Complex Analysis

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Chapter 1

The Complex Numbers

1.1 The Field of Complex Numbers

Definition. We define the set of **complex numbers** to be the collection of all ordered pairs of real numbers $\mathbb{C} = \{(a,b) : a,b \in \mathbb{R}\}$ together with the binary operations $+$ and $\cdot$ of **complex addition**, and **complex multiplication**, respectively defined by the rules

$$(a,b)+(c,d) = (a+c, b+d)$$
$$(a,b)(c,d) = (ac-bd, bc+ad)$$

Theorem 1.1.1. *The set of complex numbers $\mathbb{C}$ forms a field together with complex addition and complex multiplication.*

Corollary. *$\mathbb{C}$ is a field extension of the real numbers $\mathbb{R}$.*

Proof. The map $a \to (a,0)$ from $\mathbb{R} \to \mathbb{C}$ defines an imbedding of $\mathbb{R}$ into $\mathbb{C}$. ■

Definition. We define the element $i = (0,1)$ of $\mathbb{C}$ so that $i^2 = -1$, and the polynomial z^2+1 has as root i. We write $(a,b) = a+ib$. If $z = a+ib$, we call a the **real part** of z, and b the **imaginary part** of z and write $\operatorname{Re} z = a$ and $\operatorname{Im} z = z$.

Definition. Let $z = a+ib \in \mathbb{C}$. We define the **norm** (or **modulus**) of z to be $|z| = \sqrt{a^2+b^2}$. We define the complex **conjugate** of z to be $\overline{z} = a-ib$.

Lemma 1.1.2. *For every $z \in \mathbb{C}$, $|z|^2 = z\overline{z}$.*

Proof. Let $z = a+ib$. Then $\overline{z} = a-ib$, and so $z\overline{z} = (a+ib)(a-ib) = a^2+b^2 = (\sqrt{a^2+b^2})^2 = |z|^2$. ■

Corollary. *If $z \neq 0$, then $z^{-1} = \frac{1}{z} = \frac{\overline{z}}{|z|^2}$.*

Proof. The relation follows from above, and it remains to show that it is indeed the inverse element. Notice that if $z \in \mathbb{C}$ is nonzero, then $z\frac{\overline{z}}{|z|^2} = \frac{z\overline{z}}{|z|^2} = \frac{|z|^2}{|z|^2} = 1$. ■

Example 1.1. (1) Let $z = a+ib$. Then we get that $\frac{1}{z} = \frac{\overline{z}}{|z|}$ has real part $\operatorname{Re}\frac{1}{z} = \frac{a}{a^2+b^2}$ and imaginary part $\operatorname{Im}\frac{1}{z} = -\frac{b}{a^2+b^2}$.

(2) Let $z = a + ib$, and $c \in \mathbb{R}$. Then $\frac{z-c}{z+c} = \operatorname{Re} \frac{z-c}{z+c}$, so $\operatorname{Im} \frac{z-c}{z+c} = 0$.

(3) Let $z = a + ib$, then $z^3 = a^3 - 3ab^2 + i(3a^2b - b^3)$ So that $\operatorname{Re} z^3 = a^3 - 3ab^2$ and $\operatorname{Im} z = 3a^2b - b^3$.

(4) $\frac{3+i5}{1+i7} = \frac{19}{25} - i\frac{18}{25}$.

(5) $(-\frac{1}{2} + i\frac{\sqrt{3}}{2})^3 = 1$, and hence $(-\frac{1}{2} + i\frac{\sqrt{3}}{2})^6 = 1$.

(6) Notice that $i^n = 1, i, -1, -i$ whenever $n \equiv 0 \mod 4$, $n \equiv 1 \mod 4$, $n \equiv 2 \mod 4$, and $n \equiv 3 \mod 4$. respectively.

(7) $|-2 + i| = \sqrt{5}$, and $|(2 + i)(4 + i3)| = |5 + i10| = 5\sqrt{5}$.

Lemma 1.1.3. *The following are true for all $z, w \in \mathbb{C}$.*

(1) $\operatorname{Re} z = \frac{1}{2}(z + \overline{z})$ *and* $\operatorname{Im} z = \frac{1}{2i}(z - \overline{z})$.

(2) $\overline{(z + w)} = \overline{z} + \overline{w}$ *and* $\overline{zw} = \overline{z}\,\overline{w}$.

(3) $|\overline{z}| = |z|$.

Proof. Let $z = a + ib$ and $w = c + id$. Then notice that

$$\frac{(a + ib) + (a - ib)}{2} = \frac{2a + (ib - ib)}{2} = \frac{2a}{2} = a = \operatorname{Re} z$$

and

$$\frac{(a + ib) - (a - ib)}{2i} = \frac{(a - a) + 2ib}{2} = \frac{2ib}{2i} = b = \operatorname{Im} z$$

Moreover

$$\overline{(a + ib) + (c + id)} = \overline{(a + c) + i(b + d)} = (a + c) - i(b + d) = (a - ib) + (c - id)$$

And

$$\overline{(a + ib)(c + id)} = \overline{(ac - bd) + i(bc + ad)} = (ac - bd) - i(bc + ad) = (a - ib)(c - id)$$

so that $\overline{z + w} = \overline{z} + \overline{w}$ and $\overline{zw} = \overline{z}\,\overline{w}$.

Now, we have that $|zw|^2 = (zw)\overline{zw} = (zw)(\overline{z}\,\overline{w}) = (z\overline{z})(w\overline{w}) = |z|^2|w|^2$. Taking square roots, we get the result

$$|zw| = |z||w|$$

Finally, notice that $|z|^2 = z\overline{z} = \overline{z}\,\overline{\overline{z}} = |\overline{z}|$. ■

Corollary. *The following are also true; provided $w \neq 0$.*

(1) $\overline{(\frac{z}{w})} = \frac{\overline{z}}{\overline{w}}$.

(2) $|\frac{z}{w}| = \frac{|z|}{|w|}$

Corollary. *If $z = z_1 + \cdots + z_n$, and $w = w_1 \dots w_n$, with $z_i, w_i \in \mathbb{C}$ for all $1 \leq i \leq n$, then*

(1) $\overline{z} = \overline{z_1} + \cdots + \overline{z_n}$.

(2) $|w| = |w_1| \ldots |w_n|$.

Proof. We prove both results by induction on n. For $n = 2$, we have already shown that $\overline{z} = \overline{z_1} + \overline{z_2}$ and $|w| = |w_1||w_2|$. Now, for all $n \geq 2$, suppose that both

$$\overline{z} = \overline{z_1} + \cdots + \overline{z_n}$$
$$|w| = |w_1| \ldots |w_n|$$

Then let $z' = z + z_{n+1}$ and $w' = ww_{n+1}$ for $z_{n+1}, w_{n+1} \in \mathbb{C}$. Then we have that

$$z' = z + z_{n+1} = z_1 + \cdots + z_n + z_{n+1}$$
$$w' = ww_{n+1} = w_1 \ldots w_n w_{n+1}$$

so by the induction hypothesis, we have

$$\overline{z'} = \overline{(z + z_{n+1})} = \overline{z} + \overline{z_{n+1}} = \overline{z_1} + \cdots + \overline{z_n} + \overline{z_{n+1}}$$

and that

$$|w'| = |ww_{n+1}| = |w||w_{n+1}| = |w_1| \ldots |w_n||w_{n+1}|$$

which completes the proof. ■

Lemma 1.1.4. *Let $z \in \mathbb{C}$. Then z is a real number if, and only if $z = \overline{z}$.*

Proof. If z is real, then $z = a + i0$, for some $a \in \mathbb{R}$, and hence $\overline{z} = a - i0 = z$. COnversely, suppose that $z = \overline{z}$. Then we have

$$\operatorname{Re} z = \frac{1}{2}(z + \overline{z}) = \frac{1}{2}(z + z) = z$$

so z has only a real part, and hence must be a real number. ■

Lemma 1.1.5. *The following are true for all $z, w \in \mathbb{C}$.*

(1) $|z + w|^2 = |z|^2 + 2 \operatorname{Re} z\overline{w} + |w|^2$.

(2) $|z - w|^2 = |z|^2 - 2 \operatorname{Re} z\overline{w} + |w|^2$.

(3) $|z + w|^2 + |z - w|^2 = 2(|z|^2 + |w|^2)$.

Proof. We first notice that $|z+w|^2 = (z+w)\overline{(z + w)} = (z+w)(\overline{z}+\overline{w}) = z\overline{z}+z\overline{w}+w\overline{z}+w\overline{w} = |z|^2 + z\overline{w} + w\overline{z} + |w|^2$. Now, let $z = a + ib$ and $w = c + id$. Then we have

$$(a + ib)(c - id) = (ac + bd) - i(ad - bc)$$
$$(c + id)(a - ib) = (ac + bd) + i(ad - bc)$$

so that $z\overline{w} + w\overline{z} = 2(ac + bd) = 2 \operatorname{Re} z\overline{w}$, and we are done. To get the identity for $|z - w|^2$, we simply replace w by $-w$, and use the above argument.

Now, we have that $|z + w|^2 = |z^2| + 2 \operatorname{Re} z\overline{w} + |w|^2$, and $|z - w|^2 = |z^2| - 2 \operatorname{Re} z\overline{w} + |w|^2$, so that adding them together, the terms $2 \operatorname{Re} z\overline{w}$ cancel out and we are left with

$$|z + w|^2 + |z - w|^2 = 2(|z|^2 + |w|^2)$$

■

Lemma 1.1.6. *Let $R(z) \in \mathbb{C}(z)$ a rational function in z. Then if R has coefficients in $\mathbb{R}$, then $\overline{R(z)} = R(\overline{z})$.*

Proof. We first observe the polynomial $f \in \mathbb{C}[z]$, of finite degree $\deg f = n$, and of the form

$$f(z) = a_0 + a_1 z + \cdots + a_n z^n$$

Then if f has all coefficients in $\mathbb{R}$; i.e. $f \in \mathbb{R}[z]$, where $z \in \mathbb{C}$ is treated as indeterminant, then we have that since each $a_i \in \mathbb{R}$, then $\overline{a_i z^i} = \overline{a_i}\,\overline{z^i} = a_i \overline{z}^i$. So that

$$\overline{f(z)} = \overline{(a_0 + a_1 z + \cdots + a_n z^n)} = a_0 + a_1 \overline{z} + \cdots + a_n \overline{z}^n$$

which makes $\overline{f(z)} = f(\overline{z})$. Now, one can also extend f to a polynomial of infinite degree by taking $n \to \infty$, and the same holds.

Now, let $R(z) \in \mathbb{C}(z)$ a rational function. Recall that $R(z)$ is of the form

$$R(z) = \frac{f(z)}{g(z)} \text{ with } g \neq 0$$

for some polynomials $f, g \in \mathbb{C}[z]$. Then if R has all real coefficients, so do f and g, and hence we get

$$\overline{R(z)} = \frac{\overline{f(z)}}{\overline{g(z)}} = \frac{f(\overline{z})}{g(\overline{z})} = R(\overline{z})$$

which completes the proof. ■

1.2 The Complex Plane

Definition. We define the **complex plane** to be the space of points (x, y) of $\mathbb{R}^2$ for which $z = x + iy$.

Lemma 1.2.1. *For every $z, w \in \mathbb{C}$ $|z + w| \leq |z| + |w|$.*

Proof. Observe that $-|z| \leq \operatorname{Re} z \leq |z|$ for all $z \in \mathbb{C}$, so that $\operatorname{Re} z\overline{w} \leq |z\overline{w}| = |z||w|$. So we get

$$|z + w|^2 = |z|^2 + \operatorname{Re} z\overline{w} + |\overline{w}| \leq |z|^2 + |z||w| + |\overline{w}| = (|z| + |w|)^2$$

Taking square roots gives us the result. ■

Corollary. $|z + w| = |z| + |w|$ *if $z = tw$ for some $t \geq 0$.*

Corollary. *If $z_1, \dots, z_n \in \mathbb{C}$, then $|z_1 + \cdots + z_n| \leq |z_1| + \cdots + |z_n|$.*

Proof. By induction on n. ■

Corollary. *For all $z, w \in \mathbb{C}$, $\big||z| - |w|\big| \leq |z - w|$.*

Proof. We have that $|z| \leq |z-w| + |w|$, and $|w| \leq |z-w| + |z|$. So we get $|z| - |w| \leq |z-w|$ and $-|z-w| \leq |w| - |z|$, so that $\big||z| - |w|\big| \leq |z-w|$. ■

Definition. We define the **polar form** of a complex number $z \in \mathbb{C}$ to be the polar coordinates (r, θ) where $r = |z|$ and θ is the angle between the line segment from 0 to z and the positive real axis. We call r the **modulus** of z, and θ the **argument** of z. We write $\theta = \arg z$.

