Advanced Calculus

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Text

An Introduction to Analysis (3^{rd} edition)

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Chapter 1

Chapter 1: The Real Number System.

1.1 The Ordered Field Axioms

We want to study the algebraic structure of the real numbers; the so called set $\mathbb{R}$.We assume standard knowlege of set theory. We display the field axioms below:

Postulate 1. *There are binary operations* $+$ *and* $\cdot$ *defined on the set* $\mathbb{R} \times \mathbb{R}$*, satisfying the following properties* $\forall a, b, c \in \mathbb{R}$:

(1) $a + b \in \mathbb{R}$.

(2) $a + (b + c) = (a + b) + c$ *and* $a \cdot (b \cdot c) = (a \cdot b) \cdot c$.

(3) $a + b = b + a$ *and* $a \cdot b = b \cdot a$.

(4) $a \cdot (b + c) = (ab) + (a \cdot c)$.

(5) There exists a unique element $0 \in \mathbb{R}$ *such that* $\forall a \in \mathbb{R}$, $a + 0 = a$.

(6) There exists a unique element $1 \neq 0 \in \mathbb{R}$ *such that* $\forall a \in \mathbb{R}$, $a \cdot 1 = a$.

(7) For each $a \in \mathbb{R}$*, there exists a unique element* $-a \in \mathbb{R}$ *such that* $a + (-a) = 0$.

(8) For each $a \in \mathbb{R} \backslash \{0\}$*, there exists a unique element* $a^{-1} \in \mathbb{R}$ *such that* $a \cdot a^{-1} = 1$.

We call postulate 1 the **field axioms** for real numbers; in essence what it says is that the pairs $(\mathbb{R}, +)$ and $(\mathbb{R}, \cdot)$ form abelian groups, and that $\cdot$ distributes over $+$. W now get the following properties from the field axioms:

(1) $(-1)(-1) = 1$.

(2) $0a = 0$.

(3) $-(a - b) = b - a$.

(4) For $a, b \in \mathbb{R}$ and $ab = 0$, then either $a = 0$ or $b = 0$.

The proofs for these are relatively easy, and can be done at anytime as an exercise. We now introduce the second postulalte called the **order axioms**:

Postulate 2. *There is a relation $<$ on $\mathbb{R} \times \mathbb{R}$ satisfying the following properties:*

(1) Given $a, b \in \mathbb{R}$, one and only one of the following hold: either $a < b$, $b < a$, or $a = b$.

(2) If $a < b$ and $b < c$, then $a < c$.

(3) If $a < b$ and $c \in \mathbb{R}$, then $a + c < b + c$.

(4) If $a < b$ and $c > 0$, then $ac < bc$; if $c < 0$, then $bc < ac$.

We may may the following remarks: by $b > a$ we mean that $a < b$, by $a \leq b$, we mean that $a < b$ or $a = b$, and by $a < b < c$, we mean that $a < b$ and $b < c$.

Definition. For $a \in \mathbb{R}$, a is called **nonnegative** if $a \geq 0$, a is called **nonpositive** if $a \leq 0$. We call a **positive** if $a > 0$ and we call a **negative** if $a < 0$.

Example 1.1. If $a \in \mathbb{R}$, show that $a \neq 0$ implies $a^2 > 0$; in particular that $-1 < 0 < 1$.

Proof. If $a \neq 0$, then either $a > 0$ or $a < 0$. If $a > 0$, then clearly $a^2 > 0$ and we are done. If $a < 0$, then $-a > 0$, hence $(-a)(-a) > (-a)0$, and so $a^2 > 0$.

Now we have that $1 \neq 0$, and that $1^2 = 1 > 0$, subtracting we get $1 - 1 > 0 - 1$ so $0 > -1$.s ■

Example 1.2. If $a \in \mathbb{R}$, show that $0 < a < 1$ implies that $0 < a^2 < a$, and $a > 1$ implies that $a^2 > a$.

Proof. We have already that if $a > 0$, then $a^2 > 0$. Now suppose that $a < 1$ then multiplying by a we get $(a)(a) < 1(a)$, hence $a^2 < 1$. Likewise, by the same reasoning, if $a > 1$ we get $a^2 > a$. ■

Example 1.3. Prove that:

(1) $0 \leq a < b$ and $0 \leq c < d$ imply that $ac < bd$.

(2) $0 \leq a < b$ imply that $0 \leq a^2 < b^2$ and $0 \leq \sqrt{a} < \sqrt{b}$.

(3) $0 < a < b$ implies $0 < \frac{1}{b} < \frac{1}{a}$.

Proof. (1) Let $0 \leq a < b$ and $0 \leq c < d$. Then $0 \leq ac < bc$, and since $c < d$, then $bc < bd$, hence we have that $0 \leq ac < bd$.

(2) Let $0 \leq a < b$. Then $0 \leq aa < ab$, and notice that $ab < bb$, so $0 \leq aa < bb$, i.e. $0 \leq a^2 < b^2$. Now notice that $a = (\sqrt{a})(\sqrt{a})$ and $b = (\sqrt{b})(\sqrt{b})$, by the previous result, we have $0 \leq (\sqrt{a})(\sqrt{a}) < (\sqrt{b})(\sqrt{b})$, henve we have that $0 \leq \sqrt{a} < \sqrt{b}$.

(3) Let $0 < a < b$, multiplying by $\frac{1}{b}$, we get that $0 < a\frac{1}{b} < b\frac{1}{b}$ and get $0 < a\frac{1}{b} < 1$, now multiplying again by $\frac{1}{a}$, we get $0 < \frac{1}{a}a\frac{1}{b} < \frac{1}{a}$, thus we have that $0 < \frac{1}{b} < \frac{1}{a}$.

■

Definition. The **absolute value** of an element $a \in \mathbb{R}$ is a real number $|a|$ defined such that:

$$|a| = \begin{cases} a, \text{ if } a \geq 0 \\ -a, \text{ if } a < 0 \end{cases} \tag{1.1}$$

Remark. The absolute value is multiplicative, i.e. $|ab| = |a||b|$ for all $a, b \in \mathbb{R}$.

Proof. We do a casewise evaluation.

(1) If $a = 0$, or $b = 0$, then $|ab| = 0 = |a||b|$ (since $|a| = a = 0$ or $|b| = b = 0$).

(2) Let $a, b > 0$. then $|a| = a$ and $|b| = b$. Then $|ab| = ab$ as $ab > 0$. Hence $|ab| = ab = |a||b|$.

(3) Let $a > 0$ and $b < 0$, then $|a| = a$ and $|b| = -b$ si $|a||b| = a(-b) = -ab = |ab|$ as $ab < 0$.

(4) Let $a, b < 0$, then $|a| = -a$ and $|b| = -b$ and $|a||b| = (-a)(-b) = ab = |ab|$, since $ab > 0$.

■

Theorem 1.1.1. *Let $a \in \mathbb{R}$ and let M be nonnegative. Then $|a| \leq M$ if and only if $-M \leq a \leq M$.*

Proof. Notice that if $|a| \leq M$, then $-|a| \geq -M$. Suppose then that $a \geq 0$, then $|a| = a \leq M$, and since $-M \leq 0$, we have $-M \leq a \leq M$. Now if $a < 0$, $-|a| = a$, then $a \geq -M$, and since $a < 0$, $a \geq M$.

Conversely suppose that $-M \leq a \leq M$. Then $-M \leq a$ and $a \leq M$. For $-M \leq a$, multiplying by -1 we have $M \geq -a$. If $a \leq 0$ then $|a| = a$ and $|a| \leq M$. If $a < 0$ then $|a| = -a \leq M$. Hence we have in both cases that $|a| \leq M$. ■

Theorem 1.1.2. *The absolute value satisfies the following three properties For all $a, b \in \mathbb{R}$:*

(1) $|a| \geq 0$ *with* $|a| = 0$ *if and only if* $a = 0$.

(2) $|a - b| = |b - a|$.

(3) $|a + b| \leq |a| + |b|$.

Proof. If $a \leq 0$, then clearly $|a| \geq 0$, if $a < 0$, then $|a| = -a \geq 0$. Now if $a = 0$ then $|0| = 0$, and if $|a| = 0$, then $\pm a = 0$; then $a = 1a = (\pm 1)^2 a = (\pm 1)(\pm 1)a = (\pm 1)(\pm a)$, since $\pm a = 0$, $(\pm 1)0 = 0$ hence $a = 0$.

Now we have $(a - b) = -(b - a)$, so $|a - b| = |-1(b - a)| = |-1||b - a| = 1|b - a| = |b - a|$.

Notice for all $x \in \mathbb{R}$, $|x| \leq |x|$. So $-|x| \leq x \leq |x|$, then we have for $a, b \in \mathbb{R}$, $-|a| \leq a \leq |a|$ and $-|b| \leq b \leq |b|$, adding we get $-(|a| + |b|) \leq a + b \leq |a| + |b|$, thus $|a + b| \leq |a| + |b|$. ■

Corollary. *For all $a, b \in \mathbb{R}$, $|a - b| \geq |a| - |b|$ and $||a| - |b|| \leq |a - b|$.*

Proof. Now $|a|-|b| = |a+b-b|-|b|$, by the above theorem, $|a|-|b| \leq |a-b|+|b|-|b| = |a-b|$.

Now $||a|-|b|| \leq |a-b|$ implies $-|a-b| \leq |a|-|b| \leq |a-b|$, we need to show that $-|a-b| \leq |a|-|b|$. Then $|a-b| \geq |b|-|a|$ hence $|b-a| \geq |b|-|a|$ which reduces to the second equality. ∎

Example 1.4. Show that if $-2 < x < 1$ then $|x^2 - x| < 6$.

Proof. We have that $-2 < x < 2$, so $|x| < 2$. Then $|x^2 + x| \leq |x^2| + |x| = |x||x| + |x| < (2)(2) + 2 = 6$. ∎

Theorem 1.1.3. *Let $x, y, a \in \mathbb{R}$.*

(1) $x < y + \epsilon$ for all $\epsilon > 0$ if and only if $x \leq y$.

(2) $x > y - \epsilon$ for all $\epsilon > 0$ if and only if $x \geq y$.

(3) $|a| < \epsilon$ for all $\epsilon > 0$ if and only if $a = 0$.

Proof. (1) Suppose that $x < y + \epsilon$ for all $\epsilon > 0$, but that $x > y$. Let $\epsilon = \epsilon_0 = x - y$, then $x = y + \epsilon_0$. Then it is not true that $x < y + \epsilon_0$. A contradiction.

Conversely, suppose that $x \leq y$ and that $\epsilon > 0$. Then either $x < y$ or $x = y$. For $x < y$, $x + 0 < y + 0 < y + \epsilon$, hence $x < y + \epsilon$. Similarly, for $x = y$, $x < y + \epsilon$.

(2) This proof is analgous to the previous. Suppose that $x > y - \epsilon$ for $\epsilon > 0$. Then let $\epsilon = \epsilon_0 = y - x$. Then $x = y - \epsilon_0$, which contradictos our assumption that $x > y - \epsilon$.

Conversely, let $x \geq y$ and let $\epsilon > 0$. Then either $x > y$ or $x = y$. For $x > y$, we have $x - 0 > y - 0 > y - \epsilon$. For $x = y$, we see clearly that $x > y - \epsilon$.

(3) Notice first that if $\epsilon > 0$ then $0 > -\epsilon$. Now suppose that for $a \in \mathbb{R}$, $|a| \leq \epsilon$. Then $-\epsilon < a < \epsilon$, thus by the transitivity of $<$ we have that $0 < a$ and $a < 0$, which cannot happen, so it must be that $a = 0$.

Now let $a = 0$, then clearly, by our assumptions, we see that $a < \epsilon$ and $-\epsilon < a$, thus $|a| \leq \epsilon$.

∎

Definition. For all $a, b \in \mathbb{R}$, we define the **closed interval** to be the set $[a, b] = \{x : a \leq x \leq b\}$. We define the **open interval** to be the set $(a, b) = \{x : a < x < b\}$. We denote a **half open interval** to be a set of the form $[a, b) = \{x : a \leq x < b\}$ or $(a, b] = \{a < x \leq b\}$.

We denote $(a, \infty) = \{x : a < x\}$, $(-\infty, a) = \{x : x < a\}$ and $(-\infty, \infty) = \{x : x \in \mathbb{R}\}$.

Definition. An interval I is **bounded** if and only if it has the form: $[a, b], [a, b), (a, b]$, or (a, b). We call a and b the **endpoints**, or **bounds** of the interval. We call all other intervals **unbounded**.

Now if $a = b$, the two bounds co incide and we call the interval **degenerate**, and if $a < b$, it is called **nondegenerate**. Also notice that an interval $(1, 1) = \emptyset$, but $[1, 1] = \{1\}$. So we notice that a degenerate open interval is the empty set, and a degenerate closed interval is a point. Now for bounded intervals, we call $|a - b|$ is called the **length** of the interval, and sometimes denoted $|I|$.

Homework. Do exercises 1, 5, 7, 9, and 10.

1.2 The Well Ordering Principle.

So far we have the set of all natrual numbers $\mathbb{N}$, the set of all integers $\mathbb{Z}$, the rationals $\mathbb{Q}$ and the reals $\mathbb{R}$. Now $\mathbb{N}$ is speacial from these, as it has a "minimal" element. We clarify this below.

Definition. An element $x \in \mathbb{R}$ is a **least element** in a set $E \subseteq \mathbb{R}$ if and onlt if $x \in E$ and $x \leq \alpha$ for all $\alpha \in E$.

Postulate 3 (The Well Ordering Principle.)**.** *Every nonempty subset of* $\mathbb{N}$ *has a least element.*

Now this propertie does not apply to $\mathbb{Z}$, $\mathbb{Q}$, and $\mathbb{R}$; if one takes the subset $\mathbb{Q}$, one sees immediately that $\mathbb{Q}$ has no least element. Another thing is that the well ordering principle implies the principle of mathematical induction.

Theorem 1.2.1. *Suppose that for each $n \in \mathbb{N}$, that $A(n)$ is a propostision such that:*

(1) $A(1)$ is true.

(2) For every $k \in \mathbb{N}$ for which $A(n)$ is true, then $A(k+1)$ is also true.

Then $A(n)$ is true for all $n \in \mathbb{N}/$

Proof. Suppose there is some n for whcih $A(n)$ is false. Then the set $E = \{n \in \mathbb{N} : A(n) \text{ is false.}\} \neq \emptyset$. Then by the well ordering principle, E has a least element x. Then $A(x)$ is false. However, since $A(1)$ is true, $x \neq 1$, then $x - 1 \in \mathbb{N}$ and $x - 1 < x$, since x is the least element, then $A(x-1)$ is true. By the second condition, we get that $A(x)$ is true, contradicting that $x \in E$. Therefore, E must be empty, and $A(n)$ is true for all $n \in \mathbb{N}$. ■

Example 1.5. Show that $\sum_{k=1}^{n}(3k-1)(3k+2) = 3n^3 + 6n^2 + n$.

Proof. For $k = 1$ we see that $(3(1)-1)(3(1)+2) = 10 = 2(1)^3 + 6(1)^2 + 1$.

Now suppose that the proposition is true for $n \geq 1$. Then:

$$\begin{aligned}\sum_{k=1}^{n+1}((3k-1)(3k+2)) &= \sum_{k=1}^{n}((3k-1)(3k+2)) + (3(n+1)-1)(3(n+1)+2)\\ &= 3n^3 + 6n^2 + n + (3(n+1)-1)(3(n+1)+2)\\ &= 3(n+1)^3 + 6(n+1)^2 + (n+1)\end{aligned}$$

■

Remark. If $m, n \in \mathbb{N}$, then $m + n \in \mathbb{N}$ and $mn \in \mathbb{N}$. This also implies that $m + n, mn \in \mathbb{Z}$.

Definition. Let $a, b \in \mathbb{Z}$. We call a **binomial** an expression of the form $(a+b)^n$ for some $n \in \mathbb{N}$.

We wish to study binomials further.

Definition. We define, for $n, k \in \mathbb{N}$ the **binomial coefficient** $\binom{n}{k}$ is defined such that:

(1) $\binom{0}{0} = 1$

(2) $\binom{n}{k} = \frac{n!}{(n-k!)k!}$.

The binomial coefficient is itself a natural number, and can be visualized via pascal's triangle.

Lemma 1.2.2. $\binom{n}{k} + \binom{n}{k-1} = \binom{n+1}{k}$ *For* $n, k \in \mathbb{N}$ *and* $1 \leq k \leq n$.

Proof.

$$\binom{n}{k} + \binom{n}{k-1} = \frac{n!}{(n-k)!k!} + \frac{n!}{(n-k+1)!(k-1)!} \tag{1.2}$$

$$= n!(\frac{1}{(n-k)!k!} + \frac{1}{(n-k-1)!(k-1)!}) \tag{1.3}$$

$$= n!(\frac{1}{(k-1)!}(\frac{1}{(n-k)!k} + \frac{1}{(n-k-1)!})) \tag{1.4}$$

$$= \frac{n!}{(k-1)!(n-k)!}(\frac{1}{k} + \frac{1}{n-k+1}) \tag{1.5}$$

$$= \frac{(n+1)!}{(n-k+1)!k!} \tag{1.6}$$

$$= \binom{n+1}{k} \tag{1.7}$$

■

Theorem 1.2.3 (The Binomial Theorem)**.** *If* $a, b \in \mathbb{R}$, *and* $n \in \mathbb{N}$, *then:*

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k \tag{1.8}$$

Proof. We use induction in the proof. We see that for $n = 1$, $(a+b)^1 = a+b = \binom{1}{0}a^{1-0} + \binom{1}{1}b^1$. Now suppose that the theorem is true for all $n \geq 1$, that is $(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k$; and now consider $n+1$.Then:

$$(a+b)^{n+1} = (a+b)(a+b)^n \tag{1.9}$$

$$= (a+b)(\sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k) \tag{1.10}$$

$$\tag{1.11}$$

by the distributive law:

$$= a\sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k) + b\sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k) \quad = \sum_{k=0}^{n} \binom{n}{k} a^{n-k+1} b^k) + \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^{k+1}). \tag{1.12}$$

Listing the terms we get $\binom{n}{0}a^{n+1} + \binom{n}{1}a^n b + \binom{n}{2}a^{n-1}b^2 + \cdots + \binom{n}{n+1}ab^n + \binom{n+1}{n+1}b^n$; adding like terms, and by lemma 1.2.2 we get: $\binom{n+1}{0}a^{n+1} + \binom{n+1}{1}a^n b + \binom{n+1}{2}a^n b^2 + \cdots + \binom{n+1}{n}ab^n + \binom{n+1}{n+1}b^n = \sum_{k=0}^{n+1} \binom{n+1}{k} a^{n-k+1} b^k$. ■

Homework. Exercises: 1,2,3,5,6 on page 17. These exercises practice the principle of mathematical induction.

1.3 The Axiom of Completeness.

Definition. Let $E \subseteq \mathbb{R}$ be nonempty. The set E is **bounded above** if and only if there is an $M \in \mathbb{R}$ such that $a \leq M$ for all $a \in E$; we call M the **upperbound**. A number s is called a **supremum** or **least upperbound** of E of s is an upper bound, and for all other upper bounds M, $s \leq M$. We denote the least upper bound as $\sup E$.

Example 1.6. If $E = [0, 1]$ prove that $\sup E = 1$.

Proof. By definition of $[0, 1]$, 1 is an upperbound. Now let $M \in E$ be an upper bound, then $x \leq M$ for all $x \in E$. Now $1 \in E$, so $1 \leq M$, so by definition, we get that $1 = \sup E$. ■

Remark. If a set has one upperbound, then it has infinitely many upper bounds.

Proof. If M_0 is an upperbound, then for any $M > M_0$, M is an upper bound. ■

Remark. If a set has a least upperbound, then that least upperbound is unique.

Proof. Asuume that $s_1, s_2 \in \mathbb{E}$ are least upperbounds. Then s_1 is an upperbound and so $s_2 \leq s_1$. Likewise, s_2 is an upperbound, so $s_1 \leq s_2$, hence it must be that $s_1 = s_2$. ■

Theorem 1.3.1 (Approximation Property for Least Upperbounds)**.** *If E has a least upperbound, and $\epsilon > 0$ is a positive number, then there is an element $a \in E$ such that $\sup E - \epsilon < a \leq \sup E$.*

Proof. Suppose not, that is there is some $\epsilon_>0$ for all $a \in E$ for which a does not lie between $\sup E - \epsilon_0$ and $\sup E$. Now $a \leq \sup E$, hence $a \leq E - \epsilon_0$, so $\sup E -_0$ is an upperbound of E. So $\sup E \leq \sup E - \epsilon_0$, implying $\epsilon_0 \leq 0$, a contradiction. ■

Now it is not always true that for some set E, that $\sup E \in E$.

Remark. If $E \subseteq \mathbb{N}$ has a least upperbound, then $\sup E \in E$.

Proof. Let $s = \sup E$. By the approximation property, for $\epsilon = 1$, there is an x_0 in E such that $s - 1 < x_0 \leq s$. Now if $x_0 = s \in E$, then we are done.

Otherwise, we have $s - 1 < x < s$. Then appying the approximation property again for $\epsilon = s - x_0$, $s - \epsilon = s - s - x_0 = x_0$, then there is an $x_1 \in E$ for which $x_0 < x_1 \leq s$. Again if $x_1 = s$ we are done. Now if $x_1 < s$ we get $0 < x_1 - x_0 < s - x_0 < s - (s - 1) = 1$. We also know that $x_1 - x_0 \geq 1$ for different integers $x_1 > x_0$, a contradiction. ■

This brings us to the axiom of completeness.

