Real Analysis

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Chapter 1

The Real Numbers

1.1 The Field of Real Numbers

1.2 The Topology of $\mathbb{R}$

Definition. We call a set U of $\mathbb{R}$ **open** proveded for all $x \in U$, there exists an $r > 0$ for which the open interval $(x - r, x + r) \subseteq U$.

Example 1.1. For $a < b$ the open interval (a, b) is open in $\mathbb{R}$. Let $x \in (a, b)$ and take $r = \min\{x - a, b - x\}$, then $(x - r, x + r) \subseteq (a, b)$. Similarly the intervals (a, ∞), $(-\infty, b)$, and $(-\infty, \infty)$ are also open in $\mathbb{R}$.

Lemma 1.2.1. *The set $\mathbb{R}$ of real numbers forms a topology under the open sets of $\mathbb{R}$.*

Lemma 1.2.2. *Every nonempty open set in $\mathbb{R}$ is the disjoint union of a countable collection of open sets in $\mathbb{R}$.*

Proof. Let U be a nonempty open set in $\mathbb{R}$, and take $x \in U$. There there is a $y > x$ for which $(x, y) \subseteq U$, and a $z < x$ for which $(z, x) \subseteq U$. Now, define

$$a_x = \inf\{z : (z, x) \subseteq U\}$$
$$b_x = \sup\{y : (x, y) \subseteq U\}$$

and take

$$I_x = (a_x, b_x)$$

Then I_x is an open interval containing x. Now, we claim that $I_x \subseteq U$, but that $a_x, b_x \notin U$. Indeed, take $w \in I_x$, with $x < w < b_x$, then there is a $y > w$ for which $(x, y) \subseteq U$, so that $w \in U$.

Now, suppose that $b_x \in U$, then for some $r > 0$, $(b_x - r, b_x + r) \subseteq U$, so that $(x, b_x + r) \subseteq U$, which contradicts that b_x is a least upper bound. Similar reasoning yields that $a_x \notin U$.

Now, consider the collection $\{I_x\}_{x \in U}$. Then we have that

$$U = \bigcup I_x$$

moreover, this union is disjoint since $a_x, b_x \notin U$ for each x. Now, observe that by the density of $\mathbb{Q}$ in $\mathbb{R}$, there exists a rational $q_x \in \mathbb{Q}$ for which $q_x \in I_x$. This gives us a 1–1 correspondence of the collection $\{I_x\}$ onto $\mathbb{Q}$, which makes $\{I_x\}$ countable. ■

Definition. For a set E of real numbers, we call a point $x \in \mathbb{R}$ a **limit point** of E provided every open interval containing x contains a point in E. We call the set of all limit points of E, together with E the **closure** of E and denote it $\operatorname{cl} E$. We call E **closed** if $E = \operatorname{cl} E$.

Lemma 1.2.3. *For every set E of $\mathbb{R}$, the closure of E is closed. Morevoer, $\operatorname{cl} E$ is the smallest closed set containing E.*

Proof. Let x be a linit point of $\operatorname{cl} E$, and consider an open interval I_x containing x. Then there exists an $x' \in \operatorname{cl} E \cap I_x$. Since x' is a limit point of E, and $x' \in I_x$, we get a $x \in E \cap I''_x$. Therefore every open interval that contains x also contains a point of E. This makes $x \in \operatorname{cl} E$, and hence $\operatorname{cl} E$ is closed. ■

Lemma 1.2.4. *A set of $\mathbb{R}$ is open if and only if its complement in $\mathbb{R}$ is closed.*

Proof. Suppose that $E \subseteq \mathbb{R}$ is open , and let x be a limit point of $\mathbb{R} \backslash E$. Then $x \notin E$, since otherwise there is an open interval containing x, contained in E, and hence disjoint from $\mathbb{R} \backslash E$. Therefore $x \in \mathbb{R} \backslash E$ which makes $\mathbb{R} \backslash E$ closed. ■

Corollary. *A set $\mathbb{R}$ is closed if, and only if its complement in $\mathbb{R}$ is open.*

Proof. By DeMorgan's laws. ■

Definition. We call a collection $\{E_\lambda\}$ of sets of $\mathbb{R}$ a **cover** for a set E of $\mathbb{R}$ if $E \subseteq_\lambda$. If each E_λ is open, we call the collection $\{E_\lambda\}$ an **open cover**. We call a set E of $\mathbb{R}$ **compact** if each open cover of E has a finite subcover of E.