Lemma 1.2.2. *Let $z = r_1 \cos\theta_1 + r_1 i \sin\theta_1$ and $w = r_2 \cos\theta_2 + r_2 i \sin\theta_2$. Then*

$$zw = r_1 r_2 \cos(\theta_1 + \theta_2) + r_1 r_2 i \sin(\theta_1 + \theta_2)$$

so that $\arg zw = \arg z + \arg w$.

Proof. We multiply the expanded forms of z and w together and use the trigonometric identities to get the result. ■

Corollary. *If $z_k = r_k \cos\theta_k + r_k i \sin\theta_k$, then*

$$z_1 \dots z_n = (r_1 \dots r_n)\cos(\theta_1 + \cdots + \theta_n) + (r_1 \dots r_n) i \sin(\theta_1 + \cdots + \theta_n)$$

Proof. By induction on n. ■

Theorem 1.2.3 (DeMoivre's Theorem)**.** *For all integers $n \geq 0$, if $z = \cos\theta + i\sin\theta$, then*

$$z^n = \cos n\theta + i \sin n\theta$$

Proof. We use the corollory to lemma 1.2.2 recursively on z^n. ■

Lemma 1.2.4. *FOr each nonzero $a \in \mathbb{C}$, and integer $n \geq 2$, the polynomial $z^n - a$ has has roots all z of the form*

$$z = |a|^{\frac{1}{n}}(\cos\frac{\alpha + 2k\pi}{n} + i \sin\frac{\alpha + 2k\pi}{n}) \text{ for all } 0 \leq k \leq n-1$$

where $a = |a|\cos\alpha + |a| i \sin\alpha$

Proof. Let $a = |a|\cos\alpha + |a| i \sin\alpha$. Then we have $z^n - a = 0$ has as solution

$$z' = |a|^{\frac{1}{n}}(\cos\frac{\alpha + 2\pi}{n} + i \sin\frac{\alpha + 2\pi}{n})$$

The rest of the solutions are obtained by noting that $(z')^n - a = 0$. ■

Definition. Let $a \in \mathbb{C}$ a nonzero complex number. We call the roots of the the polynomial $z^n - a \in \mathbb{C}[z]$ the n**-th roots** of a. We call the roots of $z^n - 1 \in \mathbb{C}[z]$ the n**-th roots of unity**.

Example 1.2. The n-th roots of unity are all complex numbers of the form

$$z = \cos\frac{2k\pi}{n} + i\sin\frac{2k\pi}{n} \text{ for all } 0 \leq k \leq n-1$$

Lemma 1.2.5. *Let $L \subseteq \mathbb{C}$ a straight line in $\mathbb{C}$. Then $L = \{z \in \mathbb{C} : \operatorname{Im}\frac{z-a}{b} = 0\}$, where $z = a + tb$ for some $t \in \mathbb{R}$.*

Proof. Let a be any point in L, and b the direction vector of L. Then if $z \in L$ $z = a + tb$ for some $t \in \mathbb{R}$. Since $b \neq 0$, $\operatorname{Im}\frac{z-a}{b} = 0$, since $t = \frac{z-a}{b}$, and $t \in \mathbb{R}$. ■

Corollary. *Let $H_a = \{z \in \mathbb{C} : \operatorname{Im}\frac{z-a}{b} > 0\}$ and $K_a = \{z \in \mathbb{C} : \operatorname{Im}\frac{z-a}{b} < 0\}$. Then $H_a = a + H_0$ and $K_a = a - K_0$.*

Proof. Suppose that $|b| = 1$, and let $a = 0$, then $H_0 = \{z \in \mathbb{C} : \operatorname{Im}\frac{z}{b} > 0\}$. Now, $b = \cos\beta + i\sin\beta$. If $z = r\cos\theta + ri\sin\theta$, then $\frac{z}{b} = r\cos(\theta - \beta) + ri\sin(\theta - \beta)$. So $z \in H_0$ if, and only if $\sin(\theta - \beta) > 0$; that is $\beta < \theta < \pi + \beta$, which makes H_0 the upper half plane about L.

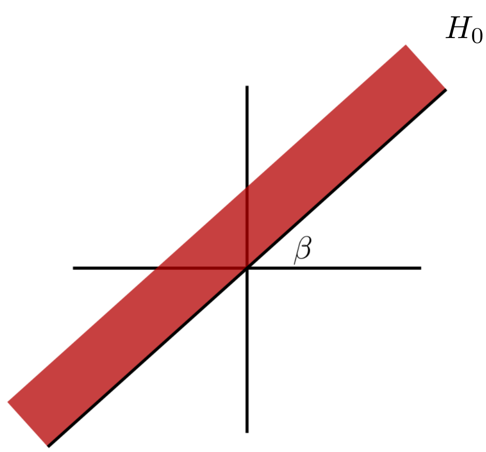

Figure 1.1:

Putting $H_a = \{z \in \mathbb{C} : \operatorname{Im}\frac{z-a}{b} > 0\}$, we get $H_a = a + H_0$. By similar reasoning, we get $K_a = a - K_0$, where $K_a = \{z \in \mathbb{C} : \operatorname{Im}\frac{z-a}{b} < 0\}$. ■

1.3 The Extended Complex Numbers

Definition. We define the **extended complex numbers** to be the set $\mathbb{C}_\infty = \mathbb{C} \cup \{\infty\}$.

Lemma 1.3.1. *$\mathbb{C}_\infty$ is homeomorphic to the unit sphere S^2 of $\mathbb{R}^3$.*

Proof. Identify $\mathbb{C}$ with the plane $\mathbb{R}^2$ as a subset of $\mathbb{R}^3$. Then $\mathbb{C}$ cuts the sphere S^2 along the equator. Now, let $N = (0, 0, 1)$ be the noth pole of S^2. For $z \in \mathbb{C}$, let L_z the line passing through z and N, and hence cuts S^3 at exactly one point $Z \neq N$. If $|z| > 1$, Z is in the northern hemisphere of S^2, and if $|z| < 1$, then Z is in the southern hemisphere. If $|z| = 1$, then $Z = z$. Then notice that as $|z| \to \infty$, then $Z \to N$; and so identify N with ∞ in $\mathbb{C}_\infty$.

Now, let $z = x + iy$ and $Z = (x_1, x_2, x_3)$ a point on S^2. Then $L_z = \{tN + (1-t)z : t \in \mathbb{R}\}$. Observe then that

$$L_z = \{((1-t)x, (1-t)y, t) : t \in \mathbb{R}\}$$

Then we get

$$1 = (1-t)^2|z|^2 + t^2$$

Taking $t \neq 1$ so that $z \neq \infty$

$$Z = \Big(\frac{2x}{|z|^2+1}, \frac{2y}{|z|^2+1}, \frac{|z|^2-1}{|z|^2+1}\Big)$$

additionally

$$Z = \Big(\frac{z+\overline{z}}{|z|^2+1}, -i\frac{z-\overline{z}}{|z|^2+1}, \frac{|z|^2-1}{|z|^2+1}\Big)$$

Taking $Z \neq N$ and $t = x_1$, we also get by definition of L_z, that

$$z = \frac{x_1 + ix_2}{1 - x_3}$$

Define now, the metric d on $\mathbb{C}_\infty$ by $d(z, w)$ is the distance between the points $Z = (x_1, x_2, x_3)$ and $W = (y_1, y_2, y_3)$ on S^2. Then we get

$$d(z, w) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2 + (x_3 - y_3)^2}$$

Squaring both sides we ovserve tha

$$d(z, w) = 2 - 2(x_1y_1 + x_2y_2 + x_3y_3)$$

Using the previous derivations of the components of Z, we finally obtain

$$d(z, w) = \frac{z|z-w|}{\sqrt{(|z|^2+1)(|w|^2+1)}} \text{ for all } z, w \in \mathbb{C}$$

When $w = \infty$, we have

$$d(z, \infty) = \frac{z}{\sqrt{|z|^2+1}}$$

Then d is the required homeomorphism. ■

Definition. We call the correspondence between S^2 and $\mathbb{C}_\infty$ the **stereographic projection** of S^2 onto $\mathbb{C}_\infty$.

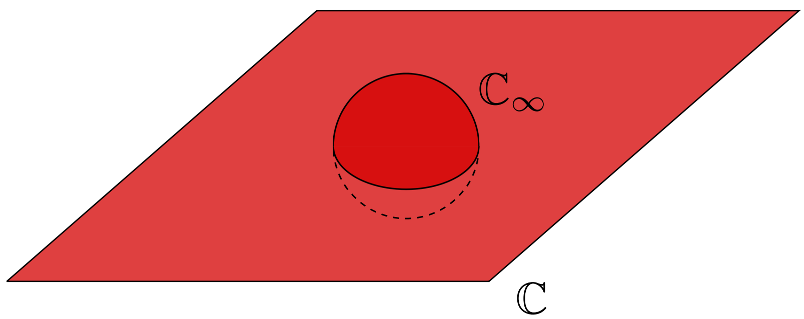

Figure 1.2: The Extended Complex Numbers.

Chapter 2

The Topology of $\mathbb{C}$.

2.1 Metric Spaces

Definition. A **metric space** is a set X together with a map $d : X \times X \to \mathbb{R}$ such that for all $x, y, z \in X$

(1) $d(x, y) \geq 0$ and $d(x, y) = 0$ if, and only if $x = y$.

(2) $d(x, y) = d(y, x)$.

(3) $d(x, y) \leq d(x, z) + d(z, y)$ (The Triangle Inequality).

We call d a **metric** on X. If $x \in X$, and $r > 0$, we define the **open ball** centered about x of **radius** r to be the set $B(x, r) = \{y \in X : d(x, y) < r\}$. We define the **closed ball** centered about x of radius r to be the set $\overline{B}(x, y) = \{y \in X : d(x, y) \leq r\}$.

Example 2.1. (1) The metric $d(x, y) = \|z - w\|$ defined by $\|z - w\| = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)}$, where $z = x_1 + iy_1$ and $w = x_2 + iy_2$ makes $\mathbb{R}$ and $\mathbb{C}$ into metric spaces. In fact, d defines the norm on $\mathbb{C}$, i.e. $\|z\| = d(z, 0)$. In $\mathbb{R}$, $\|\cdot\|$ is the absolute value. We denote howerver $\|\cdot\| = |\cdot|$ is $\mathbb{C}$ as well, when we are talking about the norm of a complex number.

(2) If X is a metric space with metric d, and $Y \subseteq X$, then d makes Y into a metric space.

(3) Define $d(x + iy, a + ib) = |x - a| + |y - b|$. Then $(\mathbb{C}, d)$ is a metric space. We call d the **taxicab metric**.

(4) Define $d(x + iy, a + ib) = \max\{|x - a|, |y - b|\}$. Then $(\mathbb{C}, d)$ is a metric space. We call d the **square metric**.

(5) Let X be any set, and define $d : X \times X \to \mathbb{R}$ by

$$d(x, y) = \begin{cases} 1, & \text{if } x \neq y \\ 0, & \text{if } x = y \end{cases}$$

Then d is a metric on X. Notice also that for any $\varepsilon > 0$, that $B(x, \varepsilon) = \{x\}$ if $\varepsilon \leq 1$, and $B(x, \varepsilon) = X$ if $\varepsilon > 1$.

(6) Define d on $\mathbb{R}^n$ by

$$d(x, y) = \sqrt{\sum_{i=1}^{n} (x_i - y_i)^2}$$

Then d is a metric on $\mathbb{R}^n$ defining the general norm. That is $\|x\| = d(x, 0)$.

(7) Let S and let $B(S)$ the set of all complex valued functions $f : S \to \mathbb{C}$ such that $\|f\|_\infty = \sup \{|f(s|) : s \in S\}$ is finite. That is, $B(S)$ is the set of all complex valued functions whose image is contained within a disk of finite radius. Define d on $B(S)$ by $d(f, g) = \|f - g\|_\infty$. Let $f, g, h \in B(S)$. Then

$$\|f(s) - g(s)\| = \|(f(s) - h(s)) - (h(s) - g(s))\| \leq \|f(s) - h(s)\| + \|h(s) - g(s)\|$$

taking least upper bounds, we get

$$\|f - g\|_\infty \leq \|f - h\|_\infty + \|h - g\|_\infty$$

Definition. Let X be a metric space together with metric d. We call a subset U of X **open** if there exists an $\varepsilon > 0$ for which $B(x, \varepsilon) \subseteq U$ for every $x \in U$.

Example 2.2. $U = \{z \in \mathbb{C} : a < \operatorname{Re} z < b\}$ is open in $\mathbb{C}$, but $U = \{z \in \mathbb{C} : \operatorname{Re} z < 0\}$ is not, as $B(0, \varepsilon) \notin U$ no matter how small we make ε.

Theorem 2.1.1. *Let X be a metric space with metric d. Then X is a topological space whose open sets are those subsets of X containing ε-balls for every element, and for $\varepsilon > 0$.*

Definition. We call a subset V of a metrix space (X, d) **closed** if $X \backslash V$ is open in X.

