Real Analysis

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July 5, 2023

Contents

Chapter 1

The Real and Complex Numbers

1.1 Ordered Sets

Definition. Let S be any set. An **order** on S is a relation $<$ such that:

[label=(0)]

1. For $x, y \in S$, one and only one of the following hold:

$$x < y \qquad\qquad x = y \qquad\qquad y < x$$

   We call this property the **trichotomy law**

2. $<$ is transitive over S.

We denote the relations $>$ and $\leq$ to mean $x > y$ if and only if $y < x$, and $x \leq y$ if and only if $x < y$, or $x = y$. We call S together with $<$ an **ordered set**.

Example 1.1. Define $<$ on $\mathbb{Q}$ such that for $r, s \in \mathbb{Q}$, $r < s$ implies $< 0s - r$.

Definition. Let S be an ordered set, and let $E \subseteq S$. We say that E is **bounded above** there is some $\beta \in S$ for which $x \leq \beta$, for all $x \in E$. We say that E is **bounded below** if $\beta \leq x$, for call $x \in E$. We say an $\alpha \in S$ is a **least upperbound** of E, if α is an upperbound of E, and for all other upperbounds, γ, of E, $\alpha \leq \gamma$. Likewise, α is a **greatest lowerbound** of E if α is a lowerbound of E, and for all other lowerbounds γ of E, $\gamma \leq \alpha$. We denote the least upperbound, and greatest lowerbound by $\sup E$ and $\inf E$, respectively.

Lemma 1.1.1. *Let S be an ordered set, and let $E \subseteq S$. Then E has (if they exist) a unique least upperbound, and a unique greatest lowerbound.*

Proof. Let $\alpha, \beta \in S$ be least upperbounds of E. Then by definition, we have that $\alpha \leq \beta$, and $\beta \leq \alpha$; thus by the trichotomy law, $\alpha = \beta$. The proof is the same for greatest lowerbounds. ∎

Example 1.2. [label=(0)]

1. Let $A = \{p \in \mathbb{Q} : p^2 < 2\}$, and $B = \{p \in \mathbb{Q} : p^2 > 2\}$. Clearly, we have that every element of B is an upperbound of A, and every element of A is a lowerbound of B. Now take $p \in \mathbb{Q}$ a positive rational, and take $q \in \mathbb{Q}$ such that $q = p - \frac{p^2-2}{p+2}$. Then $q^2 - 2 = \frac{2(p^2-2)}{(p+2)^2}$. Now if $p \in A$, then $p^2 - 2 < 0$, which implies that $p < q$, and $q^2 < 2$; thus A has no largest element; similarly, if $p \in B$, then $p^2 - 2 > 0$, which implies that $q < p$ and $q^2 > 2$, which shows that B has no least element. Thus $\sup A$ and $\inf B$ do not exist in $\mathbb{Q}$.

2. If $\alpha = \sup E \in S$, it may or may not be that $\alpha \in E$. Take $E_1 = \{r \in \mathbb{Q} : r < 0\}$, and $E_2 = \{r \in \mathbb{Q} : r \leq 0\}$. Then $\sup E_1 = \sup E_2 = 0$, but $0 \notin E_1$, where as $0 \in E_2$

3. Consider the set $\frac{1}{\mathbb{Z}^+} = \{\frac{1}{n} : n \in \mathbb{Z}^+\}$. By the well ordering principle, 1 is the least element, and is also an upperbound of all $\frac{1}{n}$ for $n > 1$. Now also notice that as n gets arbitrarily large, then $\frac{1}{n}$ gets arbitratirly small; that is to say $\frac{1}{n}$ "tends" to 0, so $\sup \frac{1}{\mathbb{Z}^+} = 1 \in \frac{1}{\mathbb{Z}^+}$, and $\inf \frac{1}{\mathbb{Z}^+} = 0 \notin \frac{1}{\mathbb{Z}^+}$.

Definition. We say an ordered set S has the **least upperbound property**, if whenever $E \subseteq S$, nonempty, and bounded above, then $\sup E \in S$ exists; likewise, S has the **greatest lowerbound property** if whenever E is nonempty, bounded below then $\inf E \in S$ exists.

Example 1.3. [label=(0)]

1. The set of all rationals $\mathbb{Q}$ does not have the least upperbound property, nor the greatest lowerbound property, take A, B as in the previous example. Letting $E = \{1, \frac{1}{2}, \frac{1}{4}\} \subseteq \frac{1}{\mathbb{Z}^+}$, we see that $\frac{1}{\mathbb{Z}^+}$ satisfies both properties, with $\sup E = 1$, and $\inf E = \frac{1}{4}$.

2. Let $A \subseteq \mathbb{R}$ be nonempty, and be bounded below. Then by the greatest lowerbound property, $\alpha = \inf A \in \mathbb{R}$ exists; Then for all $x \in A$, $\alpha \leq x$, and for all other lowerbounds γ, $\gamma \leq \alpha$. Then $-x \leq -\alpha$, and $-\alpha \leq -\gamma$, then we see that $-\gamma$ and $-\alpha$ are upperounds of $-A$, and that $-\alpha$ is the least upperbound of $-A$

Theorem 1.1.2. *If S is an ordered set with the least upperbound property, then S also inherits the greatest lowerbound property.*

Proof. Let $B \subseteq S$, and let $L \subseteq S$ be the set of all lowerbounds of B. Then we have for any $y \in L$, $x \in B$, $y \leq x$. So every element of B is an upperbound of L, and L is nonempty, hence $\alpha = \sup L \in S$ exists. Now if $\gamma \leq \alpha$, then γ is not an upperbound of L, hence $\gamma \notin B$; thus $\alpha \leq x$ for all $x \in B$, so $\alpha \in L$, and by definition of the greatest lowerbound, we get $\alpha = \inf B$. ∎

1.2 Fields

Definition. A **field** is a set F, together with binary operations $+$ and $\cdot$ (called **addition** and **multiplication**, respectively) such that:

[label=(0)]

1. F forms an abelian group under $+$.

2. $F\backslash\{0\}$ forms an abelian group under $\cdot$ (where 0 is the additive identity of F).

3. $\cdot$ distributes over $+$.

We now state the following propositions without proof.

Proposition 1.2.1. *For all $x, y, x \in F$:*

[label=(0)]$x + y = x + y$ implies $y = z$ $x + y = x$ implies $y = 0$ $x + y = 0$ implies $y = -x$ $-(-x) = x$.

Proposition 1.2.2. *For all $x, y, x \in F\backslash\{0\}$:*

[label=(0)]$xy = xy$ implies $y = z$ $xy = x$ implies $y = 1$ $xy = 1$ implies $y = x^{-1}$ $(x^{-1})^{-1} = x$.

Proposition 1.2.3. *For all $x, y, x \in F$:*

[label=(0)]$0x = 0$ $x \neq 0$ and $y \neq 0$ implies $xy \neq 0$ $(-x)y = -(xy) = x(-y)$ $(-x)(-y) = xy$.

Definition. An **ordered field** is a field F that is also an ordered set, such that:

[label=(0)]$x + y < x + z$ whenever $y < z$, for $x, yz, z \in F$ $xy > 0$ whenever $x > 0$ and $y > 0$, for $x, y \in F$.

Proposition 1.2.4. *Let F be an ordered field, then for any $x, y, z \in F$, the following hold:*

[label=(0)]$x > 0$ implies $-x < 0$. If $x > 0$ and $y < z$, then $xy < xz$. If $x < 0$ and $y < z$, then $xz < xy$. If $x \neq 0$, then $x^2 > 0$, in particular, $1 > 0$. $0 < x < y$ implies that $0 < y^{-1} < x^{-1}$.

Proof. [label=(0)]

1. If $x > 0$, then $0 = x + (-x) > 0 + (-x)$, so $-x < 0$.

2. We have $0 < z - y$, so $0 < x(z - y) = xz - xy$, so $xy < xz$.

3. Do the same as (2),, multiplying $z - y$ by $-x$.

4. If $x > 0$, we are done. Now suppose that $x < 0$, then $-x > 0$, so $(-x)(-x) = xx = x^2 > 0$; in particular, we also have that $1 \neq 0$, and $1 = 1^2$, so $1 > 0$.

5. We have $0 < xy^{-1} < yy^{-1} = 1$, then $0 < x^{-1}xy^{-1} = y^{-1} < x^{-1}1 = x^{-1}$

∎

1.3 The Field of Real Numbers

Theorem 1.3.1. *There exists an ordered field $\mathbb{R}$ with the least upperbound property, such that $\mathbb{Q} \subseteq \mathbb{R}$.*

Definition. We call the field $\mathbb{R}$ the **field of real numbers**,and we call the elements of $\mathbb{R}$ **real numbers**.

Definition. Let S be an ordered field, and let $E \subseteq S$. We say that E is **dense** in S, if for all $r, s \in S$, with $r < s$, there is an $\alpha \in E$ such that $r < \alpha < s$.

Theorem 1.3.2 (The Archimedean Principle)**.** *If $x, y \in \mathbb{R}$, and $x > 0$, then there is an $n \in \mathbb{Z}^+$ such that $nx > y$.*

Proof. Let $A = \{nx : n \in \mathbb{Z}^+\}$, and suppose that $nx \leq y$. Then y is an upperbound of A, abd since A is nonempty, $\alpha = \sup A \in \mathbb{R}$, since $x > 0$, we have $\alpha - x < \alpha$, so $\alpha - x$ is not an upperbound of A. Hence $\alpha - x < mx$ for some $m \in \mathbb{Z}^+$. Then $\alpha < (1 - m)x \in A$, contradicting that α is an upperbound of A. ■

Theorem 1.3.3 (The density of $\mathbb{Q}$ in $\mathbb{R}$)**.** *$\mathbb{Q}$ is dense in $\mathbb{R}$.*

Proof. Let $x < y$ be realnumbers, then $y - x > 0$, so by the Archimedean principle, there is an $n \in \mathbb{Z}^+$ fir which $n(y-x) > 1$. By the Archimedean principle again, we have $m_1, m_2 \in \mathbb{Z}^+$ for which $m_1 > nx$ and $m_2 > -nx$, thus $-m_2 < nx < m_1$, and we also have that there is an $m \in \mathbb{Z}^+$ for which $-m_2 < m < m_1$, and $m - 1 \leq nx < m$. Thus combining inequalities, we get $nx < m < ny$, thus $x < \frac{m}{n} < y$. ■

Theorem 1.3.4 (The existence of n^th roots of positive reals)**.** *For every real number $X > 0$, and for every $n \in \mathbb{Z}^+$, there is one, and only one positive real number y for which $y^n = x$.*

Proof. Let $y > 0$ be a real number; then $y^n > 0$, so there is atmost one such y for which $y^n = x$. Now let $E = \{t : \mathbb{R} : t^n < x\}$, choosing $t = \frac{x}{1+x}$, we see that $0 \leq t < 1$, hence $t^n < t < x$, so E is nonempty. Now if $1 + x < t$, then $t^n \geq x$, so $t \notin E$, and E has $1 + x$ as an upperbound. Therefore, $\alpha = \sup E \in \mathbb{R}$ exists.

Now suppose that $y^n < x$, choose $0 \leq h < 1$ such that $h < \frac{x-y^n}{n(y+1)^{n-1}}$, then $(y+h)^n - y^n < hn(y+h)^{n-1} < hn(y+1)n - 1 < x - y^n$, thus $(y+h)^n < x$, so $y + h \in E$, contraditing that y is an upperbound. On the other hand, if $y^n > x$, choosing $k = \frac{y^n - x}{ny^{n-1}}$, then $0 \leq k < y$, and letting $t \geq y - k$, we get that $y^n - t^n \leq y^n + (y-k)^n < kny_{n-1} = y^n - x^n$, so $t^n \geq x$, making $y - k$ an uppeerbound of E, which contradicts $y = \sup E$. ■

Remark. We denote y as $\sqrt[n]{x}$, or as $x^{\frac{1}{n}}$.

Corollary. *If $a, b \in \mathbb{R}$, with $a, b > 0$, and $n \in \mathbb{Z}^+$, then $\sqrt[n]{ab} = \sqrt[n]{a}\sqrt[n]{b}$.*

Proof. Let $\alpha = \sqrt[n]{a}$, and $\beta = \sqrt[n]{b}$. Then $\alpha^n = a$, and $\beta^n = b$, so $ab = \alpha^n\beta^n = (l\alpha\beta)^n$, we are done. ■

Definition. We define the **extended real number system** to be the field $\mathbb{R}$, together with symbols ∞, and $-\infty$, called **positive infinity** and **negative infinity**, such that $-\infty < x < \infty$ for all $x \in \mathbb{R}$. We call elements of the extended real numbers **infinite**, and every other element not in the extended real numbers **finite**.

Lemma 1.3.5. ∞ *is an upperbound for every subset* E*, of* $\mathbb{R}$*, and* $-\infty$ *is a lowerbound for every subset* E *of* $\mathbb{R}$*. Moreover, if* E *is not bounded above, then* $\sup E = \infty$*, and if* E *is not bounded below, then* $\inf E = -\infty$*.*

Remark. We make the following assumptions for extended real numbers:

[label=(0)]If $x \in \mathbb{R}$, then $x + \infty = \infty$, $x - \infty = -\infty$, and $\frac{x}{\infty} = \frac{x}{-\infty} = 0$. If $x > 0$, then $x(\infty) = \infty$ and $x(-\infty) = -\infty$. If $x < 0$, then $x(\infty) = -\infty$ and $x(-\infty) = \infty$.

1.4 The Complex Field

Definition. We define a **complex number** to be a pair of real numbers (a, b). We denote the set of all comlex numbers by $\mathbb{C}$. We define the **addition** and **multiplication** of complex numbers to be the binary operations $+ : \mathbb{C} \to \mathbb{C}$ and $\cdot : \mathbb{C} \to \mathbb{C}$ such that

$$\begin{aligned}(a, b) + (c, d) &= (a + c, b + d) \\ (a, b)(c, d) &= (ac - bd, ad + bc)\end{aligned}$$

Lastly, we define i to be the complex number such that $i = (0, 1)$.

Theorem 1.4.1. $\mathbb{C}$ *forms a field together with* $+$ *and* $\cdot$*.*

Theorem 1.4.2. *For* $(a, 0), (b, 0) \in C$*,* $(a, 0) + (b, 0) = (a + b, 0)$*, and* $(a, 0)(b, 0) = (ab, o)$*.*

3. *Proof.* This is a straightforward application of the addition and multiplication of complex numbers. ∎

Theorem 1.4.3. $i^2 = -1$*.*

Proof. $i^2 = (0, 1)(0, 1) = (0 - 1, 1 - 1) = (-1, 0) = -1$. ∎

Theorem 1.4.4. *Let* $(a, b) \in \mathbb{C}$*, then* $(a + b) = a + ib$*.*

Proof. $(a, b) = (a, 0) + (0, b) = (a, 0) + (0, 1)(b, 0) = a + ib$. ∎

Definition. Let $a, b \in \mathbb{R}$, and let $z \in \mathbb{C}$ such that $z = a + ib$. We define the **complex conjugate** of z to be the complex number $\overline{z} = a - ib$. Moreover, we define the **real part** of z to be a, and the **imaginary part** of z to be b, and we denote them $a = \operatorname{Re} z$, $b = \operatorname{Im} z$

Theorem 1.4.5. *Let* $z, w \in \mathbb{C}$*. Then*

[label=(0)]$\overline{z + w} = \overline{z} + \overline{w}$*.* $\overline{zw} = \overline{z}\overline{w}$*.* $z + \overline{z} = 2 \operatorname{Re} z$ *and* $z - \overline{z} = 2i \operatorname{Im} z$*.* $z\overline{z}$ *is a nonegative real number.*

4. *Proof.* Let $z = a + ib$, and let $w = c + id$. Then $z + w = (a + c) + i(b + d$, so $\overline{z + w} = (a + b) - i(b + d) = (a - ib) + (c - id) = \overline{z} + \overline{w}$; similarly, we get $\overline{zw} = \overline{z}\overline{w}$. Moreover, we have $(a + ib) + (a - ib) = 2a$, and $(a + ib) - (a - ib) = 2ib$, we also have that $z\overline{z} = (a + ib)(a - ib) = a^2 + b^2 \geq 0$, and $z\overline{z} = 0$ if and only if $a = b = 0$. ∎

Definition. Let $z \in \mathbb{C}$. We define the **modulus** of z to be $|z| = \sqrt{z\overline{z}}$.

Remark. $|z|$ exists and is unique.

Theorem 1.4.6. *Let $z, w \in \mathbb{C}$, then:*

[label=(0)]$|z| \geq 0$ and $|z| = 0$ if and only if $z = 0$. $|\overline{z}| = |z|$. $|zw| = |z||w|$. $\operatorname{Re} z \leq |z|$. $|z + w+ \leq |z| + |w|$.

4. *Proof.* Let $z = a + ib$, and $w = c + id$. Then $|z| = \sqrt{a^2 + b^2} \geq 0$, and $|z| = 0$ if and only if $a, b = 0$. Moreover, $|\overline{z}| = |a + i(-b)| = \sqrt{a^2 + (-b)^2} = \sqrt{a^2 + b^2} = |z|$. We also habe $|zw|^2 = (a^2 + b^2)(c^2 + d^{)} = |z|^2|w|^2$, likewise, $\||rez| = |a + i0| = \sqrt{a^2} \leq \sqrt{a^2 + b^2}$. Finally we prove (5).