Postulate 4 (The Axiom of Completeness)**.** *If E is a nonempty subset of $\mathbb{R}$ that is bounded above, then E has a finite least upperbound.*

This does not apply to $\mathbb{Q}$, for example, take π. We can take the set E to be $E = \{3.14, 3.141, 43.1415, \}$. We see that π is an upperbound, so is 4; however, π is the least upperbound. If we only consider $E \in \mathbb{Q}$, then E has upperbounds, but no least upperbound (as $\pi \notin \mathbb{Q}$). In essence, $\mathbb{Q}$ has a "hole"; infact it has many "holes" that is $\mathbb{Q}$ is not complete. But if $E \in \mathbb{R}$, we see that $\sup E = \pi$. In essence, $\mathbb{R}$ is complete as $\mathbb{R}$ has no "holes".

Theorem 1.3.2 (The Archimedean Principle)**.** *Given positive real numbers a, and b; there is an integer n such that $b < na$.*

Proof. We want to build a nonempty set of integers such that it is bounded above with a largest integer. The set $E = \{k \in \mathbb{Z} : ka \leq b\}$ has an upper bound; hence it has a least upperbound k_0. Then $k_0 + 1 \notin E$, hence $(k_0 + 1)a > b$. ■

Example 1.7. Let $A = \{1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \ldots\}$ and $B = \{\frac{1}{2}, \frac{3}{4}, \frac{7}{8}, \ldots\}$ Show that $\sup A = \sup B = 1$.

Proof. First, 1 is an upperbound for both A and B. Now for A it is clear that 1 is the least upperbound. Now let M be an upperbound for B and $M < 1$, so $1 - M > 0$ and $\frac{1}{1-M} > 0$. Then by the Archimedean principle, for $\frac{1}{1-M}$ and 1, there is an $n \in \mathbb{Z}$ such that $\frac{1}{1-M} < n$. Now we have that $n < 2^n$; hence it follows that $1 - \frac{1}{2^n} > M$. So M is not an upperbound; a contradiction. Hence $\sup B = 1$. ■

Theorem 1.3.3. *If $a, b \in \mathbb{R}$ such that $a < b$, then there exists a rational $q \in \mathbb{Q}$ such that $a < q < b$.*

Proof. If $b - a > 0$, we have a pari of positive numbers 1 and $b - a$; so by the Archimedean principle, there is an $n \in \mathbb{N}$ such that $n(b - a) > 1$.

Now suppose that $b > 0$. Consider the set $E = \{k \in N : b \leq \frac{k}{n}\}$. Clearly $E \neq \emptyset$ as $1 \in E$. Then by the well ordering principle, E has a least element k_0. Now let $m = k_0 - 1$ and let $q = \frac{m}{n}$. Now $m < k_0$, so $m \notin E$; now there are twon possible cases: either $m \leq 0$ or $b > \frac{m}{n} = q$, either way we have that $b > q$. And $k_0 \in \mathbb{E}$, so $b \leq \frac{k_0}{n}$ so $a = b - (b - a) < \frac{k_0 - 1}{n} = q$

Now suppose that $b \leq 0$, then $-b > 0$; applying the Archimedean principle for 1 and b, there is a $k \in \mathbb{N}$ such that $-b < (k)(1)$, thus $k + b > 0$. Now if $b = 0$, clearly $1 + 0 > 0$. By the first case, there is a rational $q \in \mathbb{Q}$ such that $a + k < q < b + k$. Subtracting k, we get $a < q - k < b$, which finishes the proof as $q - k \in \mathbb{Q}$. ■

Remark. If $x > 1$ and $x \notin \mathbb{N}$, then there is a natrual number $n \in \mathbb{N}$ such that $n < x < n + 1$.

Proof. For the pair $(x, 1)$ by the Archimedean principle, there is a $k \in \mathbb{N}$ such that $E = \{m \in \mathbb{N} : x < m\}$ is nonempty. Then by the well ordering principle, there is a least element m_0, hence $x < m_0$, Let $n = m_0 - 1$, then $n \notin E$. Now either $n \leq 0$, or $n < x$. As $x \notin \mathbb{N}$, $x \neq n$. Now if $n \leq 0$, then $n < x < m_0 = n + 1$; and if $n < x$, then $n < x < m_0 = n - 1$. ■

Remark. If $n \in \mathbb{N}$ is not a perfect square, then $\sqrt{n}$ is irrational.

Proof. Assume that $\sqrt{n}$ is rational for a nonperfect square $n \in N$. Then $\sqrt{n} = \frac{p}{q}$ for $p, q \in \mathbb{Z}$. Now since 1 is a perfect square, let $n \geq 2$. Then $\sqrt{n} > 1$. Then there is an $m_0 \in \mathbb{N}$ such that $m_0 < \sqrt{n} < m_0 + 1$. Consider the set $E = \{k \in \mathbb{N} : k\sqrt{n} \in \mathbb{N}\}$. Now $q\sqrt{n} = p \in \mathbb{N}$, so $E \neq \emptyset$. By the well ordering principle, E has a least element n_0. Then $n_0\sqrt{n} \in \mathbb{N}$ and $n_0 m_0 \in \mathbb{N}$. Now $n_0(\sqrt{n} - m_0) = x \in \mathbb{N}$. Now $0 < \sqrt{n} - m_0 < 1$, then $0 < x < n_0$. Now $x \notin E$, on the other hand, $x\sqrt{n} = n_0(\sqrt{n} - m_0)\sqrt{n}$, so $x \in E$; a contradiction. ■

Definition. Let $E \subseteq \mathbb{R}$ be nonempty. The set E is **bounded below** if and only if there is an $m \in \mathbb{R}$ such that $a \geq m$ for all $a \in E$. We call m a **lowerbound** of E. A nuumber t is called an **infimum** or a **greatest lowerbound** if and only if t is a lowerbound of E and $t \geq m$ for all lowerbounds m of E. We denote the greatest lowerbound as $t = \inf E$.

It should be worthwhile to say that the least element of a set, and it's greatest lowerbound are not the same.

Definition. A nonempty set $E\mathbb{R}$ is said to be **bounded** if it is bounded above and bounded below. That is there exist $m, M \in \mathbb{R}$ such that $m \leq a \leq M$ for all $a \in E$. Is $\sup E \in E$ then we write $\sup E = \max E$ and call it the **maximum** of E. If $\inf E \in E$, then we write $\inf E = \min E$ and call it the **minimum** of E.

Definition. The **reflection** of a set $E \in \mathbb{R}$ is defined to be the set $-E = \{xx = -a : \text{ for some } a \in E\}$.

Theorem 1.3.4. *Let $E \subseteq \mathbb{R}$ be a nonempty set. Then E has a greatest upperbound if and only if $-E$ has a greates lowerbound; in which case we have that* $\inf(-E) = -\sup E$. *Likewise, E has a greatest lowevound if and only if $-E$ has a least upperbound, in which case* $\sup(-E) = -\inf E$.

Proof. Suppose that E has a least upperbound s, and let $t = -s$. Now we have that $-a \geq -s = t$, for all $a \in E$, so t is a lowerbound of E. Now suppose m is any lowerbound of E. Then $m \leq -a$ for all $a \in E$. Now, $-m \geq a$ for all $a \in E$, so $-m$ is an upperbound,; now since s is the least upperbound, $s \leq -m$, hence $t = -s \geq m$

Conversely, suppose that $-E$ has a greatest lowerbound t; then $tleq - a$ for all $a \in E$, hence $-t \geq a$ for all $a \in E$, so $-t$ is an upperbound. Let M be any upperbound of E, then $M \geq a$ for all $a \in E$. Then $-M \leq -a$ and $-M$ is a lowerbound. Then $-M \leq t$, hence $M \geq -t = s$, so s is the least upperbound of E. ■

Homework. Exercises 2,3,4,5(a), and 6 on page 23.

Theorem 1.3.5 (Monotone property)**.** *Let $A \subseteq B$ be nonempty subsets of* $\mathbb{R}$. *Then:*

(1) If B has a least upper bound, then $\sup A \leq \sup B$.

(2) If B has a greatest lower bound then $\inf A \geq \inf B$

Proof.

1. We have that $A \subseteq B$, so any upper bound of B is an uppper bpund pf A. Therefore $\sup A$ is an upper bound of A; By completeness, $\sup A$ exisists, and more over, $\sup A \leq \sup B$

2. Consider $-A \subseteq -B$, by part (1), we have $\sup -A \leq \sup -B$, and we also have that $\sup -A = -\inf A$ and $\sup -B = -\inf B$; thus $-\inf A \leq -\inf B$ therefore, $\inf A \geq \inf B$.

■

To consider $\sup A$ and $\inf A$, we need nonempty and bounded sets of $\mathbb{R}$. Now we would like to talk about the least upper and greates lower bounds for any subset of $\mathbb{R}$; not just the bounded ones.

Definition. An **extended real number** x is real number such that either $x = \infty$ or $x = -\infty$.

Now if E is not bounded above, then we define the least upper bound of E to be $\sup E = \infty$. If E is not bounded below we define $\inf E = -\infty$. Now for the empty set $\emptyset$, we define $\sup \emptyset = -\infty$ and $\inf \emptyset = \infty$. It is worth considering why this phenomena occurs with the empty set.

1.4 Functions, Countability, and The Algebra of Sets.

Definition. Given a mapping $f : E \to \mathbb{R}$, we call f a **real valued function** of a **real variable**. If we are given $f : \mathbb{R}^n \to \mathbb{R}$ we call the mapping a real valued **multivariable fucntion**. If we are given $f : \mathbb{R}^m \to \mathbb{R}^m$, we call it a **vector valued** multivariable function.

We are interested in a whole slew of functions; the trigonometric functions: sin, cos, tan, cot, sec, and csc. Other functions of interest are the natural logarithmic function, and exponential function: log and e^x, as well as arbitrary power functions x^α. We can define the power function to be $x^\alpha = e^{\alpha \log x}$; where $x > 0$ and $\alpha \in \mathbb{R}$.

The derivitives of these functions carry over, and will be examined with more scrutiny later on.

Definition. A mapping $f : X \to Y$ is called **1-1** if and only if for $x_1, x_2 \in X$, $f(x_1) = f(x_2)$ implies that $x_1 = x_2$. f is called **pnto** if for every $y \in Y$, there exists an $x \in X$ such that $y = f(x)$.

Example 1.8. $f(x) = x^2$ is 1-1 on the interval $[0, \infty)$, but it is not 1-1 on $(0, \infty)$.

Theorem 1.4.1. *Let X, and Y and let $f : X \to Y$ be a mapping. The f is 1-1 from X onto X if there is a unique mapping $g : Y \to X$ that satisfies: $f(g(y)) = y$ and $g(f(x)) = x$.*

Proof. Suppose that f is 1-1 and onto. Then for each $y \in Y$, choose the unique $x \in X$ such that $f(x) = y$. We define $g(y) = x$, Then it is clear to see that g takes Y onto X, and g is also 1-1. Then $g(y) = g(f(x)) = x$ and $f(x) = f(g(y)) = y$.

Conversely, suppose that there exists a mapping $g : Y \to X$ satisfying the $g(y) = g(f(x)) = x$ and $f(x) = f(g(y)) = y$. Now let $x_1, x_2 \in X$ and let $f(x_1) = f(x_2)$. Then $x_1 = g(f(x_1)) = g(f(x_2)) = x_2$; so f is 1-1. For any $y \in Y$, let $x = g(y)$, then $f(x) = f(g(y)) = y$; thus f is onto.

Now suppose ther is another mapping $h : Y \to X$satisfying the conditions that g satisfies. Let $y \in Y$ and choose $x \in X$ such that $f(x) = y$. Then $h(yh(f(x)) = x = g(f(x)) = g(y)$; then it must be that $g = h$. ∎

Definition. if $f : X \to Y$ is 1-1 and onto, we say that f has a **inverse mapping** and denote it f^{-1}.

If $f : E \to \mathbb{R}$ is a real valued function, then the pair $(x, f(x))$ is on the graph of f; $(f^{-1}(y), y)$ is also on the graph of f. However the pair $(y, f^{-1}(y))$ is on the graph of f^{-1}. The graphs of f and f^{-1} are symmetric with respect to the line $y = x$.

Remark. If f is defferentiable on an open interval I and $f'(x) \neq 0$ for all $x \in I$.

This remark is a sufficient, but not a necessary condition. The concept of "differentiability" must also be discussed. We can prove this remark with the "mean value theorem" which will be discussed later.

Example 1.9. Prove that $f(x) = e^x - e^{-x}$ is 1-1 and find f^{-1}.

Proof. By the chain rule, we have $f'(x) = e^x + e^{-x} > 0$ fr all $x \in \mathbb{R}$. Now let $y = e^x - e^{-x}$, then $e^x y = e^{2x} - 1$, by the quadratic formula we get:

$$e^x = \frac{y}{2} + \frac{\sqrt{y^2+4}}{2}$$

So taking the natural logarithm:

$$y = \frac{x}{2} + \frac{\sqrt{x^2}+4}{2} = f^{-1}(x)$$

■

Definition. A set E is **finite** if and only if $e =$ or there is a 1-1 mapping f between $\{, \dots, n\}$ onto E. E is said to be **countable** if and only if there is a 1-1 mapping from $\mathbb{N}$ onto E. E is said to be **at most countable** if and only if E is either finite or countable. E is said to be **uncountable** if and only if E is neither finite, nor countable.

To show a set to be countable, we need a 1-1 mapping from $\mathbb{N}$ onto E. For example, let $2\mathbb{N} = \{2, 4, 6, 8, \dots\}$, and defining $f(n) = 2n$, then $f : \mathbb{N} \to 2\mathbb{N}$ is 1-1 and onto. Therefore the set $2\mathbb{N}$ is countable. We know that if a set B is finite, then any proper subset A of B has strictly less numbeer of elements.

Definition. A set B is an **infinite** set if and only if there is a proper subset A so that A and B have the same number of elements.

Now all countable sets are infinite, but not all infinite sets are countable.

Theorem 1.4.2 (Cantor's Diagonalization Argument)**.** *The open interval* $(0, 1)$ *is uncountable.*

Proof. Suppose that the interval $(0, 1)$ is countable; then by definintion, there exists a mapping $f : \mathbb{N} \to (0, 1)$ that is 1-1 and onto. Then this mapping exhausts all the elements of $(0, 1)$. We wish to construct an element $xin(0, 1)$ such that $x \neq f(n)$ for all $n \in \mathbb{N}$. For each $f(n)$ we have:

$$f(n) = 0.\alpha_{n_1}\alpha_{n_2}\dots$$

This expansion may not be unique, consider $0.1 = 0.0999\dots$. We require that the decimal expansion does not terminate in 9's. Then $f(n)$ and $0.\alpha_{n_1}\alpha_{n_2}\dots$ are 1-1:

$$f(1) = 0.\alpha_{11}\alpha_{12}\alpha_{13}\alpha_{14}\dots \tag{1.13}$$

$$f(2) = 0.\alpha_{21}\alpha_{22}\alpha_{23}\alpha_{24}\dots \tag{1.14}$$

$$f(3) = 0.\alpha_{31}\alpha_{32}\alpha_{33}\alpha_{34}\dots \tag{1.15}$$

$$f(4) = 0.\alpha_{41}\alpha_{42}\alpha_{43}\alpha_{44}\dots \tag{1.16}$$

$$\vdots \tag{1.17}$$

So we get an infinite matrix; an all α_{ij} are digits in $\{0, 1, \dots, 9\}$, and none of them terminate in 9's. Now let $x = 0.\beta_1\beta_1\beta_1\dots$ with:

$$\beta_k = \begin{cases} \alpha_{kk} + 1 \text{ if } \alpha_{kk} \leq 5 \\ \alpha_{kk} - 1 \text{ if } \alpha_{kk} > 5 \end{cases}$$

By this construction, we have that $\beta_1 \neq \alpha_{11}$, $\beta_2 \neq \alpha_{22}$, $dots$, $\beta_n \neq \alpha_{nn}$; so $x \neq f(n)$ for all $n \in \mathbb{N}$ and where $x \in (0, 1)$. This is a contradiction of the fact that we assumed f to be onto. Therefore $(0, 1)$ is uncountable. ■

Now we want to study the countability of the sets $\mathbb{Z}$, $\mathbb{Q}$, and $\mathbb{R}$. We sstudy some general properties first:

Lemma 1.4.3. *A nonempty set E is at most countable if and only if there is a mapping $g : \mathbb{N} \to E$ that is onto.*

Proof. Assume that E is at most countable, then by definition, we are done. If E is finite, by definition there is an $n \in \mathbb{N}$ and a 1-1 mapping f taking $\{1, \dots, n\}$ onto E. Now define:

$$g(j) = \begin{cases} f(j) \text{ if } j \leq n \\ f(1) \text{ if } j > n \end{cases}$$

Then g takes $\mathbb{N}$ onto E.

Conversely, suppose that g takes $\mathbb{N}$ onto E, we need to construct a 1-1 f from $\mathbb{N}$ or a subset of $\mathbb{N}$ onto E. Let $k_1 = 1$ and $E_! = \{k \in \mathbb{N} : g(k) \neq g(k_1)\}$. Now if $E = \emptyset$, we are done. Else, by the well ordering principle, there is a least element k_2 and define $E_2 = \{k \in \mathbb{N} : g(k) \neq g(k_1),\text{ and } g(k) \neq g(k_2)\}$. Now we have that $k_2 > k_1$ and $k_2 \geq 2$. If $E_2 = \emptyset$ then $E = \{g(k_1), g(k_2)\}$ is finite and we are done. Now if $E_2 \neq \emptyset$, then we continue along eoth our methpd until we exhaust all possible set until we reach $E_n = \emptyset$ for some n.

If this process ever terminates, then $E = \{g(k_1), \dots, g(k_n)\}$ and is finite; now if it never terminates, then we have a sequence $k_1 < k_2 < k_2, \dots$, and so k_{j+1} is the least element of E_j and $k_j \geq j$. We define $f(j) = g(k_j)$ and we show that f is 1-1. For $j < l$ we have $k_j < k_l$, so $k_j \leq k_{l-1}$; by construction, $g(k_l) \in E_{l-1}$, so $g(k_l) \neq g(k_j)$, f is 1-1. Now for any $x \in E$ there is an $n \in \mathbb{N}$ such that $x = g(n)$, and there is an l such that $x = g(n) = g(k_l)$; and so $E = \bigcup_{l=1}^{\infty}$, and so f is onto. Therefore, E is countable. ■

Theorem 1.4.4. *Suppose that A and B are sets and that $A \subseteq B$ and B is at most countable. Then A is at most countable. Likewise if $A \subseteq B$ and A is uncountable, then B is uncountable; in particular, $\mathbb{R}$ is uncountable.*

Proof. If B is at most countable, then there is a function $g : \mathbb{N} rightarrow B$. If A is empty, we are done. If A is not empty, then choose an element $a_0 \in A$ and define $f : \mathbb{N} \to A$ such that $f(n) = g(n)$ if $h(n) \in A$, and $f(n) = a_0$ if $g(n) \notin A$. Therefore, f is onto, and A is atmost countable.

Assume that B is atmost countable, then by the above, since $A \subseteq B$, A is also atmost countable. This is a contradiction by assumption, thus, B must also be uncoutable.

Now, since the set $(0, 1)$ is uncountable, and $(0, 1) \subseteq \mathbb{R}$, then by the above $\mathbb{R}$ is uncountable. ■

Theorem 1.4.5. *Let $A_1, A_2, \ldots$ be atmost countable sets, then:*

(1)

$$A_1 A_2$$

is atmost countable.

(2) If $E = \bigcup_{j\in\mathbb{N}} A_j$, then E is atmost countable.

Proof. A_1 and A_2 are atmost countable, then there exist onto functions $\phi_1 : \mathbb{N} \to A_1$, and $\phi : \mathbb{N} \to A_2$, (more over we notice that since A_n is also atmost countable, then there exists an onto function $\phi_n : \mathbb{N} \to A_n$).

Now define $f : \mathbb{N} \times \mathbb{N} A_1 \times A_2$ by $f(n, m) = (\phi_1(m), \phi_2(n))$, clearly f is onto, now if we can define another onto function $g : \mathbb{N} \to \mathbb{N} \times \mathbb{N}$, then the compostion $f \circ g : \mathbb{N} \to A_1 \times A_2$ is also onto; now $g(1) = (1, 1)$, $g(2) = (2, 1)$, $g(3) = (1, 2)$, $g(4 = (3, 1)$, $\ldots$, by observing the behaviour f g as g moves through $\mathbb{N}$, we can deduce a formula. Assume that $g(j)$ lies on the n-th line, so $g(1) = (1, n + 1)$, $g(2) = (2, n - 1)$, $g(3) = (3, n - 2)$, and so on. Thus we deduce that for $l \in \mathbb{N}$ that $g(j) = (l, n + 1 - 1)$. Now what is the relation between j and n? We have $j > 1 + 2 + 3 + \cdots + (n - 1) = \frac{n(n-1)}{2}$. Now $j \geq 1$, so $j + 1 \geq 2$, hence $\frac{j+1}{2} \geq 1$, to $\frac{f(f+1)}{2} \geq j$. Now let $E = \{k \imath \mathbb{N} : j \leq \frac{k(k+1)}{2}\}$ which is nonempty, hence E has a least element n. so $j \leq \frac{n(n+1}{2}$ Thus $l = j - \frac{n(n-1)}{2}$, therefore $j = l + \frac{n(n+1)}{2}$.

Now we have for arbitrary $j \in \mathbb{N}$ that A_j is countable, thus there is an onto function $\phi_j : \mathbb{N} \to A_j$. Define $f : \mathbb{N} \times n \to \bigcup_{j\in\mathbb{N}} A_j$ by $f(m, n) = \phi_n(m)$, then f is also onto. Therefor $\bigcup_{j\in\mathbb{N}} A_j$ is atmost countable. ■

Remark. We have $\mathbb{Z} = \mathbb{N} \cup -\mathbb{N}$ is countable, and $\mathbb{Q} = \bigcup_{n\in\mathbb{N}}\{\frac{p}{n} : n \in \mathbb{Z}\}$ is also countable. Now $\mathbb{R}$ is uncountable, if $\mathbb{Q}^*$ is also countable, then we get $\mathbb{Q}^* \cup \mathbb{Q} = \mathbb{R}$ is countable, contradicting the uncountability of $\mathbb{R}$, hence $\mathbb{Q}^*$ must also be uncountable.

