Algebraic Topology

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Chapter 1

Categories.

1.1 Categories and Subcategories.

Definition. A **category** $\mathcal{C}$ is a collection of a class of **objects**, denoted $\text{obj}\,\mathcal{C}$ a collection of sets of **morphisms** $\text{Hom}\,(A, B)$ for each $A, B \in \text{obj}\,\mathcal{C}$ and a binary operation $\circ : \text{Hom}\,(A, B) \times \text{Hom}\,(B, C) \to \text{Hom}\,(A, C)$, defined by $(f, g) \to g \circ f$, called **composition** such that:

(1) Each $\text{Hom}\,(A, B)$ is pairwise disjoint for all $A, B \in \text{obj}\,\mathcal{C}$.

(2) $\circ$ is associative when defined; that is if either $(g \circ f) \circ h$ or $g \circ (f \circ h)$ are defined, then $(g \circ f) \circ h = g \circ (f \circ h)$, for morphisms f, g, h.

(3) For each $A \in \text{obj}\,\mathcal{C}$, there exists an **identity** morphism $1_A \in \text{Hom}\,(A, A)$ such that for each $B, C \in \text{obj}\,\mathcal{C}$, $1_A \circ f = f$ and $g \circ 1_A = g$ for each morphism $f \in \text{Hom}\,(B, A)$ and $g \in \text{Hom}\,(A, C)$.

We denote morphisms by $f : A \to B$ instead of $f \in (A, B)$.

Definition. Let $\mathcal{C}$ be a category and $f : A \to B$ a morphism in $\mathcal{C}$. We call A and B the **domain** and **codomain** of f, respectively, and we call the set $G_f = \{(a, f(a)) : a \in A\} \subseteq B$ the **graph** of f.

Example 1.1. (1) The category of all sets Set has as onjects the class of all sets. The morphisms in Set are all functions $f : A \to B$ where A and B are sets. The composition of Set is the usual composition of functions.

(2) The category of all topological spaces Top has as objects all topological spaces, and as morphisms all continuous maps $f : Y \to Y$ from a space X to a space Y. The composition is the usual composition.

(3) The category of all groups, Grp has as objects all groups and as morphisms all homomorphisms $f : G \to H$, under the usual composition.

(4) The category of rings with unit Rng has as objects all rings with unit, along with all ring homomorphisms $f : R \to K$ to be the morphisms under the usual composition.

Definition. We call a category a **subcategory** of a category $\mathcal{C}$ if $\operatorname{obj}\mathcal{A} \subseteq \operatorname{obj}\mathcal{C}$,$\operatorname{Hom}_{\mathcal{A}}(A,B) \subseteq \operatorname{Hom}_{\mathcal{C}}(A,B)$ for all $A, B \in \mathcal{C}$, and $\mathcal{A}$ inherits the composition of $\mathcal{C}$.

Example 1.2. (1) The category of all pairs of topological spaces (X, A) where A is a subspace of X, whose morphisms are pairs of continuous maps $f = (f_1, f_2)$ such that $f_1 i = j f_2$ where $i : A \to X$ and $j : B \to Y$ are inclusions, is a subcategory of Top. We denote this category Top^2.

$$\begin{tikzcd}
A \arrow[r, hook, "i"] \arrow[d, "f_2"'] & X \arrow[d, "f_1"] \\
B \arrow[r, hook, "j"'] & Y
\end{tikzcd}$$

(2) The category of all **pointed spaces**, Top^* is defined with the objects being all pairs $(X, \{x_0\})$, where $x_0 \in X$ with the morphisms of Top^2. Top^* is a subcategory of Top^2. We call x_0 the **base point**, and we call the morphisms of Top^* **pointed maps**.

(2) The category of all Abelian groups, Ab is a subcategory of Grp. Likewise, the category of all commutative rings with unit is a subcategory of Rng.

1.2 Commutative Diagrams and Congruences.

Definition. A **diagram** in a category is a directed graph with its vertex set a subset of objects, and whose edge set are morphisms between those objects. We call a diagram **commutative** if for any pair of objects, every pair of morphisms between those objects are equal. That is if A, A' and B, B' are pairs of objects with pairs of morphisms $f : A \to B$, $f : A \to A'$ and $f' : A' \to B'$, $g' : B \to B'$ we have that $g \circ f' = f \circ g'$

$$\begin{tikzcd}
A \arrow[r, "g"] \arrow[d, "f"'] & A' \arrow[d, "f'"] \\
B \arrow[r, "g"'] & B'
\end{tikzcd}$$

Definition. A **congruence** on a category $\mathcal{C}$ is an equivalence relation $\sim$ on morphisms in $\mathcal{C}$ such that:

(1) If $f \in \operatorname{Hom}(A, B)$, and $f \sim f'$, then $f' \in \operatorname{Hom}(A, B)$.

(2) If $f \sim g$ and $f' \sim g'$, then $g \circ f \sim g' \circ f'$.

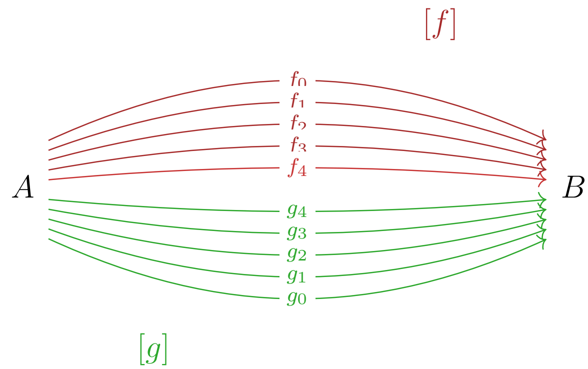

Figure 1.1: An equivalence relation between morphisms.

Theorem 1.2.1. *Let $\mathcal{C}$ be a category with congruence $\sim$. Define ${}^{\mathcal{C}}\!/\!_{\sim}$ as follows:*

(1) $\operatorname{obj} {}^{\mathcal{C}}\!/\!_{\sim} = \operatorname{obj} \mathcal{C}$.

(2) $\operatorname{Hom}_{\mathcal{C}/\sim}(A, B) = \{[f] : f \in \operatorname{Hom}_{\mathcal{C}}(A, B)\}$.

(3) $[g] \circ [f] = [g \circ f]$

Then ${}^{\mathcal{C}}\!/\!_{\sim}$ is a category.

