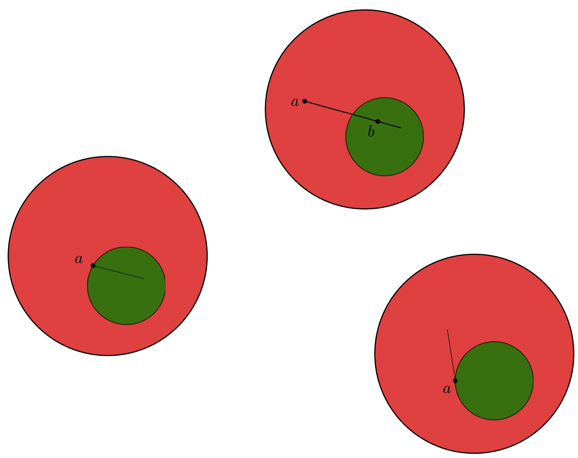

Figure 1.7: An illustration of theorem 1.5.6.

Bibliography

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