Measure Theory

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Contents

Chapter 1

Measure and Measure Spaces

1.1 σ-Algebras

Definition. Let X be a nonempty set. An **algebra** of sets on X is a nonempty collection $\mathcal{A}$ of subsets of X which are closed under finite unions and complements in X. We call $\mathcal{A}$ a **σ-algebra** if it is closed under countable unions.

Lemma 1.1.1. *Let X be a set and $\mathcal{A}$ an algbra on X. Then $\mathcal{A}$ is closed under finite intersections.*

Proof. Let $\{E_\lambda\}$ be a collection of sets of $\mathcal{A}$. Then by finite union $E = \bigcup E_\lambda \in \mathcal{A}$. Then by complements, $X \backslash E = \bigcap X \backslash E_\lambda \in \mathcal{A}$. ∎

Corollary. *σ-algebras are closed under countable disjoint unions.*

Proof. Let $\mathcal{A}$ a σ-algebra, and let $\{E_n\}$ a collection of (not necessarily disjoint) sets in $\mathcal{A}$. Then take

$$F_n = E_n \backslash \Big(\bigcup_{k=1}^{n-1} E_k \Big) \tag{1.1}$$

Then each F_n is a set in $\mathcal{A}$, and are pairwise disjoint. Moreover, $\bigcup E_n = \bigcup F_n$. ∎

Lemma 1.1.2. *Let X be a set, and $\mathcal{A}$ an algebra on X. Then $\emptyset \in \mathcal{A}$ and $X \in \mathcal{A}$.*

Proof. By closure of finite unions, notice that if $E \in \mathcal{A}$, then $E \cup X \backslash E = X \in \mathcal{A}$ lemma **??** gives us that $E \cap X \backslash E = \emptyset \in \mathcal{A}$. ∎

Example 1.1. (1) The collections $\{\emptyset, X\}$ and 2^X are σ-algebras on any set X.

(2) Let X be an uncountable set. Then the collection

$$\mathcal{C} = \{E \subseteq X : E \text{ is countable or } X \backslash E \text{ is countable}\}$$

defines a σ-algebra of sets on X, since countable unions of countable sets are countable, and $\mathcal{C}$ is closed under complements. We call $\mathcal{C}$ the **σ-algebra of countable or co-countable sets**.

Lemma 1.1.3. *Let $\{A_\lambda\}$ be a collection of σ-algebras on a set X. Then the intersection*

$$\mathcal{A} = \bigcap \mathcal{A}_\lambda$$

is a σ-algebra on X. Moreover, if $F \subseteq X$, then there exists a unique smallest σ-algebra containing F; in particular, it is the intersection of all σ-algebras containing F.

Proof. Notice that since each $\mathcal{A}_\lambda$ is a σ-algebra, they are closed under countable unions and complements. Hence by definition, $\mathcal{A}$ must also be closed under countable unions and complements.

Now, let $F \subseteq X$ and let $\{\mathcal{A}_\lambda\}$ be the collection of all σ-algebras containing F. Then the intersection $\mathcal{A} = \bigcap \mathcal{A}_\lambda$ is also a σ-algebra containing F; by above. Now, suppose that there is a smallest σ-algebra $\mathcal{B}$ containing F. Then we have that $\mathcal{B} \subseteq \mathcal{A}$. Now, by definition of $\mathcal{A}$ as the intersection of all σ-algberas containing F, we get that $\mathcal{A} \subseteq \mathcal{B}$; so that $\mathcal{B} = \mathcal{A}$. ■

Definition. Let X be a nonempty set and $F \subseteq X$. We define the σ-algebra **generated** by F to be the smallest such σ-algebra $\mathcal{M}(F)$ containing F.

Lemma 1.1.4. *Let X be a set and let $E, F \subseteq X$. Then if $E \subseteq \mathcal{M}(F)$, then $\mathcal{M}(E) \subseteq \mathcal{M}(F)$.*

Proof. We have that since $E \subseteq \mathcal{M}(F)$, and $\mathcal{M}(E)$ is the intersection of all σ-algebras containing E, then $\mathcal{M}(E) \subseteq \mathcal{M}(F)$. ■

Definition. Let X be a topological space. We define the **Borel σ-algebra** on X to be the σ-algebra $\mathcal{B}(X)$ generated by all open sets of X; that is

$$\mathcal{B}(X) = \mathcal{M}(\mathcal{T})$$

where $\mathcal{T}$ is the topology on X. We call the elements of $\mathcal{B}(X)$ **Borel-sets**

Definition. Let X be a topological space. We call a countable intersection of open sets of X a G_δ**-set** of X. We call a countable union of closed sets of X an F_σ**-set** of X.

Theorem 1.1.5. *The Borel σ-algebra on $\mathbb{R}$, $\mathcal{B}(\mathbb{R})$, is generated by the following.*

(1) All open intervals of $\mathbb{R}$.

(2) All closed intervals of $\mathbb{R}$.

(3) All half-open intervals of $\mathbb{R}$.

(4) All open rays of $\mathbb{R}$.

(5) All closed rays of $\mathbb{R}$.

Definition. Let X_α be a collection of non-empty sets, and let $X = \prod X_\alpha$. If $\mathcal{M}_\alpha$ is a σ-algebra on X_α, then we define the **product σ-algebra** on X to be the smallest σ-algebra generated by all $\pi_\alpha^{-1}(E_\alpha)$, where $E_\alpha \in \mathcal{M}_\alpha$, and $\pi_\alpha : X \to X_\alpha$ is the projection map onto the α-th coordinate. We denote the product σ-algebra by $\bigotimes \mathcal{M}_\alpha$.

Lemma 1.1.6. *Let $\{X_n\}$ be a countable collection of sets, each with a σ-algebra $\mathcal{M}_n$, and let $X = \prod X_n$. Then the product σ-algebra $\bigotimes \mathcal{M}_n$ on X is generated by all $\prod E_n$, where $E_n \in \mathcal{M}_n$.*

Proof. Let $E_n \in \mathcal{M}_n$, then by definition of the projection map, $\pi_n^{-1}(E_n) = \prod E_k$ where $E_k = X_k$ for all $k \neq n$. On the otherhand, we can see that $\prod E_n = \bigcap \pi_n^{-1}(E_n)$. ■

Lemma 1.1.7. *Let $\{X_\alpha\}$ be a collection of sets, each together with a σ-algebra $\mathcal{M}_\alpha$. If each $\mathcal{M}_\alpha$ is generated by some $\mathcal{E}_\alpha$, then$\bigotimes \mathcal{M}_\alpha$ is generated by all $\pi_\alpha^{-1}(E_\alpha)$, where $E_\alpha \in \mathcal{E}_\alpha$.*

Proof. Let $\mathcal{F} = \{\pi_\alpha^{-1}(E_\alpha) : E_\alpha \in \mathcal{E}_\alpha\}$. Then by lemma 1.1.4, $\mathcal{M}(\mathcal{F}) \subseteq \bigotimes \mathcal{M}_\alpha$. On the otherhand, for any α, the collection of all $\pi_\alpha^{-1}(E) \in \mathcal{M}(\mathcal{F})$ is a σ-algebra on X_α, containing $\mathcal{E}_\alpha$; and hence, $\mathcal{M}_\alpha$. That is, $\pi_\alpha^{-1}(E) \in \mathcal{M}(\mathcal{F})$ for all $E \in \mathcal{M}_\alpha$, which gives us the reverse inclusion. ■

Corollary. *If $\{X_\alpha\}$ is a countable collection, theb $\bigotimes \mathcal{M}_\alpha$ is generated by all $\prod E_\alpha$, where $E_\alpha \in \mathcal{E}_\alpha$.*

Lemma 1.1.8. *Let $X_1, \ldots, X_n$ be metric spaces, and $X = \prod_{i=1}^n X_i$ on the product topology. Then*

$$\bigotimes(\mathcal{B}(X_i)) \subseteq \mathcal{B}(X)$$

Moreover, if each X_i is seperable, then equality is established.

