Algebraic Geometry.

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Contents

Chapter 1

Preliminaries

1.1 Affine Varieties

Definition. Let k be an algebraically closed field. We define **affine n-space** over k to be the set $\mathbb{A}^n(k)$ of n-tuples of elements of k. We write simply $\mathbb{A}^n$ when k is understood. We call the elements of $\mathbb{A}^n$ **points** and if $P = (a_1, \ldots, a_n)$ is a point of $\mathbb{A}^n$, we call each a_i the **coordinates** of P.

Example 1.1. Let k be any algebraically closed field, and consider the multivariate polynomial ring $k[x_1, \ldots, x_n]$. We can interperate the elements of $k[x_1, \ldots, x_n]$ as functions from affine space $\mathbb{A}^n(k)$ to k by taking $f(P) = f(a_1, \ldots, a_n)$, where $f \in k[x_1, \ldots, x_n]$ and $P \in \mathbb{A}^n(k)$. This leads us to be able to talk about the set of zeros of a polynomial over k.

Definition. Let k be an algebraically closed field, and $f \in k[x_1, \ldots, x_n]$ a multivariate polynomial over k. We define the **set of zeros** of f to be the set

$$Z(f) = \{P \in \mathbb{A}^n(k) : f(P) = 0\}$$

Let T be a subset of $k[x_1, \ldots, x_n]$. Then we define the **set of zeros** of T to be

$$Z(T) = \bigcap_{f \in T} Z(f)$$

Now, if $\mathfrak{a} = (f_1, \ldots, f_r)$ is an ideal of $k[x_1, \ldots, x_n]$ generated by T, then we write $Z(T) = Z(\mathfrak{a}) = Z(f_1, \ldots, f_r)$.

Example 1.2. We have $\mathbb{A}^n = Z(0)$, and by convention, we also take $\mathbb{A}^n = Z(\emptyset)$. In similar fashion, we have $\emptyset = Z(1)$, and $\emptyset = Z(k[x_1, \ldots, x_n])$.

Definition. Let k be an algebraically closed field. We call a subset Y of $\mathbb{A}^n$ an **algebraic set** if there exists some $T \subseteq k[x_1, \ldots, x_n]$ for which Y is the set of zeros of T; i.e. $Y = Z(T)$.

Lemma 1.1.1. *Let k be an algebraically closed field. Then algebraic sets of $\mathbb{A}^n$ make $\mathbb{A}^n$ into a topology under closed sets.*

Proof. Notice that $\mathbb{A}^n = Z(0)$ and $\emptyset = Z(1)$. Then $\mathbb{A}^n$ and $\emptyset$ are both algebraic. Now, let X and Y be algebraic, then there are S, T such that $X = Z(S)$ and $Y = Z(T)$. Now, let $P \in X \cup Y$, then P is a zero of any polynomial $f \in ST$, conversly, suppose that $P \in Z(ST)$ where $P \notin Y$. There exists a polynomial $f \in S$ with $f(P) \neq 0$. Now, for any $g \in T$, we have that if $fg(P) = 0$, then $g(P) = 0$, so that $P \in S$. Therefore we have $X \cup Y = Z(ST)$, making $X \cup Y$ algebraic. So that the collection of algebraic sets is closed under finite intersection.

Lastly, consider a collection $\{Y_\alpha\}$ of algebraic sets, where $Y_\alpha = Z(T_\alpha)$ for some T_α. Let

$$Y = \bigcap Y_\alpha \text{ and } T = \bigcup T_\alpha$$

and let $P \in Y$. Then P is in every Y_α making it a zero of some $f_\alpha \in T_\alpha$, thus $P \in Z(T)$. Similarly, if $P \in Z(T)$, then $P \in Y$, making $Y = Z(T)$, and making the collection of algebraic sets closed under arbitrary intersections. ■

Definition. We define the **Zariski topology** on affine n-space $\mathbb{A}^n$ to be the topology on $\mathbb{A}^n$ whose closed sets are the algebraic sets of $\mathbb{A}^n$.

Example 1.3.

