Algebraic Topology

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09.12.2022

Contents

Chapter 1

Categories.

1.1 Categories and Subcategories.

Definition. A **category** $\mathcal{C}$ is a collection of a class of **objects**, denoted $\text{obj}\,\mathcal{C}$ a collection of sets of **morphisms** $\text{Hom}\,(A,B)$ for each $A,B \in \text{obj}\,\mathcal{C}$ and a binary operation $\circ : \text{Hom}\,(A,B) \times \text{Hom}\,(B,C) \to \text{Hom}\,(A,C)$, defined by $(f,g) \to g \circ f$, called **composition** such that:

(1) Each $\text{Hom}\,(A,B)$ is pairwise disjoint for all $A,B \in \text{obj}\,\mathcal{C}$.

(2) $\circ$ is associative when defined; that is if either $(g \circ f) \circ h$ or $g \circ (f \circ h)$ are defined, then $(g \circ f) \circ h = g \circ (f \circ h)$, for morphisms f, g, h.

(3) For each $A \in \text{obj}\,\mathcal{C}$, there exists an **identity** morphism $1_A \in \text{Hom}\,(A,A)$ such that for each $B, C \in \text{obj}\,\mathcal{C}$, $1_A \circ f = f$ and $g \circ 1_A = g$ for each morphism $f \in \text{Hom}\,(B,A)$ and $g \in \text{Hom}\,(A,C)$.

We denote morphisms by $f : A \to B$ instead of $f \in (A,B)$.

Definition. Let $\mathcal{C}$ be a category and $f : A \to B$ a morphism in $\mathcal{C}$. We call A and B the **domain** and **codomain** of f, respectively, and we call the set $G_f = \{(a, f(a)) : a \in A\} \subseteq B$ the **graph** of f.

Example 1.1. (1) The category of all sets Set has as onjects the class of all sets. The morphisms in Set are all functions $f : A \to B$ where A and B are sets. The composition of Set is the usual composition of functions.

(2) The category of all topological spaces Top has as objects all topological spaces, and as morphisms all continuous maps $f : Y \to Y$ from a space X to a space Y. The composition is the usual composition.

(3) The category of all groups, Grp has as objects all groups and as morphisms all homomorphisms $f : G \to H$, under the usual composition.

(4) The category of rings with unit Rng has as objects all rings with unit, along with all ring homomorphisms $f : R \to K$ to be the morphisms under the usual composition.

Definition. We call a category a **subcategory** of a category $\mathcal{C}$ if $\mathrm{obj}\,\mathcal{A} \subseteq \mathrm{obj}\,\mathcal{C}$,$\mathrm{Hom}_{\mathcal{A}}(A, B) \subseteq \mathrm{Hom}_{\mathcal{C}}(A, B)$ for all $A, B \in \mathcal{C}$, and $\mathcal{A}$ inherits the composition of $\mathcal{C}$.

Example 1.2. (1) The category of all pairs of topological spaces (X, A) where A is a subspace of X, whose morphisms are pairs of continuous maps $f = (f_1, f_2)$ such that $f_1 i = j f_2$ where $i : A \to X$ and $j : B \to Y$ are inclusions, is a subcategory of Top. We denote this category Top^2.

$$\begin{tikzcd} A \arrow[r, hook, "i"] \arrow[d, "f_2"'] & X \arrow[d, "f_1"] \\ B \arrow[r, hook, "j"'] & Y \end{tikzcd}$$

(2) The category of all **pointed spaces**, Top^* is defined with the objects being all pairs $(X, \{x_0\})$, where $x_0 \in X$ with the morphisms of Top^2. Top^* is a subcategory of Top^2. We call x_0 the **base point**, and we call the morphisms of Top^* **pointed maps**.

(2) The category of all Abelian groups, Ab is a subcategory of Grp. Likewise, the category of all commutative rings with unit is a subcategory of Rng.

1.2 Commutative Diagrams and Congruences.

Definition. A **diagram** in a category is a directed graph with its vertex set a subset of objects, and whose edge set are morphisms between those objects. We call a diagram **commutative** if for any pair of objects, every pair of morphisms between those objects are equal. That is if A, A' and B, B' are pairs of objects with pairs of morphisms $f : A \to B$, $f : A \to A'$ and $f' : A' \to B'$, $g' : B \to B'$ we have that $g \circ f' = f \circ g'$

$$\begin{tikzcd} A \arrow[r, "g"] \arrow[d, "f"'] & A' \arrow[d, "f'"] \\ B \arrow[r, "g"'] & B' \end{tikzcd}$$

Definition. A **congruence** on a category $\mathcal{C}$ is an equivalence relation $\sim$ on morphisms in $\mathcal{C}$ such that:

(1) If $f \in \mathrm{Hom}(A, B)$, and $f \sim f'$, then $f' \in \mathrm{Hom}(A, B)$.

(2) If $f \sim g$ and $f' \sim g'$, then $g \circ f \sim g' \circ f'$.

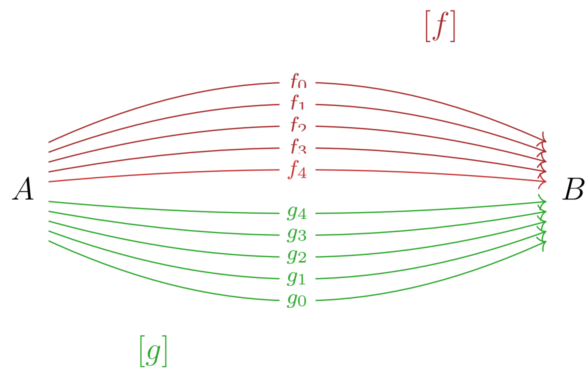

Figure 1.1: An equivalence relation between morphisms.

Theorem 1.2.1. *Let $\mathcal{C}$ be a category with congruence $\sim$. Define ${}^{\mathcal{C}}\!/\!_{\sim}$ as follows:*

(1) $\operatorname{obj} {}^{\mathcal{C}}\!/\!_{\sim} = \operatorname{obj} \mathcal{C}$.

(2) $\operatorname{Hom}_{\mathcal{C}/\sim}(A, B) = \{[f] : f \in \operatorname{Hom}_{\mathcal{C}}(A, B)\}$.

(3) $[g] \circ [f] = [g \circ f]$

Then ${}^{\mathcal{C}}\!/\!_{\sim}$ *is a category.*

Proof. We have by equivalence that $\operatorname{obj} {}^{\mathcal{C}}\!/\!_{\sim}$ is a class. Moreover, since $\sim$ partitions $\mathcal{C}$, it partions all of the $\operatorname{Hom}(A, B)$ for each A, B. So each $\operatorname{Hom}(A, B)$ is a set, moreover, they are pariwise disjoint by definition of $\sim$. Now, notice that by hypothesis, composition in ${}^{\mathcal{C}}\!/\!_{\sim}$ is well defined, so $[1_A] \circ [f] = [1_A \circ f] = [f]$ and $[g] \circ [1_A] = [g \circ 1_A] = [g]$. This makes ${}^{\mathcal{C}}\!/\!_{\sim}$ a category. ∎

Remark. On can think of the cateegory ${}^{\mathcal{C}}\!/\!_{\sim}$ as taking all morphisms with they same domain and codomain, and collapsing them into a single morphism.

Definition. Let $\mathcal{C}$ be a catogory and $\sim$ a congruence of $\mathcal{C}$. We call the category ${}^{\mathcal{C}}\!/\!_{\sim}$ induced by $\sim$ the **quotient category**.

1.3 Functors.

Definition. Let $\mathcal{A}$ and $\mathcal{C}$ be categories. We deine a **covariant functor** to be a map $F : \mathcal{A} \to \mathcal{C}$ such that:

(1) $A \in \operatorname{obj} \mathcal{A}$ implies $F(A) \in \operatorname{obj} \mathcal{C}$.

(2) If $f : A \to B$ is a morphism in $\mathcal{A}$, then $F(f) : F(A) \to F(B)$ is a morphism in $\mathcal{C}$.

$$\begin{tikzcd} A \arrow[r, bend left, "{[f]}"] \arrow[r, bend right, "{[g]}"'] & B \end{tikzcd}$$

Figure 1.2: Morphisms in a category with the same domain and codomain get collapsed onto a single morphism in the correspinding quotient category.

$$\begin{tikzcd} A & A \arrow[l, "1_A"'] \arrow[r, "g"] & B & {} \arrow[r, Rightarrow, bend left, "F"] & {} & F(A) & F(A) \arrow[l, "1_{F(A)}"'] \arrow[r, "F(g)"] & F(B) \\ & C \arrow[u, "f"] \arrow[ur, "g \circ f"'] & & & & & F(C) \arrow[u, "GFf)"] \arrow[ur, "F(g \circ f)"'] & \end{tikzcd}$$

Figure 1.3: A covariant functor taking a diagram in on category to a diagram in the other.

(3) For all morphisms f and g in $\mathcal{A}$, for which $g \circ f$ is defined, we have that $F(g \circ f) = F(g) \circ F(f)$, and $F(1_A) = 1_{F(A)}$.

Example 1.3. (1) We define the **forgetful functor** the map $F : \mathcal{C} \to \text{Set}$ that takes all objects in $\mathcal{C}$ to their underlying sets, and morphisms in $\mathcal{C}$ to themselves considered as functions under the usual composition. For example the forgetful functor $F : \text{Top} \to \text{Set}$ takes topological spaces X to their underlying sets, and continuous maps to themselves, considered just as functions.

(2) The **identity functor** is the functor $I : \mathcal{C} \to \mathcal{C}$ that takes objects and morphisms in $\mathcal{C}$ to themselves.

(3) Let M be a topological space. Define $F_M : \text{Top} \to \text{Top}$ by $F_M : X \to X \times M$, and for each continuous map $f : X \to Y$, $F(f) : X \times M \to Y \times M$ is defined by $(x, m) \to (f(x), m)$. Then F_M is a functor.