Proof. We have that $\bigotimes \mathcal{B}(X_i)$ is generated by each $\pi_i^{-1}(U_i)$, where U_i is an open set in X_i. Since these sets are open, again by lemma 1.1.4, $\bigotimes \mathcal{B}(X_i) \subseteq \mathcal{B}(X)$.

Now, suppose that each X_i is seperable, and let C_i a countable dense set in X_i, and let $\mathcal{E}_i$ be the collection of all open balls in X_i with rational radius r, and center in C_i. Then every open set in X_i is a countable union of members of $\mathcal{E}_i$. Moreover, the set of points in X whose i-th coordinate is in C_i, for all i, is countable dense in X_i. Hence, $\mathcal{B}(X_i)$ is generated by $\mathcal{E}_i$, and since (X) is generated by all $\prod_{i=1}^n E_i$, where $E_i \in \mathcal{E}_i$, we get $\mathcal{B}(X) \subseteq \mathcal{B}(X_i)$, and equality is established. ■

Corollary. $\mathcal{B}(\mathbb{R}^n) = \bigotimes_{i=1}^n \mathcal{B}(\mathbb{R})$.

Definition. We define an **elementary family** on a set X to be a collection $\mathcal{E}$ of subsets of X such that:

(1) $\emptyset \in \mathcal{E}$.

(2) If $E, F \in \mathcal{E}$, then $E \cap F \in \mathcal{E}$.

(3) If $E \in \mathcal{E}$, then $X \backslash E$ is a finite disjoint union of members of $\mathcal{E}$.

Lemma 1.1.9. *Let X be a set and $\mathcal{E}$ an elementary family on X. Let $\mathcal{A}$ be the collection of all finite disjoint unions of members of $\mathcal{E}$. Then $\mathcal{A}$ is an algebra on X.*

Proof. Let $A, B \in \mathcal{E}$, and let $X \backslash B = \bigcup_{i=1}^n C_i$, where each $C_i \in \mathcal{E}$ for all $1 \leq i \leq n$, and are disjoint. Then we have

$$A \cup B = (A \backslash V) \cup B \text{ and } A \backslash B = \bigcup_{i=1}^n (A \cap C_i)$$

so that $A \cup B \in \mathcal{A}$, and $A \backslash B \in \mathcal{A}$. Now, by induction on n, suppose that $A_1, \ldots, A_n \in \mathcal{A}$ are disjoint, then

$$\bigcup_{i=1}^{n+1} A_i = A_{n+1} \cup \bigcup_{i=1}^{n} A_i \backslash A_{n+1}$$

is also a disjoint union. Moreover, we have that if $X \backslash A_n = \bigcup_{i=1}^{N_m} B_m^i$, where the union is disjoint, then

$$X \backslash \Big(\bigcup_{m=1}^n A_m \Big) = \bigcap_{m=1}^n \Big(\bigcup_{i=1}^{N_m} B_m^i \Big)$$

is also a disjoint union. This makes $\mathcal{A}$ an algebra on X. ■

1.2 Measures

Definition. Let X be a set together with a σ-algebra $\mathcal{M}$. We define a **measure** on $\mathcal{M}$ to be a function $\mu : \mathcal{M} \to [0, \infty)$ for which the following hold:

(1) $m(\emptyset) = 0$.

(2) If $\{E_n\}$ is a countable disjoint collection of members of $\mathcal{M}$, then

$$m\Big(\bigcup E_n \Big) = \sum m(E_n) \tag{1.2}$$

We call m a **finitely additive measure** if instead of (2), m satisfies:

(2') If $\{E_i\}_{i=1}^n$ is a finite disjoint collection of members of $\mathcal{M}$, then

$$m\Big(\bigcup_{i=1}^n E_i \Big) = \sum_{i=1}^n m(E_i) \tag{1.3}$$

Definition. We call a set X together with a σ-algebra $\mathcal{M}$ a **measurable space**, and we call the members of $\mathcal{M}$ **measurable sets**. If $m : \mathcal{M} \to [0, \infty)$ is a measure on $\mathcal{M}$, then we call X together with $\mathcal{M}$ a **measure space**.

Definition. Let X together with a σ-algebra be a measure space with measure m. If $m(X) < \infty$, then we call m a **finite measure**, and if $\{E_n\}$ is a covering of X by measurable sets, each with $m(E_n) < \infty$ for all n, then we call m **σ-finite**. We also call the set $E = \bigcup E_n$ **σ-finite**. We call m **semi-finite** if for any measurable set E, of $m(E) = \infty$, there is a measurable set F contained in E such that $0 < m(F) < \infty$.

Lemma 1.2.1. *σ-finite measures are semi-finite.*

Example 1.2. (1) LEt X be a non-empty set, and let $f : X \to [0, \infty)$ be any function on X. Then f defines a measure m on 2^X by the rule

$$m(E) = \sum_{x \in E} f(x)$$

Now, m is semi-finite if, and only if $f(x) < \infty$ for all $x \in X$, and m is σ-finite if, and only if m is semi-finite, and the pre-image $f^{-1}((0, \infty))$ is countable.

(2) Consider the measure m of example (1) above, where $f(x) = 1$ for all $x \in X$. Then we call m the **counting measure** on 2^X. Indeed, observe that

$$m(E) = \sum_{x \in E} 1 = |E|$$

which counts the elements of E.

(3) Consider the measure m of example (1) above, where f is defined for any $x_0 \in X$ to be:

$$f(x) = \begin{cases} 1, & \text{if } x = x_0 \\ 0, & \text{if } x \neq x_0 \end{cases}$$

We call this measure the **Dirichlet measure**.

(4) Let X be an uncountable set, and let $\mathcal{M}$ the σ-algebra of all countable or co-countable sets. Define m on $\mathcal{M}$ by $m(E) = 0$ if E is countable, and $m(E) = 1$ if E is co-countable. Then m defines a measure on $\mathcal{M}$.

(5) Let X be an infinite set, and define m on 2^X by $m(E) = 0$ if E is finite, and $m(E) = \infty$ if E is infinite. Then m is a finitely subadditive measure on 2^X, but not a measure on 2^X.