Theorem 1.2.5 (Heine-Borel)**.** *If F is a closed bounded set in $\mathbb{R}$, then F is compact.*

Proof. Consider first the case where $F = [a, b]$, for $a < b$, the closed bounded interval from a to b. Let $\mathcal{F}$ be an open cover of $[a, b]$, and define

$$E = \{x \in [a, b] : [a, x] \text{ can be covered by a finite subcollection of } \mathcal{F}\}$$

Notice then that $a \in E$, so that E is nonempty. Moreover, E is bounded above, so by the completeness of $\mathbb{R}$, $c = \sup E$ exists in $[a, b]$. Now, then, there exists a set $U \, in \mathcal{F}$ such that $c \in U$. Since U is open (well $\mathcal{F}$ is an open cover), there exists an $\varepsilon > 0$ for which the interval $(c - \varepsilon, c + \varepsilon) \subseteq U$. Now, $c - \varepsilon$ is not an upperbound of E by definition of c, so there is an $x \in E$ with $c - \varepsilon < x$. Now, threre is a finite subcollection $\{U_i\}_{i=1}^k$ of open sets in $\mathcal{F}$ covering $[a, x]$, consequently the collectuion $\{U_i\} \cup U$ covers $[a, c + \varepsilon]$, so that $c = b$. That is $[a, b]$ has a finite subcover of $\mathcal{F}$, so that $[a, b]$ is compact.

Now, let F be any closed and bounded set, and let $\mathcal{F}$ be an open cover of F. Since F is bounded, we have $F \subseteq [a, b]$ for some $a < b$, and the set $U = \mathbb{R} \backslash F$ is open. Now, let $\mathcal{F}' = \mathcal{F} \cup U$. Since $\mathcal{F}$ covers F, $\mathcal{F}'$ covers $[a, b]$. By the compactness of $[a, b]$, we obtain the compactness of F. ■

Theorem 1.2.6 (The Nested Set Theorem)**.** *Let $\{F_n\}$ a countable descending collection of closed sets of $\mathbb{R}$, for which F_1 is bounded. Then the intersection*

$$\bigcap F_n$$

is nonempty.

Proof. Suppose to the contrary that the intersection $F = \bigcap F_n$ is empty. Then for every $x \in \mathbb{R}$, there is an $n \in \mathbb{Z}^+$ for which $x \notin F_n$. That is, $x \in U_n = \mathbb{R} \backslash F_n$, and $\mathbb{R} = \bigcup U_n$. Now, since each F_n is closed, each U_n is open, making $\{U_n\}$ an open cover of $\mathbb{R}$, and hence F_1. Then by the theorem of Heine-Borel, F_1 is compact, and there is an $N \in \mathbb{Z}^+$ for which

$$F_1 \subseteq \bigcup_{n=1}^{N} U_n$$

since $\{F_n\}$ is a descending collection, the collection of open sets $\{U_n\}$ is an ascending collection. Thus we have

$$\bigcup_{n=1}^{N} U_n = U_n = \mathbb{R} \backslash F_N$$

making $F_1 \subseteq \mathbb{R} \backslash F_N$, which contradicts that $F_n \subseteq F_1$ is nonempty ■

Definition. Let X be a set. We call a collection $\mathcal{A}$ of subsetes of X a **σ-algebra** of X provided

(1) $X \in \mathcal{A}$ and $\emptyset \in \mathcal{A}$.

(2) $\mathcal{A}$ is closed under complements in X.

(3) $\mathcal{A}$ is closed under countable unions.

Example 1.2. The collections $\{\emptyset, X\}$ and 2^X are σ-algebras on X.

Lemma 1.2.7. *Let $\mathcal{F}$ be a collection of subsets of a set X. Then the intersection $\mathcal{A}$ of all σ-algebras of X containing $\mathcal{F}$ is a σ-algebra containing $\mathcal{F}$. Moreover, it is the smallest such σ-algebra of X containing $\mathcal{F}$.*

Definition. We define the collection $\mathcal{B}$ of **Borel sets** of $\mathbb{R}$ to be the smallest σ-algebra of $\mathbb{R}$ containing all open sets of $\mathbb{R}$.

Bibliography

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