(4) Let $A \in \operatorname{obj} \mathcal{C}$ and take the map $\operatorname{Hom}(A, *) : \mathcal{C} \to \text{Set}$ that takes $A \to \operatorname{Hom}(A, B)$ and for each morphism $f : B \to B'$, $\operatorname{Hom}(A, f) : \operatorname{Hom}(A, B) \to \operatorname{Hom}(A, B')$ is given by $g \to f \circ g$. With call this functor the **covariant Hom functor**, and denote it f_*.

Definition. Let $\mathcal{A}$ and $\mathcal{C}$ be categories. We deine a **contravariant functor** to be a map $G : \mathcal{A} \to \mathcal{C}$ such that:

(1) $A \in \operatorname{obj} \mathcal{A}$ implies $G(A) \in \operatorname{obj} \mathcal{C}$.

(2) If $f : A \to B$ is a morphism in $\mathcal{A}$, then $G(f) : G(B) \to G(A)$ is a morphism in $\mathcal{C}$.

(3) For all morphisms f and g in $\mathcal{A}$, for which $g \circ f$ is defined, we have that $G(g \circ f) = G(f) \circ G(g)$, and $G(1_A) = 1_{G(A)}$.

$$\begin{tikzcd}
A & A \arrow[l, "1_A"'] \arrow[r, "g"] & B \arrow[rr, Rightarrow, bend left, "G"] & & G(A) \arrow[r, "1_{G(A)}"] & G(A) \arrow[d, "G(f)"'] & G(B) \arrow[l, "G(g)"'] \arrow[dl, "G(g\circ f)"] \\
& C \arrow[u, "f"] \arrow[ur, "g\circ f"'] & & & & G(C) &
\end{tikzcd}$$

Figure 1.4: A contravariant functor taking a diagram in on category to a diagram in the other.

Example 1.4. (1) Let F be a field, and Vec the category of all finite dimensional vector spaces over F, whose morphisms are linear transformations. Define the map $T : \text{Vec} \to \text{Vec}$ by taking $T : V \to V^{\perp}$, and $T : f \to f^T$. That is T takes vector spaces to their dual spaces, and linear transformation to their transpose. T is a contravariant functor called the **dual space functor.**

(2) Define $\text{Hom}\,(*, B) : \mathcal{C} \to \mathcal{C}$ by taking $\text{Hom}\,(*, B) : A \to \text{Hom}\,(A, B)$ and for each morphism $g : A \to A'$ in $\mathcal{C}$, $\text{Hom}\,(f, B) : \text{Hom}\,(A', B) \to \text{Hom}\,(A, B)$ is defined by taking $h \to h \circ g$. This is analogous to the covariant Hom functor, and we call it the **contravariant Hom functor.**

Definition. We call a morphism $f : A \to B$ an **equivalence** if there exists a morphism $g : B \to A$ such taht $f \circ g = 1_B$ and $g \circ f = 1_A$

Theorem 1.3.1. *Let $\mathcal{A}$ and $\mathcal{C}$ be categories, and $F : \mathcal{A} \to \mathcal{C}$ be a functor. If f is an equivalence in $\mathcal{A}$, then $F(f)$ is an equivalence in $\mathcal{C}$.*

Proof. Suppose that F is a covariant functor. Notice that if $f : A \to B$ is an equivalence, then there is a $g : B \to A$ with $f \circ g = 1_B$ and $g \circ f = 1_A$. Then $F(f \circ g) = F(f) \circ F(g) = F(1_B) = 1_{F(B)}$, and $F(g \circ f) = F(g) \circ F(f) = F(1_A) = 1_{F(A)}$.

Likewise, if F is contravariant, notice that $F(f) : B \to A$ and $F(g) : A \to B$. Then $F(f \circ g) = F(g) \circ F(f) = 1_{F(A)}$, and $F(g \circ f) = F(f) \circ F(g) = 1_{F(B)}$. In eithe case, we find that $F(f)$ is an equivalence in $\mathcal{C}$. ∎

Chapter 2

Homotopy, Convexity, and Connectedness.

2.1 Homotopy

Definition. If X and Y are topological spaces, and $f_0 : X \to Y$ and $f_1 : X \to Y$ are continuous maps, we say that f_0 is **homotopic** to f_1 if there exists a continuous map $F : X \times I \to Y$ with $F(x,0) = f_0(x)$ and $F(x,1) = f_1(x)$. We write $f_0 \simeq f_1$ and call F a **homotopy**. We also write $F : f_0 \simeq f_1$ to denote a homotopy between f_0 and f_1.

Lemma 2.1.1 (The Pasting Lemma)**.** *Let X is a topolgoical space that is covered by open sets $\{X_n\}$. If Y is some topological space for which there exist unique maps $f_n : X_n \to Y$ that coincide in the intersections of their domains, then there exists a unique map $f : X \to Y$ such that $f|_{X_n} = f_n$, for all n.*

Lemma 2.1.2. *Homotopy between continuous maps is an equivalence relation.*

Proof. Let $f : X \to Y$ be a continouos map. Define $F : X \times I \rangle Y$ by $(x,t) \to f(x)$ for all $(x,t) \in X \times I$. Then F is continuous by definition; moreover, $F(x,0) = F(x,1) = f(x)$, making $f \simeq f$.

Now suppose there exist a homotopy $F : f \simeq g$ for maps $f : X \to Y$ and $g : X \to Y$. Define the map $G : X \times I \to Y$ by $(x,t) \to F(x, 1-t)$. G is the composition of continuous maps, so G is continuous, moreover, $G(x,0) = F(x,1) = g(x)$ and $G(x,1) = F(x,0) = f(x)$, so that $g \simeq f$.

Lastly, suppose that $F : f \simeq g$ and $G : g \simeq h$ for maps f, g, h. Define the map $H : X \times I \to Y$ by:

$$H(x,t) = \begin{cases} F(x, 2t), & \text{if } 0 \leq t \leq \frac{1}{2} \\ G(x, 2t-1), & \text{if } \frac{1}{2} \leq t \leq 1 \end{cases}$$

Notice that F and G conicide in their domains which cover X. Therefore, by the pasting lemma, H is continuous. Now notice also that $H(x,0) = F(x, 2 \cdot 0) = F(x,0) = f(x)$ and $H(x,1) = G(x, 2 \cdot 1 - 1) = G(x,1) = h(x)$. This makes $f \simeq h$. ■

Definition. For any continuous map $f : X \to Y$ we define the **homotopy class** of f to be the equivalence class of all continuous maps homotopic to f. That is:

$$[f] = \{g : X \to Y : g \text{ is continous and } g \simeq f\}$$

Lemma 2.1.3. *Let $f_0 : X \to Y$, $f_1 : X \to Y$ and $g_0 : X \to Y$, $g_1 : X \to Y$ be continuous maps. If $f_0 \simeq f_1$ and $g_0 \simeq g_1$, then $g_0 \circ f_0 \simeq g_1 \circ f_1$. That is $[g_0 \circ f_0] = [g_1 \circ f_1]$.*

Proof. Let $F : f_0 \simeq f_1$ and $G : g_0 \simeq g_1$ be the homotopies of f_0 into f_1 and g_0 into g_1, respectively. Define the map $H : X \times I \to Y$ by taking $(x, t) \to G(f_0(x), t)$. Then we have that H is continuous by composition, and that $H(x, 0) = G(f_0(x), 0) = g_0(f_0(x))$, and $H(x, 1) = G(f_0(x), 1) = g_1(f_0(x))$. Thus we see that $g_0 \circ f_0 \simeq g_1 \circ f_0$.

Now define the map $K : X \times I \to Y$ by $K = g_1 \circ F$. We have that K is continuous by composition, and that $K(x, 0) = g_1 \circ f_0$ and $K(x, 1) = g_1 \circ f_1$, making $g_1 \circ f_0 \simeq g_1 \circ f_1$. Therefore, by transitivity of homotopy, $g_0 \circ f_0 \simeq g_1 \circ f_1$. ■

Theorem 2.1.4. *Homotopy is a congruence on the category* Top.

Proof. The proof follows by lemmas 2.1.2 and 2.1.3. ■

Definition. We call the quotient category of Top induced by homotopy the **homotopy category** and denote it hTop.

Definition. A continuous map $f : X \to Y$ is a **homotopy equivalence** if there exists a continuous map $g : Y \to X$ such that $f \circ g \simeq 1_Y$ and $g \circ f \simeq 1_X$. We say that the spaces X and Y have the same **homotopy type** if there exists a homotopy equivalence.

Definition. We call a continuous map **nullhomotopic** if it is homotopic to a constant map.

Example 2.1. The space of complex numbers $\mathbb{C}$ and the unit circle S^1 have the same homnotopy type.

Definition. Let Y and Z be topological spaces, and $X \subseteq Y$ a subspace of Y. If $f : X \to Z$ is a continuous map, then we call the map $g : Y \to Z$ defined by $g \circ i = f$ an **extension** of f, where $i : X \to Y$ is the inclusion map.

Theorem 2.1.5. *Let $f : S^n \to Y$ be a continuous map into a topological space Y. The following are equivalent:*

(1) f is nullhomotopic.

(2) f can be extended to a continuous map $B^{n+1} \to Y$.

(3) There exists a constant map $k : S^n \to Y$, taking $x \to f(x_0)$, for all $x \in S^n$, such that $f \simeq k$, for $x_0 \in S^n$.

Proof. Notice that (3) implies (1) immediately. Now suppose that f is nullhomotopic. Then there exists a constant map $k : X \to Y$, such that for some $x_0 \in S^n$, $k : x \to x_0$ for all $x \in S^n$ implies that $f \simeq k$. Now, define the map $g : B^{n+1} \to Y$ by:

$$g(x) = \begin{cases} y_0, & \text{if } 0 \leq \|x\| \leq \frac{1}{2} \\ F(\frac{x}{\|x\|}, 2 - 2\|x\|), & \text{if } \frac{1}{2} \leq \|x\| \leq 1 \end{cases}$$

Notice, that if $\|x\| = \frac{1}{2}$, then $g(x) = F(2x, 1) = y_0$. Therefore, by the pasting lemma, g is continuous. Moreover, if $\|x\| = 1$, $g(x) = F(x, 0) = f$, which makes g an extension of f.

