Algebraic Geometry.

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Contents

Chapter 1

Affine Algebraic Sets

1.1 Affine n-Space and Algebraic Sets

Definition. Let k be a field. We define **affine n-space** over k to be the cartesian product $\mathbb{A}^n(k) = \underbrace{k \times \cdots \times k}_{n\text{-times}}$. If the field k is understood, we write $\mathbb{A}^n$. We call the elements of $\mathbb{A}^(k)$ **affine points**. We call $\mathbb{A}^1(k)$ and $\mathbb{A}^2(k)$ the **affine line** and **affine plane** over k, respectively.

Definition. Let k be a field, and let $f \in k[x_1, \ldots, x_n]$. We call an affine point $P \in \mathbb{A}^n(k)$ a **zero**, or **root** of f if $f(P) = 0$, where $f(P)$ is understood to be $f(a_1, \ldots, a_n)$, where $P = (a_1, \ldots, a_n)$. We call the set of zeros of f, $V(f)$ the **hypersurface** defined by f. We call hypersurfaces in $\mathbb{A}^2(k)$ **affine plane curves**. If $\deg f = 1$, we call $V(f)$ a **hyperplane**. We call hypersurfaces in $\mathbb{A}^1(k)$ **lines**.

Figure 1.1: Affine Algebraic Sets in $\mathbb{A}^2(\mathbb{R})$ and $\mathbb{A}^3(\mathbb{R})$.

Example 1.1.

Definition. Let k be a field, and S any set of polynomials in $k[x_1, \ldots, x_n]$. We define the **set of zeros** of S to be the set $V(S) = \{P \in \mathbb{A}^n(k) : f(P) = 0 \text{ for all } f \in S\}$. We call a subset X of $\mathbb{A}^n(k)$ an **affine algebraic set** if $X = V(S)$ for some set S of polynomials.

Lemma 1.1.1. *The following are true for any field k.*

(1) If $\mathfrak{a}$ is an ideal in $k = [x_1, \ldots, x_n]$ generated by a set $S \subseteq k[x_1, \ldots, x_n]$, then $V(\mathfrak{a}) = V(S)$.

(2) If $\{\mathfrak{a}_\alpha\}$ is a collection of ideals of $k[x_1, \ldots, x_n]$, then

$$V\Big(\bigcup \mathfrak{a}_\alpha\Big) = \bigcap V(\mathfrak{a}_\alpha)$$

(3) If $\mathfrak{a} \subseteq \mathfrak{b}$ are ideals, then $V(\mathfrak{b}) \subseteq V(\mathfrak{a})$.

(4) If $f, g \in k[x_1, \ldots, x_n]$, then $V(fg) = V(f) \cup V(g)$.

(5) $V(0) = \mathbb{A}^n(k)$ and $V(1) = \emptyset$.

Proof. First, let S be a set of polynomials in $k[x_1, \ldots, x_n]$. Let $\mathfrak{a} = (S)$ the ideal generated by S. Then if $f \in S$ is a polynomia, $f \in I$. Then if $P \in \mathbb{A}^n$ is a zero of f in S, it is a zero of f in $\mathfrak{a}$, hence $V(S) \subseteq V(\mathfrak{a})$. Conversely, we have that if $f \in \mathfrak{a}$, then by suppostion, $f(x_1, \ldots, x_n) = f_1(x_1, \ldots, x_n) + \cdots + f_n(x_1, \ldots, x_n) + \ldots$. Now, if $f(P) = 0$ in I, then we have $f_i(P) = 0$ for every i. This makes $f(P) = 0$ in S, so that $V(\mathfrak{a}) \subseteq V(S)$.

Now, consider the collection $\{\mathfrak{a}_\alpha\}$ of ideals in $k[x_1, \ldots, x_n]$. Let $P \in V(\bigcup \mathfrak{a}_\alpha)$. Then for every $f \in \bigcup \mathfrak{a}_\alpha$, $f(P) = 0$ for each α. So that $P \in \bigcap V(\mathfrak{a}_\alpha)$. Again, on the otherhand, if $P \in \bigcap V(\mathfrak{a}_\alpha)$, $P \in V(\mathfrak{a}_\alpha)$ for all α so that $P \in V(\bigcup \mathfrak{a}_\alpha)$.

Let $\mathfrak{a}$ and $\mathfrak{b}$ ideals in $k[x_1, \ldots, x_n]$, where $\mathfrak{a} \subseteq \mathfrak{b}$. Let $P \in V(\mathfrak{b})$. Then for every polynomial $f \in \mathfrak{b}$, $f(P) = 0$, so that $f(P) = 0$ when $f \in \mathfrak{a}$, hence $P \in V(\mathfrak{a})$. This makes $V(\mathfrak{b}) \subseteq V(\mathfrak{a})$.

Consider now the polynomials $f, g \in k[x_1, \ldots, x_n]$. Certainly if $P \in V(fg)$ it is a root of fg; ie.e. $fg(P) = 0$. This makes $f(P) = 0$ or $g(P) = 0$ so that $V(fg) \subseteq V(f) \cup V(g)$. On the otherhand if P is a root of f, or a root of g, it is a root of fg making $V(f) \cup V(f) \subseteq V(fg)$, and equality is established.

Finally, observe that the zero polynomial $0(x_1, \ldots, x_n)$ has all its coefficients 0, so that any point $P \in \mathbb{A}^n$ is a zero. This makes $V(0) = \mathbb{A}^n$. Likewise, the constant polynomial $1(x_1, \ldots, x_n)$ has its 0-th coefficient 1 so that it has not points $P \in \mathbb{A}^n$ as roots. That is $V(1) = \emptyset$. ■

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