Lemma 2.1.2. *If (X, d) is a metric space, then it is a topology by closed sets.*

Definition. Let $A \subseteq X$ where X is a metric space. We define the **interior** of A to be the union of all open sets contained in A, and write $\operatorname{int} A$. We define the **closure** of A to be the intersection of all closed sets containing A and write $\operatorname{cl} A$. We define the **boundry** of A to be $\partial A = \operatorname{cl} A \cap \operatorname{cl}(X \backslash A)$.

Example 2.3. We have $\operatorname{int} \mathbb{Q}(i) = \emptyset$ and $\operatorname{cl} \mathbb{Q}(i) = \mathbb{C}$.

Lemma 2.1.3. *Let X be a metric space and A, BX. Then the following are true*

(1) A is open if, and only if $A = \operatorname{int} A$.

(2) A is closed if, and only if $A = \operatorname{cl} A$.

(3) $\operatorname{int} A = X \backslash \operatorname{cl}(X \backslash A)$, $\operatorname{cl} A = X \backslash \operatorname{int}(X \backslash A)$, and $\partial A = \operatorname{cl} A \backslash \operatorname{int} A$.

(4) $\operatorname{cl}(A \cup B) = \operatorname{cl} A \cup \operatorname{cl} B$.

(5) $x_0 \in \operatorname{int} A$ if, and only if there is an $\varepsilon > 0$ for which $B(x_0, \varepsilon) \subseteq A$.

(6) $x_0 \in \operatorname{cl} A$ if, and only if for every $\varepsilon > 0$, $B(x_0, \varepsilon) \cap A = \emptyset$.

Definition. A subset A of a metric space X is **dense** in X if $\operatorname{cl} A = X$.

Example 2.4. $\mathbb{Q}$ is dense in $\mathbb{R}$, notice that $\operatorname{cl} \mathbb{Q} = \mathbb{R}$. Moreover, $\mathbb{Q}(i)$ is dense in $\mathbb{C}$.

2.2 Connectedness in $\mathbb{C}$

Definition. We say a metric space X is connected provided there are no disjoint nonempty open sets $A, B \subseteq X$ for which $X = A \cup B$.

Lemma 2.2.1. *A metric space X is connected if its only closed and open sets are the emtpyset and itself.*

Example 2.5. Consider the space $X = \{z \in \mathbb{C} : |z| < 1\} \cup \{z \in \mathbb{C} : |z-3| < 1\}$. Let $A = \{z \in \mathbb{C} : |z| < 1\}$ and $B = \{z \in \mathbb{C} : |z-3| < 1\}$. Then then both A and B are open in X. Moreover, A is also closed in X as $B = X \backslash A$. So X is not connected.

Lemma 2.2.2. *A space $X \subseteq X$ is connected if, and only if it is an interval.*

Proof. Suppose that $X = [a,b]$, where $a, b \in \mathbb{R}$ and $a < b$. Let $A \subseteq X$ be open, with $a \in A$ and $b \in B$ and where $X \neq A$. Then there is an $\varepsilon > 0$ for which $[a, a+\varepsilon) \subseteq A$. Let $r = \sup\{\varepsilon : [a, a+\varepsilon) \subseteq A\}$. If $a \leq x < a+r$, putting $h = a + (r-x) > 0$ there is an $\varepsilon > 0$ for which $r - h < \varepsilon < r$ and $[a, a+\varepsilon) \subseteq A$. However, $a \leq a + (r-h) < a + \varepsilon$ putting $x \in A$. So that $[a, a+r) \subseteq A$. Now, if $a + r \in A$, then by the openness of A, there is a $\delta > 0$ with $[a+r, a+r+\delta] \subseteq A$, which puts $[a+r, a+r+\delta) \subseteq A$. But that contradicrts that r is a least upper boundl; so $a + r \notin A$.

Now, if A were closed, then $a + r \in B = X \backslash A$, which is open, so that there is a $\delta > 0$ such that $(a+r-\delta, a+r) \subseteq B$, which contradicts that $[a, a+r) \subseteq A$. ∎

Remark. Note that the first part of this proof lacks the proof for the other types of intervals.

Definition. Let $z, w \in \mathbb{C}$. We define the **staight line segment** $[z, w]$ from z to w to be the set

$$[z, w] = \{tw + (1-t)z : 0 \leq t \leq 1\}$$

A **polygon** from z to w is defined to be the set

$$P[z, w] = \bigcup_{k=1}^{n} [z_k, w_k]$$

where $z_1 = z$, $w_n = w$, and $z_{k+1} = w_k$ for all $1 \leq k \leq n-1$. When the endpoints of the polygon are understood, we may simply just write P, or we enumerate the points of P as $P = [z, z_2, \ldots, z_n, w]$.

Theorem 2.2.3. *An open set U of $\mathbb{C}$ is connected if, and only if for all $z, w \in U$, there exists a polygon $P[z, w]$ from z to w contained in U.*

Proof. Let $P[z, w] \subseteq U$ be the given polygon. Suppose that U were not connected. Then there exist disjoint nonempty open sets Z and W of U (as a subspace of $\mathbb{C}$) for which $U = Z \cup W$. Let $z \in Z$ and $w \in W$. Consider the case for when $P[z, w] = [z, w]$. Define $S = \{s \in [0,1] : sw + (1-s)z \in A\}$ and $T = \{s \in [0,1] : sw + (1-s)z \in B\}$. Then notice that S and T are disjoint, and that $S \cup T = [0,1]$. Moreover, they are open subsets of the interval $[0,1] \subseteq \mathbb{R}$; but $[0,1]$ is connected in $\mathbb{R}$, which is a contradiction. Therefore U must be connected.

On the otherhand, let $w \in Z$ and let $P = [z, z_2, \cdots z_n, w] \subseteq U$ SInce U is open, there is an $\varepsilon > 0$ such that $B(w, \varepsilon) \subseteq U$. Now, if $u \in B(w, \varepsilon)$, then $[w, u] \subseteq B(w, \varepsilon) \subseteq U$, so the polygon $Q = P \cup [w, u] \subseteq U$. Hence $B(w, \varepsilon) \subseteq Z$, which makes Z oepn. On the otherhand, consider $u \in U \backslash Z$, and let $\varepsilon > 0$ such that $B(u, \varepsilon) \subseteq U$. Then there is a $w \in Z \cap B(u, \varepsilon)$. Construct, then a polygon $P[z, u]$ so that $B(u, \varepsilon) \cap Z$ is empty. That is, $B(u, \varepsilon) \subseteq U \backslash Z$ making $U \backslash Z$ open, and hence Z closed. ■

Corollary. *If $U \subseteq \mathbb{C}$ is an open and connected set, then for all $z, w \in U$, there is a polygon $P[z, w]$ in U made up of straight line segments parallel to either the real axis, or the imaginary axis.*

Definition. Let X be a metric space. We call a subset $C \subset X$ a **connected component** if it is maximally connected in X.

Example 2.6. (1) A and B in example 2.5 are connected components.

(2) Let $X = \{\frac{1}{k} : k \in \mathbb{Z}^+\} \cup \{0\}$. Then every connected component is a point of x, and vise versa; with, the exception of 0.

Lemma 2.2.4. *Let X be a metric space with $x_0 \in X$. If $\{D_j\}$ is a collection of connected subsets of X, such that $x_0 \in D_j$, then the union $D = \bigcup D_j$ is connected.*

Proof. Let $A \subseteq D$, which is a metric space, for which A is both open and closed, and nonempty. Then $A \cap D_j$ is open and closed for all j. Now, since D_j is connected, either $A \cap D_j =$, or $A \cap D_j = D_j$. Since A is nonempty, we must have the latter case. Then there exists at least one index k for which $A \cap D_k = D_k$. Then if $x_0 \in A$, $x_0 \in A \cap D_k$ so that $x_0 \in D_k$ making $A \cap D_j = D_j$ for all j or $D_j \subseteq A$. In either case, we get $D = A$. ■

Theorem 2.2.5. *The connected components of a metric space partition the space.*

Proof. Let $\mathcal{D}$ the collection of all connected subsets of X containing a point $x_0 \in X$. Then $\mathcal{D}$ is nonempty by definition, and by hypothesis, we have that $C = \bigcup D_j$ is connected, and that $x_0 \in C$.

Now, suppose that $C \subseteq D$ for some connected st D. Then $x_0 \in D$ so that $D \in \mathcal{D}$, and hence $D \subseteq C$. This makes $C = D$, and hence C is a connected component of X. This then implies that $X = \bigcup C_j$ where $\{C_j\}$ is the collection of connected components of X.

Now, consider $\{C_j\}$, and suppose that for distinct components C_1 and C_2, that there is an $x_0 \in C_1 C_2$. Then $x_0 \in C_1$, and $x_0 \in C_2$ so that $C_1 = C_1 \cup C_2 = C_2$, which is a contradiction. Therefore the connected components are pairwise disjoint. ■

Lemma 2.2.6. *If X is a connected metric space with $A \subseteq X$, and $A \subseteq B \subseteq \operatorname{cl} A$, then B is also connected.*

Corollary. *Connected components of a metric space are closed.*

Theorem 2.2.7. *If U is open in $\mathbb{C}$, then U has countably many connected components; each of which is open.*

Proof. Let $C \subseteq U$ a connected component, with $x_0 \in C$. Since U is open, there is an $\varepsilon > 0$ for which $B(x_0, \varepsilon) \subseteq U$. Then $B(x_0, \varepsilon) \cup C$ is connected so that $B(x_0, \varepsilon) \cup C = C$, so that $B(x_0, \varepsilon) \subseteq C$. This makes each C open.

Now, let $S = \{a + ib \in \mathbb{Q}(i) : a + ib \in U\}$. Then S is countable by the density of $\mathbb{Q}(i)$ in $\mathbb{C}$, and each connected component of U contains a point of S. This implies there are countably many such components. ∎

2.3 Completeness in $\mathbb{C}$

Definition. We say a sequence $\{x_n\}$ of points of a metric space X **converges** to a point $x \in X$ if for every $\varepsilon > 0$, there is anb $N \in \mathbb{Z}^+$ for which

$$d(x, x_n) < \varepsilon \text{ whenever } n \geq N$$

If $\{x_n\}$ converges to x, we write $\{x_n\} \to x$, or $\lim x_n = x$.

Lemma 2.3.1. *Let X be a metric space. A set $V \subseteq X$ is closed if, and only if for every sequece $\{x_n\}$ of points in V, $\{x_n\}$ converges to a point $x \in V$.*

Proof. If V is closed, and $\{x_n\} \to x$, then for every $\varepsilon > 0$ and $x_n \in B(x, \varepsilon)$, we get that $B(x, \varepsilon) \cap V \neq \emptyset$ so that $x \in \operatorname{cl} F = F$.

Conversly, suppose that V is not closed. Then there exists a point $x_0 \in \operatorname{cl} V \backslash V$. Then we get that for every $\varepsilon > 0$, the set $B(x_0, \varepsilon) \cap F \neq \emptyset$ so that for all $n \in \mathbb{Z}^+$, there is an $x_n \in B(x_0, \frac{1}{n}) \cap F$. This makes $d(x_0, x_n) < \frac{1}{n}$, so that $\{x_n\} \to x_0$. Since $x_0 F$, the condition fails. ∎

Definition. We call a point $x \in X$ of a metric space X a **limit point** of a subset $A \subseteq X$ if there exists a sequence of points $\{x_n\}$ in A such that $\{x_n\} \to x$.

Example 2.7. Consider $\mathbb{C}$ and let $A = [0, 1] \cup \{i\}$. Then each point of $[0, 1]$ is a limit point of A, but i is not a limit point of A.

Lemma 2.3.2. *A subset of a metric space is closed if, and only if it contains all its limit points. Moreover, if A is a subset of a metric space X, then $\operatorname{cl} A = A \cup A'$, where A' is the collection of all limit points of A.*

Definition. We call a sequence $\{x_n\}$ of points of a metric space **Cauchy** if for every $\varepsilon > 0$ there is an $N \in \mathbb{Z}^+$ for which

$$d(x_m, x_m) < \varepsilon \text{ for all } m, n \geq N$$

If X is a metric space in which every Cauchy sequence converges in to a point in X, then we say X is **complete**.

Theorem 2.3.3. *The field $\mathbb{C}$ of complex numbers is complete.*

Proof. Let $\{z_n\}$ a Cauchy sequence of complex numbers with $z_n = x_n + iy_n$. Then the sequences $\{x_n\}$ and $\{y_n\}$ are Cauchy in $\mathbb{R}$. Since $\mathbb{R}$ is a complete metric space, we observe that there exist $x, y \in \mathbb{R}$ for which $\{x_n\} \to x$ and $\{y_n\} \to y$. This makes $\{z_n\} \to z$ with $z = x + iy \in \mathbb{C}$. ∎

Definition. Let X be a metric space and $A \subseteq X$. We define the **diameter** of A to be the least upper bound:

$$\operatorname{diam} A = \sup \{d(x, y) : x, y \in A\}$$

of all distances of points in A.