Proof. We have by equivalence that $\operatorname{obj} {}^{\mathcal{C}}\!/\!_{\sim}$ is a class. Moreover, since $\sim$ partitions $\mathcal{C}$, it partions all of the $\operatorname{Hom}(A, B)$ for each A, B. So each $\operatorname{Hom}(A, B)$ is a set, moreover, they are pariwise disjoint by definition of $\sim$. Now, notice that by hypothesis, composition in ${}^{\mathcal{C}}\!/\!_{\sim}$ is well defined, so $[1_A] \circ [f] = [1_A \circ f] = [f]$ and $[g] \circ [1_A] = [g \circ 1_A] = [g]$. This makes ${}^{\mathcal{C}}\!/\!_{\sim}$ a category. ∎

Remark. On can think of the cateegory ${}^{\mathcal{C}}\!/\!_{\sim}$ as taking all morphisms with they same domain and codomain, and collapsing them into a single morphism.

Definition. Let $\mathcal{C}$ be a catogory and $\sim$ a congruence of $\mathcal{C}$. We call the category ${}^{\mathcal{C}}\!/\!_{\sim}$ induced by $\sim$ the **quotient category**.

1.3 Functors.

Definition. Let $\mathcal{A}$ and $\mathcal{C}$ be categories. We deine a **covariant functor** to be a map $F : \mathcal{A} \to \mathcal{C}$ such that:

(1) $A \in \operatorname{obj} \mathcal{A}$ implies $F(A) \in \operatorname{obj} \mathcal{C}$.

(2) If $f : A \to B$ is a morphism in $\mathcal{A}$, then $F(f) : F(A) \to F(B)$ is a morphism in $\mathcal{C}$.

$$\begin{tikzcd} A \arrow[r, bend left, "{[f]}"] \arrow[r, bend right, "{[g]}"'] & B \end{tikzcd}$$

Figure 1.2: Morphisms in a category with the same domain and codomain get collapsed onto a single morphism in the correspinding quotient category.

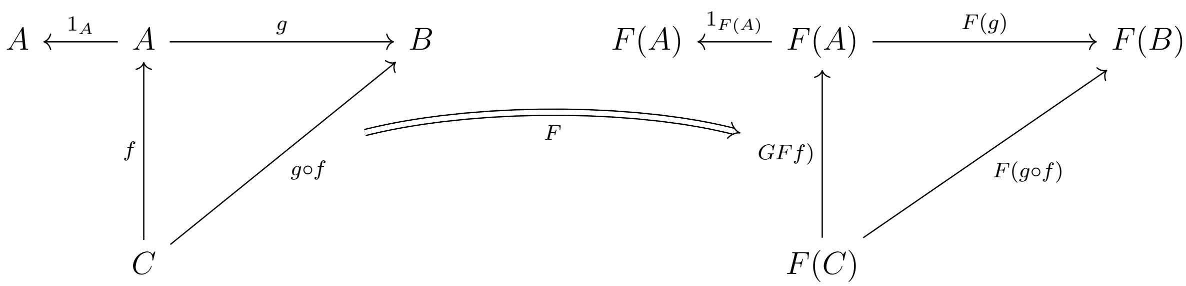

Figure 1.3: A covariant functor taking a diagram in on category to a diagram in the other.

(3) For all morphisms f and g in $\mathcal{A}$, for which $g \circ f$ is defined, we have that $F(g \circ f) = F(g) \circ F(f)$, and $F(1_A) = 1_{F(A)}$.

Example 1.3. (1) We define the **forgetful functor** the map $F : \mathcal{C} \to \text{Set}$ that takes all objects in $\mathcal{C}$ to their underlying sets, and morphisms in $\mathcal{C}$ to themselves considered as functions under the usual composition. For example the forgetful functor $F : \text{Top} \to \text{Set}$ takes topological spaces X to their underlying sets, and continuous maps to themselves, considered just as functions.

(2) The **identity functor** is the functor $I : \mathcal{C} \to \mathcal{C}$ that takes objects and morphisms in $\mathcal{C}$ to themselves.

(3) Let M be a topological space. Define $F_M : \text{Top} \to \text{Top}$ by $F_M : X \to X \times M$, and for each continuous map $f : X \to Y$, $F(f) : X \times M \to Y \times M$ is defined by $(x, m) \to (f(x), m)$. Then F_M is a functor.

(4) Let $A \in \text{obj}\, \mathcal{C}$ and take the map $\text{Hom}\,(A, *) : \mathcal{C} \to \text{Set}$ that takes $A \to \text{Hom}\,(A, B)$ and for each morphism $f : B \to B'$, $\text{Hom}\,(A, f) : \text{Hom}\,(A, B) \to \text{Hom}\,(A, B')$ is given by $g \to f \circ g$. With call this functor the **covariant Hom functor**, and denote it f_*.

Definition. Let $\mathcal{A}$ and $\mathcal{C}$ be categories. We deine a **contravariant functor** to be a map $G : \mathcal{A} \to \mathcal{C}$ such that:

(1) $A \in \text{obj}\, \mathcal{A}$ implies $G(A) \in \text{obj}\, \mathcal{C}$.

(2) If $f : A \to B$ is a morphism in $\mathcal{A}$, then $G(f) : G(B) \to G(A)$ is a morphism in $\mathcal{C}$.

(3) For all morphisms f and g in $\mathcal{A}$, for which $g \circ f$ is defined, we have that $G(g \circ f) = G(f) \circ G(g)$, and $G(1_A) = 1_{G(A)}$.

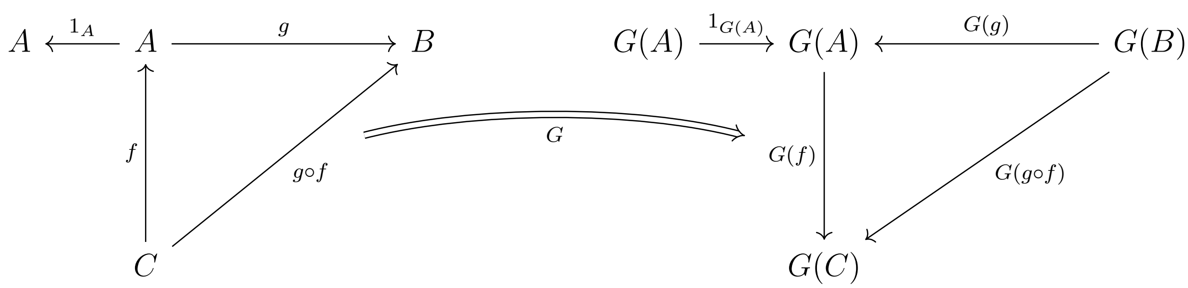

Figure 1.4: A contravariant functor taking a diagram in on category to a diagram in the other.