Theorem 1.2.2. *Let X be a measure space with measure m. The following are true.*

(1) If E and F are measurable with $E \subseteq F$, then

$$m(E) \leq m(F)$$

(2) If $\{E_n\}$ is a countable collection of measurable sets, then

$$m\Big(\bigcup E_n\Big) \leq \sum m(E_n)$$

(3) If $\{E_n\}$ is a countable collection of measurable sets, in which $E_1 \subseteq E_2 \subseteq \ldots$, then

$$m\Big(\bigcup E_n\Big) = \lim_{n \longrightarrow \infty} m(E_n)$$

(4) If $\{E_n\}$ is a countable collection of measurable sets, in which $\ldots, \subseteq E_2 \subseteq E_1$ and $m(E_1) < \infty$, then

$$m\Big(\bigcap E_n\Big) = \lim_{n \longrightarrow \infty} m(E_n)$$

Proof. For the first statement, let $E \subseteq F$ be measurable sets, then observe that

$$m(E) \leq m(E) + m(F \backslash E) = m(E \cup F \backslash E) = m(F)$$

For the second statement, define $F_1 = E_1$, and $F_i = E_i \backslash \bigcup_{i=1}^{i-1} E_i$ for all $i > 1$. Then $\{F_n\}$ is a finite disjoint collection of measurable sets, with $\bigcup_{i=1}^{n} F_i = \bigcup_{i=1}^{n} E_i$. By the above argument, we get

$$m\Big(\bigcup_{i=1}^{n} E_i\Big) = m\Big(\bigcup_{i=1}^{n} F_i\Big) = \sum_{i=1}^{n} m(F_i) \leq \sum_{i=1}^{n} m(E_i)$$

Now, for (3), let $E_0 = \emptyset$, then

$$m(\bigcup E_n) = \sum m(E_i \backslash E_{i-1}) = \lim_{n \longrightarrow \infty} \sum_{i=1}^{n} m(E_i \backslash E_{i-1}) = \lim_{n \longrightarrow \infty} m(E_n)$$

Additionally, consider when the collection $\{E_n\}$ is decreasing with $m(E_1) < \infty$. Take $F_i = E_1 \backslash E_i$, then $\{F_n\}$ is an increasing collection of measurable sets, and hence we apply the above argument. We get that $m(E_1) = m(F_n) + m(E_n)$, and

$$\bigcup F_n = E_1 \backslash \bigcap E_n$$

therefore, we get

$$m(E_1) = m\Big(\bigcap E_n\Big) + \lim_{n \longrightarrow \infty} m(F_i) = m\Big(\bigcap E_n\Big) + \lim_{n \longrightarrow \infty} (m(E_1) - m(E_n))$$

Subtracting $m(E_1)$ from both sides of the equation yields the result. ■

Definition. Let X be a measure space with measure m. We say that a statement about points in X holds **almost everywhere** (with respect to m) if it holds for all $x \in X \backslash E$, where $m(E) = 0$. We call the measure m **complete** if its domain contains all subsets of sets with measure 0.

Theorem 1.2.3. *Let X be a measure space with s-algebra $\mathcal{M}$, and measure m. Let $\mathcal{N} = \{N \in \mathcal{M} : m(N) = 0\}$, and define*

$$\overline{\mathcal{M}} = \{E \cup F : E \in \mathcal{M} \text{ and } F \subseteq N \text{ for some } N \in \mathcal{N}\}$$

Then $\overline{\mathcal{M}}$ is a σ-algebra, and there exists a unique complete measure $\overline{m}$ on $\overline{\mathcal{M}}$.

Proof. Since $\mathcal{M}$ is a σ-algebra, then so is $\mathcal{N}$, moreover, since both are closed under countable unions, so is $\overline{\mathcal{M}}$. Additionally, let $E \cup F \in \overline{\mathcal{M}}$, then we get $E \cup F = (E \cup N) \cap ((X \backslash N) \cup F)$, so that $X \backslash (E \cup F) = X \backslash (E \cup N) \cup N \backslash F$. Since $X \backslash (E \cup N) \in \overline{\mathcal{M}}$, and $N \backslash F \subseteq F$, then we get $X \backslash (E \cup F) \subseteq \overline{\mathcal{M}}$. This makes $\mathcal{M}$ a σ-algebra.

Now, for $E \cup F \in \overline{\mathcal{M}}$, define $\overline{m}$ on $\overline{\mathcal{M}}$ by $\overline{m}(E \cup F) = m(E)$. Then $\overline{m}$ is well defined. Let $E_1 \cup F_1 = E_2 \cup F_2$, where $F_i \subseteq N_i$, with $N_i \in \mathcal{N}$, for $i = 1, 2$. Then $E_1 \subseteq E_2 \cup N_2$, so that $m(E_1) \leq m(E_2) + m(N_1) = m(E_2)$. Similarly, we also get $m(E_2) \leq m(E_1)$.

Now, let $E \in \overline{\mathcal{M}}$, such that $\overline{m}(E) = 0$. Now, we have $E = A \cup B$, where $A \in \mathcal{M}$ and $B \subseteq N$, for some $N \in \mathcal{N}$. Moreover, $\overline{m}(E) = m(A) = 0$. Now, we get $E \subseteq A \cup N \in \mathcal{N}$, since $m(A) = 0$. Now, let $F \subseteq E$. Then observe that $F \subseteq A \cup N$, so that $F \in \mathcal{N}$. Then $F = \emptyset \cup F$, so that $F \in \overline{\mathcal{M}}$. Moreover, $\overline{m}(F) = m(\emptyset) = 0$.

Lastly, suppose there is another complete meaure $\overline{n}$ on $\overline{\mathcal{M}}$ for which $\overline{n}(E \cup F) = m(E)$. Let $E \in \overline{\mathcal{M}}$. Then $E = A \cup B$ where $A \in \mathcal{M}$, and $B \subseteq N$, $N\mathcal{N}$. Then $\overline{n}(E) = \overline{n}(A \cup B) = m(A) \leq m(A) + m(B) = m(A \cup B) = \overline{m}(E)$. By similar reasoning, we get $\overline{m}(E) \leq \overline{n}(E)$, which establishes uniqueness. ∎

Definition. Let X be a measure space with s-algebra $\mathcal{M}$, and measure m. Let $\mathcal{N} = \{N \in \mathcal{M} : m(N) = 0\}$, and define

$$\overline{\mathcal{M}} = \{E \cup F : E \in \mathcal{M} \text{ and } F \subseteq N \text{ for some } N \in \mathcal{N}\}$$

We call $\overline{\mathcal{M}} the \textbf{completion} of$ M$with respect to$m, $and we call the unique complete measure,$ $\overline{m}$ on $\overline{\mathcal{M}}$ the **completion** of m.

1.3 Outer Measures

Definition. Let X be a set. An **outer-measure** on X is a function $m^* : 2^X \to [0, \infty)$ for which the following are true:

(1) $m^*(\emptyset) = 0$.

(2) If $A \subseteq B$, then $m^*(A) \leq m^*(B)$.

(3) If $\{A_n\}$ is a countable collection of subsets of X, then

$$m^*\Big(\bigcup A_n\Big) \leq \sum m^*(A_n)$$

Lemma 1.3.1. *Let X be a set, and $\mathcal{E}$ a collection of subsets of X for which $\in \mathcal{E}$ and $X \in \mathcal{E}$, and let $l : \mathcal{E} \to [0, \infty]$ a function for which $l(\emptyset) = 0$. For any $A \subseteq X$, define*

$$m^*(A) = \inf\Big\{\sum l(E_n) : E_n \in \mathcal{E}, \text{ and } A \subseteq \bigcup E_n\Big\} \tag{1.4}$$

Then m^* defines an outer-measure.