(1) Consider the Zariski topology on affine 1-space $\mathbb{A}^1$. Now, since $k[x]$ is a PID, every algebraic set of $\mathbb{A}^1$ is the set of zeros of preciesly one polynomial. Moreover, by the algebraic closure of k, for any nonzero polynomial f over k, we have

$$f(x) = c(x - a_1) \dots (x - a_n)$$

where $c, a_1, \dots, a_n \in k$. Then $Z(f) = \{a_1, \dots, a_n\}$, so that the algebraic sets of $\mathbb{A}^1$ are the emptyset, itself, and finite subsets. Thus the Zariski topology on $\mathbb{A}^1$ consists of finite sets, the emptyset, and $\mathbb{A}^1$ itself. Notice that this topology is not Hausdorff.

(2) Consider the field $\mathbb{C}$ of complex numbers. Then the Zariski topology has as closed sets all algebraic sets of $\mathbb{A}^n(\mathbb{C})$; i.e. all sets of zeros of polynomials in $\mathbb{C}$. Now, by the fundamental theorem of algebra, these closed stets are finite. Consider then the polynomials $f(z) = z^2+1$, and $g(z) = z^3 - iz^2 + 25z + -i25$ in $\mathbb{C}$. Then notice that $(\mathbb{C} \backslash Z(f)) \cap (\mathbb{C} \backslash Z(g)) = Z(f) \cup Z(g) = \{i, -i, i25, -i25\}$. In general for an algebraically closed fiedl k, zero sets of polynomials over k are not disjoint; so that the Zariski topology on $\mathbb{A}^n(k)$ is not Hausdorff.

Definition. Let X be a topological space, and Y a subspace of X. We call Y **irreducible** if it cannot be written as the union $Y = Y_1 \cup Y_2$ of two sets Y_1 and Y_2 closed in Y. We make the convention that the emptyset is not irreducible.

Example 1.4. (1) Notice that the affine space $\mathbb{A}^1$ is irreducible. We have the only closed sets are finite sets, and since k is algebraically closed, and hence infinite, then $\mathbb{A}^1$ must be infinite.

(2) Subspaces of irreducible spaces are irreducible and dense.

(3) If Y is an irreducible space of a topological space X, then the closure $\operatorname{cl} Y$ is also irreducible in X.

Definition. We define an **algebraic affine variety** to be an irreducible closed subset of $\mathbb{A}^1$ under the Zariski topology. We define an open set of an affine variety to be a **quasi-affine variety**.

Definition. We define the **ideal** of a subset Y in $\mathbb{A}^n$, to be the set

$$I(Y) = \{f \in k[x_1, \ldots, x_n] : f(P) = 0 \text{ for all } P \in Y\}$$

where k is an algebraically closed field.

Theorem 1.1.2 (Hilbert's Nullstellensatz)**.** *Let k be an algebraically closed field and $\mathfrak{a}$ an ideal of $k[x_1, \ldots, x_n]$, and let $f \in k[x_1, \ldots, x_n]$ be a polynomial vanishing at all points of $Z(\mathfrak{a})$. Then there exists an $r \in \mathbb{Z}^+$ for which $f^r \in \mathfrak{a}$.*

Lemma 1.1.3. *The followin are true for any algebraically closed field k.*

(1) If $T_1, T_2 \subseteq k[x_1, \ldots, x_n]$ with $T_1 \subseteq T_2$, then $Z(T_2)Z(T_2)$.

(2) If $Y_1, Y_2 \subseteq \mathbb{A}^n$ with $Y_1 \subseteq Y_2$, then $I(Y_2) \subseteq I(Y_1)$.

(3) For any $Y_1, Y_2 \subseteq \mathbb{A}^n$, $I(Y_1 \cup Y_2) = I(Y_1) \cap I(Y_2)$.

(4) For any ideal $\mathfrak{a}$ of $k[x_1, \ldots, x_n]$, $I(Z(J)) = \sqrt{\mathfrak{a}}$, where

$$\sqrt{\mathfrak{a}} = \{f \in k[x_1, \ldots, x_n] : f^r \in J \textit{ for some } r \in \mathbb{Z}^+\}$$

(5) For every $Y \subseteq \mathbb{A}^n$, $Z(I(Y)) = \operatorname{cl} Y$ in the Zariski topology.