Example 1.4. (1) Let F be a field, and Vec the category of all finite dimensional vector spaces over F, whose morphisms are linear transformations. Define the map $T : \text{Vec} \to \text{Vec}$ by taking $T : V \to V^{\perp}$, and $T : f \to f^T$. That is T takes vector spaces to their dual spaces, and linear transformation to their transpose. T is a contravariant functor called the **dual space functor.**

(2) Define $\text{Hom}\,(*, B) : \mathcal{C} \to \mathcal{C}$ by taking $\text{Hom}\,(*, B) : A \to \text{Hom}\,(A, B)$ and for each morphism $g : A \to A'$ in $\mathcal{C}$, $\text{Hom}\,(f, B) : \text{Hom}\,(A', B) \to \text{Hom}\,(A, B)$ is defined by taking $h \to h \circ g$. This is analogous to the covariant Hom functor, and we call it the **contravariant Hom functor.**

Definition. We call a morphism $f : A \to B$ an **equivalence** if there exists a morphism $g : B \to A$ such taht $f \circ g = 1_B$ and $g \circ f = 1_A$

Theorem 1.3.1. *Let $\mathcal{A}$ and $\mathcal{C}$ be categories, and $F : \mathcal{A} \to \mathcal{C}$ be a functor. If f is an equivalence in $\mathcal{A}$, then $F(f)$ is an equivalence in $\mathcal{C}$.*

Proof. Suppose that F is a covariant functor. Notice that if $f : A \to B$ is an equivalence, then there is a $g : B \to A$ with $f \circ g = 1_B$ and $g \circ f = 1_A$. Then $F(f \circ g) = F(f) \circ F(g) = F(1_B) = 1_{F(B)}$, and $F(g \circ f) = F(g) \circ F(f) = F(1_A) = 1_{F(A)}$.

Likewise, if F is contravariant, notice that $F(f) : B \to A$ and $F(g) : A \to B$. Then $F(f \circ g) = F(g) \circ F(f) = 1_{F(A)}$, and $F(g \circ f) = F(f) \circ F(g) = 1_{F(B)}$. In eithe case, we find that $F(f)$ is an equivalence in $\mathcal{C}$. ■

Chapter 2

Homotopy, Convexity, and Connectedness.

2.1 Homotopy

Definition. If X and Y are topological spaces, and $f_0 : X \to Y$ and $f_1 : X \to Y$ are continuous maps, we say that f_0 is **homotopic** to f_1 if there exists a continuous map $F : X \times I \to Y$ with $F(x, 0) = f_0(x)$ and $F(x, 1) = f_1(x)$. We write $f_0 \simeq f_1$ and call F a **homotopy**. We also write $F : f_0 \simeq f_1$ to denote a homotopy between f_0 and f_1.

Lemma 2.1.1 (The Pasting Lemma)**.** *Let X is a topolgoical space that is covered by open sets $\{X_n\}$. If Y is some topological space for which there exist unique maps $f_n : X_n \to Y$ that coincide in the intersections of their domains, then there exists a unique map $f : X \to Y$ such that $f|_{X_n} = f_n$, for all n.*

Lemma 2.1.2. *Homotopy between continuous maps is an equivalence relation.*

Proof. Let $f : X \to Y$ be a continouos map. Define $F : X \times I \rangle Y$ by $(x, t) \to f(x)$ for all $(x, t) \in X \times I$. Then F is continuous by definition; moreover, $F(x, 0) = F(x, 1) = f(x)$, making $f \simeq f$.

Now suppose there exist a homotopy $F : f \simeq g$ for maps $f : X \to Y$ and $g : X \to Y$. Define the map $G : X \times I \to Y$ by $(x, t) \to F(x, 1 - t)$. G is the composition of continuous maps, so G is continuous, moreover, $G(x, 0) = F(x, 1) = g(x)$ and $G(x, 1) = F(x, 0) = f(x)$, so that $g \simeq f$.

Lastly, suppose that $F : f \simeq g$ and $G : g \simeq h$ for maps f, g, h. Define the map $H : X \times I \to Y$ by:

$$H(x,t) = \begin{cases} F(x, 2t), & \text{if } 0 \leq t \leq \frac{1}{2} \\ G(x, 2t - 1), & \text{if } \frac{1}{2} \leq t \leq 1 \end{cases}$$

Notice that F and G conicide in their domains which cover X. Therefore, by the pasting lemma, H is continuous. Now notice also that $H(x, 0) = F(x, 2 \cdot 0) = F(x, 0) = f(x)$ and $H(x, 1) = G(x, 2 \cdot 1 - 1) = G(x, 1) = h(x)$. This makes $f \simeq h$. ∎

Definition. For any continuous map $f : X \to Y$ we define the **homotopy class** of f to be the equivalence class of all continuous maps homotopic to f. That is:

$$[f] = \{g : X \to Y : g \text{ is continous and } g \simeq f\}$$

Lemma 2.1.3. *Let $f_0 : X \to Y$, $f_1 : X \to Y$ and $g_0 : X \to Y$, $g_1 : X \to Y$ be continuous maps. If $f_0 \simeq f_1$ and $g_0 \simeq g_1$, then $g_0 \circ f_0 \simeq g_1 \circ f_1$. That is $[g_0 \circ f_0] = [g_1 \circ f_1]$.*

Proof. Let $F : f_0 \simeq f_1$ and $G : g_0 \simeq g_1$ be the homotopies of f_0 into f_1 and g_0 into g_1, respectively. Define the map $H : X \times I \to Y$ by taking $(x, t) \to G(f_0(x), t)$. Then we have that H is continuous by composition, and that $H(x, 0) = G(f_0(x), 0) = g_0(f_0(x))$, and $H(x, 1) = G(f_0(x), 1) = g_1(f_0(x))$. Thus we see that $g_0 \circ f_0 \simeq g_1 \circ f_0$.