Proof. For all $A \subseteq X$, there is a collection $\{E_n\}$ of sets of $\mathcal{E}$ for which $A \subseteq \bigcup E_n$. Observe first, that since $l(E_n) \geq 0$ for all n, that $\sum l(E_n) \geq 0$. This makes $m^*(A) \geq 0$. Now, choose $E_n = \emptyset$ for all n, and we get $m^*(\emptyset) = 0$.

Now, let $A \subseteq B$ subsets of X, and let $\{E_n\}$ a countable cover of B. Then $\{E_n\}$ is also a countable cover of A. Define then $E = \{\sum l(E_n) : A \subseteq \bigcup E_n\}$ and $F = \{\sum l(E_n) : B \subseteq \bigcup E_n\}$. Since $A \subseteq B$, $F \subseteq E$. Therefore, by least upper bounds, we have $\inf F \leq \inf B$, that is $m^*(A) \leq m^*(B)$.

Lastly, let $\{A_n\}$ be a countable collection of sets of X, and let $A = \bigcup A_n$. Now, if at least one of the $m(A_n) = \infty$, then we are done. Suppose then that $m(A_n) < \infty$ for all n. Now, there exists a cover of A_n, $\{E_{n,k}\}_k$ for which

$$\sum_k l(E_{n,k}) < m^*(A) + \frac{1}{2^k}$$

consider now the countable collection $\{E_{n,k}\}_{n,k} = \bigcup_n \{E_{n,k}\}_k$. Then $\{E_{n,k}\}_{n,k}$ is a countable cover for A, and we get

$$m^*(A) \leq \sum_n \sum_k l(E_{n,k}) < \sum_n m^*(A_n) + \frac{1}{2^k} = \sum_n m^*(A_n) + \varepsilon$$

Take then $\varepsilon > 0$ small, and we get the result. ■

Corollary. *If E is a set of $\mathcal{E}$, then $m^*(E) = l(E)$.*

Proof. Observe that E covers itself, so that $m^*(E) = \inf\{\sum_{i=1}^1 E\} = \inf l(E) = l(E)$. ■

Definition. Let X be a set. We call a subset A of X **m^*-measurable** if for any subset E of X,

$$m^*(E) = m^*(E \cap A) + m^*(E \cap X \backslash A) \tag{1.5}$$

Lemma 1.3.2. *Let X be a set. A subset A of X is m^*-measurable if, and only if*

$$m^*(E) \geq m^*(E \cap A) + m^*(E \cap X \backslash A) \text{ for all } E \subseteq X$$

Theorem 1.3.3 (Carathéodory's Theorem)**.** *Let X be a set, and m^* an outer-measure on X. Then the collection of all m^*-measurable sets forms a σ-algebra. Moreover, m^* is a complete measure on this σ-algebra.*

Proof. Let $\mathcal{M}$ be the collection of all m^*-measurable sets. Observe first that if $A \in \mathcal{M}$, then so is $X \backslash A$, by symetry of equation 1.5. So $\mathcal{M}$ is closed under complements. Now, let $A, B \in \mathcal{M}$. Then we have

$$\begin{aligned} m^*(E) &= m^*(E \cap A) + m^*(E \cap X \backslash A) \\ &= m^*(E \cap A \cap B) + m^*(E \cap A \cap X \backslash B) + m^*(E \cap B \cap X \backslash A) + m^*(E \cap X \backslash A \cap X \backslash B) \end{aligned}$$

Now, since $A \cup B = (A \cap B) \cup (A \cap X \backslash B) \cup (B \cap X \backslash A)$, so by subadditivity, we get

$$m^*(E \cap A \cap B) + m^*(E \cap A \cap X \backslash B) + m^*(E \cap X \backslash A \cap B) \geq m^*(E \cap (A \cup B))$$

i.e. $m^*(E) \geq m^*(E \cap (A \cup B)) + m^*(E \cap X \backslash (A \cup B))$. That is, $A \cup B \in \mathcal{M}$, making $\mathcal{M}$ an algebra.

Now, let $\{A_n\}$ be a countable disjoint collection of m^*-measurable sets, and take $B_n = \bigcup_{i=1}^n A_i$, and take $B = \bigcup B_n$. Then for all $E \subseteq X$

$$\begin{aligned} m^*(E \cap B_n) &= m^*(E \cap B_n \cap A_n) + m^*(E \cap B_n \cap X \backslash A_n) \\ &= m^*(EA_n) + m^*(E \cap B_{n-1}) \end{aligned}$$

an induction argument on the collection $\{B_n\}$ gives us

$$m^*(E \cap B_n) = \sum_{i=1}^{n} m^*(E \cap A_i)$$

therefore

$$m^*(E) = m^*(E \cap B_n) + m^*(E \cap X \backslash B_n) \geq \sum_{i=1}^{n} m^(E \cap A_i) + m^*(E \cap X \backslash B_n)$$

letting $n \to \infty$,

$$m^*(E) \geq \sum m^(E \cap A_n) + m^*(E \cap X \backslash B_n)$$

so that $B \in \mathcal{M}$. Taking $E = B$, we get $m^*(B) = \sum m^*(A_n)$ so that m^* is countably additive, and $\mathcal{M}$ is a σ-algebra.

Finally, let $m^*(A) = 0$, then for any $E \subseteq X$, we have

$$m^*(E) \leq m^*(E \cap A) + m^*(E \cap X \backslash A) = m^*(E \cap X \backslash A) \leq m^*(E)$$

so that $A \in \mathcal{M}$, which makes m^* complete on $\mathcal{M}$. ■

Definition. Let X be a set, and $\mathcal{A}$ an algebra on X. We define a **pre-measure** on $\mathcal{A}$ to be a function $m_0 : \mathcal{A} \to [0, \infty]$ for which

(1) $m_0(\emptyset) = 0$.

(2) If $\{A_n\}$ is a countably disjoint collection of sets in $\mathcal{A}$, for which $\bigcup A_n \in \mathcal{A}$, then

$$m_0\Big(\bigcup A_n\Big) = \sum m_0(A_n) \tag{1.6}$$

Lemma 1.3.4. *Pre-measures on algebras define outer-measures on the overlying sets.*

Proof. Consider the definition of the outer measure m^* from equation 1.4, simply take $l = m_0$, and $\mathcal{E} = \mathcal{A}$. ■

Lemma 1.3.5. *Let X be a set, and $\mathcal{A}$ an algebra on X. If m_0 is pre-measure on $\mathcal{A}$, and the measure m^* is define by*

$$m^*(A) = \inf\Big\{\sum m_0(E_n) : E_n \in \mathcal{A},\ \textit{and } A \subseteq \bigcup E_n\Big\}$$

then the following are true.

(1) $m_0 = m^*$ on $\mathcal{A}$.

(2) Every set in $\mathcal{A}$ is m^*-measurable.

Proof. For (1), suppose that $A \in \mathcal{A}$, and that $A \subseteq \bigcup E_n$ for $E_n \in \mathcal{A}$. Take

$$F_n = A \cap A_n \backslash \Big(\bigcup_{i=1}^{n-1} A_i \Big)$$

then $\{F_n\}$ is a disjoint countable collection of sets of $\mathcal{A}$ for which $A = \bigcup F_n$. Hence

$$mo(A) = \sum m_0(F_n) \leq \sum m_0(E_n)$$

it follows from hypothesis that $m_0(A) \leq m^*(E)$. For the reverse inclusion, simply take $A \subseteq \bigcup E_n$ with $A = E_1$ and $E_n = \emptyset$ for all $n > 1$.

