Complex Analysis

Alec Zabel-Mena

Text

Complex Analysis (4^{th} edition)

Serge Lang

May 3, 2023

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Chapter 1

Complex Numbers and Funtions

1.1 Complex Numbers

1.2 Complex Valued Functions

Definition. We define a **complex valued function** to be a function $f : S \to \mathbb{C}$, where $S \subseteq \mathbb{C}$. Writing $f(z) = f(x+iy) = u(x,y) + iv(x,y)$, where $u : U_1 \times U_1 \to \mathbb{R}$ and $v : V_1 \times V_2 \to \mathbb{R}$ are real valued functions (with U_1, U_2, V_1, V_2 open in $\mathbb{R}$), we define the **real part** of f to be $\operatorname{Re} f = u(x,y)$, and the **imaginary part** of f to be $\operatorname{Im} f = v(x,y)$.

Remark. It should be noted that the domain of a complex valued function f depends on the domain of its real and imaginary parts, and vice versa.

Example 1.1. (1) The real and imaginary parts of the complex valued function $f(z) = x^3y + i\sin(x+y)$ to be $u(x,y) = x^3y$ and $v(x,y) = \sin(x+y)$, respectively.

(2) Consider the complex valued function $f(z) = z^n$, for $n \in \mathbb{Z}^+$. Writing $z = re^{i\theta}$, we get $f(z) = r^n \cos n\theta + ir^n \sin n\theta$. The real part of f is then $u(x,y) = r^n \cos n\theta$, and the imaginary part of f to be $v(x,y) = r^n \sin n\theta$.

Lettinh $\overline{B^1} = \{z \in \mathbb{C} : |z| \leq 1\}$ the closed unit ballm notice if $z \in \overline{B^1}$, then $|z^n| = |z|^n \leq 1^n = 1$, so that $z^n \in \overline{B^1}$, and hence $f(\overline{B^1}) = \overline{B^1}$.

Definition. We call the solutions to the polynomial $z^n - 1$ over $\mathbb{C}$ the complex n**-th roots of unity**.

Theorem 1.2.1. *Let ξ be a complex n-th root of unity. Then $\xi = e^{\frac{2i\pi}{n}}$.*

Corollory. *If ξ is an n-th root of unity, then so is ξ^k for all $k \in {}^{\mathbb{Z}}\!/_{n\mathbb{Z}}$.*

1.3 Complex Differentiation and Holomorphic Functions

Definition. Let U be an open set of $\mathbb{C}$, and let $w \in U$. We call a complex valued function $f : U \to \mathbb{C}$ **complex differentiable** at w if the limit

$$f'(w) = \lim_{h \longrightarrow 0} \frac{f(w+h) - f(w)}{h} = \lim_{z \longrightarrow w} \frac{f(z) - f(w)}{z - w}$$

exists. We call $f'(w)$ the **complex derivative** of f at w.

Theorem 1.3.1. *Let $f : U \to \mathbb{C}$ and $g : U \to \mathbb{C}$ be complex valued functions. If f and g are complex differentiable at a point $z \in U$, then following are true*

(1) $f + g$ is complex differentiable at z, with

$$(f+g)'(z) = f'(z) + g'(z)$$

(2) $(fg)'$ is complex differentiable at z, with

$$(fg)'(z) = f'(z)g(z) + f(z)g'(z)$$

Corollary. *The function $\frac{f}{g}$ is complex differentiable at z, provided $g(z) \neq 0$, with*

$$\left(\frac{f}{g}\right)'(z) = \frac{g(z)f'(z) - f(z) - g'(z)}{(g(z))^2}$$

Example 1.2. For all $n \in \mathbb{Z}^+$, the function $f(z) = z^n$ is complex differentiable on all of $\mathbb{C}$, with $f'(z) = nz^{n-1}$. In fact, z^n is what we call a "holomorphic" function.

