

Topology

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Chapter 1

Topological Spaces and Continuous Functions.

1.1 Topological Spaces.

Definition. A **topology** on a set X is a collection \mathcal{T} of subsets of X such that:

- (1) $\emptyset \in \mathcal{T}$ and $X \in \mathcal{T}$
- (2) For any collection $\{U_\alpha\}$ of subsets of X , $\bigcup_\alpha U_\alpha \in \mathcal{T}$.
- (3) For any finite collection $\{U_i\}_{i=1}^n$ of subsets of X , $\bigcap_{i=1}^n U_i \in \mathcal{T}$.

We call the pair (X, \mathcal{T}) a **topological space**, and we call the elements of \mathcal{T} **open sets**.

Example 1.1. (1) Let X be any set, the collection of all subsets of X , 2^X is a topology on X , which we call the **discrete topology**. We call the topology $\mathcal{T} = \{\emptyset, X\}$ the **indiscrete topology**.

- (2) The set of three points $\{a, b, c\}$ has the 9 following topologies in figure 1.1.

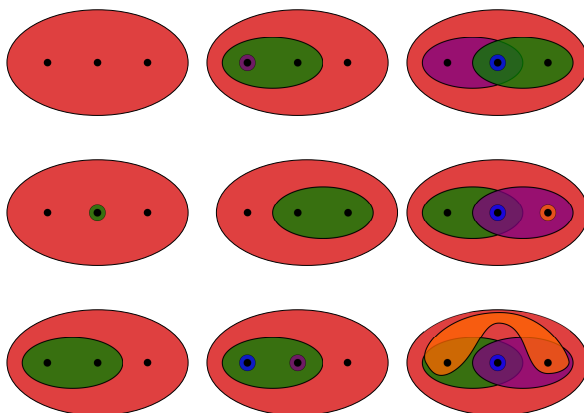


Figure 1.1: The Topologies on $\{a, b, c\}$.

- (3) Let X be any set, and let $\mathcal{T}_f = \{U \subseteq X : X \setminus U \text{ is finite, or } X \setminus U = X\}$. Then \mathcal{T}_f is a topology and called the **finite complement topology**.
- (4) Let X be any set, and let $\mathcal{T}_c = \{U \subseteq X : X \setminus U \text{ is countable, or } X \setminus U = X\}$. Then \mathcal{T}_c is a topology on X called the **countable complement topology**.
- (5) Let X be any set and consider the collection $\mathcal{T}_\infty = \{U \subseteq X : X \setminus U = \emptyset, X \setminus U = X, \text{ or } X \setminus U \text{ is infinite}\}$. Certainly, we have that $\emptyset, X \in \mathcal{T}_\infty$ as $X \setminus X = \emptyset$ and $X \setminus \emptyset = X$. However, if $X = \mathbb{R}$, and we have the sets $(0, 1)$, and $(-\infty, 0) \cup (1, \infty)$, then $\mathbb{R} \setminus (0, 1) = (-\infty, 0] \cup [1, \infty)$, and $\mathbb{R} \setminus ((-\infty, 0) \cup (1, \infty)) = [0, 1]$, both of which are infinite, but, $[0, 1] \cap ((-\infty, 0] \cup [1, \infty)) = \{0, 1\}$, which is finite. So it is not true in general that the collection \mathcal{T}_∞ is a topology.

Lemma 1.1.1. *Let X be a topological space. If $A \subseteq X$ is such that for each $x \in A$, there exists an open set U with $x \in U \subseteq A$, then A is also open in X .*

Proof. We have that for each $x \in A$, there is an open set U_x such that $x \in U_x \subseteq A$. Now, let $U = \bigcup_{x \in A} U_x$, which is open in X by definition. Then, we have $U \subseteq A$ by hypothesis; moreover, since $x \in A$ implies $x \in U_x$, then $x \in U$. This makes $A \subseteq U$, so $A = U$. ■

Definition. Let X be a set, and let \mathcal{T} and \mathcal{T}' be topologies on X . We say that \mathcal{T} is **coarser** than \mathcal{T}' , and \mathcal{T}' **finer** than \mathcal{T} if $\mathcal{T} \subseteq \mathcal{T}'$. If two topologies are either coarser, or finer than each other, we call them **comparable**.

Example 1.2. The topologies \mathcal{T}_f and \mathcal{T}_c are comparable, and we see that $\mathcal{T}_f \subseteq \mathcal{T}_c$, so \mathcal{T}_f is coarser than \mathcal{T}_c , and \mathcal{T}_c is finer than \mathcal{T}_f .

Lemma 1.1.2. *If $\{\mathcal{T}_\alpha\}$ is a collection of topologies on a set X , then the intersection of all \mathcal{T}_α , $\bigcap \mathcal{T}_\alpha$ is also a topology on X .*

Proof. Let $\mathcal{T} = \bigcap \mathcal{T}_\alpha$. We have that $\emptyset, X \in \mathcal{T}_\alpha$ for each α , so that $\emptyset, X \in \mathcal{T}$.

Now let $\{U_\alpha\}$ be a collection of open sets such that $U_\alpha \in \mathcal{T}_\alpha$ for each α . Then $U_\alpha \in \mathcal{T}$, for each α , so that $\bigcup U_\alpha \in \mathcal{T}$. Lastly, take a finite subcollection $\{U_i\}_{i=1}^n$ of $\{U_\alpha\}$, then $\bigcap_{i=1}^n U_i \in \mathcal{T}$ by similar reasoning. ■

Example 1.3. If X is any set, and $\{\mathcal{T}_\alpha\}$ is a collection of topologies in X , it is not in general true that $\bigcup \mathcal{T}_\alpha$ is also a topology on X . Consider the 9 topologies on the set $X = \{a, b, c\}$ in the preceding examples. Let $\mathcal{T}_1 = \{\emptyset, \{a\}, \{a, b\}, X\}$, $\mathcal{T}_2 = \{\emptyset, \{a, b\}, \{b\}, \{b, c\}, \{c\}, X\}$, and let $\mathcal{T}_3 = \mathcal{T}_1 \cup \mathcal{T}_2$. The sets $\{a\}$ and $\{c\}$ have union $\{a, c\}$, however, $\{a, c\} \notin \mathcal{T}_3$.

1.2 The Basis and Subbasis for a Topology.

Definition. If X is a set, the **basis** for a topology on X is a collection \mathcal{B} of subsets of X , called **basis elements**, such that:

- (1) For every $x \in X$, there is a $B \in \mathcal{B}$ such that $x \in B$.
- (2) For $B_1, B_2 \in \mathcal{B}$, if $x \in B_1 \cap B_2$, then there is a $B_3 \in \mathcal{B}$ such that $x \in B_3 \subseteq B_1 \cap B_2$

We define the topology \mathcal{T} **generated** by \mathcal{B} to be collection of open sets: $\mathcal{T} = \{U \subseteq X : \text{for all } x \in U, \text{ there exists a } B \in \mathcal{B} \text{ such that } x \in B\}$.

Theorem 1.2.1. *Let X be a set, and \mathcal{B} a basis of X , then the collection of subsets of X , $\mathcal{T} = \{U \subseteq X : \text{for all } x \in U, \text{ there exists a } B \in \mathcal{B} \text{ such that } x \in B\}$ is a topology on X .*

Proof. Let \mathcal{B} be a basis for a topology in X , and consider \mathcal{T} as defined above. Clearly, $\emptyset \in \mathcal{T}$ and so is X .

Now let $\{U_\alpha\}$ be a collection of subsets of X , and let $U = \bigcup U_\alpha$. Then if $x \in U$ for some α , there is a B_α such that $x \in B_\alpha \subseteq U_\alpha$, thus $x \in B_\alpha \subseteq U$.

Now let $x \in U_1 \cap U_2$, and choose $B_1, B_2 \in \mathcal{B}$ such that $x \in B_1 \subseteq U_1$ and $x \in B_2 \subseteq U_2$. Then by definition, there is a B_3 for which $x \in B_3 \subseteq B_1 \cap B_2$. Now suppose for arbitrary n , that $U = \bigcap_{i=1}^n U_i \in \mathcal{T}$, for some finite collection $\{U_i\}$ of subsets of X . Then by let $B_n, B_{n+1} \in \mathcal{B}$ such that $x \in B_n \subseteq U$ and $x \in B_{n+1} \subseteq U_{n+1}$. Then by our hypothesis, there is a B for which $x \in B \subseteq B_n \cap B_{n+1}$, thus $U \cap U_{n+1} = \bigcap_{i=1}^{n+1} U_i \in \mathcal{T}$. This make \mathcal{T} a topology on X . ■

Example 1.4. (1) Let \mathcal{B} be the set of all circular regions in the plane $\mathbb{R} \times \mathbb{R}$, then \mathcal{B} satisfies the conditions needed for a basis.

(2) The collection \mathcal{B}' in $\mathbb{R} \times \mathbb{R}$ of all rectangular region also forms a basis for a topology on $\mathbb{R} \times \mathbb{R}$.

(3) For any set X , the set of all 1-point subsets of X forms a basis for a topology on X .

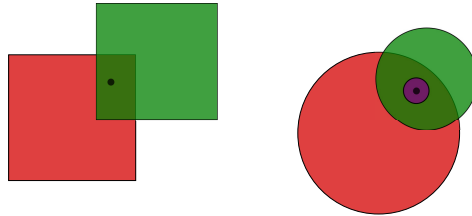


Figure 1.2: The basis for \mathcal{B} and \mathcal{B}' in $\mathbb{R} \times \mathbb{R}$ (see example (2)).

Lemma 1.2.2. *Let X be a set, and \mathcal{B} be a basis for a topology \mathcal{T} on X . Then $\mathcal{T} = \{\bigcup B : B \in \mathcal{B}\}$.*

Proof. Given a collection $\{B_i\}_{i=1}^\infty$ of basis elements in \mathcal{B} , since they are all in \mathcal{T} , their unions are also in \mathcal{T} . Conversely, given $U \in \mathcal{T}$, then for every point $x \in U$, choose a $B_x \in \mathcal{B}$ such that $x \in B_x \subseteq U$, then $U = \bigcup_{x \in U} B_x$. ■

Lemma 1.2.3. *Let (X, \mathcal{T}) be a topological space, and let $\mathcal{C} \subseteq \mathcal{T}$ be a collection of open sets of X such that for every $x \in U$, there is a $C \in \mathcal{C}$ such that $x \in C \subseteq U$. Then \mathcal{C} is the basis for a \mathcal{T} on X .*

Proof. Take any $x \in X$, then there is a $C \in \mathcal{C}$ such that $x \in C \subseteq U$, thus the first condition for a basis is satisfied. Now let $x \in C_1 \cap C_2$ for $C_1, C_2 \in \mathcal{C}$, since $C_1 \cap C_2$ is open in X , there is a $C_3 \in \mathcal{C}$ such that $x \in C_3 \subseteq C_1 \cap C_2$. Therefore \mathcal{C} is a basis for a topology on X .

Now let $\mathcal{T}_{\mathcal{C}}$ be the topology generated by \mathcal{C} , now for $U \in \mathcal{T}$, we have by the hypothesis, that $U \in \mathcal{T}_{\mathcal{C}}$; and by lemma 1.2.2, $W \in \mathcal{T}_{\mathcal{C}}$ is the union of elements of \mathcal{C} , which is a subcollection of \mathcal{T} , thus $W \in \mathcal{T}$. Therefore $\mathcal{T}_{\mathcal{C}} = \mathcal{T}$. ■

Lemma 1.2.4. *Let \mathcal{B} and \mathcal{B}' be bases for topologies \mathcal{T} and \mathcal{T}' on X . Then the $\mathcal{T} \subseteq \mathcal{T}'$ if and only if for all $x \in X$, and all $B \in \mathcal{B}$, there is a $B' \in \mathcal{B}'$ such that $x \in B' \subseteq B$.*

Proof. Suppose first that $\mathcal{T} \subseteq \mathcal{T}'$, and let $x \in X$, and choose $B \in \mathcal{B}$ such that $x \in B$, then B is open in \mathcal{T} , thus it is open in \mathcal{T}' , thus there is a $B' \in \mathcal{B}'$ such that $x \in B' \subseteq B$. Conversely, suppose there is a $B' \in \mathcal{B}'$ for which $x \in B' \subseteq B$ for all $x \in X$, $B \in \mathcal{B}$. Take $x \in U \in \mathcal{T}$, since \mathcal{B} generates \mathcal{T} , $x \in B \subseteq U$, since $B' \subseteq B$, this implies that $U \in \mathcal{T}'$ and $\mathcal{T} \subseteq \mathcal{T}'$. ■

Definition. If \mathcal{B} is the collection of open intervals (a, b) in \mathbb{R} , we call the topology generated by \mathcal{B} the **standard topology** on \mathbb{R} , and we denote it simply by \mathbb{R} .

Definition. If \mathcal{B} is the collection of half open intervals $[a, b)$ in \mathbb{R} , we call the topology generated by \mathcal{B} the **lower limit topology** on \mathbb{R} , and we denote it simply by \mathbb{R}_l . If \mathcal{B}' is the collection of all half open intervals $(a, b]$ in \mathbb{R} , then we call the topology generated by \mathcal{B}' the **upper limit topology** on \mathbb{R} , and denote it \mathbb{R}_L .

Definition. If \mathcal{B} is the collection of all open intervals of the form $(a, b) \setminus \frac{1}{\mathbb{Z}^+}$, where $\frac{1}{\mathbb{Z}^+} = \{\frac{1}{n} : n \in \mathbb{Z}^+\}$, we call the topology generated by \mathcal{B} the **$\frac{1}{\mathbb{Z}^+}$ -topology** on \mathbb{R} , and we denote it $\mathbb{R}_{\frac{1}{\mathbb{Z}^+}}$.

Lemma 1.2.5. *The topologies \mathbb{R}_l , \mathbb{R}_L , and $\mathbb{R}_{\frac{1}{\mathbb{Z}^+}}$ are all strictly finer than \mathbb{R} , but are not comparable with each other.*

Proof. Let (a, b) be a basis element for \mathbb{R} , and let $x \in (a, b)$, the basis element $[x, b) \in \mathbb{R}_l$ lies in (a, b) and contains x , however, there can be no interval (a, b) in $[x, b)$ as $x \leq a$, thus $\mathbb{R} \subset \mathbb{R}_l$; a similar argument holds for \mathbb{R}_L .

Similarly, for $(a, b) \in \mathbb{R}$, the basis element $(a, b) \setminus \frac{1}{\mathbb{Z}^+}$ of $\mathbb{R}_{\frac{1}{\mathbb{Z}^+}}$ lies in (a, b) , however, choose the basis $B = (-1, 1) \setminus \frac{1}{\mathbb{Z}^+}$, and choose $0 \in B$, since \mathbb{Z}^+ is dense in \mathbb{R} , there is no interval (a, b) containing 0 and lying in B , thus $\mathbb{R} \subseteq \mathbb{R}_{\frac{1}{\mathbb{Z}^+}}$.