For (2), if $A \in \mathcal{A}$, and $E \subseteq X$, and $\varepsilon > 0$, there is a collection $\{B_n\}$ of sets of $\mathcal{A}$ with $A \subseteq \bigcup B_n$, and

$$\sum m_0(B_n) < m^*(A) + \varepsilon$$

by additivity of m_0 on $\mathcal{A}$, we get

$$m^*(E) + \varepsilon \geq \sum m_0(B_n \cap A) + \sum m_0(B_n \cap X \backslash A) \geq m^*(E \cap A) + m^*(E \cap X \backslash A)$$

∎

Theorem 1.3.6. *Let X be a set, and $\mathcal{A}$ an algerba on X. Let m_0 be a pre-measure on $\mathcal{A}$, and let $\mathcal{M}$ the σ-algebra generated by $\mathcal{A}$. Then there exists a measure m on $\mathcal{M}$ whose restriction to is m_0. Moreover, if n is another measure extending from m_0, then*

$$n(E) \leq m(E) \text{ for all } E \in \mathcal{M}$$

where equality holds when $m(E) < \infty$. Lastly, if m_0 is σ-finite, then m is the unique extension of m_0 to $\mathcal{M}$.

Proof. Define again,

$$m^*(A) = \inf \Big\{ \sum m_0(E_k) : E_k \in \mathcal{A}, \text{ and } A \subseteq \bigcup E_k \Big\}$$

then by Carathéodory's theorem, lemma 1.3.5, the first result follows, since the σ-algebra of all m^*-measurable sets contains $\mathcal{A}$, and as consequence, also contains $\mathcal{M}$.

Now, let $E \in \mathcal{M}$ with $E \subseteq \bigcup A_k$, where $A_k \in \mathcal{A}$. Then

$$n(E) \leq \sum n(A_n) = \sum m_0(A_n)$$

which gives us $n(E) \leq m(E)$. Now, set $A = \bigcup A_n$, and observe that

$$n(A) = \lim_{k \longrightarrow \infty} n\Big(\bigcup_{i=1}^{k} A_i\Big) = \lim_{k \longrightarrow \infty} m\Big(\bigcup_{i=1}^{k} A_i\Big) = m(A)$$

if $m(E) < \infty$, choose A_k such that $m(A) < m(E) + \varepsilon$ for $\varepsilon > 0$. Then $m(A \backslash E) < \varepsilon$, and

$$m(E) \leq m(A) = n(A) = n(E) + n(A \backslash E) \leq n(E) + m(A \backslash E) \leq n(E) + \varepsilon$$

taking ε small, we get $n(E) = m(E)$.

Finally, suppose that m_0 is σ-finite, and let $X = \bigcup A_k$ for s me0disjoint collection $A_n\}$, then $m_0 m_0) < \infty$. Then for every $E \in \mathcal{M}$,

$$m(E) = \sum m(E \cap A_k) = \sum n(E \cap A_k) = n(E)$$

so that $m = n$, making m unique. ∎

1.4 Borel Measures on $\mathbb{R}$

Definition. We call measures defined on the σ-algebra $\mathcal{B}(\mathbb{R})$ of all Borel sets of $\mathbb{R}$ **borel measures**. If m is a finite Borel meausre on $\mathbb{R}$, we define the **distribution function** of m to be the function $F : \mathbb{R} \to \mathbb{R}$ by the rule

$$F(x) = m((-\infty, x])$$

Lemma 1.4.1. *Let m be a finite Borel measure on $\mathbb{R}$. Then the distribution function o of m is an increasing, right-continuous function. Moreover, if $b > a$ are extended real numbers, then $m((a,b]) = F(b) - F(a)$.*

Proof. By the monotonicity of m, F is an increasing function. Now, let $\{x_n\}$ a sequence of points for which $\{x_n\} \to x$ from the right. For $x \geq 0$, the collection $\{(-\infty, x_n]\}$, then by continuity from below, we haev

$$m\Big(\bigcup(-\infty, \times_n]\Big) = \lim_{n \to \infty} m((-\infty, x_n]) = \lim_{x_n \to x+} m((-\infty, x_n]) = m((-\infty, x])$$

That is $\lim F(x_n) = F(x)$ as $x_n \to x+$. A similar argument holds for $x < 0$, using continuity from above. Lastly, observe that

$$(-\infty, b] = (-\infty, a] \cup (a, b] \tag{1.7}$$

■

Lemma 1.4.2. *Let $F : \mathbb{R} \to \mathbb{R}$ an increasing right-continuous function. If $\{(a_i, b_i]\}_{i=1}^n$ is a finite collection of disjoint half-open intervals, and m_0 is defined by*

$$m_0\Big(\bigcup_{i=1}^n (a_i, b_i]\Big) = \sum_{i=1}^n F(b_i) - F(a_i) \text{ and } m_0(\emptyset) = 0 \tag{1.8}$$

then m_0 is a pre-measure on the algebra of all finite unions of half-open intervals.

Proof. Denote the algebra of all finite unions of half-open intervals by $\mathcal{A}$. Notice then by theorem 1.1.5, that $\mathcal{B}(\mathbb{R})$ is the smallest σ-algebra generated by $\mathcal{A}$. Now, let $\{(a_i, b_i]\}_{i=1}^n$ a finite disjoint collection of half-open intervals, and take $(a, b] = \bigcup_{i=1}^n (a_i, b_i]$ Then $(a, b]$ is partitioned by the points $P = \{a = a_1 < b_1 = a_1 < b_2 = \cdots < b_n = b \ldots\}$ Therefore

$$\sum_P F(b_i) - F(a_i) = \sum_{i=1}^n F(b_i) - F(a_i) = F(b) - F(a)$$

That is, if $\{I_j\}_{j=1}^n$ and $\{J_i\}_{i=1}^n$ are finite collections of disjoint half-open intervals, where $\bigcup_{j=1}^n I_j = \bigcup_{i=1}^n J_i$, then

$$\sum_{j=1}^n m_0(I_j) = \sum_{i=1,j=1}^{n,n} m_0(I_j \cap J_i) = \sum_{i=1}^n m_0(J_i)$$

this makes m_0 well defined, and finitely additive, by construction.