Proof. Let $T_1 \subseteq T_2$ be subsets of $k[x_1, \ldots, x_n]$, and choose a polynomial $f \in T_1$, and a point $P \in Z(T_2)$. We have by inclusion that $f \in T_2$, and that $f(P) = 0$. Now since $f \in T_1$, this puts $P \in Z(T_1)$ and we get the required inclusion. The proof for statement (2) is identical.

Now, let $P \in Y_1 \cup Y_2$, and choose a polynomial $f \in I(Y_1) \cap I(Y_2)$. Then we have that the point P is either contained in Y_1 or Y_2 (or both), so that $f(P) = 0$, which makes $I(Y_1 \cup Y_2) \subseteq I(Y_1) \cap I(Y_2)$. Conversely, if $f \in I(Y_1) \cap I(Y_2)$, then for any points $P \in Y_1 \cup Y_2$, $f(P) = 0$, which puts $f \in I(Y_1 \cup Y_2)$.

For part (4), notice this is a direct consequence of Hilbert's Nullstellensatz. Now, for part (5), notice that $Y \subseteq Z(I(Y))$, which is a closed set in the Zariski topology, so that $\operatorname{cl} Y \subseteq Z(I(Y))$. Now, let W be a closed set in $\mathbb{A}^n$, containing Y. Then we have $W = Z(\mathfrak{a})$, for some ideal af of $k[x_1, \ldots, x_n]$, so that $Y \subseteq Z(\mathfrak{a})$. Then by part (2), observe that $I(Y) \subseteq I(Z(\mathfrak{a}))$, but $\mathfrak{a} \subseteq I(Z(\mathfrak{a}))$, so by part (1), we have $Z(I(Y)) \subseteq Z(\mathfrak{a})$. This makes $Z(I(Y)) = \operatorname{cl} Y$. ∎

Corollary. *There exists a 1–1, inclusion reversing, corresponedece of algebraic sets of $\mathbb{A}^n$ onto radical ideals in $k[x_1, \ldots, x_n]$; given by the maps*

$$\begin{array}{rcl} Y & \to & I(Y) \\ \mathfrak{a} & \to & Z(\mathfrak{a}) \end{array}$$

Moreover, an algebraic set in $\mathbb{A}^n$ is irreducible if, and only if its ideal in $k[x_1, \ldots, x_n]$ is prime.

Proof. Notice that parts (1), (2), and (3) of the above lemma provide the required correspondence.

Now, suppose that Y is irreducible in $\mathbb{A}^n$, and take $fg \in I(Y)$. Then $Y \subseteq Z(fg) = Z(f) \cup Z(g)$, so that

$$Y = (Y \cap Z(f)) \cup (Y \cap Z(g))$$

which is the union of two closed sets in $\mathbb{A}^n$. Now, since Y is irreducible, we get that either $Y = Y \cap Z(f)$, or $Y = Y \cap Z(g)$; in either case, $Y \subseteq Z(f)$, or $Y \subseteq Z(g)$. This puts either $f \in I(Y)$, or $g \in I(Y)$, which makes the ideal $I(Y)$ prime.

Conversely if $\mathfrak{p}$ is a prime ideal, let $Z(\mathfrak{p}) = Y_1 \cup Y_2$, then $\mathfrak{p} = I(Y_1) \cap I(Y_2)$, so that either $\mathfrak{p} = I(Y_1)$ or $\mathfrak{p} = I(Y_2)$, since $\mathfrak{p}$ is prime. This makes $Z(\mathfrak{p}) = Y_1$ or $Z(\mathfrak{p}) = Y_2$, which makes $Z(\mathfrak{p})$ irreducible in $\mathbb{A}^n$. ■

Example 1.5. Consider k to be an algebraically closed field.