Theorem 2.3.4 (Cantor's theorem)**.** *A metric space X is complete if, and only if for every decreasing sequence $\{F_n\}$ of nonempty closed sets, with $\operatorname{diam} F_n \to 0$ for all n, then the intersection*

$$F = \bigcap F_n$$

consists of a single point.

Proof. Suppose that X is complete. Let $\{F_n\}$ a sequence of closed sets such that

(1) $F_{n+1} \subseteq F_n$; i.e. $\{F_n\}$ is a decreasing sequence.

(2) $\lim \operatorname{diam} F_n \to 0$.

Let $x_n \in F_n$. If $n, m \geq N$ then $x_m, x_n \in F_N$ so that $d(x_m, x_n) \leq \operatorname{diam} F_n$ by definition. By hypothesis, choose an N large enough such that $\operatorname{diam} F_N < \varepsilon$ for some $\varepsilon > 0$. This makes the sequence $\{x_n\}$ Cauchy. Then by the completeness of X $\{x_n\} \to x$ for some $x \in X$. Since $x_n \in F_n$ for all $n \geq N$, we get that $F_n \subseteq F_N$ and hence $x \in F_N$ which puts

$$x \in F = \bigcap F_n$$

Now, if $y \in F$, then $x, y \in F_n$ for all n which gives $d(x, y) \leq \operatorname{diam} F_n \to 0$. So $d(x, y) = 0$ which makes $x = y$ and so $F = \{x\}$.

Conversely, let $\{x_n\}$ be Cauchy in X, and take $F_n = \operatorname{cl}\{x_n, x_{n+1}, \ldots\}$. Then $F_{n+1} \subseteq F_n$, making $\{F_n\}$ decreasing sequence. If $\varepsilon > 0$, choose an $N > 0$ such that $d(x_m, x_n) < \varepsilon$ for any $m, n \geq N$. Then $\operatorname{diam} F_n \leq \varepsilon$. By hypothesis, there is an $x_0 \in X$ such that $F = \bigcap F_n = \{x_0\}$. Moreover, $x_0 \in F_n$ so that $d(x_0, x_m) \leq \operatorname{diam} F_n \to 0$, which puts $\{x_n\} \to x \in X$ which makes X complete. ∎

Lemma 2.3.5. *If X is a complete metric space, and $Y \subseteq X$, then Y is complete if, and only if Y is closed in X.*

Proof. Suppose that Y is complete and let y a limit point of Y. Then there exists a sequence $\{y_n\}$ of points of Y for which $\{y_n\} \to y$. This makes $\{y_n\}$ Cauchy, and so $\{y_n\} \to x_0 \in Y$. It follows that $y = x_0$, so that $Y' \subseteq Y$ and hence Y is closed. ∎

2.4 Compactness in $\mathbb{C}$

Definition. Let X be a metric space. We say an collection $\{U_n\}$ of open sets of X **covers** a subset K of X if $K \subseteq \bigcup U_n$. We call $\{U_n\}$ an **open cover** of K. We call K **compact** if every open cover of K has a finite open subcover.

Lemma 2.4.1. *If K is compact in a metric space X, then K is closed. Moreover, if $F \subseteq K$ is closed, then F is also compact.*

Proof. Certainly, we have $K \subseteq \operatorname{cl} K$. Now, let $x_0 \in \operatorname{cl} K$, then $B(x_0, \varepsilon) \cap K$ is nonempty for every $\varepsilon > 0$. Let $G_n = X \backslash \overline{B}(x_0, \frac{1}{n})$, and suppose that $x_0 \notin K$. Then each G_n is open in X, and $K \subseteq \bigcup G_n$. Since K is compact, then ther is an $m \in \mathbb{Z}^+$ for whcih $K \subseteq \bigcup_{n=1}^{m} G_n$. Notice, however that $G_1 \subseteq G_2 \subseteq \cdots \subseteq G_m \subseteq \ldots$ so that $K \subseteq G_m = X \backslash \overline{B}(x_0, \frac{1}{m})$, so that $B(x_0, \frac{1}{n}) \cap K = \emptyset$; a contradiction! Therefore $x_0 \in K$ and $K = \operatorname{cl} K$. ∎

Definition. Let X be a set. We say a collection $\{F_n\}$ of subsets of X has the **finite intersection property** (**FIP**) if the intersection of any finite subcollection of $\{F_n\}$ is nonempty.

Lemma 2.4.2. *A set K of a metric space X is compact if, and only if for every collection of closed sets $\{F_n\}$ satisfying the finite intersection property, the intersection*

$$F = \bigcap F_n$$

is nonempty.

Proof. Let K be compact in X, and $\{F_n\}$ a collection of closed sets of X with the FIP. Suppose that $F = \bigcap F_n = \emptyset$. Now, take $\mathcal{G} = \{X \backslash F_n\}$ the collecton of open sets. Then observe that

$$\bigcup X \backslash F_n = X \backslash \bigcap F_n = X \backslash F = X$$

by hypothesis. SInce $K \subseteq K$, $\mathcal{G}$ covers K, and since K is compact, there is a finite subcover $\{X \backslash F_i\}_{i=1}^{n}$ of K. That is

$$K \subseteq \bigcup_{i=1}^{n} X \backslash F_i = X \backslash \bigcap_{i=1}^{n} F_i \subseteq X$$

since $\bigcap_{i=1}^{n} F_n \neq \emptyset$. But then $\bigcap_{i=1}^{n} F_i \subseteq X \backslash K$, and since $F_i \subseteq K$ for all $1 \leq i \leq n$, this makes $\bigcap_{i=1}^{n} F_i =$; a contradiction! ∎

Corollary. *Compact metric spaces are complete.*

Corollary. *If X is compact, then every infinite set in X has a limit point in X.*

Proof. Let $S \subseteq X$ infinite, and suppose the set of all limit points of S in X, S', is empty. Consider the sequence $\{a_n\}$ of distinct points of S, and take $F_n = \{a_n, a_{n+1}, \ldots\}$. Then F_n has no limit points in X so that $F_n' = \emptyset$. Then $F_n' \subseteq F_n$ so that F_n is closed. Thus $\{F_n\}$ has the finite intersection property. But since $a_1 \neq \ldots \neq a_n \neq$, we get $\bigcap F_n = \emptyset$; which contradicts the above. Therefore S' is nonempty. ∎

Definition. We call a metric space **sequentially compact** if every sequence of point in the space has a convergent subsequence.

Lemma 2.4.3 (Lebesgue's Covering Lemma)**.** *If X is a sequentially compact metric space, and $\mathcal{G}$ is an open cover of X, then there is an $\varepsilon > 0$ such that if $x \in X$ there is a $G \in \mathcal{G}$ with $B(x, \varepsilon) \subseteq G$.*

Proof. Suppose by contradiction that for every open cover $\mathcal{G}$ of X there is no ε for which the statement holds. Then for every $n \in \mathbb{Z}^+$, there is an $x_n \in X$ for which $B(x_n, \frac{1}{n}) \not\subseteq G$. Now, since X is sequentially compact, there is a point $x_0 \in X$ and s subsequence $\{x_{n_k}\}$ of a sequence $\{x_n\}$ for which $\{x_{n_k}\} \to x_0$. Let $G_0 \in \mathcal{G}$ such that $x_0 \in G_0$. Choose $\varepsilon > 0$ such that $n_k \geq N$ and $n_k > \frac{1}{\varepsilon}$. Let $y \in B(x_{n_k}, \frac{1}{n_k})$. THen $d(x_0, y) \leq d(x_0, x_{n_k}) + d(x_{n_k}, y) < \frac{\varepsilon}{2} + \frac{1}{n_k} < \varepsilon$. So that $B(x_{n_k}, \frac{1}{n_k}) \subseteq B(x_0, \varepsilon)$. But that contradicts our choice of $\{x_{n_k}\}$. ∎

Definition. We say a subset K of a metric space X is **totally bounded** if for any $\varepsilon > 0$ there exist a sequence $\{x_n\}$ of points of X for which $K = \bigcup_{k=1}^{n} B(x_k, \varepsilon)$.

Theorem 2.4.4. *The following are equivalent in every metric space X.*

(1) X is compact.

(2) Every infinite set of X has a limit point in X.

(3) X is sequentially compact.

(4) X is complete, and totally bounded.

Proof. We have that if X is compact, then every infinite set of X has their limit points in X, by the above corollory.

Suppopse every infinite set of X has a limit point in X. Let $\{x_n\}$ a sequence, and suppose without loss of generality, that all the points are distinct. Then $\{x_n\}$ has a limit point x_0. Then there exist an $x_{n_1} \in B(x_0, 1)$. Similarly, there is an $n_2 > n_1$ with $x_{n_2} \in B(x_0, \frac{1}{2})$. Continuiing in this manner, we get for some $n_k > n_{k-1}$, that $x_{n_k} \in B(x_0, \frac{1}{k})$, so that $\{x_{n_k}\} \to x_0$; and so X is sequentially compact.

Suppose now that X is sequentially compact, and let $\{x_n\}$ be a Cauchy sequence. By the sequential compactness of $\{x_n\}$, it has a convergent subsequence, which makes X complete. Now, let $\varepsilon > 0$ and fix $x_1 \in X$. If $X = B(x_1, \varepsilon)$, we are done. Otherwise, choose an $x_2 \in X \backslash B(x_1, \varepsilon)$. If $X = B(x_1, \varepsilon) \cup B(x_2, \varepsilon)$ we are done. Otherwise, continuing in this manner, we find a sequence $\{x_n\}$ of points with $x_{n+1} \in X \backslash \bigcup_{k=1}^{n} B(x_k, \varepsilon)$. Which implies for $m \neq n$, that $d(x_m, x_n) \geq \varepsilon > 0$. Contradictiong that X is sequentially compact. So we have that X must be totally bounded.

Conversely, suppose that X is complete and totally bounded. Let $\{x_n\}$ a sequence of distint points. Then there is a $y_1 \in X$ and a subsequence $\{x_n^{(1)}\}$ of $\{x_n\}$ for which $\{x_n^{(1)}\} \subseteq B(y_1, 1)$. There also exists a $y_2 \in X$ and s a subsequence $\{x_n^{(2)}\}$ of $\{x_n^{(1)}\}$ such that $\{x_n^{(2)}\} \subseteq B(y_2, \frac{1}{2})$. Continuing in this manner, for all $k \geq 2$, there is a $y_k \in X$ and a subsequence $\{x_n^{(k)}\}$ of $\{x_n^{(k-1)}\}$ for which $\{x_n^{(k)}\} \subseteq B(y_k, \frac{1}{k})$. Take $K_k \operatorname{cl} \{x_n^{(k)}\}$. Then

$$\operatorname{diam} F_k \leq \frac{1}{k}$$

and $\{F_k\}$ is a decreasing collection of closed sets. Thus the intersection $F = \{x_0\}$ is a single point. So $x_0 \in F_k$, so that

$d(x_0, x_n^{(k)}) \leq F_k \leq \frac{1}{k}$ so that $\{x_n^{(k)}\} \to x_0$, making X sequentially compact.

Finally, if X is sequentially compact, and $\mathcal{G}$ is an open cover of X, then there exists an $\varepsilon > 0$ such that for every $x \in X$, there is a $G \in \mathcal{G}$, with $B(x, \varepsilon) \subseteq G$. Hence there is a sequence $\{x_n\}$ of points of X for which $X = \bigcup B(x_n, \varepsilon)$ (i.e. X is totally bounded). Then there is a $G_n \in \mathcal{G}$ for all $1 \leq k \leq n$ for which $B(x_k, \varepsilon) \subseteq G_k$. So tghat $X = \bigcup G_k$ which makes X compact. ■

Theorem 2.4.5 (Heine-Borel)**.** *A subset K of $\mathbb{R}^n$ is compact if, and only if it is closed and bounded.*

Proof. Suppose that K is compact, then K is closed by lemma 2.4.1, and K is also totally bounded, which makes K bounded. So K is closed and bounded in $\mathbb{R}^n$.

