Measure Theory

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Contents

Chapter 1

Measure and Measure Spaces

1.1 σ-Algebras

Definition. Let X be a nonempty set. An **algebra** of sets on X is a nonempty collection $\mathcal{A}$ of subsets of X which are closed under finite unions and complements in X. We call $\mathcal{A}$ a **σ-algebra** if it is closed under countable unions.

Lemma 1.1.1. *Let X be a set and $\mathcal{A}$ an algbra on X. Then $\mathcal{A}$ is closed under finite intersections.*

Proof. Let $\{E_\lambda\}$ be a collection of sets of $\mathcal{A}$. Then by finite union $E = \bigcup E_\lambda \in \mathcal{A}$. Then by complements, $X \backslash E = \bigcap X \backslash E_\lambda \in \mathcal{A}$. ∎

Corollary. *σ-algebras are closed under countable disjoint unions.*

Proof. Let $\mathcal{A}$ a σ-algebra, and let $\{E_n\}$ a collection of (not necessarily disjoint) sets in $\mathcal{A}$. Then take

$$F_n = E_n \backslash \Big(\bigcup_{k=1}^{n-1} E_k \Big) \tag{1.1}$$

Then each F_n is a set in $\mathcal{A}$, and are pairwise disjoint. Moreover, $\bigcup E_n = \bigcup F_n$. ∎

Lemma 1.1.2. *Let X be a set, and $\mathcal{A}$ an algebra on X. Then $\emptyset \in \mathcal{A}$ and $X \in \mathcal{A}$.*

Proof. By closure of finite unions, notice that if $E \in \mathcal{A}$, then $E \cup X \backslash E = X \in \mathcal{A}$ lemma **??** gives us that $E \cap X \backslash E = \emptyset \in \mathcal{A}$. ∎

Example 1.1. (1) The collections $\{\emptyset, X\}$ and 2^X are σ-algebras on any set X.

(2) Let X be an uncountable set. Then the collection

$$\mathcal{C} = \{E \subseteq X : E \text{ is countable or } X \backslash E \text{ is countable}\}$$

defines a σ-algebra of sets on X, since countable unions of countable sets are countable, and $\mathcal{C}$ is closed under complements. We call $\mathcal{C}$ the **σ-algebra of countable or co-countable sets**.

Lemma 1.1.3. *Let $\{A_\lambda\}$ be a collection of σ-algebras on a set X. Then the intersection*

$$\mathcal{A} = \bigcap \mathcal{A}_\lambda$$

is a σ-algebra on X. Moreover, if $F \subseteq X$, then there exists a unique smallest σ-algebra containing F; in particular, it is the intersection of all σ-algebras containing F.

Proof. Notice that since each $\mathcal{A}_\lambda$ is a σ-algebra, they are closed under countable unions and complements. Hence by definition, $\mathcal{A}$ must also be closed under countable unions and complements.

Now, let $F \subseteq X$ and let $\{\mathcal{A}_\lambda\}$ be the collection of all σ-algebras containing F. Then the intersection $\mathcal{A} = \bigcap \mathcal{A}_\lambda$ is also a σ-algebra containing F; by above. Now, suppose that there is a smallest σ-algebra $\mathcal{B}$ containing F. Then we have that $\mathcal{B} \subseteq \mathcal{A}$. Now, by definition of $\mathcal{A}$ as the intersection of all σ-algberas containing F, we get that $\mathcal{A} \subseteq \mathcal{B}$; so that $\mathcal{B} = \mathcal{A}$. ■

Definition. Let X be a nonempty set and $F \subseteq X$. We define the σ-algebra **generated** by F to be the smallest such σ-algebra $\mathcal{M}(F)$ containing F.