Theorem 1.3.2 (The Chain Rule)**.** *Let U and V be open sets of $\mathbb{C}$, and let $f : U \to \mathbb{C}$, and $g : V \to \mathbb{C}$ be complex valued functions, with $f(U) \subseteq V$. If f is complex differentiable at a point $z \in Z$, and g is complex differentiable at the point $f(z) \in f(U)$, then $g \circ f$ is complex differentiable at z with*

$$(g \circ f)'(z) = (g' \circ f)(z)f'(z) = g'(f(z))f'(z)$$

Definition. We call a complex valued function $f : U \to \mathbb{C}$ **holomorphic** on U if it is complex differentiable at every point of U.

Remark. It is convention to simply say that f is "holomorphic" when it is holomorphic on all of $\mathbb{C}$.

Definition. Let $f : U \to \mathbb{C}$ a complex valued function with $f(z) = u(x, y) + iv(x, y)$. We define the **vector field** of f to be the map $F : U \to V \to \mathbb{R} \times \mathbb{R}$ defined by

$$F(x, y) = (u(x, y), v(x, y))$$

Where U and V are open in $\mathbb{R}$.

Theorem 1.3.3. *If f is holomorphic on its domain, then F is real differentiable on its domain (resepctively to the domain of f) and has derivative*

$$\operatorname{Jac} F = \begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{pmatrix}$$

Where $\operatorname{Jac} F$ is the Jacobian of F.

Corollory. $f'(z) = \frac{\partial u}{\partial x} - i\frac{\partial v}{\partial y}$*, and the we have the following of equations*

$$\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} = 0$$
$$\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = 0$$

Theorem 1.3.4. *If $u : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, and $v : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are continuously real differentiable realvalued functions satisfying the equations*

$$\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} = 0$$
$$\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = 0$$

Then the function $f(z)u(x, y) + iv(x, y)$ is holomorphic on its domain.

Definition. Let $u : U_1 \times U_2 \to \mathbb{R}$ and $v : V_1 \times V_2 \to \mathbb{R}$ be continuously real differentiable real valued functions. We define the **Cauchy-Riemann equations** to be the set of equations

$$\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} = 0$$
$$\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = 0$$

Chapter 2

Power Series

2.1 Formal Power Series

Definition. Let F be a field, we define the set $F[[x]]$ of all series of the form

$$f(x) = \sum_{n=0}^{\infty} a_0 x^n, \text{ where } a_0, \dots, a_n, \dots \in F$$

the set of **formal power series** over F. We call the elements of $F[[x]]$ **formal power series**.

Definition. Let $f(x) = \sum_{n=0}^{\infty} a_0 x^n$ a formal power series over a field F. We define the **order** of f to be the smallest integer n for which $a_n \neq 0$, and write $\operatorname{ord} f = n$. We call the term a_0 of f the **constant term** of f.

Lemma 2.1.1. *Let F be a field, and define the operations $+$ and $\cdot$ on F by*

$$f(x) + g(x) = \sum_{n=0}^{\infty} c_n x^n \text{ where } c_n = a_n + b_n$$
$$f(x)g(x) = \sum_{n=0}^{\infty} d_n x^n \text{ where } c_n = \sum_{k=0}^{n} a_k b_{n-k}$$

Where $f(x) = \sum_{n=0}^{\infty} a_n x^n$ and $g(x) = \sum_{n=0}^{\infty} b_n x^n$ are formal power series over F. Then $F[[x]]$ forms a commutative ring under $+$ and $\cdot$.

Corollory. *Define the action $F \times F[[x]] \to F[[x]]$ by*

$$\alpha f(x) = \sum_{n=0}^{\infty} (\alpha a_n) x^n$$

Then $F[[x]]$ is an F-module under this action.

Lemma 2.1.2. *Let $f(x) = \sum_{n=0}^{\infty} a_n x^n$ and $g(x) = \sum_{n=0}^{\infty} b_n x^n$ be formal power series over a field F. Then $\operatorname{ord} fg = \operatorname{ord} f + \operatorname{ord} g$.*

Definition. Let $f \in F[[x]]$ be a formal power series over a field F. We say that a formal power series $g \in F[[x]]$ is an **inverse** of f if $fg = 1$.