Definition. A collection of sets $\mathcal{E}$ is said to be indexed by a set A if and only if there is a function F from A onto $\mathcal{E}$. We call A the **index set** of $\mathcal{E}$. We may write $\mathcal{E} = \{E_\alpha\}_{\alpha\in A}$.

Definition. Let $\{E_\alpha\}_{\alpha\in A} = \mathcal{E}$ be a collection of sets. Then:

(1) $\bigcup E_{\alpha\in A} = \{x : x \in E_\alpha \text{ for some } \alpha \in A\}$

(2) $\bigcap E_{\alpha\in A} = \{x : x \in E_\alpha \text{ for all } \alpha \in A\}$

Theorem 1.4.6 (DeMorgan's Law)**.** *Let $\mathcal{X}$ be a set and let $\{E_\alpha\}_{\alpha\in A}$ be a collection of subsets of $\mathcal{X}$. For each $E \subseteq \mathcal{X}$*

then:

$$(\bigcap E_\alpha)^C = \bigcup E_\alpha^C$$

where $E^C = \mathcal{X} \backslash E$.

Definition. Let X and Y be sets and let $f : X \to Y$ be function. Then the **image** of X under f is the set $f(X) = \{f(x) : x \in X\}$. The **inverse image** is the set $f^{-1}(E) = \{x \in X : y = f(x) \text{ for some } y \in E\}$.

For the inverse image, we do not require that the mapping f be $1-1$ as we require it for the existence of f^{-1}. Now let X, Y be sets and let $fX \to Y$. If $\{E_\alpha\}_{\alpha \in A}$ for some index set A then:

$$f(\bigcup_{\alpha \in A} E_\alpha) = \bigcup_\alpha f(E_\alpha) \tag{1.18}$$

and

$$f(\bigcap_{\alpha \in A} E_\alpha) = \bigcap_\alpha f(E_\alpha) \tag{1.19}$$

Now if B and C are subsets of X, then:

$$f(B) \backslash f(C) \subseteq f(B \backslash C) \tag{1.20}$$

and if $B, C \subseteq Y$ then:

$$f^{-1}(B \backslash C) = f{-}1(B) \backslash f^{-1}(C) \tag{1.21}$$

and if $\{E_\alpha\}_{\alpha \in A}$ then:

$$f^{-1}(\bigcup_{\alpha \in A} E_\alpha) = \bigcup_\alpha f^{-1}(E_\alpha) \tag{1.22}$$

and

$$f^{-1}(\bigcap_{\alpha \in A} E_\alpha) = \bigcap_\alpha f^{-1}(E_\alpha) \tag{1.23}$$

Finally, if $E \subseteq f(X)$, then $f(f^{-1}(E)) = E$, but if $E \subseteq X$, then $E \subseteq f^{-1}(f(E))$. We accept these properties witout proof as they can be demonstrated trough elementary se theory.

Now to illustrate an example of these properties, t $f : \mathbb{R} \to \mathbb{R}$ be defined by $x \to x^2$, let $E_1 = \{1\}$ and $E = \{-1\}$. Then $E_1 \cap E_2 = \emptyset$, then $f(E_1 \cap E_2) =, emptyset$; but $f(E_1) = \{1\}$ and $f(E_2) = \{1\}$, hence $f(E_1) \cap f(E_2) = \{1\}$. It is intersting to note that if f is $1-1$, then all the relations estabku=ished above are equalities.

Homework. Exercises 6, 9, 10, and 11 on page 33 of the book.

Chapter 2

Sequences in $\mathbb{R}$.

2.1 Limits of Sequences.

Definition. A **sequence** is a mapping $f : \mathbb{N} \to X$ for some nonempty set X, whose terms are $x_n = f(n)$. We denote a sequence as $\{x_n\}_n \in \mathbb{N}$ or just simply $\{x_n\}$. We say that a sequence is **real valued** if $X = \mathbb{R}$.

Definition. A sequence of real numbers $\{x_n\}$ is said to **converge** to a real number $a \in \mathbb{R}$ if and only if for every $\epsilon > 0$, there is an $N \in \mathbb{N}$ (in general depending on ϵ) such that for all $n \geq N$, $|x_n - a| < \epsilon$. We say that $\{x_n\}$ (or x_n) **converges** to a, and we write:

$$\lim_{n \to \infty} x_n = a \tag{2.1}$$

or simply, $x_n \to a$ as $n \to \infty$. We call a the **limit** of the sequence. A sequence which does not cpnverge is said to **diverge**.

We can consider x_n as a sequence of "approximations" to a and ϵ as an upperbound for the "error" of those approximations. $N \to \infty$ as $\epsilon \to 0$, thay is N gets larger as ϵ gets smaller.

Prove that $\lim_{n \to \infty} \frac{1}{n} = 0$

Proof. Let $\epsilon > 0$, by the Archimedean principle, there is an $N \in \mathbb{N}$ such that $\frac{1}{\epsilon} < N$. Then $\frac{1}{N} < \epsilon$. For all $n \geq N$, $\frac{1}{n} < \frac{1}{N} < \epsilon$, hence, $|\frac{1}{n} - 0| < \epsilon$. ∎

The sequence $\{(-1)^n\}_{n \in \mathbb{N}}$ does not converge

Proof. Assume that $(-1)^n \to a$ as $n \to \infty$. By definition, we have that for any $\epsilon > 0$, there is an $N \in \mathbb{N}$ such that $|(-1)^n - a| < \epsilon$ for $n \geq N$. Choose $\epsilon = \frac{}{2}$. then for n od we have $(-1)^n = -1$, and for n even, $(-1)^n = 1$. Then $|(-1)^n - (-1)^{n+1}| = |((-1^n) - a) - ((-1)^{n+1}) - a|$; by the triangle inequality, we have $|(-1)^n - (-1)^{n+1}| \leq |(-1)^n - a| + |(-1)^{n+1} - a| < \frac{1}{2} < 1$, when $n \geq N$. But $|(-1)^n - (-1)^{n+1}| = 2$, a contradiction. Thus $\{(-1)^n\}$ does not converge. ∎

Remark. A sequence can have at most one limit.

Proof. Suppose it has atleast two limits, that is $x_n \to a$ and $x_n \to b$ (with $a \neq b$) as $n \to \infty$. For $\epsilon > 0$ there is an N_1 such that for all $n \geq N_1$ $|x_n - a| \leq \epsilon$, and there is an N_2 such that $|x_n - b| \leq \epsilon$. Choose $N = \max(N_1, N_2)$, then for $n \geq N$ we have:

$$|x_n - a| < \epsilon \qquad \text{and} \qquad |x_n - b| < \epsilon$$

Then $|a - b| = |(a - x_n) - (b - x_n)| \leq |a - x_n| + |b - x_n| < 2\epsilon$ for every ϵ. Now choose $\epsilon = |a - b|/4 > 0$, then $2\epsilon = |a - b|/2$ hence $|a - b| < |a - b|/2$, which is a contradiction. Then x_n converges to at most one limit. ∎

Definition. A **subsequence** of a sequence $\{x_n\}_{n\in\mathbb{N}}$ is a sequence of the form $\{x_{n_k}\}_{k\in\mathbb{N}}$ where $n_k \in \mathbb{N}$ and $n_1 < n_2 < \ldots$.

Recall that the sequence $\{-1)^n\}$ does not converge, however, suppose we only select even terms and form $\{(-1)^{2n}\}$, then this latter sequence converges. Like wise the sequence $\{\frac{1}{n}\} \to 0$ rather slowly, but forming the subsequence $\{\frac{1}{2^n}\}$, the latter converges rather quickly. So the immediate use of subsequences is in the correction of sequences that behave badly (i.e they don't converge), or to make them converge quikly. Now if $n_k = k$, then the subsequence is the original sequence, if $n_k > k$, then the subsequence is a **proper** subsequence.

Remark. If $\{x_n\}$ converges to a and $\{x_{n_k}\}$ is any subsequence of $\{x_n\}$, then $\{x_{n_k}\}$ also converges to a.

Proof. For any $\epsilon > 0$, there is an $N > 0$ such that for all $n \geq N$, $|x_n - a| < \epsilon$. Now if $n_k = k$, we are done, so suppose that $n_k > k$, so for $k = N$, $n_k > N$ then $|x_{n_k} - a| < \epsilon$. ∎

Definition. Let $\{x_n\}$ be a sequence of real numbers. Then we say that $\{x_n\}$ is **bounded above** if there is an $M \in \mathbb{R}$ such that $x_n \leq M$ for all $n \in \mathbb{N}$. Similarly, $\{x_n\}$ is **bounded below** if there is an $m \in \mathbb{R}$ such that $m \leq x_n$ for all $n \in \mathbb{N}$. We say $\{x_n\}$ is **bounded** if it is both bounded above and bounded below

Now $\{x_n\}$ is bounded if and only if there is some $C \in \mathbb{R}$ with $C > 0$ such that $|x_n| \leq C$ for all $n \in \mathbb{N}$.

Theorem 2.1.1. *Every convergent sequence is bounded.*

Proof. Let $\{x_n\}$ converge to a. Then for every $\epsilon > 0$, there is an $N > 0$ such that for every $n \geq N$, $|x_n - a| < \epsilon$. Choosing $\epsilon = 1$, we have $|x_n - a| < 1$, so $|x_n| = |x_n - a + a| \leq |x_n - a| + |a| < 1 + |a|$. Now let $C = |x_1| + |x_2| + \cdots + |x_{N-1}| + |a| + 1$. Then we see that $|x_n| \leq C$ for all $n \in \mathbb{N}$. Thus $\{x_n\}$ is bounded. ∎

The converse of this theorem is not true; take $\{(-1)^n\}$ which is bounded by 1, but $\{(-1)^n\}$ diverges.

Homework. Exercises 1, 4, 5, and 6 on page 38.

2.2 Limit Theorems

There are two problems that are worth discussing, the first is that if we have a sequence, how do we know if it converges? The second problem is determining to where it converges, that is to say, what is the limit; if we cannot find the concrete value, how can we approximate it?

Theorem 2.2.1 (Sandwhich Theorem). *Consider real valued sequences $\{x_n\}$, $\{y_n\}$, and $\{w_n\}$. Suppose that $\lim x_n = \lim y_n = a$ and that there is an $N_0 \in \mathbb{N}$ sucht hat $x_n \leq w_n \leq y_n$ for all $n \geq N_0$. Then $\lim_{n\to\infty} w_n = a$.*

Proof. Let $\epsilon > 0$ and let $\{x_n\}$ and $\{y_n\}$ both converge to a. Then by definition there are $N_1, N_2 \in \mathbb{N}$ such that $|x_n - a| < \epsilon$ and $|y_n - a| < \epsilon$ for $n \geq N_1, n_2$. Now choose $N = \max N_0, N_1, N_2$, if $n \geq N$, we have $-\epsilon < x_n - a < \epsilon$, and we also have $x_n - a < w_n - a < y_n - a$, thus we have that:

$$-\epsilon < x_n - a < w_n - s < y_n - a < \epsilon$$

Thus we have that $|w_n - a| < \epsilon$. ∎

Corollary. *If $x_n \to \infty$ as $n \to \infty$, and $\{y_n\}$ is bounded, then $x_n y_n \to 0$ as $n \to \infty$.*

Proof. We have that $\{y_n\}$ is bounded, hence, there is $M > 0$ such that $|y_n| < M$ for all $n \in \mathbb{N}$. And since $\{x_n\}$ converges to 0 we have that for any ϵ there is an $N \in \mathbb{N}$ such that for $n \geq N$, $|x_n - 0| < \frac{\epsilon}{M}$. For $|x_n y_n - 0| = |x_n y_n| < M|x_n| < M\frac{\epsilon}{M} = \epsilon$. Therefore, $x_n y_n \to 0$ as $n \to \infty$. ∎

Example 2.1. Find $\lim_{n\to\infty} 2^{-n}\cos(n^3 - n^2 + n - 13)$.

solution. We have that cos is bounded by 1 for any value it takes, and we also know that $2^n > n$ hence $0 < 2^{-n} < \frac{1}{n}$. We know that $\frac{1}{n} \to 0$ as $n \to \infty$, thus by the sandwhich theorem, $2^{-n} \to 0$. Then by the corollary, since cos is bounded, we get that $\lim_{n\to\infty} 2^{-n}\cos(n^3 - n^2 + n - 13) = 0$.

Theorem 2.2.2. *Let $E \subseteq \mathbb{R}$. If E has a finite least upper bound (respectively a finite greatest lower bpund), then there is a sequence $\{x_n\} \subseteq E$ such that $x_n \to \sup E$ as $n \to \infty$*

Proof. If the least upper bound of E is finite, then for every $\epsilon > 0$, there is an $x \in E$ such that $\sup E - \epsilon < x \leq \sup E$. For all $n \geq N$, choose $\epsilon = \frac{1}{n}$. Then there is one element $x_n \in E$ such that $\sup E - \frac{1}{n} < x_n \leq \sup E$. Now construct sequences $\{w_n\}$, and $\{y_n\}$ such that $w_n = \sup E - \frac{1}{n}$ and $y_n = \sup E$. Then $\lim w_n = \lim y_n = \sup E$ and $w_n < x_n \leq y_n$; thus by the sandwhich theorem, $\lim x_n = \sup E$. Analogously, we get the same result for $\inf E$. ∎

Theorem 2.2.3. *Suppose that $\{x_n\}$ and $\{y_n\}$ are real valued sequence, and let $\alpha \in \mathbb{R}$. If $\{x_n\}$ and $\{y_n\}$ converge,and suppose that $n \to \infty$. Then:*

(1) $\lim x_n + y_n = \lim x_n + \lim y_n$.

(2) $\lim \alpha x_n = \alpha \lim x_n$.

(3) $\lim x_n y_n = \lim x_n \lim y_n$.

(4) $\lim \frac{x_n}{y_n} = \frac{\lim x_n}{\lim y_n}$, *provided that* $y_n \neq 0$ *does not converge to* 0.

Proof. Suppose that $\{x_n\} \to x$ and $\{y_n\} \to y$ as $n \to \infty$, for $x, y \in \mathbb{R}$. Then:

(1) For any $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that for $n \geq N$ we have $|x_n - x| < \frac{\epsilon}{2}$ and $|y_n - n| < \frac{\epsilon}{2}$. Then $|(x_n - x) + (y_n - y)| = |(x_n + y_n) + (x - y)| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$.

(2) For $\epsilon > 0$, let $N \in \mathbb{N}$ such that for $n \geq N$, $|x_n - x| < \frac{\epsilon}{|\alpha|}$. Then multiplying the inequality by $|\alpha|$, we get $|\alpha||x_n - x| = |\alpha x_n - \alpha x| < \epsilon$.

(3) Notice that $y_n - y \to 0$, and x_n is bounded (by theorem 2.1.1), so $x_n(y_n - y) \to 0$. Similarly, $y_n(x_n - x) \to 0$. Also notice that $|x_n y_n - xy| = |x_n(y_n - y) + y_n(x_n - x)| \leq |x_n(y_n - y)| + |y_n(x_n - x)|$.

Now for $\epsilon > 0$, there is some $N \in \mathbb{N}$ such that for $n \geq N$, $|x_n - y| < \frac{\epsilon}{2}$ and $|y_n - y| < \frac{\epsilon}{2}$. Then $|x_n(y_n - y)| + |y_n(x_n - x)| < \epsilon$. Therefore, $|x_n y_n - xy| < \epsilon$.

(4) Provided that $y_n \neq 0$ does not converge to 0, notice that:

$$|\frac{x_n}{y_n} - \frac{x}{y}| = |\frac{x_n y - x y_n}{y y_n}| = \frac{1}{|y y_n|}|x_n(y - y_n) + y_n(x_n - y)|$$

By the triangle inequality we have:

$$\frac{1}{|y y_n|}|(x_n(y - y_n)| + |y_n(x_n - x)|) = |\frac{x_n}{y y_n}(y - y_n)| + |\frac{1}{y}(x_n - x)| \tag{2.2}$$

Now we see that $\frac{1}{y_n}(x_n - x) \to 0$ we also have that $y - y_n \to 0$, and x_n converges, so x_n is bounded. Now since $y_n, y \neq 0$, we must show that $\frac{1}{y_n}$ is bounded. We have that $\lim |y_n| = |y|$, then $|y| > 0$, for $\epsilon = \frac{1}{2}y$, there is an $N \in \mathbb{N}$ such that for all $n \geq N$

$$||y_n| - |y|| < \frac{1}{2}y = \epsilon$$

then we get that $\frac{1}{|y_n|} < \frac{2}{|y|}$; hence $\frac{1}{y_n}$. For $k = 1, \ldots, N-1$ let $M = \frac{2}{|y|} + \max \frac{1}{|y_1|}, \ldots, \frac{1}{y_{N-1}}$, thus $\frac{1}{|y_n|}$ is bounded by M.

Now since $\frac{1}{|y_n|}$ is bounded, so is $\frac{x_n}{|y y_n|}$, thus we get that $\frac{x_n}{y y_n}(y - y_n) \to 0$, thus we get that

$$\frac{x_n y - x y_n}{y y_n} = \frac{x_n}{y_n} - \frac{x}{y} \to 0$$

This completes the proof.

■

Example 2.2. Find the limit: $\lim_{n \to \infty} \frac{n^3 + n^2 - 1}{1 - 3n^3}$.

solution. We divide the numerator and the denominator by the highest power of n to get:

$$\frac{1+\frac{1}{n}-\frac{1}{n^3}}{\frac{1}{n^3}-3} \tag{2.3}$$

Then we get that as $n \to \infty$ then

$$\frac{1+\frac{1}{n}-\frac{1}{n^3}}{\frac{1}{n^3}-3} \to \frac{1+0-0}{0-3} = -\frac{1}{3} \tag{2.4}$$

Hence, $\lim_{n\to\infty} \frac{n^3+n^2-1}{1-3n^3} = -\frac{1}{3}$

A sequence $\{n\}$ diverges, so does the sequence $\{(-1)^n\}$, however, they diverge differently; the sequence $\{n\}$ just gets bigger and bigger as $n \to \infty$, however, $\{(-1)^n\}$ oscilates as $n \to \infty$.

Definition. Let $\{x_n\}$ be a real valued sequence. Then

(1) $\{x_n\}$ is said to **diverge** to $+\infty$ if and only if for every $M \in \mathbb{R}$, there is an $N \in \mathbb{N}$ such that for $n \geq N$, $x_n > M$.

(2) $\{x_n\}$ is said to **diverge** to $-\infty$ if and only if for each $M \in \mathbb{R}$, there is an $N \in \mathbb{N}$ such that for $n \geq N$, $x_n < M$.

Theorem 2.2.4. *Suppose that $\{x_n\}$ and $\{y_n\}$ are real valued sequence such that $x_n \to +\infty$ as $n \to \infty$.*

(1) If $\{y_n\}$ is bounded below, then $x_n + y_n \to +\infty$ as $n \to \infty$. Similarly if $\{y_n\}$ is bounded above, then $x_n + y_n \to -\infty$.

(2) If $\alpha > 0$, then $\alpha x_n \to +\infty$ as $n \to \infty$. Similarly, if $\alpha < 0$, then $\alpha x_n \to -\infty$.

(3) If $y_n > M_0$ for some $M_0 > 0$ abd for all $n \in \mathbb{N}$. Then $x_{y\to} + \infty$ as $n \to \infty$.

(4) If $\{y_n\}$ is bounded, and $x_n \neq 0$, then $\frac{y_n}{x_n} \to 0$ as $n \to \infty$.

We defer the proof.

Corollary. *Let $\{x_n\}$ and $\{y_n\}$ be real valued sequences and α, x, y be extended real numbers. If $x_n \to x$ and $y_n \to y$ as $n \to \infty$, then $x_n + y_n \to x + y$ as $n \to \infty$. and $\alpha x_n \to \alpha x$ and $x_n y_n \to xy$; We disclude of course the indeterminate forms for the extended real numbers.*

Theorem 2.2.5 (The Comparison Theorem)**.** *Suppose that $\{x_n\}$ and $\{y_n\}$ are convergent sequences, if there is an $N_0 \in \mathbb{N}$ such that $x_n \leq y_n$ for $n \geq N_0$, then $\lim x_n \leq \lim y_n$ as $n \to \infty$. In partucular, if $x_n \in [a, b]$ converges to c, then $c \in [a, b]$.*

Proof. Suppose that the statement is false, then $\lim x_n > \lim y_n$ as $n \to \infty$. Let $\lim x_n = x$ and let $\lim y_n = y$, and let $\epsilon = \frac{x-y}{4}$, then there is an $N \in N$ such that for $n \geq N$, $|x_n - x| < \epsilon$, and $|y_n - y| < \epsilon$ Now we have thay $x_m > x - \epsilon > y+ > y_n$, a contradiction. Therefore $\lim x_n \leq \lim y_n$.

Now in partucular, we have that if $x_n \in [a, b]$, then $a \leq x_n \leq b$, letting $\lim_{n\to\infty} x_n = c$, we get from above that $a \leq c \leq b$, hence $v \in [a, b]$. ■

Homework. Exercises 3, 5, and 6 on page 44.

2.3 The Bolzano-Weierstrass Theorem.

The squence $\{(-1)^n\} = (-1, 1, -1, 1, \dots)$, has the properties that it is a bounded divergent sequence that contains convergent subsequences. This turns out to be a special case of the "Bolzano-Weierstrass theorem". The rest of this section is devoted to stating and proving the theorem.