Now, suppose that there exists an extension $g : B^{n+1} \to Y$ of f. Since S^n is a subspace of B^{n+1}, we have that $g \circ i = g|_{S^n} = f$, where $i : Y \to S^n$ is an inclusion. Now, let $x_0 \in S^n$ and define the constant map $k : S^n \to Y$ by taking $x \to f(x_0)$ for all $x \in S^n$. Additionally, define the map $F : S^N \times I \to Y$ given by $F(x,t) = g((1-t)x + x_0 t)$. We have that F is continuous by composition of continuous maps, and that $F(x, 0) = g(x) = f(x)$, since F has the domain $S^n \times I$, and that $F(x, 1) = g(x_0) = f(x_0)$, since F has the domain $S^n \times I$. This makes $f \simeq k$ with F as the associated homotopy. ∎

2.2 Quotient Spaces

Definition. Let X be a topological space, and $X' = \{X_\alpha\}$ a partion of X. We define the **natrual map** $q : X \to X'$ by taking $x \to X_\alpha$ where $x \in X_\alpha$. We define the **quotient topology** on X' to be the family:

$$\mathcal{T} = \{U' \subseteq X' : q^{-1}(U') \text{ is open in } X\}$$

We denote quotient spaces by ${}^{X}\!/_{q}$, ${}^{X}\!/_{X'}$, or ${}^{X}\!/_{\sim}$ where $\sim$ is an equivalence relation partitioning X into X'.

Example 2.2. (1) Consider the space $I = [0, 1]$ and let $A = \{0, 1\}$. The the quotient space ${}^{I}\!/_{A}$ identifies 0 to 1, and hence, under the quotient topology, is homeomorphic to S^1.

(2) Consider the space $I \times I$ and define an equivalence relation $(x, 0) \sim (x, 1)$ for all $x \in I$. Then the quotient topology formed on ${}^{I \times I}\!/_{\sim}$ is homeomorphic to the cylinder $S^1 \times I$. Defining another equivalence $(0, y) \sim (1, y)$ for all $y \in I$, we get the quotient space on ${}^{S^1 \times I}\!/_{\sim}$ under this equivalence relations is homeomorphic to the torus $S^1 \times S^1$.

(3) Let $h : X \to Y$ be a map, and define $\ker h$ the equivalence relation on X such that $x \ker h x'$ if, and only if $h(x) = h(x')$. The quotient space ${}^{X}\!/_{\ker h}$ has the following relation to the natural map on X via the commutative diagram

$$\begin{tikzcd}
X \arrow[r, "h"] \arrow[d, "q"'] & Y \\
{}^{X}\!/_{\ker h} \arrow[ur, "\phi"'] &
\end{tikzcd}$$

Where $\phi : {}^{X}\!/_{\ker h} \to Y$ is a 1–1 map defined by $\phi([x]) = h(x)$.

Definition. A continuous map $f : X \to Y$ of a topological space X onto a topological space Y is call an **identification** if a subset U of Y is open if, and only if $f^{-1}(U)$ is open in X. We denote the quotient space on X induced by f by ${}^{X}\!/_{f}$.

Example 2.3. (1) The natural map $q : X \to X/\sim$ is an identification, where $\sim$ is an equivalence relation on X inducing the quotient topology.

(2) If $f : X \to Y$ takes spaces X onto Y, is open or closed, then f is an identification.

(3) If $f : X \to Y$ is a continuous map such that there exists a map $s : Y \to X$ such that $f \circ s = 1_Y$, then f is an identification. We call the map s a **section** of f.

Theorem 2.2.1. *Let $f : X \to Y$ be a continuous map of a topological space X onto a topological space Y. f is an identification if, and only if for any toopological space Z, and all maps $g : Y \to Z$, then g is continuous if, and only if $g \circ f$ is continuous.*

$$\begin{tikzcd} X \arrow[r, "g\circ f"] \arrow[d, "f"'] & Z \\ Y \arrow[ur, "g"'] & \end{tikzcd}$$

Proof. Suppos that f is an identification. If g is continuous, then so is $g \circ f$, by continuity of f. On the other hand, if $g \circ f$ is continuous, letting V be open in Z we have $(g \circ f)^{-1}(V) = f^{-1}(g^{-1}(V))$ which is open in X. By hypothesis, $g^{-1}(V)$ is open in Y, which makes g continuous.

Now, suppose that g is continuous if, and only if $g \circ f$ is continuous. Let $Z = X/\ker f$, and $q : X \to X/\ker f$ the natural map. Additionally, define the 1–1 map $\phi : X/\ker f \to Y$ by $\phi([x]) = f(x)$. Since f is onto, we get that so is ϕ. Consider the following commutative diagram:

$$\begin{tikzcd} X \arrow[r, "q"] \arrow[d, "f"'] & Z \\ Y \arrow[ur, "\phi^{-1}"'] & \end{tikzcd}$$

Then $\phi^{-1} \circ f = q$ is continuous which implies that ϕ^{-1} is continuous. ϕ is also continuou since q is an identification. Therefore ϕ is a homeomorphism between Y and Z. Notice now, that $f = \phi \circ q$. Then since q and ϕ are continuous, this makes f continuous by composition. Moreover, $f^{-1}(U) = q^{-1}(\phi^{-1}(U))$. Since q is an identification, $\phi^{-1}(U)$ is open in Z, which makes $f^{-1}(U)$ open in X. This makes f an identification. ■

f

Corollary. *Let $f : X \to Y$ be an identification, and for some space Z, define $h : X \to Z$ to* *be the continuous map constant on each fiber of f. Then $h \circ f^{-1} : Y \to Z$ is continuous.*

$$\begin{tikzcd} X \arrow[r, "h"] \arrow[d, "f"'] & Z \\ Y \arrow[ur, "h\circ f^{-1}"'] & \end{tikzcd}$$

Moreover $h \circ f^{-1}$ is open or closed if, and only if $h(U)$ is open or closed in Z whenever $U = f^{-1}(f(U))$ is open or closed in X.

Corollary. *If $h : X \to Z$ is an identification, then the map $\phi : X/_{\ker h} \to Z$ defined by $[x] \to h(x)$ is a homeomorphism.*

2.3 Convexity and Contracibilty

Definition. We call a subset X of $\mathbb{R}^n$ **convex** if for every $x, y \in X$, the line segment joining x to y is convex. That is the line $tx + (1-t)y \in X$ for all $t \in [0, 1]$.

Example 2.4. The sets $\mathbb{R}^n$, I^n, B^n and $\Delta(\mathbb{R}^n)$ are all convex. The sphere S^{n-1} is not convex.

Definition. We call a topological space X **contracible** if 1_X is nullhomotopic.

Example 2.5. (1) Let $X = \{x, y\}$ together with the topolgoy $\mathcal{T} = \{\emptyset, \{x\}, X\}$. Then X is contractible under the topology $\mathcal{T}$. We call X together with $\mathcal{T}$ the **Sierpinski space**.

(2) The space $\mathbb{R}^n$ is contractible, but the sphere S^{n-1} is not contractible.

(3) Continuous images of contractible spaces need not be contractible.

Theorem 2.3.1. *Every convex set is contractible.*

Proof. Choose $x_0 \in X$ and consider the constant map $c : X \to X$ by $x \to x_0$ for all $x \in X$. Define $F : X \times I \to X$ by $F(x, t) = tx_0 + (1-t)x$. This map is continuous, with $F(x, 0) = x = 1_X(x)$ and $F(x, 1) = x_0 = c(x)$. Therefore $1_X \simeq c$. ■

Lemma 2.3.2. *If X is a contractible space, and homeomorphic to a space Y, then Y is also contractible.*

Example 2.6. If X and Y are subspaces of $\mathbb{R}^n$, with X homeomorphic to Y, and X convex, then Y is contractible by lemma 2.3.2, however, Y may not be convex. This shows that not all contractible spaces are convex spaces.

Lemma 2.3.3. *Contractible spaces are connected.*

Corollary. *Convex sets are connected.*

Proof. This follows from theorem 2.3.1. ■

Definition. If X is a topological space, define the equivalence relation $\sim$ on $X \times I$ by $(x, t) \sim (x', t')$ if, and only if $t = t' = 1$. Denote the equivalence classes of (x, t) as $[x, t]$. We call the quotient space $X \times I/\sim$ the **cone** over X, and denote it CX. We call the equivalence class $[x, 1]$ the **vertex** of CX.

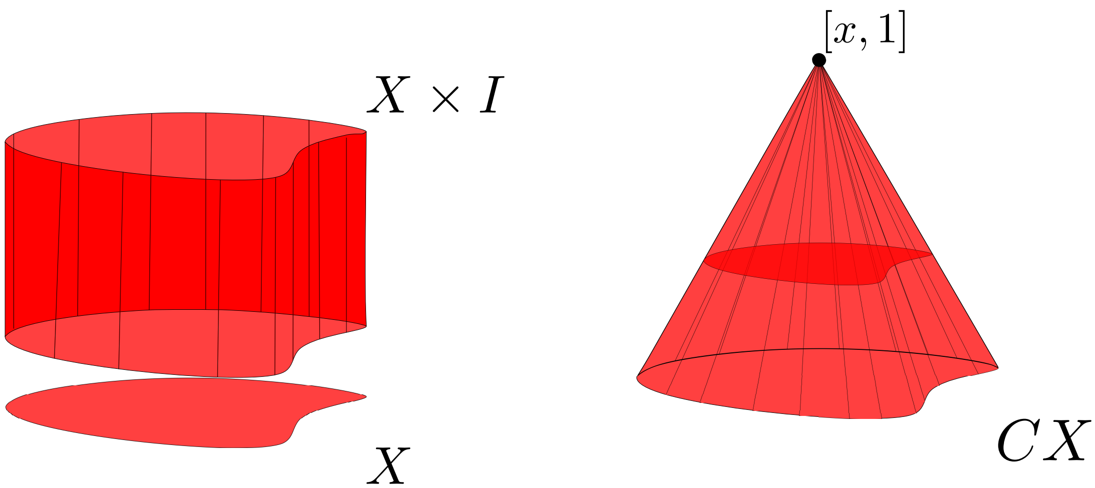

Figure 2.1: The space X and the cone CX formed by identifying all $t = 1$ of $X \times I$ to a point.