Lemma 2.1.3. *If $f(x) = \sum_{n=0}^{\infty} a_n x^n$ is a formal power series over a field F, with nonzero constant term, then there exists an inverse of f.*

Proof. Consider the series $a_0^{-1} f(x)$ instead of f. Reacall also that the geometric series

$$\frac{1}{1-r} = 1 + r + r^2 + \dots$$

is a formal power series in r over F. Then $(1-r)(1+r+r^2+\dots) = 1$. Now, let $f(x) = 1-h(x)$, where $h(x) = -(a_1 x + a_2 x^2 + \dots)$ and consider $\phi(h) = 1 + h + h^2 + \dots$. Observbe that $\operatorname{ord} h^n \geq n$ sicne $h^n = (-1)a_1^n x^n + \dots$. Thus, if $m > n$, then h^m has all coefficients of order less than n equal to 0, and the n-th coefficient of ϕ is the n-th coefficient of the sum

$$1 + h + h^2 + \dots + h^n$$

Then, we get by the above geometric series that

$$(1 - h(x))\phi(h) = (1 - h(x))(1 + h + h(x)^2 + \dots) = 1 + \dots = 1$$

■

Example 2.1. Let $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots$. By lemma 2.1.3, since $\cos x$ has nonzero constant term, it has an invers $g(x) = \frac{1}{\cos x}$. Notice that

$$\begin{aligned} \frac{1}{1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots} &= 1 + (\frac{x^2}{2!} - \frac{x^4}{4!} + \dots) + (\frac{x^2}{2!} - \frac{x^4}{4!} + \dots)^2 + \dots \\ &= 1 + \frac{x^2}{2!} - \frac{x^4}{4!} + \dots + \frac{x^4}{(2!)^2} \\ &= 1 + \frac{x^2}{2!} + (-\frac{1}{24} + \frac{1}{4})x^2 + \dots \end{aligned}$$

Which gives coefficients of $g(x)$ up to order 4.

Definition. Let $f(x) = \sum_{n=0}^{\infty} a_n x^n$ a power series over a field F, and let $h(x) = c_1 x + \dots$ a power series of order greater than 1. We define the **substitute** of h in f to be the power series

$$f \circ h(x) = a_0 + a_1 h(x) + a_2 h(x)^2 + \dots$$

Definition. Let $f(x) = \sum_{n=0}^{\infty} a_n x^n$ and $g(x) = \sum_{n=0}^{\infty} b_n x^n$ be power series over a field F. We call f **congruent** to g **modulo** x^n if $a_k = b_k$ for all $k \in \mathbb{Z}/n\mathbb{Z}$. That is, f and g have the same coeficients of terms of order up to $n-1$. We write $f \equiv g \mod x^2$.

Lemma 2.1.4. *Congruence of power series modulo x^n defines an equivalence relation.*

Lemma 2.1.5. *If $f_1 \equiv f_2 \mod x^n$ and $g_1 \equiv g_2 \mod x^n$, then $f_1 + g_1 \equiv f_2 g_2 \mod x^n$ and $f_1 g_1 \equiv f_2 g_2 \mod x^n$. Moreover, if h_1 and h_2 are formal power series with zero constant term, and $h_1 \equiv h_2 \mod x^n$, then $f_1 \circ h_1 \equiv f_1 \circ h_2 \mod x^n$.*

Proof. We prove for substitutions of h_1 in f_1 only. Let p_1 and p_2 polynomials of degree $\deg = n - 1$ such thta $f_1 \equiv p_1(x) \mod x^n$ and $f_2 \equiv p_2(x) \mod x^n$. By hypothesis, we get $p_1 \equiv p_2 \mod x^n$, and since $\deg p_1, \deg p_2 = n - 1$, this makes $p_1 = p_2$. Then $f_1 \circ h \equiv p_1 \circ h = p_2 \circ h \equiv f_2 \circ h$. Now, let $q(x)$ the polynomial of degree $n - 1$ such that $h_1 \equiv h_2 \equiv q(x) \mod x^n$Writing $p(x) = a_0 + a_1 x + \cdots + a_{n-1} x^{n-1}$. Then we get $p_1 \circ h_1 \equiv p_2 \circ h_2 \mod x^n$ and we are done. ■

Corollary. *Two power series f and g are equal if, and only if $f \equiv g \mod x^n$ for all $n \in \mathbb{Z}^+$.*