Now, consider $\{I_n\}$ a countable collection of disjoint half-open intervals. And let $I = \bigcup I_n$. Then $I \in \mathcal{A}$, and $m_0(I) = \sum m_0(I_n)$. Now, since I is a finite union of disjoint half-open intervals, partition $\{I_n\}$ into finitely many subcollections $\{I_{n_k}\}$ for which $\bigcup I_{n_k}$ is a single half-open interval. Then $I = (a, b]$ a single half-open interval. Thus, by the finite additivity of m_0, we get

$$m_0(I) = m_0\Big(\bigcup_{j=1}^{n} I_j\Big) + m_0(I\backslash\Big(\bigcup_{j=1}^{n-1} I_j\Big)) \geq m_0(I) = m_0\Big(\bigcup_{j=1}^{n} I_j\Big) = \sum_{i=1}^{n} m_0(I_j)$$

Taking $n \to \infty$, we get $m_0(I) \geq \sum m_0(I_n)$

Now, suppose that $a, b \in \mathbb{R}^\infty$ are finite, and take $\varepsilon > 0$. By hypothesis, we get F is right-continuous, so there is a $\delta > 0$ for which $F(a+\delta) - F(a) < \varepsilon$. If $I_n = (a_n, b_n]$, there is a $\delta_n > 0$ for which $F(b_n + \delta_n) - F(b_n) < \frac{\varepsilon}{2^{n+1}}$. Now, $[a+\delta, b]$ is compact, by the finite collection $\{(a_i, b_i + \delta_i)\}_{i=1}^{N}$. Now, refine this subcover by discaring any $(a_i, b_i + \delta_i)$ contained in another of that cover, and reindex i to j by letting $b_j \in (a_{j+1}, b_{j+1} + \delta_{j+1})$ for all $1 \leq j \leq N-1$. Then

$$\begin{aligned}
m_0(I) &= F(b) - F(a+\delta) + \varepsilon \\
&\leq F(b_N + \delta_N) - F(a_1) + \varepsilon \\
&= F(b_N + \delta_N) - F(a_N) + \sum_{j=1}^{N-1} F(a_{j+1}) - F(a_j) + \varepsilon \\
&\leq F(b_N + \delta_N) - F(a_N) + \sum_{j=1}^{N-1} F(b_j + \delta_j) - F(a_j) + \varepsilon \\
&< \sum_{i=1}^{N} F(b_i) + \frac{\varepsilon}{2^{i+1}} - F(a_i) + \varepsilon \qquad\qquad < \sum m_0(I_n) + \varepsilon
\end{aligned}$$

Since a and b are finite, taking ε small enough gives us the m_0. as a pre-measure on $\mathcal{A}$. Now, if $a = -\infty$, for all $M < \infty$, there is a cover $\{(a_i, b_i + \delta_i)\}$ of $[-M, b]$, so that by the pervious argument, $F(b) - F(-M) \leq \sum m_0(I_n) + \varepsilon$. If $b = \infty$, then for all $M < \infty$, by a similar argument, $F(M) - F(a) \leq \sum m_0(I_n) + \varepsilon$. Taking ε small then gives us the same result. ■

Theorem 1.4.3. *If $F : \mathbb{R} \to \mathbb{R}$ is any increasing right-continuous function, then there exists a unique Borel measure on $\mathbb{R}$, m_F such that $m((a, b]) = F(b) - F(a)$ for all $a, b \in \mathbb{R}^\infty$. Moreover if G is another increasing right-continuous function, then $m_F = m_G$ if, and only if $F - G$ is a constant function. Lastly, if m is a Borel measure on $\mathbb{R}$, finite on all bounded Borel sets of $\mathbb{R}$, and if F is defined by*

$$F(x) = \begin{cases} m((0, x]), & \text{if } x > 0 \\ 0, & \text{if } x = 0 \\ -m((0, x]), & \text{if } x < 0 \end{cases} \tag{1.9}$$

Then F is the distribution function associated with m.

Proof. We have by lemma 1.4.2, that F defines pre-measures on the σ-algebra $\mathcal{A}$ of finite unions of open half-intervals, and moreover that $m_F = m_G$ if, and only if $F - G$ is constant. Moreover, each m_F is σ-finite since $\mathbb{R} = \bigcup_{-n}^{n}(n, n+1]$.

Now, let m be a Borel measure on $\mathbb{R}$, and define F by equation 1.9. By lemma 1.4.1, F is increasing and right continuous. Lastly, since $m = m_F$ on $\mathcal{A}$, and $\mathcal{B}(\mathbb{R})$ is generated by $\mathcal{A}$, then $m = m_F$ on all $\mathcal{B}(\mathbb{R})$. ■

Definition. We call Borel measures on $\mathbb{R}$, with dsitribution functions defined by

$$F(x) = \begin{cases} m((0,x]), \text{ if } x > 0 \\ 0, \text{ if } x = 0 \\ -m((0,x]), \text{ if } x < 0 \end{cases}$$

Lebesgue-Stieltjes measures.

Lemma 1.4.4. *Let m be a Lebesgue-Stieljes measure on $\mathbb{R}$. Then for any m-measurable set E,*

$$m(E) = \inf\left\{\sum m((a_n, b_n)) : E \subseteq \bigcup (a_n, b_n)\right\} \tag{1.10}$$

Proof. Observe that for all m-measurable sets E, that

$$m(E) = \inf\left\{\sum F(b_k) - F(a_k) : E \subseteq \bigcup (a_n, b_n)\right\}$$

Let

$$n(E) = \inf\left\{\sum m((a_k, b_k)) : E \subseteq \bigcup (a_k, b_k)\right\}$$

and let $E \subseteq \bigcup (a_k, b_k)$. Then $\{(a_k, b_k)\}$ is a countable collection of disjoint half-open intervals I_j^l. Specifically, $I_k^l = (c_j^l, c_j^{l+1}]$, where $\{c_j^l\}$ is an increasing sequence, with $c_j^1 = a_j$, and $\{c_j^l\} \to b_j$ as $l \to \infty$. Then

$$E \subseteq \bigcup_{j,l} I_j^l$$

We get

$$\sum m((a_k, b_k)) = \sum_{j,l} m(I_j^l) \geq m(E)$$

so that $n(E) \geq m(E)$. On the other hand, letting $\varepsilon > 0$, there exists a countable collection $\{(a_k, b_k]\}$ of disjoint half-open intervals where $E \subseteq \bigcup (a_k, b_k]$, and

$$\sum m((a_k, b_k)) \leq m(E) + \varepsilon$$

Thus, for every j, there is a δ_j for which $F(b_j + \delta_j) - F(b_j) < \frac{\varepsilon}{2^{j+1}}$. Therefore $E \subseteq \bigcup ((a_j, b_j + \delta_j))$ and

$$\sum m((a_j, b_j + \delta_j)) \leq \sum m((a_k, b_k]) + \varepsilon \leq m(E) + \varepsilon$$

so that $n(E) \leq m(E)$. ■

Theorem 1.4.5. *Let m be a Lebesgue-Stieltjes measure on $\mathbb{R}$. If E is m-measurable, then*

$$m(E) = \inf\{m(U) : U \text{ is open and } E \subseteq U\} = \sup\{m(K) : K \text{ is compact and } K \subseteq\} \tag{1.11}$$

Proof. By lemma 1.4.4, for every $\varepsilon > 0$, there is a countable collection $\{(a_n, b_n)\}$ of disjoint intervalscovering E, for which

$$\sum m((a_n, b_n)) \leq m(E) + \varepsilon$$

Let $U = \bigcup (a_n, b_n)$, then U is open, and $E \subseteq U$, with $m(U) \leq m(E) + \varepsilon$. On the other hand, by monotonicity of m, $m(E) \leq m(U)$, so we get the first equality.