Now define the map $K : X \times I \to Y$ by $K = g_1 \circ F$. We have that K is continuous by composition, and that $K(x, 0) = g_1 \circ f_0$ and $K(x, 1) = g_1 \circ f_1$, making $g_1 \circ f_0 \simeq g_1 \circ f_1$. Therefore, by transitivity of homotopy, $g_0 \circ f_0 \simeq g_1 \circ f_1$. ■

Theorem 2.1.4. *Homotopy is a congruence on the category* Top*.*

Proof. The proof follows by lemmas 2.1.2 and 2.1.3. ■

Definition. We call the quotient category of Top induced by homotopy the **homotopy category** and denote it hTop.

Definition. A continuous map $f : X \to Y$ is a **homotopy equivalence** if there exists a continuous map $g : Y \to X$ such that $f \circ g \simeq 1_Y$ and $g \circ f \simeq 1_X$. We say that the spaces X and Y have the same **homotopy type** if there exists a homotopy equivalence.

Definition. We call a continuous map **nullhomotopic** if it is homotopic to a constant map.

Example 2.1. The space of complex numbers $\mathbb{C}$ and the unit circle S^1 have the same homnotopy type.

Definition. Let Y and Z be topological spaces, and $X \subseteq Y$ a subspace of Y. If $f : X \to Z$ is a continuous map, then we call the map $g : Y \to Z$ defined by $g \circ i = f$ an **extension** of f, where $i : X \to Y$ is the inclusion map.

Theorem 2.1.5. *Let $f : S^n \to Y$ be a continuous map into a topological space Y. The following are equivalent:*

(1) f is nullhomotopic.

(2) f can be extended to a continuous map $B^{n+1} \to Y$.

(3) There exists a constant map $k : S^n \to Y$, taking $x \to f(x_0)$, for all $x \in S^n$, such that $f \simeq k$, for $x_0 \in S^n$.

Proof. Notice that (3) implies (1) immediately. Now suppose that f is nullhomotopic. Then there exists a constant map $k : X \to Y$, such that for some $x_0 \in S^n$, $k : x \to x_0$ for all $x \in S^n$ implies that $f \simeq k$. Now, define the map $g : B^{n+1} \to Y$ by:

$$g(x) = \begin{cases} y_0, & \text{if } 0 \leq \|x\| \leq \frac{1}{2} \\ F(\frac{x}{\|x\|}, 2 - 2\|x\|), & \text{if } \frac{1}{2} \leq \|x\| \leq 1 \end{cases}$$

Notice, that if $\|x\| = \frac{1}{2}$, then $g(x) = F(2x, 1) = y_0$. Therefore, by the pasting lemma, g is continuous. Moreover, if $\|x\| = 1$, $g(x) = F(x, 0) = f$, which makes g an extension of f.

Now, suppose that there exists an extension $g : B^{n+1} \to Y$ of f. Since S^n is a subspace of B^{n+1}, we have that $g \circ i = g|_{S^n} = f$, where $i : Y \to S^n$ is an inclusion. Now, let $x_0 \in S^n$ and define the constant map $k : S^n \to Y$ by taking $x \to f(x_0)$ for all $x \in S^n$. Additionally, define the map $F : S^N \times I \to Y$ given by $F(x, t) = g((1-t)x + x_0 t)$. We have that F is continuous by composition of continuous maps, and that $F(x, 0) = g(x) = f(x)$, since F has the domain $S^n \times I$, and that $F(x, 1) = g(x_0) = f(x_0)$, since F has the domain $S^n \times I$. This makes $f \simeq k$ with F as the associated homotopy. ■

2.2 Quotient Spaces

Definition. Let X be a topological space, and $X' = \{X_\alpha\}$ a partion of X. We define the **natrual map** $q : X \to X'$ by taking $x \to X_\alpha$ where $x \in X_\alpha$. We define the **quotient topology** on X' to be the family:

$$\mathcal{T} = \{U' \subseteq X' : q^{-1}(U') \text{ is open in } X\}$$

We denote quotient spaces by X/q, X/X', or $X/\sim$ where $\sim$ is an equivalence relation partitioning X into X'.

Example 2.2. (1) Consider the space $I = [0, 1]$ and let $A = \{0, 1\}$. The the quotient space I/A identifies 0 to 1, and hence, under the quotient topology, is homeomorphic to S^1.

(2) Consider the space $I \times I$ and define an equivalence relation $(x, 0) \sim (x, 1)$ for all $x \in I$. Then the quotient topology formed on $I \times I/\sim$ is homeomorphic to the cylinder $S^1 \times I$. Defining another equivalence $(0, y) \sim (1, y)$ for all $y \in I$, we get the quotient space on $S^1 \times I/\sim$ under this equivalence relations is homeomorphic to the torus $S^1 \times S^1$.

(3) Let $h : X \to Y$ be a map, and define $\ker h$ the equivalence relation on X such that $x \ker h x'$ if, and only if $h(x) = h(x')$. The quotient space $X/\ker h$ has the following relation to the natural map on X via the commutative diagram

$$\begin{tikzcd} X \arrow[r, "h"] \arrow[d, "q"'] & Y \\ X/\ker h \arrow[ur, "\phi"'] & \end{tikzcd}$$

Where $\phi : X/\ker h \to Y$ is a 1–1 map defined by $\phi([x]) = h(x)$.

Definition. A continuous map $f : X \to Y$ of a topological space X onto a topological space Y is call an **identification** if a subset U of Y is open if, and only if $f^{-1}(U)$ is open in X. We denote the quotient space on X induced by f by X/f.

Example 2.3. (1) The natural map $q : X \to {}^{X}\!/\!_{\sim}$ is an identification, where $\sim$ is an equivalence relation on X inducing the quotient topology.

(2) If $f : X \to Y$ takes spaces X onto Y, is open or closed, then f is an identification.

(3) If $f : X \to Y$ is a continuous map such that there exists a map $s : Y \to X$ such that $f \circ s = 1_Y$, then f is an identification. We call the map s a **section** of f.

