Algebraic Geometry.

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Contents

Chapter 1

Affine Algebraic Sets

1.1 Affine n-Space and Algebraic Sets

Definition. Let k be a field. We define **affine n-space** over k to be the cartesian product $\mathbb{A}^n(k) = \underbrace{k \times \cdots \times k}_{n\text{-times}}$. If the field k is understood, we write $\mathbb{A}^n$. We call the elements of $\mathbb{A}^(k)$ **affine points**. We call $\mathbb{A}^1(k)$ and $\mathbb{A}^2(k)$ the **affine line** and **affine plane** over k, respectively.

Definition. Let k be a field, and let $f \in k[x_1, \ldots, x_n]$. We call an affine point $P \in \mathbb{A}^n(k)$ a **zero**, or **root** of f if $f(P) = 0$, where $f(P)$ is understood to be $f(a_1, \ldots, a_n)$, where $P = (a_1, \ldots, a_n)$. We call the set of zeros of f, $V(f)$ the **hypersurface** defined by f. We call hypersurfaces in $\mathbb{A}^2(k)$ **affine plane curves**. If $\deg f = 1$, we call $V(f)$ a **hyperplane**. We call hypersurfaces in $\mathbb{A}^1(k)$ **lines**.

Example 1.1. The following curves in figure 1.1 define algebraic sets.

Definition. Let k be a field, and S any set of polynomials in $k[x_1, \ldots, x_n]$. We define the **set of zeros** of S to be the set $V(S) = \{P \in \mathbb{A}^n(k) : f(P) = 0 \text{ for all } f \in S\}$. We call a subset X of $\mathbb{A}^n(k)$ an **affine algebraic set** if $X = V(S)$ for some set S of polynomials.

Lemma 1.1.1. *The following are true for any field k.*

(1) If $\mathfrak{a}$ is an ideal in $k = [x_1, \ldots, x_n]$ generated by a set $S \subseteq k[x_1, \ldots, x_n]$, then $V(\mathfrak{a}) = V(S)$.

(2) If $\{\mathfrak{a}_\alpha\}$ is a collection of ideals of $k[x_1, \ldots, x_n]$, then

$$V\Big(\bigcup \mathfrak{a}_\alpha\Big) = \bigcap V(\mathfrak{a}_\alpha)$$

(3) If $\mathfrak{a} \subseteq \mathfrak{b}$ are ideals, then $V(\mathfrak{b}) \subseteq V(\mathfrak{a})$.

(4) If $f, g \in k[x_1, \ldots, x_n]$, then $V(fg) = V(f) \cup V(g)$.

(5) $V(0) = \mathbb{A}^n(k)$ and $V(1) = \emptyset$.