Now choose $[0, 1)$ in \mathbb{R}_l , and choose $\frac{1}{k} \in [0, 1)$ such that $k \in \mathbb{Z}^+$. Now $(0, 1) \subseteq [0, 1)$, so we cannot say that $[0, 1)$ is a basis for \mathbb{R} , and moreover, $[0, 1) \setminus \frac{1}{\mathbb{Z}^+}$ cannot be said to be a basis in $\mathbb{R}_{\frac{1}{\mathbb{Z}^+}}$, thus \mathbb{R}_l and $\mathbb{R}_{\frac{1}{\mathbb{Z}^+}}$ are incomparable, a similar argument holds for \mathbb{R}_L .

Lastly, let (a, b) be in \mathbb{R} and choose $x \in (a, b)$. Then $(a, x]$ and $[x, b)$ are both in (a, b) , however it is clear that $(a, x]$ and $[x, b)$ cannot be contained in each other, thus \mathbb{R}_l and \mathbb{R}_L are incomparable. ■

Definition. A **subbasis**, \mathcal{S} , for a topology on X is a collection of subsets of X whose union equals X . We call the **topology generated by \mathcal{S}** to be the collection of all unions of finite intersections of elements of \mathcal{S} , that is:

$$\mathcal{T} = \left\{ \bigcup_{i=1}^n S_i : S_i \in \mathcal{S} \text{ for } 1 \leq i \leq n \right\}$$

Theorem 1.2.6. *Let \mathcal{S} be a subbasis for a topology on X . Then the collection $\mathcal{T} = \{\bigcup \bigcap_{i=1}^n S_i : S_i \in \mathcal{S} \text{ for } 1 \leq i \leq n\}$ is a topology on X .*

Proof. It is sufficient to show that the collection \mathcal{B} of all finite intersections of elements of \mathcal{S} is a basis for a topology on X . By lemma 1.2.1, for $x \in X$, it belongs to an element S of \mathcal{S} , and therefore, to an element of \mathcal{B} . Now let $B_1 = \bigcap_{i=1}^m S_i$ and $B_2 = \bigcap_{j=1}^n S'_j$ be basis elements of \mathcal{B} . The intersection $B_1 \cap B_2$ is a finite intersection of elements of \mathcal{S} , and hence also belongs in \mathcal{B} , and hence we can take another basis element B_3 such that $x \in B_3 \subseteq B_1 \cap B_2$. ■

1.3 The Order Topology.

Definition. Let X be a set with a simple order relation, and suppose that $|X| > 1$. Let \mathcal{B} be the collection of sets of the following forms:

- (1) All open intervals $(a, b) \in X$.
- (2) All half open intervals $[a_0, b)$ where a_0 is the least element (if any) of X .
- (3) All half open intervals of the form $(a, b_0]$ where b_0 is the greatest element (if any) of X .

Then \mathcal{B} forms the basis for a topology on X called the **order topology**

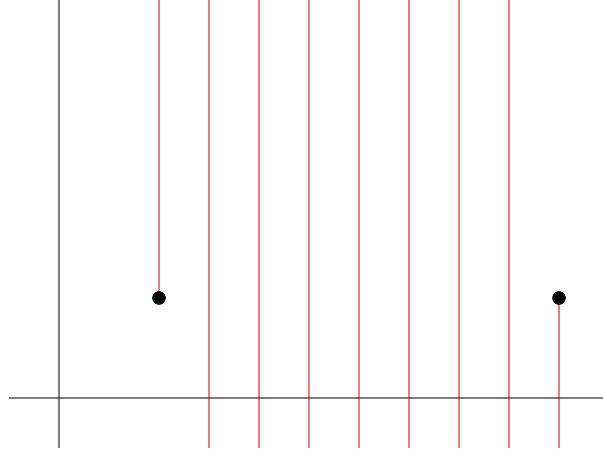
Theorem 1.3.1. *The collection \mathcal{B} forms a basis.*

Proof. Consider $x \in X$, if x is the least element of X , then it lies in all intervals of type (2), if it is the largest, then it lies in all intervals of type (3). If x is neither the least nor largest element, then $x \in (a_0, b_0)$ with a_0 and b_0 the least and largest elements (if any) of X . If no such elements exist, then $x \in (a, b)$, for some lowerbound a and upperbound b . Thus, in all three cases, there is a basis element containing x .

Now suppose $B_1, B_2 \in \mathcal{B}$ such that $x \in B_1 \cap B_2$. If B_1 and B_2 are both of type (1), then let $B_1 = (a, b)$, $B_2 = (c, d)$, then $B_1 \cap B_2$ is an open interval of type (1), now fix B_1 to be of type one. If B_2 is of type (2), then letting $B_2 = [a_0, c)$, then $x \in [a_0, d)$ for some $d \in X$. Likewise, if $B_2 = (c, b_0]$, is of type (3), we get a similar result. Moreover, the results are analogous if we fix B_2 and let B_1 range between intervals of the three types. Thus in all cases, there is a $B_3 \in \mathcal{B}$ such that $x \in B_3 \subseteq B_1 \cap B_2$. ■

Example 1.5. (1) The standard topology on \mathbb{R} is the order topology on \mathbb{R} induced by the usual order relation. We have that \mathbb{R} under this topology has no intervals of type (2), nor (3), so all bases elements in the standard topology are open intervals in \mathbb{R} .

- (2) Consider the dictionary order on $\mathbb{R} \times \mathbb{R}$. Since $\mathbb{R} \times \mathbb{R}$ has no intervals of type (2), nor (3), the bases of $\mathbb{R} \times \mathbb{R}$ under the dictionary order are the open intervals of the form $(a \times b, c \times d)$ Where $a \leq c$, and $b < d$.
- (3) The positive integers \mathbb{Z}^+ with the least element 1 form an ordered set under the usual order. Taking $n > 1$, we see the bases of \mathbb{Z}^+ under the order topology are of the form $(n-1, n+1) = \{n\}$ and $[1, n) = \{1, \dots, n-1\}$. Thus the order topology on \mathbb{Z}^+ is the discrete topology.

Figure 1.3: The order topology on $\mathbb{R} \times \mathbb{R}$.

- (4) The set $X = \{1, 2\} \times \mathbb{Z}^+$ over the dictionary order is also an ordered set, with the least element 1×1 . Denote $1 \times n$ as a_n and $2 \times n$ as b_n . Then X consist of the elements $a_1, a_2, \dots, b_1, b_2, \dots$.

Now take $\{b_1\}$, then any open set containing b_1 must have a basis about b_1 , and also contains points a_i with $i \in \mathbb{Z}^+$; thus the order topology on X is not the discrete topology.

Definition. Let X be an ordered set, and let $a \in X$. There are two subsets in X , $(a, \infty) = \{x \in X : x > a\}$ and $(-\infty, a) = \{x \in X : x < a\}$ called **open rays** of X . There are also two sets $[a, \infty) = \{x \in X : x \geq a\}$ and $(-\infty, a] = \{x \in X : x \leq a\}$ called **closed rays** of X .

Theorem 1.3.2. Let X be an ordered set. Then the collection of all open rays in X form a subbasis for the order topology on X .

Proof. Let \mathcal{S} be the collection of all open rays of X , let $a < b$ and $(a, \infty), (-\infty, b) \in \mathcal{S}$, then $(a, b) = (a, \infty) \cap (-\infty, b)$. Now take:

$$S = \bigcup_{a, b \in X} (a, b)$$

then $S \subseteq X$, likewise, since S runs through all intersections of open rays of X , it contains all open intervals in X , hence $X \subseteq S$, and so $X = S$ as required. ■

1.4 The Product Topology.

Definition. Let X and Y be topological spaces. We define the **product topology** on $X \times Y$ to be the topology having as basis the collection

$$\mathcal{B} = \{U \times V \subseteq X \times Y : U \text{ is open in } X \text{ and } V \text{ is open in } Y\}$$

Theorem 1.4.1. The collection $\mathcal{B} = \{U \times V \subseteq X \times Y : U \text{ is open in } X \text{ and } V \text{ is open in } Y\}$ forms a basis for the product topology on $X \times Y$.

Proof. Clearly, we have that $X \times Y$ is a basis element of \mathcal{B} . Now take $U_1 \times V_1$ and $U_2 \times V_2$ in \mathcal{B} . Since $U_1 \times V_1 \cap U_2 \times V_2 = U_1 \cap U_2 \times V_1 \cap V_2$, since $U_1 \cap U_2$ and $V_1 \cap V_2$ are open in X and Y respectively, then we have that $U_1 \times V_1 \cap U_2 \times V_2$ is a basis element as well. ■

Theorem 1.4.2. *If \mathcal{B} is the basis for a topology on X , and \mathcal{C} is the basis for a topology on Y , then the collection:*

$$\mathcal{D} = \{B \times C : B \in \mathcal{B} \text{ and } C \in \mathcal{C}\}$$

Is a basis for the topology on $X \times Y$.

Proof. By lemma 1.2.3, let W be an open set of $X \times Y$, and let $x \times y \in W$. Then there is a basis $U \times V$ such that $x \times y \in U \times V \subseteq W$. Since \mathcal{B} and \mathcal{C} are bases of X and Y respectively, choosing $B \in \mathcal{B}$ and $C \in \mathcal{C}$, we have that $x \in B \subseteq U$, and $y \in C \subseteq V$, thus $x \times y \in B \times C \subseteq U \times V \subseteq W$. Therefore, \mathcal{D} is the basis for a topology on $X \times Y$. ■

Example 1.6. The product of the standard topology on \mathbb{R} with itself is called the **standard topology on $\mathbb{R} \times \mathbb{R}$** , and has as basis the collection of all products of open sets in \mathbb{R} . By theorem 1.4.2, if we take the collection of all open intervals $(a, b) \times (c, d)$ in $\mathbb{R} \times \mathbb{R}$, we form a basis. Constructing this basis geometrically gives the interior of a rectangle, whose boundaries are the intervals (a, b) and (c, d) .

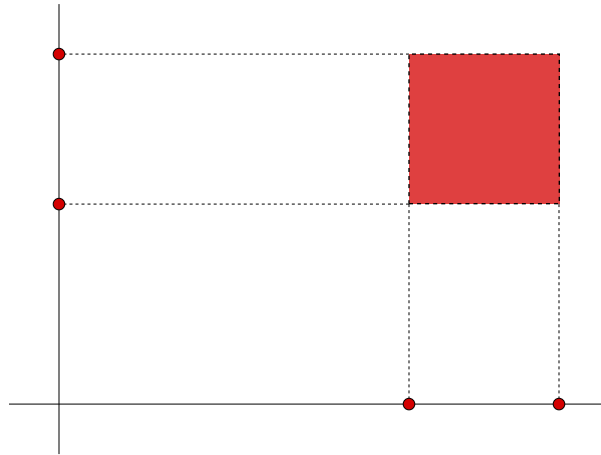


Figure 1.4: A basis element for $\mathbb{R} \times \mathbb{R}$

Definition. Let $\pi_1 : X \times Y \rightarrow X$ be defined such that $\pi_1(x, y) = x$, and define $\pi_2 : X \times Y \rightarrow Y$ such that $\pi_2(x, y) = y$. We call π_1 and π_2 **projections** of $X \times Y$ onto its first and second **factors**; that is onto X and Y , respectively.

Clearly, π_1 and π_2 are both onto. Now let U be open in X , then $\pi_1^{-1}(U) = U \times Y$ is open in $X \times Y$; similarly, $\pi_2^{-1}(V) = X \times V$ is also open in $X \times Y$, for V open in Y .

Example 1.7. The maps $\pi_1 : x \times y \rightarrow x$ and $\pi_2 : x \times y \rightarrow y$ of $X \times Y$ onto X and Y , respectively, are what we call “open maps”. Let U and V be open in X and Y respectively, then $U \times V$ is open in $X \times Y$, and for every $x \times y \in U \times V$, we have $\pi_1(x \times y) = x$ and $\pi_2(x \times y) = y$, so that $\pi_1(U \times V) = U$ and $\pi_2(U \times V) = V$ are open in X and Y respectively.

Theorem 1.4.3. *The collection $\mathcal{S} = \{\pi_1^{-1}(U) : U \text{ is open in } X\} \cup \{\pi_2^{-1}(V) : V \text{ is open in } Y\}$ is a subbasis for the product topology on $X \times Y$.*

Proof. Let \mathcal{T} be the product topology on $X \times Y$, and let \mathcal{T}' be the topology generated by \mathcal{S} . Since every element of \mathcal{S} is open in \mathcal{T} , $\mathcal{T} \subseteq \mathcal{T}'$. Conversely, consider the basis element $U \times V$ of \mathcal{T} , then $\pi_1^{-1}(U) \cap \pi_2^{-1}(V) = U \times Y \cap X \times V = U \times V$, thus $\mathcal{T} \subseteq \mathcal{T}'$. Therefore, \mathcal{S} is a subbasis for the product topology. ■

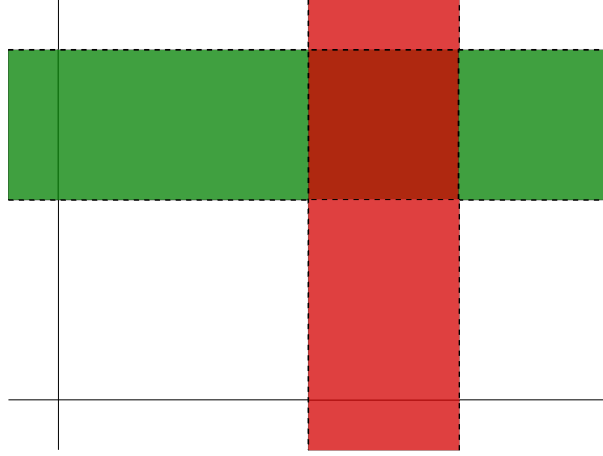


Figure 1.5: The inverse images, $\pi_1^{-1}(U)$ and $\pi_2^{-1}(V)$, of the projections π_1 and π_2 onto the $X \times Y$ plane.

1.5 The Subspace Topology.

Theorem 1.5.1. *Let X be a topological space with topology \mathcal{T} , and let $Y \subseteq X$. Then the collection:*

$$\mathcal{T}_Y = \{Y \cap U : U \in \mathcal{T}\}$$

forms a topology on Y .

Proof. Clearly, $Y \cap \emptyset = \emptyset \in \mathcal{T}_Y$ and $Y \cap X = Y \in \mathcal{T}_Y$. Now consider the collection $\{U_\alpha\}$. Then $\bigcup (Y \cap U_\alpha) = Y \cap \bigcup U_\alpha$, similarly, for $\{U_i\}_{i=1}^n$, $\bigcap (Y \cap U_i) = Y \cap \bigcap U_i$, hence \mathcal{T}_Y is a topology on Y . ■

Definition. Let X be a topological space, and let $Y \subseteq X$. We call the \mathcal{T}_Y defined in theorem 1.5.1 the **subspace topology** on Y . We say that $U \subseteq Y$ is **open in Y** if $U \in \mathcal{T}_Y$.