Lemma 1.1.4. *Let X be a set and let $E, F \subseteq X$. Then if $E \subseteq \mathcal{M}(F)$, then $\mathcal{M}(E) \subseteq \mathcal{M}(F)$.*

Proof. We have that since $E \subseteq \mathcal{M}(F)$, and $\mathcal{M}(E)$ is the intersection of all σ-algebras containing E, then $\mathcal{M}(E) \subseteq \mathcal{M}(F)$. ■

Definition. Let X be a topological space. We define the **Borel σ-algebra** on X to be the σ-algebra $\mathcal{B}(X)$ generated by all open sets of X; that is

$$\mathcal{B}(X) = \mathcal{M}(\mathcal{T})$$

where $\mathcal{T}$ is the topology on X. We call the elements of $\mathcal{B}(X)$ **Borel-sets**

Definition. Let X be a topological space. We call a countable intersection of open sets of X a G_δ**-set** of X. We call a countable union of closed sets of X an F_σ**-set** of X.

Theorem 1.1.5. *The Borel σ-algebra on $\mathbb{R}$, $\mathcal{B}(\mathbb{R})$, is generated by the following.*

(1) All open intervals of $\mathbb{R}$.

(2) All closed intervals of $\mathbb{R}$.

(3) All half-open intervals of $\mathbb{R}$.

(4) All open rays of $\mathbb{R}$.

(5) All closed rays of $\mathbb{R}$.

Definition. Let X_α be a collection of non-empty sets, and let $X = \prod X_\alpha$. If $\mathcal{M}_\alpha$ is a σ-algebra on X_α, then we define the **product σ-algebra** on X to be the smallest σ-algebra generated by all $\pi_\alpha^{-1}(E_\alpha)$, where $E_\alpha \in \mathcal{M}_\alpha$, and $\pi_\alpha : X \to X_\alpha$ is the projection map onto the α-th coordinate. We denote the product σ-algebra by $\bigotimes \mathcal{M}_\alpha$.

Lemma 1.1.6. *Let $\{X_n\}$ be a countable collection of sets, each with a σ-algebra $\mathcal{M}_n$, and let $X = \prod X_n$. Then the product σ-algebra $\bigotimes \mathcal{M}_n$ on X is generated by all $\prod E_n$, where $E_n \in \mathcal{M}_n$.*

Proof. Let $E_n \in \mathcal{M}_n$, then by definition of the projection map, $\pi_n^{-1}(E_n) = \prod E_k$ where $E_k = X_k$ for all $k \neq n$. On the otherhand, we can see that $\prod E_n = \bigcap \pi_n^{-1}(E_n)$. ■

Lemma 1.1.7. *Let $\{X_\alpha\}$ be a collection of sets, each together with a σ-algebra $\mathcal{M}_\alpha$. If each $\mathcal{M}_\alpha$ is generated by some $\mathcal{E}_\alpha$, then$\bigotimes \mathcal{M}_\alpha$ is generated by all $\pi_\alpha^{-1}(E_\alpha)$, where $E_\alpha \in \mathcal{E}_\alpha$.*

Proof. Let $\mathcal{F} = \{\pi_\alpha^{-1}(E_\alpha) : E_\alpha \in \mathcal{E}_\alpha\}$. Then by lemma 1.1.4, $\mathcal{M}(\mathcal{F}) \subseteq \bigotimes \mathcal{M}_\alpha$. On the otherhand, for any α, the collection of all $\pi_\alpha^{-1}(E) \in \mathcal{M}(\mathcal{F})$ is a σ-algebra on X_α, containing $\mathcal{E}_\alpha$; and hence, $\mathcal{M}_\alpha$. That is, $\pi_\alpha^{-1}(E) \in \mathcal{M}(\mathcal{F})$ for all $E \in \mathcal{M}_\alpha$, which gives us the reverse inclusion. ■

Corollary. *If $\{X_\alpha\}$ is a countable collection, theb $\bigotimes \mathcal{M}_\alpha$ is generated by all $\prod E_\alpha$, where $E_\alpha \in \mathcal{E}_\alpha$.*

Lemma 1.1.8. *Let $X_1, \ldots, X_n$ be metric spaces, and $X = \prod_{i=1}^n X_i$ on the product topology. Then*

$$\bigotimes(\mathcal{B}(X_i)) \subseteq \mathcal{B}(X)$$

Moreover, if each X_i is seperable, then equality is established.