Theorem 2.2.1. *Let $f : X \to Y$ be a continuous map of a topological space X onto a topological space Y. f is an identification if, and only if for any toopological space Z, and all maps $g : Y \to Z$, then g is continuous if, and only if $g \circ f$ is continuous.*

$$\begin{tikzcd} X \arrow[r, "g \circ f"] \arrow[d, "f"'] & Z \\ Y \arrow[ur, "g"'] & \end{tikzcd}$$

Proof. Suppos that f is an identification. If g is continuous, then so is $g \circ f$, by continuity of f. On the other hand, if $g \circ f$ is continuous, letting V be open in Z we have $(g \circ f)^{-1}(V) = f^{-1}(g^{-1}(V))$ which is open in X. By hypothesis, $g^{-1}(V)$ is open in Y, which makes g continuous.

Now, suppose that g is continuous if, and only if $g \circ f$ is continuous. Let $Z = {}^{X}\!/\!_{\ker f}$, and $q : X \to {}^{X}\!/\!_{\ker f}$ the natural map. Additionally, define the 1–1 map $\phi : {}^{X}\!/\!_{\ker f} \to Y$ by $\phi([x]) = f(x)$. Since f is onto, we get that so is ϕ. Consider the following commutative diagram:

$$\begin{tikzcd} X \arrow[r, "q"] \arrow[d, "f"'] & Z \\ Y \arrow[ur, "\phi^{-1}"'] & \end{tikzcd}$$

Then $\phi^{-1} \circ f = q$ is continuous which implies that ϕ^{-1} is continuous. ϕ is also continuou since q is an identification. Therefore ϕ is a homeomorphism between Y and Z. Notice now, that $f = \phi \circ q$. Then since q and ϕ are continuous, this makes f continuous by composition. Moreover, $f^{-1}(U) = q^{-1}(\phi^{-1}(U))$. Since q is an identification, $\phi^{-1}(U)$ is open in Z, which makes $f^{-1}(U)$ open in X. This makes f an identification. ■

f

Corollary. *Let $f : X \to Y$ be an identification, and for some space Z, define $h : X \to Z$ to* *be the continuous map constant on each fiber of f. Then $h \circ f^{-1} : Y \to Z$ is continuous.*

$$\begin{tikzcd} X \arrow[r, "h"] \arrow[d, "f"'] & Z \\ Y \arrow[ur, "h\circ f^{-1}"'] & \end{tikzcd}$$

Moreover $h \circ f^{-1}$ is open or closed if, and only if $h(U)$ is open or closed in Z whenever $U = f^{-1}(f(U))$ is open or closed in X.

Corollary. *If $h : X \to Z$ is an identification, then the map $\phi : X/_{\ker h} \to Z$ defined by $[x] \to h(x)$ is a homeomorphism.*

2.3 Convexity and Contracibilty

Definition. We call a subset X of $\mathbb{R}^n$ **convex** if for every $x, y \in X$, the line segment joining x to y is convex. That is the line $tx + (1-t)y \in X$ for all $t \in [0,1]$.

Example 2.4. The sets $\mathbb{R}^n$, I^n, B^n and $\Delta(\mathbb{R}^n)$ are all convex. The sphere S^{n-1} is not convex.

Definition. We call a topological space X **contracible** if 1_X is nullhomotopic.

Example 2.5. (1) Let $X = \{x, y\}$ together with the topolgoy $\mathcal{T} = \{\emptyset, \{x\}, X\}$. Then X is contractible under the topology $\mathcal{T}$. We call X together with $\mathcal{T}$ the **Sierpinski space**.

(2) The space $\mathbb{R}^n$ is contractible, but the sphere S^{n-1} is not contractible.

(3) Continuous images of contractible spaces need not be contractible.

Theorem 2.3.1. *Every convex set is contractible.*

Proof. Choose $x_0 \in X$ and consider the constant map $c : X \to X$ by $x \to x_0$ for all $x \in X$. Define $F : X \times I \to X$ by $F(x,t) = tx_0 + (1-t)x$. This map is continuous, with $F(x,0) = x = 1_X(x)$ and $F(x,1) = x_0 = c(x)$. Therefore $1_X \simeq c$. ■

Lemma 2.3.2. *If X is a contractible space, and homeomorphic to a space Y, then Y is also contractible.*

Example 2.6. If X and Y are subspaces of $\mathbb{R}^n$, with X homeomorphic to Y, and X convex, then Y is contractible by lemma 2.3.2, however, Y may not be convex. This shows that not all contractible spaces are convex spaces.

Lemma 2.3.3. *Contractible spaces are connected.*

Corollary. *Convex sets are connected.*

Proof. This follows from theorem 2.3.1. ■

Definition. If X is a topological space, define the equivalence relation $\sim$ on $X \times I$ by $(x,t) \sim (x',t')$ if, and only if $t = t' = 1$. Denote the equivalence classes of (x,t) as $[x,t]$. We call the quotient space ${}^{X \times I}\!/\!_{\sim}$ the **cone** over X, and denote it CX. We call the equivalence class $[x,1]$ the **vertex** of CX.

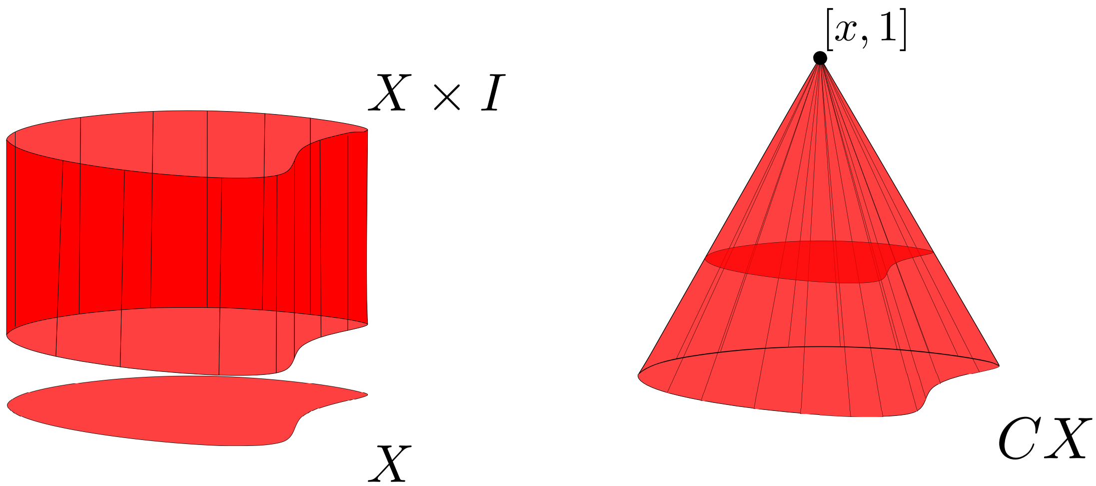

Figure 2.1: The space X and the cone CX formed by identifying all $t = 1$ of $X \times I$ to a point.