Lemma 1.5.2. *Let \mathcal{B} be the basis for a topology on X . Then the collection $\mathcal{B}_Y = \{B \cap Y : B \in \mathcal{B}\}$, where $Y \subseteq X$, is a basis for the subspace topology on Y .*

Proof. Let U be open in X , and let $y \in Y \cap U$, and choose $B \in \mathcal{B}$ such that $y \in B \subseteq U$, then $y \in B \cap Y \subseteq U \cap Y$, then by lemma 1.2.2, \mathcal{B}_Y is the basis for the subspace topology on Y . ■

Lemma 1.5.3. *Let Y be a subspace of X , If $U \subseteq Y$ is open in Y and Y is open in X , then U is open in X .*

Proof. Let $U \in \mathcal{T}_Y$, then for some $V \subseteq X$, $U = Y \cap V$. Now since Y is open in X , and so is V , then it follows that U is also open in X . ■

Remark. What this lemma says is that given a topological space X , and a subspace Y of X , then the subspace topology of Y is coarser than the topology on X , i.e. $\mathcal{T}_Y \subseteq \mathcal{T}$.

Theorem 1.5.4. *If A is a subspace of X , and B is a subspace of Y , then the product topology on $A \times B$ is the topology that $A \times B$ inherits as a subspace of $X \times Y$.*

Proof. We have that $U \times V$ is the basis element for $X \times Y$, with U open in X , and V open in Y . Thus $(U \times V) \cap (A \times B) = (U \cap A) \times (V \cap B)$ is a basis element for the subspace topology on $X \times Y$. Since $U \cap A$ and $V \cap B$ are open in the subspace topologies of A and B respectively, then $(U \cap A) \times (V \cap B)$ is a basis for the product topology on $A \times B$. ■

Example 1.8. (1) Consider $[0, 1] \subseteq \mathbb{R}$. In the subspace topology of $[0, 1]$, we have as basis elements of the form $(a, b) \cap [0, 1]$, with $(a, b) \subseteq \mathbb{R}$. If we have that $(a, b) \subseteq [0, 1]$, then $(a, b) \cap [0, 1] = (a, b)$. On the other hand, if $a \in [0, 1]$ or $b \in [0, 1]$, then we get $(a, b) \cap [0, 1] = (a, 1]$ or $(a, b) \cap [0, 1] = [0, b)$, lastly if neither a nor b are in $[0, 1]$, then we have $(a, b) \cap [0, 1] = [0, 1]$ only if $[0, 1] \subseteq (a, b)$, and $(a, b) \cap [0, 1] = \emptyset$ otherwise.

The only one of these sets open in \mathbb{R} under the standard topology is $(0, 1)$.

(2) For $[0, 1) \cup \{2\} \subseteq \mathbb{R}$, the singleton $\{2\}$ is open in the subspace topology on $[0, 1) \cup \{2\}$; for observe, that $(\frac{3}{5}, \frac{5}{2}) \cap ([0, 1) \cup \{2\}) = \{2\}$, however, in the order topology, on that same set, $\{2\}$ is not open. Any basis element on $[0, 1) \cup \{2\}$ containing 2 is of the form $(a, 2]$, where $a \in [0, 1) \cup \{2\}$.

(3) The dictionary order on $[0, 1] \times [0, 1]$ is a restriction of the dictionary order on $\mathbb{R} \times \mathbb{R}$. Now the set $\{\frac{1}{2}\} \times (\frac{1}{2}, 1]$ is open in the subspace topology on $[0, 1] \times [0, 1]$, but it is not open in the dictionary order on the same set.

Definition. We call the set $[0, 1] \times [0, 1]$ on the dictionary order the **ordered square**, and we denote it by I_0^2 .

Lemma 1.5.5. *Let Y be a subspace of a topological space X , and let $A \subseteq Y$. Then the topology of A as a subspace of Y is the same as the topology of A as a subspace of X .*

Proof. Let V be open in Y , then $V \cap A$ is open in A as a subspace of Y , however, since Y is a subspace of X , $V = U \cap Y$ for some U open in X , so $V \cap A = (U \cap Y) \cap A = U \cap (A \cap Y) = U \cap A$, making $V \cap A$ open in A as a subspace of X .

Conversely, if U is open in X , then $U \cap A$ is open in A as a subspace of X , additionally, $U \cap Y$ is open in Y as a subspace of Y , so that $(U \cap Y) \cap A = U \cap (A \cap Y) = U \cap A$, which makes $U \cap A$ open in A as a subspace of Y . ■

Example 1.9. Suppose that \mathcal{T} and \mathcal{T}' are topologies on a set X , with $\mathcal{T} \subseteq \mathcal{T}'$. Then if $Y \subseteq X$, then the topology of Y as a subspace of X under \mathcal{T}' is finer than the topology of Y as a subspace of X under \mathcal{T} ; for, if U is open in X under \mathcal{T} , then $U \cap Y$ is open in Y under \mathcal{T} . But $\mathcal{T} \subseteq \mathcal{T}'$, which implies that $U \cap Y$ is open in Y under \mathcal{T}' .

Definition. Let X be an ordered set. We say that a nonempty subset $Y \subset X$ is **convex** in X if for each pair of points $a, b \in Y$, with $a < b$, then the open interval $(a, b) \subseteq X$ is also contained in Y .

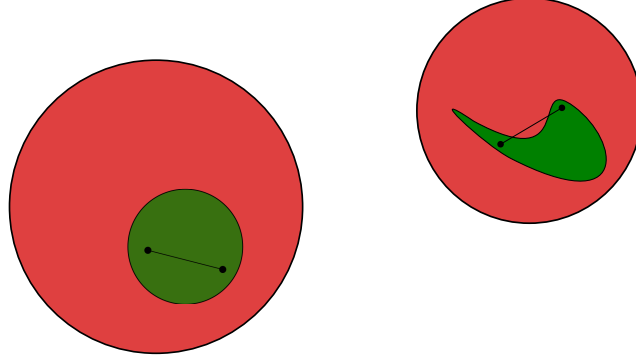


Figure 1.6: A convex set, and a nonconvex set.

Example 1.10. Let X be any ordered set. Then by definition, all open intervals and rays in X are convex in X .

Theorem 1.5.6. Let X be an ordered set on the order topology, and let $Y \subseteq X$ be convex in X . Then the order topology on Y is the same as the subspace topology on Y .

Proof. Consider $(a, \infty) \subseteq X$. If $a \in Y$, then $(a, \infty) \cap Y = \{x \in Y : x > a\}$, which is by definition an open ray on Y . Now if $a \notin Y$, then a is either a lowerbound, or an upperbound. Then $(a, \infty) \cap Y = \emptyset$ and $(-\infty, a) \cap Y = Y$ if a is an upperbound, similarly, if a is a lowerbound we get $(a, \infty) \cap Y = Y$ and $(-\infty, a) \cap Y = \emptyset$.

Since $(a, \infty) \cap Y$ and $(-\infty, a) \cap Y$ form a subbasis on the subspace topology on Y , and since they are also open in the order topology, then the order topology contains the subspace topology.

Now if (a, ∞) is an open ray in Y , then $(a, \infty) = (b, \infty) \cap Y$, with (b, ∞) some open ray in X , hence (a, ∞) is open in the subspace topology of Y , and since it also forms the subbasis for the order topology, we have that the order topology is contained within the subspace topology. Thus both topologies are equal. ■

1.6 Closed Sets and Limit Points.

Definition. A subset A of a topological space X is said to be **closed** if $X \setminus A$ is open.

Example 1.11. (1) Consider $[a, b] \subseteq \mathbb{R}$, we have that $\mathbb{R} \setminus [a, b] = (-\infty, a) \cup (b, \infty)$ which is open in \mathbb{R} . So $[a, b]$ is closed.

(2) In $\mathbb{R} \times \mathbb{R}$, the set $A = \{x \times y : x, y \geq 0\}$ (i.e the first quadrant of the plane) is closed, for $\mathbb{R} \times \mathbb{R} \setminus A = ((-\infty, 0) \times \mathbb{R}) \cup (\mathbb{R} \times (-\infty, 0))$, which is open in $\mathbb{R} \times \mathbb{R}$.

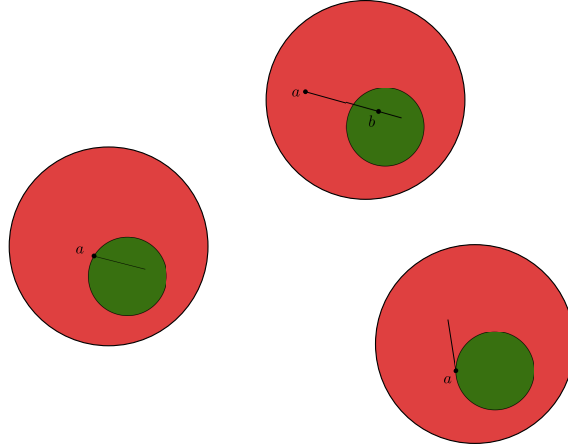


Figure 1.7: An illustration of theorem 1.5.6.

- (3) Consider the finite complement topology \mathcal{T}_f on a set X . We have that $X \setminus X = \emptyset \in \mathcal{T}_f$, so X is closed, similarly, \emptyset is also closed. Likewise, if $A \subseteq X$ is a finite set, then $X \setminus A$ is also finite, and hence A is also closed. Thus, we have that all the closed sets of \mathcal{T}_f are those finite subsets of X . As a consequence, this example also illustrates that sets can be both closed and open.
- (4) In the discrete topology 2^X , every open set is closed. This is another example where open sets are also closed sets.
- (4) Consider $[0, 1] \cup (2, 3)$ in the subspace topology on \mathbb{R} . We have that $[0, 1]$ is open in the subspace topology on \mathbb{R} ; $[0, 1] = [0, 1] \cup (2, 3) \cap (-\frac{2}{3}, \frac{3}{2})$, similarly, $(2, 3)$ is also open. Now taking $[0, 1] \cup (2, 3) \setminus (2, 3) = [0, 1]$, which is open, so $[0, 1]$ is closed in the subspace topology on \mathbb{R} , by the same reasoning, so is $(2, 3)$.

Theorem 1.6.1. *Let X be a topological space. Then:*

- (1) \emptyset and X are closed.
- (2) Arbitrary intersections of closed sets are closed.
- (3) Finite unions of closed sets are closed.

Proof. We have that $X \setminus \emptyset = X$ and $X \setminus X = \emptyset$, both of which are open in X , so they are also closed in X . Now let $\{U_\alpha\}$ be a collection of closed sets of X . We have that:

$$X \setminus \bigcap_{\alpha} U_{\alpha} = \bigcup_{\alpha} X \setminus U_{\alpha}.$$

Similarly, for $\{U_i\}_{i=1}^n$, we have

$$X \setminus \bigcup_{i=1}^n U_i = \bigcap_{i=1}^n X \setminus U_i.$$

Both of which are open in X . This completes the proof. ■

Example 1.12. For the following examples, let X and Y be topological spaces.

- (1) If Y is a subspace of X , and A is closed in Y , and Y is closed in X , then by theorem 1.6.1, $A = U \cap Y$, where U is closed in X . Then by theorem 1.6.1, we get that A is closed in X .
- (2) Suppose that A is closed in X and B is closed in Y . Then $X \setminus A$ and $Y \setminus B$ are open in X and Y , respectively, so $(X \setminus A) \times (Y \setminus B) = (X \times Y) \setminus (A \times B)$ is open in $X \times Y$. Therefore $A \times B$ is closed in $X \times Y$.

Definition. If Y is a subspace of X , we say that A is **closed in Y** if $A \subseteq Y$ and A is closed in the subspace topology on Y .

Theorem 1.6.2. *Let Y be a subspace of X . Then A is closed in Y if and only if A equals the intersection of a closed set of X with Y .*

Proof. Suppose that A is closed in Y , then $Y \setminus A$ is open in Y , hence we have that $Y \setminus A = U \cap Y$ for some open set U of X . Now $X \setminus U$ is closed in X , and with $A \subseteq Y$, we have that $A = Y \cap (X \setminus U)$.

Conversely, suppose that $A = C \cap Y$, with C closed in X . Then $X \setminus C$ is open in X , hence $(X \setminus C) \cap Y$ is open in Y , now since $(X \setminus C) \cap Y = Y \setminus A$, which is open, we have that A is closed in Y . ■

Theorem 1.6.3. *Let Y be a subspace of X . If A is closed in Y , and Y is closed in X , then A is closed in X ; that is, closure is transitive.*

Proof. By theorem 1.6.2, if A is closed in Y , then $A = C \cap Y$ with C closed in X , now since Y is closed in X , then $Y = D \cap X$ with D closed in X . Thus $A = (C \cap D) \cap X$, therefore, A is closed in X . ■

We now go over the concepts of the closure, and the interior of a set.

Definition. Let $A \subseteq X$, with X a topological space. The **interior** of A is defined to be the union of all open sets in A . The **closure** of A is defined to be the intersection of all closed sets containing A . We denote the interior and the closure of A as $\text{Int } A$ and $\text{cl } A$ respectively

We have by the very definitions that $\text{Int } A \subseteq A \subseteq \text{cl } A$

Lemma 1.6.4. *$\text{Int } A = A$ only when A is open, and $\text{cl } A = A$ only when A is closed.*

Proof. Now, if A is open, then it is in the union of all open sets of A , hence $A \subseteq \text{Int } A$, likewise, if A is closed, then since $\text{cl } A$ is the intersection of all closed sets containing A , we get $\text{cl } A \subseteq A$. ■

Corollary. *A is closed and open if and only if $\text{Int } A = \text{cl } A$.*

Theorem 1.6.5. *Let Y be a subspace of X , and let $A \subseteq Y$. Then $\text{cl } A \cap Y$ is the closure of A in Y .*

Proof. Let $\text{cl } A$ be the closure of A in X . Since $\text{cl } A$ is closed in X , by theorem 1.6.2, $\text{cl } A \cap Y$ is closed in Y , now we have that $A \subseteq \text{cl } A \cap Y$, and since $\text{cl } A = \bigcap U$, then $\text{cl } A \subseteq \text{cl } A \cap Y$.

Conversely, suppose that $\text{cl } A$ is closed in Y , again by theorem 1.6.2, we have that $\text{cl } A = C \cap Y$, where C is closed in X , since $A \subseteq \text{cl } A$, then $A \subseteq C$, and since C is closed, then $\text{cl } A \subseteq C$, thus $\text{cl } A \cap Y \subseteq \text{cl } A$. ■

Definition. Let X be a topological space, and let $x \in X$. We call an open set U of X a **neighborhood** of x if $x \in U$.