Conversely, suppose that K is closed and bounded. Then there are sequences $\{a_k\}_{k=1}$ and $\{b_k\}_{k=1}^n$ for which $K \subseteq [a_1, b_1] \times [a_n, b_n]$. Now, since $\mathbb{R}^n$ is complete, and K is closed, K is also complete. Hence it remains to show that K is totally bounded. Let $\varepsilon > 0$, and write K as the union of n-dimensional rectangles of diameters less than ε. Then $K \subseteq \bigcup_{k=1}^m B(x_k, \varepsilon)$ where x_k is contained in one of the rectangles, for all $1 \leq k \leq m$. This makes K totally bounded, and therefore, compact. ■

2.5 Continuity and Uniform Convergence in $\mathbb{C}$

Definition. Let (X, d) and (Y, ρ) be metric spaces, and $f : X \to Y$ a function. We say that f is **continuous** at a point $a \in X$ if for every $\varepsilon > 0$, there is a $\delta > 0$ for which

$$\rho(f(x), y) < \varepsilon \text{ whenever } 0 < d(x, a) < \delta$$

for some $y \in Y$ and we write $\lim_{x \longrightarrow a} f(x) = y$, or simply $f \to y$. If f is continuous at every point in X, we say that f is **continuous** on X (or simply that f is **continuous**).

Lemma 2.5.1. *Let X and Y be metric spaces. If $f : X \to Y$ is a function, then the following statements are equivalent for any $a \in X$ with $y = f(a)$.*

(1) f is continuous at a.

(2) For any $\varepsilon > 0$ $f^{-1}(B(y, \varepsilon))$ contains a ball centered about a.

(3) If $\{x_n\}$ is a sequence of points of X converging to a, then the sequence $\{f(x_n)\}$ converges to y.

Lemma 2.5.2. *Let X and Y be metric spaces, and $f : X \to Y$ a function. The following statements are equivalent.*

(1) f is continuous on X.

(2) For any open set U of Y, $f^{-1}(U)$ is open in X.

(3) For any closed set V of Y, $f^{-1}(V)$ is closed in X.

Lemma 2.5.3. *Let $f : X \to \mathbb{C}$ and $g : X \to \mathbb{C}$ be complex-valued functions. If f and g are continuous, then for every $\alpha, \beta \in \mathbb{C}$, we have*

(1) $\alpha f + \beta g$ is continuous.

(2) fg is continuous, and $\frac{f}{g}$ is continuous provided $g(z) \neq 0$ for all $z \in X$.

Lemma 2.5.4. *If $f : X \to Y$ and $g : Y \to Z$ are continuous, then $g \circ f : X \to Z$ is continuous.*

Definition. We call a function $f : X \to Y$ **uniformly continuous** if for every $\varepsilon > 0$, there is a $\delta > 0$, depending on ε, such that

$$\rho(f(x), f(y)) < \varepsilon \text{ whenever } d(x, y) < \delta$$

We call f **Lipschitz continuous** if there exists an $M > 0$ such that

$$\rho(f(x), f(y)) = Md(x, y) \text{ for all } x, y \in X$$

Lemma 2.5.5. *Lipschitz continuous functions are uniformly continuous, and uniformly contiuous functions are continuous.*

Definition. Let X be a metric space, and $A \subseteq X$ a nonempty subset. We define the **distance** from a point $x \in X$ to A to be

$$d(x, A) = \inf \{d(x, a) : a \in A\}$$

Lemma 2.5.6. *Let X a metric space, and $A \subseteq X$ nonempty. The following are true.*

(1) $d(x, A) = d(x, \operatorname{cl} A)$.

(2) $d(x, A) = 0$ *if, and only if* $x \in \operatorname{cl} A$.

(3) $|d(x, A) - d(y, A)| \leq d(x, y)$ *for all* $x, y \in X$.

Proof. Let $A \subseteq B$. Then by definition, $d(x, B) \leq d(x, A)$, so that $d(x, \operatorname{cl} A) \leq d(x, A)$. Now, if $\varepsilon > 0$, there is a $y \in \operatorname{cl} A$ for which $d(x, y) \leq d(x, \operatorname{cl} A) + \frac{\varepsilon}{2}$, and there exists an $a \in A$ with $d(y, a) < \frac{\varepsilon}{2}$. Then

$$|d(x, y) - d(x, a)| < d(y, a) < \frac{\varepsilon}{2}$$

by the triangle inequality. Then $d(x, a) < d(x, y) + \frac{\varepsilon}{2}$ so that $d(x, A) < d(x, \operatorname{cl} A) + \frac{\varepsilon}{2}$. That is $d(x, A) \leq d(x, \operatorname{cl} A)$.

Now, if $x \in \operatorname{cl} A$, then $d(x, \operatorname{cl} A) = d(x, A) = 0$. Conversly, if $d(x, A) = 0$, then consider the decreasing sequence $\{a_n\}$ of A such that $\lim d(x, a_n) = d(x, A)$. Then $\lim d(x, a_n) = 0$ so that $\lim a_n = x$, so that $x \in \operatorname{cl} A$.

Finally, we have for $a \in A$ that $d(x, a) \leq d(x, y) + d(y, a)$, so that $d(x, A) \leq \inf \{d(x, y) + d(y, a) : a \in A\}$ $d(x, y) + d(y, A)$. This gives $d(x, A) - d(y, A) \leq d(x, y)$. Similar reasoning also gives $d(y, A) - d(x, A) \leq d(x, y)$ so that

$$|d(x, A) - d(y, A)| \leq d(x, y) \text{ for all } x, y \in X$$

∎

Corollary. *The function $f : X \to \mathbb{R}$ defined by $f(x) = d(x, A)$ is Lipschitz continuous.*

Theorem 2.5.7. *Let $f : X \to Y$ be continuous. Then following are true.*

(1) If X is compact, then so is $f(X)$.

(2) If X is connected, so is $f(X)$.

Proof. Without loss of generality, suppose $f(X) = Y$. If X is compact, et $\{y_n\}$ a sequence in Y. Then for every $n \geq 1$, there is a sequence of points $\{x_n\}$ of X with $f(x_n) = y_n$, and $\{x_{n_k}\} \to x$. If $y = f(x)$, then by continuity, $\{y_{n_k}\} \to y$ so that Y is also compact.

Now, if X is connected, let $S \subseteq Y$ a nonempty set wich is both open and closed. Then $f^{-1}(S) \neq \emptyset$ and $f^{-1}(S)$ is also open and closed, so that $X = f^{-1}(S)$ by connectivity. This makes $S = Y$, and so Y must also be continuous. ■

Corollary. *If K is compact or connected in X, then $f(K)$ is compact or connected in Y.*

Corollary. *If $f : X \to \mathbb{R}$ is continuous, and X is connected, then $f(X)$ is an interval.*

Theorem 2.5.8 (The Intermediate Value Theorem)**.** *If $f[a, b] \to \mathbb{R}$ is continuous, with $f(a) \leq c \leq f(b)$, then there is an $x \in [a, b]$ with $f(x) = c$.*

Corollary. *If $K \subseteq X$ is compact, then there exist $x_0, y_0 \in K$ with $f(x_0) = \sup\{f(x) : x \in K\}$ and $f(y_0) = \inf\{f(y) : y \in K\}$.*

Corollary. *If $K \subseteq X$ is nonempty, and $x \in X$, there is a $y \in K$ for which $d(x, y) = d(x, K)$.*

Proof. Define $f : X \to \mathbb{R}$ by $f(y) = d(x, y)$. Then f is continuous, and by above, assumes a minimum value $yinK$. Then $f(y) \leq f(x)$ for all $x \in K$, so that $d(x, y) = d(x, K)$ by definition. ■

Theorem 2.5.9. *Let $f : X \to Y$ be continuous. If X is compact, then f is uniformly continuous.*

Proof. Let $\varepsilon > 0$ and suppose there is no such $\delta > 0$ for which the statement holds. Then each $\delta = \frac{1}{n}$ in particular fais. Then there exist $x_n, y_n \in X$ with $d(x_n, y_n) < \frac{1}{n}$, but where $\rho(f(x_n), f(y_n)) \geq \varepsilon$. Now, since X is compact, there is a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ converging to a point $x \in X$. Now, $d(x, y_{n_k}) \leq d(x, x_{n_k}) + \frac{1}{n_k}$ which goes to 0 as $k \to \infty$. SO $\{y_{n_k}\} \to x$. But if, $y = f(x)$, and $y = \lim f(x_{n_k}) = \lim f(y_{n_k})$, then we get

$$\varepsilon \leq \rho(f(x_{n_k}), f(y_{n_k})) \leq \rho(f(x_{n_k}), y) + \rho(y, f(y_{n_k})) = 0$$

which is a contradiction since $\varepsilon > 0$. ■

Definition. If $A, B \subseteq X$ are nonempty subsets of a metric space X, we define the **distance** between A and B to be

$$d(A, B) = \inf\{d(a, b) : a \in A, b \in B\}$$

Theorem 2.5.10. *let A and B be disjoint subsets of a metric space X; with B closed, and A compact. Then $d(A, B) > 0$.*

Proof. Define $f : X \to \mathbb{R}$ by $f(x) = d(x, B)$. Since A and B are disjoint, and B is closed, $f(a) > 0$ for all $a \in A$. Moreover, since A is compact, there is an $a \in A$ for which $0 < f(a) = \inf\{f(x) : x \in A\} = d(A, B)$. ■

Definition. Let X be a set, and (Y,ρ) a metric space; and let $\{f_n\}$ a sequence of functions from X to Y. We say that $\{f_n\}$ **converges uniformly** if for every $\varepsilon > 0$, there is an $N > 0$, dependent on ε such that

$$\rho(f(x), f_n(x)) < \varepsilon \text{ whenever } n \geq N$$

for all $x \in X$. We write $\{f_n\} \xrightarrow{\text{uniformly}} f$, or just $\{f_n\} \to f$.

Theorem 2.5.11. *If $f_n : X \to Y$ is continuous for each $n \geq 1$, and $\{f_n\} \xrightarrow{\textit{uniformly}} f$, then f is also continuous.*

Proof. Fix $x_0 \in X$ and let $\varepsilon > 0$. Since $\{f_n\} \to f$, there is a function f_n for which $\rho(f(x), f_n(x)) < \frac{\varepsilon}{3}$ for every $x \in X$. Since f_n is continuous, there is a $\delta > 0$ such that

$$\rho(f_n(x_0), f_n(x)) < \frac{\varepsilon}{3} \text{ whenever } d(x, x_0) < \delta$$

Therefore, if $d(x_0, x) < \delta$ we have

$$\rho(f(x_0), f(x)) \leq \rho(f(x_0), f_n(x_0)) + \rho(f_n(x_0), f_n(x)) + \rho(f_n(x), f(x)) < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon$$

so that f is continuous. ∎

Theorem 2.5.12 (The Weierstrass M-test)**.** *Let $u_n : X \to \mathbb{C}$ be a function such that $|u_n(x)| \leq M_n$, for all $x \in X$, and suppose that the sum $\sum M_n$ is finite. THen $\sum u_n$ is uniformly convergent.*

Proof. Let $f_n(x) = u_1(x) + \cdots + u_n(x)$. Then for $n > m$, $|f_n(x) - f_m(x)| = |u_{m+1}(x) + \cdots + u_n(x)| \leq \sum_{k=m+1}^{n} M_k$. Since $\sum M_k$ is finite, this sum converges, so that $\{f_n\}$ is Cauchy in $\mathbb{C}$. That is, there exists a $\xi \in \mathbb{C}$ for which $\{f_n(x)\} \to \xi$. Define then $f(x) = \xi$, then $f : X \to \mathbb{C}$ is a function with

$$|f(x) - f_n(x)| = |u_{m+1}(x) + \cdots + u_n(x)| \leq \sum_{k=m+1}^{n} |u_k(x)| \leq \sum_{k=m+1}^{n} M_k$$

Then for every $\varepsilon > 0$, there is an $N > 0$ such that $\sum M_k < \varepsilon$, whenever $n \geq N$. THus $|f(x) - f_n(x)| < \varepsilon$ for all $x \in X$. ∎

Chapter 3

Analytic Functions

3.1 Convergent Power Series

Definition. For a sequence $\{a_n\}$ of points of $\mathbb{C}$, the series $\sum_{n=0}^{\infty} a_n$ is said to **converge** to a point $z \in \mathbb{C}$ if for all $\varepsilon > 0$, there is an $N \in \mathbb{Z}^+$ such that $|s_m - z| < \varepsilon$, whenever $m \geq N$; where

$$s_m = \sum_{n=0}^{m} = a_n$$

is the n**-th partial sum**. Se say that the series $\sum a_n$ **converges absolutely** if the series $\sum |a_k|$ converges.

Lemma 3.1.1. *Let $\{a_n\}$ a sequence of points in $\mathbb{C}$. If the series $\sum a_n$ converges absolutely, then it converges.*

Proof. Let $\varepsilon > 0$ and put $z_n = a_0 + a_1 + \cdots + a_n$. Since the series $\sum |a_n|$ converges, there is an $N \in \mathbb{Z}^+$ such that $\sum_{n=N}^{\infty} |a_n| < \varepsilon$. Tus, if $m > k \geq N$, we have

$$|z_m - z_k| = \Big| \sum_{n=k+1}^{m} |a_n| \Big| \leq \sum_{n=k+1}^{m} |a_n| \leq \sum_{n=N}^{m} |a_n| < \varepsilon$$

This makes $\{z_n\}$ a Cauchy sequence in $\mathbb{C}$, si that $\{z_n\} \to z$. Therefore $\sum a_n = z$. ■

Definition. Let $\{a_n\}$ a sequence of points of $\mathbb{C}$. A **power series** about a point $z_0 \in \mathbb{C}$ is a series of the form

$$\sum a_n (z - z_0)^n$$

We say the power series is **convergent**, if the series converges.