Example 2.7. (1) For topological spaces X and Y, every continuous map $f : X \times I \to Y$ with $f(x, 1) = y_0$ for some $y_0 \in Y$ induces a continuous map $Cf : CX \to Y$ by taking $[x, t] \to f(x, t)$.

(2) The cone over S^{n-1} is $CS^{n-1} = D^n$ and has the vertex 0.

Theorem 2.3.4. *For any topological space X, the cone over X is contractible.*

Proof. Define the map $F : CX \times I \to CX$ by taking $([x, t], s) \to [x, (1-s)t+s]$. This map is continuous by composition, moreover $F([x, t], 0) = [x, t]$ and $F([x, t], 1) = [x, 1]$ which makes $1_{CX} \simeq c$ where $c : CX \to CX$ is the constant map taking $[x, t] \to [x, 1]$ for all $x \in X$. ■

Theorem 2.3.5. *A topological space has the same homotopy type as a point if, and only if X is contractible.*

Proof. Let $\{a\}$ be a point space, and suppose that $X \simeq \{a\}$ have the same homotopy type. Then there are maps $f : X \to \{a\}$ and $g : \{a\} \to X$ with $a \xrightarrow{g} x_0$ such that $g \circ f \simeq 1_X$ and $f \circ g \simeq 1_{\{a\}}$. Notice that $g \circ f(x) = g(a) = x_0$, for all $x \in X$, so $g \circ f$ is constant. This makes 1_X (and 1_Y) nullhomotopic. Therefore X is contractible.

On the otherhand, supposing that X is contractible, let $1_X \simeq c$ where $c : X \to X$ is the constant map defined by $x \to x_0$ for all $x \in X$. Define the maps $f : X \to \{x_0\}$ and $g : \{x_0\} \to X$ by $x \xrightarrow{f} x_0$ and $x_0 \xrightarrow{g} x_0$. Observe that $g \circ f = 1_X$, and that $f \circ g \simeq 1_{\{x_0\}}$ Therefore X is of the same homotopy type as $\{x_0\}$. ■

Remark. This theorem shows that the simplest objects in hTop are the contractible spaces.

Theorem 2.3.6. *If Y is a contractible space, then any two maps $X \to Y$ are homotopic.*

Proof. Suppose that $1_Y \simeq c$ where $c : Y \to Y$ takes $y \to y_0$ for all $y \in Y$. Defie $g : X \to Y$ by taking $x \to y_0$ for all $x \in X$. If $f : X \to Y$ is any continuous map, then $f \simeq g$. Consider the diagram

$$\begin{tikzcd} X \arrow[r] & Y \arrow[r, bend left, "k"] \arrow[r, bend right, "1_Y"'] & Y \end{tikzcd}$$

Since $1_Y \simeq k$, we get that $f = 1_Y \circ f \simeq k \circ f = g$. ■

Corollary. *Any two maps $X \to Y$ are nullhomotopic.*

2.4 Path Connectedness.

Definition. A **path** in a topological space X is a continuous map $f : [0,1] \to X$ such that $f(0) = a$ and $f(1) = b$ for some $a, b \in X$. We call a and b the **endpoints** of f, we say f goes **from** a **to** b.

Definition. We call a topological space X **path connected** if there exists a path from a to b for all $a, b \in X$.

Example 2.8. The sphere S^n is path connected.

Lemma 2.4.1. *If $f : X \to Y$ is a continuous map and X is a path connected space, then $f(X)$ is also path connected.*

Theorem 2.4.2. *If X is a path connected space, then X is a connected space.*

Proof. Suppose that X is disconnected. Then there exists a seperation of X into disjoint open sets U and V. That is $X = U \cup V$. Suppose however that X is path connected. Then for points $a \in U$ and $b \in V$, there is a path $f : [0,1] \to X$ from a to b. Since $[0,1]$ is a connected space, so is $f([0,1])$; however notice that $f([0,1]) = (U \cap f([0,1])) \cup (f([0,1]) \cap v)$, which is a seperation of $f([0,1])$, since U and V form a seperation. ■

Example 2.9. The converse of theorem 2.4.1 is not true in general. Consider the following two examples:

(1) Consider the subspace $X = (0 \times [0,1]) \cup G$ where G is the graph of $\sin \frac{1}{x}$ on the interval $(0, 2\pi]$. We have that X is connected, since the component containing G is closed, and $0 \times [0,1] \subseteq \operatorname{cl} G$. However, X is not path connected. We call the space X the **topologists sine curve**.

(2) Another example of a connected space in $\mathbb{R}^2$ that is not path connected is the **topologist's whirlpool**.

Lemma 2.4.3. *Every contractible space is path connected.*

Lemma 2.4.4. *A topological space X is path connected if, and only if any two constant maps $X \to X$ are homotopic.*

Lemma 2.4.5. *If X is a contractible space and Y a path connected space, then any two continuous maps $X \to Y$ are homotopic.*

Corollary. *The continuous maps are nullhomotopic.*

Lemma 2.4.6. *If X and Y are path connnected spaces, then so is $X \times Y$.*

Lemma 2.4.7. *If $f : X \to Y$ is a continuous map and X is a path connected space, then $f(X)$ is also path connected.*

Theorem 2.4.8. *If X is a topological space, then the relation $\sim$ defined on X by $a \sim b$ if, and only if there is a path from a to b, is an equivalence relation.*

Proof. Consider the constant path $c : [0,1] \to X$ where $c(x) = a$ for all $x \in A$. c is continuous, and $c(0) = c(1) = a$. So $a \sim a$.

Now suppose that for $a, b \in X$, that $a \sim b$. Then there is a path $f : [0,1] \to X$ with $f(0) = a$ and $f(1) = b$. Consider the map $g : [0,1] \to X$ defined by $g(t) = f(1-t)$. g is continuous by composition, and $g(0) = f(1) = b$ and $g(1) = f(0) = a$, which makes $b \sim a$.

Lastly, suppose that $a \sim b$ and $b \sim c$ for some $a, b, c \in X$. Then there exist paths $f : [0,1] \to X$ and $g : [0,1] \to X$ with $f(0) = a$, $f(1) = b$, and $g(0) = b$, $g(1) = a$. Now, consider the map $h : [0,1] \to X$ defined by:

$$h(t) = \begin{cases} f(2t), \text{ if } 0 \leq t \leq \frac{1}{2} \\ g(2t-1), \text{ if } \frac{1}{2} \leq t \leq 1 \end{cases}$$

Notce that $f(\frac{1}{2}) = g(\frac{1}{2}) = f(1) = g(0) = b$, so the domains of f and g coincide. Therefore by the pasting lemma, h is continuous. Now, observe that $h(0) = f(0) = a$, and that $h(1) = g(1) = c$. This makes $a \sim c$. ■

Definition. We define the equivalence classes of X under path connectedness to be called **path components** of X.

Definition. We denote the collection of all path components of a topological space X to be $pi_0(X)$; that is $pi_0(X) = {}^{X}\!/\!_{\sim}$ (not necesarily as a quotient space), Moreover, we define the map $pi_0(f) : pi_0(X) \to pi_0(Y)$ to be the map taking the path component C to the unique path component of Y containing $f(C)$.

Theorem 2.4.9. $pi_0 : \text{Top} \to \text{Set}$ *is a funtor.*

Proof. Consider $1_X : X \to X$ the identity on X. Let $\pi_0(X) = \{X_\alpha\}$ where X_α is a path component of X. We have that $pi_0(1_X) : \pi_0(X) \to \pi_0(X)$ sends $X_\alpha \to X_\beta$ where X_β is the unique path component of X containing $1_X(X_\alpha) = X_\alpha$. However, since X_α and X_β are equivalence classes, we have $X_\alpha \subseteq X_\beta$ if and only if $\alpha = \beta$, i.e. $X_\alpha = X_\beta$. This makes $pi_0(1_X) = 1_{\pi_0(X)}$.