Corollary. *$(f_1 + f_2) \circ h = (f_1 \circ h) + (f_2 \circ h)$, and $(f_1 f_2) \circ h = (f_1 \circ h)(f_2 \circ h)$. That is, composition of power series distributes over the addition and multiplication of power series.*

Corollary. *Provided that* $\operatorname{ord} f_2 = 0$*, then*

$$\left(\frac{f_1}{f_2}\right) \circ h = \frac{f_1 \circ h}{f_2 \circ h}$$

Example 2.2. Considre the power series for $\frac{1}{\sin x}$. We have by definition that

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots = x(1 - \frac{x^2}{3!} + \frac{x^4}{5!})$$

so that

$$\frac{1}{\sin x} = \frac{1}{x(1 - \frac{x^2}{3!} + \frac{x^4}{5!})} = \frac{1}{x}(1 + \frac{x^2}{3!} - \frac{x^4}{5!} + \left(\frac{x^2}{3!}\right)^2 + \dots) = \frac{1}{x} + \frac{x}{3!} + \left(\frac{1}{(3!)^2} - \frac{1}{5!}\right) x^3 + \dots$$

2.2 Convergent Power Series

Definition. Let $\{z_n\}_{n \in \mathbb{Z}^+}$ a sequence of complex numbers, and consider the series $\sum_{n=0}^{\infty} z_n$. We define the n**-th partial sum** to be

$$s_n = \sum_{k=1}^{n} z_k$$

and we say that the series **converges** if there exists a $w \in \mathbb{C}$ for which $\lim \{s_n\} = w$ as $n \to \infty$. We call w the **sum** of the series.

Lemma 2.2.1. *Let $A = \sum \alpha_n$ and $B = \sum \_n$ be convergent series with n-th partial sums s_n and t_n. Then the sum and product of A and B converge, with*

$$A + B = \sum (\alpha_n + \_n) \text{ and } AB = \lim_{n \to \infty} \{s_n t_n\}$$

Definition. Let $\sum \alpha_n$ a series of complex numbers. We say that $\sum \alpha_n$ **converges absolutely** if the series of real numbers $\sum |\alpha_n|$ converges.

Lemma 2.2.2. *If $\sum \alpha_n$ is a series of complex numbers which converges absolutely, then it converges.*

Proof. Let $s_n = \sum_{k=1}^{n} \alpha_k$, then for $m \leq n$, notice that $s_n - s_m = \alpha + m_1 + \cdots + \alpha_n$, hence $|s_n - s_m| \leq \sum_{k=m+1}^{n} |\alpha_k|$. By absolute convergence, let $\varepsilon > 0$ then there exists an $N > 0$ such that $\sum |\alpha_k| < \varepsilon$ whenever $m, n \geq N$. Thus $|s_n - s_m| < \varepsilon$ which makes $\sum \alpha_n$ converge. ∎

Lemma 2.2.3. *Let $\sum c_n$ be a convergent series of real numbers greater than 0. If $\{\alpha_n\}$ is a sequence of complex numbers such that $|\alpha_n| < c_n$ for all $n \in \mathbb{Z}^+$, then $\sum \alpha_n$ converges absolutely.*

Proof. Notice that the partial sums $\sum_{k=1}^{n} c_n$ are bounded, hence $\sum |\alpha_n| \leq \sum c_k$. ∎

Lemma 2.2.4. *Let $\{\alpha_n\}$ a sequence of complex numbers. Then the following are true*

(1) If $\sum \alpha_n$ is absolutely convergent, then the series obtained by permuting terms is absolutely convergent, with the same limit.

(2) If $\sum_{n=1}^{\infty} (\sum_{m=1}^{n} \alpha_{mn})$ is absolutely convergent, then so is the series $\sum_{m=1}^{n} (\sum_{n=1}^{\infty} \alpha_{mn})$, and they converge to the same limit.