We have $|z + w|^2 = (x + w)(\overline{z} + \overline{w}) = z\overline{z} + \overline{z}w + \overline{w}z + w\overline{w} = |z|^2 + w \operatorname{Re} z\overline{w} + |w|^2 \leq |z|^2 + 2|s\overline{w}| + |w|^2 = (|z| + |w|)^2$. ■

Theorem 1.4.7 (The Cauchy Schwarz Inequality)**.** *Let $a_i, b_i \in \mathbb{C}$, for $1 \leq i \leq n$. Then:*

$$|\sum_{i=1}^{n} a_i\overline{b_i}| \leq \sum_{i=1}^{n} |a_i|^2 \sum_{i=1}^{n} |b_j|^2 \tag{1.1}$$

Proof. Let $A = \sum a_j|^2$, $B = \sum |b_i|^2$, and $C = \sum a_i\overline{b_i}$. If $B = 0$, then $b_i = 0$ for $1 \leq i \leq n$, and we are done; so suppose that $B > 0$. Then

$$\begin{aligned}
\sum |Ba_j - Cb_j|^2 &= \sum (Ba_j - Cb_j)(B\overline{a_j} - \overline{Cb_j}) \\
&= B \sum |a_j|^2 - B\overline{C} \sum a_j\overline{b_j} - BC \sum \overline{a_j}b_j + |C^2| \sum |b_j|^2 \\
&= (B^2A - B|C|^2) = B(AB - |C|^2) \geq 0
\end{aligned}$$

Since $B > 0$, we get $|C|^2 \leq AB$ as required. ■

1.5 Euclidean Spaces

Definition. Let $k \in \mathbb{Z}^+$, and let $\mathbb{R}^k$ be the set of all ordered k-tuples $(x_1, x_2, \ldots, x_k)$, with $x_i \in \mathbb{R}$ for $1 \leq i \leq k$. We call $\mathbb{R}^k$ the **Euclidean space** of **dimension** k; more simply the **Euclidean k-space**. We call elements of $\mathbb{R}^k$ **vectors** or **points**; and we define **vector addition** and **scalar multiplication** to be:

$$\begin{aligned}
(x_1, \ldots, x_k) + (y_1, \ldots, y_k) &= (x_1 + y_1, \ldots, x_k + y_k) \\
\alpha(x_1, \ldots, x_k) &= (\alpha x_1, \ldots, \alpha x_k)
\end{aligned}$$

for $(x_1, \ldots, x_k), (y_1, \ldots, y_k) \in \mathbb{R}^k$ and $\alpha \in \mathbb{R}$.

Theorem 1.5.1. $\mathbb{R}^k$ *forms a vector space together with vector addition and scalar multiplication.*

Definition. Let $x, y \in \mathbb{R}^k$. We define the **inner product** of x and y to be the binary operation $\langle , \rangle : \mathbb{R}^k \mathbb{R}^k \to \mathbb{R}$ such that

$$\langle x, y \rangle = \sum_{i=1}^{k} x_i y_i$$

We define the **norm** of x to be $||x|| = \sqrt{\langle x, x \rangle} = \sqrt{\sum x_i^2}$.

Theorem 1.5.2. *Let $x, y \in \mathbb{R}^k$, and $\alpha \in \mathbb{R}$. Then:*

[label=(0)]$||x|| \geq 0$ and $||x|| = 0$ if and only if $x_i = 0$ for all $1 \leq i \leq k$. $||\alpha x|| = |\alpha| ||x||$. $||\langle x, y \rangle|| \leq ||x|| ||y||$. $||x + y|| \leq ||x|| + ||y||$, and $||x - z|| \leq ||x - y|| + ||y - z||$

4. *Proof.* (1) follows by definition of the norm. We also have that $||\alpha x|| = \sqrt{\sum \alpha^2 x_i^2} = \sqrt{\alpha^2} \sqrt{\sum x_i^2} = |\alpha| ||x||$.

Now by the Cauchy Schwarz inequality, we have that $||\langle x, y \rangle||^2 = \sum x_i^2 y_i^2 \leq \sum x_i^2 \sum y_i^2 = ||x|| ||y||$. Finally we have that $||x + y|| = \langle x + y, x + y \rangle = \langle x, x \rangle + 2\langle x, y \rangle + \langle y, y \rangle \leq ||x||^2 + 2||x|| ||y|| + ||y^2|| = (||x|| + ||y||)^2$, the last result follows immediately. ■

Chapter 2

Topological Foundations

2.1 Finite, Countable, and Uncountable Sets

Definition. Let A be a set, and let $E \subseteq \mathbb{N}$. We say that A is **finite** if there exists a 1-1 mapping of A ont E, we say A is **countable** if $E = \mathbb{N}$, and we say A is **atmost countable** if A is either finite or countable.

Example 2.1. The set of all integers $\mathbb{Z}$ is countable. Take $f : \mathbb{N} \to \mathbb{Z}$ such that $f(n) = 2$ if n is even, and $f(n) = -n$ if n is odd.

Definition. Let A be a set, and let $E \subseteq \mathbb{N}$. A **sequence** in A is a mapping $f : E \to A$ such that $f(n) = x_n$, for $x_n \in A$. We call the values of f **terms** of the sequence. We denote sequences by $\{x_n\}_{i=1}^{n}$, and when $E = \mathbb{N}$, we denote them simply by $\{x_n\}$.

Theorem 2.1.1. *Every infinite subset of a countable set is countable.*

Proof. Let A be countable, and let $E \subseteq A$ be infinite. Arrange the elements of A into a sequence $\{x_n\}$, and construct a sequence $\{n_k\}$ such that n_1 is the least term for which $\{x_{n_k}\} \in E$, and n_k is the least term greater than n_{k-1} for which $x_{n_k} \in E$. Let $f(k) = x_{n_k}$, and we get a 1-1 mapping of $\mathbb{N}$ onto E. ■

Theorem 2.1.2. *Let $\{E_n\}$ be a sequence of countable sets. Then $S = \bigcup E_n$ is also countable.*

Proof. Arrange every set E_n in a sequence $\{x_{nk}\}$, and consider the infinite array (x_{ij}), in which the elements of E_n form the n-th row. Then (x_{ij}) contains all the elements of S, and we can arrange them is a sequence

$$x_{11}, (x_{21}, x_{12}), (x_{31}, x_{22}, x_{13}), \ldots$$

Moreover, if $E_j \cap E_j \neq \emptyset$, for $i \neq j$, then the elements of $E_i \cap E_j$ appear more than once in the sequence of S; so taking $T \subseteq \mathbb{N}$, we get a 1-1 mapping of T onto S, hence S is atmost countable, and since $E_i \subseteq S$ for $i \in \mathbb{N}$, is infinite, by theorem 2.1.1, S is infinite, thus S is countable. ■

▫Figures/diagonalizationArray.png

Figure 2.1: The infinite array (x_{ij})

Corollary. *Let A be atmost countable, and suppose for all $\alpha \in A$ that the sets B_α are atmost countable. Then*

$$T = \bigcup_{\alpha \in A} B_\alpha$$

is atmost countable.

Theorem 2.1.3. *Let A be countable, and let B_n be the set of all n-tuples $(a_1, \ldots, a_n)$ such that $a_i \in A$ for $1 \leq i \leq n$. Then B_n is countable.*

Proof. By induction on n, we have that $B_1 = A$, which is countable. Now suppose that B_n is countable, and consider B_{n+1} whose elements are of the form (b, a) where $b \in B_n$ and $a \in A$. Fixing b, we get a 1-1 correspondence between the elements of B_{n+1} and A; therefore B is countable. ■

Corollary. $\mathbb{Q}$ *is countable.*

Proof. For every rational $\frac{p}{q} \in \mathbb{Q}$, represent $\frac{p}{q}$ as (p, q). Then the countability of $\mathbb{Q}$ follows from theorem 2.1.3. ■

Theorem 2.1.4. *Let A be the set of all sequences of* 0 *and* 1*; then A is uncountable.*

Proof. Let EA be countable, and let E consist of all the sequences of 0 and 1, $s_1, s_2, s_3, \ldots$. Construct the sequence s such that if the n-th term of the sequence s_i is 0, then the n-th term of s is 1, and vice versa, for $i \in \mathbb{Z}^+$. Then the sequence s differs from the sequence s_i at atleast one place; thus $s \notin E$, but $s \in A$. Therefore $E \subset A$, which establishes the uncountablitiy of A. ■

2.2 Metric Spaces

Definition. A set X, whose elements we will call **points**, is said to be a **metric space** if there exists a mapping $d : X \times X \to \mathbb{R}$, called a **metric** (or **distance function**) such that for $x, y \in X$

[label=(0)]

1. $d(x, y) \geq 0$, and $d(x, y) = 0$ if and only if $x = y$.
2. $d(x, y) = d(y, x)$.
3. $d(x, y) \leq d(x, z) + d(z, y)$ (The Triangle Inequality).

Example 2.2. The absolute value, $|\cdot|$ for real numbers, the modulus $|\cdot|$ for complex numbers, and the norm $||\cdot||$ for vectors are all metrics. They turn $\mathbb{R}$, $\mathbb{C}$, and $\mathbb{R}^k$ into metric spaces respectively.

Definition. An **open interval** in $\mathbb{R}$ (or **segment**) is a set of the form $(a, b) = \{a, b \in \mathbb{R} : a < x < b\}$, a **closed interval** in $\mathbb{R}$ is a set of the form $[a, b] = \{x \in \mathbb{R} : a \leq x \leq b\}$; and **half open intervals** in $\mathbb{R}$are sets of the form $[a, b) = \{x \in \mathbb{R} : a \leq x < b\}$ and $(a, b] = \{x \in \mathbb{R} : a < x \leq b\}$.

If $a_i < b_i$, for $1 \leq i \leq k$, the set of all points $(x_1, \ldots, x_k) \in \mathbb{R}^k$ which satisfy the Inequalities $a_i \leq x_i \leq b_i$ is called a **k-cell** in $\mathbb{R}^k$. If $x \in \mathbb{R}^k$, and $r > 0$, we call the set $B_r(x) = \{y \in \mathbb{R}^k : ||x - y|| < r\}$ an **open ball** in $\mathbb{R}^k$, and we call the set $B_r[x] = \gamma \in \mathbb{R}^k : ||x - y|| \leq r\}$ a **closed ball** in $\mathbb{R}^k$.

Definition. We call a set $E \subseteq \mathbb{R}^k$ **convex**, if whenever $x, y \in E$, $\lambda x + (1 - \lambda)y \in E$ for $0 < \lambda < 1$.

Lemma 2.2.1. *Open and closed balls, along with k-cells are convex.*

Proof. Let $B_r(x)$ be an open ball; let $y, x \in B_r(x)$, and $0 < \lambda < 1$. Then $||x - (\lambda y + (1 - \lambda)z|| = ||\lambda(x - y) - (1 - \lambda)(x - z)|| \leq \lambda||x - y|| + (1 - \lambda)||x - z|| < \lambda r + (1 - \lambda)r$. The proof is analogous for closed ball.

Now let K be a k-cell for $a_i < b_i$, for $1 \leq i \leq k$, let $x, y \in K$, then $a_i \leq x_i, y_i \leq b_i$, so $\lambda a_i \leq \lambda x_i \leq \lambda b_i$, and $(1 - \lambda)a_i \leq (1 - \lambda)y_i \leq (1 - \lambda)b_i$, since $0 < \lambda < 1$, $a_i \leq a_i + (1 - \lambda)a_i \leq \lambda x_i + (1 - \lambda)y_i \leq \lambda b_i + (1 -)b_i \leq b$. ■

Corollary. *Open and closed intervals, along with half open intervals are convex.*

Proof. We just notice that open and closed intervals are open and closed balls in $\mathbb{R}^1 = \mathbb{R}$, we also notice that half open intervals $[a, b)$ and $(a, b]$ are subsets of the closed interval $[a, b]$, and hence inherit convexity. ■

For the following definitions, let X be a metric space with metric d.

Definition. A **neighborhood** of a point $x \in X$ is the set $N_r(x) = \{y \in X : d(x, y) < r\}$ for some $r > 0$ called the **radius** of the neighborhood. We call x a **limit point** of a set $E \subseteq X$ if every neighborhood of x contains a point $y \neq x$ such that $y \in E$. If $y \in E$, and y is not a limit point, we call y an **isolated point**.

Definition. We call a set $E \subseteq X$ **closed** if every limit point of E is in E. A point $x \in X$ is an **interior point** of E if there is a neighborhood N of x such that $N \subseteq E$. We call E **open** if every point of E is an interior point of E.

Definition. $E \subseteq X$ is called **prefect** if E is closed, and every point of E is a limit point of E. We call E **dense** if every point of X is either a limit point of E, or a point of E, or both.

Lemma 2.2.2. *If $E \subseteq X$, then E is perfect in X if and only if $\overline{E} = E$.*

Lemma 2.2.3. *If EX is dense in X, then either E is perfect in X, or $X = E$, or both.*

Definition. We call $E \subseteq X$ **bounded** if there is a real number $M > 0$, and a point $y \in X$ such that $d(x, y) < M$ for all $x \in E$.

Theorem 2.2.4. *Let X be a metric space and $x \in X$. Every neighborhood of x is open.*

Proof. Consider the neighborhood $N_r(x)$, and $y \in E$, there is a positive real number h such that $d(x, y) = r - h$, then for $z \in X$ such that $d(y, s) < h$, we have $d(x, s) \leq d(x, y) + d(y, s) < r - h + h = r$, thus $s \in E$, so y is an interior point of E. ■

Theorem 2.2.5. *If x is a limit point of a set E, then every neighborhood of x contains infinitely many points of E.*

Proof. Let N be a neighborhood of x contai ning only a finite number points of E. Let $y_1, \ldots, y_n$ be points of $N \cap E$ distinct from x and let $r = \min\{d(x, y_i)\}$ for $1 \leq i \leq n$, then $r > 0$, and the neighborhood $N_r(x)$ contains no point y of E for which $y \neq x$, so x is not a limit point; which is a contradiction. ■

Corollary. *A finite point set has no limit points.*

Proof. By theorem 2.2.5, if x is a limit point in the finite point set E, then evry neoghborhood of contains infinitely many points of E; contradicting its finiteness. ■

Example 2.3. [label=(0)]

1. The set of all $z \in \mathbb{C}$ such that $|z| < 1$ is open, and bounded.

2. The set of all $z \in \mathbb{C}$ for which $|z| \leq 1$ is closed, perfect, and bounded.

3. Any nonempty finite set is closed, and bounded.

4. $\mathbb{Z}$ is closed, but it is not open, perfect, or bounded.

5. The set $\frac{1}{\mathbb{Z}^+}$ is neither closed, nor open, it is not perfect; but it is bounded..

6. $\mathbb{C}$ is closed, open, and perfect, but it is not bounded.

7. The open interval in (a, b) is open (only in $\mathbb{R}$), and bounded.

Theorem 2.2.6. *Let X be a metric space, a set $E \subseteq X$ is open if and only if $X \backslash E$ is closed.*

Proof. Suppose that $X \backslash E$ is closed, let $x \in E$, then $x \notin X \backslash E$, and x is not a limit point of $X \backslash E$. Thus there is a neighborhood N of x such that $N \cap E = \emptyset$, thus $N \subseteq E$, and so x is an interior point of E.

Conversely, suppose that E is open, and let x be a limit point of $X \backslash E$, then every neighborhood of of x contains a point of $X \backslash E$, so x is not an interior point of E, since E is open, it follows that $x \in X \backslash E$, thus $X \backslash E$ is closed. ■

Corollary. *E is closed if and only if $X \backslash E$ is open.*

Proof. This is the converse of theorem 2.2.5. ■

Theorem 2.2.7. *Let X be a metric space. The following are true:*

[label=(0)]If $\{G_\alpha\}$ is a collection of open sets, then $\bigcup G_\alpha$ is open. If $\{G_i\}_{i=1}^n$ is a finite collection of open sets, then $\bigcap_{i=1}^n G_i$ is open. if $\{G_\alpha\}$ is a collection of closed sets, then $\bigcap G_\alpha$ is closed. If $\{G_i\}_{i=1}^n$ is a finite collection of closed sets, then $\bigcup_{i=1}^n G_i$ is closed.

Proof. Let $G = \bigcup G_\alpha$, then if $x \in G$, $x \in G_\alpha$ for some α, then x is an interior point of G_α, hence an interior point of G, so G is open. Now let $G = \bigcap_{i=1}^{n} G_i$ For $x \in G$, there are neighborhoods N_i of x, with radii r_i such that $N_i \subseteq G_i$ for $1 \leq i \leq n$. Then let $r = \min\{r_1, \dots, r_n\}$, and let N be the neighborhood of x with radius r, then $N \subseteq G_i$, hence $N \subseteq G$, so G is open.

The proofs of (3) and (4) are just the converse of the proofs of (1) and (2). ■

Definition. Let X be a metric space, and let $E \subseteq X$, and let E' be the set of all limit points of E. We define the **closure** of E to be the set $\overline{E} = E \cup E'$.

Theorem 2.2.8. *If X is a metric space, and $E \subseteq X$, then the following hold*

[label=(0)]$\overline{x}$ is closed. E is closed if and only if $E = \overline{E}$. If $F \subseteq X$ such that $E \subseteq F$, and F is closed, then $\overline{E} \subseteq F$.

Proof. If $x \in X$, and $x \notin \overline{E}$, then $x \notin E$, nor is it a limit point of E, thus there is a neighborhood of x that is disjoint from E, hence $X \backslash \overline{E}$ is open.