Example 3.1. The **geometric series** $\sum z^n$ is a power series. Notice that

$$1 - z^{n+1} = (1 - z)(1 + z + \cdots + z^n)$$

so that

$$1 + z + \cdots + z^n = \frac{1 - z^{n+1}}{1 - z}$$

Now, when $|z| < 1$, $z^n \to 0$ and the series

$$\sum z^n = \frac{1}{1-z}$$

When $|z| > 1$, the series diverges.

Theorem 3.1.2. *Let $S = \sum a_n(z-z_0)^n$ be a power series, and define R such tht $0 \le R \le \infty$ by*

$$\frac{1}{R} = \limsup \sqrt[n]{|a_n|}$$

Then the following hold

(1) If $|z - z_0| < R$, then S converges absolutely.

(2) If $|z - a| > R$, then S diverges.

(3) If r is such that $0, r < R$, then S converges uniformly on the open ball $B(z_0, r)$.

Proof. Suppose without loss of generality, that $z_0 = 0$. If $|z| < R$, then there exists an r with $|z| < r < R$ and hence an $N \in \mathbb{Z}^+$ such that $\sqrt[n]{|a_n|} = \frac{1}{r}$ for all $n \ge N$; since $\frac{1}{r} < \frac{1}{R}$. Then we get

$$|a_n| < \frac{1}{r^n}$$

and so $|a_n z^n| < (\frac{|z|}{r})^n$. Hence, the tail, $\sum_{n=N}^{\infty} a_n z^n$ is dominated by the sum $\sum (\frac{|z|}{r})^n$, and since $\frac{|z|}{r} < 1$, we get that S converges absolutely for all $|z| < R$; i.e. the ball $B(0, R)$.

Now, suppose that $r < R$ and choose a $r < \rho < R$ as above. Take $N \in \mathbb{Z}^+$ such that $|a_n| < \frac{1}{\rho^n}$ for all $n \ge N$. Then if $|z| \le r$, $|a^n_z| \le (\frac{z}{\rho})^n$ and $\frac{r}{\rho} < 1$. By the Weierstrass M-test, we get that the series S converges uniformly on the ball $B(0, r)$.

Now, let $|z| > R$ and choose an r with $|z| > r > R$ so that $\frac{1}{r} < \frac{1}{R}$. Then $\sqrt[n]{|a_n|}$ gives infinitely many integers n with $\frac{1}{r} < \sqrt[n]{|a_n|}$. Hence

$$|a_n z^n| > \left(\frac{|z|}{r}\right)^n$$

and since $\frac{|z|}{r} > 1$, the terms become unbounded, making S diverge. ■

Definition. We define the **radius of convergence** of a power series $\sum a_n(z_- z_0)^n$ to be a number R such that $0 \le R \le \infty$ and the following hold

(1) If $|z - z_0| < R$, then S converges absolutely.

(2) If $|z - a| > R$, then S diverges.

(3) If r is such that $0, r < R$, then S converges uniformly on the open ball $B(z_0, r)$.

Lemma 3.1.3. *If $\sum a_n(z - z_0)^n$ is a power series with radius of convergence $R > 0$, then*

$$R = \lim_{n \longrightarrow \infty} \left| \frac{a_n}{a_{n+1}} \right|$$

Proof. Without loss of generality, let $z_0 = 0$ and take $\alpha = \lim |\frac{a_n}{a_{n+1}}|$, and suppose that this limit does indeed exist. Suppose that $|z| < r < \alpha$ and take $N \in \mathbb{Z}^+$ such that $r < |\frac{a_n}{a_{n+1}}|$ for all $n \geq N$. Take $B = |a_N|r^N$. Then $|a_{N+1}r^{N+1} = |a_{N+1}|rr^N < |a_N|r^N = B$. That $|a_{N+2}|r^{N+2} = |a_{N+2}|rr^{N+2} < |a_{N+1}|r^{N+1} < B$. By induction we get $|a_n|r^N \leq B$ for all $n \geq N$. Then $|a_nz^n| = |a_nr^n|\frac{|z|^n}{r}$ for all $n \geq N$. Since $|z| < r$, we get that the series $\sum |a_nz^n|$ is dominated by a convergent series and hence is convergent itself.

Now, if $|z| > r > \alpha$. then $|a_n| < r|a_{n+1}|$ for all $n \geq N$, for some $N \in \mathbb{Z}^+$. We find that

$$|a_nr^n| \geq B = |a_Nr^N|$$

so we get

$$|a_nz^n| \geq B\frac{|z|^n}{|r|^n}$$

and $B\frac{|z|^n}{|r|^n} \to \infty$ as $n \to \infty$. Therefore the series $\sum a_nz^n$ diverges so that $R \leq \alpha$. THis makes $R = \alpha$ and we are done. ■

Example 3.2. The **exponential series** defined by

$$\exp z = \sum \frac{z^n}{n!}$$

converges on all $\mathbb{C}$ and has radius of convergence $R = \infty$.

Lemma 3.1.4. *LEt $\sum a_n(z - z_0)^n$ and $\sum b_n(z - z_0)^n$ be convergent power series with radi of convergence greater than some $r > 0$. Let $c_n = \sum_{k=0}^n a_kb_{n-k}$. THen the series*

$$\sum (a_n + b_n)(z - z_0)^n \text{ and } \sum c_n(z - z_0)^n$$

are convergent power series with radi of convergent greater than r.

3.2 Analytic Functions

Definition. Let U be an open set in $\mathbb{C}$, and $f : U \to \mathbb{C}$ a complex valued function. We call f **complex differentiable** at a point $z_0 \in U$ if

$$f'(z_0) = \lim_{h \longrightarrow 0} \frac{f(z_0 + h) - f(z_0)}{h} = \lim_{z \longrightarrow z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

exists, and we call $f'(z_0)$ the **complex derivative** of f at z_0. We call f **complex differentiable** on U if it is complex differentiable at every point $z_0 \in U$, and the value $f'(z_0)$ defines a function $f' : U \to \mathbb{C}$ called the **complex derivative** of f on U. If f is complex differentiable on U, and f' exists on U and is continuous, then we call f **continuously differentiable**, and say it is of class C^1.

Definition. Let U be an open set of $\mathbb{C}$, and let $f : U \to \mathbb{C}$ be a continuously differentiable complex valued function; i.e. of class C^1. We define the n**-th derivative** of f recuresively as

(1) $f^{(0)}(z) = f$ and $f^{(1)} = f'(z)$.

(2) $f^{(n+1)}(z) = (f^{(n)})'(z)$, provided $f^{(n)}(z)$ exists.

We say that f is **n-th differentiable** if $f^{(n)}$ exists. We say that f is **q-smooth** if $f^{(q)}(z)$ exists for some $q \geq 0$, and $f^{(q)}$ is continuous, and we call f of class C^q. We call f **smooth** if $f^{(n)}(z)$ exists and is continuous for any n, and we call f of class C^∞.

Lemma 3.2.1. *If $f : U \to \mathbb{C}$ is a complex valued function, complex differentiable at a point $z_0 \in U$, then f is continuous on U.*

Proof. We have

$$|f(z) - f(z_0)| = \left|\frac{f(z) - f(z_0)}{z - z_0}\right||z - z_0|$$

Taking $z \to z_0$, we see that $|f(z) - f(z_0)| \to |f'(z_0)| \cdot 0 = 0$. ∎

Corollary. *If f is n-th differentiable, for all $n \in \mathbb{Z}^+$, then f is of class C^∞.*

Definition. We call a complex valued function $f : U \to \mathbb{C}$, on an open set U of $\mathbb{C}$, **analytic** on U if it is of class C^1 on U.

Theorem 3.2.2. *Let U be open in $\mathbb{C}$, and let $f : U \to \mathbb{C}$ and $g : U \to \mathbb{C}$ analytic on U. Then*

(1) $f + g$ is analytic on U with $(f + g)'(z) = f'(z) + g'(z)$.

(2) fg is analytic on U, with $(fg)'(z) = f'(z)g(z) + f(z)g'(z)$.

(3) For any $\alpha \in \mathbb{C}$, αf is analytic on U with $(\alpha f)'(z) = \alpha f'(z)$.

Corollary. *$\frac{f}{g}$ is analytic on U, provided that $g(z) \neq 0$ for all $z \in U$, and*

$$\left(\frac{f}{g}\right)'(z) = \frac{f'(z)g(z) - f(z)g'(z)}{g(z)^2}$$

Theorem 3.2.3 (The Chain Rule)**.** *Let U and V be open in $\mathbb{C}$. Let $f : U \to \mathbb{C}$ be analytic on U and let $g : V \to \mathbb{C}$ be analytic on V, with $f(U) \subseteq V$. Then $g \circ f$ is analytic on U, with*

$$(g \circ f)'(z) = g' \circ f(z)f'(z)$$

Proof. Fix $z_0 \in U$, and let $r > 0$ such that the open ball $B(z_0, r)$ is contained in U; i.e. that $B(z_0, r) \subseteq U$. It suffices to show that if $0 < |h_n| < r$, where the sequence $\{h_n\} \to 0$ as $n \to \infty$, then

$$\lim \frac{g \circ f(z_0 + h_n) - g \circ f(z_0)}{h_n}$$

exists and equals $g' \circ f(z_0)f'(z_0)$.

Suppose first that $f(z_0) \neq f(z_0 + h_n)$ for all $n \in \mathbb{Z}^+$. Then we have

$$\frac{g \circ f(z_0 + h_n) - g \circ f(z_0)}{h_n} = \frac{g \circ f(z_0 + h_n) - g \circ f(z_0)}{f(z_0 + h_n) - f(z_0)} \cdot \frac{f(z_0 + h_n) - f(z_0)}{h_n}$$

Since $\lim (f(z_0 + h) - f(z_0)) = 0$, we get

$$\lim \frac{g \circ f(z_0 + h_n) - g \circ f(z_0)}{h_n} = g' \circ f(z_0) f'(z_0)$$

Now, suppose that $f(z_0) = f(z_0 + h_n)$ for infinitely many n. Write $\{h_n\} = \{k_n\} \cup \{l_n\}$, where $f(z_0) \neq f(z_0 + k_n)$ and $f(z_0) = f(z_0 + l_n)$ for all n. Since f is analytic, and hence complex differentiable

$$f'(z_0) = \lim \frac{f(z_0 + h_n) - f(z_0)}{l_n} = 0$$

so that

$$\lim \frac{g \circ f(z_0 + l_n) + g \circ f(z_0)}{l_n} = 0$$

Now, by above we also get that

$$\lim \frac{g \circ f(z_0 + k_n) + g \circ f(z_0)}{k_n} = g' \circ f(z_0) f'(z_0)$$

so that $g' \circ f(z_0) f'(z_0) = 0$. ∎

Definition. Let $A \subseteq \mathbb{C}$ an arbitrary set of $\mathbb{C}$. We call a complex valued function f **analytic** on A if it is analytic on some open set of $\mathbb{C}$ containing A.

Theorem 3.2.4. *Let $f(z) = \sum a_n (z - z_0)^n$ a convergent power series with radius of convergence $R > 0$. Then the following are true.*

(1) The series $\sum \frac{n!}{(n-k)!} a_n (z - z_0)^{n-1}$ converges with radius of convergence R.

(2) f is smooth on the ball $B(z_0, R)$ with

$$f^{(n)}(z) = \sum \frac{n!}{(n-k)!} a_n (z - z_0)^{n-1}$$

(3) For all $n \geq 0$

$$a_n = \frac{f^{(n)}(z_0)}{n!}$$

Proof. Suppose without loss of generality that $z_0 = 0$. Then it is sufficient to show that f' exists and has the power series $\sum n a_n z^{n-1}$. We have by definition that

$$\frac{1}{R} = \limsup \sqrt[n]{|a_n|}$$

Now, by L'ôpital's rule, we have

$$\lim \frac{\log n}{n - 1} = 0$$

so that $\lim \sqrt[n-1]{n} = 1$. Let

$$\frac{1}{R'} = \limsup \sqrt[n-1]{|a_n|}$$

Then R' is the radius of convergence of the series

$$\sum na_n z^{n-1}$$

Notice that $\sum a_n z^{n-1} = \sum a_{n+1} z^n$, so that

$$z \sum a_{n+1} z^n + a_0 = \sum a_n z^n$$

Now, if $|z| < R$, and $z \neq 0$, then $\sum |a_n z^n| = \frac{1}{|z|}$. Moreover,

$$\sum |a_n z^n| + \frac{1}{|z|} a_0 < \infty$$

which makes $R \leq R'$. Therefore $R = R'$, and so the series $\sum na_n z^{n-1}$ converges and has radius of convergence R.