Definition. Let $S \subseteq \mathbb{C}$, and let f be a bounded complex valued function on S. We define the **sup norm** of f on S to be

$$\|f\|_S = \sup_{z \in S} \{|f(z)|\}$$

Lemma 2.2.5. *Let $S \subseteq \mathbb{C}$. The sup norm of a complex valued function on S defines a metric on $\mathbb{C}$.*

Definition. Let $\{f_n\}_{n \in \mathbb{Z}^+}$ a sequence of complex valued functions on a set $S \subseteq \mathbb{C}$. We say that the $\{f_n\}$ **converges uniformly** on S if there exists a bounded complex valued function f on S such that for all $\varepsilon > 0$, there is an $N > 0$ for which

$$\|f_n - f\|_S < \varepsilon \text{ whenever } n \geq N$$

We call $\{f_n\}$ **Cauchy** if for every $\varepsilon > 0$ there is an $N > 0$ for which

$$\|f_n - f_m\|_S < \varepsilon \text{ whenever } n, m \geq N$$

Theorem 2.2.6. *Let $\{f_n\}$ be a sequence of complex valued functions on a set $S \subseteq \mathbb{C}$. If $\{f_n\}$ is Cauchy, then it converges uniformly.*

Proof. We have for all $z \in S$, take $f(z) = \lim f_n(z)$ as $n \to \infty$. Then for $\varepsilon > 0$ there is an $N > 0$ for which $|f_n(z) - f_m(z)| < \varepsilon$ for al $z \in S$ and $m, n \geq N$. Now, for $n \geq N$, take $m(n) \geq N$ large enough so that $|f(z) - f_{m(n)}(z)| < \varepsilon$. Then we get that

$$|f(z) - f_n(z)| \leq |f(z) - f_{m(n)}(z)| + |f_{m(n)}(z) - f_n(z)| < \varepsilon + \|f_{m(n)} - f_n\| < 2\varepsilon$$

∎

Corollory. *If $\{f_n\}$ is bounded for all $n \in \mathbb{Z}^+$, then so is f.*

Definition. We say a series of complex valued functions on a domain $S \subseteq \mathbb{C}$, $\sum f_n$ **converges uniformly** if the sequence $\{s_n\}$ of n-th partial sums converges uniformly. We say that $\sum f_n$ **converges absolutely** if for all $z \in S$, $\sum |f_n(z)|$ converges.

Theorem 2.2.7 (The Comparison Test)**.** *Let $\{c_n\}$ be a sequence of real numbers greater than 0 such that $\sum c_n$ converges. Let $\{f_n\}$ a sequence of complex valued functions on a domain $S \subseteq \mathbb{C}$ such that $\|f_n\|_S \leq c_n$ for all $n \in \mathbb{Z}^+$. Then the series $\sum f_n$ converges uniformly, and converges absolutely.*

Proof. Let $m \leq n$. Then $\|s_n - s_m\| \leq \sum_{k=m+1}^{n} \|f_k\|_S \leq \sum c_k$. Since $\sum c_k$ converges, the uniform and absolute convergnce of $\sum f_n$ follows. ∎

Theorem 2.2.8. *Let $S \subseteq \mathbb{C}$ and $\{f_n\}$ a sequence of continuous complex valued functions on S. If $\{f_n\}$ converges unifromly to a complex valued function f on S, then f is also continuous.*

Proof. let $\alpha \in S$ and n be large enough such that $\|f - f_n\|_S < \varepsilon$ for some $\varepsilon > 0$. By the continuity of f_n at α, choose ¿0 such that $|f_n(z) - f_n(\alpha)| < \varepsilon$ whenever $|z - \alpha| <$.$Thenobservethat |f(z) - f(\alpha)| \leq |f(z) - f_n(z)| + |f_n(z) - f_n(\alpha)| + |f_n(\alpha) - f(\alpha)| < 2\|f - f_n\| + \varepsilon < 3\varepsilon$ ∎

Theorem 2.2.9. *Let $\{a_n\}$ a sequence of complex numbers, and let $r > 0$ such that $\sum |a_n| r^n$ converges. Then the power series $\sum a_n z^n$ converges absolutely and converges uniformly whenever $|z| \leq r$.*

Example 2.3. (1) Let $r > 0$ and consider the series

$$\exp z = \sum_{n=0}^{\infty} \frac{z^n}{n!}$$

Then $\exp z$ converges absolutely and uniformly whenever $|z| \leq r$. Indeed, let $c_n = \frac{r^n}{n!}$, then

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