Now if E is closed, then $E' \subseteq E$, so $\overline{E} = E$, conversely, if $E = \overline{E}$, then clearly E is closed. Now if F is closed and $E \subseteq F$, then $F' \subseteq F$, and $E' \subseteq F$, therfore $\overline{E} \subseteq F$. ■

Theorem 2.2.9. *Let $E\mathbb{R}$ be nonempty and bnounded above, let y $supE$, then $y \in \overline{E}$, hence $y \in E$ if E is closed.*

Proof. Suppose that $y \notin E$, then for every $h > 0$, there exists a point $x \in E$ such that $y - h < x < y$, then y is a limit point of E, thus $y \in \overline{E}$. ■

Theorem 2.2.10. *Let $Y \subseteq X$; a subset E of Y is open in Y if and only if $E = Y \cap G$ for some open subset G of X.*

Proof. Suppose E is open in Y, then for each $x \in E$, there is a $r_p > 0$ such that $d(x, y) < r_p$, if $y \in Y$, that implies that $y \in E$; hence let V_x be the set of all $y \in X$ such that $d(x, y) < r_p$, and define

$$G = \bigcup_{x \in E} V_p$$

Then by theorems 2.2.2 and 2.2.6, G is open in X, and $EG \cap Y$. Now we also have that $V_p \cap YE$, thus $G \cap YE$, thus $E = G \cap Y$. Conversely, if G is open in X, and $E = G \cap Y$, then every $x \in E$ has a neighborhood $v_p \in G$, thus $V_p \cap Y \subseteq E$, hence E is open in Y. ■

2.3 Compact Sets

Definition. Let X be a metric space, and let $E \subseteq X$. An **open cover** of E is a collection $\{G_\alpha\}$ of subsets of X such that $E \subseteq \bigcup G_\alpha$. We call a collection $\{E_\beta\}$ of subsets of X an **open subcover** of E if $\{E_\beta\}$ is a cover of E, and $\bigcup E_\beta \subseteq \bigcup G_\alpha$. We call E **compact** if every open cover of E contains a finite open subcover.

Lemma 2.3.1. *Every finite set is compact.*

Proof. Let K be finite, and let $\{G_\alpha\}$ be an open subcover of K. Since K is finite, there is a 1-1 mapping of K onto the set $\{1, \ldots, n\}$. Let $\{E_i\}_{i=1}^n$ be the finite collection of all subsets of K, clearly, $\{E_i\}$ is an open cover of K. Moreover, if $\bigcup E_i \subseteq \bigcup G_\alpha$, we are done, and if $\bigcup G_\alpha \subseteq \bigcup E_i$, then $\{G_i\}$ is a finite subcollection that covers K, so in either case, K is compact. ■

Theorem 2.3.2. *Let X be a metric space, and let $K \subseteq Y \subseteq X$. Then Y is compact in X if and only if K is compact in Y.*

Proof. Suppose K is compact in Y, and let $\{G_\alpha\}$ be a collection of subsets of Y X that cover K, and let $V_\alpha = Y \cap G_\alpha$, then $\{V_\alpha\}$ is a collection of subsets of X covering K, in which $V_\alpha \subseteq G_\alpha$ for all α, therefore K is compact in Y

conversely, suppose that K is compact in X, and let $\{V_\alpha\}$ be a collection of open sets in Y sucht that $K \subseteq \bigcup V_\alpha$, by theorem 2.2.10, there is a collection $\{G_\alpha\}$ of open sets in Y such that $V_\alpha = Y \cap G_\alpha$, for all α. Then $K \subseteq \bigcup_{i=1}^n G_{\alpha_i}$; therefore, K is compact in Y. ■

Theorem 2.3.3. *Compact subsets of metric spaces are closed.*

Proof. Let X be a metric space, and let K be compact in X and let $x \in X \backslash K$, if $y \in K$, let U and V be neighborhoods of x and y respectively, each of radius $r < \frac{1}{2}d(x,y)$. Since K is compact, there are finitely many points $y_1, \ldots y_n$ such that $K \bigcup_{i=1}^n V_i = V$, where V_i is a neighborhood of y_i for $1 \leq i \leq n$. Let $U = \bigcap_{i=1}^n U_i$, then $V \cap W$ is empty, hence $UX \backslash V$, therefore, $x \in X \backslash K$, therefore K is closed. ■

Theorem 2.3.4. *Closed subsets of compact sets are compact.*

Proof. Let X be a metric space with $F \subseteq KX$, with F closed in X, and K compact. Let $\{V_\alpha\}$ be an open cover of F. If we append $X \backslash F$ to $\{V_\alpha\}$, we get an open cover Θ of K, and since K is compact, there is a finite subcollection Φ which covers K, so Φ is an open cover of F, $X \backslash F \Phi$, then $\Phi \backslash (X \backslash F)$ still covers F, therfore F is compact. ■

Theorem 2.3.5. *Let $\{K_\alpha\}$ be a collection of compact sets of a metric space X, such that every finite subcollection of $\{K_\alpha\}$ is nonempty. Then $\bigcap K_\alpha$ is nonempty.*

Proof. Fix $K_1 \subseteq \{K_\alpha\}$, and let $G_\alpha = X \backslash K_\alpha$. Suppose no point of K_1 is in $\bigcap K_\alpha$, then $\{G_\alpha\}$ covers K_1, and since K is compact, we have $K_1 \bigcup_{i=1}^n G_{\alpha_i}$, for $1 \leq i \leq n$, which implies that $\bigcap K_\alpha$ is empty, a contradiction. ■

Corollary. *If $\{K_\alpha\}$ is a sequence of nonempty compact sets, such that $K_{n+1} \subseteq K_n$, then $\bigcap_{i=1}^\infty K_n$ is nonempty.*

Theorem 2.3.6. *If E is a infinite subset of a compact set K, then E has a limit point in K.*

Proof. Suppose no point of K is a limit point of E, then for all $x \in K$, the neighborhood U_x contains atmost one point in E. Then no finite subcollection of $\{U_x\}$ covers E, which contradicts the compactness on K. ■

Theorem 2.3.7 (The Nested Interval Theorem)**.** *if $\{I_n\}$ is a sequence of intervals in $\mathbb{R}$ such that $I_{n+1} \subseteq I_n$, then $\bigcap_{j=1}^\infty I_n$ is nonempty.*

Proof. We let $I_n = [a_n, b_n]$. Letting E be the set of all a_n, E is nonempty and bounded above by b_1. Letting $x = \sup E$, and $m \geq n$, we have $[a_m, b_m] \subseteq [a_n, b_n]$, thus $a_m \leq x \leq b_m$ for all m, thus $x \in I_m = \bigcap_{j=i}^{n} I_j$ ■

Theorem 2.3.8. *Let $k \in \mathbb{Z}^+$, and $\{I_n\}$ be a nonempty sequence of k-cells of $\mathbb{R}^k$ such that $I_{n+1}I_n$. Then $\bigcap_{j=1}^{\infty} I_n$ is nonempty.*

Proof. Let I_n be the set of all points $x \in \mathbb{R}^k$ such that $a_{n,j} \leq x_j \leq b_{n,j}$, and let $I_{n,j} = [a_{n,j}, b_{n,j}]$. Then for each $1 \leq j \leq k$, by the nested interval theorem, $\bigcap_{l=1}^{\infty} I_{l,j}$ is nonempty, hence there are real numbers x'_j such that $a_{n,j} \leq x'_j \leq b_{n,j}$. Letting $x' = (x'_1, \ldots, x'_k)$, we get that $x' \in I \bigcap_{l=1}^{\infty} I_l$ ■

Theorem 2.3.9. *Every k-cell is compact.*

Proof. Let I be a k-cell, and let $\delta = \sqrt{\sum_{j=1}^{k} a(b_j - a_j)^2}$ we get for $x, y \in I$, $||x - y|| \leq \delta$. Now suppose there is an open cover $\{G_\alpha\}$ of I for which no finite subcover is contained. Let $c_j = \frac{a_j + b_j}{2}$, then the closed intervals $[a_j, c_j]$, $[c_j, b_j]$ determine the 2^k k-cells Q_i such that $\bigcup Q_i = I$. Then atleast one Q_i cannot be covered by any finite subcollectio of $\{G_\alpha\}$. Subdividing Q_1, we get a sequence $\{Q_n\}$ such that $Q_{n+1} \subseteq Q_n$, Q_n is not covered by any finite subcollection of $\{G_\alpha\}$, and $||x - y|| \leq \frac{\delta}{2^n}$ for $x, y \in Q_n$. Then by theorem 2.3.8, there is a point $x' \in Q_n$, and for some α, $x' \in G_\alpha$; since G_α is open, there is an $r > 0$ for which $||x - || < r$ implies $y \in G_\alpha$. THen for n sufficiently large, we have that $\frac{\delta}{2^n} < r$, then we get that $Q_n \in G_\alpha$, which is a contradiction. ■

Theorem 2.3.10 (The Heine-Borel Theorem)**.** *If E is a subset of $\mathbb{R}^k$, then the following are equivalent:*

> *[label=(0)]E is closed and bounded. E is compact. Every infinite subset of E has a limit point in E.*

2. *Proof.* Suppose that E is closed and bounded, then $E \subseteq I$ for some k-cell I in $\mathbb{R}^k$, and hence it is compact. By theorem 2.3.4, E is compact. Now suppose that E is compact, then by theorem 2.3.6, every infinite subset of E has a limit point in E.

Now suppose that every infinite subset of E has a limit point in E. If E is not bounded, then $||x_n|| > n$ for some $x_n \in E$ and $n \in \mathbb{Z}^+$. Then the set of all such x_n is infinite, and has no limit point in E, a contradiction; moreover suppose that E is not closed. Then there is a point $x_0 \in \mathbb{R}^k \backslash E$, which is a limit point of E. Then there are points $x_n \in E$ for which $||x_n - x_0|| < \frac{1}{n}$, let S be the set of all such points. Then S is infinite and has x_0 as its only limit point; for if $y \neq x_0 \in \mathbb{R}^k$, then $\frac{1}{2}||x_0 - y|| \leq ||x_0 - y|| - \frac{1}{n} \leq ||x_0 - y|| - ||x_n - x_0|| \leq ||x_n - y||$ for only some n. Thus by theorem 2.2.3, y is not a limit point of S Therefore, if every infinite subset of E has a limit point in E, E must be closed. ■

Theorem 2.3.11 (The Bolzano-Weierstrass Theorem)**.** *Every bounded infinite subset E of $\mathbb{R}^k$ has a limit point in $\mathbb{R}^k$.*

Proof. We have that $E \subseteq I$, for some k-cell I in $\mathbb{R}^k$. Since k-cells are compact, by the Heine-Borel theorem, E is also compact and has a limit point in I. ■

2.4 Perfect Sets

Theorem 2.4.1. *If $P \subseteq \mathbb{R}^k$ is a nonempty perfect set, then P is uncountable.*

Proof. Since every point of P is a limit point of P, we gave that P must be infinite. Then suppose that P is countable. For points $x_n \in P$, construct the sequence $\{U_n\}$ of neighborhoods of x_n, for $n \in \mathbb{Z}^+$; now by induction, if U_1 is a neighborhood of x_1, then for $y \in \hat{U_1}$, $||x_1 - y|| \leq r$ for some $r > 0$. Now suppos the neighborhood U_n of x_n has been constructed such that $U_n \cap P$ is nonempty. Then there is a neighborhood U_{n+1} fo x_{n+1} such that $\hat{U_{n+1}} \subseteq U_n$, $x_n \notin \hat{U_{n+1}}$, and $U_{n+1} \cap P$ is nonempty. Therefore there is a nonempty $K_n = U_n \cap P$. Since $\hat{U_n}$ is close and bounded, $\hat{U}$ is compact, and since $x_n \notin K_{n+1}$, $x_n \notin \bigcap_{i=1}^{\infty} K_i$, and since $K_n \subseteq P$, $\bigcap K_i$ is empty, a contradiction. ∎

Corollary. *Let $a < b$ be real numbers. Then the closed interval $[a, b]$ is uncountable. Moreover, $\mathbb{R}$ is uncountable.*

Proof. We have $[a, b]$ is closed, and perfect (since $(a, b)[a, b] is perfect$), thus $[a, b]$ is uncountable. Moreover, take $f : \mathbb{R} \to [a, b]$, by $f(x) = \frac{a+b}{2}x$; then f is a 1-1 mapping of $\mathbb{R}$ onto $[a, b]$, which makes $\mathbb{R}$ uncountable. ∎

Theorem 2.4.2 (The construction of the Cantor set)**.** *There exists a perfect set in $\mathbb{R}$ which contains no open interval.*

Proof. Let $E_0 = [0, 1]$, and remove $(\frac{1}{3}, \frac{2}{3})$, and let $E_1 = [0, \frac{1}{3}] \cup [\frac{2}{3}, 1]$. Now remove the open intervals $(\frac{1}{9}, \frac{2}{9})$ $(\frac{3}{9}, \frac{6}{9})$, $(\frac{7}{9}, \frac{8}{9})$, and let $E_2 = [0, \frac{2}{9}] \cup [\frac{2}{9}, \frac{3}{9}] \cup [\frac{6}{9}, \frac{7}{9}] \cup [\frac{7}{8}, \frac{8}{9}]$. Continuig the remove the middle third of each interval, we obtain the sequence of compact sets $\{E_n\}$, such that $E_{n+1}E_n$, and E_n is the union of 2^n closed intervals of length $\frac{1}{3^n}$. Then let:

$$P = \bigcap_{i=1}^{\infty} E_i \tag{2.1}$$

Then P is nonempty, and compact.

Now let I be the open interval of the form $(\frac{3k+1}{3^m}, \frac{3k+2}{3^m})$, with $k, m \in \mathbb{Z}^+$. Then by the construction of P, I has no point in P, we also see that every other open interval contains a subinterval of the form of I; them P contains no open interval.

Now let $x \in P$, and let S be any open interval for which $x \in S$. LEt I_n be the closed interval of E_n such that $x \in I_n$. Choose n sufficiently large such that $I_n S$. If $x_n \neq x$ is an endpoint of I_n, then $x_n \in P$, and so x is a limit point of P. Therefore P is perfect. ∎

Definition. The we call the set P constructed in the proof of theorem 2.4.2 the **Cantor set**.

2.5 Connected Sets

Definition. Two subsets A and B of a metric space X are **seperated** if $A \cap \hat{B}$ and $\hat{A} \cap B$ are both empty. We say a subset E of X is **connected**, if E is not the union of two nonepmty speperated sets.

Theorem 2.5.1. *A subset E of $\mathbb{R}$ is connected if and only if $x, y \in E$ and $x < z < y$ imply $z \in E$.*

Proof. Let $x, y \in E$ such that for some $z \in (x, y))$, $z \notin E$. Then $E = A \cup B$, with $A = E \cup (-\infty, z)$ and $B = E \cup (z, \infty)$. Then A and B are seperated, which contradicts the connectedness of E.

Conversely suppose for $x, y \in E$, that $z \in E$ for $z \in (x, y)$. Then there are nonempty seperated sets A and B such that $A \cup B = E$. Choose $x \in A$, $y \in B$ such that $x < y$, and let $z = \sup(A \cap [x, y])$. Then by theorem 2.2.8, $z \in \hat{A}$, so $z \; notinB$. In particular, $x \leq x < y$. Now if $z \notin A$, then $x < z < y$, with $z \notin E$. Now if $z \in A$, then $z \notin \hat{B}$, hence there is a z' such that $z < z' < y$, and $z' \notin B$. Then $x < z' < y$ and $z' \not inE$. ■

Chapter 3

Sequences

3.1 Convergent Sequences

Definition. A sequence $\{x_n\}$ in a metric space X is said to **converge** if there is a point $x \in X$ such that for every $\varepsilon > 0$, there is an $N \in \mathbb{Z}^+$ such that $d(x_n, x) < \varepsilon$ whenever $n \geq N$. We say $\{x_n\}$ **converges** to x, and we call x the **limit** of $\{x_n\}$ as n approaches ∞. We write $x_n \to x$ as $n \to \infty$, and $\lim_{n\to\infty} x_n = x$ (or $\lim x_n = x$). If $\{x_n\}$ does not converge, we say the $\{x_n\}$ **diverges**, or **is divergent**.

Example 3.1. Consider the following sequences in $\mathbb{C}$.

[label=(0)]$\{\frac{1}{n}\}$ is bounded, and $\lim_{n\to\infty} \frac{1}{n} = 0$. The sequence $\{n^2\}$ us unbounded and diverges. $1 + \frac{(-1)^n}{n} \to 1$ as $n \to \infty$, and $\{1 + \frac{(-1)^n}{n}\}$ is bounded. $\{i^n\}$ is bounded and divergent. $\{1\}$ is bounded and converges to 1.

Theorem 3.1.1. *Let $\{x_n\}$ be a sequence in a metric space, then:*

[label=(0)]$\{x_n\}$ converges to $x \in X$ if and only if every every neighborhood of x contains x_n for all but finitely many n. If $\{x_n\}$ converges to x, and x', then $x = x'$. If $\{x_n\}$ converges, then x_n is bounded. If $E \subseteq X$, and x is a limit point of E, then there is a sequence in E that converges to x.

1. *Proof.* Suppose $x_n \to x$, and let U be a neighborhood of x. For some $\varepsilon > 0$, there is an $N \in \mathbb{Z}^+$ for which $d(x_n, x) < \varepsilon$, whenever $n \geq N$, thus $x_n \in U$ for finitely many n. Conversely, suppose that $x_n \in U$ for some $n \geq N$, then letting $\varepsilon > 0$, we havae $d(x, x_n) < \varepsilon$, hence $x_n \to x$.

Let > 0, then there are $N_1, N_2 \in Z^+$ such that $d(x_n, x) < \frac{\varepsilon}{2}$, and $d(x_n, x') < \frac{\varepsilon}{2}$. Then choosing $N = \max\{N_1, N_2\}$, and letting ε be arbitrarily small, we have $d(x, x') \leq d(x, x_n) + d(x_n, x') < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$; and so we get that $x = x'$.

Let $x_n \to x$, then there is an $N \in \mathbb{Z}^+$ for which $d(x_n, x) < 1$ whenever $n \geq N$. Letting $r = \max\{1, d(x_N, x)\}$, then $d(x_n, x) \leq r$.