Proof. We have that $\bigotimes \mathcal{B}(X_i)$ is generated by each $\pi_i^{-1}(U_i)$, where U_i is an open set in X_i. Since these sets are open, again by lemma 1.1.4, $\bigotimes \mathcal{B}(X_i) \subseteq \mathcal{B}(X)$.

Now, suppose that each X_i is seperable, and let C_i a countable dense set in X_i, and let $\mathcal{E}_i$ be the collection of all open balls in X_i with rational radius r, and center in C_i. Then every open set in X_i is a countable union of members of $\mathcal{E}_i$. Moreover, the set of points in X whose i-th coordinate is in C_i, for all i, is countable dense in X_i. Hence, $\mathcal{B}(X_i)$ is generated by $\mathcal{E}_i$, and since (X) is generated by all $\prod_{i=1}^n E_i$, where $E_i \in \mathcal{E}_i$, we get $\mathcal{B}(X) \subseteq \mathcal{B}(X_i)$, and equality is established. ■

Corollary. $\mathcal{B}(\mathbb{R}^n) = \bigotimes_{i=1}^n \mathcal{B}(\mathbb{R})$.

Definition. We define an **elementary family** on a set X to be a collection $\mathcal{E}$ of subsets of X such that:

(1) $\emptyset \in \mathcal{E}$.

(2) If $E, F \in \mathcal{E}$, then $E \cap F \in \mathcal{E}$.

(3) If $E \in \mathcal{E}$, then $X \backslash E$ is a finite disjoint union of members of $\mathcal{E}$.

Lemma 1.1.9. *Let X be a set and $\mathcal{E}$ an elementary family on X. Let $\mathcal{A}$ be the collection of all finite disjoint unions of members of $\mathcal{E}$. Then $\mathcal{A}$ is an algebra on X.*

Proof. Let $A, B \in \mathcal{E}$, and let $X\backslash B = \bigcup_{i=1}^n C_i$, where each $C_i \in \mathcal{E}$ for all $1 \leq i \leq n$, and are disjoint. Then we have

$$A \cup B = (A\backslash V) \cup B \text{ and } A\backslash B = \bigcup_{i=1}^{n} (A \cap C_i)$$

so that $A \cup B \in \mathcal{A}$, and $A\backslash B \in \mathcal{A}$. Now, by induction on n, suppose that $A_1, \ldots, A_n \in \mathcal{A}$ are disjoint, then

$$\bigcup_{i=1}^{n+1} A_i = A_{n+1} \cup \bigcup_{i=1}^{n} A_i \backslash A_{n+1}$$

is also a disjoint union. Moreover, we have that if $X\backslash A_n = \bigcup_{i=1}^{N_m} B_m^i$, where the union is disjoint, then

$$X\backslash\Big(\bigcup_{m=1}^{n} A_m\Big) = \bigcap_{m=1}^{n} \Big(\bigcup_{i=1}^{N_m} B_m^i\Big)$$

is also a disjoint union. This makes $\mathcal{A}$ an algebra on X. ■

1.2 Measures

Definition. Let X be a set together with a σ-algebra $\mathcal{M}$. We define a **measure** on $\mathcal{M}$ to be a function $\mu : \mathcal{M} \to [0, \infty)$ for which the following hold:

(1) $m(\emptyset) = 0$.

(2) If $\{E_n\}$ is a countable disjoint collection of members of $\mathcal{M}$, then

$$m\Big(\bigcup E_n\Big) = \sum m(E_n) \tag{1.2}$$

We call m a **finitely additive measure** if instead of (2), m satisfies:

(2') If $\{E_i\}_{i=1}^n$ is a finite disjoint collection of members of $\mathcal{M}$, then

$$m\Big(\bigcup_{i=1}^{n} E_i\Big) = \sum_{i=1}^{n} m(E_i) \tag{1.3}$$

Definition. We call a set X together with a σ-algebra $\mathcal{M}$ a **measurable space**, and we call the members of $\mathcal{M}$ **measurable sets**. If $m : \mathcal{M} \to [0, \infty)$ is a measure on $\mathcal{M}$, then we call X together with $\mathcal{M}$ a **measure space**.