Definition. Let $\{x_n\}$ be a real valued sequence. Then:

(1) $\{x_n\}$ is **increasing** if and only if $x_i \leq x_{i+1}$; and $\{x_n\}$ is **strictly increasing** if $x_i < x_{i+1}$ for all $i \in \mathbb{N}$

(2) $\{x_n\}$ is **decreasing** if and only if $x_{i+1} \leq x_i$; and $\{x_n\}$ is **strictly decreasing** if and only if $x_{i+1} < x_i$ for all $ui \in \mathbb{N}$.

(3) $\{x_n\}$ is **monotone** if and only if it is either increasing or decreasing.

If $\{x_n\}$ is increasing, and converges to a, we may write $x_n \uparrow a$, if $\{x_n\}$ is decreasing and convergent to a, we may write $x_n \downarrow a$.

Claim. (1) If a sequence is strictly increasing, then it is increasing.

(2) If a sequence is strictly decreasing, then it is decreasing.

(3) If $\{x_n\}$ is a real valued sequence, then $\{x_n\}$ is increasing if and only if $\{-x_n\}$ is decreasing. Similarly $\{x_n\}$ is decreasing if and only if $\{-x_n\}$ is increasing.

We know that all convergent sequences are bounded, however, the converse is not true. However, what if we only consider monotone sequences?

Theorem 2.3.1 (The Monotone Convergence Theorem)**.** *If $\{x_n\}$ is increasing and bounded above, or decreasing and bounded below, then $\{x_n\}$ converges to a finite limit.*

Proof. We show that an increasing sequence bounded above will converge to its least upperbound. Suppose $\{x_n\}$ is increasing and bounded above. By the axiom of completeness, $\{x_n\}$ has a finite least upperbound a. Now for any $\epsilon > 0$ there is an $N \in \mathbb{N}$ such that $a - \epsilon < x_N \leq a$. Then for any $n \geq N$, $\{x_n\}$ is increasing, hence $x_n \geq x_N$, so $a - \epsilon < x_N \leq x_n \leq a$. Hene $|x_n - a| < \epsilon$. Therefore, $x_n \to a$. By a similar agrument, we get that if $\{x_n\}$ is and bounded below, then $x_n \inf \{x_n\}$.

We can also by direct application of the first part, note that by the axiom of completeness, $\{-x_n\} \to -b$, and by the limit laws, we get that $x_n \to b$, hence we get that $\{-x_n\}$ is increasing and bounded above, hence $-b = \sup \{-x_n\}$, therefore $b = \inf \{x_n\}$. ■

Example 2.3. If $|a| < 1$, then $a^n \to 0$ as $n \to \infty$.

solution. If $\{x_n\}$ is a sequence with the limit $\lim |x_n| = 0$, then $\lim x_n = 0$. Then it suffices to show that $\lim |a|^n = 0$. We have that $|a|^n < |a|^{n-1}$, so $\{|a^n|\}$ is decreasing, and we see that 0 is a lowerbound, hence by theorem 2.3.1, $\lim |a^n| = L$, we must show that $L = 0$. Suppose it is not 0, then $|a|^{n+1} = |a||a|^n$, so we consider the sequences $\{|a|^{n+1}\} = |a|\{|a|^n\}$, then $\{|a|^{n+1}\}$ is a subsequence of $\{|a|^n\}$. Hence we get $\lim |a||a|^n = |a| \lim |a|^n = |a|L$, hence we get that $L = |a|L$, implying that $|a| = 1$. A contradiction, so $L = 0$.

Example 2.4. If $a > 0$, then $\lim a^{\frac{1}{n}} = 1$

solution. We consider 3 cases. Suppose first that $a = 1$, then $a^{\frac{1}{n}} = 1$ for all n, and hence we have the limit of a constant sequence. Now assume that $a > 1$. Then it suffices to show that $\{a^{\frac{1}{n}}\}$ is decreasing. For any $n \in \mathbb{N}$, $a^{n+1} > a^n > 1$, taking the nth root, we have $(a^n)^{\frac{1}{n}} > a$, hence we have $(a^{\frac{1}{n}})^n > a$, so we get that $a^{\frac{1}{n}} > a^{\frac{1}{n+1}}$ which shows that $\{a^{\frac{1}{n}}\}$ is decreasing. Therefore $a^{\frac{1}{n}} \geq 1$. Therefore, by theorem 2.3.1, the limit exists aand $\lim a^{\frac{1}{n}} = L$.

Consider then $a^{\frac{1}{2n}}$ and produce the appropriate subsequence. Notice then that $a^{\frac{1}{n}} = (a^{\frac{1}{n}})^{\frac{1}{2}}$ and the limit is L, We also see that the limit of $(a^{\frac{1}{n}})\frac{1}{2} = L^{\frac{1}{2}}$. So we have that $L = L^{\frac{1}{2}}$. Therefore $L^2 - L = 0$, and so $L = 0$ or $L = 1$. However, since $a^{\frac{1}{n}} \geq 1$, $L \neq 0$, so $L = 1$.

Now suppose that $0 < a < 1$, then we know that $\frac{1}{a} > 1$. Thus by the previous case, we get that $\lim \frac{1}{a}^{\frac{1}{n}} = 1$. Using some algebra, we get that

$$\lim a^{\frac{1}{n}} = \frac{1}{\frac{1}{a^{\frac{1}{n}}}} = \frac{1}{1} = 1$$

Definition. A sequence of sets $\{I_n\}_{n\in\mathbb{N}}$ is said to be **nested** if and only if $I_{i+1} \subseteq I_i$ for all $i \in \mathbb{N}$.

Theorem 2.3.2. *If $\{I_n\}$ is a nested sequence of nonempty, closed, and bounded intervals then:*

$$E = \bigcap_{n\in\mathbb{N}} I_n$$

contains at least one point. Moreover, if the length of these intervals satisfiy $|I_n| \to 0$ as $n \to \infty$, then E contains exactly one point.

It is interesting to oserve some things before we being the proof. First note that if $\{I_n\}$ is a sequence of nested sets, and $E = \bigcap I_n$ for all n, then $E = [a, b]$ for some $a, b \in \mathbb{R}$. Moreover, we see that $E \subseteq \dots I_n \subseteq \cdots \subseteq I_2 \subseteq I_1$, hence E is a nonempty, closed and bounded interval, and $E \in \{I_n\}$. What this illustrates, is that we can make E as arbitrarily small an interval we need it to be, and it will always contain atleast one element. Moreover, if we construct $\{I_n\}$ right, then we can make E have exactly one element.

Proof. Let $I_n = [a_n, b_n]$, and that $\{I_n\}$ is nested. Then $a_n \leq a_{n+1}$ and $b_{n+1} \leq b_n$. Thus $\{a_n\}$ is increasing, and $\{b_n\}$ is decreasing, with an upperbound b_1 and a lowebound a_1, respectively. Then we have a and b with $a_n \to a$ and $b_n \to b$ as $n \to \infty$. Now since $a_n < b_n$ for all n, we get $a \leq b$. Therefore, $E = [a, b]$ contains at least one point. Moreover, if the length of $|I_n| \to 0$ as $n \to \infty$, then $b_n - a_n \to 0$, So $b - a = 0$. Therefore $b = a$ and we get that E contains exactly one point. ∎

Remark. The nested interval property does not hold if "closure" is omitted. The interval $I_n = (0, \frac{1}{n})$ is nested,, nonempty, and bounded, however $\bigcap I_n = \emptyset$.

Remark. The neseted interval property may not hold if the interval is not bounded. Consider the interval $I_n = [n, \infty)$, which is nested and closed, but not bounded. Then $\bigcap I_n = \emptyset$.

Theorem 2.3.3 (The Bolzano-Weierstrass Theorem)**.** *Every bounded sequence of real numbers has a convergent subsequence.*

Proof. We begin with a general boservation. Let $\{x_n\}$ be a sequence, and let E be a set. Then $E = A \cup B$ for sets A and B, and E contains x_n for infinitely many values of n; then at least one of the sets A or B contains $\{x_n\}$ for infinitely many values of n.

Now let $\{x_n\}$ be a bounded sequences, choose $a, b \in \mathbb{R}$ such that all $x_n \in [a, b]$. Set $I_0 = [a, b]$; now divide I_0 into two subintervals $[a, \frac{1+b}{2}]$ and $[\frac{1+b}{2}, b]$. Now I_0 contains x_n for infinitely many values of n, and $I_0 = [a, \frac{1+b}{2}] \cup [\frac{1+b}{2}, b]$. Then atleast one of the aformentioned intervals contains x_n for infinitely many values of n. Call that set I_1. and we know that $|I_1| = \frac{1}{2}|I_0| = \frac{b-a}{2}$. Now divide I_1 equally into twon sub intervals I_1' and I_1'', then as before, atleast one of these intervals contains x_n for infinitely many values of n. Call that interval I_2, and so $|I_2| = \frac{1}{2}|I_1| = \frac{b-1}{4}$. Repeating the process, we get a nested sequence of intervals $\{I_n\}$ where $|I_n| = \frac{b-a}{2^n}$ and I_n is nonempty, closed, and bounded; so the intersection $E = \bigcap I_n = \{x\}$ by theorem 2.3.2. Now we have that I_n is npnempty, so we will have $x_{n_k} \imath I_n$ because I_n contains x_n for infinitely many values of n. Choose $n_k < n_{k+1}$. So $\{x_{n_k}\}$ is a subsequence of $\{x_n\}$ with $x_{n_k} \imath I_n$.

Now observe that $x_{n_k}, x \in I_n$, so $0 \leq |x_{n_k} - x| \leq |I_k| \leq \frac{b-a}{2^k}$ for $n_k \geq k$. Hence, by definition of the limit, when $k \to \infty$, $\lim x_{n_k} = x$, choosing $\epsilon > \frac{b-a}{2^k}$. ∎

Homework. Problems 1, 2, 4, 5, and 6 on page 48.

2.4 Cauchy Sequences.

Whenever we say a sequences converges, we would like to find the limit. However, the limit may lie outside of the sequence. It is here that building up the notion of a "Cauchy sequence" which establishes a property that is independent form the limit, but that guarantees that a sequence will be convergent to said limit (which of course we may or may not know).

Definition. A realvalued sequence $\{x_n\}$ is said to be a **Cauchy sequence** if and only if for every $\epsilon > 0$, there is an $N \in \mathbb{N}$ such that $n, m \geq N$ implies that $|x_n - x_m| < \epsilon$.

Remark. If $\{x_n\}$ is a convergent sequence, then $\{x_n\}$ is cauchy.

Proof. Suppose that $\{x_n\} \to x$ as $n \to \infty$. For any $\epsilon > 0$ there is an $N \in \mathbb{N}$ such that forall $n \geq N$, $|x_n - x| < \frac{\epsilon}{2}$. Now the, for any $m \geq N$ we get $|x_m - x| < \frac{\epsilon}{2}$. Then by the triangle inequality: $|x_n - x_m = |(x_n - x) - (x_m - x)| \leq |x_n - x| + |x_m - x| < \epsilon$. we see that $\{x_n\}$ is a Cauchy sequence. ∎

Theorem 2.4.1 (Cauchy's Theorem)**.** *Let $\{x_n\}$ be a realvalued sequence. Then $\{x_n\}$ is a Cauchy sequence if and only if $\{x_n\}$ converges to some point in $\mathbb{R}$.*

Proof. We need only to show that Cauchy sequences converge (the remark before hand proves the converse implication). Let $\{x_n\}$ be a Cauchy sequence, then $\{x_n\}$ is bounded. LEt $\epsilon = 1$, then there is some $K \in \mathbb{N}$ such that for all $m, n \geq K$, $|x_n - x_n| < 1$. We have that $|x_1|, |x_2|, \ldots |x_K|$, and $|x_{K+1} - x_K| < 1$. Now $|x_{K+1}| = |x_{K+1} - x_K + x_K| \leq |x_{K+1}| + |x_K| < 1 + |x_K|$. Hence choose $M = \max\{|x_1|, |x_2|, \ldots, |x_K|, 1 + |x_K|\}$. Then $|x_n| \leq M$ for all $n \in \mathbb{N}$. Hence every Cauchy sequence is bounded, hence by the Bolzano-Weierstrass theorem, $\{x_n\}$ has a convergent subsequence $\{x_{n_k}\}$. So there is an $x \in \mathbb{R}$, such that for all $\epsilon > 0$, there exists $K \in \mathbb{N}$ such that for all $k \geq K$, $|x_{n_k} - x| < \frac{\epsilon}{2}$.

$\{x_n\}$ is a Cauchy sequense, hece for this same $\epsilon > 0$, there is an $N_1 \in \mathbb{N}$ such that for $m, n \geq N_1$, $|x_n - x_m| < \frac{\epsilon}{2}$. Chossing $K \geq N_1$, we have that $k \geq K$, hence $n_k \geq k \geq K \geq N_1$. For all $n \geq K$, For all $n \geq K$, $|x_n - x| = |x_n - x_{n_k} + x_{n_k} - x| \leq |x_n - x_{n_k}| + |x_{n_k} - x| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$. Thus $x_n \to x$ as $n \to \infty$, hence all Cauchy sequences are convergent. ■

Example 2.5. Prove that any real valued sequence $\{x_n\}$ satisfying $|x_n - x_{n+1}| < \frac{1}{2^n}$ is convergent.

solution. To prove that it is convergent, first prove that it is a Cauchy seqence. For $n < m$, we have $|x_n - x_m| = |x_n - x_{n+1} + x_{n+1} - x_{n+2} + x_{n+2} + \cdots + x_{m-1} - x_m| \leq |x_n - x_{n+1}| + |x_{n+1} - x_{n+2}| + \cdots + |x_{m-1} - x_m| < \frac{1}{2^n} + \frac{1}{2^{n+1}} + \cdots + \frac{1}{2^{m-1}} < \frac{1}{2^{n-1}}$. Then for $\epsilon > 0$, there exists $N \in \mathbb{N}$ with $\frac{1}{2^{N-1}} < \epsilon$. Hence, for $m, n \geq N$ we have that $|x_n - x_m| < \frac{1}{2^{n-1}} < \frac{1}{2^{N-1}} < \epsilon$. Therefore, $\{x_n\}$ is a Cauchy sequence, and hence converges to a point in $\mathbb{R}$.

Remark. If $|x_{n+1} - x_n| \to 0$, then it is not sufficient to deduce that it is a Cauchy sequence. For example, consides $x_n = \log n$. Then $\{x_n\}$ is unbounded and divergent. However, $\log n + 1 - n = \frac{logn+1}{\log n} = \log 1 = 0$.

Homework. Exercises 1, 2, 3, and 6 on page 51.

2.5 Limits, Least Upperbounds and Greatest Lowerbounds.

We know that a sequence converges to a point if and only if it is a Cauchy sequence. We would like to talk about related concepts of sequences in general. Considet the sequence $x_1, x_2, x_3, \ldots$. And construct the following seqeunce: $S_1 = \sup\{x_1, x_2, x_3, \ldots\}$, $T_1 = \inf\{x_1, x_2, x_3, \ldots\}$. Cntinuing, constructs $S_2 = \sup\{x_2, x_3, x_4, \ldots\}$ and $T_2 = \inf\{x_2, x_3, x_4, \ldots\}$. Continuing allong this line, we get the terms $S_n = \sup\{x_n, x_{n+1}, \ldots\}$ and $T_n = \inf\{x_n, x_{n+1}, dots\}$. We see that $\{S_n\}$ is a decreasing sequence of extended real numbers, now it could turn out that $\{S_n\}$ is a complete sequence of ∞; if at a certain point, the term $s_n \neq \infty$, then the rest of the sequence is not ∞.

Similarly, $\{T_n\}$ is an increasing sequence of extended real numbers. It could happen that $\{T_n\}$ is $-\infty$ everywhere, however, if at a certain point $t_n \neq -\infty$, then the wholw of the sequence is not $-\infty$.

Definition. Let $\{x_n\}$ be a real valued ssequence. Then the **limit supremum** is the extended real number, $\limsup x_n = \lim(\sup_{k \geq n}\{x_k\})$ as , $n \to \infty$. And the **limit infimum** is the extended real number $\liminf x_n = \lim(\inf_{k \geq n}\{x_k\})$, as $n \to \infty$.

We discuss some cases. If $s_n = \infty$ for all $n \in \mathbb{N}$, then $\limsup x_n = \infty$. Now if $s_n \neq \infty$, and if $\{S_n\}$ is bounded below, then $\limsup x_n$ is a real number. Now if $S_n \neq \infty$ for all $n \in \mathbb{N}$, and $\{S_n\}$ is not bounded below, then $\limsup x_n = -\infty$.

Similarly, if $t_n = -\infty$ for all $n \in \mathbb{N}$, then $\liminf x_n = -\infty$. Now if $t_n \neq -\infty$ fot all $n \in \mathbb{N}$, and if $\{T_n\}$ is bounded above, then $\liminf x_n$ is a real number. Now if $t_n \neq -\infty$ for all $n \in \mathbb{N}$ and $\{T_n\}$ is not bounded above, then $\liminf x_n = \infty$.

Theorem 2.5.1. *Let $\{x_n\}$ be a real valued sequence, and let $s = \limsup x_n$ and let $t = \liminf x_n$. Then there are subsequences $\{x_{n_k}\}$ and $\{x_{l_j}\}$ such that $x_{n_k} \to s$ as $k \to \infty$, and $x_{l_j} \to t$ as $j \to \infty$.*

Proof. We prove it only for the lim inf. We know that $t_n = \inf x_k$, and that $t_n \to t$ as $n \to \infty$ (for t_n increasing). Now if $t = -\infty$, then $t_n = -\infty$ for all $n \in \mathbb{N}$. Then $t_1 = -\infty$, so there is an n_1 such that $x_{n_1} < -1$. Now $t_{n_1+1} = -\infty$, then there is an n_2 such that $x_{n_2} < -2$ and $n_2 > n_1$. Continuing along this process, we have $n_1 < n_2 < n_3 < \dots$ and $x_{n_k} < -k$. Then we have that $\lim x_{n_k} = -\infty$.

Now if $t = \infty$, then $t_n \neq -\infty$ for all $n \in \mathbb{N}$, and $\{T_n\}$ is not bounded above. Now without loss of generality, assume that $t_n \neq -\infty$ for all $n \in \mathbb{N}$. For $M = 1$, there is an n_1 such that $t_{n_1} > 1$. So $x_{n_1} \geq t_{n_1} > 1$. Now for $M = 2$, there is an n_2 such that $n_2 \geq n_1$, and $t_{n_2} > 2$. Continuing this process, we get that $n_1 \leq n_2 \leq \dots$ and $x_{n_k} > k$.

Suppose that $- < t < \infty$, then t is not $t_n \neq -\infty$ for all $n \in \mathbb{N}$, and $\lim t_n = t$. So $t_n \to t$ (where t_n is increasing). Then for every $\epsilon > 0$, there is an $N \in \mathbb{N}$ such that for $n \geq N$, we have $|t - t_n| = t - t_n < \frac{\epsilon}{2}$. Now if $t_n = \inf\{x_k\}$, sor the same ϵ, there is an $n_k \geq n$ such that $t_n \leq x_{n_k} < t_n + \frac{\epsilon}{2}$. Then ny the traingle inequality, we have $|t - t_n| = |t - t_n| + |t_n - x_{n_k}| < \epsilon$. Now consider $\epsilon = \frac{1}{n}$, if $n = 1$, then there is an $N_1 \in \mathbb{N}$, such that for $n \geq N_1$, $|t - t_n| = t - t_n < \frac{1}{2}$. Then for $t_{N_1} = \inf x_k$, there is an $n_1 \geq N_1$, with $t_{N_1} \leq x_{n_1} < t_{N_1} + \frac{1}{2}$. Combining these two relations we have $|t - x_{n_1}| < \frac{1}{2}$. Now for $n = 2$, there is an $N_2 \in \mathbb{N}$ with $N_2 > n_1$ such that for $n \geq N_2$, we have $|t - x_{n_2}| < \frac{1}{4}$. Continuing this process, we find a sequence $\{n_k\}$ such that $n_1 < n_2 < \dots$, and $|t - x_{n_k}| < \frac{1}{k}$. Thus we have that $\lim_{k\to\infty} x_{n_k} = t$. ∎

Theorem 2.5.2. *Let $\{x_n\}$ be a eal valued sequence and let x be an extended real number. Then $\{x_n\} \to x$ as $n \to \infty$ if and only if:*

$$\limsup_{n\to\infty} x_n = \liminf_{n\to\infty} x_n = x$$

Proof. Suppose that $x_{\to}x$ as $n \to \infty$. Then all subsequences $\{x_{n_k}\}$ will converge to x as $k \to \infty$. Then by theorem 2.5.1, we know that $\limsup x_n = \liminf x_n = x$. COnversely, suppose that $\limsup x_n = \liminf x_n = x$, then we have three cases. If $x = \infty$, then $s_n = \infty$, and $\{T_n\}$ is not bounded above, Thus for all $M \in \mathbb{R}$, there is an $N \in \mathbb{N}$ such that for $n \geq N$, $t_n = M$, so $x_n \geq t_n \geq M$. Thus $\lim x_n = \infty$.