Finally, let x be a limit point of E, then for each $n \in Z^+$, there is an $x_n \in E$ such that $d(x, x_n) < \frac{1}{n}$, choose $N > \frac{1}{\varepsilon}$, then whenever $n \ geqN$, $d(x, x_n) < \varepsilon$; hence $x_n \to x$. ■

Theorem 3.1.2. *Suppose $\{x_n\}$ and $\{y_n\}$ are sequences in $\mathbb{C}$, and that $limx_n = x$, $\lim y_n = y$ as $n \to \infty$. Then the following hold as $n \to \infty$:*

[label=(0)] $\lim(x_n + y_n) = \lim x_n + \lim y_n = x + y$. *$lim x_n y_n$* $= \lim x_n \lim y_n = xy$. *$lim \frac{x_n}{y_n}$* $= \frac{\lim x_n}{\lim y_n} = \frac{x}{y}$*; given that* $y_n, y \neq 0$.

1. *Proof.* [label=(0)]

   Let > 0, then for $N_1, N_2 \in \mathbb{Z}^+$, $|x_n - x| < \frac{\varepsilon}{2}$ and $|y_n - y| < \frac{\varepsilon}{2}$. Then choose $N = \max\{N_1, N_2\}$, then whenever $n \geq N$, we have $|(x_n+y_n)-(x+y)| \leq |x_n-x|+|y_n-y| < \varepsilon$.

2. Notice that $x_n y_n - xy = (x_n - x)(y_n - y) + x(y_n - y) + y(x_x - x)$, then for $N_1, N_2 \in \mathbb{Z}^+$, $|x_n - x| < \sqrt{\varepsilon}$, and $|y_n - y| < \sqrt{\varepsilon}$. Then choosing $N = \max\{N_1, N_2\}$, then $|(x_n - x)(y_n - y)| < \varepsilon$, thus we have $|x_n y_n - xy| \leq |(x_n - x)(y_n - y)| + |x(y_n - y)| + |y(x_x - x)| < \varepsilon$.

3. We first show that $\frac{1}{y_n} \to \frac{1}{y}$, given that $y_n, y \neq 0$. Choose m such that $|y_n - y| < \frac{1}{2}|y|$ whenever $n \geq m$, then $|y_n| > \frac{1}{2}|y|$. Then for $\varepsilon > 0$, there is an $N > m$ such that whenever $n \geq N$, $|y_n - y| < \frac{1}{2}|y|^2\varepsilon$. Then $|\frac{1}{y_n} - \frac{1}{y}| \leq \frac{|y_n - y|}{|y_n y|} < \frac{2}{|y|^2}|y_n - y| < \varepsilon$. Then choosing the sequences $\{x_n\}$ and $\{\frac{1}{y_n}\}$, the rest follows.

■

Corollary. *[label=(0)]*

1. *For any $c \in \mathbb{C}$, and a sequene $x_n \to x$, we have* $\lim cx_n = c \lim x_n = cx$ *and* $\lim(c + x_n) = c + \lim x_n = c + x$ *as $n \to \infty$.*

2. *Provided that $x, x_n \neq 0$, we have* $\lim \frac{1}{x_n} = \frac{1}{\lim x_n} = \frac{1}{x}$*, as $n \to \infty$.*

Proof. We choose $\{x_n\}$ and $\{y_n\} = \{c\}$ for all n, then the results follow. ■

Theorem 3.1.3. *[label=(0)]*

1. *Let $x_n = (\alpha_{1n}, \ldots \alpha_{kn}) \in \mathbb{R}^k$. Then $\{x_n\}$ converges to x if and only if* $\lim \alpha_{jn} = \alpha_j$ *for $1 \leq j \leq k$, as $n \to \infty$.*

2. *Let $\{x_n\}, \{y_n\}$ be sequences in $\mathbb{R}^k$, and let $\{\beta_n\}$ be a sequence in $\mathbb{R}$ such that $x_n \to x$, $y_n \to y$, and $\beta_n \to \beta$. Then* $\lim(x_n + y_n) = x + y$, $\lim x_n y_n = xy$*, and $lim \beta_n x_n = \beta x$.*

Proof. If $x_n \to x$, then $|\alpha_{jn} - \alpha_j| \leq ||x_n - x|| < \varepsilon$, thus $\lim \alpha_{jn} = \alpha_j$. Conversely, suppose that $\alpha_{jn} \to \alpha_j$. Then for $\varepsilon > 0$ there is an $N \in \mathbb{Z}^+$ such that $n \geq N$ implies $|\alpha_{jn} - \alpha_j| < \frac{\varepsilon}{\sqrt{k}}$. Then for $n \geq N$,

$$||x_n - x|| = \sqrt{\sum |\alpha_{jn} - \alpha_j|^2} < epsilon$$

To prove (2), we appy part (1) of this theorem together with theorem 3.1.2. ■

Theorem 3.1.4 (The Sandwhich Theorem). *Let $\{x_n\}$, $\{y_n\}$, and $\{w_n\}$ be sequences in $\mathbb{R}$, and Suppose that* $\lim x_n = \lim y_n = a$ *and that there is an $N \in \mathbb{Z}^+$ such hat $x_n \leq w_n \leq y_n$ for all $n \geq N$. Then* $\lim_{n\to\infty} w_n = a$.

Proof. Let $\varepsilon > 0$ and let $\{x_n\}$ and $\{y_n\}$ both converge to a. Then by definition there are $N_1, N_2 \in \mathbb{Z}^+$ such that $|x_n - a| < \varepsilon$ and $|y_n - a| < \varepsilon$ for $n \geq N_1, N_2$. Now choose $N = \max\{N_0, N_1, N_2\}$, if $n \geq N$, we have $-\varepsilon < x_n - a < \varepsilon$, and we also have $x_n - a < w_n - a < y_n - a$, thus we have that:

$$-\varepsilon < x_n - a < w_n - a < y_n - a < \varepsilon$$

Thus we have that $|w_n - a| < \varepsilon$. ■

Corollary. *If $x_n \to \infty$ as $n \to \infty$, and $\{y_n\}$ is bounded, then $x_n y_n \to 0$ as $n \to \infty$.*

Proof. We have that $\{y_n\}$ is bounded, hence, there is $M > 0$ such that $|y_n| < M$ for all $n \in \mathbb{Z}^+$. And since $\{x_n\}$ converges to 0 we have that for any ε there is an $N \in \mathbb{Z}^+$ such that for $n \geq N$, $|x_n - 0| < \frac{\varepsilon}{M}$. For $|x_n y_n - 0| = |x_n y_n| < M|x_n| < M\frac{\varepsilon}{M} = \varepsilon$. Therefore, $x_n y_n \to 0$ as $n \to \infty$. ■

Corollary. *Let $\{x_n\}$, $\{y_n\}$ be sequences such that $0 \leq x_n \leq y_n$ for $n \geq N > 0$. Then if $y_n \to 0$, then $x_n \to 0$ as $n\infty$.*

Proof. This is a special case of the sandwhich theorem. ■

3.2 Subsequences

Definition. Let $\{x_n\}$ be a sequence, and let $\{n_k\}\mathbb{Z}^+$ such that $n_k < n_{k+1}$. We call the sequence $\{x_{n_k}\}$ a **Subsequence** of $\{x_n\}$. If $\{x_{n_k}\}$ converges, we call its limit the **subsequential limit** of $\{x_n\}$.

Theorem 3.2.1. *A sequence $\{x_n\}$ converges to a point x if and only if every subsequence $\{x_{n_k}\}$ converges to x.*

Proof. Clearly if $x_n \to x$, then every subsequence $x_{n_k} \to x$, (since subsequences can be thought of as subsets of thier parent sequences). On the other hand, let $x_{n_k} \to x$ for $\{\eta_k\} \subseteq \mathbb{Z}^+$. Then for $\varepsilon > 0$, there is a $K \in \mathbb{Z}^+$ for which $d(x_{n_k}, x) < \frac{\varepsilon}{2}$ for $k \geq K$. Let $N \in \mathbb{Z}^+$, and choose $n \geq max\{N, K\}$, then $d(x_n, x) \leq d(x_n, x_{n_k}) + d(x_{n_k}, d) < \varepsilon$. ■

Theorem 3.2.2. *If $\{x_n\}$ is a sequence in a compact metric space X, then some subsequence of $\{x_n\}$ converges to a point x.*

Proof. If $\{x_n\}$ is finite, then thre is an $x \in \{x_n\}$ and a sequence $\{n_k\}$ with $n_k < n_{k+1}$ such that $x_{n_i} = x$ for $1 \leq i \leq k$, then the subsequence converges to x.

Now if $\{x_n\}$ is infinite, there is a limit point $x \in X$ of $\{x_n\}$, then choose n_i such that $d(x, x_i) < \frac{1}{i}$ for $1 \leq i \leq k$. Obtaining $\{n_k\}$ from this, we see that $n_k < n_{k+1}$, and so we get that $\{x_{n_k}\}$ converges to x. ■

Corollary. *Every bounded sequence in $\mathbb{R}^k$ contains a convergent subsequence.*

Theorem 3.2.3. *The subsequential limits of $\{x_n\}$ is a metric space X form a closed subset of X.*

Proof. Let E be the set of all subsequential limits of $\{x_n\}$, and let x be a limit point of E. Choose n_i such that $x_{n_i} \neq x$ and let $\delta = d(x, x_{n_i})$, for $1 \leq i \leq k$. Then consier the sequence $\{n_k\}$, since x is a limit point of E, there is an $x' \in E$ for which $d(x, x') < \frac{\delta}{2^i}$. Thus there is an $N_I > n_i$ such that $d(x', x_{n_i}) < \frac{\delta}{2^i}$, thus $d(x, x_{n_i}) < \frac{\delta}{2^i}$. So $\{x_n\}$ converges to x and $x \in E$. ■

3.3 Cauchy Sequences

Definition. We call a sequence $\{x_n\}$ in a metric space X a **Cauchy sequence** in X, or more simply, **Cauchy** in X if for all $\varepsilon > 0$, there is an $N \in \mathbb{Z}^+$ such that $d(x_n, x_m) < \varepsilon$ whenever $m, n \geq N$.

Definition. Let E be a nonempty subset of a metrix space X, and lelt $S \subseteq \mathbb{R}$ be the all real numbers $d(x, y)$, with $x, y \in E$. We call $\sup S$ the **diameter** of E, and denote it $\operatorname{diam} E$.

Theorem 3.3.1. *Let $\{x_n\}$ be a sequence, and let E_N be the set of all points p_N such that $_N < p_{n+1}$. Then $\{x_n\}$ is Cauchy if and only if* $\lim \operatorname{diam} E_N = 0$ *as $N \to \infty$.*

Proof. Let $\{x_n\}$ be Cauchy, Let $x_{N_1}, x_{N_2} \in E$ such that $d(x_n, x_{N_1}) < \frac{\varepsilon}{2}$, and $d(x_{N_2}, x_m) < \frac{\varepsilon}{2}$. Then we see that $d(x_{N_1}, x_{N_2}) \leq d(x_{N_1}, x_n) + d(x_m, x_{N_2}) < \varepsilon$, so $\{x_{N_k}\}$ is Cauchy and we see that $\lim \operatorname{diam} E_N = 0$. Now suppose that $\lim \operatorname{diam} E = 0$, then for any $x_n, x_m \in S$, $d(x_n, 0) < \frac{\varepsilon}{2}$ and $d(0, x_m) < \frac{\varepsilon}{2}$ implies that $d(x_n, x_m) \leq d(x_n, 0) + d(0, x_m) < \varepsilon$, whenever $n, m > N$, for $\varepsilon > 0$. ■

Theorem 3.3.2. *[label=(0)]*

1. *If $E \subseteq X$, then* $\operatorname{diam} \hat{E} = \operatorname{diam} E$.
2. *If $\{K_n\}$ is a sequence of compact sets in X, such that $K_{n+1} \subseteq K_n$, and if* $\lim \operatorname{diam} K_n =$ 0 *as $n \to \infty$, then $\bigcap_{i=1}^{infty} K_i$ contains exactly one point.*

Proof. Clearly $\operatorname{diam} E \leq \operatorname{diam} \hat{E}$. Now let $\varepsilon > 0$, and choose $x, y \in \hat{E}$, then there are points $x', y' \in \hat{E}$ such that $d(x, x') < \frac{\varepsilon}{2}$ and $d(y, y') < \frac{\varepsilon}{2}$. Hence, $d(x, y) \leq d(x, x') + d(x', y') + d(y'y) < \varepsilon \operatorname{diam} E$, then choosing ε arbitrarily small, $\operatorname{diam} \hat{E} \leq \operatorname{diam} E$.

Now, we also have that by the nested interval theorem that $K = \bigcap K_i$ is nonempty. Now suppose that K contains more that one point. then $\operatorname{diam} K > 0$, and since $K \subseteq K_n$ for all n, $diam K \leq \operatorname{diam} K_n$, a contradiction. Thus K contains exactly one element. ■

Theorem 3.3.3. *[label=(0)]*

1. *In any metric space X, every convergent sequence is a Cauchy sequence.*
2. *If X is compact, and $\{x_n\}$ is Cauchy in X, then $\{x_n\}$ converges to a point in X.*

Proof. [label=(0)]

If $x_n \to x$, and $\varepsilon > 0$ such that there is an $N \in \mathbb{Z}^+$ such that $d(x_n, x) < \frac{\varepsilon}{2}$ for all $n \geq N$, then for $m \geq N$, we have $d(x_n, x_m) \leq d(x_n, x) + d(x, x_m) < \varepsilon$. Thus $\{x_n\}$ is Cauchy.

2. Let $\{x_n\}$ be Cauchy, and let E_N be the set of all points x_N for which $x_N < x_{N+1}$. Then $\lim \operatorname{diam} \hat{E} = 0$, then being closed in X, each $\hat{E_N}$ is compact in X, and $\hat{E_{N+1}} \subseteq \hat{E_N}$, so by theorem 3.3.2, there is a unique $x \in X$ in all of $\hat{E_N}$. Now for $\varepsilon > 0$, there is an $N_0 \in \mathbb{Z}^+$ for which $\operatorname{diam} \hat{E} < \varepsilon$. Then for all $x_n \in \hat{E}$, $d(x_n, x) < \varepsilon$ whenever $n \geq N_0$. ■

Corollary (The Cauchcy Criterion)**.** *Every Cauchy sequence in $\mathbb{R}^k$ converges to a point in $\mathbb{R}^k$.*

Proof. Let $\{x_n\}$ be Cauchy in $\mathbb{R}^k$, define E_N as in (2), then for some $N \in \mathbb{Z}^+$, $\operatorname{diam} E < 1$, and so $\{x_n\}$ us the union of all E_n, and ther set of points $\{x_1, \ldots, x_{N-1}\}$, so $\{x_n\}$ is bounded, and thus has a compact closure, it follows then that $x_n \to x$ for some $x \in \mathbb{R}^k$. ■

Definition. We call a metric space **complete** if every Cauchy sequence in the space converges.

Theorem 3.3.4. *All compact metric spaces, and all Euclidean spaces are complete.*

Example 3.2. Consider $\mathbb{Q}$ together with the metric $|x - y|$. The metric space induced on $\mathbb{Q}$ by $|\cdot|$ is not complete.

Definition. A sequence $\{x_n\}$ in $\mathbb{R}$ is said to be **monotonically increasing** if $x_n \leq x_{n+1}$, $\{x_n\}$ is said to be **monotonically decreasing** if $x_{n+1} < x_n$. We call $\{x_n\}$ **monotonic** if it is either monotonically increasing or monotonically decreasing.

Theorem 3.3.5. *A monotonic sequence converges if and only if it is bounded.*

Proof. Suppose, without loss of generality, that $\{x_n\}$ is monotonically increasing. If $\{x_n\}$ is bounded, then $x_n \leq x$, then for all $\varepsilon > 0$, there is an $N \in \mathbb{Z}^+$ for which $x - \varepsilon < x_N \leq x$. Then for $n \geq N$, $x_n \to x$. The converse follows from theorem 3.1.2. ■

3.4 Upper and Loweer Limits.

Let $\{\xi_n\}$ be a sequence in $\mathbb{R}$ such that for all $M > 0$, there is an $N \in \mathbb{Z}^+$ for which $n \geq N$ implies that either $x_n \geq M$, or $x_n \leq M$. Then we write $x_n \to \infty$ and $x_n \to -\infty$, respectively.

Definition. Let $\{x_n\}$ be a sequence in $\mathbb{R}$, and let E tbe the set of all extended real numbers x such that $x_{n_k} \to x$ for some subseqence $\{x_{n_k}\}$. Then E contains all subsequential limits of $\{x_n\}$, and possible $\pm\infty$. We then call $\sup E$ the **upper limit** of E, and $\inf E$ the **lower limit** of E.

Theorem 3.4.1. *Let $\{x_n\}$ be a sequence in $\mathbb{R}$, and let E be the set of all extended real numbers x, let $s = \sup E$ and $s' = \inf E$. Then the following hold:*

[label=(0)]s,s' $\in$ E.Ifx¿s,andx'¿s',thereisanN $\in \mathbb{Z}^+$ such that $n \geq N$ implies that $x' < x_n < x$.

1. *Proof.* We prove the theorem for the case of s, since it is analogous for s'.

   [label=(0)]If $s = \infty$, then E is not bounded above, so neither is $\{x_n\}$, and there is a subsequence for which $x_n \to \infty$. Now if $s \in \mathbb{R}$, then E is bounded above, and has at least one subsequential limit. Then $s \in E$. Now if $s = -\infty$, then E contains only $-\infty$, and so by definition $x_n \to -\infty$. Suppose there is an $x > s$, such that $x_n \geq x$ for all n. Then there is a $y \in E$ such that $y \geq x \geq s$, a contradiction of the definition of s.

