3-Manifolds

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Contents

Chapter 1

Manifolds

1.1 Topological Manifolds

Definition. A **topological n-manifold** is a second countable Hausdorff space M, together with a collection $\{(M_\alpha, \phi_\alpha)\}$ for which

(1) $\{M_\alpha\}$ is a collection of open sets of M covering M ; that is, $M_\alpha \subseteq M$ is open and $M = M_\alpha$.

(2) ϕ_a is a homeomorphism of M_α onto an open subset U of $\mathbb{R}^n$.

We call the pairs (M_α, ϕ_α) **charts** of M, and we call the collection of all such charts of M an **atlas** of M. We define the **dimension** of M to be $\dim M = n$.

Example 1.1. (1) Every subset of $\mathbb{R}^N$ is second countable and Hausdorff, so that a subset $\mathbb{R}^N$ is an n-manifold uf every point of M has a neighborhood homeomorphic to $\mathbb{R}^n$, for $n \leq N$. In particular, $\mathbb{R}^n$ is an n-manifold.

(2) The n-sphere $S^n = \{x \in \mathbb{R}^{n+1} : \|x\| = 1\}$ (see figure 1.1) is an n-manifold. It is a second countable Hausdorff space, since it is a subspace of $\mathbb{R}^n$. Moreover, the stereographic projection $h : S^n \backslash (0, \ldots, 0, 1) \to \mathbb{R}^n$ is a homeomrphism. So for $x \in S^n$, $x \neq (0, \ldots, 0, 1)$, x has a neighborhood homeomorphic to $\mathbb{R}^n$. Now, if we take the composition of $\mathbb{R}^n \times \{0\}$ with h to obtain the map $h' : S^n \backslash (0, \ldots, 0, -1) \to \mathbb{R}^n$, then we get that $S^n \backslash (0, \ldots, 0, -1)$ is a neighborhood of $(0, \ldots, 0, 1)$ homeomorphic to $\mathbb{R}^n$.

(3) The n-torus $T = \underbrace{S_1 \times \cdots \times S^1}_{n \text{ times}}$ (see figure 1.2) is the quotient space obtained fron $\mathbb{R}^n$ by identfying two points $x, y \in \mathbb{R}^n$ if, and only if there is some $g \in G$ for which $g(x) = y$, where G is the group generated by all translations by distance 1 along the coordinate axes. Let $x \in T^n$ and $U = \partial B(x, \frac{1}{4})$ the sphere centered about x of radius $\frac{1}{4}$ and let $q : \mathbb{R}^n \to \mathbb{R}^n$ the quotient map of the quotient space of T^n. Then $q^{-1}|_{q(U)}$ is a homoemorphism. This makes T an n-manifold, with atlas $\{(U, q^{-1}|_{q(U)})\}$.

(4) Identify the antipodal points of S^n, then the resulting quotient space is an n-manifold called **n-dimensional real projective space** which we denote by $\mathbb{PR}^n$. Let $x \in$

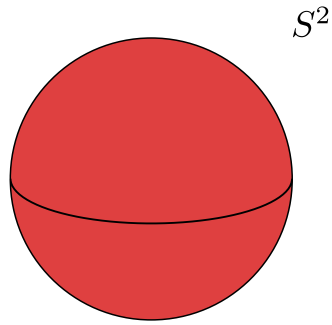

Figure 1.1: The 2-Sphere of $\mathbb{R}^3$ is a 2-manifold.

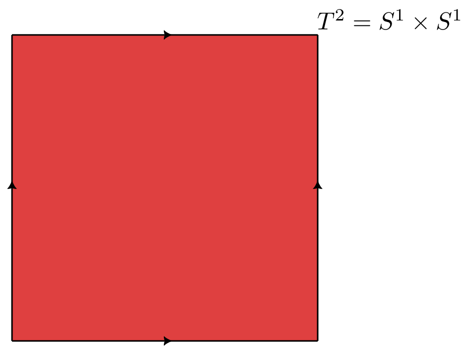

Figure 1.2: The 2-torus is a 2 manifold of $\mathbb{R}^2$.