(1) Notice that $\mathbb{A}^n$ maps to the ideal (0) in $k[x_1, \ldots, x_n]$, which is a prime ideal. This makes $\mathbb{A}^n$ irreducible by the corollary to lemma 1.1.3.

(2) Let $f \in k[x, y]$ be an irreducible polynomial. Then f generates the prime ideal (f) in $k[x, y]$, since $k[x, y]$ is a UFD. Thus the zero set $Y = Z(f)$ is irreducible, and closed in $\mathbb{A}^n$; hence it is an affine variety. We call Y an **affine curve** in $\mathbb{A}^n$ defined by the equation $f(x, y) = 0$ of **degree** $\deg f = d$. Now, moregenerally, if f is an irreducible polynomial in $k[x_1, \ldots, x_n]$, then we call the affine variety $Y = Z(f)$ a **surface** in $n = 3$ and a **hypersurface** in $n > 3$.

(3) A maximal ideal $\mathfrak{m}$ of $k[x_1, \ldots, x_n]$ correspondes to a minimal affine variety of $\mathbb{A}^n$, which are the point-sets $\{P\}$ of $\mathbb{A}^n$; where $P = (a_1, \ldots, a_n)$. Thus every maximal ideal of $k[x_1, \ldots, x_n]$ is of the form $M = (x_1 - a_1, \ldots, x_n - a_n)$ for some $a_1, \ldots, a_n \in k$.

(4) Consider the field $\mathbb{R}$ which is not algebraically closed, and the curve defined by $x^2 + y^2 + 1 = 0$ in $\mathbb{A}^2(\mathbb{R})$. Notice that this curve is irreducible, but has no points in $\mathbb{A}^2$ (i.e. no roots in $\mathbb{R}$). This shows that if the field k is no algebraically closed, then the above results do not hold in general. Notice that the paraboloid $f(x, y) = x^2 + y^2 + 1$ (figure 1.1) does not intersect the real xy-plane

Definition. Let k be an algebraically closed field, and Y an affine algebraic set of $\mathbb{A}^n$. We definen the **affine coordinate ring** of Y to be the factor ring

$$A(Y) = {}^{k[x_1, \ldots, x_n]}\!\big/_{I(Y)}$$

Lemma 1.1.4. *If Y is an affine variety, then $A(Y)$ is an integral domain. Moreover, there exists a 1–1 correspondence of finitely generated k-algebras onto affine coordinate rings of affine varieties.*

Definition. We call a topological space X **Noetherian** if it satisfies the descending chain condition on closed sets; that is, if

$$\cdots \subseteq Y_2 \subseteq Y_1$$

is a descending chain of closed sets in X, then there exists an $r \in \mathbb{Z}^+$ for which $Y_r = Y_{r+1} = \cdots$

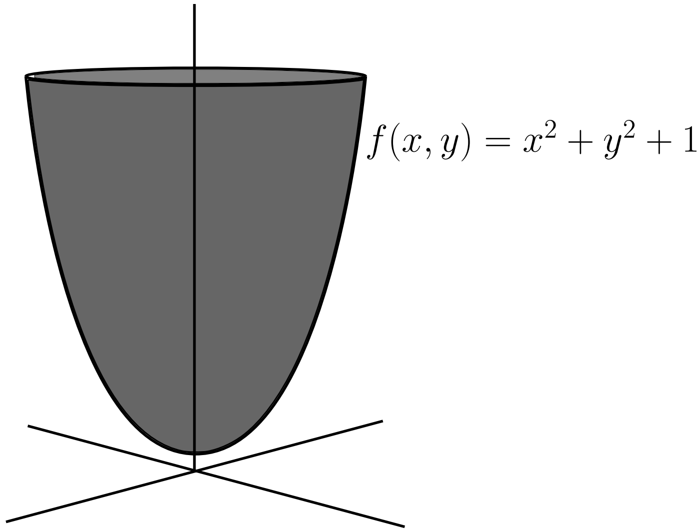

Figure 1.1: The curve $x^2 + y^2 + 1 = 0$ does not describe an affine variety.