   ■

Example 3.3. [label=(0)]

1. Let $\{x_n\}$ be a sequence in $\mathbb{Q}$, then every real number is a subsequential limit, and $\limsup x_n = \infty$ and $\liminf x_n = -\infty$.
2. Let $\{x_n\} = \{\frac{(-1)^n}{1+\frac{1}{n}}\}$; then $\limsup x_n = 1$ and $\liminf X_n = -1$ as $n \to \infty$.
3. For a sequence $\{x_n\}$ in $\mathbb{R}$, $\lim x_n = x$ if and only if $\limsup x_n = \liminf x_n = x$ as $n \to \infty$.

Theorem 3.4.2. *If $x_n \leq y_n$, for $n \geq N > 0$, then $\liminf x_n \leq \liminf y_n$ and $\limsup x_n \leq \limsup y_n$ as $n \to \infty$.*

3.5 Special Sequences

Theorem 3.5.1. *Let $n, p \in \mathbb{Z}^+$. Then the following hold as $n \to \infty$.*

[label=(0)]

1. $\lim \frac{1}{n^p} = 0$.
2. $\lim \sqrt[p]{n} = 1$.
3. $\lim \sqrt[n]{n} = 1$.
4. *If $\alpha \in \mathbb{R}$, then* $\lim \frac{n^\alpha}{(1+p)^n} = 0$.
5. *If $|x| < 1$, then* $\lim x^n = 0$.

Proof. [label=(0)]

Let $n > [p]\frac{1}{\varepsilon}$; then $|\frac{1}{n^p}| < \varepsilon$.

1. If $p = 1$, we are done. If $p > 1$, let $x_n = \sqrt[n]{p} - 1$, then $x_n > 0$. By the binomial theorem, $1 + nx_n \leq (1 + x_n)^p = p$, hence $0 \leq x_n \leq \frac{p-1}{p}$. Now if $1 > p > 0$, then $\frac{1}{p} > 0$, so we notice that $0 \leq \frac{1}{x_n} \leq \frac{1}{\frac{p-1}{n}}$.
3. Let $x_n = \sqrt[n]{n} - 1$, then $x_n \geq 0$, then by the binomial theorem again, $n = (1 + x_n)^n \geq \frac{n(n-1)}{2}x_n^2$, then $0 \leq x_n \leq \sqrt{\frac{2}{n-1}}$.

4. Let $k \in \mathbb{Z}^+$ such that $k > \alpha$. Then $n > 2k$,let $(1+p)^n > \binom{n}{k}p^k > \frac{n^k p^k}{2^k k!}$. So $0 < \frac{n^\alpha}{(1+p)^n} < \frac{2^k k!}{p^k} n^{-k}$, since $\alpha - k < 0$, $n^{\alpha-k} \to 0$ and we are done.

5. Take $\alpha = 0$, and let $x = \frac{1}{1+p}$, then the result follow.

■

Chapter 4

Series.

4.1 Convergent Series.

Chapter 5

Continuity

5.1 Limits of Functions.

Definition. Let X, and Y be metric spaces, and let $E \subseteq X$, and let $f : E \to Y$ be a function. We say that f **converges** to a point $q \in Y$, as x **approaches** a limit point $p \in X$ if for every $\varepsilon > 0$, there is a $\delta > 0$ for which $d_Y(f(x), q) < \varepsilon$, whenever $0 < d_X(x, p) < \delta$. We say that q is the **limit** of f at p and we write $f \to q$ as $x \to p$, and $\lim_{x \to p} f(x) = q$, or more simply, $\lim f = q$.

Example 5.1. [label=(0)]

1. Let $X = Y = \mathbb{R}$, under the absolute value $|\cdot|$, and let $I \subseteq \mathbb{R}$ be an interval, and $f : I \to \mathbb{R}$. Then f has a limit L as x approaches a limit point $c \in \mathbb{R}$ if for every $\varepsilon > 0$, there is a $\delta > 0$ such that $|f(x) - L| < \varepsilon$ whenever $0 < |x - c| < \delta$. We call functions that map into $\mathbb{R}$ **real valued**.

2. Let $X = Y = \mathbb{C}$, under the modulus $|\cdot|$, and let $D \subseteq \mathbb{R}$ be an domain, and $f : D \to \mathbb{R}$. Then f has a limit L as z approaches a limit point $w \in \mathbb{R}$ if for every $\varepsilon > 0$, there is a $\delta > 0$ such that $|f(x) - L| < \varepsilon$ whenever $0 < |z - w| < \delta$. We call functions that map into $\mathbb{C}$ **complex valued**.

3. Let $X = Y = \mathbb{R}^k$, under the norm $||\cdot||$, and let $D \subseteq \mathbb{R}^k$ be an domain, and $f : D \to \mathbb{R}^k$. Then f has a limit L as x approaches a limit point $c \in \mathbb{R}^k$ if for every $\varepsilon > 0$, there is a $\delta > 0$ such that $||f(x) - L|| < \varepsilon$ whenever $0 < ||x - c|| < \delta$. We call functions that map into $\mathbb{R}^k$ **vector valued**.

Theorem 5.1.1 (The Sequential Criterion)**.** *Let X and Y be metric spaces, and let $E \subseteq X$, and $f : E \to Y$ be a function, and $p \in E$ be a limit point. Then $\lim f(x) = q$ as $x \to p$ if and only if $\lim f(x_n) = q$ as $n \to \infty$ for any sequence $\{x_n\} \in E$, such that $x_n \neq p$ and $\lim x_n = p$.*

Proof. Suppose that $\lim f(x) = q$ as $x \to p$, and choose $\{x_n\} \subseteq E$ such that $x_n \neq p$ and $\lim x_n = p$ as $n \to \infty$. Then for $\varepsilon > 0$, there is a $\delta > 0$ such that $d_Y(f(x), q) < \varepsilon$ whenever $0 < d_X(x, p) < \delta$, and since $d_X(x_n, p) < \delta$ whenever $n \geq N$ for some $N > 0$, we have $d_Y(f(x_n), q) < \varepsilon$ whenever $d_X(x_n, p) < \delta$.

Conversely, suppose that $\lim f \neq q$, that is for some $\varepsilon > 0$, $d_Y(f(x), q) > \geq \varepsilon$ whenevr $d_X(x, p) < \delta$ for all $\delta > 0$. Then choose $\delta = \frac{1}{n}$, for $n \in \mathbb{Z}^+$, then we have $\lim x_n = p$, but $\lim f(x_n) \neq q$. ■

The importance of the sequential criterion is that it lets us translate theorems about limits of sequences into theorems about limits of functions.

Corollary. *If f has a limit at p, then the limit of f is unique.*

Definition. Letting $f, g : E \to Y$, we define the **sum**, **product**, **scalar product** and the **quotient** of f and g to be the functions from E into Y:

[label=(0)]$f + g(x) = f(x) + g(x)$. $fg(x) = f(x)g(x)$. $(\lambda f)(x) = \lambda f(x)$ for $\lambda \in X$. $\frac{f}{g}(x) = \frac{f(x)}{g(x)}$, provided that $g(x) \neq 0$.

It is well known that the set of all functions from E into Y form an algebra under these operations.

Theorem 5.1.2. *Let $E \subseteq X$ a metric space, and let $p \in E$ be a limit point. Let $f, g : E \to Y$ be functions, such that $\lim f = A$ and $\lim g = B$ as $x \to p$. Then the following hold as $x \to p$.*

[label=(0)]$\lim(f + g) = \lim f + \lim g = A + B$. $\lim fg = \lim f \lim g = AB$. $\lim \frac{f}{g} = \frac{\lim f}{\lim g} = \frac{A}{B}$, provided that $B \neq 0$.

Corollary. *The following hold:*

[label=(0)]$\lim \lambda f = \lambda \lim f = \lambda A$, and $\lim(\lambda + f) = \lambda + \lim f = \lambda + A. \lim \frac{1}{f(x)} = \frac{1}{\lim f} = \frac{1}{A}, provided that A \neq 0.$

Theorem 5.1.3 (The Sandwich Theorem)**.** *Let f, g, and h be real valued functions defined on $\mathbb{R}$ such that $lim f = \lim g = A$ as $x \to p$, and suppose that $f(x) \leq h(x) \leq g(x)$ for all $x \in \mathbb{R}$. Then $\lim h = A$ as $x \to p$.*

Corollary. *Let f, g be real valued functions defined on $\mathbb{R}$ such that $0 \leq f(x) \leq g(x)$ for all $x \in \mathbb{R}$. Then if $g \to 0$ as $x \to p$, then $f \to 0$.*

The proofs of all these are the result of appling the sequential criterion.

5.2 Continuous Functions.

Definition. Let X and Y be metric spaces and let $p \in E \subseteq X$, and $f : E \to Y$ be a function. We say that f is **continuous** at p if for every $\varepsilon > 0$, there is a $\delta > 0$ such that $d_Y(f(x), f(p)) < \varepsilon$ whenever $0 < d_X(x, p) < \delta$. If f is continuous at every point in X, we say that f is **continuous on** X.

Theorem 5.2.1. *If $E \subseteq X$ a metric space, and if f is a function defined on X, and $p \in E$ is a limit point, then f is continuous if and only if $\lim f(x) = f(p)$ as $x \to p$.*

Theorem 5.2.2. *Suppose X, Y, and Z are metric spaces, and that $f : E \to Y$, $g : Y \to Z$, are functions (with $E \subseteq X$) such that f is continuous at p and g is continuous at $f(p)$. Then $g \circ f$ is continuous at p.*

2. *Proof.* For every $\varepsilon > 0$, we have $\delta_1, \delta_2 > 0$ such that $d_Y(f(x), f(p)) < \varepsilon$, when $0 < d_X(x,p) < \delta_1$, and $d_Z(g(y), g(f(p))) < \varepsilon$ whenever $d_Y(y, f(p)) < \delta_2$. Then choose $\delta = \min\{\delta_1, \delta_2\}$,a nd we see that $d_Z(g(f(x)), g(f(p))) < \varepsilon$ whenever $0 < d_X(x,p) < \delta$. ■

Theorem 5.2.3. *A mapping f of a metric space X into a metric space Y is continuous if and only if for every open set $V \subseteq Y$, $f^{-1}(V)$ is open in X.*

Proof. Let f be continuous on X, and let V be open in Y. For $p \in X$, $f(p) \in V$, and since V is open, there is an $\varepsilon > 0$ such that $y \in V$ when $d_Y(y, f(p)) < \varepsilon$. Since f is continuous, there is a $\delta > 0$ such that $d_Y(f(x), f(p)) < \varepsilon$, whenever $0 < d_x(x,p) < \delta$. Thus $f^{-1}(V)$ is open in X.

Conversly, suppose that $f^{-1}(V)$ is open in X for V open in Y. Let $p \in X$ and $\varepsilon > 0$, and let $V = \{y \in Y : d_Y(y, f(p)) < \varepsilon\}$; V is open in Y, so $f^{-1}(V)$ is open in X, thus there is a $\delta > 0$ such that $x \in f^{-1}(V)$ when $0 < d_X(x,p) < \delta$), then $f(x) \in V$, so $d_Y(f(x), f(p)) < \varepsilon$; therefore, f is continuous at p. ■

Corollary. *A mapping f from X into Y is continuous if and only if $f^{-1}(C)$ is closed in X, whenever C is closed in Y.*

Proof. This is the converse of the previous theorem. ■

Theorem 5.2.4. *Let $f, g : X \to \mathbb{C}$ be continuous complex valued functions defined on a metric space X, then $f + g$, fg, and $\frac{f}{g}$ are continuous.*

Proof. This follows from theorem 5.1.2 and the sequential criterion. ■

Theorem 5.2.5. *Let $f_1, \ldots, f_k$ be realvalued functions defined on a metric space X, and define $f : X \to \mathbb{R}^k$ by $f(x) = (f_1(x), \ldots, f_k(x))$ for all $x \in X$. Then f is continuous if and only if f_i is continuous for $11 \leq i \leq k$. Moreover, if $g : X \to \mathbb{R}^k$ and f are continuous, then so is $f + g$ and fg.*

Proof. Notice that $|f_i(x) - f_i(y)| \leq ||f(x) - f(y)|| = \sqrt{\sum |f_i(x) - f_i(y)|^2}$ for $1 \leq i \leq k$. If follows then that f is contiuous if and only f_i is. Moreover, if $g : X \to \mathbb{R}^k$ is also contiuous, then by the previous theorem, so is $f + g$ and fg. ■

Example 5.2. [label=(0)]

1. Let $x \in \mathbb{R}^k$, define the functions $\phi_i : \mathbb{R}^k \to \mathbb{R}$ by $\phi_i(x) = x_i$ for all $1 \leq i \leq k$, then ϕ_i is contiuous on $\mathbb{R}^k$

2. The monomials $x_1^{n_1} x_2^{n_2} \ldots x_k^{n_k}$, with $n_i \in \mathbb{Z}^+$ for $1 \leq i \leq k$ are contiuous on $\mathbb{R}^k$. So are all constant ultiples, thus the polynomial $\sum c_{n_1,\ldots,n_k} x_1^{n_1} x_2^{n_2} \ldots x_k^{n_k}$ is also continuous on $\mathbb{R}^k$.

3. We have $||||x|| - ||y|||| \leq ||x - y||$ for all $x, y \in \mathbb{R}^k$, thus the mapping $x \to ||x||$ is contiuous on $\mathbb{R}^k$.

5.3 Continuity and Compactness.

Definition. A mappinf $f : E \to \mathbb{R}^k$ is said to be **bounded** if there is a real number $M > 0$ such that $||f|| \le M$ for all $x \in E$.

Theorem 5.3.1. *Let f be a cn=ontiuous mapping of a compact metric space X into a metric space Y. Then $f(X)$ is compact in Y.*

Proof. Let $\{V_\alpha\}$ be an open cover of $f(X)$, since f is contiuous, then $f^{-1}(V_\alpha)$ is open in X, and since X is compact, $X \subseteq \bigcup_{i=1}^n V_{\alpha_i}$, and $f(f^{-1}(E)) \subseteq E$, we have that $f(X) \subseteq \bigcup_{i=1} 6nV_{\alpha_i}$. ■

Theorem 5.3.2. *If $f : X \to \mathbb{R}^k$ is continuous, where X is a compact metric space, then $f(X)$ is closed and bounded; in particular, f is bounded.*

Proof. From theorem 5.3.1, we have that $f(X)$ is compact in $\mathbb{R}^k$, therefore, it is closed and bounded. ■

Theorem 5.3.3 (The Extreme Value Theorem)**.** *Suppose f is a continuous, realvalued function on a metric space X, and that $M = \sup f$, and $m = \inf f$. Then there exist points $p, q \in X$ such that $f(p) = M$ and $f(q) = m$.*

Proof. By theorem 5.3.2, $f(X)$ is closed and bounded, thus $M, m \in f(X)$. ■

Theorem 5.3.4. *Suppose f is a continuous 1-1 mapping of a compact metric space X onto a metric space Y. Then the inverse mapping $f^{-1} : Y \to X$ is a Continuous mapping of Y onto X.*

Proof. By theorem 5.2.3, it suffices to show that $f(V)$ is open in Y whenever V is open in X. We have that $X \backslash V$ is closed in X, and compact, thus $f(X \backslash V)$ is closed and compact in Y, thus $f(V) = Y \backslash f(X \backslash V)$ is open in Y. ■

Definition. Let f be a mapping of a metric space X into a metric space Y. We say that f is **uniformly continuous** on X if for every $\varepsilon > 0$, there is a $\delta > 0$ such that $d_Y(f(q), f(p)) < \varepsilon$, for all $p, q \in X$ for which $d_X(p, q) < \delta$.

Lemma 5.3.5. *If f is uniformly continuous, then f is contiuous.*

Theorem 5.3.6. *Let f be a continuous mapping of a compact metric space X into a metric space Y. Then f is uniformly continuous on X*

Proof. Let $\varepsilon > 0$, by the continuity of f, we can associate for each $p \in X$ a number $\phi(p) > 0$ such that for $q \in X$, $d_X(p, q) < \phi(p)$ implies $d_Y(f(p), f(q)) < \frac{1}{2}\phi(p)$. Now let $J(p) = \{q \in X : d_X(p, q) < (p)\}$. Clearly, $p \in J(p)$, so $J(p)$ is an open cover of X, and since X is compact, there are $p_1, \dots, p_n$ for which $X \subseteq \bigcup_{i=1}^n J(p_i)$, then take $\delta = \min\{\phi(p_1), \dots, \phi(p_n)\}$; we have $\delta > 0$. Now let $p, q \in X$ such that $d_X(p, q) < \delta$. Then there is an $m \in \mathbb{Z}^+$ with $1 \le m \le n$ such that $p \in J(p_m)$, thus $d_X(p, q) < \frac{1}{2}\phi(p_m)$, by the triangle inequality, we get $d_(q, p_m) \le d_X(q, p) + d_X(p, p_m) < \delta + \frac{1}{2}\phi(p_m) = \phi(p_m)$, for $1 \le m \le n$. Therefore, $d_Y(f(p), f(q)) \le d_Y(f(p), f(p_m)) + d_Y(f(p_m), f(q)) < \varepsilon$. Thus, f is uniformly contiuous. ■

Remark. What this theorem says, is that in any compact metric space, continuity and uniform continuity are equivalent.