Now let $f : X \to Y$ and $g : Y \to Z$ be continuous maps. Let $\pi_0(X) = \{X_\alpha\}$, $\pi_0(Y) = \{Y_\beta\}$, $\pi_0(Z) = \{Z_\gamma\}$ the collection of path components of X, Y, and Z, respectively. Now consider X_α and Z_γ such that $\pi_0(g \circ f)(X_\alpha) = Z_\gamma$. Then Z_γ is the unique path component of Z containing $g(f(X_\alpha))$. Now, if Y_β is the unique path component of Y containing X_α, then $\pi_0(f)(X_\alpha) = Y_\beta$ and we see that $g(f(X_\alpha)) \subseteq g(Y_\beta)$. Moreover, if $Z_\prime$ is the unique path component of Z containing $g(Y_\beta)$, then $\pi_0(g)(Y_\beta) = Z_{\gamma'}$, and $g(Y_\beta) \subseteq Z_{\gamma'}$. But $g(f(X_\alpha)) \subseteq g(Y_\beta) \subseteq Z_{\gamma'}$; by above, and since path components partition their spaces, this makes $\gamma = \gamma'$. Thus $Z_\gamma = Z_{\gamma'}$ and we have that $g(f(X_\alpha)) \subseteq g(Y_\beta) \subseteq Z_\gamma$. Therefore Z_γ is the unique path component of Z containing both $g(f(X_\alpha))$ and $g(Y_\gamma)$; that is $\pi_0(g)(Y_\beta) = Z_\gamma$, where $\pi_0(f)(X_\alpha) = Y_\beta$. This implies that $pi_0(g \circ f) = \pi_0(g) \circ \pi_0(f)$, which makes π_0 a functor. ∎

Corollary. *If $f \simeq g$, then $\pi_0(f) = \pi_0(g)$.*

Proof. Suppose that $F : f \simeq g$ is a homotopy between the maps $f : X \to Y$ and $g : X \to Y$. Let C be a path component of X, then $C \times I$ is path connected by lemma 2.4.6. Thus by lema 2.4.1, $F(C \times I)$ is also path connected. Notice then that:

$$f(C) = F(C \times 0) \subseteq F(C \times I)$$

and

$$g(C) = F(C \times 1) \subseteq F(C \times I)$$

So the unique path connected component of Y containing $F(C \times I)$ contains both $f(C)$ and $g(C)$. Therefore $\pi_0(f) = \pi_0(g)$. ∎

Corollary. *If X and Y are topological spaces with the same homotopy type, then they have the same number of path components.*

Proof. Suppose that $f : X \to Y$ and $g : Y \to X$ are continuous maps with $g \circ f = 1_X$ and $f \circ g = 1_Y$. Since f is a homotopy equivalence, then $[f]$ is an equivalence in hTop. Restricting π_0 to hTop, this also gives use that $\pi_0([f])$ is an equivalence in Set. That is f is 1–1 and onto. ∎

Definition. A topological space X is **locally path connected** if, for each $x \in X$, and every open neighborhood U of x there is an open set V with $x \in V \subseteq U$ such that any two points in V can be joined by a path in U.

Example 2.10. Form the subspace X of $\mathbb{R}^2$ by asjoining a curve from $(0, 1)$ to $(\frac{1}{2\pi}, 0)$ on the topologist's sine curve. Then X is path connected, but not locally path connected.

Theorem 2.4.10. *A topological space is locally path connected if, and only if path components of open sets are open.*

Proof. Suppose that X is locally path connectd, and lwet U be open in X. Let $x \in C$, where C s a path component of U. Then there is an open V with $x \in V \subseteq U$ such that very point of V can be joined to x by a path in U. Thus each point of V lies in the path component of x, which is C. Thus $V \subseteq C$, which makes C open.

Conversely, suppose that path components of open sets in X are open. Let U be an open set of X, and for some $x \in U$, let C be the path component of x in U. Then we have $x \in C \subseteq U$. Since C is open, this makes X locally path connected. ∎

Corollary. *If X is locally path connected, then its path components are open.*

Corollary. *X is locally path connected if, and only if for every $x \in X$, and each open neighborhood U of x, there is an open path connected set V with $x \in V \subseteq U$.*

Corollary. *If X is locally path connected, then the connected components of every open set coincide with its path components. In particular the connected components of X coincide with the path components of X.*

Corollary. *If X is connected, and locally path connected, then X is connected.*

Definition. Let A be a subspace of a topological space X, and let $i : A \to X$ be the inclusion. Then A is a **deformation retract** of X if there is a continuous map $r : X \to A$ such that r is a retraction of X; i.e. $r \circ i = 1_A$ and $i \circ r = 1_X$.

Lemma 2.4.11. *Every deformation retract is a retract.*

Theorem 2.4.12. *If A is a deformation retract of a topological space X, then X and A have the same homotopy type.*

Corollary. *S^1 is a deformation retract of $\mathbb{C}\backslash 0$.*

Proof. For every $z \in \mathbb{C}\backslash 0$, we can write z as $z = \rho e^{i\theta}$, where $\rho > 0$, and $0 \leq \theta \leq 2\pi$. Now, define $F : (\mathbb{C}\backslash 0) \times I \to \mathbb{C}\backslash 0$ by taking $(\rho e^{i\theta}, t) \to ((1-t)\rho + t)e^{i\theta}$. Notce that F is never 0, and that F is continuous, with $F(\rho e^{i\theta}, 0) = \rho e^{i\theta}$, $F(e^{i\theta}, 1) = e^{i\theta}$. Moreover $F(\rho e^{i\theta}, 1) = F(e^{i\theta}) = e^{i\theta}$. Writing S^1 as $S^1 = \{e^{i\theta} : 0 \leq \theta \leq 2\pi\}$. We see that F makes S^1 into a deformation retract of $\mathbb{C}\backslash 0$. ∎

Corollary. *S^1 has the same homotopy type as $\mathbb{C}\backslash 0$.*

Definition. Let $f : X \to Y$ be a continuous map from a topological space X to a topological space Y. Define

$$M_f = {(X \times I) \cup Y}/{\sim}$$

Where $(X \times I) \cup Y$ is a disjoint union, and $\sim$ is an equivalence relation defined by $(x,t) \sim y$ if $y = f(x)$ and $t = 1$. Denote the equivalece classes of (x,t) by $[x,t]$. We call the quotient spce M_f the **mapping cylinder** of f.

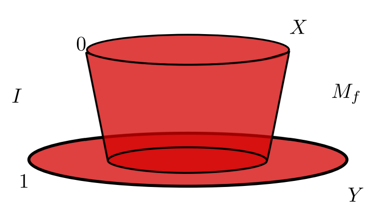

Figure 2.2: The mapping cylinder of a continuous map $f : X \to Y$.

Chapter 3

Simplexes.

3.1 Affine Spaces.

Definition. We call a subset $X \subseteq \mathbb{R}^n$ **affine** if for every $x, y \in X$, the line $l(x, y)$ passing through x and y is contained in X.

Lemma 3.1.1. *Affine sets are convex.*

Proof. Note that the line $l(x, y)$ contains the segment $l[x, y]$ which is in X for every $x, y \in X$. ■

Theorem 3.1.2. *If $\{X_\alpha\}$ is a collection of affine (or convex) sets in $\mathbb{R}^n$, then the intersection of all X_α is affine (or convex) in $\mathbb{R}^n$.*

Proof. Let $X = \bigcap X_\alpha$ and let $x, y \in X$. let $l(x, y)$ be the line passing through x and y, then $l(x, y) \in X_\alpha$ for every α, since $x, y \in X_\alpha$ which is affine. This makes $l(x, y) \in X$, which makes X affine in $\mathbb{R}^n$. The proof for convexity of X is the same except using the line segment $l[x, y]$. ■

Definition. An **affine combination** of points $x_0, \ldots, x_m \in \mathbb{R}^n$ is a point $x \in \mathbb{R}^n$ such that

$$x = t_0 x_1 + \cdots + t_m x_m$$

Where $\sum t_i = 1$. A **convex combination** is an affine combination in which each $t_i \geq 0$ for $o \leq i \leq m$.

Example 3.1. The line $tx + (1 - t)y$ is a convex combination in $\mathbb{R}^n$.

Definition. We say a subset $X \subseteq \mathbb{R}^n$ **spans** an affine set $[X]$ if $[X]$ is the intersection of all affine subsets containing X. Similarly, we say X **spans** a convex set $[X]$ if $[X]$ is the intersection of all convex subsets containing X. We call these the affine and convex **hulls**, respectively.

Theorem 3.1.3. *If $x_0, \ldots, x_m \in \mathbb{R}^n$, then the convex hull $[x_0, \ldots, x_m]$ is the set of all convex combinations of $x_0, \ldots, x_m$.*

Proof. Let S be the set of all convex combinations of $x_0, \ldots, x_m$, then $[x_0, \ldots, x_m] \subseteq S$. Now, let $t_j = 1$ and $t_i = 0$, then $x_i \in S$ for all j. Moreovoer, let $\alpha = \sum a_i x_i$ and $\beta = \sum b_i x_i$ where $\sum a_i = \sum b_i = 1$. Then for $t \in [0, 1]$ we have

$$t\alpha + (1-t)\beta = t \sum a_i x_i + (1-t) \sum b_i x_i = \sum (t(a_i x_i) + (1-t) b_i x_i)$$

moreover, $t \sum a_i + (1-t) \sum b_i = 1$ and $ta_i + (1-t)b_i \geq 0$ for all $0 \leq i \leq m$, so $t\alpha + (1-t)\beta$ is a convex combination in S.

Now, let X be any convex set containing $\{x_0, \ldots, x_m\}$. By induction on m, for $m = 0$, $S = \{x_0\}$. Now let $m \geq 0$ and $t_i \geq 0$ with $\sum t_i = 1$. Assume without loss of generality that $t_0 \neq 1$. Then

$$y = (\frac{t_1}{1-t_0})x_0 + \cdots + (\frac{t_m}{1-t_0})x_m \in X$$

which makes $x = t_0 x_0 + (1-t_0)y \in X$ This makes $S \subseteq [x_0, \ldots, x_m]$. ■

Definition. We call points $x_0, \ldots, x_m \in \mathbb{R}^n$ **affinely independent** if $\{x_1 - x_0, \ldots, x_m - x_0\}$ is linearly independent in $\mathbb{R}^n$ as a vector space. We say the points $x_0, \ldots, x_m$ are **affinely dependent** if they are not affinely independent.

Theorem 3.1.4. *For any points $x_0, \ldots, x_m \in \mathbb{R}^n$, the following are equivalent:*

(1) $x_0, \ldots, x_m$ are affinely independent.

(2) If $s_0, \ldots, s_m \in \mathbb{R}$ such that $\sum s_i x_i = 0$ and $\sum s_i = 0$, then $s_0 = \cdots = s_m = 0$.

(3) If A is an affine set spanned by $x_0, \ldots, x_m$, then every $x \in A$ can be written as a unique affine combination of $x_0, \ldots, x_m$.