Theorem 1.6.6. *If $A \subseteq X$, with X a topological space, then $\text{cl } A$ is a neighborhood of $x \in X$ if and only if for every neighborhood U of x , $A \cap U \neq \emptyset$.*

Proof. We prove the contrapositive. If $x \notin \text{cl } A$, then $U = X \setminus \text{cl } A$ is an open set containing x , disjoint from A . Conversely, suppose there is a neighborhood U of x , with U disjoint from A , then $X \setminus U$ is closed, and therefore contains the closure of A , thus $x \notin \text{cl } A$. ■

Corollary. *$\text{cl } A$ is a neighborhood of x if and only if for every basis element B of X , containing x , intersects A .*

Proof. This is a direct application of theorem 1.6.6, since basis elements are open sets. ■

Example 1.13. (1) We have the closure of $(0, 1]$ in \mathbb{R} is the closed interval $[0, 1]$, since every neighborhood of 0 intersects $(0, 1]$. Now every point outside of $[0, 1]$ has a neighborhood disjoint from $[0, 1]$ (take the neighborhood $(2, 3)$ of 2).

$$(2) \text{cl } \frac{1}{\mathbb{Z}^+} = \{0\} \cup \frac{1}{\mathbb{Z}^+} \text{ and } \text{cl } \{0\} \cup (1, 2) = \{0\} \cup [1, 2].$$

$$(3) \text{cl } \mathbb{Q} = \mathbb{R}, \text{cl } \mathbb{Z}^+ = \mathbb{Z}^+, \text{cl } \mathbb{R}^+ = \mathbb{R}^+ \cup \{0\}. \text{ This first follows from the density of } \mathbb{Q} \text{ in } \mathbb{R}. \text{ Every neighborhood } n \in \mathbb{Z}^+ \text{ intersects } \mathbb{Z}^+, \text{ so } \text{cl } \mathbb{Z}^+ \subseteq \mathbb{Z}^+, \text{ and we have that the neighborhood } (0, 1) \text{ of } 0 \text{ intersects } \mathbb{R}^+, \text{ so } \text{cl } \mathbb{R}^+ \subseteq \mathbb{R}^+ \cup \{0\}.$$

Definition. If $A \subseteq X$, with X a topological space, and if $x \in X$, we say that x is a **limit point** of A if every neighborhood of x intersects A at some distinct point. That is: $x \in \text{cl } X \setminus \{x\}$.

Example 1.14. (1) Consider $(0, 1]$, we have that $0 \in [0, 1] = \text{cl } (0, 1] = \text{cl } [0, 1] \setminus \{0\}$, so 0 is a limit point of $(0, 1]$, the same can be said for any $x \in (0, 1]$.

(2) For $\frac{1}{\mathbb{Z}^+}$, 0 is once again a limit point. Let $x \in \mathbb{R}$ be nonzero, and let $[x, b)$ be the neighborhood of x in the lower limit topology. Then $[x, b) \cap \frac{1}{\mathbb{Z}^+} = \emptyset$ or $\{x\}$, hence, 0 is the only limit point of $\frac{1}{\mathbb{Z}^+}$.

(3) $\text{cl } \{0\} \cup (1, 2) = \{0\} \cup [1, 2]$ has all of its limit points in $[1, 2]$. Likewise, every point in \mathbb{R} is a limit point of \mathbb{Q} . \mathbb{Z}^+ has no limit points in \mathbb{R} , and the limit points of \mathbb{R}^+ are all the points of $\text{cl } \mathbb{R}^+$.

Theorem 1.6.7. *Let $A \subseteq X$, X a topological space, and let A' be the set of all limit points in A . Then $\text{cl } A = A \cup A'$.*

Proof. Let $x \in A'$, then every neighborhood of x intersects A at some distinct point x' , by definition, so by theorem 1.6.6, $x \in \text{cl } A$, hence $A' \subseteq \text{cl } A$, so $A \cup A' \subseteq \text{cl } A$. Now, let $x \in \text{cl } A$. If $x \in A$, we are done. Otherwise, since every neighborhood of x intersects A , we have that they intersect at distinct points, thus $x \in A'$, therefore $\text{cl } A \subseteq A \cup A'$. ■

Corollary. $A \subseteq X$ is closed if and only if $A' \subseteq A$.

Proof. If A is closed, then $\text{cl } A = A = A \cup A'$, thus $A' \subseteq A$. The converse is obvious. ■

Example 1.15. Let X and Y be topological spaces, and let $A \subseteq X$ and $B \subseteq Y$. Then $\text{cl}(A \times B) = \text{cl } A \times \text{cl } B$. It suffices to show that limit points of one closure is in the other and vice versa as $A \times B \subseteq \text{cl}(A \times B)$ and $A \times B \subseteq \text{cl } A \times \text{cl } B$.

Suppose $x' \times y' \in (A \times B)'$ is a limit point of $A \times B$, then for a neighborhood $U \times V$ of $x' \times y'$, we have $(U \times V) \cap (A \times B) = (U \cap A) \times (V \cap B) \neq \emptyset$. Therefore $U \cap A \neq \emptyset$ and $V \cap B \neq \emptyset$. This makes $x' \in A'$ and $y' \in B'$. Therefore $\text{cl}(A \times B) \subseteq \text{cl } A \times \text{cl } B$. By similar reasoning, we have the reverse inclusion so that equality is established.

Definition. Let X be a topological space. A sequence of points of X $\{x_n\}$ is said to **converge** to a point $x \in X$ if for every neighborhood U of x , there is an $N \in \mathbb{Z}^+$ such that $x_n \in U$ for all $n \geq N$.

Example 1.16. (1) Consider the following topological space on $\{a, b, c\}$ in figure ??, and

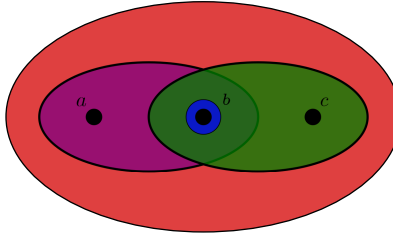


Figure 1.8: A topology on $\{a, b, c\}$, is not a Hausdorff space.

define the sequence $\{x_n\}$ by $x_n = b$ for all $n \in \mathbb{Z}^+$. The neighborhoods of a , b , and c are $U_a = \{a, b\}$, $U_b = \{b\}$, and $U_c = \{b, c\}$. Now let $N > 0$, then we see that for all $n \geq N$, that $b \in U_a$, $b \in U_b$, and $b \in U_c$ thus b converges to a and to c , and itself.

- (2) Consider the sequence of points $\{\frac{1}{n}\}_{n \in \mathbb{Z}^+}$ of \mathbb{R} in the finite complement topology. Then for any neighborhood of $x \in \mathbb{R}$, which is of the form $\mathbb{R} \setminus U$, there are finitely many $\frac{1}{n}$ not included. Thus we get that $\lim \frac{1}{n} = x$ as $n \rightarrow \infty$ for all $x \in \mathbb{R}$.

Definition. A topological space X is called a **Hausdorff space** if for each pair of distinct points x, y , there are neighborhoods U and V of x and y respectively such that U and V are disjoint.

Example 1.17. The topology of the previous example in figure 1.8 is not a Hausdorff space.

Theorem 1.6.8. Every finite point set in a Hausdorff space is closed.

Proof. Let X be a Hausdorff space, and let $x_0 \in X$. We have that $\text{cl}\{x_0\} = \bigcap_{\{x_0\} \in U} U$. Now let $x \neq x_0 \in X$. Since $x \notin \{x_0\}$, and X is Hausdorff, the intersections of the neighborhoods of x and x_0 is empty, thus $x \notin \text{cl}\{x_0\}$, therefore $\text{cl}\{x_0\} = \{x_0\}$.

We can then extend this proof to finite point sets of size n by induction. ■

Now the condition that finite point sets be closed need not depend on whether or not X is a Hausdorff space. In fact, we can assume the following for some topological spaces.

Axiom 1.6.1 (The T_1 Axiom). *In any topological space, every finite point set of X is closed.*

Theorem 1.6.9. *Let X be a topological space satisfying the T_1 axiom, and let $A \subseteq X$. Then a point x is a limit point of A if and only if every neighborhood of x contains infinitely many points of A .*

Proof. Let U_x be a neighborhood of x . If U_x intersects A at infinitely many points of A , then it intersects A at a point distinct from x , thus x is a limit point of A .

Conversely suppose that x is a limit point of A , and let $U_x \cap A$ be finite, then $U_x \cap (A \setminus \{x\})$ is also finite. Now let $U_x \cap (A \setminus \{x\}) = \{x_1, \dots, x_m\}$. By the T_1 axiom, $\{x_1, \dots, x_m\}$ is closed, so $X \setminus \{x_1, \dots, x_m\}$ is open, thus $U_x \cap (X \setminus \{x_1, \dots, x_m\})$ is a neighborhood of x that does not intersect $A \setminus \{x\}$, which contradicts that x is a limit point. ■

Theorem 1.6.10. *If X is a Hausdorff space, then a sequence of points of X converges to at most one point in X .*

Proof. Let $\{x_n\}$ be a sequence of points converging to x , and let $y \neq x$ and let U_x and U_y be neighborhoods of x and y respectively. Then $U_x \cap U_y = \emptyset$. Now since $\{x_n\}$ converges to x , we have that for $N > 0$, $x_n \in U_x$ whenever $n \geq N$. Then $x_n \notin U_y$, and so $\{x_n\}$ cannot converge to y . ■

Definition. Let $\{x_n\}$ be a sequence in a Hausdorff space X . If $\{x_n\}$ converges to a point $x \in X$, we call x the **limit** of $\{x_n\}$ as n **approaches** ∞ and we write $\lim_{n \rightarrow \infty} x_n = x$ or $\{x_n\} \rightarrow x$ as $n \rightarrow \infty$.

Theorem 1.6.11. *The following are true:*

- (1) *Every simply ordered set under the order topology is Hausdorff.*
- (2) *The product of two Hausdorff spaces is Hausdorff.*
- (3) *The subspace of a Hausdorff space is Hausdorff.*

Proof. (1) Let X be an ordered set under the order topology. Take $x, y \in X$ distinct, and suppose without loss of generality that $x < y$. Then consider the neighborhoods $(-\infty, x]$ and $[y, \infty)$ of x and y respectively. Then $(-\infty, x] \cap [y, \infty) = \emptyset$.

- (2) Let X and Y be Hausdorff, and consider $X \times Y$ in the product topology. Let $x_1 \times y_1$ and $x_2 \times y_2$ be distinct points, and let $U_{x_1}, U_{x_2}, V_{y_1}$ and V_{y_2} be basis elements of x_1, x_2, y_1 , and y_2 respectively. Then they are neighborhoods of those elements respectively.

Now we have that $U_{x_1} \times V_{y_1}$ and $U_{x_2} \times V_{y_2}$ are basis elements of $x_1 \times y_1$ and $x_2 \times y_2$, respectively, and hence neighborhoods of those elements respectively. Then we have $(U_{x_1} \times V_{y_1}) \cap (U_{x_2} \times V_{y_2}) = (U_{x_1} \cap U_{x_2}) \times (V_{y_1} \cap V_{y_2}) = \emptyset \times \emptyset = \emptyset$.

- (3) Let X be Hausdorff, and let Y be a subspace of X . Let x_1 and x_2 be distinct points, and let U_{x_1} and U_{x_2} be their neighborhoods. Since Y is open in X , then so are $Y \cap U_{x_1}$ and $Y \cap U_{x_2}$, so they are also neighborhoods of x_1 and x_2 respectively. Then $(Y \cap U_{x_1}) \cap (Y \cap U_{x_2}) = Y \cap (U_{x_1} \cap U_{x_2}) = \emptyset$.

■

Definition. For any set X , we define the **diagonal** of X to be the set

$$\Delta(X) = \{x \times x : x \in X\} \quad (1.1)$$

Theorem 1.6.12. *A topological space X is Hausdorff if, and only if the diagonal $\Delta(X)$ is closed.*

Proof. Suppose that X is Hausdorff, then for any $x, y \in X$ distinct, there are neighborhoods U_x and U_y of x and y respectively with U_x and U_y disjoint. Notice then that $X \setminus U_x$ and $X \setminus U_y$ are closed in X , hence $(X \times X) \setminus (U_x \times U_y)$ is also closed. Now, since $x \neq y$ in the neighborhood of $U_x \times U_y$ we get that the intersection:

$$\bigcap_{x, y \in X} (X \times X) \setminus (U_x \times U_y) = \Delta(X)$$

So $\Delta(X)$ is closed.

Conversely, suppose that $\Delta(X)$ is closed. Then $(X \times X) \setminus \Delta(X) = \{x \times y : x \neq y\}$ is open. Now, let $U \times V$ be a neighborhood for some $x \times y \in (X \times X) \setminus \Delta(X)$ as a subspace of $X \times X$. Then U is a neighborhood for x and V is a neighborhood for y . Now, if $U \cap V \neq \emptyset$, there is some $z \in U$ and $z \in V$, so that $z \times z \in U \times V$. But $U \times V$ is a neighborhood contained in $(X \times X) \setminus \Delta(X)$, which contradicts $z \times z \in U \times V$. Therefore, U and V must be disjoint, which makes X into a Hausdorff space. ■

We conclude this section by looking at another characteristic of sets in topological spaces, closely related to the interior and closure of sets.

Definition. Let X be a topological space, and $A \subseteq X$. We define the **boundary** of A

$$\partial A = \text{cl } A \cap \text{cl } (X \setminus A) \quad (1.2)$$

Example 1.18. The following are true about the boundary:

- (1) $\text{int } A$ and ∂A are disjoint and $\text{cl } A = \text{int } A \cup \partial A$.
- (2) If A is open and closed, then $\partial A = \emptyset$.
- (3) If A is open, then $\partial A = \text{cl } A \setminus A$.

1.7 Continuous Functions.

Definition. Let X and Y be topological spaces. We say that a mapping $f : X \rightarrow Y$ is **continuous** if for each open set V in Y , $f^{-1}(V)$ is open in X .

Now if $f : X \rightarrow Y$ is continuous, then for every open set V of Y , $f^{-1}(V)$ is open in X . Now suppose that \mathcal{B} is a basis of Y , then $V = B_\alpha$, hence $f^{-1}(B_\alpha)$ is open in X .

Similarly, if \mathcal{S} is a subbasis of Y , then for any basis element B of Y , $B = \bigcap_{i=1}^n S_i$, which then implies that $f^{-1}(B) = \bigcap_{i=1}^n f^{-1}(S_i)$, thus $f^{-1}(S_i)$ is also open in X for $1 \leq i \leq n$.

Example 1.19. (1) Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be a continuous realvalued function. Then for each open interval $I \subseteq \mathbb{R}$, $f^{-1}(I)$ is an open interval in \mathbb{R} , so take $x_0 \in \mathbb{R}$ and $\varepsilon > 0$, and let $I = (f(x_0) - \varepsilon, f(x_0) + \varepsilon)$, then since $x_0 \in f^{-1}(I)$, there is a basis $(a, b) \subseteq f^{-1}(I)$ about x_0 . Then take $\delta = \min\{x_0 - a, x_0 - b\}$, then $x \in (a, b)$ whenever $0 < |x - x_0| < \delta$, and we get that $f(x) \in I$, that is, $|f(x) - f(x_0)| < \varepsilon$. This is the definition of continuity defined in the real analysis. We can prove that the converse holds also.

If $f : \mathbb{R} \rightarrow \mathbb{R}$ is continuous at a point x_0 , then for every $\varepsilon > 0$, there is a $\delta > 0$ such that $|f(x) - f(x_0)| < \varepsilon$ whenever $0 < |x - x_0| < \delta$. Then we notice that x and x_0 are distinct, furthermore, $x_0 - \delta < x < x_0 + \delta$, hence $x \in (x_0 - \delta, x_0 + \delta)$ implies that $f(x) \in (f(x_0) - \varepsilon, f(x_0) + \varepsilon)$. Letting $V_\delta(x_0) = (x_0 - \delta, x_0 + \delta)$ and $V_\varepsilon(f(x_0)) = (f(x_0) - \varepsilon, f(x_0) + \varepsilon)$, we have that whenever $x \in V_\delta(x_0)$, then $f(x) \in V_\varepsilon(f(x_0)) \subseteq f^{-1}(V_\delta(x_0))$. And so the topological definition of continuity is equivalent to the real analytic definition of continuity.