Now, suppose that E is bounded. If E is closed, then E is compact, and there is nothing else to prove. Otherwise, let $\varepsilon > 0$ and choose an open set U contained in $(\operatorname{cl} E)\backslash E$ (where $\operatorname{cl} E$ is the topological closure of E) such that $m(U) \leq m((\operatorname{cl} E)\backslash E) + \varepsilon$. Now, let $K = (\operatorname{cl} E)\backslash U$. Then K is compact, and $K \subseteq E$, and $m(K) = m(E) - m(E \cap U) = m(E) - m(U) - m((U)\backslash E) \geq m(E) - m(U) + m((\operatorname{cl} E)\backslash E) \geq m(E) - \varepsilon$

Now, if E is unbounded, let $E_n = E \cap (n, n+1]$. Then by the preceding argument, for every $\varepsilon > 0$, there is a K_n compact, contained in E_n for which $m(K_n) \geq m(E_n) + \frac{\varepsilon}{2^{|n|}3}$. Let $H_n = \bigcup_{-n}^{n} K_n$. Then H_n is compact, and contained in E, and

$$m(H_n) = m\Big(\bigcup_{i=-n}^{n} E_i\Big) - \varepsilon$$

By continuity from above, we are done. ∎

Theorem 1.4.6 (Inner and Outer Approximation)**.** *Let m be a Lebesgue-Stieltjes measure on $\mathbb{R}$, and let E any subset of $\mathbb{R}$ of finite m-measure. Then the following are equivalent.*

(1) E is m-measurable.

(2) There exists a G_δ-set V for which $E = V\backslash N_1$, and $m(N_1) = 0$.

(3) There exists an F_σ-set H for which $E = H \cup N_2$, and $m(N_2) = 0$.

Proof. Since m is complete on all m-measurable sets, statements (2) and (3) imply (1). Now, let E be m-measurable, with $m(E) < \infty$. Choose U_n open, containing E for all n, and choose K_n compact, conatined in E for all n, such that,

$$m(U_n) - \frac{1}{2^n} \leq m(E) \leq m(K_n) - \frac{1}{2_n}$$

Let $V = \bigcap U_n$, and $H = \bigcup K_n$. Then V is a G_δ-set, H is an F_σ-set, and $H \subseteq E \subseteq V$. Moreover, $m(V) = m(H) = m(E) < \infty$, so that $m(V\backslash E) = m(E\backslash H) = 0$. ∎

Lemma 1.4.7. *Let m be a Lebesgue-Stieltjes measure. If E is m-measurable, of finite m-measure, then for every $\varepsilon > 0$, there exists a finite collection $\{I_j\}_{j=1}^n$, such that*

$$m(E\backslash A \cup A\backslash E) < \varepsilon \text{ where } A = \bigcup_{j=1}^{n} I_j \tag{1.12}$$

Definition. We define the **Lebesgue measure** on $\mathbb{R}$ to be the complete Lebesgue-Stieljes measure m associated to the distribution funtion $l(x) = x$. We call all m-measurable sets **Lebesgue measurable**, and denote the domain of m by $\mathcal{L}$.

Theorem 1.4.8. *If E is Lebesgue measurable, then so is $E + s$ and rE, for all $s, r \in \mathbb{R}^\infty$, and where $E + s = \{x + s : x \in E\}$, and $rE = \{rx : x \in E\}$. Moreover*

$$m(E + s) = m(E) \text{ and } m(rE) = |r|m(E)$$

Theorem 1.4.9. *Countable sets of $\mathbb{R}$ have Lebesgue measure 0.*

Example 1.3. Let m be the Lebesgue measure on $\mathbb{R}$.

(1) $m(\mathbb{Q}) = 0$, since $\mathbb{Q}$ is countable.

(2) Here is a pathological example where topologically "large" sets can be measured to be as small as one likes, and where topologically "small" sets can be measured as large as one likes. Let $\{r_n\}$ be an enumeration of $\mathbb{Q}$ in the interval $[0,1]$. Take $\varepsilon > 0$, and let $I_r = (r - \frac{1}{2^n}, r + \frac{1}{2^n})$ the open interval centered around r of length $\frac{1}{2^n}$. Then take $U = (0,1) \cap \bigcup I_r$. Then U is open and dense in $[0,1]$, but $m(U) \leq \sum \frac{1}{2^n} < \varepsilon$. Now, let $K = [0,1] \backslash U$. Then K is closed and nowhere dense in $[0,1]$, however $m(K) \geq 1 - \varepsilon$. Notice however that non-empty open sets of $\mathbb{R}$ cannot have Lebesgue measure 0.

1.5 The Cantor Set and the Cantor-Lebesgue Function

Definition. We define the **Cantor set** $\mathcal{C}$ to be the intersection

$$\mathcal{C} = \bigcap C_k$$

where $\{C_k\}$ is a descending collection of closed sets in which for each $k \in \mathbb{Z}^+$, C_k is the disjoint union of $2^k - 1$ closed intervals in $\mathbb{R}$ each of length $\frac{1}{3^k}$.

Theorem 1.5.1. *The Cantor set is a compact and uncountable set of Lebesgue measure $m(\mathcal{C}) = 0$.*

Proof. Since $\mathcal{C}$ is an arbitrary intersection of closed sets in $\mathbb{R}$, $\mathcal{C}$ is closed in $\mathbb{R}$. We also have that $\mathcal{C} \subseteq [0,1]$, which is compact. Therefore, by the theorem of Heine-Borel, $\mathcal{C}$ is compact. Moreover, by the definition of each C_k, C_k is Lebesgue measurable and

$$m(C_k) = \left(\frac{2}{3}\right)^k \text{ for each } k \in \mathbb{Z}^+$$

Then $\mathcal{C}$ is also Lebesgue measurable, and we have by monotonicity of the Lebesgue measure

$$m(\mathcal{C}) \leq m(C_k) = \left(\frac{2}{3}\right)^k$$

Taking $k \to \infty$, gives us that $m(\mathcal{C}) = 0$.

Now, suppose that $\mathcal{C}$ is countable, and let $\mathcal{C} = \{c_k\}$ be an enumeration of $\mathcal{C}$. Now, one of the disjoint intervals whose union is C_1 fails to contain c_1; call it F_1. Proceeding, one of the disjoint intervals whose union is F_1 fails to contain c_2; call it F_2. Proceeding inductively, we get a descending collection $\{F_k\}$ of closed sets, for which $c_k \notin F_k$ for each $k \in \mathbb{Z}^+$. Now, by the nested set theorem, the intersection

$$F = \bigcap F_k$$

is nonempty. Moreover, $F \subseteq \mathcal{C}$. This implies that there is an element $x \in \mathcal{C}$ which is not equal to any c_k; but $\{c_k\}$ is an enumeration of $\mathcal{C}$, which is absurd! Therefore, $\mathcal{C}$ fails to be countable. ∎

Corollary. *$\mathcal{C}$ is perfect.*

Proof. Let $x \in \mathcal{C}$, so that $x \in C_k$ for all k. Now, observe by construction that $\mathcal{C}$ contains all the endpoints of all subintervals of each C_k; i.e. contains the points $\{0, 1, \frac{1}{3}, \frac{2}{3}, \dots\}$. Take $\varepsilon > 0$, and choose k large enough so that $\frac{1}{3^k} < \varepsilon$. Since $x \in C_k$, $x \in [a, b]$, where a, b are one of the aforementioned endpoints. Since $m([a, b]) = \frac{1}{3^k} < \varepsilon$, we get $|x - a| < \varepsilon$, and $|x - b| < \varepsilon$. That is, there is some $y \in \mathcal{C}$ for which $y \neq x$, and $y \in (x - \varepsilon, x + \varepsilon) \cap \mathcal{C}$. Therefore, $\mathcal{C}$ contains no isolated points. Since $\mathcal{C}$ is also closed, $\mathcal{C}$ is perfect. ∎

Corollary. *$\mathcal{C}$ is nowhere dense, and totally disconnected.*

Proof. Since $\mathcal{C}$ has Lebesgue measure 0, it cannot contain any open set (see example 1.3(2)). Hence $\mathcal{C} = \emptyset$, so that $\mathcal{C}$ is nowhere dense.