Definition. Let X together with a σ-algebra be a measure space with measure m. If $m(X) < \infty$, then we call m a **finite measure**, and if $\{E_n\}$ is a covering of X by measurable sets, each with $m(E_n) < \infty$ for all n, then we call m **σ-finite**. We also call the set $E = \bigcup E_n$ **σ-finite**. We call m **semi-finite** if for any measurable set E, of $m(E) = \infty$, there is a measurable set F contained in E such that $0 < m(F) < \infty$.

Lemma 1.2.1. *σ-finite measures are semi-finite.*

Example 1.2. (1) LEt X be a non-empty set, and let $f : X \to [0, \infty)$ be any function on X. Then f defines a measure m on 2^X by the rule

$$m(E) = \sum_{x \in E} f(x)$$

Now, m is semi-finite if, and only if $f(x) < \infty$ for all $x \in X$, and m is σ-finite if, and only if m is semi-finite, and the pre-image $f^{-1}((0, \infty))$ is countable.

(2) Consider the measure m of example (1) above, where $f(x) = 1$ for all $x \in X$. Then we call m the **counting measure** on 2^X. Indeed, observe that

$$m(E) = \sum_{x \in E} 1 = |E|$$

which counts the elements of E.

(3) Consider the measure m of example (1) above, where f is defined for any $x_0 \in X$ to be:

$$f(x) = \begin{cases} 1, & \text{if } x = x_0 \\ 0, & \text{if } x \neq x_0 \end{cases}$$

We call this measure the **Dirichlet measure**.

(4) Let X be an uncountable set, and let $\mathcal{M}$ the σ-algebra of all countable or co-countable sets. Define m on $\mathcal{M}$ by $m(E) = 0$ if E is countable, and $m(E) = 1$ if E is co-countable. Then m defines a measure on $\mathcal{M}$.

(5) Let X be an infinite set, and define m on 2^X by $m(E) = 0$ if E is finite, and $m(E) = \infty$ if E is infinite. Then m is a finitely subadditive measure on 2^X, but not a measure on 2^X.

Theorem 1.2.2. *Let X be a measure space with measure m. The following are true.*

(1) If E and F are measurable with $E \subseteq F$, then

$$m(E) \leq m(F)$$

(2) If $\{E_n\}$ is a countable collection of measurable sets, then

$$m\Big(\bigcup E_n\Big) \leq \sum m(E_n)$$

(3) If $\{E_n\}$ is a countable collection of measurable sets, in which $E_1 \subseteq E_2 \subseteq \dots$, then

$$m\Big(\bigcup E_n\Big) = \lim_{n \longrightarrow \infty} m(E_n)$$

(4) If $\{E_n\}$ is a countable collection of measurable sets, in which $\dots, \subseteq E_2 \subseteq E_1$ and $m(E_1) < \infty$, then

$$m\Big(\bigcap E_n\Big) = \lim_{n \longrightarrow \infty} m(E_n)$$

Proof. For the first statement, let $E \subseteq F$ be measurable sets, then observe that

$$m(E) \leq m(E) + m(F \backslash E) = m(E \cup F \backslash E) = m(F)$$

For the second statement, define $F_1 = E_1$, and $F_i = E_i \backslash \bigcup_{i=1}^{i-1} E_i$ for all $i > 1$. Then $\{F_n\}$ is a finite disjoint collection of measurable sets, with $\bigcup_{i=1}^{n} F_i = \bigcup_{i=1}^{n} E_i$. By the above argument, we get

$$m\Big(\bigcup_{i=1}^{n} E_i\Big) = m\Big(\bigcup_{i=1}^{n} F_i\Big) = \sum_{i=1}^{n} m(F_i) \leq \sum_{i=1}^{n} m(E_i)$$

Now, for (3), let $E_0 = \emptyset$, then

$$m(\bigcup E_n) = \sum m(E_i \backslash E_{i-1}) = \lim_{n \longrightarrow \infty} \sum_{i=1}^{n} m(E_i \backslash E_{i-1}) = \lim_{n \longrightarrow \infty} m(E_n)$$