$\mathbb{PR}^n$,since S^n is an n-manifold, there is a neighborhood U of x and a homeomorphism $h : U \to \mathbb{R}^n$. Let $-U = a(U)$, where $a : S^n \to S^n$ is the antipodal map. Then $-U$ is a neighborhood of $-x$, and $-h = h \circ a$ is a homeomorphism of U onto $\mathbb{R}^n$. Then the collection $\{(U, h)\}$ is an atlas for $\mathbb{PR}^n$.

(5) Consider the unit circle S^2 and the following "trefoil knot" K_3 pictured below in figure 1.3 Both of these are 1-manifolds, and are homeomorphic to each other. Consider the

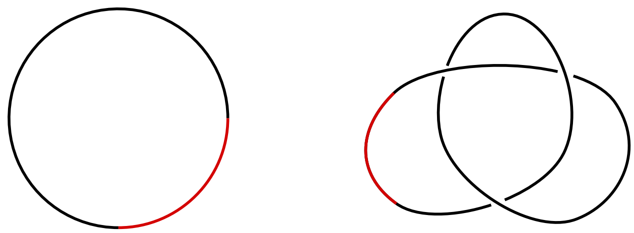

Figure 1.3: The unit circle S^2 and the trefoil knot.

neighborhoods on either space (colored in red). Notice that each of these neighborhoods is homeomorphic to an interval in $\mathbb{R}_1$, and hence homeomorphic to each other.

Definition. Let M be an n-dimensional manifold. A p-**dimensional submanifold** of M is a closed subset L of M for which there exists an atlas $\{(M_\alpha, \phi_a)\}$ of M such that for all $x \in L$, there exists a chart (M_α, ϕ_α) in where $x \in M_\alpha$ and $\phi_\alpha(L \cap M_\alpha) = \{0\} \times \mathbb{R}^p$.

Lemma 1.1.1. *Submanifolds of manifolds are manifolds.*

Lemma 1.1.2. *Let M be an m-manifold, and N an n-manifold. Then the product $M \times N$ is an $(n+m)$-manifold.*

Proof. We have that both M and N are Hausdorff, which makes $M \times N$ Hausdorff. Moreover, since M and N are second countable, they have countable bases $\mathcal{B}_M$ and $\mathcal{B}_N$. Then the product $\mathcal{B}_M \times \mathcal{B}_N$ serves as a countable basis for $M \times N$.

Now, let $\{(M_\alpha, \phi_\alpha)\}$ and $\{(N_\beta, \psi_\beta)\}$ be atlases for M and N respectively. Then since each M_α is open in M, and each N_β is open in N, $M_\alpha \times N_\beta$ is open in $M \times N$. Moreover we also have that $M = \bigcup M_\alpha$, $N = \bigcup N_\alpha$ so that

$$M \times N = (\bigcup M_\alpha) \times (\bigcup N_\beta) = \bigcup M_\alpha \times N_\beta$$

Now, we also have that ϕ_α is a homeomorphism of M_α onto an open subset of $\mathbb{R}^m$, and ψ_β is a homeomorphism of N_β onto an open subset of $\mathbb{R}^n$. Since ϕ_α and ψ_β are homeomorphisms, they are continuous with continuous inverses ϕ_α^{-1} and ψ_β^{-1}. This makes the map $\phi_\alpha \times \psi_\beta$ continuous with continuous inverse $(\phi_\alpha \times \psi_b)^{-1}$, which makes $\phi_\alpha \times \psi_b$ a homeomorphism of $M_\alpha \times N_\beta$ onto a subset of $\mathbb{R}^m \times \mathbb{R}^n \simeq \mathbb{R}^{m+n}$. Therefore $M \times N$ is an $(m+n)$-manifold. ∎

Example 1.2. The equator, S^1 of S^2 is a submanifold of S^2 (see figure 1.1).

Definition. We define the **boundry** of an n-manifold M to be the set ∂M, of all points of M for which there is a neigborhood homeomorphic to a neighborhood of H^n, but no neighborhood homeomorphic to a neighborhood of $\mathbb{R}^n$; where $H^n = \{x \in \mathbb{R}^n : x_1 \geq 0\}$.