Suppose now that $x = -\infty$, then $t_n = -\infty$ and $\{S_n\}$ is not bounded below. So for all $M \in \mathbb{R}$, there is an $N \in \mathbb{N}$ such that for $n \geq N$, $s_n \leq M$. So $\lim x_n = -\infty$. Now suppose that $-\infty < x < \infty$. Then for all $\epsilon > 0$, ther is an $N \in \mathbb{N}$, such that $0 \leq s_n - x < \frac{\epsilon}{2}$ and $0 \leq x - t_n < \frac{\epsilon}{2}$. Then fo $m, n \geq N$, assume that $|x_n - x_m| = x_m - x_n \leq s_m - t_n \leq \epsilon$. Hence, $\{x_n\}$ is a Cauchy sequence, so $x_n \to x'$ in $\mathbb{R}$. So all its subsequences converge to x'. But we know that $\limsup x_n = \liminf x_n = x$, thus $x' = x$ and we are done. ∎

Theorem 2.5.3. *Let $\{x_n\}$ be a real valued sequence. Then $\limsup x_n$ is the largest value to which some subsequence of $\{x_n\}$ converges, as $n \to \infty$. Similiarly, $\liminf x_n$ is the smallest value for which some subsequence of $\{x_n\}$ converges. In particular, if $x_{n_k} \to x$ as $k \to \infty$, then:*

$$\liminf_{n\to\infty} x_n \leq x \leq \limsup_{n\to\infty} x_n.$$

Proof. Assume that a subsequence $\{x_{n_k}\}$ converges to x as $k \to \infty$. Then fix $N \in \mathbb{N}$, and choose K so that for $k \geq K$, $n_k \geq N$. Now under this condition, $\inf x_j \leq x_{n_k} \leq \sup x_j$ for $j \geq N$ for all $k \geq K$. Now we let $k \to \infty$, then $x_{n_k} \to x$, then by the comparison of the limit, we have $\inf x_j \leq x \leq \sup x_j$ for $j \geq N$, again, by the comparison of limits, letting $N \to \infty$, we have that $\inf x_j$ is increasing, and $\sup x_j$ is decrasing, so we get that $\liminf x_n \leq x \leq \limsup x_n$. ■

Corollary. *If $\{x_n\}$ is any real valued sequence, then :*

$$\liminf_{n\to\infty} x_n \leq \limsup_{n\to\infty} x_n.$$

Proof. The proof follows form theorem 2.5.3. ■

Remark. A real valued sequence $\{x_n\}$ is bounded above if and only if $\limsup x_n < \infty$ as $n \to \infty$; and $\{x_n\}$ is bounded below if and only if $\liminf x_n > -\infty$.

Proof. If $\{x_n\}$ is bounded above, there is an M such that $x_n \leq M$ for all n. THen $\sup x_j \leq M$ for $j \geq N$. Then we get that $\limsup x_n \leq M$ as $n \to \infty$. Conversely, if $\limsup x_n = M$ then for every $\epsilon > 0$, there exists an $N \in \mathbb{N}$ such that:

$$|\sup_{j\geq N} x_j - M| < \epsilon$$

Thus $M - \epsilon < \sup x_j < M + \epsilon$. Then choose $M' = |x_1| + |x_2| + \cdots + |x_{n-1}| + M + \epsilon$. Then $x_j \leq M'$ for all $j \in \mathbb{N}$. The proof is similar for $\liminf x_n$. ■

Theorem 2.5.4. *If $x_n \leq y_n$ for suffieciently large N, then $\limsup x_n \leq \limsup y_n$, and $\liminf x_n \leq \liminf y_n$ as $n \to \infty$.*

Proof. We have $\sup x_j \leq \sup y_j$ for $j \leq n$ for n sufficiently large. Then by the comparison theorem of the limit, we will have $x_n \leq \limsup y_n$ as $n \to \infty$. The same holds for $\liminf$. ■

Homework. Problems 2, 3, 4, 5, and 6 on page 35.

Chapter 3

Continuity on $\mathbb{R}$.

3.1 Two Sided Limits.

Definition. Let $f : A \to B$ be a function. We call f a **real valued function** if $B \subseteq \mathbb{R}$.

It need not be that $A \subseteq \mathbb{R}$ for f to be a real valued function, as long as B is a subset of $\mathbb{R}$, then the definition is satisfied. However, it is also common to take $A \subseteq \mathbb{R}$ (either an interval, or one of the sets $\mathbb{N}, \mathbb{Z}, \mathbb{Q}$ or $\mathbb{Q}^*$). $\mathbb{B}$ necessarily, is any of those sets.

Definition. Let $a \in \mathbb{R}$ and let I be an open interval containing a, and let f be a real valued function defined everywhere on I except possibly at a. Then $f(x)$ is said to **converge** to L as $x \to a$ if and only if for every $\epsilon > 0$, there is a $\delta > 0$ (in general depending on ϵ, f, I, and a) such that: Whenever $0 < |x - a| < \delta$, $|f(x) - L| < \epsilon$. We write $\lim f(x) = L$ as $x \to a$, and we call L the **limit** of f as x approaches a.

Example 3.1. Suppose that $f(x) = mx + b$ where $m, b \in \mathbb{R}$, then $\lim f(x) = f(a)$ as $x \to a$ for all $a \in \mathbb{R}$.

Example 3.2. If $f(x) = x^2 + x - 3$, show that $f(x) \to -1$ as $x \to 1$.

solution. We check $f(x) - L$, to be $x^2 + x - 3 - (-1) = x^2 + x - 2 = (x + 2)(x - 1)$, we choose δ sufficiently small such that $0 < \delta \leq 1$. Then $|x - 1| < \delta$, then $-1 < x - 1 < 1$, hence $0 < x < 2$. Then by the triangle inequality $|x + 2| \leq |x| + 2 < 2 + 2 = 4$. In order to habe $|f(x) - L| = |x + 2||x - 1| < 4\delta$, if $4\delta < \epsilon$, we require $\delta = \frac{\epsilon}{4}$, hence for $\delta < \frac{1}{4}$ and $\delta < 1$, we choose $\delta = \min\{\frac{\epsilon}{4}, 1\}$, Then we have for $|x - 1| < \delta$, we get $|f(x) - L| < 4\delta < \epsilon$. Hence by definition $\lim f(x) = -1$ as $x \to 1$.

Remark. Let $a \in \mathbb{R}$ and let I be an openinterval containing a. Let f and g be real valued functions defined everyewhere on I except possibly at a. If $f(x) = g(x)$ for all $x \in I \backslash \{a\}$ and $f(x) \to a$ as $x \to a$, then g also has a limit as $x \to a$, and

$$\lim_{x \to a} g(x) = \lim_{x \to a} f(x)$$

Proof. For every $\epsilon > 0$, there is a $\delta > 0$ such that $0 < |x - a| < \delta$ implies that $|f(x) - L| < \epsilon$. Now since $|x - a| > 0$, $x \neq a$, then $f(x) = g(x)$. Thus we get that for $0 < |x - a| < \delta$, that $|g(x) - L| < \epsilon$, so by definition of the limit, we see that g has a limit, and moreover, $\lim g = \lim f$. ∎

What this remark says is that those two functions are equal to each other, except (possibly) at a, however if f has a limit, then g must have the same limit.

Example 3.3. Prove that $g(X) = \frac{x^3+x^2-x-1}{x^2-1}$ has a limit as x approaches 1.

solution. First, $g(1)$ is undefined, simplifying $g(x)$ to $(x+1)$ for $x \neq \pm 1$. Letting $f(x) = x+1 = g(x)$, we have $f(1) \neq g(1)$, so by the remark, we have $\lim f = \lim g = 2$ as $x \to 1$.

We now set up the connection between the limit of a function and the limit of a sequence. This will allow us to translate results from limits of sequences, to those of functions and vice versa.

Theorem 3.1.1 (The Sequential Characterization of Limits.). *Let $a \in \mathbb{R}$, and let I be an open interval containing a. Let f be a real valued function defined everywhere on I except possibly at a. Then:*

$$\lim_{x \to a} f(x) = L$$

exists if and only if $f(x_n) \to L$ as $n \to \infty$ for every sequence $\{x_n\} \subseteq I \backslash \{a\}$ convergent to a as $n \to \infty$.

Proof. Suppose that f has limit L, then by definition, for every $\epsilon > 0$ there is a $\delta > 0$ such that $0 < |x-a| < \delta$ implies $|f(x) - L| < \epsilon$. For the same δ, there is an $N \in \mathbb{N}$ such that for $n \geq N$ implies $|x_n - a| < \delta$ Hence, for every $\epsilon > 0$ there is an N such that for $n \geq N$, $|x_n - a| < \delta$ implies $|f(x_n) - L| < \epsilon$. So $lim f(x_n) = L$. as $n \to \infty$.

Conversely suppose that $f(x_n) \to L$ as $n \to \infty$ for all sequences $\{x_n\}$ in I converging to a. We show that $\lim f = L$ as $x \to a$; for suppose to the contrary. Then there is some $\epsilon > 0$ such that for any $\delta > 0$, there is some x_δ for which $0 < |x_\delta - a| < \delta$ and $|f(x_\delta) - L| \geq \epsilon$. Choosing $\delta = \frac{1}{n}$, denote x_δ as x_n. Then $0 < |x_n - a| < \frac{1}{n}$ implies that $\lim x_n = a$, but $|f(x_n) - L| \geq \epsilon$ for all $n \in \mathbb{N}$, thus $\lim f(x_n) \neq L$ as $n \to \infty$. A contradiction. Hence $\lim f = L$ as $x \to a$. ■

What this theorem means if x_n is a point of a sequence in I convergent to a, then when f is evaluated at x_n, then f approaches L. This theorem gives us a way to show that some functions have no limit at certain points. We can construct 2 sequences converging to a, but the sequence of f values do not equal a.

Example 3.4. Prove that $f(x) = \sin \frac{1}{x}$ for $x \neq 0$ and $f(x) = 0$ otherwise has no limit as $x \to 0$.

solution. We know that $\sin 2k\pi + \frac{\pi}{2} = 1$, so let $x_n = \frac{1}{2n\pi + \frac{\pi}{2}}$, then $f(x_n) = \sin \frac{1}{x_n} = -1$, and $\lim x_n = 0$ as $\to \infty$. Similarly, we have that $\sin 2k\pi - \frac{\pi}{2} = 1$, so constructing $y_n = \frac{1}{2n\pi - \frac{\pi}{2}}$, then $f(y_n) = \sin y_n = -1$ and $\lim y_n = 0$ as $n \to \infty$. So, f has no limit as $x \to 0$.

We define the algebra of functions.

Definition. A vector space with **vector multiplication** defined over it is called an **albgebra**.

Theorem 3.1.2. *Suppose that f and g are real valued functions. Define $+$ of functions to be $f+g(x) = f(x)+g(x)$, define αf by $(\alpha f)(x) = \alpha f(x)$. Now define $\cdot$ by $fg(x) = f(x)g(x)$, and finally define the inverse of $\cdot$ to be $f/g(x) = f(x)g(x)$ for $g(x) \neq 0$ for all x. Then the space of all realvalued functions, together with these operations is an algebra.*

Theorem 3.1.3. *Suppose that $a \in \mathbb{R}$, and I is an open interval containing a, and that f and g are realvalued functions defined everywhere on I, except possibly at a. If f and g converge as $x \to a$, then so do $f+g$, αf for $\alpha \in \mathbb{R}$, fg, and f/g.*

Proof. We apply theorem 3.1.1, and then repeat the proof analogous to that for the sequences. ∎

Theorem 3.1.4 (The Squeeze Theorem for Limits of Functions)**.** *Let $a \in \mathbb{R}$, and let I be an open interval containing a, and let f, g, and h be realvlaued functions defined every where on I, except possibly at a. If $g(x) \leq h(x) \leq f(x)$ for all $x \neq a \in I$, and $\lim f = \lim g = L$ as $x \to a$, then the limit of h exists, and $\lim h = L$ as $x \to a$.*

Corollary. *If $|g| \leq M$ for all $x \neq a \in I$, and $f \to 0$ as $x \to a$, then $\lim fg = 0$ as $x \to a$.*

The proof of theorem 3.1.4 and its corollary are identical to the analogous proofs for sequences, with the added caveat that we are applying theorem 3.1.1.

Theorem 3.1.5 (Comparison Theorem for Limits of Functions)**.** *Suppose that $a \in \mathbb{R}$ and that I is an open interval containing a. Let f, and g be realvalued functions defined everwhere on I, except possibly at a. If f and g have limits as $x \to a$ and $f \leq g$ for all $x \in I\backslash\{a\}$, then $\lim f \leq \lim g$ as $x \to a$.*

This theorem is also a direct application of theorem 3.1.1. We can now easily evaluate limits of functions with these theorems out of the way.

Example 3.5.

$$\lim_{x\to 1} \frac{x-1}{3x-1} = \frac{1-1}{3+1} = \frac{0}{4} = 0.$$

Homework. Exercises 1, 2, 3, and 4 on page 63.

3.2 One Sided Limits.

If we have a realvalued function $f(x) = \sqrt{x-1}$, we know that $x \geq 1$, and we would like to talk about $\lim f$ at $x = 1$. Then we can only talk about $x \to 1$ from the righthand side of the function since because of our restriction on x.

Definition. Let $a \in \mathbb{R}$. A realvalued function is said to **converge** to L as x **approaches** a **from the right** if and only if f is efied on some open interval I containing a as a left endpoint and for every > 0, there is a $\delta(\epsilon) > 0$ such taht $a + \delta \in I$, and $a < x < a + \delta$ implies $|f(x) - L| < \epsilon$. We then call L the **righthanded limit** of f as x approaches a and denote it $\lim f$ as $x \to a^+$.

Definition. Let $a \in \mathbb{R}$. A realvalued function is said to **converge** to L as x **approaches** a **from the left** if and only if f is efied on some open interval I containing a as a right endpoint and for every > 0, there is a $\delta(\epsilon) > 0$ such taht $a + \delta \in I$, and $a - \delta < x < a$ implies $|f(x) - L| < \epsilon$. We then call L the **lefthanded limit** of f as x approaches a and denote it $\lim f$ as $x \to a^-$.

In general, we may also write $\lim_{x\to a+} f$ and $\lim_{x\to a-} f$ respectively.

Example 3.6. Let $f(x) = x + 1$ for $x \geq 0$ and $f(x) = x - 1$ for $x < 0$. We see that $\lim_{x\to 0^+} f = \lim x + 1 = 1$ and $\lim_{x\to 0^-} f = \lim x - 1 = -1$. Does the limit of f in general exist? Since the limit of f must be unique, then it may be either of these two limits, however, since the righthanded and left handed limits converge at two different values, that would imply two different limits at the same x, which is impossible. So in general we cannot say if $\lim f$ exists as $x \to 0$.

Theorem 3.2.1. *Let f be a realvalued function. Then the limit* $\lim f$ *as $x \to a$ exists and equals L if and only if:*

$$\lim_{x\to a^+} f = \lim_{x\to a^-} f = L$$

Proof. Suppose that f has limit L as $x \to a$. Then for any $\epsilon > 0$, there is a $\delta > 0$ such that $0 < |x - a| < \delta$ implies $|f(x) - L| < \epsilon$. Then $a - \delta < x < a + \delta$ implies $|f(x) - L| < \epsilon$. Thus we have that $x - \delta < x < a$ and $a < x < a + \delta$ both imply $|f(x) - L| < \epsilon$. Hence we get that $\lim_{x\to a^-} f = \lim_{x\to a^+} f = L$.

Conversely Suppose that $\lim_{x\to a^-} f = \lim_{x\to a^+} f = L$. then by definition fo the left and righthanded limits we get that $a - \delta_1 < x < a$ and $a < x < a + \delta_2$ imply $|f(x) - L| < \epsilon$, thus we have that $a - \delta_1 < x < a - \delta_2$, then choosing $\delta = \min\{\delta_1, \delta_2\}$, and noting that $x \neq a$, we get that $0 < |x - a| < \delta$ implies $|f(x) - L| < \epsilon$. Thus $\lim f = L$ as $x \to a$. ∎

Another thing we would like to study is the limit of a function f (with domain A) as x gets arbitrarily large, and as x gets arbitrarily small, i.e. as $x \to \infty$ and $x \to -\infty$. We say that $f \to L$ as $x \to \infty$ if and only if there exists a $C > 0$ such that $(C, \infty) \subseteq A$ and $\epsilon > 0$ such that tere is an $M \in \mathbb{R}$ such that $x > M$ implies $|f(x) - L| < \epsilon$; likewise, $f \to L$ as $x \to -\infty$ if and only if there is a $C > 0$ such that $(-\infty, C) \subseteq A$ and $\epsilon > 0$ such that there is an $M \in \mathbb{R}$ such that $x < M$ implies $|f(x) - L| < \epsilon$.

Similarly, we can go on to define "infinite limits" of a function f. We say that $f \to \infty$ as $x \to a^+$ if and only if there is an open interval I with left endpoint a such that for any $M > 0$, there is is a $\delta > 0$ such that $a < x < a + \delta$ implies $f(x) > M$. Other infinite limits can be defined similarly. We also have infinite limits at infinity, for example $\lim f = \infty$ as $x \to \infty$ (take $f(x) = x$). What we mean by "infinity" is just to say that for x sufficiently large (or small), f tends to L (in the case for limits at infinity), and that f grows without bound in either direction as x approaches some value a (for infinite limits). In the case of infinite limits at infinity, then we are saying that f grows without bound, (in either direction), for x sufficiently large (or small). The symbol ∞ should not be taken as a literal number, but as a concept denoting something as arbitrarily big, or without bound.

Example 3.7. Show that $\lim \frac{1}{x} = \infty$ as $x \to 0^+$ and that $\lim \frac{1}{x} = 0$ as $x \to \infty$.

solution. For any $M > 0$, we need $\frac{1}{x} > M$, then $x < \frac{1}{M}$, so choose $\delta = \frac{1}{M}$, then for $0 < x < \delta$, $\frac{1}{x} > M$, so by definition, $lim \frac{1}{x} = \infty$ as $x \to 0^+$.

Similarly, for any $\epsilon > 0$, we need $|\frac{1}{x}| < \epsilon$, hence $x > \frac{1}{\epsilon}$, so choose $M = \frac{1}{\epsilon}$. Then by definition, the limit follows.

3.3 Continuity.

In general, we can say that realvalued function f is **continuous** at a if f is defined at a, and $\lim f(x) = f(a)$ as $x \to a$. We give the formal definition below.

Definition. Let $E \subseteq \mathbb{R}$ be nonemoty, and let $f : E \to \mathbb{R}$ be a realvalued funtion. f is said to be **continuous** at a point $a \in E$, if given $\epsilon > 0$, there is a $\delta(\epsilon) > 0$ such that $|x - a| < \delta$, and $x \in E$ imply $|f(x) - f(a)| < \epsilon$. f is said to be **continuous**, on E if and only if f is continuous at x for all $x \in E$.

Remark. we have that a realvalued function f is continuous at a point a on its domain if and only if $\lim f = f(a)$ as $x \to a$. We see that $\lim f(x) = f(\lim x) = f(a)$ as $x \to a$. That is to say, if you can commute f and lim, then f is a continuous function.

Theorem 3.3.1. *Suppose that $E \subseteq \mathbb{R}$ is nonempty, and that $a \in E$, and $f : E \to \mathbb{R}$ is a realvalued function. THen the following are equivalent:*

(1) f is continuous at a.

(2) If $x_n \to a$, and $x_n \in E$, then $f(x_n) \to f(a)$ as $n \to a$.

Proof. We apply the sequential characterization of limits of functions. ■

Theorem 3.3.2. *Let $E \subseteq \mathbb{R}$ be nonempty, and let $f : E \to \mathbb{R}$ and $g : E \to \mathbb{R}$ be realvalued functions. If f and g are continuous at a point $a \in \mathbb{R}$, then so is $f + g$, fg, αf for $\alpha \in \mathbb{R}$. Moreover, f/g is continuous at $a \in E$ when $g(a) \neq 0$ (specifically, when $g(x) \neq 0$ for all $a \in E$).*

We omit the proof. What this theorem says algebraically, is the the set of all contnuous functions over a domain E form an algebra. Now consider for realvalued functions continuous at a point $a \in E$, then

$$f^+(x) = \frac{f(x) + |f(x)|}{2} = \max\{f(x), 0\}$$
$$f^-(x) = \frac{|f(x)| - f(x)}{2} = \max\{-f(x), 0\}$$

Then notice that $f = f^+ - f^-$, and so any function is the difference of two non negative functions, moreover $|f| = f^+ + f^-$. Moreover, all polynomial functions are continuous over $\mathbb{R}$.

Definition. We define the composition of functions as the binary operation $\circ$ such that if $f : A \to B$ are fubctions and $g : B \to C$, then $f \circ g : A \to C$ is the function defined by $f \circ g(x) = f(g(x))$.

It is well known that $\circ$ is not commutative, i.e. $f \circ g \neq g \circ f$.

Theorem 3.3.3. *Suppose that $A, B \subseteq \mathbb{R}$ and that $f : A \to \mathbb{R}$ and $g : B \to \mathbb{R}$ are realvalued functions, with $F(A) \subseteq B$. Then if $A = I \backslash \{a\}$ where I is a nondegenerate interval, that either contains a, or has a as one of its endpoints. If $L = \lim f$ as $x \to a$ for $x \in I$ exists and belongs to B, and if g is continuous at $L \in B$, then $\lim g \circ f = g(\lim f)$ as $x \to a$ for $x \in I$.*

Proof. Let $\{x_n\} \subseteq I \backslash \{a\}$, such that $x_n \to a$ as $n \to \infty$. Then if $f(A) \subseteq B$ with $f(x_n) \in B$, then $L = \lim f$ as $x \to a$, then by the sequential characterization of limits of functions, for $x \in I$, so $f(x_n) \to L$ as $n \to \infty$. This implies that $g(fx_n)) \to g(L)$ as $n \to \infty$, and we are done. ∎

Corollary. *If f is continuous at $a \in A$, and g is continuous at $f(a) \in B$, then $g \circ f$ is continuous at $a \in A$.*

Definition. Let $E \subseteq \mathbb{R}$ be a nonempty subset of $\mathbb{R}$. A realvalued function $f : E \to \mathbb{R}$ is said to be **bounded** if and only if there is an $M \in \mathbb{R}$ sucht that $|f| \leq M$ for all $x \in E$.