- (2) Let $f : \mathbb{R} \rightarrow \mathbb{R}_l$ be defined such that $f(x) = x$ for all $x \in \mathbb{R}$. Take $[a, b) \subseteq \mathbb{R}_l$, we have that $f^{-1}([a, b)) = [a, b)$, which is not open in \mathbb{R} (under the standard topology), hence f is not continuous. However, the map $g : \mathbb{R}_l \rightarrow \mathbb{R}$ defined the same way is continuous since $g^{-1}([a, b))$ is open in \mathbb{R}_l .
- (3) Consider the map $f : \mathbb{R} \rightarrow \mathbb{R}$ defined by $f(x) = x$ if $x \in \mathbb{Q}$ and $f(x) = 0$ if $x \in \mathbb{R} \setminus \mathbb{Q}$. Then f is continuous only at $x = 0$, for the limits

$$\lim_{x \rightarrow x_0^+} f(x)$$

and

$$\lim_{x \rightarrow x_0^-} f(x)$$

both fail to exist for $x_0 \in \mathbb{Q}$ and $x_0 \neq 0$.

- (4) Let $f : A \rightarrow B$ and $g : C \rightarrow D$ be continuous and define $f \times g : A \times C \rightarrow B \times D$ by $f \times g : a \times c \rightarrow f(a) \times g(c)$. Then $f \times g$ is continuous. We have that U and V open in B and D make $f^{-1}(U)$ and $g^{-1}(V)$ open in A and C , respectively. Then it follows that $U \times V$ is open in $B \times D$, and that $(f \times g)^{-1}(U \times V) = f^{-1}(U) \times g^{-1}(V)$ is also open by the definition if $f \times g$. This makes $f \times g$ continuous.

Theorem 1.7.1. Let X and Y be topological spaces, and let $f : X \rightarrow Y$ be a mapping of X into Y . Then the following are equivalent:

- (1) f is continuous.
- (2) For every $A \subseteq X$, $f(\text{cl } A) \subseteq \text{cl } f(A)$.
- (3) For every closed set $B \subseteq Y$, $f^{-1}(B)$ is closed in X .
- (4) For each $x \in X$ and each neighborhood V of $f(x)$, there is a neighborhood U of x such that $f(U) \subseteq V$.

Proof. Let f be continuous and let $A \subseteq X$. Consider the neighborhood V of $f(x)$, then $f^{-1}(V)$ is open in X , and intersects A at a point y . Then $V \cap f(A) = f(y)$, thus $f(x) \in \text{cl } f(A)$.

Now let B be closed in Y , and let $A = f^{-1}(B)$. Then we have that $f(A) = f(f^{-1}(B)) \subseteq B$, thus $x \in \text{cl } A$.

Now let V be open in Y , so that $B = Y \setminus V$ is closed in Y , and $f^{-1}(B) = f^{-1}(Y) \setminus f^{-1}(V) = X \setminus f^{-1}(V)$ which is closed in X , hence $f^{-1}(V)$ is open in X .

Now let $x \in X$, and let V be a neighborhood of $f(x)$. Then $U = f^{-1}(V)$ is a neighborhood of x for which $f(U) \subseteq V$. Finally let V be open in Y , and let $x \in f^{-1}(V)$, then $f(x) \in V$, so there is a neighborhood U_x of x for which $f(U_x) \subseteq V$, then $U_x \subseteq f^{-1}(V)$, then $f^{-1}(V)$ is a union of open sets, and hence open in X . ■

Example 1.20. Suppose $f : X \rightarrow Y$ is continuous, and that AX . Let $x \in A'$ be a limit point of A . Certainly, we have $x \in \text{cl } A$ so then $f(x) \in f(\text{cl } A) \subseteq \text{cl } f(A)$. It remains to see if $x \in (f(A))'$ then. Let V be open in Y such that $x \in U = f^{-1}(V)$. Then $U \cap (A \setminus x) \neq \emptyset$. So then we get $V \cap (f(A) \setminus f(x)) \cap f(A) \neq \emptyset$. If $z \in U \cap A$, then $f(z) = f(x)$, so that $f(U \cap A) = \{f(x)\}$, which means that $x \notin (f(A))'$, in general. It is the case that $f(x)$ is a limit point of $f(A)$ whenever f is 1-1.

Definition. Let X and Y be topological spaces, and $f : X \rightarrow Y$ be a 1-1 mapping of X onto Y . We call f a **homeomorphism** if both f and f^{-1} are continuous.

Lemma 1.7.2. Let X and Y be topological spaces and let $f : X \rightarrow Y$ be a homeomorphism. Then $f(U)$ is open if and only if U is open.

Proof. We have that both $f : X \rightarrow Y$ and $f^{-1} : Y \rightarrow X$ are continuous 1-1 of X and Y onto each other (respectively). Now let U be open in X , then $U = f^{-1}(V)$, for some set V open in Y . Notice then, that $f(U) = f(f^{-1}(V)) = V$, thus $f(U)$ is open in Y . Conversely, let $V = f(U)$ be open in Y for some open set U in X , then $U = f^{-1}(V)$, so by definition of continuity, U is open in X . ■

We can also characterize now when the topology of one set is finer or coarser than the topology of another given another given a continuous mapping.

Lemma 1.7.3. If X and X' are two sets with topologies \mathcal{T} and \mathcal{T}' , respectively, then if $f : X' \rightarrow Y$ is the identity mapping, then f is continuous if, and only if $\mathcal{T} \subseteq \mathcal{T}'$

Proof. Suppose that f is continuous, then for every V open in X under \mathcal{T} , $f^{-1}(V)$ is open in X' under \mathcal{T}' . Since f is the identity, we get that $V = f^{-1}(V)$, so that V is open in X' .

Conversely, if $\mathcal{T} \subseteq \mathcal{T}'$, then for every V open in X , V is open in X' . Notice then $V = f^{-1}(V) \subseteq X'$. Therefore, f is continuous. ■

Corollary. $f : X' \rightarrow X$ is a homomorphism if, and only if $\mathcal{T} = \mathcal{T}'$.

Definition. Let X and Y be topological spaces and let $f : X \rightarrow Y$ be a continuous 1 – 1 mapping of X into Y , and consider $f(X)$ as a subspace of Y . We call $f : X \rightarrow f(X)$ a **topological imbedding** if f is a homeomorphism of X onto $f(X)$.

Example 1.21. (1) The map $f : \mathbb{R} \rightarrow \mathbb{R}$ defined by $f(x) = 3x + 1$ is a homeomorphism whose inverse is $f^{-1}(y) = \frac{1}{3}(y - 1)$, both f and f^{-1} are continuous.

(2) The map $f : (-1, 1) \rightarrow \mathbb{R}$ defined by $f(x) = \frac{x^2}{1-x^2}$ has as its inverse the map $f : \mathbb{R} \rightarrow (-1, 1)$ defined by $f^{-1}(y) = \frac{2y}{1+\sqrt{1+4y^2}}$. Both f and f^{-1} are continuous, so f is a homeomorphism.

(3) The map $g : \mathbb{R}_l \rightarrow \mathbb{R}$ defined by $g(x) = x$ is not a homeomorphism, despite being continuous, as $g^{-1}(1)$ is undefined.

(4) Let S^1 be the unit circle in \mathbb{R}^2 , which is a subspace of \mathbb{R}^2 , and define $f : [0, 1) \rightarrow S^1$ by $f(t) = (\cos(2t\pi), \sin(2t\pi))$. Clearly f is 1 – 1 onto S^1 , and continuous, however f^{-1} is not continuous as $f([0, \frac{1}{4}))$ is not open in S^1 as $f(0)$ is in no open set of \mathbb{R}^2 such that $U \cap S^1 = f([0, 1))$.

(5) Consider the mappings $g : [0, 1) \rightarrow \mathbb{R}^2$ by $g(t) = (\cos(2t\pi), \sin(2t\pi))$. Now g is 1 – 1 and continuous, and we have that $g([0, 1)) \subseteq S^1$, however since g is not a homeomorphism, g fails to be a topological embedding.

(6) Let X and Y be topological spaces, and consider the maps $f : x \rightarrow x \times y_0$ and $g : y \rightarrow y_0 \times x$ for some $x_0 \in X$ and $y_0 \in Y$. We see that f and g are both onto $f(X)$ and $g(Y)$, respectively, and that $x \times y_0 = x' \times y_0$ and $x_0 \times y = x_0 \times y'$ both imply that $x = x'$ and $y = y'$, so f and g are also 1 – 1.

Lastly, we see that if $U \times y_0$ and $x_0 \times V$ are open in $X \times Y$, then $f^{-1}(U \times y_0)$ and $g^{-1}(x_0 \times V)$ are open in X and Y , respectively. This makes f and g continuous, and by the same argument, so are f^{-1} and g^{-1} . Thus f and g are imbeddings of X onto $f(X) = X \times y_0$ and $g(Y) = x_0 \times Y$.

Theorem 1.7.4 (Constructions for continuous functions.). *Let X and Y be topological spaces, then:*

- (1) (Constant construction) If $f : X \rightarrow Y$ maps $x \rightarrow y_0$ for all $x \in X$, then f is continuous.
- (2) (Inclusion) If $A \subseteq X$ is a subspace, then the inclusion mapping $\iota : A \rightarrow X$ is continuous.
- (3) (Construction by composition) If $f : X \rightarrow Y$ and $g : Y \rightarrow Z$ are continuous, then $f \circ g : X \rightarrow Z$ is also continuous.
- (4) (Domain restriction) If $f : X \rightarrow Y$ is continuous and $A \subseteq X$, then $f : A \rightarrow Y$ is continuous.

- (5) (*Range restriction*) if $f : X \rightarrow Y$, and $Z \subseteq Y$ such that $f(X) \subseteq Z$, then $f : X \rightarrow Z$ is continuous.
- (6) (*Range expansion*) If $f : X \rightarrow Y$ is continuous, and $Y \subseteq Z$ is a subspace of Z , then $f : X \rightarrow Z$ is continuous.
- (7) (*Local Formulation*) The map $f : X \rightarrow Y$ is continuous if X can be written as the union of open sets U_α such that $f : U_\alpha \rightarrow Y$ is continuous for all α .

Proof. (1) Let $f(x) = y_0$ for all $x \in X$, and let V be open in Y , then $f^{-1}(V) = X$ or \emptyset depending on if $y_0 \in V$ or not. In either case, $f^{-1}(V)$ is open.

(2) If U is open in X , then $f^{-1}(U) = U \cap A$ which is open in the subspace topology of X .

(3) If U is open in Z , $g^{-1}(U)$ is open in Y , hence $f^{-1}g^{-1}(U)$ is open in X .

(4) Notice that $f_A = \iota \circ f = f : A \rightarrow Y$ which is continuous by (2) and (3).

(5) Let $f : A \rightarrow Y$ be continuous and let $f(X) \subseteq Z \subseteq Y$. Let B be open in Z , so $B = Z \cap U$ for some U open in Y . Now by hypothesis, we have that $f^{-1}(U) \subseteq f^{-1}(B)$, hence $f^{-1}(B)$ is open in X , thus $f : X \rightarrow Z$ is continuous.

(6) Let f be as in (5), and let $Y \subseteq Z$ be a subspace of Z . Then the mapping $h : X \rightarrow Z$ defined by $h = \iota \circ f$ is continuous.

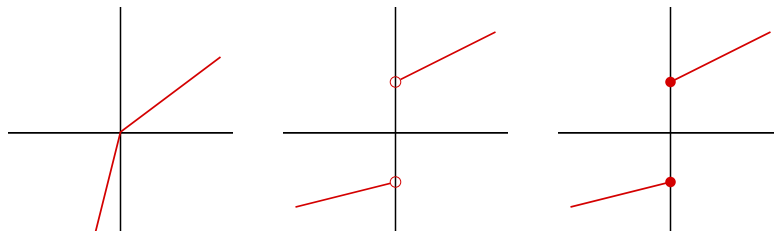
(7) Let $X = \bigcup U_\alpha$ where U_α is open in X , and $f : U_\alpha \rightarrow Y$ is continuous for all α . Let V be open in Y , then $f^{-1}(V) \cap U_\alpha = f_U^{-1}(V)$, and since f is continuous on U_α , then $f^{-1}(V) = \bigcup f_U^{-1}(V)$ is open in X . ■

Theorem 1.7.5 (The pasting lemma). *Let $X = A \cup B$ with A and B closed in X , and let $f : A \rightarrow Y$ and $g : B \rightarrow Y$ be continuous. If $f(x) = g(x)$ for all $x \in A \cap B$, then we can construct a mapping $h : X \rightarrow Y$ defined by $h(x) = \begin{cases} f(x), & x \in A \\ g(x), & x \in B \end{cases}$. Then h is continuous.*

Proof. Let C be closed in Y , then $h^{-1}(C) = f^{-1}(C) \cup g^{-1}(C)$. Since f and g are continuous, then $f^{-1}(C)$ and $g^{-1}(C)$ are closed in A and B , respectively. Thus $h^{-1}(C)$ is closed in X . ■

Example 1.22. Define $h : \mathbb{R} \rightarrow \mathbb{R}$ by $h(x) = \begin{cases} x, & x \leq 0 \\ \frac{x}{2}, & x \geq 0 \end{cases}$. We have that x and $\frac{x}{2}$ are continuous on their respective domains, intersecting at 0, i.e. $x : (-\infty, 0] \rightarrow \mathbb{R}$, $\frac{x}{2} : [0, \infty) \rightarrow \mathbb{R}$, and $\{0\} = (-\infty, 0] \cap [0, \infty)$. Thus h is continuous on \mathbb{R} .

However, $k, l : \mathbb{R} \rightarrow \mathbb{R}$ defined by $k(x) = \begin{cases} x - 2, & x \leq 0 \\ x + 2, & x \geq 0 \end{cases}$ and $l(x) = \begin{cases} x - 2, & x < 0 \\ x + 2, & x \geq 0 \end{cases}$ are not continuous. We have that their domains intersect at 0, but that $k(0) = \pm 2$, (so k isn't even a function). Likewise, $(-\infty, 0) \cap [0, \infty) = \emptyset$, which is open in \mathbb{R} see 1.9.

Figure 1.9: The mappings h , k , and l .

Theorem 1.7.6. Let $f : A \rightarrow X \times Y$ be defined by $f(a) = (f_1(a), f_2(a))$, where $f_1 : A \rightarrow X$ and $f_2 : A \rightarrow Y$. Then f is continuous if and only if f_1 and f_2 are continuous.

Proof. Let $\pi_1 : X \times Y \rightarrow X$ and $\pi_2 : X \times Y \rightarrow Y$ be projections onto X and Y respectively. Since $\pi_1^{-1}(U) = U \times Y$ and $\pi_2^{-1}(V) = X \times V$ are both open in $X \times Y$, π_1 and π_2 are continuous. Then notice that $f_1(a) = \pi_1 \circ f(a)$ and $f_2(a) = \pi_2 \circ f(a)$, both of which are continuous.