Definition. Let $H^n = \{(x_1, \ldots, x_n) \in \mathbb{R}^n : x_1 \geq 0\}$. We define an **$n$-manifold with boundry** to be a second countable Hausdorff space M with atlas $\{(M_\alpha, \phi_\alpha)\}$ such that ϕ_α is a homeomorphism from M_α to an open subset of $\mathbb{R}^n$, or H^n.

Example 1.3. (1) The unit ball $B^n = \{x \in \mathbb{R}^n : \|x\| \leq 1\}$ is an n-dimensional manifold with boundry $\partial B^n = S^{n-1}$. For interior points of B^n, this is clear. For points in S^{n-1}, extending the stererographic projection gives the required homeomorphism.

(2) The **pair of pants** (see figure 1.4) Is a 2-manifold with boundry.

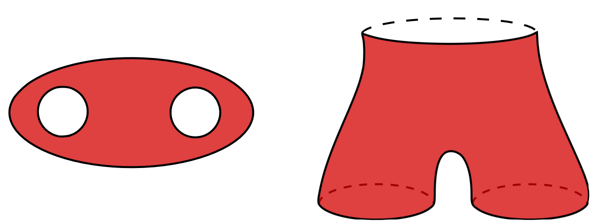

Figure 1.4:

(3) The 1-holed torus is a 2-manifold with boundry.

Definition. A **p-dimensional submanifold with boundry** of an n-dimensional manifold M is a closed subset L of M for which there is an atlas $\{(M_\alpha, \phi_a)\}$ of M and $0 \le p \le n$, such that for every $x \in L$ in the interior of M, there is a chart (M_α, ϕ_a) such that $x \in M_\alpha$, and $\phi_\alpha(L \cap M_\alpha) = \{0\} \times \mathbb{R}^p$, and for every $x \in L$ in the boundry of M, there is a chart (M_α, ϕ_α) such that $x \in M_\alpha$, and with $\phi_\alpha(L \cap M_\alpha) = \{0\} \times \mathbb{R}^p$, and for which $\phi_\alpha(x) \in \{0\} \times \partial H^p$.

Lemma 1.1.3. *The boundry of an n-manifold is an $(n-1)$-submanifold with boundry.*

Example 1.4. The diameter of the ball B^2 is a submanifold with boundry.

Definition. We call an n-manifold M **closed** if M is compact with nonempty boundry ∂M.

Example 1.5. The n-sphere and n-torus are closed manifolds. Additionally, the projection map $\pi_y : T^2 \to S^1$ fo $T^2 = S^1 \times S^1$ onto the second factor is a continuous map between manifolds.

Lemma 1.1.4. *If M is an n-manifold with boundry, then its boundry ∂M is an $(n-1)$-manifold without boundry.*

Proof. We first notice by our definition of boundry that $\partial M \subseteq M$. Since M is second countable and Hausdorff, ∂M inherits these properties as a subspace of M.

Now, consider a point $x \in \partial M$. Then by definition, there is a neighborhood U of x for which U is homeomorphic to a neighborhood V of H^n. Let $\phi : U \to V$ be the given homeomorphism. Notice, that $\partial H^n = \{x \in \mathbb{R}^n : x_1 = 0\}$, and that $\phi(x) \in V$ implies $\phi(x) \in V \cap \partial H^n$. Since V is open in H^n, $V \cap \partial H^n$ is open in ∂H^n as a subspace of H^n. So that $\phi(U) \simeq V \cap \partial H^n$ and U is homeomprphic to an open set in ∂H^n.

Now, take the projection map $\pi_y : \partial H^n \to \mathbb{R}^{n-1}$ onto the second factor, defined by $\pi_y : (0, (x_2, \ldots, x_n)) \to (x_2, \ldots, x_n) = (y_1, \ldots, y_{n-1})$. Then π_y defines a homeomorphism of ∂H^n onto $\mathbb{R}^{n-1}$, so that $\pi_y \circ \phi$ is a homeomorphism of U onto an open set of $\mathbb{R}^{n-1}$. This makes ∂M an $(n-1)$-manifold. Moreover, since $x \in \partial M$ was arbitrary, there is no $x \in \partial M$ with neighborhood homeomorphic to an open set in H^{n-1}, so that $\partial(\partial M) = \emptyset$; i.e. ∂M is without boundry. ■

1.2 Smooth Manifolds

Definition. We call a map $f : \mathbb{R}^n \to \mathbb{R}^n$ **q-smooth**, or C^q, if it has continuous partial derivatives of order q. We call f **smooth**, or C^∞, if it has continuous partial derivatives of all orders.