Example 1.6. Let $\cdots \subseteq Y_2 \subseteq Y_1$ be a descending chain of closed sets in $\mathbb{A}^n$. Then $I(Y_1) \subseteq I(Y_2) \subseteq \ldots$ is an ascending chain of ideals of $k[x_1, \ldots, x_n]$. Since $k[x_1, \ldots, x_n]$ is Noetheria, we get an $r \in \mathbb{Z}^+$ for which $I(Y_r) = I(Y_{r+1}) = \ldots$. Since $Y_r = Z(I(Y_r))$, this makes $Y_r = Y_{r+1} = \ldots$. This makes $\mathbb{A}^n$ a Noetherian space.

Lemma 1.1.5. *If X is a Noetherian space, then every nonempty closed set Y in X can be written as a finite union of irreducible closed sets of X; i.e.*

$$Y = \bigcup_{i=1}^{r} Y_i$$

where each Y_i is closed and irreducible in X. Moreover, if $Y_i \nsubseteq Y_j$ for all $i \neq j$, then this representation is unique.

Proof. Let $\mathcal{C}$ be the collections of nonempty closed sets in X, which are not expressable as a finite union of closed irreducible sets in X. Suppose that $\mathcal{C}$ is nonempty. Since X is Noetherian, $\mathcal{C}$ contains a minimal elements Y. Then by definition, we have $Y = Y_1 \cup Y_2$ where Y_1 and Y_2 are closed sets; and by the minimality of Y, they can be expressed as a finite union of closed irreducible sets in X. This makes Y a finite union of closed irreducible sets of X, which means $Y \notin \mathcal{C}$; a contradiction of our assumption that Y is the minimal element. Therefore $\mathcal{C}$ must be empty, and every closed set Y is the finite union of closed irreducible sets in X.

Now, let $Y = \bigcup_{i=1}^{r} Y_i$ where each Y_i is closed and irreducible in X; and suppose that for each $i \neq j$, $Y_i Y_j$. Let

$$Y = \bigcup_{j=1}^{s} Z_j$$

another representation of Y as a finite union of closed irreducible sets in X. Notice that that

$Z_1 \subseteq Y_1 \cup \dots Y_r$, so that

$$Z_1 = \bigcup_{i=1}^{r} (Z_1 \cap Y_i)$$

since Z_1 is irreducible, we get $Z_1 \subseteq Y_i$ for some i, and $Y_1 \subseteq Z_j$ for some j. This gives that $Y_1 = Z_j$. Now, let $Z = (Y \backslash Y_1)^-$ then $Z = Y_2 \cup \cdots \cup Y_r$, and $Z = Z_2 \cup \cdots \cup Z^s$. Proceeding inductively gives us the desired uniqueness. ■

Corollary. *Every affine algebraic set in $\mathbb{A}^n$ can be uniquely expressed as a finite union of affine varieties, no one containing the other.*

Definition. We define the **dimension** of a topological space X to be the suprememum on all integers $n \geq 0$ for which there is a chain

$$Z_0 \subseteq \cdots \subseteq Z_n$$

of distinct irreducible closed sets in X, we write $\dim X = n$.

Example 1.7. The dimension of $\mathbb{A}^1$ under the Zariski topology is $\dim \mathbb{A}^1 = 1$, since the only irreducible closed sets are $\mathbb{A}^1$ and point-sets.

Definition. Let A be a ring. We define the **height** of a prime ideal $\mathfrak{p}$ to be the suprememum on all integeres $n \geq 0$ for which there exists a chain

$$\mathfrak{p}_0 \subseteq \dots \mathfrak{p}_n = \mathfrak{p}$$

of distinct prime ideals, and write height $\mathfrak{p} = n$. We define the **Krull-dimension**of A to be

$$\dim A = \sup_{\mathfrak{p} \subseteq A} \{\text{height}\, \mathfrak{p}\}$$

where the suprememum is taken over all prime ideals of A.