Let $x, y \in \mathcal{C}$, and suppose without loss of generality that $x < y$. Now, suppose also that $y - x > \frac{1}{3^k}$, so that x and y belong to completely different closed intervals in some C_k of $\mathcal{C}$. Now, let $z \notin C_k$ for which $x < z < y$. Then there is a seperation of the set $\{x, y\}$ into

$$\{x, y\} = (\{x, y\} \cap (-\infty, z)) \cup (\{x, y\} \cap (z, \infty))$$

so that $\{x, y\}$ is disconnected. Now, if $S \subseteq \mathcal{C}$ with $|S| > 2$, then the previous argument follows. Taking $x, y \in S$, where $x < y$, there is a z such that $x < z < y$. Then $S = (S \cap (-\infty, z)) \cup (S \cap (z, \infty))$ is a seperation, making $\mathcal{C}$ totally disconnected. ∎

Definition. Let $\mathcal{C}$ be the Cantor set, and define U_k such that $C_k = [0, 1] \backslash U_k$, and define $\mathcal{U} = \bigcup U_k$; so that $[0, 1] = \mathcal{C} \cup \mathcal{U}$. Fix $k \in \mathbb{Z}^+$ and define

$$\phi : U_k \to \mathbb{R}$$

to be the increasing function which is constant on each of the $2^k - 1$ intervals of U_k, and whose image is

$$\phi([0, 1]) = \left\{ \frac{1}{2^k}, \frac{2}{2^k}, \dots, \frac{2^k - 1}{2^k} \right\}$$

We define the **Cantor-Lebesgue function** to be the extension Φ of ϕ to $[0, 1]$; defined in terms of $\mathcal{C}$ to be

$$\Phi(0) = \phi(0) = 0 \text{ and } \Phi(x) = \sup \{\phi(t) : t \in \mathcal{U} \cap [0, x]\} \text{ for all } x \in \mathcal{C} \backslash \{0\}$$

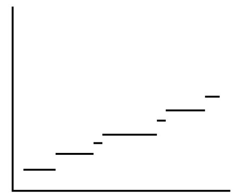

Figure 1.1: The Canotr Lebesgue Function on U_3.

Theorem 1.5.2. *The Cantor-Lebesgue function is an increasing continuous finction mapping* $[0, 1]$ *onto* $[0, 1]$*, and differentiable on* $\mathcal{U}$ *with*

$$\Phi'(x) = 0 \textit{ on } \mathcal{U} \textit{ where } m(\mathcal{U}) = 1$$

Proof. Since ϕ is an increasing fucntion, and Φ is the extension of ϕ to $[0, 1]$, then Φ must also be increasing. Now, ϕ is continuous at each point of $\mathcal{U}$, since it is constant on each U_k. Now, consider $x_0 \in \mathcal{C}$ with $x_0 \neq 0, 1$. Then $x_0 \notin U_k$ for all $k \in \mathbb{Z}^+$. So for k large enought, x_0 lies in between consecutive intervals of U_k. Choose a_k to be the lowerbound of the lower interval, and b_k the upperbound of the upper interval. Then by definition of ϕ, we get

$$a_k < x_0 < b_k \text{ and } \phi(b_k) - \phi(a_k) = \frac{1}{2^k}$$

Since k is arbitrarily large, Φ fails to have a jump discontinuity at x_0, which is the only possible discontinuity it can have. Therefore Φ is continuous at x_0. A similar argument shows that Φ is continuous at $x_0 = 0, 1$.

Now, since ϕ is constant on each U_k, ϕ is differentiable on U_k and has $\phi' = 0$ on each U_k, hence $\phi' = 0$ on $\mathcal{U}$. Since $m(\mathcal{C}) = 0$ and $[0, 1] = \mathcal{C} \cup \mathcal{U}$, then $m(\mathcal{U}) = 1$. Since Φ is an extension, we get that

$$\Phi' = 0 \text{ on } \mathcal{U} \text{ where } m(\mathcal{U}) = 1$$

Finally, since $\Phi(0) = 0$ and $\Phi(1) = 1$, and Φ is increasing, by the intermediate value theorem $\Phi([0, 1]) = [0, 1]$. ∎

Lemma 1.5.3. *Let* Φ *be the Cantor-Lebesgue function, and define* $\Psi : [0, 1] \to \mathbb{R}$ *by*

$$\Psi(x) = \Phi(x) + x \textit{ for all } x \in [0, 1]$$

Then Ψ *is strictly increasing, and maps* $[0, 1]$ *onto* $[0, 2]$*. Moreover,* Ψ *maps the Cantor set onto a set of positive Lebesgue measure, and maps a measurable subset of the Cantor set onto a nonmeasurable set.*

Proof. Since Ψ is the sum of the strictly increasing function $f(x) = x$ and the increasing function $\Phi(x)$, Ψ is strictly increasing. Moreover, Ψ is continuous since it is the sum of continuous functions, and $\Psi(0) = 0$, and $\Psi(1) = 2$, so by the intermediate value theorem, $\Psi([0, 1]) = [0, 2]$. Finally, notice that since Φ and $f(x) = x$ are 1–1, then Ψ is also 1–1. Therefore Ψ has a continuous inverse Ψ^{-1}. This makes $\Psi(\mathcal{C})$ closed, and $\Psi(\mathcal{U})$ open. Therefore both $\Psi(\mathcal{C})$ and $\Psi(\mathcal{U})$ are measurable.

Now, let $\{I_k\}$ be the collection of intervals removed from $[0,1]$ to form $\mathcal{C}$; i.e.

$$\mathcal{U} = \bigcup I_k$$

Since Φ is constant on each of these intervals, we get that Ψ maps I_k onto the translate $I_k + x$. Since Ψ is 1–1, we have that $\{\Psi(I_k)\}$ is a disjoint collection of measurable sets. Therefore, by countable additivity

$$m\Big(\bigcup \Psi(I_k)\Big) = \sum m(\Psi(I_k)) = \sum m(I_k + x) = \sum m(I_k) = m(\mathcal{U}) = 1$$

Since $[0,2] = \Psi(\mathcal{C}) \cup \Psi(\mathcal{U})$, we get $\Psi(\mathcal{C}) = 1$.

Now, by Vitali's theorem, there exists a nonmeasurable subset $W \subseteq \Psi(\mathcal{C})$. Notice then that $\Psi^{-1}(W) \subseteq \mathcal{C}$ is Lebesgue measurable of measure $m(^{-1}(W)) = 0$, since $m(\mathcal{C}) = 0$. That is, we have mapped a measurable subsete of $\mathcal{C}$ to a nonmeasurable set. This concludes the proof. ■

Theorem 1.5.4. *There exists a measurable subset of the Cantor set which is not Borel.*

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