Now suppose that f_1 and f_2 are continuous. We have that $a \in f^{-1}(U \times V)$ if and only if $f_1(a) \in U$ and $f_2(a) \in V$, then $f_1^{-1}(U)$ and $f_2^{-1}(V)$ are open in A , hence so is $f^{-1}(U \times V) = f_1^{-1}(U) \cap f_2^{-1}(V)$. ■

We now conclude the section with some more definitions and examples.

Definition. We define the **parametrized curve** of the plane \mathbb{R}^2 to be the continuous function $f : [a, b] \rightarrow \mathbb{R}^2$ defined by $f(t) = (x(t), y(t))$. If f is in a vector field, then we define $f(t) = x(t)i + y(t)j$ where $i = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $j = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

Example 1.23. The function $f(t) = ((\cos(t)), \sin(t))$ is a parametrization of the curve $x^2 + y^2 = 1$, i.e. the unit circle S^1 .

Example 1.24. Suppose $\{A_\alpha\}$ is a finite collection of subsets of a topological space X , and that $X = \bigcup_\alpha A_\alpha$. Suppose $f : X \rightarrow Y$ is defined such that the maps $f_\alpha = f|_{A_\alpha} : A_\alpha \rightarrow Y$ are continuous for each α . Then f is also continuous provided that each A_α is closed. For, if each A_α is closed, then X is the finite union of closed sets. Moreover, for any $V \subseteq Y$ closed, we get $f_\alpha^{-1}(V)$ is closed in A_α as a subspace of X . So $f^{-1}(V) = \bigcup f_\alpha^{-1}(V)$ is closed in $X = \bigcup A_\alpha$, via local formulation, Making f continuous.

This fact is not true however if $\{A_\alpha\}$ is countable, but not finite. Consider the collection $\{[\frac{1}{n}, 1]\}_{n \in \mathbb{Z}^+}$ of the space $[0, 1]$. We have $[0, 1] = \bigcup_{n \in \mathbb{Z}^+} [\frac{1}{n}, 1]$, and each $[\frac{1}{n}, 1]$ is closed in $[0, 1]$ as a subspace of \mathbb{R} . However, the function $f(x) = \frac{1}{n}$ if $x \in [\frac{1}{n}, 1]$ and $f(x) = 1$ otherwise is continuous on each $[\frac{1}{n}, 1]$, but its local formulation is not continuous on $[0, 1]$.

Remark. Theorem 1.7.4 only states that the local formulation of a mapping is continuous only if its domain is the union of open sets. Thus a local formulation where the domain is a union of closed sets need not be continuous.

Definition. Let X be a topological space, and $\{A_\alpha\}$ a collection of subsets of X . We call $\{A_\alpha\}$ **locally finite** if for each $x \in X$, there is a neighborhood U of x such that $U \cap A_\alpha \neq \emptyset$ only for a finite number of α .

Lemma 1.7.7. *Let X be a topological space, and $\{A_\alpha\}$ a locally finite collection of subsets of X . If $f : X \rightarrow Y$ is defined such that $f_\alpha = f|_{A_\alpha} : A_\alpha \rightarrow Y$ is continuous for each α , then f is continuous.*

Proof. If $\{A_\alpha\}$ is locally finite, then for each $x \in X$, there is a neighborhood U_x of x such that $U_x \cap A_\alpha \neq \emptyset$ for only a finite number of α . Then $U_x \cap A_\alpha$ is open in A_α as a subspace of X . Then $V_x = A_\alpha \setminus (U_x \cap A_\alpha)$ is closed in A_α , and $V_x = f_\alpha^{-1}(W)$ for some W closed in Y . Thus $\bigcup_{x \in X} V_x = \bigcup_{\alpha} f_\alpha^{-1}(W) = f^{-1}(W)$ (which is a finite union) is closed in X . This makes f continuous. ■

Definition. Let X , Y , and Z be topological space. We say a map $F : X \times Y \rightarrow Z$ is **continuous in each variable** if the maps $F_X : X \rightarrow Z$ and $F_Y : Y \rightarrow Z$ defined by $F_X(x) = F(x \times y_0)$ and $F_Y(y) = F(x_0 \times y)$ are continuous.

Lemma 1.7.8. *If X , Y , and Z are topological spaces, and $F : X \times Y \rightarrow Z$ is continuous, then F is continuous in each variable separately.*

Proof. If F is continuous, then W open in Z implies that $F^{-1}(W)$ is open in $X \times Y$, and that $U \times V = F^{-1}(W)$ for U and V open in X and Y respectively. Then F is continuous at $U \times y_0$, for $y_0 \in V$; that is $F^{-1}(U \times y_0)$ is open in $X \times Y$. So, by definition, $F_X(U)$ is open in X , and F_X is continuous. By similar reasoning, we can also conclude that F_Y is continuous, and that F is continuous in each variable. ■

Example 1.25. Define $F : \mathbb{R} \rightarrow \mathbb{R}$ by

$$F(x \times y) = \begin{cases} \frac{xy}{x^2+y^2}, & \text{if } x \times y \neq 0 \times 0 \\ 0, & \text{otherwise} \end{cases}$$

Define $F_X(x) = F(x \times y_0)$, then $F(x \times y_0) = \frac{xy_0}{x^2+y_0^2}$ if $x \neq 0$ and $y_0 \neq 0$, and 0 otherwise. Then when $y_0 = 0$ we see that

$$\lim_{x \rightarrow 0} F(x \times y_0) = \lim_{x \rightarrow 0} F_X(x) = F_X(0) = F(0 \times 0) = 0$$

so that F_X is continuous at $x = 0$, and it is also continuous everywhere else. So F_X is continuous. Similarly, the mapping $F_Y(y) = F(x_0 \times y)$ is also continuous, and F is continuous in each variable.

F however, is not continuous in general. Define $g(x) = F(x \times x)$ so we get

$$g(x) = \begin{cases} \frac{1}{2}, & \text{if } x \neq 0 \\ 0, & \text{otherwise} \end{cases}$$

which is not continuous. So F cannot be continuous at the points $x \times x$, for it degenerates into a noncontinuous function. In fact, if we have $\frac{xy}{x^2+y^2} = \frac{1}{2}$, then we get $x^2+y^2-2xy = (x-y)^2 = 0$ and $x \times y \neq 0 \times 0$. This means $F^{-1}(\{\frac{1}{2}\}) = \{x \times x : x \neq 0\}$ is not closed.

Chapter 2

More on Topological Spaces

2.1 The Product Space and Box Topologies.

We now explore more about the product topology.

Definition. Let J be an indexed set, and let X be a set. We define a **J -tuple** of elements of X to be a map $x : J \rightarrow X$, where if $\alpha \in J$, then $x(\alpha) = x_\alpha$, and we call it the **α -th coordinate** of x . We write $(x_\alpha)_\alpha \in J$, or just simply (x_α)

Definition. Let $\{A_\alpha\}$ be an indexed family, and let $X = \bigcup_{\alpha \in J} A_\alpha$. We define the **cartesian product** of $\{A_\alpha\}$, $\prod_{\alpha \in J} A_\alpha$ to be the set of all J -tuples (x_α) of elements of X , where $x_\alpha \in A_\alpha$

Theorem 2.1.1. *Let $\{X_\alpha\}$ be a family of topological spaces, and consider the cartesian product $\prod X_\alpha$. Then the collection of all cartesian products $\prod U_\alpha$, where U_α is open in X_α , for all α , forms a basis for the topology on $\prod X_\alpha$.*

Proof. Clearly $\prod X_\alpha$ itself is a basis element by the first condition. Now consider $\prod U_\alpha$ and $\prod V_\alpha$, then $\prod U_\alpha \cap \prod V_\alpha = \prod (U_\alpha \cap V_\alpha)$, which is also a basis element. ■

Definition. Let $\{X_\alpha\}$ be a family of topological spaces, and take as basis the collection of all products $\prod U_\alpha$ where U_α is open in X_α . We call the topology generated by this basis the **box topology** on $\prod X_\alpha$.

Definition. Let $\pi_\beta : \prod X_\alpha \rightarrow X_\beta$ be defined by $\pi_\beta((x_\alpha)) = x_\beta$. We call this map the **projection mapping** of $\prod X_\alpha$ onto X_β

Theorem 2.1.2. *Let $\mathcal{S}_\beta = \{\pi_\beta^{-1}(U_\beta) : U_\beta \text{ is open in } X_\beta\}$, and let $\mathcal{S} = \bigcup \mathcal{S}_\beta$. Then \mathcal{S} forms the basis for a topology on $\prod X_\alpha$.*

Proof. Since U_β is open in X_β , $\pi_\beta^{-1}(U_\beta) \subseteq \prod X_\alpha$. Taking $\bigcup \mathcal{S}$, we get that $\bigcup \pi_\beta^{-1}(U_\beta) = \prod X_\beta$ for all β . Thus \mathcal{S} is a subbasis. ■

Definition. Let π_β be a projection mapping of $\prod X_\alpha$ onto X_β , and take as subbasis the collection of all $\pi_\beta^{-1}(U_\beta)$ where U_β is open in X_β . We call the topology generated by this subbasis the **product space topology**, or more generally the **product topology** on $\prod X_\alpha$.

Theorem 2.1.3. *The box topology on $\prod X_\alpha$ has as basis all sets of the form $\prod U_\alpha$, where U_α is open in X_α , and the product space topology has as basis all sets of the form $\prod U_\alpha$, where U_α is open in X_α , and $U_\alpha = X_\alpha$ except only for finitely many α .*

Proof. That the box topology has as a basis all sets of the form $\prod U_\alpha$ is clear. Now consider the basis \mathcal{B} that \mathcal{S} generates, and let β_1, \dots, β_n be a finite set of distinct indices and let U_{β_i} be open in X_{β_i} , and $U_\alpha = X_\alpha$ for all other α . Since $B \in \mathcal{B}$ is a finite intersection of elements of \mathcal{S} , we have that $B = \bigcap_{i=1}^n \beta_i^{-1}(U_{\beta_i})$.

Now a point $x = (x_\alpha) \in B$ if and only if the β_i -th coordinate is in U_{β_i} , for $1 \leq i \leq n$, hence membership depends only on a finite number of α , thus $B = \prod U_\alpha$ where $U_\alpha = X_\alpha$ for $\alpha \neq \beta_i$ for $1 \leq i \leq n$. ■

Corollary. *The box topology on $\prod X_\alpha$ is finer than the product topology on $\prod X_\alpha$; moreover, if $\{X_i\}_{i=1}^n$ is a finite family of topologies, then the box topology, and the product topology on $\prod_{i=1}^n X_i$ are equal.*

For convention, from now on when we consider the product $\prod X_\alpha$, we assume that it is under the product space topology.

Theorem 2.1.4. *Suppose the topology on X_α is given by a basis \mathcal{B}_α . The collection of all sets $\prod B_\alpha$ where $B_\alpha \in \mathcal{B}_\alpha$ for each α is a basis for the box topology on $\prod X_\alpha$.*

The same collection for a finite number of α , and where $B_\alpha = X_\alpha$ for all other α forms a basis for the product space topology on $\prod X_\alpha$.

Proof. Let \mathcal{B} be the collection of all $\prod B_\alpha$, where $B_\alpha \in \mathcal{B}_\alpha$. Now each X_α is already its own basis, hence so is $\prod X_\alpha$. Now let $\prod U_\alpha$ and $\prod V_\alpha$ be basis elements. Since $\prod U_\alpha \cap \prod V_\alpha = \prod U_\alpha \cap V_\alpha$, for finite alpha, and since $\prod U_\alpha \cap \prod V_\alpha = \prod X_\alpha$ for all other α (in the case of the product space topology), we get another basis element. Hence \mathcal{B} is a basis for the box topology, and, provided the necessary condition, is also a basis for the product topology. ■

Theorem 2.1.5. *Let A_α be a subspace of X_α . Then $\prod A_\alpha$ is a subspace of $\prod X_\alpha$ under both the box and product space topologies.*

Proof. Since $\prod A_\alpha \cap \prod U_\alpha = \prod (A_\alpha \cap U_\alpha)$, and $A_\alpha \cap U_\alpha$ is a basis element for X_α under the subspace topology, then it follows that $\prod (A_\alpha \cap U_\alpha)$ is a basis element for the same topology on $\prod X_\alpha$, thus $\prod A_\alpha$ is a subspace. ■

Theorem 2.1.6. *If X_α is a Hausdorff space, then so is $\prod X_\alpha$ under both the box and product space topologies.*

Proof. Since X_α is a Hausdorff space, a sequence of points of X_α , $\{x_{\alpha_n}\}$ converges to at most one point. Now construct a sequence $\{x_n\}$ where $x_i = x_{\alpha_i}$ and x_{α_i} is the i -th term of (x_α) , we see that $\{x_{\alpha_n}\}$ is a subsequence of $\{x_n\}$, by definition, and hence $\{x_n\}$ must also converge at at most one point. ■

Example 2.1. For Euclidean space \mathbb{R}^n , a basis consists of all products of the form $(a_1, b_1) \times \dots \times (a_n, b_n)$ where (a_i, b_i) is an open interval for all $1 \leq i \leq n$. Since \mathbb{R}^n is a finite product space, both the box and product topologies on \mathbb{R}^n are the same.

Theorem 2.1.7. *If $A_\alpha \subseteq X_\alpha$, then $\prod \text{cl } A_\alpha = \text{cl } \prod A_\alpha$*

Proof. Let $x = (x_\alpha) \in \prod \text{cl } A_\alpha$ and let $U = \prod U_\alpha$ be a basis element (for either topology). Choosing $y_\alpha \in U_\alpha \cap A_\alpha$, for each α , let $y = (y_\alpha)$. Then $y \in U$, and $y \in \prod A_\alpha$, hence $x \in \text{cl } \prod A_\alpha$.

Now suppose that $x \in \prod A_\alpha$ (in either topology). Let V_β be an open set of X_β containing x_β . Since $\pi_\beta^{-1}(V_\beta)$ is open in $\prod X_\alpha$ (in either topology), it contains a point $y = (y_\alpha)$ of $\prod A_\alpha$. Then $y_\beta \in V_\beta \cap A_\beta$, hence $x \in \text{cl } A_\beta$. ■

Theorem 2.1.8. *Let $f : A \rightarrow \prod X_\alpha$ be defined by $f(a) = (f_\alpha(a))$, where $f_\alpha : A \rightarrow X_\alpha$. Letting $\prod X_\alpha$ have the product space topology, f is continuous if and only if f_α is continuous for each α .*

Proof. We know that the projection mapping π_β is continuous. Now suppose that f is continuous, and notice that $f_\beta = \pi_\beta \circ f$, which makes f_β continuous for each β .