Additionally, consider when the collection $\{E_n\}$ is decreasing with $m(E_1) < \infty$. Take $F_i = E_1 \backslash E_i$, then $\{F_n\}$ is an increasing collection of measurable sets, and hence we apply the above argument. We get that $m(E_1) = m(F_n) + m(E_n)$, and

$$\bigcup F_n = E_1 \backslash \bigcap E_n$$

therefore, we get

$$m(E_1) = m\Big(\bigcap E_n\Big) + \lim_{n \longrightarrow \infty} m(F_i) = m\Big(\bigcap E_n\Big) + \lim_{n \longrightarrow \infty} (m(E_1) - m(E_n))$$

Subtracting $m(E_1)$ from both sides of the equation yields the result. ■

Definition. Let X be a measure space with measure m. We say that a statement about points in X holds **almost everywhere** (with respect to m) if it holds for all $x \in X \backslash E$, where $m(E) = 0$. We call the measure m **complete** if its domain contains all subsets of sets with measure 0.

Theorem 1.2.3. *Let X be a measure space with s-algebra $\mathcal{M}$, and measure m. Let $\mathcal{N} = \{N \in \mathcal{M} : m(N) = 0\}$, and define*

$$\overline{\mathcal{M}} = \{E \cup F : E \in \mathcal{M} \text{ and } F \subseteq N \text{ for some } N \in \mathcal{N}\}$$

Then $\overline{\mathcal{M}}$ is a σ-algebra, and there exists a unique complete measure $\overline{m}$ on $\overline{\mathcal{M}}$.

Proof. Since $\mathcal{M}$ is a σ-algebra, then so is $\mathcal{N}$, moreover, since both are closed under countable unions, so is $\overline{\mathcal{M}}$. Additionally, let $E \cup F \in \overline{\mathcal{M}}$, then we get $E \cup F = (E \cup N) \cap ((X \backslash N) \cup F)$, so that $X \backslash (E \cup F) = X \backslash (E \cup N) \cup N \backslash F$. Since $X \backslash (E \cup N) \in \overline{\mathcal{M}}$, and $N \backslash F \subseteq F$, then we get $X \backslash (E \cup F) \subseteq \overline{\mathcal{M}}$. This makes $\mathcal{M}$ a σ-algebra.

Now, for $E \cup F \in \overline{\mathcal{M}}$, define $\overline{m}$ on $\overline{\mathcal{M}}$ by $\overline{m}(E \cup F) = m(E)$. Then $\overline{m}$ is well defined. Let $E_1 \cup F_1 = E_2 \cup F_2$, where $F_i \subseteq N_i$, with $N_i \in \mathcal{N}$, for $i = 1, 2$. Then $E_1 \subseteq E_2 \cup N_2$, so that $m(E_1) \leq m(E_2) + m(N_1) = m(E_2)$. Similarly, we also get $m(E_2) \leq m(E_1)$.

Now, let $E \in \overline{\mathcal{M}}$, such that $\overline{m}(E) = 0$. Now, we have $E = A \cup B$, where $A \in \mathcal{M}$ and $B \subseteq N$, for some $N \in \mathcal{N}$. Moreover, $\overline{m}(E) = m(A) = 0$. Now, we get $E \subseteq A \cup N \in \mathcal{N}$, since $m(A) = 0$. Now, let $F \subseteq E$. Then observe that $F \subseteq A \cup N$, so that $F \in \mathcal{N}$. Then $F = \emptyset \cup F$, so that $F \in \overline{\mathcal{M}}$. Moreover, $\overline{m}(F) = m(\emptyset) = 0$.

