Complex Analysis

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Chapter 1

The Complex Numbers

1.1 The Field of Complex Numbers and the Complex Plane

Definition. We define the set of **complex numbers** to be the collection of all ordered pairs of real numbers $\mathbb{C} = \{(a,b) : a,b \in \mathbb{R}\}$ together with the binary operations $+$ and $\cdot$ of **complex addition**, and **complex multiplication**, respectively defined by the rules

$$\begin{aligned}(a,b)+(c,d) &= (a+c, b+d)\\ (a,b)(c,d) &= (ac-bd, bc+ad)\end{aligned}$$

Theorem 1.1.1. *The set of complex numbers $\mathbb{C}$ forms a field together with complex addition and complex multiplication.*

Corollary. *$\mathbb{C}$ is a field extension of the real numbers $\mathbb{R}$.*

Proof. The map $a \to (a,0)$ from $\mathbb{R} \to \mathbb{C}$ defines an imbedding of $\mathbb{R}$ into $\mathbb{C}$. ■

Definition. We define the element $i = (0,1)$ of $\mathbb{C}$ so that $i^2 = -1$, and the polynomial z^2+1 has as root i. We write $(a,b) = a+ib$. If $z = a+ib$, we call a the **real part** of z, and b the **imaginary part** of z and write $\operatorname{Re} z = a$ and $\operatorname{Im} z = z$.

Definition. Let $z = a+ib \in \mathbb{C}$. We define the **norm** (or **modulus**) of z to be $\|z\| = \sqrt{a^2+b^2}$. We define the complex **conjugate** of z to be $\overline{z} = a - ib$.

Lemma 1.1.2. *For every $z \in \mathbb{C}$, $\|z\|^2 = z\overline{z}$.*

Proof. Let $z = a+ib$. Then $\overline{z} = a-ib$, and so $z\overline{z} = (a+ib)(a-ib) = a^2+b^2 = (\sqrt{a^2+b^2})^2 = \|z\|^2$. ■

Corollary. *If $z \neq 0$, then $z^{-1} = \frac{1}{z} = \frac{\overline{z}}{\|z\|^2}$.*

Proof. The relation follows from above, and it remains to show that it is indeed the inverse element. Notice that if $z \in \mathbb{C}$ is nonzero, then $z\frac{\overline{z}}{\|z\|^2} = \frac{z\overline{z}}{\|z\|^2} = \frac{\|z\|^2}{\|z\|^2} = 1$. ■

Lemma 1.1.3. *The following are true for all $z, w \in \mathbb{C}$.*

(1) $\operatorname{Re} z = \frac{1}{2}(z + \overline{z})$ *and* $\operatorname{Im} z = \frac{1}{2i}(z - \overline{z})$.

(2) $\overline{(z + w)} = \overline{z} + \overline{w}$ *and* $\overline{zw} = \overline{z}\ \overline{w}$.

(3) $\|\overline{z}\| = \|z\|$.

Proof. Let $z = a + ib$ and $w = c + id$. Then notice that

$$\frac{(a + ib) + (a - ib)}{2} = \frac{2a + (ib - ib)}{2} = \frac{2a}{2} = a = \operatorname{Re} z$$

and

$$\frac{(a + ib) - (a - ib)}{2i} = \frac{(a - a) + 2ib}{2} = \frac{2ib}{2i} = b = \operatorname{Im} z$$

Moreover

$$\overline{(a + ib) + (c + id)} = \overline{(a + c) + i(b + d)} = (a + c) - i(b + d) = (a - ib) + (c - id)$$

And

$$\overline{(a + ib)(c + id)} = \overline{(ac - bd) + i(bc + ad)} = (ac - bd) - i(bc + ad) = (a - ib)(c - id)$$

so that $\overline{z + w} = \overline{z} + \overline{w}$ and $\overline{zw} = \overline{z}\ \overline{w}$.

Now, we have that $\|zw\|^2 = (zw)\overline{zw} = (zw)(\overline{z}\ \overline{w}) = (z\overline{z})(w\overline{w}) = \|z\|^2\|w\|^2$. Taking square roots, we get the result

$$\|zw\| = \|z\|\|w\|$$

Finally, notice that $\|z\|^2 = z\overline{z} = \overline{z}\ \overline{\overline{z}} = \|\overline{z}\|$. ■

Corollary. *The following are also true; provided $w \neq 0$.*

(1) $\overline{\left(\frac{z}{w}\right)} = \frac{\overline{z}}{\overline{w}}$.

(2) $\|\frac{z}{w}\| = \frac{\|z\|}{\|w\|}$

Bibliography

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