Definition. A **C^q-manifold**, wit $q > 0$ is a topological manifold with an atlas that is C^q. That is, for any charts (M_α, ϕ_α) and (M_β, ϕ_β), $\phi_\beta \circ \phi_\alpha^{-1}$ is C^q wherever it is defined. We call C^∞-manifolds **smooth manifolds**, or **differentiable manifolds**.

Example 1.6. (1) $\mathbb{R}^n$ is a smooth manifold, as are all its open subsets.

(2) Consider the n-manifold S^n with charts

$$(S^n \backslash (0, \dots, 0, 1), h) \qquad (S^n \backslash (0, \dots, 0, -1), h')$$

where

$$h(x_1, \dots, x_{n+1}) = \frac{1}{1 - x_{n+1}}(x_1, \dots x_n) \text{ and } h'(x_1, \dots, x_{n+1}) = \frac{1}{1 + x_{n+1}}(x_1, \dots x_n)$$

The map $h' \circ h^{-1}$ is smooth. Notice that

$$h^{-1}(y_1, \dots, y_n) = \Big(\frac{2y_1}{1 + y_1^2 + \dots + y_n^2}, \dots, \frac{2y_n}{1 + y_1^2 + \dots + y_n^2}\Big)$$

So that

$$h' \circ h^{-1} = \frac{1}{y_1^2 + \dots + y_n^2}(y_1, \dots, y_n)$$

Moreover, for all $q > 0$, $\partial^q h' \circ h^{-1}$ exists, which makes $h' \circ h^{-1}$ smooth. This makes S^n a smooth manifold.

(3) The product of smooth manifolds are smooth manifolds. In particular, the torus $T^2 = S^1 \times S^1$ is a smooth manifold.

Definition. Let M and N manifolds with atlases $\{(M_\alpha, \phi_\alpha)\}$ and $\{(N_\beta, \psi_\beta)\}$. We call a map $f : M \to N$ q**-smooth**, or C^q if $\psi_\beta \circ \phi_\alpha^{-1}$ is C^q wherever it is defined. We call C^q-maps between manifolds C^q**-diffeomorphisms**. We call C^∞-diffeomorphisms **diffeomorphisms**. We call any two C^q-manifolds **diffeomorphic** if there exists a C^q-diffeomorphism between them.

Example 1.7. (1) The map $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = x^3$ is a smooth map, but it is not a diffeomorphism, since $f'(x) = 3x^2$ has a singular point at 0 (elaborate?). It is not even a C^1-diffeomorphism.

(2) The projection map of $T^2 = S^1 \times S^1$ onto the second factor is a smooth map between manifolds.

Definition. LEt M a C^q-manifold, for some $q \geq 1$, and let $x \in M$ and (M_α, ϕ_α) a chart containing x. We call x a **critical point** of a map $f : M- < \mathbb{R}$ if it is a critical point of $f \circ \phi_\alpha^{-1}$. If $g : \mathbb{R}^n \to \mathbb{R}^n$ is a map, we call x a **nondegenerate** critical point if the Hessian of g is nonsingular at x, and we call x a **nondegenerate** critical point of f if it is a nondegenerate critical point of $f \circ \phi_\alpha^{-1}$.

Definition. We define a **Morse function** on a manifold M to be a smooth map $f : M \to \mathbb{R}$ such that

(1) f has nondegenerate critical points.

(2) Distinct critical points map to distinct values.

Example 1.8. The projection map of the Torus $T^2 \subseteq \mathbb{R}^3$ on to the thrid coordinate is a map with critical points. It has 1 maximum value, 2 minimum values, and 2 saddle points. Moreover these critical points are nondegenerate, so that the projection is a Morse function.

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