Theorem 3.3.4 (The Extreme Value Theorem)**.** *If I is closed, bounded interval, and $f : I \to \mathbb{R}$ is continuous on I,then f is bounded on I; moreover, if $M = \sup f(I)$ and $m = \inf f(I)$, then there exists points x_m and x_M such that $f(x_M) = M$ and $f(x_m) = m$.*

Proof. Suppose that f is not bounded on I, then there is some $x_n \in I$ such that $|f(x_n)| > n$ for every $n \in \mathbb{N}$. Now $I isbounded$, hence so is the sequence $\{x_n\}$ of points of I. Thus, by the Bolzano Weierstrass theorem, $\{x_n\}$ has a convegernt subsequence, $x_{n_k} \to a$ as $k \to \infty$. Now since I is closed, it contains all the limit points, so $a \in I$. Since $f(a) \in I$, and since f is continuous, then $\lim |f| = \lim |f(x_{n_k})| = \infty$ as $k \to \infty$, which is a contradiction. Hence f is bounded on I.

Now because f is bounded on I, then M, m are finite. We show that there is an $xM \in I$ with $f(x_M) = M$. Suppose that $f < M$ for all $x \in I$, define $g : I \to \mathbb{R}$ by

$$g(x) = \frac{1}{M - f(x)}$$

We have that g is contininous, and bounded on I. Then there is a $C > 0$ such that $|g| \leq C$, so $g(x) \leq C$ for all $x \in I$. Then by definition of g we have that $f \leq M - \frac{1}{C} < M$ for all $x \in I$, so $M - \frac{1}{C}$ is an upperbound of f over I which is less than $\sup f(I)$, a contradiction. Thus $f(x_M) = M$ for some $x_M \in I$. Similarly, we get that $f(x_m) = m$. ∎

We can prove the second statement alternatively by noting that $M = \sup f(I)$, so for every $n \in \mathbb{N}$, there is an $x_n \in I$ such that $M - \frac{1}{n} < f(x_n) \leq M$, then the sequence $\{x_n\} \subseteq I$ has a convergent subsequence $x_{n_k} \to a$ by the Bolzano Weierstrass theorem. Since I is closed, then we get that $a \in I$, and since f is continuous, then by the comparison theorem

$$M - \frac{1}{n_k} < f(x_{n_k}) \leq M$$

Then $M \leq f(a) \leq M$, then $f(a) = M$. The proof is also analogous for $m = \inf f(I)$.

The dowside of this proof for the second part of the extreme value theorem is that it repeats the proof of the first part; that aside, the formal proof given also illustrates the relation between the extreme values of f, and the extreme values of g.

For the extreme value theorem to work, the interval I must be closed, and bounded. That is it does not work for intervals of the type (a, b), (a, ∞) and $[a, \infty)$, or $(-\infty, b)$ and $(-\infty, b]$. f must also be a continous function. Remove any one of those properties, and the theorem falls apart.

Lemma 3.3.5 (The Sign Preserving Property)**.** *Let $f : I \to \mathbb{R}$ where I is an open nondegenerate interval. If f is continous at a point $x_0 \in I$, and $f(x_0) > 0$, then there are positive numbers ϵ, δ, such that $|x - x_0| < \delta$ implies $f(x) > \epsilon$.*

Proof. Choose $\epsilon = \frac{f(x_0)}{2}$, choose $\delta > 0$, such that $|x - x_0| < \delta$ implies that $-\frac{f(x_0)}{2} < f(x) - f(x_0) < \frac{f(x_0)}{2}$, then we see that $f(x) > \epsilon$. ■

Definition. We say that a real number a lies **between** real numbers b, c if either $c < a < b$ or $b < a < c$.

Theorem 3.3.6 (The Intermediate Value Theorem)**.** *Let I be a nondegenerate interval, and let $f : I \to \mathbb{R}$ be a continuous realvalued function. Then if $a, b \in I$, with $a < b$, and if y_0 is between $f(a)$ and $f(b)$ then there is an $x_0 \in (a, b)$ such that $f(x_0) = y_0$.*

Proof. Suppose that $f(a) < y_0 < f(b)$, and let $E = \{x \in [a, b] : f(x) < y_0\}$. We have $a, b \in E$, and $E \subseteq [a, b] \subseteq \mathbb{R}$, then by the completeness axiom, $x_0 = \sup E \in E$ is finite. Then we have that $x_0 \neq a, b$ by the sign preserving property. Now choose $x_n \in E$ such that $x_n \to x_0$ as $n \to \infty$. Then by the comparison theorem, $x \in [a, b]$, and we also have that $f(x_0) = \lim f(x_n) \leq y_0$. Now if $f(x_0) < y_0$, then $y_0 - f(x)$, is continuous, and $y_0 - f(x_0) > 0$. Then by the sign preserving property, choose ϵ, δ¿0 such that $y_0 - f(x) > \epsilon$ for $|x - x_0| < \delta$. Then, that is, $x_0 < x < x_0 + \delta$ and $f(x) < y_0$ for any x, which contradicts that $x_0 = \sup E$. ■

Definition. We say a realvalued function $f : I \to \mathbb{R}$ is **discontinuous** at a point $a \in I$ if f is not continuous at a. Then we call a the point of **point of discontiuity**.

Example 3.8. Show that $f(x) = \frac{|x|}{x}$, at $x \neq 0$ and $f(x) = 1$ at $x = 0$ is continuous on $(-\infty, 0)$ and $[0, \infty)$ and discontinuous everywhere else, and that $f(0^+)$ and $f(0^-)$ exist.

solution. We have that $f(0+) = 1$ for ≥ 0, and $f \to f(a)$ as $x \to a$. Now, $f(0^-) = -1$, and f is continous on $(-\infty, 0)$.

Example 3.9. $f(x) = \sin x$ for $x \neq 0$ and $f(x) = 1$ for $x = 0$ is continuous on $(-\infty, 0)$ and $(0, \infty)$, and discontinuous everywhere else. Now we also havee that $f(0^+)$ and $f(0^-)$ do not exist. Choose sequences $\{x_n\}$ and $\{y_n\}$ with $x_n = \frac{1}{2n\pi + \frac{\pi}{2}}$ and $y_n = \frac{1}{2n\pi - \frac{\pi}{2}}$.

Example 3.10. The **Dirichlet function** is defined on $\mathbb{R}$ such that $f(x) = 1$ if $x \in \mathbb{Q}$, and $f(x) = 0$ if $x \notin \mathbb{Q}$. Then every point $x \in \mathbb{R}$ is a point of discontiuity. We call these types of functions **nowhere continous**.

Proof. Let $x_0 \in \mathbb{R}$, and $\epsilon > 0$, and let there be a $\delta > 0$ such that $|x - x_0| < \delta$ implies that $|f(x) - f(x_0)| < \epsilon$. We consider two cases; suppose $x_0 \in \mathbb{R}$. Then $f(x_0) = 1$ and choose $\epsilon = \frac{1}{2}$. Then since there is alwas an irrational x, we have for every $\delta > 0$, $|x - x_0| < \delta$ abd $f(x) = 0$. Then $|f(x) - f(x_0)| = |f(x)| = 1 > \epsilon$. a contradiction. Now suppose that $x_0 \notin \mathbb{R}$, and choose $\epsilon = \frac{1}{2}$, and by the density of $\mathbb{Q}$ in $\mathbb{R}$, we have for every $\delta > 0$ there is a rational x such that $|x - x_0| < \delta$. Then $f(x) = 1$, and $|f(x) - f(x_0)| = |f(x)| = 1 > \epsilon$; another contradiction. ∎

Example 3.11. Define $f(x) = \frac{1}{q}$ if $x = \frac{p}{q} \in \mathbb{Q}$ (with $\gcd(p, q) = 1$); and $f(x) = 0$ if $x \notin \mathbb{Q}$. Then f is continous at irrational numbers and discontinuous at rationals over $(0, 1)$.

Proof. Let $x_0 = \frac{p}{q} \in \mathbb{Q}$. Then $f(x_0) = \frac{1}{q}$. Choose $\epsilon = \frac{1}{2q}$. Then for any $\delta > 0$, there is an irrational x with $|x - x_0| < \delta$, and $f(x) = 0$. Then $|f(x) - f(x_0)| = |f(x_0)| = \frac{1}{q} > \epsilon$; A contradiction. Now for $x \notin \mathbb{Q}$, then $f(x) = 0$. Now let $\epsilon > 0$, then there is a positive integer q such that
$frac1q < \epsilon$. Now consider $1, 2, 3, 4, \ldots, q - 1$; there are finitely many rational numbers with denominator $1, 2, 3, 4 \ldots, q - 1$. Now choose $\delta > 0$ such that $|x - x_0| < \delta$ does not include any rational numbers from the above list. In other words, if $|x - x_0| < \delta$ and $x = \frac{p}{q} \in \mathbb{Q}$, then $p \geq q$, which implies $f(x) = \frac{1}{p} <\leq \frac{1}{q} < \epsilon$, then $|f(x) - f(x_0)| = |f(x)| \leq \frac{1}{q} < \epsilon$. Thuse f is continuous at $x \notin \mathbb{Q}$. ∎

Remark. $g \circ f$ can be nowhere continuous, even though f is discontinuous only on $\mathbb{Q}$ and g is discontinuous only at a point. One possible function is taking the function in the previous example, and taking $g(x) = 1$ for $x \neq 0$, and $g(x) = 0$ for $x = 0$. Then $g \circ f$ is the Dirichlet function.

Homework. Exercises 3, 4, 6, 8 on page 79.

3.4 Uniformity.

We say a function $f : I \to \mathbb{R}$ is continous if $\lim f = f(a)$ as $x \to a$. That is for any $\epsilon > 0$, there is a $\delta(\epsilon) > 0$ such that $|x - a| < \delta$ implies that $|f(x) - f(a)| < \epsilon$. Here, δ depends on ϵ, however, δ usually also depends on the domain I of f, f istself, and even a. If f is the constan function, then δ does not depend on ϵ, as $|f(x) - f(a)| = 0$. The same is true for a.

Definition. Let $E \subseteq \mathbb{R}$ be nonempty, and let $f : E \to \mathbb{R}$ be a realvalued function. We call f **uniformly continous** on E if and only if for every $\epsilon > 0$, there is a $\delta > 0$ such that $|x - a| < \delta$ and $a, x \in E$ imply that $|f(x) - f(a)| < \epsilon$.

That is to say, a function is uniformly continous if and only if δ depends only on ϵ, E, or f.

Example 3.12. $f(x) = x^2$ is uniformly continuous on $(0, 1)$.

Proof. Given $\epsilon > 0$, let $x, a \in (0, 1)$. Then $|x^2 - a^2| = |x - a||x + a| \leq (|x| + |a|)|x - a| < 2|x - a|$. Then choose $\delta = \frac{\epsilon}{2}$, then we have $|x - a| < \delta$, for $x, a \in (0, 1)$ implies $|f(x) - f(a)| < \epsilon$. So $f(x) = x^2$ is uniformly continuous on $(0, 1)$. If we choose $E = (0, 2)$ for $f(x) = x^2$, then we require $\delta = \frac{\epsilon}{4}$. So $f(x) = x^2$ is uniformly continuous on any bounded domain. ∎

Example 3.13. $f(x) = x^2$ is not uniformly continuous on $\mathbb{R}$.

Proof. For $\epsilon > 0$, and choosing $x, a \in \mathbb{R}$, Suppose that f is uniformly continous on $\mathbb{R}$. Then there is a $\delta > 0$ such that $|x - a| < \delta$ implies $|f(x) - f(a)| < \epsilon$. Choose $\epsilon = 1$, then there is a corresponding δ for which $|f(x) - f(a)| < 1$. Letting $x = a + \frac{\delta}{2}$, we have that $|x - a| < \delta$, and $|x^2 - a^2| = |\delta + \frac{\delta^2}{4}| > 1$, if we suppose that $a > 1$. So a can be as large as possible, and this contradicts the assumption that $|f(x) - f(a)| < 1$. So f is not uniformly continous on $\mathbb{R}$. (In general, we can show that f is not uniformly continous on $\mathbb{R}$ by using the Archimedian principle for any $n \in \mathbb{N}$). ∎

Example 3.14. $f(x) = \frac{1}{x}$ is continous on $(0, 1)$, but not uniformly continuous on $(0, 1)$.

Lemma 3.4.1. *Suppose that $E \subseteq \mathbb{R}$ and $f : E \to \mathbb{R}$ is unifomrly continous. If $\{x_n\} \subseteq E$ is Cauchy, then $\{f(x_n)\} \subseteq \mathbb{R}$ is also Cauchy.*

Proof. Suppose that f is uniformly continuous. Then for evey $\epsilon > 0$, there is a $\delta > 0$ such that $|x - a| < \delta$ for $x, a \in E$ implies $|f(x) - f(a)| < \epsilon$. We also have that for this δ, there is an $N > 0$ such that for $m, n \geq N$, $|x_n - x_m| < \delta$. Then $|f(x_n) - f(x_m)| < \epsilon$ whenever $x_n, x_m \in E$. Thus we have that $\{f(x_n)\}$ is a Cauchy sequence. ∎

Theorem 3.4.2. *Suppose that I is a closed, bounded interval. If $f : I \to \mathbb{R}$ is contnuous on I; then f is uniformly continous on I.*

Proof. Suppose that $f : I \to \mathbb{R}$ is continous, but not uniformly continous on I. Then for every $\epsilon > 0$, there is a $\delta > 0$ such that $|x - a| < \delta$ implies $|f(x) - f(a)| < \epsilon$. On the other hand, we have that there is some $\epsilon > 0$ for which given any $\delta > 0$, $|x - a| < \delta$, with $x, a \in I$ implies $|f(x) - f(a)| \geq \epsilon$.

Let $\epsilon_0 > 0$, and for every $\delta > 0$, choose $x_n, y_n \in I$ such that $|x_n - y_n| < \delta$, but $|f(x_n) - f(y_n)| \geq \epsilon_0$. Now since I is bounded, then so are the sequences $\{x_n\}$ and $\{y_n\}$; so by thr Bolzanno-Weierstrass theorem, they have convergent subsequences. By the sequential criterion, $f(x_n), f(y_n)$ converge to the same point, which contradicts our assumption. Therefore, f must be uniformly continuous. ∎

Theorem 3.4.3. *Let (a, b) be a bounded open, nondegenerate interval, and let $f : (a, b) \to \mathbb{R}$ be a realvalued function. Then f is uniformly continous on (a, b) if and only if f is continous on $[a, b]$.*

Proof. Suppose that f is unifomly continuous on $[a, b]$, then clearly it is unifomly continuous on (a, b). Now suppose that f is uniformly continous on (a, b). Define $f(a)$ and $f(b)$ continuously, that is define $f(a)$ and $f(b)$ sicht that $f(x^+) = f(a)$ and $f(x^-) = f(b)$. Pick $\{x_n\} \subseteq (a, b)$ with $\lim x_n = a$, and $\{x_n\}$ is Cauchy. Then f is uniformly continous and $\{f(x_n)\}$ is also Cauchy. So define $lim f = f(a)$. Then choose $\{y_n\}$ corresponding to the upper criteria. Then we have that $x - y \to 0$. Now given $\epsilon > 0$, choose $\delta > 0$ such that $|x_n - y_n| < \delta$ implies $|f(x) - f(y)| < \epsilon$. Then there is an $N > 0$ such that when $n \geq N$, $|x_n - y_n| < \delta$, hence $|f(x_n) - f(y_n)| < \epsilon$. Then $\lim f(x_n) = \lim f(y_n)$.

Then by the sequential criterion, we have that $f(a) = f(x^+)$ and $f(b) = f(x^-)$. Thus we have a continous extention of f onto $[a, b]$. ∎

Homework. Exercises 1, 2, 3 and 4 on page 83.

Chapter 4

Differentiability on $\mathbb{R}$.

4.1 The Derivative.

Definition. A realvalued function $f : I \to \mathbb{R}$ (with I an open interval) is said to be **differentiable** at a point $a \in I$, if and only if f is defined over all of I, except possibly at a, and

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} = \lim_{x \to a} \frac{f(x) - f(a)}{x - a} \tag{4.1}$$

exists. We call $f'(a)$ the **derivative** of f at a.

This definition, illustrated geometrically, is given by the line tangent to the function f at a. That is, if $\frac{f(x)-f(a)}{x-a}$ is the slope of the secant cutting f at a, then the derivative is the limit of that slope as $x \to a$; which gives us the "instantaneous" rate of change of f at a.

It need not be that f is differentiable at every point in its domain. However, if f' is defined on every point in the domain of f, then we call f' the **derivative function**. Alternative notations for the derivative of f' include: $D_x f$, $\frac{dy}{dx}$, and $f^{(1)}$.

Now if f' is also differentiable on a domain E, then the **excond derivative** of f is the derivative of f', denoted by f''. Similarly, we define the n^{th} derivative recursively to be:

(1) $f^{(0)} = f$, $f^{(1)} = f'$.

(2) $f^{(n+1)} = (f^{(n)})'$.

Theorem 4.1.1. *A realvalued function f is differentiable at some point $a \in \mathbb{R}$ if and only if there exists an open interval I and a function $F : I \to \mathbb{R}$ such that $a \in \mathbb{R}$, F is continous at point a, and $f(x) = F(x)(x - a) + f(a)$ for all $x \in I$; in which case, $F(a) = f'(a)$.*

Proof. First, when $x \neq a$, we have $F(x) = \frac{f(x)-f(a)}{x-a}$, which satisfies $f(x) = F(x)(x-a)+f(a)$. Now suppose that f is differentiable at a. Then $\lim \frac{f(x)-f(a)}{x-a} = \lim F = F(a)$ as $x \to a$, and so F is continous at a; and so $f'(a) = F(a)$.

Now if F exists, and $f(x) = F(x)(x - a) + f(a)$, then $\lim \frac{f(x)-f(a)}{x-a} = \lim F = F(a)$ as $x \to a$. So f is differentiable at a with $f'(a) = F(a)$ ■

Theorem 4.1.2. *Let $f : \mathbb{R} \to \mathbb{R}$ be a realvalued function. Then f is differentiable at a if and only if there is a function T of the form $T(x) = mx$ such that:*

$$\lim_{h\to 0} \frac{f(a+h) - f(a) - T(h)}{h} = 0 \tag{4.2}$$

Proof. Suppose that f is differentiable at a, with $m = f'(a)$. Let $T(x) = mx$, then $\lim \frac{f(a+h)-f(a)-f'(a)h}{h} = \lim \frac{f(a+h)-f(a)}{h} - f'(a) = f'(a) - f'(a) = 0$, as $h \to 0$. Conversely if $\lim \frac{f(a+h)-f(a)}{h} - m = \lim \frac{f(a+h)-f(a)-mh}{h} = 0$ as $h \to 0$. Threrefore, f is differentiable at a, with $f'(a) = m$. ■

Theorem 4.1.3. *If f is differentiable at a, then f is continous at a.*

Proof. By theorem 4.1.1, $f(x) = F(x)(x-a) + f(a)$ is continuous. ■

What theroem 4.1.3 says that discontinuous functions are never differentiable at their points of discontinuity. Likewise, the converse of this theorem is not true; we may have continuous functions that are not differentiable at their points of continuity.

Example 4.1. The absolute value function $|x|$ is continous everywhere, but it is not differentiable at $x = 0$.

Definition. Let I be a nondegenerate interval. A function $f : I \to \mathbb{R}$ is said to be **differentiable** on I if and only if :

$$f'_I(a) = \lim_{x\to a} \frac{f(x) - f(a)}{x - a} \tag{4.3}$$

for all $x \in I$, and is finite on all of I. f is said to be **continuously differentiable** on I if and only if f'_I exists and is continous for every point in I.

Example 4.2. Consider $f(x) = x^{\frac{3}{2}}$, then $f'(x) = \frac{3}{2}\sqrt{x}$ is continous on the interval$[0, \infty)$. So f is continously differentiable on $[0, \infty)$. However, f' is not continuously differentiable on $[0, \infty)$.

We denote the set of all realvalued functions whose n^{th} derivative exists, and are contiuous on a domain I by $C^n(I)$. We denote $C^\infty(I)$ to be the set of all realvalued functions that are infinitely contiuously differentiable on a domain I.

Example 4.3. Let $f(x) = x^2 \sin\frac{1}{x}$ for $x \neq 0$, and $f(x) = 0$ when $x = 0$. Then f is differentiable on $\mathbb{R}$, but not continuously differentiable on $\mathbb{R}$ (it is not contiuously differentiable on any interval containing 0).

4.2 Differentialbility Theorems.

Theorem 4.2.1. *Let f, g be realvalued functions, and let $\alpha \in \mathbb{R}$. If f and g are differentiable on at some point a, then:*

(1) $f + g$ is differentiable at a, and $(f+g)' = f' + g'$.

(2) αf *is differentiable at* a*, and* $(\alpha f)' = \alpha f'$.

(3) fg *is differentiable at* a*, and* $(fg)' = f'g + fg'$

(4) G iven that $g(a) \neq 0$*, then* $\frac{f}{g}$ *is differentiable at* a*, and* $(\frac{f}{g})' = \frac{f'g-fg'}{g^2}$.

Proof. We only prove the Quotient rule, the others follow easily from definition. We know thiif $g(a) \neq 0$, then there is a $\delta > 0$ such that $g(x) \neq 0$ on the neigheborhood $(a - \delta, a + \delta)$. Then we have that:

$$\begin{aligned}
\frac{\frac{f(x)}{g(x)} - \frac{f(a)}{g(a)}}{x-a} &= \frac{f(x)g(a) - f(a)g(x)}{g(x)g(a)(x-a)} \\
&= \frac{f(x)g(a) - f(a)g(a) + f(a)g(a) = f(a)g(x)}{g(x)g(a)(x-a)} \\
&= \frac{f(x) - f(a)}{g(x)g(a)(x-a)}g(a) - f(a)\frac{g(x) - g(a)}{g(x)g(a)(x-a)}
\end{aligned}$$

Taking limits, we then get $\frac{f'g-fg'}{g^2}$ as required. ■

Theorem 4.2.2 (The Chain Rule)**.** *Let* f *and* g *be realvalued functions, with* f *differentiable at* a*, and* g *differentiable at* $f(a)$*. Then* $g \circ f$ *is differentiable at* a*, and* $(g \circ f)' = (g' \circ f)f'$

Proof. Let $F : I \to \mathbb{R}, G : J \to \mathbb{R}$ be realvalued functions such that $f(x) = F(x)(x-a)+f(a)$ and $g(y) = G(y)(y - f(a)) + g(f(a))$, then by theorem 4.1.1, $F(a) = f'(a)$,a nd $G(f(a)) = g'(f(a))$. Observe that $a \in I$, and $f(a) \in f(I) \subseteq J$. Letting $y = f(x)$, then we get $g(y) = g(f(x)) = G(f(x))(f(x) - f(a) + g(f(a)) = G(f(x))F(x)(x - a) + g(f(a))$, so again by theorem 4.1.1, we get that $G(f(a))F(a) = g'(f(a))f'(a)$. ■

Homework. Exercise 7, on page 94.