On the other hand, suppose that f_β is continuous for each β . Notice that $f_\beta^{-1} = f^{-1} \circ \pi_\beta^{-1}$, since $\pi_\beta^{-1}(U_\beta)$ is open in $\prod X_\alpha$, then so is $f^{-1} \circ \pi_\beta^{-1}(U_\beta) = f_\beta^{-1}(U_\beta)$. This makes f continuous. ■

Example 2.2. Theorem 2.1.8 holds only for the product space topology and fails in general for the box topology. Consider \mathbb{R}^ω and define the map $f : \mathbb{R} \rightarrow \mathbb{R}^\omega$ by $f(t) = (t, t, t, \dots)$. We have that $f_n(t) = t$ is continuous, which makes f continuous under the product topology. Now consider the box topology: let $B = (-1, 1) \times (-\frac{1}{2}, \frac{1}{2}) \times (-\frac{1}{3}, \frac{1}{3}) \times \dots$, and suppose that $f^{-1}(B)$ were open. Then it contains some interval $(-\delta, \delta)$, about 0, thus $\pi_\beta \circ f((-\delta, \delta)) = f_\beta((-\delta, \delta)) = (-\delta, \delta) \subseteq (-\frac{1}{n}, \frac{1}{n})$, which is absurd. Thus the only implication of the theorem that holds for the box topology is that f_α is continuous only when f is continuous.

2.2 The Metric Topology

Definition. A **metric** (or **distance function**) on a set X is a map $d : X \times X \rightarrow \mathbb{R}$ satisfying the following for all $x, y, z \in X$:

- (1) $d(x, y) \geq 0$ and $d(x, y) = 0$ if and only if $x = y$.
- (2) $d(x, y) = d(y, x)$.
- (3) $d(x, y) \leq d(x, z) + d(z, y)$ (The Triangle Inequality).

We call $d(x, y)$ the **distance** between x and y , and given $\varepsilon > 0$, we define the **ε -ball centered about x** to be the set $B_d(x, \varepsilon) = \{y \in X : d(x, y) < \varepsilon\}$.

Lemma 2.2.1. *Let d be a metric on X . For $x, y \in X$, and $B_d(x, \varepsilon)$ an ε -ball centered about x , there is a δ -ball centered about y , $B_d(y, \delta)$ such that $B_d(y, \delta) \subseteq B_d(x, \varepsilon)$.*

Proof. Let $y \in B_d(x, \varepsilon)$ and let $\delta = \varepsilon - d(x, y)$, and take $z \in B_d(y, \delta)$, then we have that $d(y, z) < \varepsilon - d(x, y)$, thus $d(x, z) \leq d(x, y) + d(y, z) < \varepsilon$ which completes the proof. ■

Theorem 2.2.2. *Let d be a metric on X . Then the collection of all ε -balls about x , for some $x \in X$ forms the basis for a topology on X .*

Proof. Clearly $x \in B_d(x, \varepsilon)$, by definition, so it remains to show that the intersection of two ε -balls contains an ε -ball. Let B_1 and B_2 be ε -balls about x , and let $y \in B_1 \cap B_2$. By lemma 2.2.1, there are $\delta_1, \delta_2 > 0$ such that $B_d(y, \delta_1) \subseteq B_1$ and $B_d(y, \delta_2) \subseteq B_2$. Now take $\delta = \min\{\delta_1, \delta_2\}$, then we see that $B_d(y, \delta) \subseteq B_1 \cap B_2$. ■

Definition. If d is a metric on X , we call the topology having as basis the collection of all ε -balls centered about x , for some $x \in X$ and $\varepsilon > 0$, the **metric topology** induced by d .

Corollary. *A set U is open in the metric topology induced by d if and only if for each $y \in U$, and $\delta > 0$, there is a δ -ball centered about y contained in U .*

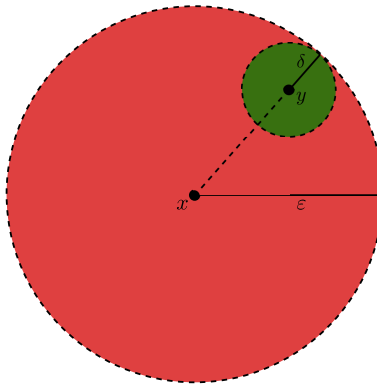


Figure 2.1: All ε -balls centered about x are open in the metric topology by lemma 2.2.1.

Example 2.3. (1) Define $d(x, y) = 1$ if $x \neq y$ and $d(x, y) = 0$ if $x = y$. Clearly d is a metric on X , and induces the discrete topology on X . The basis $B_d(x, 1) = \{x\}$

(2) The standard metric on \mathbb{R} is defined to be $d(x, y) = |x - y|$ and is a metric on \mathbb{R} (that is, the absolute $|\cdot|$ is a metric on \mathbb{R}). This metric induces the standard topology on \mathbb{R} as we see that it has basis $B_d(x, \varepsilon) = \{y \in \mathbb{R} : |x - y| < \varepsilon\} = \{y \in \mathbb{R} : y - \varepsilon < x < y + \varepsilon\} = (y - \varepsilon, y + \varepsilon)$.

Definition. If X is a topological space, we call X **metrizable** if there is a metric d which induces the topology on X . A **metric space** is a metrizable space X together with the metric inducing the topology of X .

Definition. Let X be a metric space with metric d . A subset $A \subseteq X$ is said to be **bounded** if there is an $M > 0$ such that $d(a_1, a_2) \leq M$ for all $a_1, a_2 \in A$. We define the **diameter** of a bounded set A to be $\text{diam } A = \sup\{d(a_1, a_2) : a_1, a_2 \in A\}$.

It is easy to see that boundedness of a set does not depend on the topology of X , but on the metric.

Theorem 2.2.3. Let X be a metric space with metric d and define $\bar{d} : X \times X \rightarrow \mathbb{R}$ by $\bar{d} = \min\{d(x, y), 1\}$ for all $x, y \in X$. Then \bar{d} is a metric on X that induces the same topology as d .

Proof. Clearly we have that $0 \leq \bar{d}(x, y) \leq 1$, and that $\bar{d}(x, y) = \min\{d(x, y), 1\} = \min\{d(y, x), 1\} = \bar{d}(y, x)$. It remains to show the triangle inequality.

Now if $d(x, z) \leq 1$ and $d(z, y) \leq 1$, then by the triangle inequality $d(x, y) \leq 1$ and $\bar{d}(x, y) \leq \bar{d}(x, z) + \bar{d}(z, y)$. Now if $d(x, z) < 1$ and $d(z, y) < 1$, we get the same conclusion. Thus we see that \bar{d} is a metric on X .

Now take as basis the collection of all ε -balls with $0 < \varepsilon < 1$, and any basis element of x contains such an ε -ball, thus \bar{d} induces the same topology as d . ■

Definition. Let X be a metric space with metric d . We define the **standard bounded metric** to be the metric $\bar{d} : X \times X \rightarrow \mathbb{R}$ defined by the rule $\bar{d}(x, y) = \min\{d(x, y), 1\}$ for any $x, y \in X$.

Definition. Let $x \in \mathbb{R}^n$. We define the **norm** of x , $\|x\|$, to be $\|x\| = \sqrt{x_1^2 + \cdots + x_n^2}$. We define the **square metric** ρ on \mathbb{R}^n to be $\rho(x, y) = \{|x_1 - y_1|, \dots, |x_n - y_n|\}$.

Before we show that $\|\cdot\|$ and ρ are metrics, we introduce the following:

Definition. Let $x, y \in \mathbb{R}^n$. We define the **inner product** of x and y to be:

$$\langle x, y \rangle = x_1 y_1 + \cdots + x_n y_n \quad (2.1)$$

Lemma 2.2.4. For $x, y \in \mathbb{R}^n$, $\langle x, y + z \rangle = \langle x, y \rangle + \langle x, z \rangle$ and $|\langle x, y \rangle| \leq \|x\| \|y\|$.

Proof. We have $\langle x, y + z \rangle = x_1(y_1 + z_1) + \cdots + x_n(y_n + z_n) = (x_1 y_1 + \cdots + x_n y_n) + (x_1 z_1 + \cdots + x_n z_n) = \langle x, y \rangle + \langle x, z \rangle$.

Now if $x = 0$ and $y = 0$, then $|\langle x, y \rangle| = \|x\| \|y\| = 0$, so suppose that both $x, y \neq 0$, and let $a = \frac{1}{\|x\|}$ and $b = \frac{1}{\|y\|}$. Notice that $\|ax + by\| \geq 0$ and $\|ax - by\| \geq 0$, then $\|ax + by\|^2 \|ax - by\|^2 = \|x\|^2 \|y\|^2 - |\langle x, y \rangle|^2 \geq 0$, hence $|\langle x, y \rangle| \leq \|x\| \|y\|$. ■

Remark. We call the last relation in the lemma the **Cauchy-Schwarz inequality**.

Theorem 2.2.5. Both the norm and square metrics make \mathbb{R}^n into a metric space.

Proof. We start with the norm. Now clearly, since $\sqrt{x} \geq 0$ (for real numbers), $\|x - y\| \geq 0$, and if $\|x - y\| = 0$ then $(x_1 - y_1)^2 + \cdots + (x_n - y_n)^2 = 0$ hence $x_i = y_i$ for all $1 \leq i \leq n$, and if $x_i = y_i$, then clearly $\|x - y\| = 0$. We also see that $\|x - y\| = \|y - x\|$.

Now consider $z \in \mathbb{R}^n$, notice that $\|x + y\|^2 = \langle x + y, x + y \rangle = \langle x + y, x \rangle + \langle x + y, y \rangle = \langle x, x \rangle + \langle y, y \rangle \leq \|x\|^2 + \|y\|^2$. Using this we have $\|x - z\| + \|z - y\| \geq \|x - y\|$ (square the left hand side and evaluate), so $\|\cdot\|$ is a metric on \mathbb{R}^n .

Now consider the square metric. Clearly we have that $\rho(x, y) \geq 0$ and that $\rho(x, y) = 0$ if and only if $x = y$ (since $|\cdot|$ is also a metric), we also see that $\rho(x, y) = \rho(y, x)$.

Now let $x \in \mathbb{R}^n$, and we have for all $1 \leq i \leq n$ that $|x_i - y_i| \leq |x_i - z_i| + |z_i - y_i|$, hence by definition $|x_i - y_i| \leq \rho(x, z) + \rho(z, y)$, thus $\rho(x, y) \leq \rho(x, z) + \rho(z, y)$ and ρ is a metric. ■

Lemma 2.2.6. *Let d and d' be metrics on X and let \mathcal{T} and \mathcal{T}' be the topologies induced by d and d' respectively. $\mathcal{T} \subseteq \mathcal{T}'$ if and only if for each $x \in X$, and $\varepsilon > 0$, there is a $\delta > 0$ such that $B_{d'}(x, \delta) \subseteq B_d(x, \varepsilon)$.*

Proof. Suppose that $\mathcal{T} \subseteq \mathcal{T}'$, and take $B_d(x, \varepsilon)$ in \mathcal{T} , then by lemma 2.2.1, there is a B' in \mathcal{T}' or which $B' \subseteq B_d(x, \varepsilon)$, hence there is a δ -ball about x for which $B_{d'}(x, \delta) \subseteq B'$.

Conversely, suppose for $x \in X$ and $\varepsilon > 0$, that there is a $\delta > 0$ for which $B_{d'}(x, \delta) \subseteq B_d(x, \varepsilon)$. Given a basis B of \mathcal{T} , there is an ε -ball about x contained in B , hence $B_d(x, \varepsilon)$ is also in B , thus we have that $\mathcal{T} \subseteq \mathcal{T}'$. ■

Theorem 2.2.7. *The norm and the square metric both induce the product topology on \mathbb{R}^n .*

Proof. Notice that $\rho(x, y) \leq \|x - y\| \leq \rho(x, y)\sqrt{n}$. This first inequality shows that $B_{\|\cdot\|}(x, \varepsilon) \subseteq B_\rho(x, \varepsilon)$, and the second shows that $B_\rho(x, \frac{\varepsilon}{\sqrt{n}}) \subseteq B_{\|\cdot\|}(x, \varepsilon)$, thus both $\|\cdot\|$ and ρ induce the same topology.

Now let $B = (a_1, b_1) \times \cdots \times (a_n, b_n)$ be a basis for the product topology on \mathbb{R}^n . Since, for each i , there is an $\varepsilon_i > 0$ such that $(x_i - \varepsilon_i, x_i + \varepsilon_i) \subseteq (a_i, b_i)$, choosing $\varepsilon = \min\{\varepsilon_1, \dots, \varepsilon_n\}$, we see that $B_\rho(x, \varepsilon) \subseteq B$. Conversely, given $y \in B_\rho(x, \varepsilon)$, notice that $B_\rho(x, \varepsilon) = (x_1 - \varepsilon, x_1 + \varepsilon) \times \cdots \times (x_n - \varepsilon, x_n + \varepsilon)$, thus $B \subseteq B_\rho(x, \varepsilon)$. Thus the topologies are the same. ■

Definition. Given an index set J and given points $x = (x_\alpha)$ and $y = (y_\alpha)$ of \mathbb{R}^J , we define the **uniform metric** $\bar{\rho}$ on \mathbb{R}^J by $\bar{\rho}(x, y) = \sup\{\bar{d}(x_\alpha, y_\alpha) : \alpha \in J\}$, where \bar{d} is the standard bounded metric on \mathbb{R}^J . We call the topology induced by $\bar{\rho}$ the **uniform topology**.

Theorem 2.2.8. *The uniform topology on \mathbb{R}^J is finer than the product topology on \mathbb{R}^J and coarser than the box topology on \mathbb{R}^J , and all three topologies are different if J is infinite.*

Proof. Let $x = (x_\alpha)$ and $\prod U_\alpha$ be a basis element for the product topology, and let $\alpha_1, \dots, \alpha_n$ be the indices for which $U_\alpha \neq \mathbb{R}$, and for each i , choose $\varepsilon_i > 0$ such that the ε_i -ball about x_{α_i} , $B_{\bar{d}}(x_{\alpha_i}, \varepsilon_i) \subseteq U_{\alpha_i}$. Let $\varepsilon = \min\{\varepsilon_1, \dots, \varepsilon_n\}$, and let $z = (z_\alpha) \in \mathbb{R}^J$ be such that $\bar{\rho}(x, z) < \varepsilon$. Then $\bar{d}(x_\alpha, z_\alpha) < \varepsilon$ for all α . Hence $B_{\bar{\rho}} \subseteq \prod U_\alpha$ for all α ; so the uniform topology is finer.

Likewise, consider B the ε -ball about x in the $\bar{\rho}$ metric. Then the box $U = \prod (x_\alpha - \frac{\varepsilon}{2}, x_\alpha + \frac{\varepsilon}{2})$ and $y \in U$ if $\bar{d}(x_\alpha, y_\alpha) < \frac{\varepsilon}{2}$, then $\bar{\rho}(x, y) \leq \frac{\varepsilon}{2}$, so $U \subseteq B$, and the uniform topology is coarser.

Now in the case where J is infinite, if J is uncountable, we are done, since there is no way to map the indices of J onto \mathbb{Z}^+ . So consider the case where J is countable, and map $J \rightarrow \mathbb{Z}^+$ by $\alpha_i \rightarrow i$. Let $U = \prod (x_i - \varepsilon, x_i + \varepsilon)$ and consider a base $B_{\bar{\rho}}(x, \varepsilon)$. We have that for $y \in B_{\bar{\rho}}(x, \varepsilon)$ that $\bar{d}(x_\alpha, y_\alpha) = \min\{\rho(x_i, y_i), 1\}$, we have that $\bar{d}(x_i, y_i) = \bar{\rho}(x_i, y_i)$ or 1, and if we choose $0 < \varepsilon < 1$, then $\bar{\rho}$ fails to put $B_{\bar{\rho}}(x, \varepsilon)$ inside of U . Likewise, the basis $\prod U_\alpha$ (in the product topology) fails to be contained in $B_{\bar{\rho}}(x, \varepsilon)$ by the same argument. Thus the uniform topology is not necessarily finer than the box topology, nor coarser than the product topology in \mathbb{R}^J , when J is infinite. ■

Remark. Clearly the box, product and uniform topologies on \mathbb{R}^J are the same when J is finite, as the box and product topologies are the same for finite product spaces.