Example 2.7. (1) For topological spaces X and Y, every continuous map $f : X \times I \to Y$ with $f(x,1) = y_0$ for some $y_0 \in Y$ induces a continuous map $Cf : CX \to Y$ by taking $[x,t] \to f(x,t)$.

(2) The cone over S^{n-1} is $CS^{n-1} = D^n$ and has the vertex 0.

Theorem 2.3.4. *For any topological space X, the cone over X is contractible.*

Proof. Define the map $F : CX \times I \to CX$ by taking $([x,t],s) \to [x,(1-s)t+s]$. This map is continuous by composition, moreover $F([x,t],0) = [x,t]$ and $F([x,t],1) = [x,1]$ which makes $1_{CX} \simeq c$ where $c : CX \to CX$ is the constant map taking $[x,t] \to [x,1]$ for all $x \in X$. ■

Theorem 2.3.5. *A topological space has the same homotopy type as a point if, and only if X is contractible.*

Proof. Let $\{a\}$ be a point space, and suppose that $X \simeq \{a\}$ have the same homotopy type. Then there are maps $f : X \to \{a\}$ and $g : \{a\} \to X$ with $a \xrightarrow{g} x_0$ such that $g \circ f \simeq 1_X$ and $f \circ g \simeq 1_{\{a\}}$. Notice that $g \circ f(x) = g(a) = x_0$, for all $x \in X$, so $g \circ f$ is constant. This makes 1_X (and 1_Y) nullhomotopic. Therefore X is contractible.

On the otherhand, supposing that X is contractible, let $1_X \simeq c$ where $c : X \to X$ is the constant map defined by $x \to x_0$ for all $x \in X$. Define the maps $f : X \to \{x_0\}$ and $g : \{x_0\} \to X$ by $x \xrightarrow{f} x_0$ and $x_0 \xrightarrow{g} x_0$. Observe that $g \circ f = 1_X$, and that $f \circ g \simeq 1_{\{x_0\}}$ Therefore X is of the same homotopy type as $\{x_0\}$. ■

Remark. This theorem shows that the simplest objects in hTop are the contractible spaces.

Theorem 2.3.6. *If Y is a contractible space, then any two maps $X \to Y$ are homotopic.*

Proof. Suppose that $1_Y \simeq c$ where $c : Y \to Y$ takes $y \to y_0$ for all $y \in Y$. Defie $g : X \to Y$ by taking $x \to y_0$ for all $x \in X$. If $f : X \to Y$ is any continuous map, then $f \simeq g$. Consider the diagram

$$\begin{tikzcd} X \arrow[r] & Y \arrow[r, bend left, "k"] \arrow[r, bend right, "1_Y"'] & Y \end{tikzcd}$$

Since $1_Y \simeq k$, we get that $f = 1_Y \circ f \simeq k \circ f = g$. ■

Corollary. *Any two maps $X \to Y$ are nullhomotopic.*

2.4 Path Connectedness.

Definition. A **path** in a topological space X is a continuous map $f : [0, 1] \to X$ such that $f(0) = a$ and $f(1) = b$ for some $a, b \in X$. We call a and b the **endpoints** of f, we say f goes **from** a **to** b.

Definition. We call a topological space X **path connected** if there exists a path from a to b for all $a, b \in X$.

Example 2.8. The sphere S^n is path connected.

Lemma 2.4.1. *If $f : X \to Y$ is a continuous map and X is a path connected space, then $f(X)$ is also path connected.*

Theorem 2.4.2. *If X is a path connected space, then X is a connected space.*

Proof. Suppose that X is disconnected. Then there exists a seperation of X into disjoint open sets U and V. That is $X = U \cup V$. Suppose however that X is path connected. Then for points $a \in U$ and $b \in V$, there is a path $f : [0, 1] \to X$ from a to b. Since $[0, 1]$ is a connected space, so is $f([0, 1])$; however notice that $f([0, 1]) = (U \cap f([0, 1])) \cup (f([0, 1]) \cap v)$, which is a seperation of $f([0, 1])$, since U and V form a seperation. ■

Example 2.9. The converse of theorem 2.4.1 is not true in general. Consider the following two examples:

(1) Consider the subspace $X = (0 \times [0, 1]) \cup G$ where G is the graph of $\sin \frac{1}{x}$ on the interval $(0, 2\pi]$. We have that X is connected, since the component containing G is closed, and $0 \times [0, 1] \subseteq \operatorname{cl} G$. However, X is not path connected. We call the space X the **topologists sine curve**.

(2) Another example of a connected space in $\mathbb{R}^2$ that is not path connected is the **topologist's whirlpool**.

Lemma 2.4.3. *Every contractible space is path connected.*

Lemma 2.4.4. *A topological space X is path connected if, and only if any two constant maps $X \to X$ are homotopic.*

Lemma 2.4.5. *If X is a contractible space and Y a path connected space, then any two continuous maps $X \to Y$ are homotopic.*

Corollary. *The continuous maps are nullhomotopic.*

Lemma 2.4.6. *If X and Y are path connnected spaces, then so is $X \times Y$.*

Lemma 2.4.7. *If $f : X \to Y$ is a continuous map and X is a path connected space, then $f(X)$ is also path connected.*

Theorem 2.4.8. *If X is a topological space, then the relation $\sim$ defined on X by $a \sim b$ if, and only if there is a path from a to b, is an equivalence relation.*

Proof. Consider the constant path $c : [0,1] \to X$ where $c(x) = a$ for all $x \in A$. c is continuous, and $c(0) = c(1) = a$. So $a \sim a$.

Now suppose that for $a, b \in X$, that $a \sim b$. Then there is a path $f : [0,1] \to X$ with $f(0) = a$ and $f(1) = b$. Consider the map $g : [0,1] \to X$ defined by $g(t) = f(1-t)$. g is continuous by composition, and $g(0) = f(1) = b$ and $g(1) = f(0) = a$, which makes $b \sim a$.