Theorem 5.3.7. *Let $E \subseteq \mathbb{R}$ be noncompact, then:*

> *[label=(0)]There exists a continuous function on E which is not bounded. There is a bounded, continuous function on E which has no maximum. If E is bounded, there exists a continuous function on E that is not uniformly continuous.*

3. *Proof.* Suppose first that E is bounded. Then there is a limit point $x_0 \notin E$ of E. Consider the function

$$f(x) = \frac{1}{x - x_0} \text{ for all } x \in E$$

Then f is continuous on E, but not bounded. Then let $\varepsilon > 0$ and $\delta > 0$, and choose $x \in E$ such that $|x - x_0| < \delta$, then taking t arbitrarily close to x_0, we can get $|f(x) - f(t)| \geq \varepsilon$, even though $|x - t| < \delta$. Thus f is not uniformly continuous.

Now choose

$$g(x) = \frac{1}{1 + (x - x_0)^2} \text{ for all } x \in E$$

g is continuous, and bounded on E (0¡g¡1), then $\sup g = 1$, and since $g(x) < 1$ for all x, we see that g attains no maximum.

Lastly, suppose that E is unbounded, then the functions $f(x) = x$ and $h(x) = \frac{x^2}{1+x^2}$ for all $x \in E$ establish (1) and (2). ∎

Example 5.3. Let f be the mapping of the interval $[0, 2\pi)$ onto the unit circle. That is $f(t) = (\cos t, t)$ for $0 \leq t < 2\pi$. Then f is a continuous 1-1 mapping of $[0, 2\pi)$ onto the unit circle, however, the inverse mapping, f^{-1} fails to be continuous at the point $f(0) = (1, 0)$.

5.4 Continuity and Connectedness.

Theorem 5.4.1. *If f is a continuous mapping of a metric space X into a metric space Y, and if $E \subseteq X$ is Connected, then so is $f(E)$.*

Proof. Suppose that $f(E) = A \cup B$ with $A, B \subseteq Y$ nonempty and seperated. Let $G = E \cap f^{-1}(A)$ and $H = E \cap f^{-1}(B)$, then $E = G \cup H$, and G and H are both nonempty. THen since $A \subseteq \overline{A}$, $G \subseteq f^{-1}(\overline{A})$, and since f is continuous, $f^{-1}(\overline{A})$ is closed, so $\overline{G} \subseteq f^{-1}(\overline{A})$, thus $f(\overline{G}) \subseteq \overline{A}$. Since $f(H) = B$, and $\overline{A} \cap B$ is empty, $\overline{G} \cap H$, and $H \cap \overline{H}$ are alsp empty, which contradicts the connectedness of E. ∎

Theorem 5.4.2 (The Intermediate Value Theorem)**.** *Let $f[a, b] \to \mathbb{R}$ be a realvalued function. If $f(a) < f(b)$, and $c \in \mathbb{R}$ such that $f(a) < c < f(b)$, then there is an $x \in (a, b)$ such that $f(x) = x$.*

Proof. We have that $[a, b]$ is connected in $\mathbb{R}$, thus by theorem 5.4.1, $f([a, b])$ is connected in $\mathbb{R}$, hence there is an $x \in (a, b)$ for which $f(x) = c$. ∎

Corollary. *If $f : [a, b] \to \mathbb{R}$ is a realvaulued function such that $f(a) < 0 < f(b)$, then there is an $x \in (a, b)$ such that $f(x) = 0$.*

5.5 Discontinuities.

Definition. Let X and Y be metric spaces, and let $f : E \to Y$ for $E \subseteq X$. If there is a point $x\ inE$ for which f is not continuous, we say that f is textbfdiscontinuous at x, and we say that f has a **discontinuity** at x.

Definition. Let f be defined on (a, b), and let x be such that $a \leq x < b$. We write $f(x+) = q$ if $f(t_n) \to q$ for all sequences $\{t_n\}$ in (x, b) such that $t_n \to x$. Similarly, if x is such that $a < x \leq b$, we write $f(x-) = q$ if $f(t_n) \to q$ for all sequences $\{t_n\}$ in (a, x) such that $t_n \to x$. We call $f(x+)$ and $f(x-)$ the **right handed limit** and **left handed limit** of f at x respectively, and write $\lim_{t \to x^+} f = f(x+)$ and $lim_{t \to x^-} f = f(x-)$.

Theorem 5.5.1. *If $x \in (a, b)$, then $\lim f$ exists as $t \to x$ if and only if, $f(x+) = f(x-) = lim f$.*

Proof. Suppose that $\lim f$ exists, by the uniqueness of the limit, and the sequential criterion, we get that $f(x+) = f(x-) = \lim f$. Conversely, suppose that $f(x+) = f(x-) = q$. Then $f(t_n) \to q$ for all sequences $\{t_n\}$ in (x, b) and (a, x), then $f(t_n) \to q$ for all sequences $\{t_n\}$ in (a, b), thus by the sequential criterion again, $\lim f$ exists, and $\lim f = q$. ∎

Definition. Let f be defined on (a, b). If f is discontinuous at a point x, and $f(x+)$ and $f(x-)$ exists, we say that f has a **removable discontinuity** at x, otherwise, we say the f has an **infinite discontinuity**.

Example 5.4. [label=(0)]

1. The function $f(x) = 1$ for $x \in \mathbb{Q}$ and $f(x) = 0$ for $x \in \mathbb{R} \backslash \mathbb{Q}$ has an infinite discontinuity at every point x.

2. The function $f(x) = x$ for $x \in \mathbb{Q}$ and $f(x) = 0$ for $x \in \mathbb{R} \backslash \mathbb{Q}$ is continuous at $x = 0$, and has an infinite discontinuity at every other point x.

3. The function $f(x) = \sin \frac{1}{x}$ for $x \neq 0$ and $f(x) = 0$ for $x = 0$, has an infinite discontinuity at $x = 0$.

4. The function $f(x) = x + 2$ for $-3 < x < -2$ and $0 \leq x < 1$ and $f(x) = -x - 2$ for $-2 \leq x < 0$ has a removable discontinuity at $x = 0$, and is continuous everywhere else.

Remark. The discontinuities in examples (1) and (2) are the result of $\mathbb{Q}$ and $\mathbb{R} \backslash \mathbb{Q}$ being dense in $\mathbb{R}$.

5.6 Monotonic Functions.

Definition. Let f be a realvalued function on an interval (a, b). We say that f is **monotonically increasing** on (a, b) if $a < x < y < b$ implies $f(x) \leq f(y)$. We say that f is **monotonically decreasing** on (a, b) if $a < x < y < b$ implies $f(y) \leq f(x)$. We say f is **monotonic** if it is either monotonically increasing or monotonically decreasing.

Theorem 5.6.1. *Let f be monotonic on (a,b) then $f(x+)$ and $f(x-)$ exist at every point of (a,b) and $\sup f = f(x-)$ and $\inf f = f(x+)$, and the following hold:*

[label=(0)][label=(0)]If f is monotonically increasing $f(x-) \leq f(x) \leq f(x+)$ If f is monotonically decreasing $f(x+) \leq f(x) \leq f(x-)$

(b) *Proof.* We prove only (1), since (2) is analogous. Suppose that f is monotonically increasing, clearly, f has an upperbound A for which $A \leq f$. Now let $\varepsilon > 0$, then there is a $\delta > 0$ for which $a < x - \delta < x$, and $A - \varepsilon < f(x-\delta) \leq A$. Then we have $f(x-\delta) < f(t) \leq A$ for all $x - \delta < t < x$, then we get $|f(t) - A| < \varepsilon$, hence $f(x-) = A$, Similarly, we get $f(+) = -\inf f$. Now since $\sup f \leq f \leq \inf f$, we get the desired result. ■

Corollary. *Monotonic functions have no infinite discontinuities.*

Theorem 5.6.2. *Let f be monotonic on (a,b), then the set of all points of (a,b) for which f is discontinuous is atmost countable.*

Proof. Suppose, without loss of generality that g is monotonically increasing, and let E be the set of all points of (a,b) for which f is discontinuous. By the density of $\mathbb{Q}$ in $\mathbb{R}$, for each $x \in E$ associate $r(x) \in \mathbb{Q}$ such that $f(x+) < f(x) < f(x-)$. Since $x_1 < x_2$ implies $f(x_1+) \leq f(x_2-)$, then $r(x_1) \neq r(x_2)$, thus $x_1 \neq x_2$, and so r is a 1-1 mapping of E into $\mathbb{Q}$. ■

Now, given a countable E in an interval (a,b), we can construct a monotonic function f that is discontinuous at every point in E and continuous everywhere else. Arrange the points of E into a sequence $\{x_n\}$ and let $\{c_n\}$ be a sequence such that $c_n > 0$ for all $n \in \mathbb{Z}^+$, such that $\sum c_n$ converges. Define $f(x) = \sum_{x_n < x} c_n$, for $x \in (a,b)$. Then we have that

[label=(0)]f is monotonically increasing on (a,b). f is discontinuous at every point in E with $f(x_n+) - f(x_n-) = c_n$. f is continuous at every point in $(a,b)\backslash E$.

Definition. Let f be a realvalued function defined on an interval (a,b). We say that f is **continuous form the right** if $f(x+) = f(x)$, and we say f is **continuous from the left** if $f(x-) = f(x)$.

5.7 Infinite Limits and Limits at Infinity.

Definition. For any $c \in \mathbb{R}$, the set of all real numbers x such that $x > c$ is called the **neighborhood of** ∞, and denoted (c, ∞). The set of all real numbers x such that $x > c$ is called the **neighborhood of** $-\infty$, and denoted $(-\infty, c)$.

Definition. Let $f : E \to \mathbb{R}$ be a realvalued function. We say that $f(t) \to A$ as $t \to x$, with A, and x extended real numbers if for every neighborhood of U A, there is a neighborhood V of x such that $V \cap E$ is nonempty, and $f(t) \in U$ for all $t \neq x \in V \cup E$.

Theorem 5.7.1. *Let $f, g : E \to \mathbb{R}$ be realvalued functions such that $f \to A$, and $g \to B$ as $t \to x$, for extended real nuumbers A, B, and x. Then the following hold as $t \to x$.*

[label=(0)]$f \to A'$ implies $A = A'$. $f + g \to A + B$. $fg \to AB$. $\frac{f}{g} \to \frac{A}{B}$. Provided that (1), (2), and (3) are not of the forms $\infty - \infty$, $0 \cdot \infty$, $\frac{\infty}{\infty}$, and $A_{\overline{0}}$, respectively.

4. *Proof.* This is a direct application of the sequential criterion using the appropriate definition. ■

Chapter 6

Differentiation

6.1 The Derivative of Real valued Functions.

Definition. Let $f : [a, b] \to \mathbb{R}$ be a realvalued function defined on $[a, b]$. The **derivative** of f at a point $x \in (a, b)$ is the function $f' : (a, b) \to \mathbb{R}$ defined by

$$f'(x) = \lim_{t \to x} \frac{f(t) - f(x)}{t - x} \tag{6.1}$$

If f' is defined at $x \in [a, b]$, then we say that f is **differentiable** at x, and if f' is defined for all $x \in (a, b)$, we say that f is **differentiable** on (a, b).

Theorem 6.1.1. *Let $f : [a, b] \to \mathbb{R}$ be a realvalued function. If f is differentiable at a point $x \in (a, b)$, then f is continuous.*

Proof. As $t \to x$, we get $|f(t) - f(x)| = |\frac{f(t)-f(x)}{t-x}||t - x| \to f'(x)0 = 0$, thus $f(t) \to f(x)$. ■

Theorem 6.1.2. *Suppose $f, g : [a, b] \to \mathbb{R}$ are realvalued functiond differentiable at a point $x \in (a, b)$. Then $f + g$, fg, and $\frac{f}{g}$ are differentiable at x, and as $t \to x$:*

[label=(0)]$(f + g)' = f' + g'$. $(fg)' = f'g + fg'$. $(\frac{f}{g})' = \frac{f'g-fg'}{g^2}$, provided that $g(x) \neq 0$.

3. *Proof.* (1) follows directly from the definiton. Now notice that $fg(t) - fg(x) = f(t)(g(t) - g(x)) + g(t)(f(t) + f(x))$, then dividing by $t - x$, the result follows by definition.

Now also notice that $\frac{\frac{f}{g}(t)-\frac{f}{g}(x)}{t-x} = \frac{1}{g(t)g(x)}(g(x)\frac{f(t)-f(x)}{t-x} - f(x)\frac{g(t)-g(x)}{t-x})$, and the result again follows by definition. ■

Example 6.1. [label=(0)]

1. The derivative of constant functions are alway 0, and the derivative of the identity function is always 1.
2. Let $f(x) = x^n$, for $n \in \mathbb{Z}$, and $x \neq 0$ for $n < 0$, then f is differentiable and $f'(x) = nx^{n-1}$.
3. Polynomial functions are differentiable, and so are rational functions $\frac{p}{q}$, provided that $q \neq 0$.

Theorem 6.1.3 (Caratheodory's Theorem)**.** *Let $f : [a, b] \to \mathbb{R}$ be a continuous realvalued function. Then f is differentiable at a point $x \in (a, b)$ if and only if there is a continuous function $\phi : (a, b) \to \mathbb{R}$ such that $f(t) - f(x) = \phi(t)(t - x)$; moreover, $\phi = f'$.*

Proof. Suppose f' exists at x, and define $\phi : (a, b) \to \mathbb{R}$ by $\phi(t) = \frac{f(t)-f(x)}{t-x}$ when $t \neq x$, and $\phi(t) = f'(x)$ at $t = x$. Then by the contiunity of f, ϕ is continuous at x, moreover, at $t \neq x$ we see that $f(t) - f(x) = \phi(t)(x - t)$.

Conveersesly, sup[ose there is a ϕ, continuous at x such that $f(t) - f(x) = \phi(t)(x - t)$, then clearly, $\lim \phi = f'(x)$ as $t \to x$, and since ϕ is continuous, $\phi(x) = f'(x)$. ■

Theorem 6.1.4 (The Chain Rule)**.** *Suppose that $f : [a, b] \to \mathbb{R}$ and $g : I \to \mathbb{R}$ are continuous, where $f([a, b]) \subseteq I \subseteq [a, b]$, and suppose that f is differentiable at x, and that g is differentiable at $f(x)$. Then $g \circ g$ is differentiable at x, and $(g \circ f)' = (g' \circ f)f'$.*

Proof. We have by Caratheodory's theorem that $f(t) - f(x) = (t - x)(f'(x) - u(t))$, and $g(s) - g(y) = (s - y)(g'(y) - v(s))$. Then letting $y = f(x)$, and $s \to y$ as $t \to x$, we see that $u, v \to 0$, and we get that $g(f(t)) - g(f(x)) = g'(f(t)f(t)) - g'(f(x))f(x)$, dividing by $t - x$ give the desired result. ■

Example 6.2. [label=(0)]

1. Let $f(x) = \sin \frac{1}{x}$ at $x \neq 0$, and $f(x) = 0$ at $x = 0$. We have at $x \neq 0$, that $f'(x) = \sin \frac{1}{x} - \frac{1}{x} \cos \frac{1}{x}$, but at $x = 0$, we must appeal to the definition, and we get $f(t) = \sin \frac{1}{t}$, which diverges at $t \to 0$, thus $f'(0)$ does not exist.
2. Let $f(x) = x^2 \sin \frac{1}{x}$ at $x \neq 0$, and $f(x) = 0$ at $x = 0$. For $x \neq 0$, we get $f'(x) = 2x \sin \frac{1}{x} - \cos \frac{1}{x}$, and at $x = 0$, we notice that $|t \sin \frac{1}{t}| \leq |t|$, so by the sandwhich theorem, $f'(0) = 0$ as $t \to 0$.

6.2 Mean Value Theorems.

Definition. Let $f : X \to \mathbb{R}$ be defined on a metric space X. We say that f has a **local maximum** at a point $p \in X$, if there is a $\delta > 0$ for which $f(q) \leq f(p)$ whenever $d(q, p) < \delta$. Likewise f has a **local minimum** at a point $p \in X$, if there is a $\delta > 0$ for which $f(q) \leq f(p)$ whenever $d(q, p) < \delta$. We call local maxima and local minumums **local extrema**.

Theorem 6.2.1. *Let $f : [a, b] \to \mathbb{R}$ be a realvalued function, and suppose that f has a local extremum at $x \in (a, b)$. If f' exists, then $f'(x) = 0$.*

Proof. Suppose, without loss of generality that f has a local maximum at x. Choosse $\delta > 0$ such that $a < x - \delta < x < x + \delta < b$. Then if $x - \delta < t < x$, we have $|t - x + \delta| < \delta$, so $f(t) \leq f(x)$, thus $\frac{f(t)-f(x)}{t-x} \leq 0$. Similarly, for $x < t < x + \delta$, we get $\frac{f(t)-f(x)}{t-x} \geq 0$, hence, as $t \to x$, we get $0 \leq f'(0) \leq 0$, thus $f'(x) = 0$. ■

Theorem 6.2.2 (The Generalized Mean Value Theorem)**.** *If $f, g : [a, b] \to \mathbb{R}$ are continuous on $[a, b]$, and differentiable on (a, b), then there is a point $x \in (a, b)$ such that $(f(b) - f(a))g('x) = (g(b) - g(a))f'(x)$.*

Proof. Let $h(t) = (f(b) - f(a))g('x) - (g(b) - g(a))f'(x)$, for $t \in [a, b]$, then h is continuous on $[a, b]$, and differentiable on (a, b), moreover, we have $h(b) = f(b)g(a) - f(a)g(b) = h(b)$. Now if h is constant, then $h' = 0$ for all t and we are done., Now suppose that $h(a) \leq h(t)$, and let $x \in (a, b)$, be a local minimum of h, then $h'(x) = 0$, and we are done; the same result follows for local minima of h. ∎

Corollary (The Mean Value Theorem)**.** *LEt $f : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$, and differentiable on (a, b). Then there is an $x \in (a, b)$ such that $f(b) - f(a) = (b - a)f'(x)$.*

Proof. Take $g(t) = t$. ∎

Theorem 6.2.3. *Suppose that $f : [a, b] \to \mathbb{R}$ is differentiable on (a, b). Then the following hold for all $x \in (a, b)$:*

[label=(0)]If $f' \geq 0$, then f is monotonically increasing. If $f' = 0$, then f is constant. If $f' \leq 0$, then f is monotonically decreasing.