Now, for $|z| < R$, put $g(z) = \sum na_n z^{n-1}$ and $s_n(z) = \sum_{k=0}^{n} a_k z^n$ and $R(z) = \sum_{k=n+1}^{\infty} a_k z^k$. Let $w \in B$. Let $w \in B(0, R)$, the open ball of radius R about 0, and fix r such that $|w| < r < R$. Let $\delta > 0$ such that the closed ball $\overline{B}(w, \delta)$ of radius δ about w is contained in $B(0, R)$; that is, $\overline{B}(w, \delta) \subseteq B(0, R)$. Let $z \in B(w, \delta)$, then we see that

$$\frac{f(z) - f(w)}{z - w} - g(w) = \Big(\frac{s_n(z) - s_n(w)}{z - w} - s'(w)\Big) + s_n'(w) - g(w) + \frac{R_n(z) - R_n(w)}{z - w}$$

So that

$$\frac{f(z) - f(w)}{z - w} - g(w) = \frac{1}{z - w} \sum_{k=n+1}^{\infty} a_k (z^k - w^k)$$

However, notice that

$$\Big|\frac{z^k - w^k}{z - w}\Big| = |z^{k-1} + z^{k-2} w + \cdots + z w^{k-2} + w^{k-1}| \leq k r^{k-1}$$

Hence

$$\Big|\frac{R_n(z) - R_n(w)}{z - w}\Big| \leq \sum_{k=n+1}^{\infty} |a_k| k r^{k-1}$$

Since $r < R$, $\sum |a_k| k r^{k-1}$ converges so that for any $\varepsilon > 0$, there is an $N_1 \in \mathbb{Z}^+$ such that

$$\Big|\frac{R_n(z) - R_n(w)}{z - w}\Big| < \frac{\varepsilon}{3}$$

whenever $n \geq N_1$ and for all $z \in B(w, \delta)$. Also, notice that $\lim s_n'(w) = g(w)$, and that there exists an $N_2 \in \mathbb{Z}^+$ such that

$$|s_n'(w) - g(w)| < \frac{\varepsilon}{3}$$

whenever $n \geq N_2$. Now, let $N = \max\{N_1, N_2\}$ and choose $\delta > 0$ such that

$$\Big|\frac{s_n(z) - s_n(w)}{z - w}\Big| < \frac{\varepsilon}{3}$$

whenever $0 < |z - w| < \delta$. Then

$$\left|\frac{f(z) - f(w)}{z - w} - g(w)\right| < \varepsilon$$

whenever $0 < |z - w| < \delta$. Therefore $f'(z) = g(z)$.

Finally, observe that $f(0) = f^{(0)}(0) = a_0$. Using the power series

$$f^{(n)}(z) = \sum \frac{n!}{(n-k)!} a_n z^{n-k}$$

we find that

$$f^{(n)}(0) = n! a_n$$

and we are done. ■

Corollary. *If $f(z) = \sum a_n (z - z_0)^n$ is a convergent power series with radius of convergence $R > 0$, then f is analytice on the open ball $B(z_0, R)$.*

Example 3.3. $\exp z = \sum \frac{z^n}{n!}$ is analytic on all of $\mathbb{C}$.

Lemma 3.2.5. *If U is open and connected in $\mathbb{C}$, and $f : U \to \mathbb{C}$ is complex differentiable on U with $f'(z) = 0$, then f is constant.*

Proof. Take $z_0 \in U$ and let $\omega_0 = f(z_0)$. Take $A = \{z \in U : f(z) = \omega_0\}$. And pick a point $z \in U$. Let $\{z_n\}$ a sequence of points in A converging to z; i.e. $\{z_n\} \to z$. Since $f(z_n) = \omega_0$ for all $n \geq 0$, and if f is continuous, we have $f(z) = \omega_0$ which makes $z \in A$, and A is closed in U.

Now, let $a \in A$ and take $e > A$ such that $B(a, \varepsilon) \subseteq U$. If $z \in B(a, \varepsilon)$, let

$$g(z) = f(tz + (1-t)a) \text{ where } 0 \leq t \leq 1$$

Then

$$\frac{g(t) - g(s)}{t - s} = \frac{g(t) - g(s)}{(t-s)z + (s-t)a} \frac{(t-s)z + (s-t)a}{t - a}$$

thus, if $t \to s$ we get

$$g'(s) = \lim \frac{g(t) - g(s)}{t - s} = f'(sz + (1-s)a)(z - a) = 0 \text{ for all } 0 \leq s \leq 1$$

This makes g constant, so that $f(z) = g(1) = g(0) = f(a) = \omega_0$ and hence $B(a, \varepsilon)A$ which makes A open. Therefore $A = U$ and this makes f constant on U. ■

Example 3.4. (1) Differentiating $f'(z) = \exp z$ we get

$$f'(z) = \sum \frac{n}{n!} z^{n-1} = \sum \frac{z^{n-1}}{(n-1)!} = \sum \frac{z^n}{n!}$$

which makes $f'(z) = f(z)$. Taking $e^z = \exp z$, that is

$$\frac{d}{dz} e^z = e^z$$

Now, take $g(z) = e^z e^{a-z}$. Then $g'(z) = e^z e^{a-z} - e^z e^{a-z} = 0$ so that g is constant, and $g(z) = \omega$ for all $z \in \mathbb{C}$. Taking $e^0 = 1$, we get $\omega = g(0) = e^a$; moreover that $e^z e^{a-z} = a^a$. This shows that

$$e^{a+b} = e^a e^b \text{ and } e^z e^{-z} = 1 \text{ for all } a, b, z \in \mathbb{C}$$

and that

$$e^{-z} = \frac{1}{e^z}$$

since $e^z \neq 0$ for all $z \in \mathbb{C}$.

Moreover, notice that by the power series expansion, we have

$$\overline{\sum \frac{z^n}{n!}} = \sum \frac{\overline{z}^n}{n!}$$

so that

$$\overline{\exp z} = \exp \overline{z}$$

Now, if $\theta \in \mathbb{R}$, notice that $|e^{i\theta}| = e^{i\theta} e^{-i\theta} = e^0 = 1$ and $|e^z|^2 = e^z e^{\overline{z}} = e^{z+\overline{z}} = 2\operatorname{Re} z$, and $|\exp z| = \exp 2\operatorname{Re} z$.

(2) Define the follwing series $\sin z$ and $\cos z$ by

$$\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} + \cdots + (-1)^n \frac{z^{2n-1}}{(2n-1)!}$$
$$\cos z = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} + \cdots + (-1)^n \frac{z^{2n}}{(2n)!}$$

Then it can be shown that $\sin z$ and $\cos z$ are convergent power series with radius of convergence $R = \infty$, so that they are also analytic on all of $\mathbb{C}$. Differentiating the power series, we find that

$$\frac{d}{dz}\sin z = \cos z \text{ and } \frac{d}{dz}\cos z = -\sin z$$

We can also find that

$$\sin z = \frac{e^{iz} - e^{-iz}}{2} \text{ and } \cos z = \frac{e^{iz} + e^{-iz}}{2}$$

which shows that

$$\cos^2 z + \sin^2 z = 1$$

In partiuclar if $\theta \in \mathbb{R}$, we get $e^{i\theta} = \cos\theta + i\sin\theta$, and hence, for all $z \in \mathbb{Z}$, we get $z = |z|e^{i\theta}$ where $\theta = \arg z$. Since $\exp(a+b) = (\exp a)(\exp b)$, we find that $\arg \exp z = \operatorname{Im} z$.

Definition. We call a complex valued function f on $\mathbb{C}$ **periodic**, if there exists a $c \in \mathbb{C}$ such that $f(z+c) = f(z)$ for all $z \in \mathbb{C}$. We call c the **period** of f.

Example 3.5. If c is the period of $\exp z$, then $\exp z = \exp(z+c) = e^z e^c$, implying that $\exp c = 1$. Thus $\operatorname{Re} c = 0$, and so $c = i\theta$ for some $\theta \in \mathbb{R}$. Now, observe that

$$\exp c = \exp i\theta = \sin\theta + i\sin\theta = 1$$

which makes $\theta = 2\pi k$, and hence $c = 2i\pi k$, where $k \in \mathbb{Z}$.

Example 3.6 (The Complex Logarithm)**.** Define $\log w$ such that $w = \exp z$ whenever $z = \log w$. Since $\exp z \neq 0$ for all $z \in \mathbb{C}$, $\log 0$ is undefined. Now, let $\exp z = w$, where $w \neq 0$, if $z = x + iy$, then $|w| = e^x$ and $y = \arg w + 2\pi k$ for some $k \in \mathbb{Z}$. So the set of solutions to $\exp z$ is given by all $\log|w| + (\arg w + 2\pi k)$. Notice then that $\log|w|$ defines the natural logarithm of $|w|$ by definition of $\exp z$.

Definition. We define a **region** of $\mathbb{C}$ to be an open and connected set of $\mathbb{C}$.

Definition. Let U be a region of $\mathbb{C}$, and let $f : U \to \mathbb{C}$ a continuous complex valued function such that $z = \exp f(z)$. Then we call f a **branch of the logarithm**.

Lemma 3.2.6. *If U is a region in $\mathbb{C}$, and f is a branch of the logarithm in U, then then any other branch of logarithm is of the form*

$$f(z) + 2i\pi k \textit{ where } k \in \mathbb{Z}^+$$

Proof. Let f be a branch of the logarithm in U and let $k \in \mathbb{Z}$. Take $g(z) = f(z) + 2i\pi k$. Then $\exp g = \exp f = z$ so that g is also a branch of the logarithm.

Conversely, suppose that f and g are both branches of the logarithm; then $g(z) = f(z) + 2i\pi k$, for some k (not necessarrily an integer). Now, define

$$h(z) = \frac{1}{2i\pi}(g(z) - f(z))$$

Then h is continuous on U, and $h(U) \subseteq \mathbb{Z}$. Now, since U is connected, then so is $h(U)$. Noticing that $h(z) = k$, this makes $k \in \mathbb{Z}$, and we are done. ■

Definition. Let $U = \mathbb{C}\backslash\mathbb{R}_{\leq 0}$; that is, the complex numbers $\mathbb{C}$ slit about the negative real axis. Notice that U is connected, and that $z = re^{i\theta}$ where $r = |z|$ and $-\pi < \theta < \pi$. Define $f(z) = \log r + i\theta$. Since f is the sum of two continuous functions, i.e; $\log r$ for all $r \in \mathbb{R}\backslash\{0\}$ and the map $\theta \to i\theta$, we get that f is continuous on U. Therefore f defines a branch of the logarithm

Lemma 3.2.7. *Let U and V be open in $\mathbb{C}$. Let $f : U \to \mathbb{C}$ and $g : V \to \mathbb{C}$ be continuous complex valued functions with $f(U) \subseteq V$ and $g \circ f(z) = z$. Then if g is complex differentiable on V, f is complex differentiable on U, with*

$$f'(z) = \frac{1}{g' \circ f(z)}$$

Proof. Fix $z_0 \in U$ and let $h \in \mathbb{C}$ such that $h \neq 0$ and $h + z_0 \in U$. Then notice that $z_0 = g \circ f(z_0)$ and $z_0 + h = g \circ f(z_0 + h)$. This makes $f(z_0) \neq f(z_0 + h)$. Now, observe that

$$\frac{g \circ f(z + h_0) - g \circ f(z_0)}{h} = \frac{g \circ f(z + h_0) - g \circ f(z_0)}{f(z_0 + h) - f(z_0)} \frac{f(z_0 + h) - f(z_0)}{z - z_0} = 1$$

Now, taking the limit of both sides as $h \to 0$, we get $g' \circ f(z_0) f'(z_0) = 1$, and $f'(z_0)$ exists since $g' \circ f(z_0) \neq 0$. ■

Corollary. *If g is analytic on V, then f is analyitic on U.*

Corollary. *If U is a connected; i.e. a region, and f is a branch of the logarithm on U, then f is analytic.*

Proof. Observe by definition that $z = \exp f(z)$. ■

Definition. We define the **principle branch of the logarithm** on $\mathbb{C} \backslash \mathbb{R}_{\leq 0}$ to be the complex function $\log : \mathbb{C} \backslash \mathbb{R}_{\leq 0} \to \mathbb{C}$ defined by

$$\log z = \log r + i\theta \text{ where } z = r \exp i\theta$$

Definition. If f is a branch of the logarithm on some region U, and if $b \in \mathbb{Z}$. The **branch** of $bf(z)$ to be a complex function $g : U \to \mathbb{C}$ defined by $g(z) = expbf(z)$. We write $g(z) = z^b$, when the branch f is the principle branch of the logarithm.