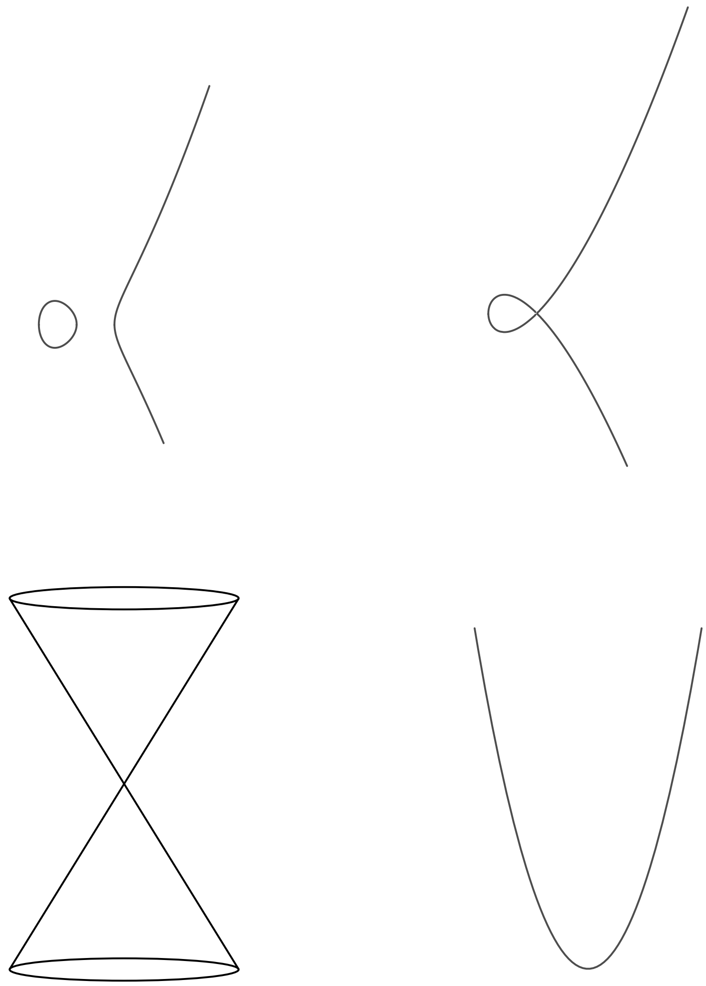

Figure 1.1: Affine Algebraic Sets in $\mathbb{A}^2(\mathbb{R})$ and $\mathbb{A}^3(\mathbb{R})$.

Proof. First, let S be a set of polynomials in $k[x_1, \ldots, x_n]$. Let $\mathfrak{a} = (S)$ the ideal generated by S. Then if $f \in S$ is a polynomia, $f \in I$. Then if $P \in \mathbb{A}^n$ is a zero of f in S, it is a zero of f in $\mathfrak{a}$, hence $V(S) \subseteq V(\mathfrak{a})$. Conversely, we have that if $f \in \mathfrak{a}$, then by suppostion, $f(x_1, \ldots, x_n) = f_1(x_1, \ldots, x_n) + \cdots + f_n(x_1, \ldots, x_n) + \ldots$. Now, if $f(P) = 0$ in I, then we have $f_i(P) = 0$ for every i. This makes $f(P) = 0$ in S, so that $V(\mathfrak{a}) \subseteq V(S)$.

Now, consider the collection $\{\mathfrak{a}_\alpha\}$ of ideals in $k[x_1, \ldots, x_n]$. Let $P \in V(\bigcup \mathfrak{a}_\alpha)$. Then for every $f \in \bigcup \mathfrak{a}_\alpha$, $f(P) = 0$ for each α. So that $P \in \bigcap V(\mathfrak{a}_\alpha)$. Again, on the otherhand, if $P \in \bigcap V(\mathfrak{a}_\alpha)$, $P \in V(\mathfrak{a}_\alpha)$ for all α so that $P \in V(\bigcup \mathfrak{a}_\alpha)$.

Let $\mathfrak{a}$ and $\mathfrak{b}$ ideals in $k[x_1, \ldots, x_n]$, where $\mathfrak{a} \subseteq \mathfrak{b}$. Let $P \in V(\mathfrak{b})$. Then for every polynomial $f \in \mathfrak{b}$, $f(P) = 0$, so that $f(P) = 0$ when $f \in \mathfrak{a}$, hence $P \in V(\mathfrak{a})$. This makes $V(\mathfrak{b}) \subseteq V(\mathfrak{a})$.

Consider now the polynomials $f, g \in k[x_1, \ldots, x_n]$. Certainly if $P \in V(fg)$ it is a root of fg; ie.e. $fg(P) = 0$. This makes $f(P) = 0$ or $g(P) = 0$ so that $V(fg) \subseteq V(f) \cup V(g)$. On the otherhand if P is a root of f, or a root of g, it is a root of fg making $V(f) \cup V(f) \subseteq V(fg)$, and equality is established.