Proof. Suppose, that $x_0, \ldots, x_m$ are affinely independent. Let $a_0, \ldots, a_m \in \mathbb{R}$ such that $\sum a_i = 0$ and $\sum a_i x_i = 0$. We see that

$$\sum a_i x_i = \sum a_i x_i - 0 \cdot x_0 = \sum a_i x_i - x_0 \sum a_i =_i (x_i - x_0) = 0$$

Now, since $x_0, \ldots, x_m$ are affinelt independent, $x_1 - x_0, \ldots, x_m - x_0$ are linearly indpendent which implies that $a_0 = \cdots = a_m = 0$.

Now, suppose that if $\sum a_i = 0$ and $\sum a_i x_i = 0$, then $a_0 = \cdots = a_m = 0$. Let A be an affine set spanned by $x_0, \ldots, x_m$ and suppose there is an $x \in A$ for which $x = \sum a_i x_i$ and $x = \sum b_i x_i$. Then

$$\sum a_i x_i = \sum (b_i x)$$

so that $\sum (a_i - b_i) x_i = 0$. Notice also that $\sum a_i - b_i = \sum a_i - \sum b_i = 1 - 1 = 0$, so that by hypothesis, $a_i - b_i = 0$ for each i. That is $a_i = b_i$.

Finally, suppose that A is an affine set spanned by $x_0, \ldots, x_m$ for which every $x \in A$ can be written uniquely as an affine combination of these points. That is $x = \sum a_i x_i$ where $\sum a_i = 1$. Now, if $m = 0$, we get each $x = a_0 x_0$ which is trivially affinely independent. Now, suppose $m \geq 0$ and by induction on m, suppose that $x_1 - x_0, \ldots, x_m - x_0$ are linearly dependent. Then there exists $a_0, \ldots, a_m \in \mathbb{R}$ not all 0 such that $\sum a_i(x_i - x_0) = 0$. Choose $r_j \neq 0$ then

$$\sum \frac{r_i}{r_j}(x_i - x_0) = 0$$

Suppose then, without loss of generality that $r_j = 1$. Then $x_j \in \{x_0. \ldots, x_m\}$ which gives x_j two affine combinations:

$$x_j = 1 \cdot x_j$$
$$x_j = -\sum a_i x_i + (1 + \sum a_i)x_0$$

This contradicts that each $x \in A$ has a unique representation as an affine combination, hence $x_1 - x_0, \ldots, x_m - x_0$ have to be linearly independent, making $x_0, \ldots, x_m$ affinely indpendent. ∎

Corollary. *Given points $x_0, \ldots, x_m \in \mathbb{R}^n$, affine independence on the points is independent of their ordering.*

Corollary. *If $A \subseteq \mathbb{R}^n$ is an affine set spanned by affinely independent points $x_0, \ldots, x_m$, then it is the translation of an m-dimensional subspace V of $\mathbb{R}^n$ as a vector space.*

Proof. Let $p_0 = x_0$ and V subspace of $\mathbb{R}^n$ as a vector space with basis $\{x_1 - x_0, \ldots, x_m - x_0\}$. If $z \in A$, then $z = \sum a_i x_i$ where $\sum a_i = 1$ Then $z = \sum a_i x_i + a_0 x_0 = \sum a_i x_i - \sum a_i x_0 + (a_0 + \sum a_i)x_0 = \sum a_i(x_i - x_0) + x_0 \in V + x_0$. By similar reasoning, if we have $z \in V + p_0$, then $z \in A$. ∎

Definition. We say a set of points $a_1, \ldots, a_k \in \mathbb{R}^n$ are in **general position** if every $n+1$ of its points are affinely independent.

Theorem 3.1.5. *Given $k \geq 0$, $\mathbb{R}^n$ contains k points in general position.*

Proof. For $0 \leq k \leq n+1$, take the origin 0 together with any $k-1$ elements of a basis of $\mathbb{R}^n$. These points are in general position.

Now, suppose that $k > n+1$, and choose $r_1, \ldots, r_k \in \mathbb{R}$ and define

$$a_i = (r_i, r_i^2, \ldots, r_i^n) \text{ for } 1 \leq i \leq k$$

Suppose additionally that the points $a_1, \ldots, a_k$ are not in general position. Then $n+1$ of the points $a_{i_0}, \ldots, a_{i_n}$ which are affinely dependent. Then $a_{i_1} - a_{i_0}, \ldots, a_{i_n} - a_{i_0}$ are linearly dependent. Then there exist $s_0, \ldots, s_n$, not all 0 such that

$$\sum s_j(a_{i_j} - a_{i_0}) = 0$$

Consider now, the $n \times n$ south east block, V^* of the $(n+1) \times (n+1)$ Vandermonde matrix obtained from $r_{i_0}, \ldots, r_{i_n}$. Let $\sigma = (s_0, \ldots, s_m)$, then the equation above give the matrix equation

$$V^* \sigma^T = 0 \tag{3.1}$$

Now, since V^* is nonsingular, and each of the r_{i_j} is distinct, we get that $\sigma = 0$, which contradicts our assumption that $a_{i_0}, \ldots, a_{i_n}$ are affinely independent. ∎

Definition. Let $x_0, \dots, x_m \in \mathbb{R}^n$ be affinely independent and let $x \in \mathbb{R}^n$ be such that $x = \sum t_i x_i$. We call the $(m+1)$-tuple $(t_0, x_0, \dots, t_m, x_m)$ the **barycentric coordinates** fo x.

Definition. Let $x_0, \dots, x_m \in \mathbb{R}^n$ be affinely independent. We call the convex set spanned by each of these points, $[x_0, \dots, x_m]$ an m-simplex. We call the n-simplex $[e_0, \dots, e_n]$ of $\mathbb{R}^{n+1}$, where $\{e_0, \cdots e_m\}$ is the standard basis of $\mathbb{R}^N n+1$ the **standard** m-simplex, and we denote it Δ^n.

Theorem 3.1.6. *If $x_0, \dots, x_m \in \mathbb{R}^n$ are affinely independent then each $x \in [x_0, \dots, x_m]$ is the unique convex combination of barycentric coordinates.*

Proof. Note that barycentric coordinates are unuqiue by theorem 3.1.4. ∎

Definition. If $x_0, \dots, x_m \in \mathbb{R}^n$ are affinely independent, the **barycenter** of $[x_0, \dots, x_m]$ is the point $\frac{\sum x_i}{m+1}$ of $[x_0, \dots, x_m]$.

Example 3.2. (1) $[x_0]$ is a 0-simplex with x_0 as its barycenter.

(2) The 1-simplex $[x_0, x_1]$ has as its barycenter the point $\frac{x_0+x_1}{2}$, which is the midpoint of a closed line segment between x_0 and x_1.

(3) The 2-simplex $[x_0, x_1, x_2]$ has barycenter $\frac{x_0+x_1+x_2}{3}$ which is the geometric barycenter of a triangle.

(4) Let Δ^n the standard n-simplex. Every point $x \in \Delta^n$ has the form $\sum t_i e_i$, which is represented as $(t_0, \dots, t_n)$ in $\mathbb{R}^{n+1}$ as a vector space. Therefore the barycentric coordinates of any point in Δ^n are precisely its cartesian coordinates.

Definition. Let $[x_0, \dots, x_m]$ be an m-simplex. We define the **face opposite** of x_i to be the set

$$[x_0, \dots \hat{x}_i, \dots x_m] = \{\sum t_j x_j : t_j \geq 0, \sum t_j = 0, \text{ and } , t_i = 0\}$$

We define a k-**face** of $[x_0, \dots, x_m]$ to be a k-simplex spanned by $k+1$ vertices of $[x_0, \dots, x_n]$. We define the **boundry** of $[x_0, \dots, x_m]$ to be the union of all faces opposite x_i for all $0 \leq i \leq m$, and we write $\partial[x_0, \dots, x_m]$.

Example 3.3. (1) Note that $\partial[e_0, \dots, e_n] = \partial\Delta^n$.

(2) Given any m-simplex, it has $\binom{m+1}{k+1}$ k-faces.

Theorem 3.1.7. *Let $S = [x_0, \dots, x_n]$ be an n-simplex. The following are true*

(1) If $u, v \in S$, then $\|u - v\| \leq \sup_i \|u - x_i\|$

(2) $\operatorname{diam} S = \sup_{i,j} \|x_i - x_j\|$

(3) If b is the barycenter of S, then $\|b - x_i\| \leq \frac{n}{n+1} \operatorname{diam} S$.

Proof. Let $u, v \in S$, and $v = \sum t_i x_i$ where $t_i \geq 0$ and $\sum t_i = 1$. Then $\|u - v\| = \|u - \sum t_i x_i\| = \|u \sum t_i - \sum t_i x_i\| \leq \sum t_i \|u_i - x_i\| \leq \sum t_i \sup \|u - x_i\| = \sup \|u - x_i\|$. It also follow that the second statement is true using the properties of least upperbounds.

Now, let $b = \frac{x_0 + \cdots + x_n}{n+1}$ the barycenter of S. Then

$$\begin{aligned}
\|b - x_i\| &= \|\frac{1}{n+1} \sum x_j - x_i\| \\
&= \|frac1n + 1 \sum x_j - \frac{1}{n+1} \sum x_i\| \\
&= \|\frac{1}{n+1} \sum x_j - x_i\| \\
&\leq \frac{1}{n+1} \sum \|x_j - x_i\| \\
&\leq \frac{n}{n+1} \sup \|x_j - x_i\| \\
&= \frac{n}{n+1} \operatorname{diam} S
\end{aligned}$$

■

3.2 Affine Maps.

Definition. Let $x_0, \ldots, x_m \in \mathbb{R}^n$ be affinely independent points and let A be the affine set spanned by these points. An **affine map** is a map $T : A \to \mathbb{R}^k$, with $k \leq n$ such that

$$T(\sum a_i x_i) = \sum a_i T(x_i)$$

whenever $\sum a_i = 1$ for $a_1, \ldots, a_m \in \mathbb{R}$.