Lastly, suppose there is another complete meaure $\overline{n}$ on $\overline{\mathcal{M}}$ for which $\overline{n}(E \cup F) = m(E)$. Let $E \in \overline{\mathcal{M}}$. Then $E = A \cup B$ where $A \in \mathcal{M}$, and $B \subseteq N$, $N\mathcal{N}$. Then $\overline{n}(E) = \overline{n}(A \cup B) = m(A) \leq m(A) + m(B) = m(A \cup B) = \overline{m}(E)$. By similar reasoning, we get $\overline{m}(E) \leq \overline{n}(E)$, which establishes uniqueness. ∎

Definition. Let X be a measure space with s-algebra $\mathcal{M}$, and measure m. Let $\mathcal{N} = \{N \in \mathcal{M} : m(N) = 0\}$, and define

$$\overline{\mathcal{M}} = \{E \cup F : E \in \mathcal{M} \text{ and } F \subseteq N \text{ for some } N \in \mathcal{N}\}$$

We call $\overline{\mathcal{M}} the \textbf{completion} of \mathrm{M} with respect to \mathrm{m}, and we call the unique complete measure, \overline{m}$ on $\overline{\mathcal{M}}$ the **completion** of m.

1.3 Outer Measures

Definition. Let X be a set. An **outer-measure** on X is a function $m^* : 2^X \to [0, \infty)$ for which the following are true:

(1) $m^*(\emptyset) = 0$.

(2) If $A \subseteq B$, then $m^*(A) \leq m^*(B)$.

(3) If $\{A_n\}$ is a countable collection of subsets of X, then

$$m^*\Big(\bigcup A_n\Big) \leq \sum m^*(A_n)$$

Lemma 1.3.1. *Let X be a set, and $\mathcal{E}$ a collection of subsets of X for which $\in \mathcal{E}$ and $X \in \mathcal{E}$, and let $l : \mathcal{E} \to [0, \infty]$ a function for which $l(\emptyset) = 0$. For any $A \subseteq X$, define*

$$m^*(A) = \inf\Big\{\sum l(E_n) : E_n \in \mathcal{E}, \text{ and } A \subseteq \bigcup E_n\Big\} \tag{1.4}$$

Then m^* defines an outer-measure.

Proof. For all $A \subseteq X$, there is a collection $\{E_n\}$ of sets of $\mathcal{E}$ for which $A \subseteq \bigcup E_n$. Observe first, that since $l(E_n) \geq 0$ for all n, that $\sum l(E_n) \geq 0$. This makes $m^*(A) \geq 0$. Now, choose $E_n = \emptyset$ for all n, and we get $m^*(\emptyset) = 0$.

Now, let $A \subseteq B$ subsets of X, and let $\{E_n\}$ a countable cover of B. Then $\{E_n\}$ is also a countable cover of A. Define then $E = \{\sum l(E_n) : A \subseteq \bigcup E_n\}$ and $F = \{\sum l(E_n) : B \subseteq \bigcup E_n\}$. Since $A \subseteq B$, $F \subseteq E$. Therefore, by least upper bounds, we have $\inf F \leq \inf B$, that is $m^*(A) \leq m^*(B)$.

Lastly, let $\{A_n\}$ be a countable collection of sets of X, and let $A = \bigcup A_n$. Now, if at least one of the $m(A_n) = \infty$, then we are done. Suppose then that $m(A_n) < \infty$ for all n. Now, there exists a cover of A_n, $\{E_{n,k}\}_k$ for which

$$\sum_k l(E_{n,k}) < m^*(A) + \frac{1}{2^k}$$

consider now the countable collection $\{E_{n,k}\}_{n,k} = \bigcup_n \{E_{n,k}\}_k$. Then $\{E_{n,k}\}_{n,k}$ is a countable cover for A, and we get

$$m^*(A) \leq \sum_n \sum_k l(E_{n,k}) < \sum_n m^*(A_n) + \frac{1}{2^k} = \sum_n m^*(A_n) + \varepsilon$$

Take then $\varepsilon > 0$ small, and we get the result. ■

Corollary. *If E is a set of $\mathcal{E}$, then $m^*(E) = l(E)$.*

Proof. Observe that E covers itself, so that $m^*(E) = \inf\{\sum_{i=1}^1 E\} = \inf l(E) = l(E)$. ■

Definition. Let X be a set. We call a subset A of X m^***-measurable** if for any subset E of X,

$$m^*(E) = m^*(E \cap A) + m^*(E \cap X \backslash A) \tag{1.5}$$

Lemma 1.3.2. *Let X be a set. A subset A of X is m^*-measurable if, and only if*

$$m^*(E) \geq m^*(E \cap A) + m^*(E \cap X \backslash A) \text{ for all } E \subseteq X$$