Lastly, suppose that $a \sim b$ and $b \sim c$ for some $a, b, c \in X$. Then there exist paths $f : [0,1] \to X$ and $g : [0,1] \to X$ with $f(0) = a$, $f(1) = b$, and $g(0) = b$, $g(1) = a$. Now, consider the map $h : [0,1] \to X$ defined by:

$$h(t) = \begin{cases} f(2t), & \text{if } 0 \leq t \leq \frac{1}{2} \\ g(2t-1), & \text{if } \frac{1}{2} \leq t \leq 1 \end{cases}$$

Notce that $f(\frac{1}{2}) = g(\frac{1}{2}) = f(1) = g(0) = b$, so the domains of f and g coincide. Therefore by the pasting lemma, h is continuous. Now, observe that $h(0) = f(0) = a$, and that $h(1) = g(1) = c$. This makes $a \sim c$. ∎

Definition. We define the equivalence classes of X under path connectedness to be called **path components** of X.

Definition. We denote the collection of all path components of a topological space X to be $pi_0(X)$; that is $pi_0(X) = {}^{X}\!/\!_{\sim}$ (not necesarily as a quotient space), Moreover, we define the map $pi_0(f) : pi_0(X) \to pi_0(Y)$ to be the map taking the path component C to the unique path component of Y containing $f(C)$.

Theorem 2.4.9. $pi_0 : \mathrm{Top} \to \mathrm{Set}$ *is a funtor.*

Proof. Consider $1_X : X \to X$ the identity on X. Let $\pi_0(X) = \{X_\alpha\}$ where X_α is a path component of X. We have that $pi_0(1_X) : \pi_0(X) \to \pi_0(X)$ sends $X_\alpha \to X_\beta$ where X_β is the unique path component of X containing $1_X(X_\alpha) = X_\alpha$. However, since X_α and X_β are equivalence classes, we have $X_\alpha \subseteq X_\beta$ if and only if $\alpha = \beta$, i.e. $X_\alpha = X_\beta$. This makes $pi_0(1_X) = 1_{\pi_0(X)}$.

Now let $f : X \to Y$ and $g : Y \to Z$ be continuous maps. Let $\pi_0(X) = \{X_\alpha\}$, $\pi_0(Y) = \{Y_\beta\}$, $\pi_0(Z) = \{Z_\gamma\}$ the collection of path components of X, Y, and Z, respectively. Now consider X_α and Z_γ such that $\pi_0(g \circ f)(X_\alpha) = Z_\gamma$. Then Z_γ is the unique path component of Z containing $g(f(X_\alpha))$. Now, if Y_β is the unique path component of Y containing X_α, then $\pi_0(f)(X_\alpha) = Y_\beta$ and we see that $g(f(X_\alpha)) \subseteq g(Y_\beta)$. Moreover, if $Z\prime$ is the unique path component of Z containing $g(Y_\beta)$, then $\pi_0(g)(Y_\beta) = Z_{\gamma'}$, and $g(Y_\beta) \subseteq Z_{\gamma'}$. But $g(f(X_\alpha)) \subseteq g(Y_\beta) \subseteq Z_{\gamma'}$; by above, and since path components partition their spaces, this makes $\gamma = \gamma'$. Thus $Z_\gamma = Z_{\gamma'}$ and we have that $g(f(X_\alpha)) \subseteq g(Y_\beta) \subseteq Z_\gamma$. Therefore Z_γ is the unique path component of Z containing both $g(f(X_\alpha))$ and $g(Y_\gamma)$; that is $\pi_0(g)(Y_\beta) = Z_\gamma$, where $\pi_0(f)(X_\alpha) = Y_\beta$. This implies that $pi_0(g \circ f) = \pi_0(g) \circ \pi_0(f)$, which makes π_0 a functor. ■

Corollary. *If $f \simeq g$, then $\pi_0(f) = \pi_0(g)$.*

Proof. Suppose that $F : f \simeq g$ is a homotopy between the maps $f : X \to Y$ and $g : X \to Y$. Let C be a path component of X, then $C \times I$ is path connected by lemma 2.4.6. Thus by lema 2.4.1, $F(C \times I)$ is also path connected. Notice then that:

$$f(C) = F(C \times 0) \subseteq F(C \times I)$$

and

$$g(C) = F(C \times 1) \subseteq F(C \times I)$$

So the unique path connected component of Y containing $F(C \times I)$ contains both $f(C)$ and $g(C)$. Therefore $\pi_0(f) = \pi_0(g)$. ■

Corollary. *If X and Y are topological spaces with the same homotopy type, then they have the same number of path components.*

Proof. Suppose that $f : X \to Y$ and $g : Y \to X$ are continuous maps with $g \circ f = 1_X$ and $f \circ g = 1_Y$. Since f is a homotopy equivalence, then $[f]$ is an equivalence in hTop. Restricting π_0 to hTop, this also gives use that $\pi_0([f])$ is an equivalence in Set. That is f is 1–1 and onto. ■

Definition. A topological space X is **locally path connected** if, for each $x \in X$, and every open neighborhood U of x there is an open set V with $x \in V \subseteq U$ such that any two points in V can be joined by a path in U.

Example 2.10. Form the subspace X of $\mathbb{R}^2$ by asjoining a curve from $(0, 1)$ to $(\frac{1}{2\pi}, 0)$ on the topologist's sine curve. Then X is path connected, but not locally path connected.

Theorem 2.4.10. *A topological space is locally path connected if, and only if path components of open sets are open.*

Proof. Suppose that X is locally path connectd, and lwet U be open in X. Let $x \in C$, where C s a path component of U. Then there is an open V with $x \in V \subseteq U$ such that very point of V can be joined to x by a path in U. Thus each point of V lies in the path component of x, which is C. Thus $V \subseteq C$, which makes C open.