3. *Proof.* Let $x_1, x_2 \in (a, b)$, then by the mean value theorem, there is an $x \in (x_1, x_2)$ such that $f(x_2) - f(x_1) = (x_2 - x_1)f'(x)$. Then if $f'(x) = 0$, we get $f(x_2) = f(x_1)$, and that f is constant. If $f'(x) \geq 0$, we get $f(x_2) \geq f(x_1)$, making f monotonically increasing, similarly, if $f'(x) \leq 0$, we get f monotonically decreasing. ∎

6.3 The Continuity of Derivatives.

Theorem 6.3.1. *Let $f : [a, b] \to \mathbb{R}$ be differentiable on all of $[a, b]$, and suppose that $f'(a) < \lambda < f'(b)$. Then there is an $x \in (a, b)$ such that $f'(x) = \lambda$.*

Proof. Let $g(t) = f(t) - \lambda t$, then $g'(a) < 0$ and $g'(b) > 0$. Then for $t_1, t_2 \in (a, b)$, $g(t_1) < g(a)$, and $g(b) < g(t_2)$. Then by the extreme value theorem, g attains a maximum at a point $x \in (t_1, t_2)$, hence $g'(x) = 0$, hence $f'(x) = \lambda$. ∎

Corollary. *If $f : [a, b] \to \mathbb{R}$ is differentiable, then f cannot have any removable discontinuities, nor jump discontinuities.*

Remark. f' may have infinite discontinuities.

6.4 L'Hosptal's Rule.

Theorem 6.4.1 (L'Hospital's Rule)**.** *Suppose f and g are realvalued functions differentiable on (a, b), and that g' neq0 for all $x \in (a, b)$, where $-\infty \leq a < b \leq \infty$, and suppose that $\frac{f'}{g'} \to A$ as $x \to a$. If $f, g \to 0$, or if $g \to \pm\infty$, as $x \to a$, then $\frac{f}{g} \to A$ as $x \to a$.*

Proof. Suppose first that $-\infty \leq A < \infty$, and choose $q, r \in \mathbb{R}$ such that $A < r < q$. By hypothesis, there is a $c \in (a, b)$ for which $a, x < c$ implies $\frac{f}{g} < r$. If $a < x < y < c$, then by the generalized mean value theorem, $\frac{f(x)-f(y)}{g(x)-g(y)} = \frac{f'(t)}{g'(t)} < r$, thus letting $x \to a$, we see hat $\frac{f(y)}{g(y)} \leq r < q$. Now suppose, without loss of generality, that $g \to \infty$. Fixing y, and choosing

$c_1 \in (a,y)$ such that $g(x) > g(y)$, and $g(x) > 0$, if $a < x < c_1$, then $\frac{f(x)}{g(x)} < r - r\frac{g(y)+f(y)}{g(x)}$, then as $x \to a$, there is a $c_2 \in (x, c_1)$ such that $\frac{f}{g} < q$.

Likewise, if we suppose that $-\infty < A \leq \infty$, by the same reasoning, we can choose a $p < A$ and $c_3 \in (a,b)$ such that $p < \frac{f}{g}$ as $x \to a$. Since $p < A < q$, by the sandwhich theorem, we get $\frac{f}{g} = A$ as $x \to a$. ■

6.5 Taylor's Theorem.

Definition. If f has a derivative f' on an interval, and f' is differentiable, we denote f'' to be $(f')'$ and call it the **second derivative** of f; likewise, if f'' is differentiable, we denote the **third derivative** by $f^{(3)} = (f'')'$. More generally, for $n \in \mathbb{Z}^+$, we define recursively the **nth derivative** to be:

[label=(0)]

1. $f^{(0)} = f$ and $f^{(1)} = f'$.
2. $f^{(n+1)} = (f^{(n)})'$, given that $f^{(n)}$ is differentiable.

We call f **nth differentiable** if $f^{(n)}$ exists.

Theorem 6.5.1 (Taylor's Theorem)**.** *Suppose $f : [a,b] \to \mathbb{R}$ is a realvalued function, that is nth differentiable, and let $n \in \mathbb{Z}^+$ be such that $f^{(n-1)}$ is continuous on $[a,b]$, and that $f^{(n)}$ exists on (a,b). LEt $\alpha, \beta \in [a,b]$ be distinct, and define:*

$$p(t) = \sum_{k=0}^{n-1} \frac{f^{(k)}(\alpha)}{k!}(t-\alpha)^k \tag{6.2}$$

Then there exists a point $x \in (\alpha, \beta)$ such that $f(\beta) = p(\beta) + \frac{f^{(n)}(x)}{n!}(\beta-\alpha)^n$.

Proof. For $n = 1$, this reduces to the mean value theorem, so suppose that $n > 1$. Let $M \in \mathbb{R}$ be such that $f(\beta) = p(\beta) + M(\beta-\alpha)^n$, and let $g(t) = f(t) - p(t) + M(\beta-\alpha)^n$, for $t \in [a,b]$. Then g is nth differentiable, and we get $g^{(n)} = f^{(n)} - n!M$ for $t \in (a,b)$. We wish to show that $f^{(n)} = n!M$.

We have that $p^{(k)} = f^{(k)}(\alpha)$ for $0 \leq k \leq n-1$, then $g(\alpha) = g'(\alpha) = \cdots = g^{(n-1)}(\alpha) = 0$, and our choice of M shows that $g(\beta) = 0$. So $g'(x_1) = 0$ for $x_1 \in (\alpha, \beta)$, so by the mean value theorem, since $g'(\alpha) = 0$, then $g''(\xi_2) = 0$ for $x_2 \in (\alpha, x_2)$. Proceeding inductively, we then get that $g_{(n)(x_n)=0}$ for $x_n \in (\alpha, x_{n-1})$, hence we get that $n!M = f^{(n)}(x)$. ■

Definition. We call the series in equation (5.2) the **Taylor series** (or **Taylor polynomial**) of f about α. We call the realnumber M such tat $n!M = f^{(n)}(x)$ the **tail**, (or **error**) of the Taylor series.

6.6 Derivatives of vector valued functions.

Definition. Let $f : [a,b] \to \mathbb{C}$ be a complex valued function, such that $f(t) = f_1(t) + if_2(t)$. We say that f is **differentiable** at a point x if and only if f_1 and f_2 are differentiable, and we denote the **derivative** of f to be the function $f : (a,b) \to \mathbb{C}$ such that $f' = f_1' + if_2'$

Definition. Let $f : [a, b] \to \mathbb{R}^k$ be a vectorvalued function for $k \in \mathbb{Z}^+$. f is said to be **differentiable** at $x \in (a, b)$ if there is some point $f'(x) \in \mathbb{R}^k$ such that:

$$\lim_{t \to x} ||\frac{f(t) - f(x)}{t - x} - f'(x)|| = 0 \tag{6.3}$$

We define the **derivative** of f at x to be the function $f' : (a, b) \to \mathbb{R}$ such that the values of f' statisfy equat ion (5.3)

Remark. If $f : [a, b] \to \mathbb{R}^k$ is defined by $f = (f_1, \ldots, f_k)$, then f is differentiable at a point $x \in (a, b)$ if and only if f_i is differentiable at x for $1 \leq i \leq k$, and we have that $f' = (f_1', \ldots, f_k')$.

Theorem 6.1.1 follows natrually, and so does theorem 6.1.2(a) and (2), where we define fg as $\langle f, g \rangle$, howeever, the mean value theorem in general does not hold.

Example 6.3. [label=(0)]

1. Define $f : \mathbb{R} \to \mathbb{C}$ by $f(x) = e^{ix} = \cos x + i \sin x$. Then $f(2\pi) - f(0) = 0$, however, $f'(x) = ie^{ix} \neq 0$ for all x (moreover, $|f'| = 1$), so the generalized mean value theorem fails here.
2. Define $f, g : (0, 1) \to \mathbb{C}$ by $f(x) = x$ and $g(x) = x + x^2 e^{\frac{i}{x^2}}$ for all x. Since $|e^{it}| = 1$, we have that $\lim \frac{f}{g} = 1$ as $x \to 0$. Now $g'(x) = 1 + (2x - i\frac{2}{x})e^{\frac{1}{x^2}}$ on $(0, 1)$, hence $|g'| = |2x - i\frac{2}{x}| - 1 \geq \frac{2}{x} - 1$, so $|\frac{f'}{g'}| \leq \frac{x}{2-x} \to 0$ as $x \to 0$, so L'Hospital's rule fails in $\mathbb{C}$ as well, and hence in $\mathbb{R}^2$ (as $\mathbb{C}$ is isomorphic to $\mathbb{R}^2$).

Theorem 6.6.1. *Suppose $f : [a, b] \to \mathbb{R}^k$, for $k \in \mathbb{Z}^+$ is continuous, and that f is differentiable on (a, b). Then there is an $x \in (a, b)$ for which $||f(b) - f(a)|| \leq (b - a)||f'(x)||$.*

Proof. Let $z = f(b) - f(a)$, and define $\phi = \langle f, g \rangle$ for all $t \in [a, b]$, then ϕ is a realvalued function continuous on $[a, b]$, moreover it is differentiable on (a, b); therefore, by the mean value theorem, $\phi(b) - \phi(a) = (b - a)\phi'(a) = (b - a)\langle z, f'(x) \rangle$ for $x \in (a, b)$. On the other hand, we have that $\phi(b) - \phi(a) = \langle z, z \rangle = ||z||^2$, hence, by the Cauchy Schwarz inequality, we have that $||z||^2 = (b - a)\langle z, f' \rangle \leq ||z||||f'||$, which gives the desired result. ∎

Chapter 7

Integration

7.1 The Riemann-Stieltjes Integral.

Definition. Let $[a, b]$ be an interval. A **partition** of $[a, b]$ is a set of points $P = \{x_0, x_1, \ldots, x_n\}$ such that $a = x_0 < x_1 < \cdots < x_n = b$, and we write $\Delta x_i = x_i - x_{i-1}$. Now let $f : [a, b] \to \mathbb{R}$ be a bounded realvalued function, and for each partition P of $[a, b]$ let $M_i = \sup f$ and $m_i = \inf_f$ for all $x_{i-1} \leq x \leq x_i$. We define the **upper Riemann sum** and the **lower Riemmann sum** to of f with respect to be:

$$U(f, P) = \sum_{i=1}^{n} M_i \Delta x_i \tag{7.1}$$

$$L(f, P) = \sum_{i=1}^{n} m_i \Delta x_i \tag{7.2}$$

respectively. We also define the **upper Riemann integral** and the **lower Riemann integral** of f over $[a, b]$ to be:

$$\overline{\int_a^b} f(x)dx = \inf U(f, P) \tag{7.3}$$

$$\underline{\int_a^b} f(x)dx = \sup L(f, P) \tag{7.4}$$

Respectively.

If $\overline{\int_a^b} f = \underline{\int_a^b} f$, then we say that f is **Riemann integrable** on $[a, b]$, and we its value the **Riemann integral**, and denote it to be:

$$\int_a^b f(x)dx = \overline{\int_a^b} f(x)dx = \underline{\int_a^b} f(x)dx \tag{7.5}$$

Lemma 7.1.1. *$\overline{\int_a^b} f$, and $\underline{\int_a^b} f$ are defined for every bounded realvalued function f over $[a, b]$.*

Proof. Let f be bounded on $[a,b]$, then there are m and M such that $m \leq f \leq M$ for all $a \leq x \leq b$. Now let P be a partition of $[a,b]$. Since $\inf f \leq \sup f$, we have that $m \leq m_i = \inf f \leq M_i = \sup f \leq M$, thus $m(b-a) \leq L(f,P) \leq U(f,P) \leq M(b-a)$, hence L and U form a bounded set, and we are done. ■

Corollary. *$L(f,P) \leq U(f,P)$ for every bounded function f.*

Now the question of the integrability of f is a very delicate matter, and requires a closer scrutiny on the concepts of upper and lower sums. Infact, it turns out that the Riemann integral is a consequence of a more general class of integrals. Developng this more general situation will allow us to discern facts about the Riemann integral.

Definition. Let α be a bounded monontonically increasing function on $[a,b]$, and let P be a partition of $[a,b]$ and let $\Delta\alpha_i = \alpha(x_i) - \alpha(x_{i-1})$. For any realvalued, bounded function on $[a,b]$, defined the **upper sum** and the **lower sum** of f with respect to P and α to be:

$$U(f,P,\alpha) = \sum_{i=1}^{n} M_i \Delta\alpha_i \tag{7.6}$$

$$L(f,P,\alpha) = \sum_{i=1}^{n} m_i \Delta\alpha_i \tag{7.7}$$

Where $M_i = \sup f$ and $m_i = \inf f$ for all $x_{i-1} \leq x \leq x_i$, and again, define the **upper integral** and **lower integral** of f with respect to α on $[a,b]$ to be:

$$\overline{\int_a^b} f(x)d\alpha = \inf U(f,P,\alpha) \tag{7.8}$$

$$\underline{\int_a^b} f(x)d\alpha = \sup L(f,P,\alpha) \tag{7.9}$$

If $\overline{\int_a^b} f d\alpha = \underline{\int_a^b} f d\alpha$, we call the value:

$$\int_a^b f(x)d\alpha = \overline{\int_a^b} f(x)d\alpha = \underline{\int_a^b} f(x)d\alpha \tag{7.10}$$

the **Riemann-Stieltjes integral** of f with respect to f on $[a,b]$. If such an integra exists, we say that f is **integrable** with respect to α on $[a,b]$.

Example 7.1. Let $\alpha(x) = \alpha$, be defined over $[a,b]$. Then α is monontonically increasing, and our defintitions reduces to those for the Riemann integral. Here $U(f,P,x) = U(f,P)$ and $L(f,P,x) = L(F,P)$.

We are now in a position to investigate the properties of integrability, in the Riemann-Stieltjes sense.

Definition. Let $a,b]$ be an interval, and let P and Q be partitions of $[a,b]$. We say that Q is a **refinment** of P if PQ, and we also say that Q is **finer** than P. Now if neither P nor Q is a refinment of the other, we say that the two partitions are **noncomparable**.

Lemma 7.1.2. *Let P and Q be partitions of and interval $[a, b]$, then $P \cup Q$ is a partition of $[a, b]$, and is a refinment of both P and Q.*

Proof. If P is a refinment of Q, or viceversa, then we are done; so suppose that P and Q are noncomparable. Let $P = \{x_0, x_1, \ldots, x_n\}$ and $Q = \{y_0, y_1, \ldots, y_m\}$ with $a = x_0 < x_1 < \ldots x_n = b$ and $a = y_0 < y_1 < \ldots y_m = b$. Then $P \cup Q = \{x_0, y_0, x_1, y_1, \ldots, x_n, y_m\}$ and $a = x_0 = y_0 < x_1, y_1 < \cdots < x_n = y_m = b$, thus $P \cup Q$ is a partition of $[a, b]$, that it is a refinment of P and Q follows trivially. ■

Theorem 7.1.3. *Let α be monontonically increasing, and bounded on $[a, b]$, and let P and Q be partitions of $[a, b]$. If Q is a refinment of P, then $L(f, P, \alpha) \leq L(f, Q, \alpha)$ and $U(f, Q, \alpha) \leq U(f, P, \alpha)$.*

Proof. Let $Q = P \cup \{x'\}$ and suppose that $x_{i-1} \leq x' \leq x_i$. Let $w_1 = \inf f$ for $x_{i-1} \leq x \leq x'$ and let $w_2 = \inf f$ for $x' \leq x \leq x_i$. Then $m_i \leq w_1, w_2$, thus $L(f, Q, \alpha) - L(f, P, \alpha) = (w_1 - m_i)(\alpha(b) - \alpha(a)) - (w_2 - m_i)(\alpha(b) - \alpha(a)) \geq 0$, we are done. The proof is analogous for U. ■

Corollary. *$L(f, P, \alpha)$ is monontonically increasing and $U(f, P, \alpha)$ is monontonically decreasing.*

Proof. We note that if Q is a refinment of P, then $|P| \leq |Q|$, the result follows by direct application. ■

Remark. If Q contains k more points than P, we can repeat the proof inductively.

Theorem 7.1.4. *$\underline{\int} f d\alpha \leq \overline{\int} f d\alpha$.*

Proof. Let $P = P_1 \cup P_2$ for partitions P_1 and P_2 of $[a, b]$. By theorem 7.1.3 and lemma 7.1.1, we have:

$$L(f, P_1, \alpha) \leq L(f, P_2, \alpha) \leq U(f, P_2, \alpha) \leq U(f, P_1, \alpha) \tag{7.11}$$

Fixing P_2 and taking the supremum over all P_1, we get $\underline{\int} f \leq U(f, P_2, \alpha)$, the infimum over P_2 we get $\underline{\int} f \leq \overline{\int} f$ ■

Theorem 7.1.5. *A realvalued function f is integrable over an interval $[a, b]$ if and only if for $\varepsilon > 0$, there is a partition P such that:*

$$U(f, P, \alpha) - L(f, P, \alpha) < \varepsilon \tag{7.12}$$

Proof. For every P, we have that $L(f, P, \alpha) \leq \underline{\int} f \leq \overline{\int} f \leq U(f, P, \alpha)$, so if for $\varepsilon > 0$, we assume that $U(f, P, \alpha) - L(f, P, \alpha) < \varepsilon$, then we get that:

$$0 \leq \overline{\int_a^b} f d\alpha - \underline{\int_a^b} f d\alpha < \varepsilon \tag{7.13}$$

implying integrablity for small enough ε.