Theorem 2.2.9. *Let $\bar{d}(a, b) = \min\{|a - b|, 1\}$ be the standard bounded metric on \mathbb{R} , and for $x, y \in \mathbb{R}^\omega$, define:*

$$D(x, y) = \sup_i \left\{ \frac{\bar{d}(x_i, y_i)}{i} \right\} \quad (2.2)$$

Then D is a metric that induces the product space topology on \mathbb{R}^ω

Proof. Since \bar{d} is a metric, D satisfies the conditions for a metric, it is worth looking into the case for the triangle inequality. Notice that for all i , $D(x, y) \leq \frac{\bar{d}(x_i, y_i)}{i} \leq \frac{\bar{d}(x_i, z_i)}{i} + \frac{\bar{d}(z_i, y_i)}{i} \leq D(x, z) + D(z, y)$. Therefore $D(x, y) \leq D(x, z) + D(z, y)$.

Now let U be open in the metric topology induced by D , and let $x \in U$. Choose an ε -ball $B_D(x, \varepsilon) \subseteq U$ and choose $N > 0$ arbitrarily large such that $\frac{1}{N} < \varepsilon$ and let $V = (x_1 - \varepsilon, x_1 + \varepsilon) \times \dots \times (x_N - \varepsilon, x_N + \varepsilon) \times \mathbb{R} \times \dots \times \mathbb{R} \dots$. Given $y \in \mathbb{R}^\omega$, we have $\frac{\bar{d}(x_i, y_i)}{i} \leq \frac{1}{N}$ for all $i \geq N$, thus $D(x, y) \leq \max\{\frac{\bar{d}(x_1, y_1)}{1}, \dots, \frac{\bar{d}(x_N, y_N)}{N}, \frac{1}{N}\}$. Now if $y \in V$, then the expression is less than ε , so $V \subseteq B_D(x, \varepsilon)$.

Conversely let $U = \prod U_i$ where U_i is open in \mathbb{R} for $i = \alpha_1, \dots, \alpha_n$, and $U_i = \mathbb{R}$ for all other indices. Given $x \in U$, for $i = \alpha_1, \dots, \alpha_n$, and letting $\varepsilon = \min\{\frac{\varepsilon_i}{i} : i = \alpha_1, \dots, \alpha_n\}$ where $\frac{1}{N} < \varepsilon_i$ for each i . Letting $B_d(x, \varepsilon)$ be the corresponding ε -ball under the metric d , we see that $\frac{\bar{d}(x_i, y_i)}{i} \leq D(x, y) < \varepsilon$. Now if $i = \alpha_1, \dots, \alpha_n$, then $\bar{d}(x_i, y_i) < \varepsilon_i \leq 1$, hence $|x_i - y_i| < \varepsilon_i$, putting $y \in U$. ■

2.3 More on Metric Spaces

We go more in depth on metric spaces here.

Theorem 2.3.1. *If A is a subspace of a metric space X , with metric d , then d restricted to $A \times A$ makes A into a metric space.*

Proof. Clearly $d : A \times A \rightarrow \mathbb{R}$ is a metric. So consider the ε -ball about x , $B_d(x, \varepsilon)$ in X ; restricting d to $A \times A$, consider $A \cap B_d(x, \varepsilon)$. For $y \in A$, there is a δ -ball about y such that $B_d(y, \delta) \subseteq B_d(x, \varepsilon)$; then $B_d(y, \delta) \subseteq A \cap B_d(x, \varepsilon)$. This makes A as a subspace, into a metric space. ■

Theorem 2.3.2. *The Hausdorff axiom is satisfied in every metric space.*

Proof. If $x, y \in X$ are distinct, let $\varepsilon = \frac{1}{2}d(x, y)$, by the triangle inequality, we have that $B_d(x, \varepsilon)$ and $B_d(y, \varepsilon)$ are disjoint. ■

Theorem 2.3.3. *Countable products of metrizable spaces are metrizable.*

Proof. Let X be a metric space with metric d . Define $\bar{d}(x, y) = \min\{d(x, y), 1\}$ on X and define $D(x, y) = \sup\{\frac{\bar{d}(x_i, y_i)}{i}\}$ on X^ω . It is clear that both \bar{d} and D are metrics on X and X^ω respectively. We would like to show that D induces the product topology on X^ω .

Let U be open and let $x \in U$. Choose $B_D(x, \varepsilon) \subseteq U$ and choose N large enough such that $\frac{1}{N} < \varepsilon$. Now let $V = (x_1 - \varepsilon, x_1 + \varepsilon) \times \dots \times (x_N - \varepsilon, x_N + \varepsilon) \times X \times \dots$ be a basis in the product topology on X^ω . Given $y \in X^\omega$, such that $\frac{\bar{d}(x_i, y_i)}{i} \leq \frac{1}{N}$, we have by definition that $D(x, y) \leq \max\{\bar{d}(x_1, y_1), \frac{\bar{d}(x_2, y_2)}{2}, \dots, \frac{\bar{d}(x_N, y_N)}{N}, \frac{1}{N}\}$. If $y \in V$, we get that $V \subseteq B_D(x, \varepsilon)$ and we are done.

Conversely let $U = \prod U_i$ be a basis of the product topology where U_i is open in X for $i = 1, \dots, n$ and $U_i = X$ for all other indices. Now let $x \in U$ and choose an interval about x_i , $(x_i - \varepsilon_i, x_i + \varepsilon_i)$ lying in U_i with $0 < \varepsilon_i \leq 1$ for all i . Choose $\varepsilon = \min\{\varepsilon_1, \frac{\varepsilon_2}{2}, \dots, \frac{\varepsilon_n}{n}\}$. Then $x \in B_D(x, \varepsilon) \subseteq U$, for if $y \in B_D(x, \varepsilon)$, we have that $\frac{\bar{d}(x_i, y_i)}{i} \leq D(x, y) < \varepsilon$, hence $\varepsilon \leq \frac{\varepsilon_i}{i}$ and $d(x_i, y_i) < \varepsilon_i$, and so $y \in U_i$. Therefore D induces the product space topology. ■

Remark. This theorem generalizes theorem 2.2.9 for any countable product space of a metric space X . Hence we can take theorem 2.2.9 as a corollary to this theorem.

We would now like to study continuous functions in metric spaces, which brings us into the realm of analysis. We show that the “ ε - δ ” definition, and the sequence definition of continuity carry over.

Theorem 2.3.4. *Let $f : X \rightarrow Y$ with X and Y metric spaces with metric d_X and d_Y respectively. Then f is continuous if and only if for $x \in X$, and $\varepsilon > 0$, there is a $\delta > 0$ such that $d_Y(f(x), f(y)) < \varepsilon$ whenever $d_X(x, y) < \delta$.*

Proof. Suppose that f is continuous and consider $f^{-1}(B(f(x), \varepsilon))$ open in X . Then it contains a δ -ball $B(x, \delta)$ about x . If $y \in B(x, \delta)$, then $f(y) \in B(f(x), \varepsilon)$, as is required.

Now suppose that for $x \in X$ and $\varepsilon > 0$, that there is a $\delta > 0$ such that $d_Y(f(x), f(y)) < \varepsilon$ whenever $d_X(x, y) < \delta$, for $x \in X$. Let V be open in Y and let $x \in f^{-1}(V)$. Then $f(x) \in V$, hence there is an ε -ball $B(f(x), \varepsilon) \subseteq V$. By hypothesis, there is a $\delta > 0$ such that $f(B(x, \delta)) \subseteq B(f(x), \varepsilon)$, hence $B(x, \delta) \subseteq f^{-1}(V)$ which makes $f^{-1}(V)$ open. ■

Lemma 2.3.5 (The Sequence Lemma). *Let X be a topological space and let $A \subseteq X$. If there is a sequence of points of A converging to $x \in X$, then $x \in \text{cl } A$. The converse holds if X is metrizable.*

Proof. Suppose for some sequence $\{x_n\} \subseteq A$ that $x_n \rightarrow x$. By theorem 1.6.6, we have every neighborhood of x contains points of A , hence $x \in \text{cl } A$. Conversely, suppose that X is metrizable with metric d , and let $x \in \text{cl } A$. For $n \in \mathbb{Z}^+$, take $B_d(x, \frac{1}{n})$ and take $\{x_n\} = B_d(x, \frac{1}{n}) \cap A$. Then $x_n \rightarrow x$, for: any open set U of x contains an ε -ball about x , $B_d(x, \varepsilon)$, so choose N large enough so that $\frac{1}{N} < \varepsilon$, hence U contains x_i for all $i \geq N$. ■

Theorem 2.3.6 (The Sequential Criterion). *Let $f : X \rightarrow Y$ be continuous, then for every convergent sequence $\{x_n\}$ converging to $x \in X$, the sequence $\{f(x_n)\}$ converges to $f(x)$. the converse holds if X is metrizable.*

Proof. Let f be continuous and suppose that $x_n \rightarrow x$. Let V be a neighborhood of $f(x)$, then $f^{-1}(V)$ is a neighborhood of x ; hence there is an $N > 0$ such that $x_n \in f^{-1}(V)$ whenever $n \geq N$, thus $f(x_n) \in V$ whenever $n \geq N$.

Conversely suppose that for every $\{x_n\}$ converging to x , that $\{f(x_n)\}$ converges to $f(x)$, and let $A \subseteq X$. if $x \in \text{cl } A$, by the sequence lemma, there is a sequence $\{x_n\} \subseteq A$ converging to x . Now since $f(x_n) \rightarrow f(x)$. and $f(x_n) \in f(A)$, by the sequence lemma again, $f(x) \in \text{cl } f(A)$. Thus $f(\text{cl } A) \subseteq \text{cl } f(A)$ and we are done. ■

We now consider methods for constructing continuous functions on metric spaces.

Lemma 2.3.7. *The additions, subtraction, and multiplication operations are continuous from $\mathbb{R} \times \mathbb{R}$ to \mathbb{R} . The quotient operation is continuous from $\mathbb{R} \times \mathbb{R} \setminus \{0\}$ to \mathbb{R} .*

Theorem 2.3.8. *If X is a topological space and if $f, g : X \rightarrow \mathbb{R}$ are continuous, then $f + g$, $f - g$, and fg are continuous; moreover if $g(x) \neq 0$ for all $x \in X$, then $\frac{f}{g}$ is also continuous.*

Proof. The map $h : X \rightarrow \mathbb{R} \times \mathbb{R}$ defined by $h(x) = f(x) \times g(x)$ is continuous. Then notice that $f + g(x) = +(f(x), g(x)) = + \circ h(x)$, so by the above lemma, we get that $f + g$ is continuous. We also have that $f - g$ is continuous for $f - g(x) = +(f(x), -g(x))$. The same argument holds for fg and $\frac{f}{g}$. ■

Definition. Let $f_n : X \rightarrow Y$ be a sequence of functions from X to the metric space Y , with metric d . We say that the sequence $\{f_n\}$ **converges uniformly** to the function $f : X \rightarrow Y$ if for $\varepsilon > 0$, there is an integer $N > 0$ such that $d(f_n(x), f(x)) < \varepsilon$ whenever $n \geq N$, for all $x \in X$.

Theorem 2.3.9. *Let $f_n : X \rightarrow Y$ be a sequence of continuous functions from the topological space X to the metric space Y . If $\{f_n\}$ converges uniformly to f , then f is continuous.*

Proof. Let V be open in Y and let $x_0 \in f^{-1}(V)$. Let $y_0 = f(x_0)$ and choose $\varepsilon > 0$ such that $B(y_0, \varepsilon) \subseteq V$. By uniform convergence, choosing $N > 0$ so that whenever $n \geq N$, $d(f_n(x), f(x)) < \frac{\varepsilon}{3}$ for all $x \in X$. By the continuity of f_N , choose a neighborhood U of x_0 such that $f_N(U) \subseteq B(f_N(x_0), \frac{\varepsilon}{3})$. Then if $x \in U$, we have $d(f(x), f_N(x)) < \frac{\varepsilon}{3}$, $d(f_N(x), f_N(x_0)) < \frac{\varepsilon}{3}$, and $d(f_N(x_0), f(x_0)) < \frac{\varepsilon}{3}$ by the triangle inequality we get $d(f(x), f(x_0)) < \varepsilon$ which completes the proof. ■

Example 2.4. (1) \mathbb{R}^ω is not metrizable in the box topology. Let $A = \{(x_1, x_2, \dots) \in \mathbb{R}^\omega : x_i > 0\}$ and consider $0 = (0, 0, \dots) \in \mathbb{R}^\omega$. $0 \in \text{cl } A$ if for any basis element $B = (a_1, b_1) \times (a_2, b_2) \times \dots$, $0 \in B$; then $B \cap A \neq \emptyset$ (take the point $\frac{1}{2}b \in \mathbb{R}^\omega$). Now let $\{a_n\}$ be a sequence of points of A with $a_n = (x_{1n}, x_{2n}, \dots, x_{in}, \dots)$, since $x_{in} > 0$, construct a basis element $B' = (-x_{11}, x_{11}) \times (-x_{22}, x_{22}) \times \dots$. Then $0 \in B'$, but $\{a_n\} \not\subseteq B'$ for the point $x_{nn} \notin (-x_{nn}, x_{nn})$. Thus $a_n \not\rightarrow 0$.

(2) An uncountable product of \mathbb{R} with itself is not metrizable. Let J be uncountable, and let $A = \{(x_\alpha) \in \mathbb{R}^J : x_\alpha = 1 \text{ for all but finitely many } \alpha\}$. Consider $0 \in \mathbb{R}^J$ and let U be a basis for containing 0 . Now $U_\alpha \neq \mathbb{R}$ for $\alpha_1, \dots, \alpha_n$, so let $(x_\alpha) \in A$ be defined by letting $x_\alpha = 0$ for $\alpha_1, \dots, \alpha_n$ and $x_\alpha = 1$ for all other indices. Then $(x_\alpha) \in A \cap U_\alpha$.

Now let $\{a_n\} \subseteq A$ and for $n \in \mathbb{Z}^+$ let $J_n = \{\alpha \in J : a_{\alpha n} \neq 1\}$. Then we see that $\bigcup J_n$ is a countable union of finite sets, and hence countable itself. Now since J is uncountable, there is a $\beta \in J$ for which $\beta \notin \bigcup J_n$, so $a_{\beta n} = 1$. Letting $U_\beta = (-1, 1)$ in \mathbb{R} let $U = \pi_\beta^{-1}(U_\beta)$ in \mathbb{R}^J . Then U is a neighborhood of 0 not containing any points of $\{a_n\}$, so $a_n \not\rightarrow 0$.

Bibliography

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