Theorem 1.3.3 (Carathéodory's Theorem)**.** *Let X be a set, and m^* an outer-measure on X. Then the collection of all m^*-measurable sets forms a σ-algebra. Moreover, m^* is a complete measure on this σ-algebra.*

Proof. Let $\mathcal{M}$ be the collection of all m^*-measurable sets. Observe first that if $A \in \mathcal{M}$, then so is $X \backslash A$, by symetry of equation 1.5. So $\mathcal{M}$ is closed under complements. Now, let $A, B \in \mathcal{M}$. Then we have

$$\begin{aligned} m^*(E) &= m^*(E \cap A) + m^*(E \cap X \backslash A) \\ &= m^*(E \cap A \cap B) + m^*(E \cap A \cap X \backslash B) + m^*(E \cap B \cap X \backslash A) + m^*(E \cap X \backslash A \cap X \backslash B) \end{aligned}$$

Now, since $A \cup B = (A \cap B) \cup (A \cap X \backslash B) \cup (B \cap X \backslash A)$, so by subadditivity, we get

$$m^*(E \cap A \cap B) + m^*(E \cap A \cap X \backslash B) + m^*(E \cap X \backslash A \cap B) \geq m^*(E \cap (A \cup B))$$

i.e. $m^*(E) \geq m^*(E \cap (A \cup B)) + m^*(E \cap X \backslash (A \cup B))$. That is, $A \cup B \in \mathcal{M}$, making $\mathcal{M}$ an algebra.

Now, let $\{A_n\}$ be a countable disjoint collection of m^*-measurable sets, and take $B_n = \bigcup_{i=1}^n A_i$, and take $B = \bigcup B_n$. Then for all $E \subseteq X$

$$\begin{aligned} m^*(E \cap B_n) &= m^*(E \cap B_n \cap A_n) + m^*(E \cap B_n \cap X \backslash A_n) \\ &= m^*(EA_n) + m^*(E \cap B_{n-1}) \end{aligned}$$

an induction argument on the collection $\{B_n\}$ gives us

$$m^*(E \cap B_n) = \sum_{i=1}^{n} m^*(E \cap A_i)$$

therefore

$$m^*(E) = m^*(E \cap B_n) + m^*(E \cap X \backslash B_n) \geq \sum_{i=1}^{n} m^(E \cap A_i) + m^*(E \cap X \backslash B_n)$$

letting $n \to \infty$,

$$m^*(E) \geq \sum m^(E \cap A_n) + m^*(E \cap X \backslash B_n)$$

so that $B \in \mathcal{M}$. Taking $E = B$, we get $m^*(B) = \sum m^*(A_n)$ so that m^* is countably additive, and $\mathcal{M}$ is a σ-algebra.

Finally, let $m^*(A) = 0$, then for any $E \subseteq X$, we have

$$m^*(E) \leq m^*(E \cap A) + m^*(E \cap X \backslash A) = m^*(E \cap X \backslash A) \leq m^*(E)$$

so that $A \in \mathcal{M}$, which makes m^* complete on $\mathcal{M}$. ■

Definition. Let X be a set, and $\mathcal{A}$ an algebra on X. We define a **pre-measure** on $\mathcal{A}$ to be a function $m_0 : \mathcal{A} \to [0, \infty]$ for which

(1) $m_0(\emptyset) = 0$.

(2) If $\{A_n\}$ is a countably disjoint collection of sets in $\mathcal{A}$, for which $\bigcup A_n \in \mathcal{A}$, then

$$m_0\Big(\bigcup A_n\Big) = \sum m_0(A_n) \tag{1.6}$$

Lemma 1.3.4. *Pre-measures on algebras define outer-measures on the overlying sets.*

Proof. Consider the definition of the outer measure m^* from equation 1.4, simply take $l = m_0$, and $\mathcal{E} = \mathcal{A}$. ■

Lemma 1.3.5. *Let X be a set, and $\mathcal{A}$ an algebra on X. If m_0 is pre-measure on $\mathcal{A}$, and the measure m^* is define by*

$$m^*(A) = \inf\Big\{\sum m_0(E_n) : E_n \in \mathcal{A},\ \text{and } A \subseteq \bigcup E_n\Big\}$$

then the following are true.