4.3 The Mean Value Theorem.

Lemma 4.3.1 (Rolle's Theorem)**.** *Let* $a, b \in \mathbb{R}$*, with* $a \neq b$*. If* f *is continous on* $[a, b]$*, and differentiable on* (a, b)*, and if* $f(a) = f(b)$*, then there is a* $c \in (a, b)$ *such that* $f'(c) = 0$.

Proof. By the extreme value theorem, f is continous, so f has a finite maximum M and a finite minimum m on $[a, b]$. If $M = m$, then f is a constant finction and $f'(c) = 0$ for all $c \in (a, b)$. Now suppose that $M \neq m$. Since $f(a) = f(b)$, then there must be some $c \in (a, b)$ such that $f(c) = M$ or $f(c) = m$. Without loss of generality, if $f(c) = M$, then we have that $f(x) \leq f(c)$ for all $x \in [a, b]$, then $f(c) - f(x) \geq 0$, so $\frac{f(c)-f(x)}{c-x} \geq 0$, on the other hand we have $f(x) - f(c) \leq 0$, and $\frac{f(x)-f(c)}{x-c} \leq 0$; taking limits as $x \to c$, we then get that $0 \leq f'(c) \leq 0$. Therefore $f'(c) = 0$. A similar argument follows for $f(c) = m$. ■

Remark. The continuity hypothesis in Rolle's theorem cannot be relaxed, neither can the assumption that $f(a) = f(b)$, and f being differentiable on (a, b).

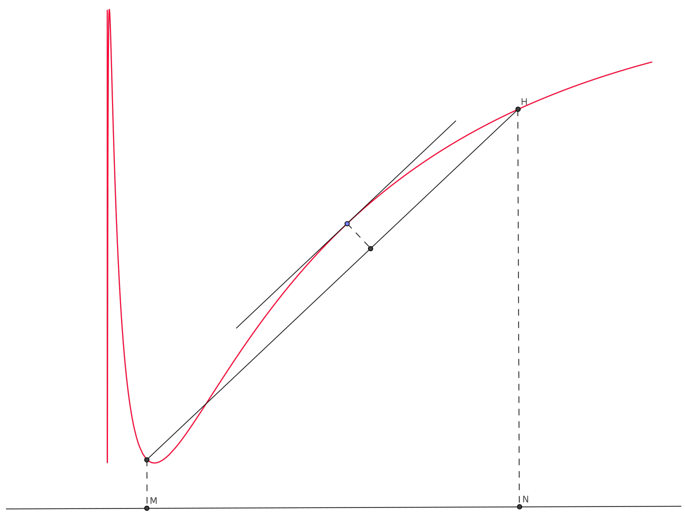

Figure 4.1: The Mean Value Theorem

Example 4.4. Consider the function $f(x) = x$ for $x \in [0, 1)$, and $f(x) = 0$ everywhere else. Rolle's theorem fails cause f is not continous at $x = 1$. Likewise, $f = |x|$ is not differentiable at $x = 0$, and so Rolle's theorem failes for that as well.

Theorem 4.3.2 (The Mean Value Theorem)**.** *Suppose that $a, b \in \mathbb{R}$, with $a < b$. Then:*

(1) If f, g are continuous on $[a, b]$, and differentiable on (a, b), then there is a $c \in (a, b)$ such that $f'(c)(g(b) - g(a)) = g'(c)(f(b) - f(a))$.

(2) If f is continuous on $[a, b]$, and differentiable on (a, b), then there is a $c \in (a, b)$ such that $f'(c)(b - a) = f(b) - f(a)$.

Proof. We need only prove (1). Let $h(x) = f(x)(g(b) - g(a)) - g(x)(f(b) - f(a))$. We have that h is continous on $[a, b]$, differentiable on (a, b), and we also see that $h(a) = h(a)$. Then by Rolle's theorem, there is some $c \in (a, b)$ for which $h'(c) = 0$. Thus we have that $0 = f'(c)(g(b) - g(a)) - g'(c)(f(b) - f(a))$. We are done. To prove (2) we take g to the identity function. ■

Example 4.5. Show that $e^x \geq 1 + x$.

solution. Notice that for $f(x) = e^x$, $f'(x) = e^x$. We also have that $e^x > 1$ when $x > 0$, and $e^x < 1$ when $x < 0$. Then consider the intervals $(x, 0)$ and $(0, x)$. By the mean value theorem, there is a $c \in (0, x)$ for which $e^x - 1 = xe^c$. Since $e^c > 1$, then we get that $e^x - 1 \geq x$. We get the same result for $(x, 0)$.

Theorem 4.3.3 (Bernoulli's Inequallity)**.** *Let $\alpha > 0 \in \mathbb{R}$ and let $\delta \geq -1$. If $0 < \alpha \leq 1$, then $(1 + \delta)^\alpha 1 + \alpha\delta$, and if $\alpha > 1$ then $(1 + \delta)^\alpha 1 + \alpha\delta$*

Theorem 4.3.4 (L'Hospital's Rule)**.** *Let a be an extended real number, and let I be an open interval containing a in it's interior, or as an endpoint. Suppose that f and g are differentiable on $I\{a\}$, and that $g(x), g'(x) \neq 0$ for all $x \in I\{a\}$. Suppose further that $A = \lim f = \lim g$ as $a \to a$, for $x \in I$ is either 0 or ∞. If $B = \lim \frac{f}{g}$ as $x \to a$ exists as an extended real number then:*

$$\lim_{x\to a} \frac{f}{g} = \lim_{x\to a} \frac{f'}{g'} \tag{4.4}$$

Proof. Let $\{x_n\} \subseteq I$, such that $x_n \to a$ as $n \to \infty$. Then by the sequential characterization, it suffices to show that $\frac{f(x_n)}{g(x_n)} \to B$ as $n \to \infty$. Assume that $B \in \mathbb{R}$. By the mean value theorem, we have that $g(x) - g(y) \neq 0$.

Now suppose that $A = 0$, and $a \in \mathbb{R}$. Extending f and g to $I \cup \{a\}$ by $f(a) = g(a) = 0$, then f and g are continuous ove $I \cup \{a\}$, and differentiable over $I \backslash \{a\}$. Then there is a c_n between x_n and a such that

$$\frac{(x_n) - f(a)}{g(x_n) - g(a)} = \frac{f'(c_n)}{g'(g(c_n)}$$

We also have that $\lim c_n = \lim x_n = a$ as $n \to \infty$ by the squeeze theorem. Then

$$\lim_{n\to\infty} \frac{f(x_n)}{g(x_n)} = \lim_{x\to\infty} \frac{f'(c_n)}{g'(c_n)} = B$$

Now suppose without loss of generality that $A = \infty$, and that $a \in \mathbb{R}$.Since $x_n \to a$, for each $n, k \in \mathbb{N}$, by the mean value theorem, we there is a $c_{k,n}$ between x_k and x_n, such that:

$$\frac{f(x_n)}{g(x_n)} = \frac{f(x_k)}{g(x_n)} - \frac{f(x_n) - f(x_k)}{g(x_n)}$$

Then we get:

$$\frac{f(x_n)}{g(x_n)} = \frac{f(x_k)}{g(x_n)} - \frac{1}{g(x_n)}\left(\frac{(g(x_n) - g(x_k))f'(c_{k,n})}{g'(c_{k,n})}\right)$$

simplifying and distributing the appropriate terms we get

$$\frac{f(x_n)}{g(x_n)} = \frac{f(x_k)}{g(x_n)} - \frac{g(x_k)f'(c_{k,n})}{g(x_n)g'(c_{k,n})} + \frac{f'(c_{k,n})}{g'(c_{k,n})}$$

Taking limits, we then get:

$$\lim_{k,n\to\infty} \frac{f(x_n)}{g(x_n)} = \lim_{k,n\to\infty} \frac{f'(c_{k,n})}{g'(c_{k,n})} = B$$

Now suppose without loss of generality that $a = \infty$. Then choose c such that $(c, \infty) \subseteq I$. Then for each $y \in (0, \frac{1}{c})$, let $\phi(y) = f(\frac{1}{y})$. and $\psi(y) = g(\frac{1}{y})$.

$$\frac{\phi'(y)}{\psi'(y)} = \frac{f'(\frac{1}{y})}{g'(\frac{1}{y})}$$

so let $x = \frac{1}{y}$, since ϕ and ψ satisfy the above cases, we are done i.e.

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{y\to 0} \frac{\phi(y)}{\psi(y)} = \lim_{y\to 0} \frac{\phi'(y)}{\psi'(y)} = \lim_{x\to\infty} \frac{f'(x)}{g'(x)}$$

∎

4.4 Monotonic Functions and Inverse Functions.

Definition. Let $E \subseteq \mathbb{R}$ be nonempty, and let $f : E \to \mathbb{R}$ be a realvalued function. We say that f is **monotonically increasing** on E if for $x < y$, $f(x) \leq f(y)$. Similarly, f is **monotonically decreasing** if $f(y) \leq f(x)$. In either case, we call f a **monotonic** function.

Example 4.6. The function $f(x) = x^2$ is not monotonoic on I, but it is monotonically increasing over $[0, \infty)$ and monotonically decreasing over $(-\infty, 0)$.

Theorem 4.4.1. *Suppose that $a, b \in \mathbb{R}$, with a and b distinct, and let f be contiuous over $[a, b]$ and differentiable over (a, b). Then:*

(1) If $f' \geq 0$, for all $x \in (a, b)$, then f monotonically increasing on $[a, b]$; respectively, if $f' \leq 0$, f is monotonically decreasing.

(2) If $f' = 0$, for all $x \in (a, b)$, then f is constant on $[a, b]$

Proof. Assume without loss of generality that $f' \leq 0$, then for any $a < x_1 < x_2 < b$, then f is continuous on $[x_1, x_2]$ and differentiable on (x_1, x_2), hence by the mean value theorem, then there is a $c \in (x_1, x_2)$ such that $f(x_2)f(x_1) = f'(c)(x_2 - x_1)$, hence $f(x_2) - f(x_1) \leq 0$, hence f is monotonically decreasing. The case is analogous for $f' > 0$.

Now suppose that $f' = 0$, then again by the mean value theorem, there is a $c \in (x_1, x_2)$ such that $f(x_2) - f(x_1) = f'(c)(x_2 - x_1)$, then $f(x_2) - f(x_2) = 0$, for $x_1, x_2 \in (a, b)$ arbitrary therefore, f is constant. ∎

Remark. If f and g are continuous on a nondegenerate inteval, $[a, b]$, and differentiable on (a, b), and $f' = g'$ for all $x \in (a, b)$, then $f - g$ is constant.

Proof. We repeat the proof for the constant case using the Cauchy mean value theorem; similarly, we can just apply the previous theorem to the function $f - g$. ∎

Suppose that f is a realvalued function, who has an inverse function f^{-1}. Then the graph of f^{-1} is symmetric to the graph of f with respect to the line $y = x$. Then visually, f^{-1} is as smooth as f. Algebraically, we can apply an operation to the graph of f to obtain the graph of f^{-1}, and we can observe that smoothness is preserved, however we would like to prove this rigorously.

Theorem 4.4.2. *If f is a 1-1 continuous function of the interval I onto $\mathbb{R}$, then f is strictly monotonic on I, and f^{-1} is continuous and strictly monotonic on $f(I)$.*

Proof. We have that since f is 1-1, then $f(x) = f(y)$ implies $x = y$, hence if $x < y$ are in I, then either $f(x) < f(y)$ or $f(x) > f(y)$, now if f is not strictly monotone, then for some $c \in I$, with $x < c < y$, we have that either $f(x)$ is inbetween $f(c)$ and $f(b)$, or $f(y)$ is between $f(x)$ and $f(c)$, hence, by the intermediate value theorem, there is an $x_1 \in I$ such that $f(x_1) = f(x)$ or $f(x_1) = f(y)$. Then either $x_1 = x$ or $x_2 = y$ which contradicts the assumption.

Now suppos that f is strictly increasing on I, since f is 1-1 ont $\mathbb{R}$, then f^{-1} exists on $f(I)$. Now suppose that there are $y_1, y_2 \in f(I)$ such that $y_1 < y_2$ but $f^{-1}(y_1) \geq f^{-1}(y_2)$, and since f is increasing, then $f(x_1) \geq f(x_2)$, which is absurd. We also have that $f(I)$ is an interval for if $I = [a, b]$, then $f([a, b]) = [f(a)f(b)]$ by the intermediate value theorem.

Now fix $y_0 \in f(I)$, and let $\epsilon > 0$, since f^{-1} is strictly increasing, on $f(I)$ by the above assumption, if y_0 is not a right endpoint of $f(I)$, then $f^{-1}(y_0)$ is not a right endpoint of I, then there is an $0 < \epsilon_0 < \epsilon$ with $\epsilon + \epsilon_0 \in I$. Let $\delta = f(x_0 + \epsilon_0) - f(x_0)$, and suppose that $0 < y - y_0 < \delta$, then $y_0 < y < y_0 + \delta = f(x_0 + \epsilon_0)$, then since f^{-1} is strictly increasing, it follows that $x_0 < x < x + \epsilon_0$, hence we have that $0 < f^{-1}(y) - f^{-1}(y_0) < \epsilon$, that is $f^{-1}(y_0+)$ exists, similatly, if y_0 is not a left endpoint of I, then $f^{-1}(y_0-)$ exists. In both cases, we have that $f^{-1}(y_0\pm) = f(y_0)$. Therefore, f^{-1} is contiuous. ■

Theorem 4.4.3 (The Inverse Function Theorem)**.** *Let f be a 1-1 continuous function of an open interval I onto $\mathbb{R}$. If $a \in f(I)$, and if $f'(f^{-1}(a)) \neq 0$ exists, then f^{-1} is differentiable at a, and*

$$(f^{-1})' = \frac{1}{f'} \tag{4.5}$$

Proof. We have that $f(f(^{-1})(x)) = x$, if f and f^{-1} are both differentiable, then the chain rule yields the appropriate result. It remains to show that f^{-1} is differentiable.

By the previous theorem, we have that f is strictly monotonic, and say, without loss of generalilty, that f is strictly increasing. Then f^{-1} exists, is continuous, and is strictly increasing on $f(I)$. Let $x_0 = f^{-1}(a)$, then there are $c, d \in I$ such that $x_0 \in (c, d) \subseteq I$, thus by the intermediate value theorem, we have that $f((c, d)) = (f(c), f(d))$ which contains $f(x_0) = a$. Then for $h \neq 0$ sufficiently small $a + h \in (f(c), f(d))$, hence $f^{-1}(a + h)$ is well defined. Now let $x = f^{-1}(a+h)$, then $f(x) = a+h = f(x_0)+h$, thus $h = f(x)-f(x_0)$. Since f is continuous, we have that as $x \to x_0$, $h \to 0$, hence $x - x_0 = f^{-1}(a + h) - f^{-1}(a) \to 0$, hence $\frac{f^{-1}(a+h)-f^{-1}(a)}{h} = \frac{x-x_0}{f(x)-f(x_0)}$ which exists by the continuity of f, thus f^{-1} is differentiable at a. ■

Example 4.7. Take $f(x) = 1$ for $x \in \mathbb{Q}$ and $f(x) = 0$ for $x \in \mathbb{R}\backslash\mathbb{Q}$, which is not continuous at any x, also notice that x is not monotonic.

Lemma 4.4.4. *Suppose that f is monotonic on $[a, b]$, if $x_0 \in [a, b)$, then $f(x_0+)$ exists and $f(x_0) \leq f(x_0+)$ or $f(x_0) \geq f(x_0+)$ (if it is increasing or decreasing respectively). Moreover, if $x_0 \in (a, b]$, then $f(x_0-)$ exists and $f(x_0-) \leq f(x_0)$ or $f(x_0-) \geq f(x_0)$.*

Proof. Fix $x_0 \in [a, b]$, by the symmetry, it suffices to show that $f(x_0-)$ exists and that $f(x_0-) \leq f(x_0)$ if f is increasing. Set $E = \{f(x) : a < x < x_0\}$ and let $s = \sup E$. We have $f(x_0)$ is an upeprbound, thus s is finite, and $s \leq f(x_0)$. It remains to show that $s = f(x_0-)$.

Now given $\epsilon > 0$, by the approximation property, there is an $x_1 \in (a, x_0)$ such that $s - \epsilon < f(x) \leq s$. Since f is increasing, then we have $s - \epsilon < f(x_1) < f(x) \leq s$, so choose $\delta = x_0 - x_1 > 0$, then for $-\delta < x - x_0 < 0$, we have that $|f(x) - s| < \epsilon$, and we are done. ■

Theorem 4.4.5. *If f is monotonic on an interval I, then f has at most countably many discontinuities on I.*

Proof. Suppose that f is monotonically increasing, without loss of generality. We know that the countable union of atmost countable sets is countable. It suffices to show that the set of all discontinuities of f is a coutnable union of atmost countable sets.

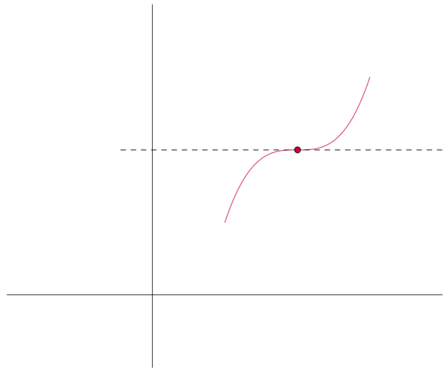

Figure 4.2: The intermediate value theorem for derviavtives.

Notice that $\mathbb{R} = \bigcup_{n=1}^{\infty} [-n, n]$, now let I be a closed, bounded, interval $[a, b]$, and let E be the set of all discontinuities of f on (a, b). By lemma 4.4.4, we have that $f(x-) \leq f(x) \leq f(x+)$, then we see that f is discontinuous at x if and only if $0 < f(x+) - f(x-)$. Let $A_j = \{x \in I : f(x+) - f(x-) \geq \frac{1}{j}\}$, then $A_j \subseteq E$, and we have $E = \bigcup_{j=1}^{\infty} A_j$. We would like to show that A_j is finite for all j.

For suppose not, then there is a j_0 such that A_{j_0} is infinite, then pick $a \leq x_1 < x_2 < \cdots \leq b \in A_{j_0}$, with $f(x_k+) - f(x_k-) > \frac{1}{j_0}$ for all k. Then $f(b) - f(a) \geq f(b) - f(x_1-) \geq f(x_1+) - f(x_1-) > \frac{1}{j_0} \geq f(x_2+) - f(x_1-) = f(x_2+) - f(x_2-) + f(x_2-) - f(x_1-) > \frac{1}{j_0}$, continuing along this line, we get that $f(b) - f(a) \geq \frac{n}{j_0}$ for all $n \geq 1$, which implies that $f(b) - f(a)$ is infinite, which is a contradiction. Thus A_j must be finite. ■

Theorem 4.4.6 (The Intermediate Value Theorem for Derivatives)**.** *Suppose that f is differentiable on a closed interval $[a, b]$, where $f'(a) \neq g('b)$. If $y_0 \in \mathbb{R}$ such that $f'(a) < y_0 < f'(b)$, then there is an $x_0 \in (a, b)$ such that $f'(x_0) = y_0$.*

Proof. Suppose that $f'(a) < y_0 < f'(b)$, now suppose without loss of generality that $f'(a) < f'(b)$. Now let $F(x) = f(x) - y_0 x$, for $x \in [a, b]$. We have that F is differentiable on (a, b), then by the extreme value theorem, F has a local maximum, and a local minumum, let $F(x_0)$ is a local minimum on $[a, b]$. We have that $F'(a) = f'(a) - y_0 < 0$, which is decreasing, so for h sufficiently small, we have $F(a + h) - F(a) < 0$, hence $F(a)$ cannot be the minumum, hence $x_0 \neq a$. Similarly, $F'(b) = f'(b) - y_0 > 0$, which is increasing, by similar reasoning, we have that $F(b + h) - F(b) > 0$ for h sufficiently small, hence $F(b)$ cannot be a local maximum, so $x_0 \neq b$. Thus $x_0 \in (a, b)$. Now since $F(x_0)$ is a local minumum, then we have that $F'(x_0) = 0$, hence $f'(x_0) = y_0$ ■

Remark. If in the proof we assume that $f'(a) > y_0 > f'(b)$, then we consider $F(x_0)$ to be a local maximum.

Chapter 5

Integrability on $\mathbb{R}$.

5.1 The Riemann Integral.

Definition. Let $a, b \in \mathbb{R}$, with $a < b$. We define a **partition** of $[a, b]$ to be a set of points $P = \{x_0, x_1, \ldots, x_n\}$ such that $a = x_0 < x_1 < \cdots < x_n = b$. We define the **norm** of a partion $P = \{x_0, x_1, \ldots, x_n\}$ to be the number:

$$||P|| = \max |x_j - x_{j-1}| \tag{5.1}$$

We call a **refiment** of a partition $P = \{x_0, x_1, \ldots, x_n\}$ to be a partition Q of $[a, b]$ such that $P \subseteq P$, and we say that Q is **finer** than P. We say that P and Q are **noncomparable** if neither is a refinment of the other if neither is a refinment of the other..