Lemma 3.2.8. *Let U be a region, and f a branch of the logarithm and let $b \in \mathbb{Z}$. Then the branch of $bf(z)$, $g(z) = \exp bf(z)$ is analytic.*

Corollary. *The branch z^b of $\log z$ is analytic.*

Definition. Let $u : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ and $v : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be realvalued functions. We define the **Cauchy Riemann equations** to be the systems of equations

$$\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} = 0$$
$$\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = 0$$

Lemma 3.2.9. *Let $f : U \to \mathbb{C}$ a complex valued function with $f(z) = f(x + iy) = u(x, y) + iv(x, y)$. Then if f is analytic, we have*

$$f'(z) = \frac{\partial u}{\partial x}(x, y) + i\frac{\partial v}{\partial x}$$

Proof. We evaluate $f'(z)$ as $h \to 0$ along the real axis $\mathbb{R}$ of $\mathbb{C}$. ■

Lemma 3.2.10. *Let $f : U \to \mathbb{C}$ a complex valued function with $f(z) = f(x + iy) = u(x, y) + iv(x, y)$. Then if f is analytic, we have*

$$f'(z) = \frac{\partial v}{\partial y}(x, y) - i\frac{\partial u}{\partial y}$$

Proof. We evaluate $f'(z)$ as $h \to 0$ along the imaginary axis $i\mathbb{R}$ of $\mathbb{C}$. ■

Theorem 3.2.11. *Let $u(z) = u(x, y)$ and $v(z) = v(x, y)$, where $z = x + iy$ be real valued functions on a region U with continuous partial derivarives. If $f : U \to \mathbb{C}$ is a complex valued function such that $f(z) = u(x, y) + iv(x, y)$, then f is analytic on U if, and only if u and v satisfy the Cauchy Riemann equations.*

Proof. Suppose that u and v satisfy the Cauchy Riemann equations, and let $B(z, r) \subseteq U$. If $h = s + itB(0, r)$, then

$$u(x + s, y + t) - u(x, y) = (u(x + s, y + t) - u(x, y + t)) + (u(x, y + t) - u(x, y))$$

By the mean value theorem for real valued functions, we have that there exist $s_1, t_1 \in B(0, r)$ with $|s_1| < |s|$ and $|t_1| < |t|$ for which

$$u(x + s, y + t) - u(x, y + t) = u_x(x + s_1, y + t)s$$

and

$$u(x, y + t) - u(x, y + t) = u_y(x, y + t)t$$

Letting

$$\phi(s, t) = (u(x + s, y + y) - u(x, y)) - (u_x(x, y)s - u_y(x, y)t)$$

Then

$$\frac{\phi(s, t)}{s + it} = \frac{s}{s + it}(u_x(x + s_1, y + t) - u_x(x, y)) + \frac{t}{s + it}(u_y(x, y + t_1) - u_y(x, y))$$

Now, since $|s| \leq |s + it|$ and $|t| \leq |s + it|$, and u_x and u_y are continuous, we get

$$\lim_{s+it \longrightarrow 0} \frac{\phi(s, t)}{s + it} = 0$$

So that

$$u(x + s, y + t) - u(x, y) = u_x(x, y)s + u_y(x, y)t + \phi(s, t)$$

Similarly, for v we get

$$v(x + s, y + t) - v(x, y) = v_x(x, y)s + v_y(x, y)t + \psi(s, t)$$

where

$$\lim_{s+it \longrightarrow 0} \frac{\psi(s, t)}{s + it} = 0$$

Notice then that

$$\frac{f(z + (s + it)) - f(z)}{s + it} = u_x(x, y) + iv(x, y) + \frac{\phi(s, t) + i\psi(s, t)}{s + it}$$

Taking $s + it \to 0$, makes f complex differentiable on U with $f'(z) = u_x(x, y) + iv(x, t)$. Since u_x and v_x are continuous, so is f, which makes f analytic.

Conversely, if we suppose that f is analytic, then by lemma 3.2.9 and lemma 3.2.10, we get

$$\frac{\partial u}{\partial x}(x, y) + i\frac{\partial v}{\partial x} = \frac{\partial v}{\partial y}(x, y) - i\frac{\partial u}{\partial y}$$

which shows that u and v satisfy the Cauch Riemann equations. ■

Definition. We call a real valued function $u : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ **harmonic** if

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

If $v : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a real valued function, and f is a complex valued function for which $f(z) = u(x, y) + iv(x, y)$, then we call v the **harmonic conjugate** of u.

Example 3.7. The function $u(x, y) = \log \sqrt{x^2 + y^2}$ is harmonic.

Theorem 3.2.12. *Let U be an open ball, or $\mathbb{C}$. If $u : U \to \mathbb{C}$ is harmonic, then u has a harmonic conjugate.*

Proof. Let $U = B(0, R)$ for $0 \leq R \leq \infty$, and let $u : U \to \mathbb{C}$ be harmonic. Define

$$v(x, y) = \int_0^y u_x(x, t)\ dt + \phi(x)$$

wheere ϕ is determined by taking $v_x + u_y = 0$. Then differentiating both sides, and by Leibniz's rule for differentiating under the integral, we have

$$\begin{aligned} v_x(x, y) &= \int_0^y u_{xx}(x, t)\ dt + \phi'(t) \\ &= -\int_0^y u_{xx}(x, t)\ dt + \phi'(x) \\ &= -u_y(x, y) + u_y(x, 0) + phi'(x) \end{aligned}$$

and $\phi'(x) = u_y(x, 0)$. Then u and $v = \int_0^y u_x(x, t)\ dt - \int_0^x u_y(s, 0)\ ds$ satisfy the Cauchy Riemann equations. This makes v a harmonic conjugate. ∎

3.3 Möbius Transformations

Definition. We define a **path** on $\mathbb{C}$ to be a continnuous complex valued functiom $\gamma : [a, b] \to \mathbb{C}$. We call $\gamma(a)$ the **initial point** of γ and $\gamma(b)$ the **end point** of γ. Moreover wecall γ **piecewise** C^1 if there exists a partition $P = \{a = t_0 < \cdots < t_n = b\}$ of $[a, b]$ such that γ is C^1 on each $[t_i, t_{i+1}]$ for all $0 \leq i \leq n - 1$.

Definition. We call a complex valued function S on C_∞, defined by

$$S(z) = \frac{az + b}{cx + d} \text{ where } a, b, c, d \in \mathbb{C}$$

a **Möbius transformation** if $ad - cd \neq 0$.

We have the following results for Möbius transformations.

Theorem 3.3.1. *The set of all Möbius transformations forms a group under function compositon.*

Proof. Since $\circ$ is associative, it suffices to show the identity, closure, and inverse laws. Indeed, notice that the function

$$I = \frac{1z+0}{0z+1} = z$$

is a Möbius transformation, and that for any Möbius transformation, S, $S = S \circ I = I \circ S$.

Now, Let S and T be Möbius transformations. Then

$$S(z) = \frac{az+b}{cz+d} \text{ and } T(z) = \frac{fz+g}{hz+l}$$

for $a, b, c, d, f, g, h, l \in \mathbb{C}$. Then we get

$$S \circ T(z) = \frac{aT(z)+b}{cT(z)+b}$$

and since $T(z) \in \mathbb{C}_\infty$, for all values of ζ, then $S \circ T$ is a Möbius transformation.

Finally, let

$$S^{-1}(z) = \frac{dz-b}{cz-a}$$

Then $S \circ S^{-1} = S^{-1} \circ S(z) = I(z)$, and we are done. ■

Definition. Let $a \in \mathbb{C}$. We call a Möbius trasnsformation of the form $T(z) = z + a$ a **translation** of z by a. We call a Möbius transformation $D(z) = az$ a **dilation** of z by a. Let $0 \leq t \leq 2\pi$. Then we call the Möbius transformation $R(z) = e^{it}z$ a **rotation** of z about t, and we call the Möbius transformation $S(z) = \frac{1}{z}$ an **inversion** of z.

Lemma 3.3.2. *If S is a Möbius transformation, then S is the composition of translations, dilations, and inversions.*

Proof. Let

$$S(z) = \frac{az+b}{cz+d}$$

Suppose that $c = 0$, so that $S(z) = \frac{a}{d}z + b$. Then $S = S_2 \circ S_1$ where $S_1(z)$ is a translation by b and $S_2(z)$ is a dilation by $\frac{a}{d}$.

Now, if $c \neq 0$, then let $S_1(z) = z + \frac{d}{c}$, $S_2(z) = \frac{1}{z}$, $S_3(z) = \frac{bc-ad}{c^2}z$ and $S_4(z) = z + \frac{a}{c}$. Then $S = S_4 \circ S_3 \circ S_2 \circ S_1$. ■

Corollary. *Rotations are compositions of translations, dilations, and inversions.*

Definition. Let S be a Möbius transformation. We cal a point $z \in \mathbb{C}$ a **fixed point** of S if $S(z) = z$.

Lemma 3.3.3. *If*

$$S(z) = \frac{az+b}{cz+d}$$

is a Möbius transformation, and z is a fixed point, then $cz^2 + (d-a)z - b = 0$ and S has at most two fixed points; unless it is the identity transformation.

Proof. Suppose that $S \neq I$, and consider the equation

$$z = \frac{az+b}{cz+d}$$

to obtain a quadratic polynomial over $\mathbb{C}$, which has at most two roots in $\mathbb{C}$. ∎

Lemma 3.3.4. *Let S be a Möbius transformation on $\mathbb{C}_\infty$, and let $a, b, c, d \in \mathbb{C}_\infty$ distinct points with $\alpha = S(a)$, $\beta = S(b)$ and $\gamma = S(c)$. If T is another Möbius transformation with this property, then $S = T$.*

Proof. Notice by hypothesis that the transformation $T^{-1} \circ S$ has a, b, and c as fixed points, which forces $T^{-1} \circ S = I$. ∎

Definition. Let $z_2, z_3, z_4 \in \mathbb{C}_\infty$ and define the Möbius transformation $S : \mathbb{C}_\infty \to \mathbb{C}_\infty$ by

$$\begin{aligned}
S(z) &= \frac{\left(\frac{z-z_3}{z-z_4}\right)}{\left(\frac{z_2-z_3}{z_2-z_4}\right)} \text{ if } z_2, z_3, z_4 \in C \\
S(z) &= \frac{z-z_3}{z-z_4} \text{ if } z_2 = \infty \\
S(z) &= \frac{z_2-z_4}{z-z_4} \text{ if } z_3 = \infty \\
S(z) &= \frac{z_2-z_3}{z_2-z_3} \text{ if } z_4 = \infty
\end{aligned}$$

and where $S(z_2) = 1$, $S(z_3) = 0$, and $S(z_4) = \infty$. Then if $z_1 \in \mathbb{C}_\infty$, we define the **cross ratio**, (z_1, z_2, z_3, z_4) of z_1 to be $S(z_1)$.

Example 3.8. $(z_2, z_2, z_3, z_4) = 1$, $(z_3, z_2, z_3, z_4) = 0$, and $(z_4, z_2, z_3, z_4) = 0$, by definition. Now, if M is aany Möbius transformation, and w_2, w_3, w_4 are points on M such that $M(w_1) = 1$, $M(w_3) = 0$, and $M(w_4) = \infty$, then $M(z) = (z, w_2, w_3, w_4)$.

Theorem 3.3.5. *IF $z_2, z_3, z_4 \in \mathbb{C}_\infty$ are distinct points, and T is a Möbius transformation, then $(z, z_2, z_3, z_4) = (T(z), T(z_2), T(z_3), T(z_4))$ for all $z \in \mathbb{C}_\infty$. That is, the cross ratio is invariant under transformations.*

Proof. Let $S = (z, z_2, z_3, z_4)$, then S is a Möbius transformation. Now, if $M = S \circ T^{-1}$, then $M(T(z_2)) = 1$, $M(T(z_3)) = 0$, and $M(T(z_4)) = \infty$, which makes $S \circ T^{-1} = (z, T(z_2), T(z_3), T(z_4))$. ∎

Lemma 3.3.6. *If $z_2, z_3, z_4 \in \mathbb{C}_\infty$ and $w_2, w_3, w_4 \in \mathbb{C}_\infty$ are all distinct points, then there exists one, and only one Möbius transformation S for which $S(z_2) = w_2$, $S(z_3) = w_3$, and $S(z_4) = w_4$.*

Proof. Let $T(z) = (z, z_2, z, 3, z_4)$ and $M = (z, w, 2, w, 3, w_4)$. Put $S = M^{-1} \circ T$. Then $s(Z_2) = w_2$, $S(z_3) = z_3$,and $S(z_4) = w_4$. Now, if R is another Möbius transformation having this property, then $R^{-1} \circ S$ has 3 fixed points, which makes $R^{-1} \circ S = I$. ∎

Lemma 3.3.7.

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