Conversely, suppose that path components of open sets in X are open. Let U be an open set of X, and for some $x \in U$, let C be the path component of x in U. Then we have $x \in C \subseteq U$. Since C is open, this makes X locally path connected. ■

Corollary. *If X is locally path connected, then its path components are open.*

Corollary. *X is locally path connected if, and only if for every $x \in X$, and each open neighborhood U of x, there is an open path connected set V with $x \in V \subseteq U$.*

Corollary. *If X is locally path connected, then the connected components of every open set coincide with its path components. In particular the connected components of X coincide with the path components of X.*

Corollary. *If X is connected, and locally path connected, then X is connected.*

Definition. Let A be a subspace of a topological space X, and let $i : A \to X$ be the inclusion. Then A is a **deformation retract** of X if there is a continuous map $r : X \to A$ such that r is a retraction of X; i.e. $r \circ i = 1_A$ and $i \circ r = 1_X$.

Lemma 2.4.11. *Every deformation retract is a retract.*

Theorem 2.4.12. *If A is a deformation retract of a topological space X, then X and A have the same homotopy type.*

Corollary. *S^1 is a deformation retract of $\mathbb{C}\backslash 0$.*

Proof. For every $z \in \mathbb{C}\backslash 0$, we can write z as $z = \rho e^{i\theta}$, where $\rho > 0$, and $0 \leq \theta \leq 2\pi$. Now, define $F : (\mathbb{C}\backslash 0) \times I \to \mathbb{C}\backslash 0$ by taking $(\rho e^{i\theta}, t) \to ((1-t)\rho + t)e^{i\theta}$. Notce that F is never 0, and that F is continuous, with $F(\rho e^{i\theta}, 0) = \rho e^{i\theta}$, $F(e^{i\theta}, 1) = e^{i\theta}$. Moreover $F(\rho e^{i\theta}, 1) = F(e^{i\theta}) = e^{i\theta}$. Writing S^1 as $S^1 = \{e^{i\theta} : 0 \leq \theta \leq 2\pi\}$. We see that F makes S^1 into a deformation retract of $\mathbb{C}\backslash 0$. ∎

Corollary. *S^1 has the same homotopy type as $\mathbb{C}\backslash 0$.*

Definition. Let $f : X \to Y$ be a continuous map from a topological space X to a topological space Y. Define

$$M_f = {(X \times I) \cup Y}\big/{\sim}$$

Where $(X \times I) \cup Y$ is a disjoint union, and $\sim$ is an equivalence relation defined by $(x, t) \sim y$ if $y = f(x)$ and $t = 1$. Denote the equivalece classes of (x, t) by $[x, t]$. We call the quotient spce M_f the **mapping cylinder** of f.

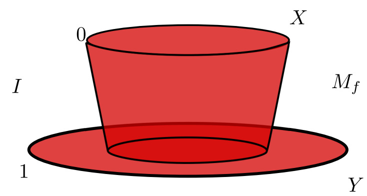

Figure 2.2: The mapping cylinder of a continuous map $f : X \to Y$.

Chapter 3

Simplexes.

3.1 Affine Spaces.

X

Definition. We call a subset $X \subseteq \mathbb{R}^n$ **affine** if for every $x, y \in X$, the line $l(x, y)$ passing through x and y is contained in X.

Lemma 3.1.1. *Affine sets are convex.*

Proof. Note that the line $l(x, y)$ contains the segment $l[x, y]$ which is in X for every $x, y \in X$. ■

Theorem 3.1.2. *If $\{X_\alpha\}$ is a collection of affine (or convex) sets in $\mathbb{R}^n$, then the intersection of all X_α is affine (or convex) in $\mathbb{R}^n$.*

Proof. Let $X = \bigcap X_\alpha$ and let $x, y \in X$. let $l(x, y)$ be the line passing through x and y, then $l(x, y) \in X_\alpha$ for every α, since $x, y \in X_\alpha$ which is affine. This makes $l(x, y) \in X$, which makes X affine in $\mathbb{R}^n$. The proof for convexity of X is the same except using the line segment $l[x, y]$. ■

Definition. An **affine combination** of points $x_0, \ldots, x_m \in \mathbb{R}^n$ is a point $x \in \mathbb{R}^n$ such that

$$x = t_0 x_1 + \cdots + t_m x_m$$

Where $\sum t_i = 1$. A **convex combination** is an affine combination in which each $t_i \geq 0$ for $o \leq i \leq m$.

Example 3.1. The line $tx + (1 - t)y$ is a convex combination in $\mathbb{R}^n$.

Definition. We say a subset $X \subseteq \mathbb{R}^n$ **spans** an affine set $[X]$ if $[X]$ is the intersection of all affine subsets containing X. Similarly, we say X **spans** a convex set $[X]$ if $[X]$ is the intersection of all convex subsets containing X. We call these the affine and convex **hulls**, respectively.

Theorem 3.1.3. *If $x_0, \ldots, x_m \in \mathbb{R}^n$, then the convex hull $[x_0, \ldots, x_m]$ is the set of all convex combinations of $x_0, \ldots, x_m$.*

Proof. Let S be the set of all convex combinations of $x_0, \dots, x_m$, then $[x_0, \dots, x_m] \subseteq S$. Now, let $t_j = 1$ and $t_i = 0$, then $x_i \in S$ for all j. Moreovoer, let $\alpha = \sum a_i x_i$ and $\beta = \sum b_i x_i$ where $\sum a_i = \sum b_i = 1$. Then for $t \in [0,1]$ we have

$$t\alpha + (1-t)\beta = t\sum a_i x_i + (1-t)\sum b_i x_i = \sum (t(a_i x_i) + (1-t)b_i x_i)$$

moreover, $t\sum a_i + (1-t)\sum b_i = 1$ and $ta_i + (1-t)b_i \geq 0$ for all $0 \leq i \leq m$, so $t\alpha + (1-t)\beta$ is a convex combination in S.

Now, let X be any convex set containing $\{x_0, \dots, x_m\}$. By induction on m, for $m = 0$, $S = \{x_0\}$. Now let $m \geq 0$ and $t_i \geq 0$ with $\sum t_i = 1$. Assume without loss of generality that $t_0 \neq 1$. Then

$$y = (\frac{t_1}{1-t_0})x_0 + \cdots + (\frac{t_m}{1-t_0})x_m \in X$$

which makes $x = t_0 x_0 + (1-t_0)y \in X$ This makes $S \subseteq [x_0, \dots, x_m]$. ∎

Definition. We call points $x_0, \dots, x_m \in \mathbb{R}^n$ **affinely independent** if $\{x_1 - x_0, \dots, x_m - x_0\}$ is linearly independent in $\mathbb{R}^n$ as a vector space.

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