Example 5.1 (Dyadic Partitions)**.** Let $P_n = \{\frac{j}{2^n} : j = 0, 1, \ldots 2^n\}$, then P_n is a partiton of $[0, 1]$, and P_m is finer than P_n if $m > n$. Dyadic are good choices for partitions.

If we have two partitions P and Q, in general, they may have no relation. But, $P \cup Q$ is a refinment of both P and Q. Now if Q is finer than P, we see that $||Q|| \leq ||P||$, that is the norm is a monotonically decreasing function of x_j.

Definition. Let $a, b \in \mathbb{R}$ with $a < b$ let $P = \{x_0, x_1, \ldots, x_n\}$ be a partition of $[a, b]$, and suppose that $f : [a, b] \to \mathbb{R}$ is bounded. We define the **upper Riemann sum** of f over P to be:

$$U(f, P) = \sum_{j=1}^{n} M_j(f)(x_j - x_{j-1}) \tag{5.2}$$

where $M_j(f) = \sup f(x)$ where $x \in [x_{j-1}, x_j]$.

We define the **lower Riemann sum** of f over P to be:

$$L(f, P) = \sum_{j=1}^{n} m_j(f)(x_j - x_{j-1}) \tag{5.3}$$

Where $m_j(f) = \inf f(x)$ with $x \in [x_{j-1}, x_j]$

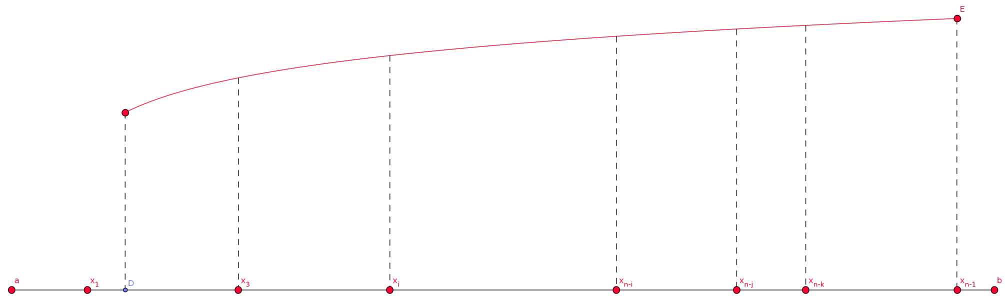

Figure 5.1: The Riemann Sum of a nonnegative bounded curve over the interval $[a, b]$.

Remark. SUppose that $f(x) = \alpha$ is a constant function on $[a, b]$. Then $M_j(f) = m_j(f) = \alpha$, hence we see that

$$U(f, P) = L(f, P) = \alpha \sum_{j=1}^{n} x_j - x_{j-1} = \alpha(b - a)$$

Remark. We have that $L(f, P) \leq U(f, P)$ for all partitions P and bounded functions f, since $\inf f \leq \sup f$, so the relevant equalities follow.

Lemma 5.1.1. *If P is any partition of $[a, b]$, and there is a refinment of P, Q, then $L(f, P) \leq L(f, Q) \leq U(f, Q) \leq U(f, P)$.*

Proof. Assume that $Q = P \cup \{x'\}$, with $x_{j-1} < x' < x_j$. Then $L(f, P) = \sum m_j(f)(x_j - x_{j-a}) + m_j(f)(x_j - x' + x' - x_{j-1}) \leq \sum m_j(f)(x_j - x_{j-1}) + m_{j_0'}(f)(x_{j_0} - x') + m_{j_0''}(f)(x' - x_{j0}) = L(f, Q)$. We see that $inf f$ is increasing while $\sup f$ is decreasing, hence we get that $U(f, Q) \leq U(f, P)$ by similar reasoning, and clearly $L(f, Q) \leq U(f, Q)$. ■

Remark. What lemma 5.1.1 says is that the lower sum is monotonically increasing while the upper sum is monotonically decreasing with respect to the refinment.

Lemma 5.1.2. *If P and Q are any partitions of $[a, b]$, then $L(f, P) \leq U(f, Q)$.*

Proof. For refinment, lower sums are monotonically increasing, and upper sums are monotonically decreasing, and the result follows from lemma 5.1.1. Now suppose that P and Q are noncomparable. Now $P \cup Q$ is a refinment of both P and Q, hence, by direct application of lemma 5.1.1, we have $L(f, P) \leq L(f, P \cup Q) \leq U(f, P \cup Q) \leq U(f, Q)$. ■

Remark. Now we have that with respect to refinment, we have that all upper sums are bounded below, and all lower sums are bounded above, hence the least upper, and greates lower bounds are finite and exist. now when they apporach a certain value, we we obtain the "integral" of f with over the interval $[a, b]$.

Definition. Let $a, b \in \mathbb{R}$, with $a < b$ a realvalued function $f : [a, b] \to \mathbb{R}$ is said to be **Riemann integrable** (or **integrable**) on $[a, b]$ if and only if f is bounded on $[a, b]$, and for every $\epsilon > 0$, there is a partition P of $[a, b]$ such that $U(f, p) - L(f, P) < \epsilon$.

Theorem 5.1.3. *Suppose that $a, b \in \mathbb{R}$, with $a < b$. If f is continuous on $[a, b]$, then f is integrable on $[a, b]$.*

Proof. We know that if a function is continuous over a closed interval, then it is uniformly continuous. Given $\epsilon > 0$, we have that f is uniformly continuous over $[a, b]$, then there is a $\delta > 0$ such that $|f(x) - f(y)| < \frac{}{b-a}$ whenever $|x - y| < \delta$, for $x, y \in [a, b]$. Let $P = \{x_0, x_1, \ldots, x_n\}$ be a partition of $[a, b]$, with $||P|| < \delta$. For the jth subinterval, by the extreme value theorem, tehre are $x_m, x_M \in [x_{j-1}, x_j]$ with $m_j(f) = f(x_m)$ and $M_j(f) = f(x_M)$. Now $|x_M - x_m| < \delta$, so $|f(x_M) - f(x_m)| = M_j(f) - m_j(f) < \frac{\epsilon}{b-a}$. Then we have:

$$U(f, P) - L(f, P) = \sum M_j(f) - m_j(f)(x_j - x_{j-1}) \leq \sum \frac{\epsilon}{b-a} = \frac{\epsilon}{b-a}(b - a) = \epsilon$$

hence, f is integrable. ∎

Example 5.2. (1) The Dirichlet function, $f(x) = \begin{cases} 1 & x \in \mathbb{Q} \\ 0 & x \notin \mathbb{Q} \end{cases}$ is not Riemann integrable on $[0, 1]$. Choose $P = \{x_0, x_1 \ldots, x_n\}$ a partition of $[0, 1]$, for $[x_j, x_{j-1}]$, we have $M_j(f) = 1$ and $m_j(f) = 0$ by definition of f. Then $U(f, P) - L(f, P) = \sum 1 \cdot (x_j - x_{j-1}) + \sum 0 \cdot (x_j - x_{j-1}) = \sum (x_j - x_{j-1}) = 1$, hence.

(2) Constant functions are always Riemann integrable.

(3) Let $f(x) = \begin{cases} 0 & 0 \leq x < \frac{1}{2} \\ 1 & \frac{1}{2} \leq x \leq 1 \end{cases}$. Now given $\epsilon > 0$, construct $P = \{0, \frac{1-\epsilon}{2}, \frac{1+\epsilon}{2}, 1\}$ for $\epsilon < \frac{1}{2}$. Then:

$$U(f, P) - L(f, P) = 1 \cdot (\frac{1+\epsilon}{2} - \frac{1-\epsilon}{2}) = \epsilon \tag{5.4}$$

So, we have that f is integrable is integrable.

Definition. Let $a, b \in \mathbb{R}$ with $a < b$, and let $f : [a, b] \to \mathbb{R}$ be a realvalued function. We define the **upper Riemann intergral** of f over $[a, b]$ to be:

$$\overline{\int_a^b} f(x)dx = \inf\{U(f, P) : P \text{ is a partition of } [a, b]\}. \tag{5.5}$$

Similarly, we define the **lower Riemann integral** of f over $[a, b]$ to be:

$$\underline{\int_a^b} f(x)dx = \sup\{U(f, P) : P \text{ is a partition of } [a, b]\}. \tag{5.6}$$

Lastly, if $\overline{\int_a^b} f = \underline{\int_a^b} f$, then we define the **Riemann Integral** to be:

$$\int_a^b f(x)dx = \overline{\int_a^b} f(x)dx = \underline{\int_a^b} f(x)dx \tag{5.7}$$

Now if $a = b$, we define $\int_a^b f = 0$ for all bounded functions f. If $b < a$, we define $\int_a^b f = -\int_b^a f$.

Now not all bounded functions may be integrable, however, the lower and upper Riemann integrals always exist.

Lemma 5.1.4. $\underline{\int_a^b} f \leq \overline{\int_a^b} f$.

Proof. Applying lemma 5.1.2, we have $L(f, P) \leq U(f, Q)$, now $L(f, P)$ is a lowerbound of all uppersums, hence $L(f, P) \leq \overline{\int_a^b} f$. Now $L(f, P) \leq \overline{\int_a^b} f$ is an upperbound of all lowersums, hence $\underline{\int_a^b} f \leq \overline{\int_a^b} f$ ■

Theorem 5.1.5. *Let $a, b \in \mathbb{R}$ with $a < b$, and let $f : [a, b]\mathbb{R}$ be a bounded realvalued function. Then f is integrable on $[a, b]$ if and only if $\underline{\int_a^b} f = \overline{\int_a^b} f$.*

Proof. Suppose thay f is integrable, then for $\epsilon > 0$, there is a partition P such that $U(f, P) - L(f, P) \leq \epsilon$ whenever $||P|| \leq \delta$. Noq we have that $L(f,) \leq \underline{\int_a^b} f \leq \overline{\int_a^b} f \leq U(f, P)$. So $\overline{\int_a^b} f - \underline{\int_a^b} f \leq U(f, P) - L(f, P) < \epsilon$. Hence $\overline{\int_a^b} f - \underline{\int_a^b} f = 0$.

Conversely, suppose that $\overline{\int_a^b} f - \underline{\int_a^b} f = 0$, then given $\epsilon > 0$, there is a partition P_1 of $[a, b]$, with:

$$\overline{\int_a^b} f \leq U(f, P) < \overline{\int_a^b} f + \frac{\epsilon}{2}$$

Likewise, there is a partition P_2 of $[a, b]$ with:

$$\underline{\int_a^b} f - \frac{\epsilon}{2} < U(f, P) \leq \underline{\int_a^b} f$$

Now let $P = P_1 \cup P_2$, then:

$$\overline{\int_a^b} f - \underline{\int_a^b} f2 < U(f, P) - L(f, P) < \overline{\int_a^b} f - \underline{\int_a^b} f < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

. Hence, f is integrable over $[a, b]$. ■

Theorem 5.1.6. *If $f(x) = \alpha$ is a constant function on $[a, b]$, then $\int_a^b f(x)dx = \alpha(b - a)$ for all $x \in [a, b]$.*

Proof. We have that f is integrable, so $\overline{\int_a^b} f = \underline{\int_a^b} f$, moreover we have $U(f, P) = L(f, P) = \alpha(b - a)$, combining these two, we get the desired result. ■

5.2 Riemann Sums.

Definition. Let $f : [a, b]\mathbb{R}$. We define the **Riemann sum** of F with respect to some partition P of $[a, b]$ to be a sum of the form:

$$\sum_{i=1}^{n} f(t_i)(x_i - t_i - 1) \tag{5.8}$$

where $t_i \in [x_{i-1}, x_i]$. Now we say that the Riemann sums of f with respect to P **converges** to a point $I(f)$ as $||P|| \to 0$ if and only if for $\epsilon > 0$, there is a partition P_ϵ of $[a, b]$ such that:

$$|\sum_{i=1}^{n} f(t_i)(x_i - t_i - 1) - I(f)| < \epsilon$$

whenever $P_\epsilon \subseteq P$, and we write:

$$I(f) = \lim_{||P|| \to 0} \sum_{i=1}^{n} f(t_i)(x_i - t_i - 1) \tag{5.9}$$

Theorem 5.2.1. *Let $a, b \in \mathbb{R}$ with $a < b$, and suppose that $f : [a, b] \to \mathbb{R}$ is a bounded realvalued function. Then f is Riemann integrable if on $[a, b]$ if and only if $I(f)$ exists, moreover:*

$$I(f) = \int_a^b f(x)dx \tag{5.10}$$

Proof. Suppose that f is integrable on $[a, b]$, and let $\epsilon > 0$. Then there is a partition P_ϵ of $[a, b]$ such that $L(f, P_\epsilon) > \int f + \epsilon$ and $U(f, P_\epsilon) < \int f + \epsilon$. By the approximation property, we have that $\int f = \sup L = \inf U$. Now for any partiton P finer than P_ϵ, we have $L(f, P) > L(f, P_\epsilon)$ and $U(f, P) < U(f, P_\epsilon)$, thus let $t_i \in [x_{i-1}, x_i]$

$$\int_a^b f dx - \epsilon < L(f, P) \leq \sum f(t_i)(x_i - x_{i-1}) \leq U(f, P) < \int_a^b f dx + \epsilon$$

Then we get:

$$|\sum f(t_i)(x_i - x_{i-1}) - \int_a^b f dx| < \epsilon$$

so $I(f) = \int f$ exists.

Conversely suppose that the Riemann sums of f converges to $I(f)$. THen for every $\epsilon > 0$, there is a partition P_ϵ such that $|\sum f(t_i)(x_i - x_{i-1}) - I(f)| <$ whenever $P_\epsilon \subseteq P$. Now for each M_i and m_i, there are $u_i, v_i \in [x_{i-1}, x_i]$ such that $M_j \geq f(u_i) > M_i - \epsilon$ and $m_i \leq f(v_j) < m_i + \epsilon$. Then $f(u_i) - f(v_i) > M_i - m_i - 2\epsilon$ hence:

$$\begin{aligned}
U(f, P) - L(f, P) &= \sum_{i=1}^{n} (M_i - m_i)(x_i - x_{i-1}) \\
&<= \sum (f(u_i) - f(v_i))(x_i - x_{i-1}) + 2\epsilon(b - a) \\
&< \sum (f(u_i) - f(v_i))(x_i - x_{i-1}) - I(f) \\
&+ I(f) - \sum (f(u_i) - f(v_i))(x_i - x_{i-1}) + 2\epsilon(b - a) \\
&< (2 + 2(b - a))\epsilon
\end{aligned}$$

Thus, f is Riemann integrable and $I(f) = \int f$. ∎

Theorem 5.2.2 (Linearity). *If f and g are integrable on an interval $[a, b]$, and $\alpha \in \mathbb{R}$, then $f + g$ and αf are integrable, and:*

(1) $\int_a^b f + g dx = \int_a^b f dx + \int_a^b g dx$

(2) $\int_a^b \alpha f dx = \alpha \int_a^b f dx$

Proof. Let $\epsilon > 0$ and let P_ϵ be a partition of $[a, b]$, and for any partition P finer than P_ϵ, an $t_i \in [x_{i-1}, x_]$, we have:

$$|\sum f(t_i)(x_i - x_{i-1}) - \int_a^b f dx| < \epsilon$$

and

$$|\sum g(t_i)(x_i - x_{i-1}) - \int_a^b g dx| < \epsilon$$

Then, adding the inequalities, we have by the triangle inequality:

$$|\sum (f(t_i) + g(t_i))(x_i - x_{i-1}) - \int_a^b f dx - \int_a^b g dx| < 2\epsilon$$

So $I(f + g)$ exists, anf $I(f + g) = I(f) + I(g)$.

The second equality is just part (1) applied α times. ∎

Theorem 5.2.3. *If f is integrable on $[a, b]$, then f is integrable on each sub interval of $[a, b]$, moreover, if $a < c < b$, then:*

$$\int_a^b f dx = \int_a^c f dx + \int_c^b f dx \tag{5.11}$$

Proof. Assume that $a < b$, given $\epsilon > 0$, there is a partition P of $[a, b]$ such that $U(f, P) - L(f, P) < \epsilon$. Then choose $P' = P \cup \{c, d\}$, then let $P_1 = P' \cap [c, d]$ where $c < d$ and $[c, d][a, b]$. Then $U(f, P_1) - L(f, P_1) \leq U(f, P') - L(f, P') \leq U(f, P) - L(f, P)$. Since f is integrable on P, we have that it is integrable over P_1, that is, it is integrable on $[c, d]$.

Moreover let P be a partition of $[a, b]$, and let $P_0 = P \cup \{c\}$, and let $P_1 = P_0 \cap [a, c]$ and $P_2 = P_0 \cap [c, b]$. Then $P_0 = P_1 \cup P_2$. Now we have that $U(f, P) \geq U(f, P_0) = U(f, P_0) + U(f, P_1) \geq \overline{\int_a^c} f + \overline{\int_c^b} f$, doing the same for lower sums, we get

$$\underline{\int_a^c} f + \underline{\int_c^b} f \leq \int_a^b f \leq \overline{\int_a^c} f + \overline{\int_c^b} f$$

Which establishes the equality. ∎

Theorem 5.2.4 (The comparison theorem). *If f and g are integrable on an interval $[a, b]$, and $f \leq g$ for all $x \in [a, b]$, then:*

$$\int_a^b f dx \leq \int_a^b g dx \tag{5.12}$$

in particular, if f is bounded between m and M for all $x \in [a,b]$, then we have:

$$m(b-a) \leq \int_a^b f dx \leq M(b-a) \tag{5.13}$$

Proof. Let P be a partition of $[a,b]$, since $f \leq g$, then we know that $M_i(f) \leq M_i(g)$, then it follows that $U(f,P) \leq U(g,P)$, hence:

$$\overline{\int} f \leq U(g,P)$$

taking infimums, we get:

$$\int f \leq \int g$$

Moreover, if $m \leq f \leq M$, we just take intergral of m, f and M to get the desired result. ■

Theorem 5.2.5. *If f is integrable on an $[a,b]$, then $|f|$ is integrable on $[a,b]$, and:*

$$|\int_a^b f dx| \leq \int_a^b |f| dx \tag{5.14}$$

Proof. If we know that $|f|$ is integrable, then this reduces to a corollary of theorem 5.2.4, since $-|f| \leq f \leq |f|$ for all $x \in [a,b]$. That is:

$$-\int |f| \leq \int f \leq \int |f|$$

It remains to show that $|f|$ is integrable. Let P be a partition of $[a,b]$. We claim that $M_j(|f|) - m_i(|f|) \leq M_i(f) - m_i(f)$ for all $1 \leq i \leq n$. Let $x, y \in [x_{i-1}, x_i]$, Now if $f(x), f(y) > 0$, then $|f(x)| - |f(y)| = f(x) - f(y) \leq M_i(f) - m_i(f)$, and we have $M_i(|f|) - m_i(|f|) \leq M_i(f) - m_i(f)$.

The proof is similar for the cases of $f(x(, f(y) \leq 0$, and $f(x) > 0$ and $f(y) < 0$ (and vice versa). Now given $\epsilon > 0$, and a partiton P of $[a,b]$ such that $U(|f|, P) - L(|f|, P) \leq U(f,p) - L(f,P) < \epsilon$, by definition, thus $|f|$ is integrable on $[a,b]$. ■

Theorem 5.2.6. *If f and g are integrable on an interval $[a,b]$, then fg is also integrable on $[a,b]$.*

Proof. We know that $fg = \frac{(f+g)^2 - f^2 - g^2}{2}$, we have that $f+g$ is integrable, and we would like to show that f^2 (consecuently g^2) is integrable.

We have that $M_i(f^2) = (M_i(|f|))^2$, $m_i(f^2) = (m_i(|f|))^2$. So $M_i(f^2) - m_i(f^2) = M_i(|f|)^2 - m_{i(}|f|)^2 = (M_i(|f|) - m_{i(}|f|))(M_i(|f|) + m_{i(}|f|) \leq 2M(M_i(|f|) - m_i(|f|))$ where $M - \sup |f$ for all $x \in [a,b]$. Then taking the upper and lower sums on this inequality, we get:

$$U(f^2, P) - L(f^2, P) \leq 2M(U(f,P) - L(f,P)) < 2M\epsilon.$$

Taking ϵ small enough, then f^2 is integrable. Hence the implication that fg is integrable follows. ■

Theorem 5.2.7 (The first mean value theorem for Integrals)**.** *Suppose that f and g are integrable on an interval $[a,b]$, with $g \geq 0$ for all $x \in [a,b]$. If $m = \inf f$ and $M = \sup f$ for all $x \in [a,b]$, then there is a number $c \in [m,M]$such that:*

$$\int_a^b fg(x)dx = c\int_a^b g(x)dx \tag{5.15}$$

in particular, if f is continuous on $[a,b]$, then there is an $x_0 \in [a,b]$ satisfying:

$$\int_a^b fg(x)dx = f(x_0)\int_a^b g(x)dx \tag{5.16}$$

Proof. We have that $m \leq f \leq M$ for all $x \in [a,b]$, and we have that $g \geq 0$. So we have that $mg \leq fg \leq Mg$. Notice that all three functions are integrable, then by the comparison theorem, we have:

$$m\int g \leq \int fg \leq M\int g$$

If $\int_a^b g = 0$, then $\int_a^b fg = 0$ by the Squeeze theorem, and any $c \in [m,M]$ is sufficient.

If $\int g \neq 0$, define $c = \frac{\int fg}{\int g}$, then we have $m\int g \leq c\int g \leq M\int g$. Then we have (since $\int g > 0$) that $m \leq c \leq M$.

Now if f is continuous on $[a,b]$, then there is an $x_0 \in [a,b]$ for which $f(x_0) = c$. ■

Bibliography

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