Theorem 3.2.1. *If $[x_0, \ldots, x_m]$ and $[y_0, \ldots, y_n]$ are m and n-simplexes, and $f : \{x_0, \ldots, x_m\} => [y_0, \ldots, x_m]$ os a map, then there exists a unique affine map $T : [x_0, \ldots, x_m] \to [y_0, \ldots, y_m]$ such that $T(x_i) = f(x_i)$ for all $1 \leq i \leq m$.*

Chapter 4

The Fundamental Group.

4.1 The Fundamental Groupoid

Definition. Let $f :\to_X$ and $g : I \to X$ be paths with $f(1) = g(0)$. We define **path multiplication** to be the operation

$$f * g = \begin{cases} f(2t), & \text{if } 0 \le t \le \frac{1}{2} \\ g(2t-1), & \text{if } \frac{1}{2} \le t \le 1 \end{cases}$$

We call $f * g$ the **path product** of f and g.

Lemma 4.1.1. *The path product of two paths is a continuous map.*

Proof. This follows from the pasting lemma. ∎

Corollary. *The path product is a path.*

Proof. Notice that if f and g are paths with $f(1) = g(0)$, then $f*g(0) = f(0)$ and $f*g(1) = g(1)$. ∎

Definition. Let X be a topological space and A a subspace of X Let $f_0 : X \to Y$ and $f_1 : X \to Y$, are continuous maps with $f_0|_A = f_1|_A$. we say that f_0 is **homotopic** to f_1 **relative** A, and write $f_0 \simeq f_1 \operatorname{rel} A$ or $f_0 \simeq_A f_1$, if there is a continuous map $F : X \times I \to Y$ that defines a homotpy between f_0 and f_1, and $F(a,t) = f_0(a) = f_1(a)$ for all $a \in A$. We call F the **relative homotopy**.

Lemma 4.1.2. *Relative homotopy is an equivalence relation.*

Definition. Let ∂I be the boundry of $I = [0,1]$ in $\mathbb{R}$. The equivalence class of $f : I \to X \operatorname{rel} \partial I$ is called the **path class** of f, and denoted $[f]$.

Theorem 4.1.3. *Let f_0, f_1 and g_0, g_1 be paths on X with $f_0 \simeq f_1 \operatorname{rel} \partial I$ and $g_0 \simeq g_1 \operatorname{rel} \partial I$. If $f_0(1) = g_0(0)$ and $f_1(1) - g_1(0)$ then $f_0 * g_0 \simeq f_1 * g_1 \operatorname{rel} \partial I$.*

Proof. Let $F : f_0 \sim_{\partial I} f_1$ and $G : g_0 \sim_{\partial I} g_1$ be the relative homotopies between f_0 and f_1, and g_0 and g_1, respectively. Define the map $H : I \times I \to Y$ by

$$H(s,t) = \begin{cases} F(2t,s), & \text{if } 0 \leq t \leq \frac{1}{2} \\ G(2t-1,s), & , \text{ if } \frac{1}{2} \leq t \leq 1 \end{cases}$$

H is continuous by the pasting lemma, moreover, $H(0,s) = F(0,s) = f_0 * g_0(s)$, and $H(1,s) = G(1,s) = f_1 * g_1(s)$. Additionally, ∂I is fixed by H. This makes H a relative homotopy. ■

Definition. Let $f : I \to X$ be a path from x_0 to x_1 on X. The **origin** of f is x_0 and we write $x_0 = \alpha(f)$, and the **end** of f is x_1 and we write $x_1 = \omega(f)$. We call f a **closed** path if $\alpha(f) = \omega(f)$. We define the maps $i_p : I \to X$ and $i_q : I \to X$ given by $i_p(t) = \alpha(f)$, and $i_q(t) = \omega(f)$ to be the **constant paths**. We define the **inverse path** of f to be the path $f(1-t)$ and denote it f^{-1}.

Definition. A set G together with an operation $*$ is called a **groupoid** if for all $a, b, \in G$ we have:

(1) $a * (b * c) = (a * b) * c$ (Associativity)

(2) There exist elements $e_1, e_2 \in G$ such that $a * e_1 = a$ and $e_1 * a = a$.

(3) For ever $a \in G$, there is an element $a^{-1} \in G$ such that $a * a^{-1} = e_1$ and $a^{-1} * a = e_2$.

Theorem 4.1.4. *For any topological space, the collection of all path classes forms a groupoid under path multiplication. More precisely, if $p = \alpha[f]$ and $q = \omega[f]$ then:*

(1) Path multiplication is associative whenever defined.

(2) $i_p * f \simeq f \operatorname{rel} \partial I$ and $f * i_q \simeq f \operatorname{rel} \partial I$.

(3) $f * f^{-1} \simeq i_p \operatorname{rel} \partial I$ and $f^{-1} * f \simeq i_q \operatorname{rel} \partial I$.

Proof. Let $[f]$ be a path with $p = \alpha[f]$ and $q = \omega[f]$. Consider the following space (see figure 4.1) of $I \times I$ together with the line joining the points $(0,1)$ and $(\frac{1}{2}, 0)$. This line has equation $2s = t - 1$s. Now, define $\theta_t : [\frac{1-t}{2}, 1] \to [0,1]$ to be the affine map:

$$\theta_t(s) = \frac{s - \frac{1-t}{2}}{1 - \frac{1-t}{2}}$$

Define the map $H : I \times I \to I$ by:

$$H(s,t) = \begin{cases} p, & \text{if } 2s \leq 1-t \\ f(\theta_t(s)), & \text{if } 2s \geq 1-t \end{cases}$$

By the pasting lemma, H is continuous. Moreover, $H(0,t) = p = \alpha(f) = i_p * f$, and $H(1,t) = f(\theta_t(1)) = f(t)$. Lastly, ∂I is fixed on H, so we have $i_p * f \simeq f \operatorname{rel} \partial I$. By similar reasoning, we also get $f * i_q \simeq f \operatorname{rel} \partial I$.

Now, conisder the space of figure **??** on $I \times I$ togehter with slanted lines. Construct a

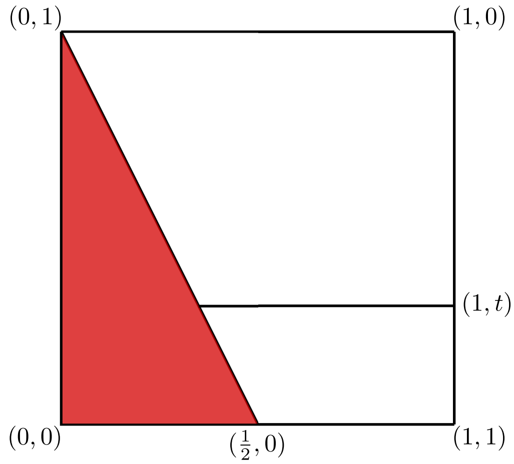

Figure 4.1:

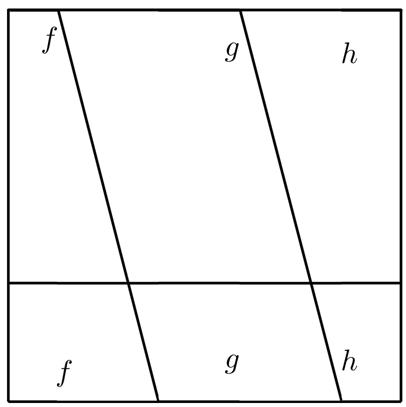

Figure 4.2:

continuous map defined by the affine map from $[0, \frac{1}{2}]$ to $[0, \frac{2-t}{4}]$. It follows that $f * (g * h) \simeq (f * g) * h \operatorname{rel} \partial I$.

Finally, subdivied $I \times I$, again, as in figure **??** and define the map $H : I \times I \to X$ by

$$H(s,t) = \begin{cases} f(2(s(1-t))), & \text{if } 0 \leq s \leq \frac{1}{2} \\ f(2(1-s)(1-t)), & \text{if } \frac{1}{2} \leq s \leq 1 \end{cases}$$

This map is continuous by the pasting lemma with $H(0,t) = f * f^{-1}$, and $H(1,t) = i_q(t)$. Therefore $f * f^{-1} \simeq i_p \operatorname{rel} \partial I$. Again by similar reasoning, we get $f^{-1} * f \simeq i_q \operatorname{rel} \partial I$. ■

Definition. Let X be a topological space and $x_0 \in X$. The **fundamental group** of X with **basepoint** x_0 is the collection of all path classes on X which are closed at x_0. We denote it $\pi_1(X, x_0)$.

Theorem 4.1.5. *The fundamental group of a topoligical space is a group under path multiplication for every basepoint in the space.*

4.2 The π_1 Functor

Theorem 4.2.1. *The map $\pi_1 : \mathrm{Top} \to \mathrm{Grp}$ is a covariant functor. Moreover, if $h : (X, x_0) \to (Y, y_0)$ and $k : (X, x_0) \to (Y, y_0)$ are continuous maps such that $h \simeq k \operatorname{rel} x_0$, then $\pi_1 h = \pi_1 k$.*

Proof. Let $[f] \in \pi_1(X, x_0)$ and define $\pi_1 h : [f] \to [h \circ f]$. Notice that the map $h \circ f : I \to Y$ is continuous by composition and well defined. Moreover $h \circ f$ is a closed path in Y at y_0. This shows that $\pi_1 h([f]) = [h \circ f] \in \pi_1(Y, y_0)$. Now, if $f \simeq f' \operatorname{rel} \partial I$, then $h \circ f \simeq h \circ f' \operatorname{rel} \partial I$, so that if f and g are closed paths in X, at x_0, then $h \circ (f * g) = (h \circ f) * (h \circ g)$. This makes $\pi_1 h$ into a homomorphism. Moreocer, $\pi_1 1_{(X,x_0)} = 1_{\pi_1(X,x_0)}$. This shows that π_1 is a functor, and that it is covariant.