(1) $m_0 = m^*$ on $\mathcal{A}$.

(2) Every set in $\mathcal{A}$ is m^*-measurable.

Proof. For (1), suppose that $A \in \mathcal{A}$, and that $A \subseteq \bigcup E_n$ for $E_n \in \mathcal{A}$. Take

$$F_n = A \cap A_n \backslash \Big(\bigcup_{i=1}^{n-1} A_i \Big)$$

then $\{F_n\}$ is a disjoint countable collection of sets of $\mathcal{A}$ for which $A = \bigcup F_n$. Hence

$$mo(A) = \sum m_0(F_n) \leq \sum m_0(E_n)$$

it follows from hypothesis that $m_0(A) \leq m^*(E)$. For the reverse inclusion, simply take $A \subseteq \bigcup E_n$ with $A = E_1$ and $E_n = \emptyset$ for all $n > 1$.

For (2), if $A \in \mathcal{A}$, and $E \subseteq X$, and $\varepsilon > 0$, there is a collection $\{B_n\}$ of sets of $\mathcal{A}$ with $A \subseteq \bigcup B_n$, and

$$\sum m_0(B_n) < m^*(A) + \varepsilon$$

by additivity of m_0 on $\mathcal{A}$, we get

$$m^*(E) + \varepsilon \geq \sum m_0(B_n \cap A) + \sum m_0(B_n \cap X \backslash A) \geq m^*(E \cap A) + m^*(E \cap X \backslash A)$$

■

Theorem 1.3.6. *Let X be a set, and $\mathcal{A}$ an algerba on X. Let m_0 be a pre-measure on $\mathcal{A}$, and let $\mathcal{M}$ the σ-algebra generated by $\mathcal{A}$. Then there exists a measure m on $\mathcal{M}$ whose restriction to is m_0. Moreover, if n is another measure extending from m_0, then*

$$n(E) \leq m(E) \text{ for all } E \in \mathcal{M}$$

where equality holds when $m(E) < \infty$. Lastly, if m_0 is σ-finite, then m is the unique extension of m_0 to $\mathcal{M}$.

Proof. Define again,

$$m^*(A) = \inf \Big\{ \sum m_0(E_k) : E_k \in \mathcal{A}, \text{ and } A \subseteq \bigcup E_k \Big\}$$

then by Carathéodory's theorem, lemma 1.3.5, the first result follows, since the σ-algebra of all m^*-measurable sets contains $\mathcal{A}$, and as consequence, also contains $\mathcal{M}$.

Now, let $E \in \mathcal{M}$ with $E \subseteq \bigcup A_k$, where $A_k \in \mathcal{A}$. Then

$$n(E) \leq \sum n(A_n) = \sum m_0(A_n)$$

which gives us $n(E) \leq m(E)$. Now, set $A = \bigcup A_n$, and observe that

$$n(A) = \lim_{k \longrightarrow \infty} n\Big(\bigcup_{i=1}^{k} A_i \Big) = \lim_{k \longrightarrow \infty} m\Big(\bigcup_{i=1}^{k} A_i \Big) = m(A)$$

if $m(E) < \infty$, choose A_k such that $m(A) < m(E) + \varepsilon$ for $\varepsilon > 0$. Then $m(A \backslash E) < \varepsilon$, and

$$m(E) \leq m(A) = n(A) = n(E) + n(A \backslash E) \leq n(E) + m(A \backslash E) \leq n(E) + \varepsilon$$

taking ε small, we get $n(E) = m(E)$.

Finally, suppose that m_0 is σ-finite, and let $X = \bigcup A_k$ for s me0disjoint collection $A_n\}$, then $m_0 m_0) < \infty$. Then for every $E \in \mathcal{M}$,

$$m(E) = \sum m(E \cap A_k) = \sum n(E \cap A_k) = n(E)$$

so that $m = n$, making m unique. ■

1.4 Borel Measures on ℝ

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