Lemma 1.1.6. *If Y is an affine algebraic set, then $\dim Y = \dim A(Y)$; that is, the dimension of Y is the dimension of the affine coordinate ring of Y.*

Proof. SUppose that Y is an affine algebraic set in $\mathbb{A}^n$, then the affine varieties of Y correspond to the prime ideals of $k[x_1, \dots, x_n]$ containing $I(Y)$. This in turn correspond to the prime ideals of $A(Y)$. Hence $\dim Y$ is the length of the longest chain of prime ideals in $A(Y)$, which is $\dim A(Y)$ by definition of the Krull-dimension. ■

Theorem 1.1.7. *Let k be a field and B an integral domain which is a finitely generated k-algebra. Then the following are true*

(1) $\dim B$ is the transcendence degree of the factor field $K(B)$ of B over k.

(2) For every prime ideal $\mathfrak{p}$ in B

$$\text{height}\, \mathfrak{p} + \dim {}^{B}\!/_{\mathfrak{p}} = \dim B$$

Lemma 1.1.8. $\dim \mathbb{A}^n = n$.

Proof. We have that $A(\mathbb{A}^n)$ is an integral domnain, so by theorem 1.1.7, the transcendence degree of $K(\mathbb{A}^n)$ of $\mathbb{A}^n$ over k is n, so that $\dim \mathbb{A}^n = \dim A(\mathbb{A}^n) = n$. ■

Lemma 1.1.9. *If Y is a quasi-affinee varaiety, then* $\dim Y = \dim(\operatorname{cl} Y)$.

Proof. Let $Z_0 \subseteq\subseteq Z_n$ be a chain of distinct closed irreducible sets of Y. Then $\operatorname{cl} Z_0 \subseteq \cdots \subseteq \operatorname{cl} Z_n$ is a chain of closed irreducible sets in $\operatorname{cl} Y$ (not necessarily distinct), so that $\dim Y \leq \dim(\operatorname{cl} Y)$.

Now, $\dim Y$ is finite, so choose a maximal chain $Z_0 \subseteq \cdots \subseteq Z_n$ for which $\dim Y = n$. Then Z_0 must be a point and $P : \operatorname{cl} Z_0 \subseteq \cdots \subseteq \operatorname{cl} Z_n$ is also a maximal chain. Now, this maximal chain corresponds to a maximal ideal $\mathfrak{m}$ of the affine coordinate ring $A(\operatorname{cl} Y)$ of $\operatorname{cl} Y$. Then $\operatorname{cl} Z_i$ corresponds to a prime ideal in $\mathfrak{m}$, so that $\operatorname{height} \mathfrak{m} = n$. Now, we have that P is a point in affine space, and since

$$A(\operatorname{cl} Y)/\mathfrak{m} \simeq k$$

we get $n = \dim A(\operatorname{cl} Y) = \dim(\operatorname{cl} Y)$, so that $\dim Y = \dim(\operatorname{cl} Y)$. ■

Theorem 1.1.10 (Krull's Hauptidealsatz)**.** *Let A be a Noetherian ring and $f \in A$ be an element which is neither a unit, nor a zero divisor. Then every minimal prime ideal $\mathfrak{p}$ containing f has* $\operatorname{height} \mathfrak{p} = 1$.

Lemma 1.1.11. *A Noetherian integral domain is a unique factorization domain if, and only if every prime ideal of* $\operatorname{height} = 1$ *is a principle ideal.*

Lemma 1.1.12. *An affine variety Y of $\mathbb{A}^n$ has dimension $n-1$ if and only if it is the zero set $Z(f)$ of a single irreducible polynomial $f \in k[x_1, \dots, x_n]$.*

Proof. Notice that if $f \in k[x_1, \dots, x_n]$ is irreducible, then the set $Y = Z(f)$ is an affine variety and its ideal is the prime ideal (f) of height 1. Hence, by theorem 1.1.7 we have

$$\dim Y = n - 1$$

Conversely, if Y is an affine variety of dimension $n-1$, it corresponds to an ideal of height 1. Now, we have that $k[x_1, \dots, x_n]$ is a UFD, so this prime ideal is a principle ideal and generated by a nonconstant polynomial f. This makes $Y = Z(f)$. ■

1.2 Projective Varieties

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