Now, let $h \simeq k \operatorname{rel} \partial I$. Then $h \circ f \simeq k \circ f \operatorname{rel} \partial I$, where f is some closed path in X at x_0. That is $\pi_1 h = \pi_1 k$. ■

Definition. We define the **pointed homotopy category** hTop to be the quotient category of Top* arising from the congruence of relative homotopy.

Theorem 4.2.2. *Let $x_0 \in X$, and X_0 the path component of X containing x_0. Then $\pi_1(X_0, x_0) \simeq \pi_1(X, x_0)$.*

Proof. Consider the inclusion $j : (X_0, x_0) \to (X, x_0)$ and let $[f] \in \ker \pi_1 j$. Then $j \circ f \simeq c \operatorname{rel} \partial I$ where $c : I \to X$ is the constant path from t to x_0. Let $F : j \circ f \simeq c$ a homotopy, then $F(0,0) = x_0$, the space $F(I \times I)$ is path connected, and $F(I \times I) \subseteq X$. It remains to show that f is nullhomotopic. Notice that $F(I \times I)$ makes $\pi_1 j$ 1–1. Now, if f is a closed path in X_0 at x_0, then $f(I) \subseteq X_0$. Define $f' : I \to X_0$ by $f'(t) = f(t)$ for all $t \in I$. Then notice that $j \circ f' = f$, this makes $\pi_1 j$ onto. ■

Theorem 4.2.3. *If X is a path connected space, then the fundamental group of X is independent of the basepoint chosen; i.e. if $x_0, x_1 \in X$, then $\pi_1(X, x_0) \simeq \pi_1(X, x_1)$.*

Proof. Let $\gamma : I \to X$ be a path from x_0 to x_1. Define the map $\phi : \pi_1(X, x_0) \to \pi_1(X, x_1)$ by $\phi : [f] \to [\gamma][f][\gamma^{-1}]$. By theorem 4.1.4, ϕ is an isomorphism. ■

Definition. Let H and K be sets. We define the **projections** $p : H \times K \to H$ and $q : H \times K \to K$ defined by $p : h \times k \to h$ and $q : h \times k \to k$. Additionally, for some set L, there exist maps $\alpha : L \to H$ and $\beta : L \to H$ such that the map $\alpha \times \beta : L \to H \times K$ satisfies $p \circ (\alpha \times \beta) = \alpha$ and $q \circ (\alpha \times \beta) = \beta$.

$$\begin{tikzcd}
 & H \times K \arrow[dl, "p"'] \arrow[dr, "q"] & \\
H & & K \\
 & L \arrow[ul, "\alpha"] \arrow[ur, "\beta"'] \arrow[uu, dashed, "\alpha\times\beta"] &
\end{tikzcd}$$

Theorem 4.2.4. *For pointed spaces (X, x_0) and (Y, y_0), we have that $\pi_1(X \times Y, x_0 \times y_0) \simeq \pi_1(X, x_0) \times \pi_1(Y, y_0)$.*

Proof. Let p and q the projections of $(X \times Y, x_0 \times y_0)$ onto (X, x_0) and (Y, y_0), respectively. Then $\pi_1 p \times \pi_1 q : \pi_1(X \times Y, x_0 \times y_0) \to \pi_1(X, x_0) \times \pi_1(Y, y_0)$ is a homomorphism. If f is a closed path, in $X \times Y$ at $x_0 \times y_0$, then $\pi_1 p \times \pi_1 q([f]) = [pf] \times [qf]$. Now, let g be a closed path in X at x_0 and h a closed path in Y at y_0 and define $\theta : [g] \times [h] \to [g \times h]$. Then θ is well defined and is the incerse of $\pi_1 p \times \pi_1 q$. ∎

Lemma 4.2.5. *Suppose that $F : \phi_0 \simeq \phi_1$ is a free homotopy, where $\phi_0 : X \to Y$ and $\phi_1 : X \to Y$ are continuous. Choose an $x_0 \in X$ and let $\lambda = F(x_0, t)$ the path in Y from $\phi_0(x_0)$ to $\phi_0(x_0)$. Then there is a commutative diagram*

$$\begin{tikzcd}
\pi_1(X, x_0) \arrow[r, "\pi_1\phi_1"] \arrow[dr, "\pi_1\phi_0"'] & \pi_1(Y, \phi_1(x_0)) \arrow[d, "\psi"] \\
& \pi_1(Y, \phi_0(x_0))
\end{tikzcd}$$

where $\psi : [g] \to [\lambda * g * \lambda^{-1}]$.

Proof. Let $f : I \to X$ a closed path at X and define $G : I \times I \to Y$ by $t \times s \to F(f(s), t)$. Then $G : \phi_0 \circ f \simeq \phi_1 \circ f$. Consider the triangulations of the box $I \times I$ and define a continuous

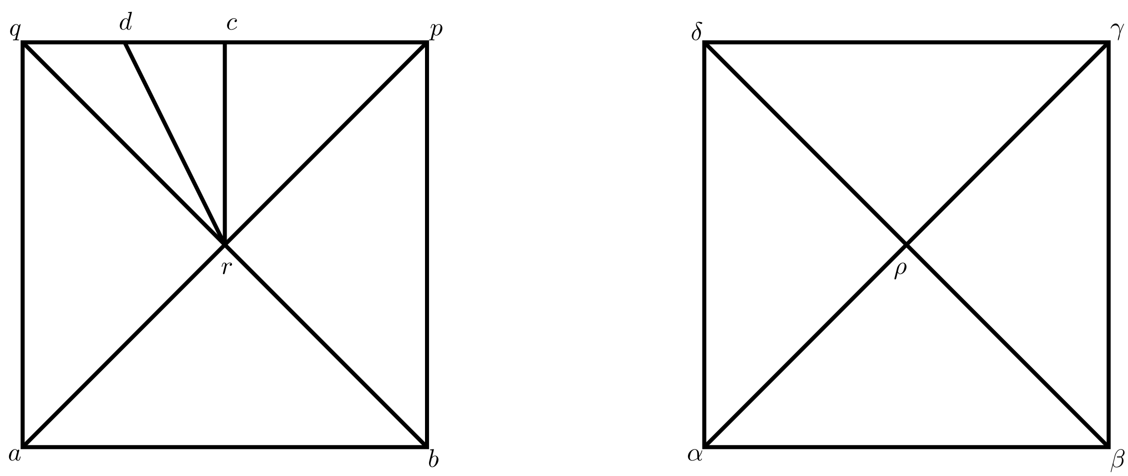

Figure 4.3:

map $H : I \times I \to I \times I$ by defining it in each triangle and applying the pasting lemma. Let

$$H(a) = H(q) = \alpha$$
$$H(b) = H(p) = \beta$$
$$H(c) = \gamma$$
$$H(d) = \delta$$
$$H(r) = \rho$$

Then the map $J = G \circ H$ is a relative homotopy with $J : \phi_0 \circ f \simeq \lambda * (\phi_1 \circ f) * \lambda^{-1} \operatorname{rel} \partial I$. So $\pi_1\phi_0([f]) = [\phi_0 \circ f] = [\lambda * (\phi_1 \circ f) * \lambda^{-1}]$ and $\psi \circ \pi_1\phi_1([f]) = \psi([\phi_1 \circ f]) = [\lambda * (\phi_1 \circ f) * \lambda^{-1}]$ as desired. ■

Corollary. *If ϕ_0 and ϕ_1 are homotopic, then*

(1) $\pi_1\phi_0$ and $\pi_1\phi_1$ are conjugate.

(2) If $\pi_1(X, y_0)$ is Abelian, then $\pi_1\phi_0 = \pi_1\phi_1$.

Theorem 4.2.6. *If $\beta : X \to Y$ is a homotopy equivalence, then the induces homomorphism $\pi_1\beta$ is an isomorphism for every $x_0 \in X$.*

Proof. Let $\alpha : Y \to X$ a continuous map with $\alpha \circ \beta = 1_X$ and $\beta \circ \alpha = 1_Y$. Then the lower triangle of the diagram

$$\begin{tikzcd}
 & \pi_1(Y, \beta(x_0)) \arrow[dr, "\pi_1\alpha"] & \\
\pi_1(X, x_0) \arrow[ur, "\pi_1\beta"] \arrow[rr, "\pi_1(\alpha\circ\beta)"] \arrow[dr, "1"'] & & \pi_1(X, \alpha \circ \beta(x_0)) \arrow[dl, "\psi"] \\
 & \pi_1(X, x_0) &
\end{tikzcd}$$

commutes. Since ψ, defined in lemma 4.2.5 is an isomporphism, then so is $\pi_1(\alpha \circ \beta)$. Moreover, the top tirangle also commutes since π_1 is a functor, and $\pi_1(\alpha \circ \beta) = \pi_1\alpha \circ \pi_1\beta$, so that $\pi_1\beta$ is 1–1 and $\pi_1\alpha$ is onto. A similar diagram arising from $\beta \circ \alpha = 1_Y$ shows that $\pi_1\beta$ is onto. ■

Corollary. *If X and Y are path connected with the same homotopy type, then for any $x_0 \in X$, $y_0 \in Y$, $\pi_1(X, x_0) \simeq \pi_1(Y, y_0)$.*

Corollary. *If X is a contractible space, then for any $x_0 \in X$, $\pi_1(X, x_0) \simeq \langle 1 \rangle$, the trivial group.*

Corollary. *If $\beta : (X, x_0) \to (Y, y_0)$ is freely nullhomotopic, then the induced isomorphism $\pi_1\beta$ is trivial.*

Proof. Let $k : X \to Y$ the constant map at y. Then $\pi_1 k$ is trivial since $\pi_1 k([f]) = [k \circ f]$ and $k \circ f$ is constant. Then if $\beta \simeq k$ by lemma 4.2.5, there is an isomoprhism ψ with $\psi \circ \pi_1\beta = \pi_1 k$, so that $\pi_1\beta = \phi^{-1}\pi_1 k$, which makes $\pi_1\beta$ trivial. ■

Definition. We call a topological space **simply connected** if it is path connected with trivial fundamental group.

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