Finally, observe that the zero polynomial $0(x_1, \ldots, x_n)$ has all its coefficients 0, so that any point $P \in \mathbb{A}^n$ is a zero. This makes $V(0) = \mathbb{A}^n$. Likewise, the constant polynomial

$1(x_1,\dots,x_n)$ has its 0-th coefficient 1 so that it has not points $P \in \mathbb{A}^n$ as roots. That is $V(1) = \emptyset$. ■

Corollary. *Finite unions of algebraic sets are algebraic.*

Example 1.2. (1) Let k be a field, and consider $\mathbb{A}^1(k)$. Let $f \in k[x]$ be a polynomial of degree n. Then f has at most n roots in k. Now, if $\mathfrak{a}$ is an ideal in k, since k is a PID, we also get $\mathfrak{a} = (f)$ for some $f \in k[x]$. That is $|V(\mathfrak{a})| \leq n$, and so any algebraic set in $\mathbb{A}^1(k)$ is necessarily finite, except, possibly $\mathbb{A}^1(k)$.

(2) Let k be a finite field with p^m elements, where $p, m \in \mathbb{Z}^+$ and p is prime. Then k is the splitting field of the polynomial $f(x_n) = x_n^{p^m} - x_n$ over the finite field $\mathbb{F}_p$. Suppose then that there is no set S of polynomials in $k[x_1,\dots,x_n]$ for which $X = V(S)$, for some $X \in \mathbb{A}^n(k)$. Choose then a point $P \in X$ and a polynomial $g \in S$. Then we have $g(x_1,\dots,x_n) = g_1(\tilde{X})x_n + \cdots + g_n(\tilde{X})x_n$. Notice that if P is a root of f; i.e. $P \in V(f)$; i.e. $P^{p^m} - P = 0$, then since $P^{p^m} - P$ is a generator for k as a multiplicative group, it generates S. That is, S must contain the point P as a root for g, notice $P^{p^m} = P$ so that $g(P) = g_1(P)P + \cdots + g_n(P)P = 0$ in k. This contradicts that $X \neq V(S)$. This makes every set of $\mathbb{A}^n(k)$ algebraic for any finite field.

(3) By the corollory to lemma 1.1.1, we have that finite unions of algebraic sets are algebraic. Now, consider the field $\mathbb{Q}$, and let $f_q(x) = x + \frac{q}{2}$ in $\mathbb{Q}[x]$. We have that there are $X \subseteq \mathbb{A}^1(\mathbb{Q})$ algebraic, ini where $X = V(f_q)$. Notice however, that the polynomial

$$f(x) = \prod_{q \in \mathbb{Q}} f_q(x)$$

has no roots in $\mathbb{Q}$, as that would imply that for some $n \in \mathbb{Z}^+$, $\sqrt[n]{2} \in \mathbb{Q}$. That is, there is no $X \subseteq \mathbb{A}^1(\mathbb{Q})$ for which $X = V(\prod f_q) = \bigcup V(f_q)$. In general, the countable union of algebraic sets need not be algebraic.

Example 1.3. (1) Let k be a field, and $X = \{(t, t^2, t^3) \in \mathbb{A}^3(k) : t \in k\}$. If k is finite, this is algebraic. Suppose that k is infinite, and consider the polynomial $f(x_1, x_2, x_3) = x_1 + x_2^2 + x_3^3$. Notice that the point $0 \in X$ is a root of f, and that if P is a root of f, then $P \in X$. That is, $X = V(f)$ making X algebraic.

(2) Let $X = \{(\cos t, \sin t) \in \mathbb{A}^2(\mathbb{R}) : t \in \mathbb{R}\}$. Consider the polynomial $f(x, y) = x^2 + y^2 - 1$. Since we have that $\cos^2 t + \sin^2 t = 1$, $X = V(f)$ and X is algebraic.

(3) Let $X = \{(r, \sin t) \in \mathbb{A}^2(\mathbb{R}) : r = \sin t, t\, in \mathbb{R}\}$. Consider the polynomial $f(x, y) = x - y$. Then $X = V(f)$.

The following sets are not algebraic.$X = \{(x, y) \in \mathbb{A}^2(\mathbb{R}) : y = \sin x\}$. $X = \{(z, w) \in \mathbb{A}^2(\mathbb{C}) : |z|^2 + |w|^2 = 1\}$, where $|x + iy|^2 = x^2 + y^2$ for all $x, y \in \mathbb{R}$.

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