Conversely, suppose that $\overline{\int} f = \underline{\int} f$. Let $\varepsilon > 0$. Then there are partitions P_1 and P_2 such that $U(f, P_2, \alpha) - \int f < \frac{\varepsilon}{2}$ and $\int f - L(f, P_1,) < \frac{\varepsilon}{2}$. Take $P = P_1 \cup P_2$, then by theorem 7.1.4, adding the inequalities we get $U(f, P, \alpha) - L(f, P, \alpha) < \varepsilon$. ■

Theorem 7.1.6. *The following hold:*

[label=(0)]If $U(f,P,\alpha)-L(f,P,\alpha)<\varepsilon$ for some $\varepsilon>0$ and some P, then it holds, with the same ε, for every refinment of P. If $U(f,P,\alpha)-L(f,P,\alpha)<\varepsilon$ for $P=\{x_0,\dots,x_n\}$, and if $s_i,t_i\in[x_{i-1},x_i]$, then:

$$\sum_{i=1}^{n}|f(s_i)-f(t_i)|\Delta\alpha_i<\varepsilon \tag{7.14}$$

If f is integrable with respect to α on $[a,b]$, and (2) holds, then:

$$|\sum_{i=1}^{n}f(t_i)\Delta\alpha_i-\int_a^b fd\alpha|<\varepsilon \tag{7.15}$$

1. *Proof.* [label=(0)]

   Theorem 7.1.3 implies this.

2. If $U(f,P,\alpha)-L(f,P,\alpha)<\varepsilon$, we have that $f(s_i),f(t_i)\in[m_i,M_i]$, hence $|f(s_i)-f(t_i)|\leq M_i=m_i$, thus we get that $\sum|f(s_i)-f(t_i)|\Delta\alpha_i\leq U(f,P,\alpha)-L(f,P,\alpha)<\varepsilon$
3. We have $L(f,P,\alpha)\leq\sum f(t_i)\Delta\alpha_i\leq U(f,P,\alpha)$, and $L(f,P,\alpha)\leq\int fd\alpha_i\leq U(f,P,\alpha)$, taking differences, we get the following.

■

Theorem 7.1.7. *If f is continuous on $[a,b]$, then f is integrable with respect to α on $[a,b]$.*

Proof. Let $\varepsilon>0$, and choose $\xi>0$ such that $(\alpha(b)-\alpha(a))\xi<\varepsilon$. Now we have that f is uniformly continuous on $[a,b]$, so there is a $\delta>0$ such that $|f(x)-f(t)|<\xi$ whenever $|x-t|<\delta$ for $x,t\in[a,b]$. Now since P is a partition of $[a,b]$, with $\Delta x_i<\delta$ for all i, then we have $M_i-m_i\leq\xi$ for $i-1,\dots,n$ thus $U(f,P,\alpha)-L(f,P,\alpha)=\sum(M_i-m_i)\Delta\alpha_i\leq\xi(\alpha(b)-\alpha(a))<\varepsilon$, therefore, f is integrable. ■

Theorem 7.1.8. *If f is monotonic on $[a,b]$, and α is monotonic and continuous on $[a,b]$, then f is integrable with respect to α on $[a,b]$.*

Proof. Let $\varepsilon>0$, and for $n>0$, construct a partition P such that $\Delta\alpha_i=\frac{\alpha(b)-\alpha(a)}{n}$ for $1\leq i\leq n$, which is guaranteed by the intermediate value theorem.

Now suppose without loss of generality that f is monoton increasing, then $M_f(x_i)$ and $m_i=f(x_{i-1})$ by the extreme value theorem, Then:

$$U(f,P,\alpha)-L(f,P,\alpha)=\frac{\alpha(b)-\alpha(a)}{n}\sum f(x_i)-f(x_{i-1})=\frac{\alpha(b)-\alpha(a)}{n}(f(b)-f(a))<\varepsilon$$

Taking n large enough implies integrability. ■

Theorem 7.1.9. *Suppose that f is bounded on $[a,b]$, with only finitely many discontinuities on $[a,b]$. Suppose that α is continuous at every discontinuity of f, then f is integrable with respect to on $[a,b]$.*

Proof. Let $\varepsilon > 0$ and $M = \sup|f|$ and let E be the set of all discontinuities of f. Since E is finite, and α is continuous on E, then E can be covered by finitely many disjoint intervls $[u_j, v_j] \subseteq [a, b]$ such that $\sum \alpha(u_j) - \alpha(v_j) < \varepsilon$.

Now suppose that we construct these intervals in such a way that every point of $E \cap [a, b]$ lies in (u_j, v_j), for some j. Taking $K = [a, b]\backslash(u_j, v_j)$, K is compact, so f is uniformly continuous on K. Thus there is a $\delta > 0$ such that $|f(s) - f(t)| < \varepsilon$ whenever $|s - t| < \delta$ for $s, t \in K$.

Now construct the partition P of $[a, b]$ such that $u_j, v_j \in P$, but $(u_j, v_j) \nsubseteq P$ for some j. If $x_i \neq u_j$, then $\Delta x_i < \delta$. Notice that $M_i - m_i \leq 2M$ for all i and that $M_i - m_i \leq \varepsilon$. Thus for ε small enough, we have $U(f, P, \alpha) - L(f, P, \alpha) \leq (\alpha(b) - \alpha(a))\varepsilon + 2M\varepsilon$, thus f is integrable. ■

Theorem 7.1.10. *Suppose that f is integrable with respect to α on an interval $[a, b]$, and that $m \leq f \leq M$, and let ϕ be continuous on $[m, M]$, and let $h = \phi \circ f$ be defined on $[a, b]$. Then h is integrable with respect to α on $[a, b]$.*

Proof. Let $\varepsilon > 0$. We have that ϕ is uniformly continuous on $[m, M]$, so there is a $\varepsilon > \delta > 0$ for which $|\phi(s) - \phi(t)| < \varepsilon$ whenever $|s - t| < \delta$, for $s, t \in [m, M]]$.

Now since f is integrable, there is a partition P of $[a, b]$ such that $U(f, P, \alpha) - L(f, P, \alpha) < \delta^2$. Let $M_i = \sup f$, $m_i = \inf f$ and $M_i' = \sup h$, $m_i' = \inf h$ for all $x_{i-1} \leq x \leq x_i$. Now divide $\{1, \ldots, n\}$ into two sets A and B such that $i \in A$ if $M_i - m_i < \delta$ and $i \in B$ if $M_i - m_i > \delta$ (then A and B are disjoint). Then for $i \in A$, we get $M_i' - m_i' \leq \varepsilon$, and for $i \in B$, we get $M_i' - m_i' \leq 2K$ with $K = \sup|\phi|$ for $m \leq t \leq M$.

$$\delta \sum_{i\ inB} \Delta\alpha_i \leq \sum_{i \in B} (M_i - m_i)\Delta\alpha_i < \delta^2$$

so $\sum \Delta\alpha_i < \delta$. Then

$$\begin{aligned} U(h, P, \alpha) - L(h, P, \alpha) &= \sum_{i \in A} (M_i' - m_i')\Delta\alpha_i + \sum_{i \in B} (M_i' - m_i')\Delta\alpha_i \\ &\leq \varepsilon(\alpha(b) - \alpha(a)) + 2K\delta \\ &< \varepsilon(\alpha(b) - \alpha(a) + 2K) \end{aligned}$$

Then for ε small enough, we get that h is integrable. ■

7.2 Properties of the Integral.

Theorem 7.2.1. *Let f and g be realvaulued functions integrable with respect to α over an interval $[a, b]$. Then:*

[label=(0)]

1. *$f + g$ and cf*

$$\int_a^b f + g d\alpha = \int_a^b f d\alpha + \int_a^b g d\alpha \tag{7.16}$$

2. *If $f \leq g$ on $[a,b]$, then:*

$$\int_a^b f d\alpha \leq \int_a^b g d\alpha \tag{7.17}$$

3. *If $c \in (a,b)$ and f then:*

$$\int_a^c f d\alpha + \int_c^b f d\alpha = \int_a^b f d\alpha \tag{7.18}$$

4. *If $|f| \leq M$ on $[a,b]$, then:*

$$|\int_a^c f d\alpha| \leq M(\alpha(b) - \alpha(a)) \tag{7.19}$$

5. *If f is intergrable with respect to α_1, and α_2 on $[a,b]$, and $c > 0$, then:*

$$\int_a^b f d(\alpha_1 + \alpha_2) = \int_a^b f d\alpha_1 + \int_a^b f d\alpha_2 \tag{7.20}$$

Proof. [label=(0)]

Let P be a partition of $[a,b]$, then $L(f,P,\alpha) + L(f,P,\alpha) \leq L(f+g,P,\alpha) \leq U(f+g,P,\alpha) \leq U(f,P,\alpha) + U(g,P,\alpha)$. Now let $\varepsilon > 0$, there are partitions P_1, P_2, with $P = P_1 \cup P_2$ such that $U(f,P_1,\alpha) - L(f,P_1,\alpha) < \frac{\varepsilon}{2}$, and $U(g,P_2,\alpha) - L(g,P_2,\alpha) < \frac{\varepsilon}{2}$. Then we get $U(f+g,P,\alpha) - L(f+g,P,\alpha) < \varepsilon$, so $f+g$ is intergrable. Moreover, we have that $U(f,P\alpha) < \int f + \frac{\varepsilon}{2}$ which guarantees $\int f + g \leq \int f + \int g$. Doing the same with L, we get $\int f + \int g \leq \int f + g$, a nd we are done.

2. If $f \leq g$, then $0 \leq g - f$, and we have by part (1) that $g - f$ is intergrable, and that $0 \leq \int g - f = \int g - \int f$, wich gives us the established inequality.

3. Let P_1 be a ppartition of $[a,c]$, and let P_2 be a partition of $[c,b]$, and let $P = P_1 \cup P_2$ be a partition of $[a,b]$. SInce f is intergrable on $[a,b]$, there is an $\varepsilon > 0$ such that $U(f,P,\alpha) - L(f,P,\alpha) < \varepsilon$, by theorem 7.1.3, we have that $L(f,P_1,\alpha) \leq L(f,P,\alpha) \leq U(f,P,\alpha) \leq U(f,P_1,\alpha)$ and $L(f,P_2,\alpha) \leq L(f,P,\alpha) \leq U(f,P,\alpha) \leq U(f,P_2,\alpha)$, thus we get $U(f,P_1,\alpha) - L(f,P_2,\alpha) < \varepsilon$ and $U(f,P_2,\alpha) - L(f,P,\alpha) < \varepsilon$, thus f is integrable on $[a,c]$ and on $[c,b]$.

   Now we have that $U(f,P_1,\alpha) < \int_a^c f + \frac{\varepsilon}{2}$ and $\int_c^b f + \frac{\varepsilon}{2}$, so $\int_a^b f \leq \int_a^c f \int_c^b f$. With L we get the reverse inequality, and we are done.

4. We have that $-M \leq f \leq M$, now let P be a partition of $[a,b]$, and let $\varepsilon > 0$. Since f is integrable, we have that

$$-M(\alpha(b) - \alpha(a)) \leq U(f,P,\alpha) - L(f,P,\alpha) < \varepsilon \leq M(\alpha(b) - \alpha(a)) \tag{7.21}$$

   which goves us for arbitrarily small ε, $|\int f| \leq M(\alpha(b) - \alpha(a))$.

5. Let P_1 and P_2 be partitions of $[a,b]$, respective to α_1 and α_2, and construct $P = P_1 \cup P_2$ respective to $\alpha_1 + \alpha_2$. We have that $U(f,P_1,\alpha_1) - L(f,P_1,\alpha_1) < \frac{\varepsilon}{2}$ and $U(f,P_2,\alpha_2) - L(f,P_2,\alpha_2) < \frac{\varepsilon}{2}$, then we have by the previous inequalities that:

$$U(f,P,\alpha_1+\alpha_2) - L(f,P,\alpha_1+\alpha_2) \leq (U(f,P_1,\alpha_1) + U(f,P_2,\alpha_2)) - (L(f,P_1,\alpha_1) + L(f,P_2,\alpha_2)) < \varepsilon$$

Thus, f is integrable with respect to $\alpha_1+\alpha_2$. Now again, by the similar reasoning used in parts (1) and (3), we get that $\int f\delta(\alpha_1+\alpha_2)=\int fd\alpha_1+\int fd\alpha_2$.

■

Corollary. *$\int cfd\alpha = c\int fd\alpha$ and $\int fd(c\alpha)=c\int fd\alpha$*

Theorem 7.2.2. *If f and g are integrable with respect to α on an interval $[a,b]$, then:*
[label=(0)]fg is integrable with respect to α on $[a,b]$. $|f|$ is integrable with respect to α on $[a,b]$ and:

$$|\int_a^b fd\alpha| \leq \int_a^b |f|d\alpha \tag{7.22}$$

2. *Proof.* [label=(0)]

   Take $\phi(t)=t^2$, then by theorem 7.1.10, $\phi\circ f=f^2$ is integrable; then notice that $4fg=(f+g)^2-(f-g)^2$ is integrable, hence, so is fg.

2. Take $\phi(t)=|t|$, then again by theorem 7.1.10, $\phi\circ f=|f|$ is integrable. Furthermore, choose $c=\pm1$ such that $0\leq c\int f$, since $cf\leq|f|$, we have that $|\int f|=c\int f=\int cf\leq \int |f|$.

■

Definition. Let ψ be a real-valued function on an interval $[a,b]$. We call ψ a **step function** provided there is a partition $P=\{a=x_0<\cdots<x_n=b\}$ of $[a,b]$, and real numbers $c_1,\ldots,c_n$ such that $\psi(x)=c_i$ if $x_i<x<x_{i+1}$ for all $0\leq i\leq n-1$. We define the **unit step function** $I:\mathbb{R}\to\{0,1\}$ to be $I(x)=1$ if $x>0$ and $I(x)=0$ if $x\leq 0$.

Theorem 7.2.3. *If $s\in(a,b)$, and f is bounded on $[a,b]$ and continuous at s, and if $\alpha(x)=I(x-s)$, then $\int f=f(s)$.*

Proof. Let $P=\{x_0,x_1,x_2,x_3\}$ be a partition of $[a,b]$, such that $a=x_0<s=x_1<x_2<x_3=b$. THen $U(f,P,\alpha)=M_2$ and $L(f,P,\alpha)=m_2$. Now since f is continuous, $M_2,m_2\to f(s)$ as $x_2\to s$, and we are done. ■

Theorem 7.2.4. *Suppose that α is monotonically increasing, and differentiable on an interval $[a,b]$, and suppose that α' is Riemann integrable on $[a,b]$. Let f be a bounded realvalued function on $[a,b]$, then f is integrable with respect to α on $[a,b]$ if and only if $f\alpha'$ is Riemann integrable.*

Proof. Let $\varepsilon>0$, and let P be a partition of $[a,b]$ such that $U(\alpha',P)-L(\alpha',P)<\varepsilon$. By the mean value theorem, there is a $t_i\in[x_{i-1},x_i]$ for which $\Delta\alpha_i=\alpha'(t_i)\Delta x_i$, for $1\leq i\leq n$. Now if $s_i\in[x_{i-1},x_i]$, then $\sum|\alpha'(s_i)-\alpha'(t_i)|\Delta x_i<\varepsilon$, and by theorem 7.1.6, with $M=\sup|f|$, since $\sum f(s_i)\Delta\alpha_i=\sum f(s_i)\alpha'(t_i)\Delta x_i$, we get

$$|\sum f(s_i)\Delta\alpha_i-\sum f(s_i)\alpha'(t_i)\Delta x_i|\leq M\varepsilon$$

in particular, $\sum f(s_i)\Delta x_i\leq U(f,\alpha',P)+M\varepsilon$, hence we get $U(f,P,\alpha)\leq U(f\alpha',P)+M\varepsilon$, hence we have:

$$|\overline{\int} fd\alpha-\overline{\int} f\alpha'dx|\leq M\varepsilon$$

for small enough ε, $\overline{\int} fd\alpha=\overline{\int} f\alpha'dx$. We get an analogoues result for lower sums. ■

Remark. Taking α to be a step function allows us to reduce the integral to be a finite or infinite series; similarly, if the derivative of α (if it exists) is integrable, then the Riemann-Stielhes integral reduces to just the Riemann integral.

Theorem 7.2.5 (Change of Variables)**.** *Of ϕ is a strictly increasing continuous function, mapping the interval $[A, B]$ onto the interval $[a, b]$, and if α is monotonically increasing on $[a, b]$, and if f is integrable with respect to α on $[a, b]$, define β and g on $[a, b]$ by $\beta = \alpha \circ \phi$ and $g = f \circ \phi$, then g is integrable with respect to β on $[A, B]$, and*

$$\int_A^B g d\beta = \int_a^b f d\alpha \tag{7.23}$$

Proof. For each partition P of $[a, b]$, construct a partitoon Q of $[A, B]$ such that $x_i = \phi(y_i)$. Then the values of f on $[x_{i-1}, x_i]$ are the same as the values of g on $[y_{i-1}, y_i]$. Then $U(g, Q, \beta) = U(f, P, \alpha)$ and $L(g, Q, \beta) = L(f, P, \alpha)$. It follows that since f is integrable, then $U(g, Q, \beta) - L(g, Q, \beta) < \varepsilon$ for $\varepsilon > 0$, thus g is integrable; and the equality is established by the equality of their sums. ■

Corollary. *Let $\alpha(x) = x$ and $\beta = \phi$, and suppose that ϕ' is Riemann integrable on $[A, B]$. Then:*

$$\int_a^b f dx = \int_A^B (f \circ \phi)\phi' dy \tag{7.24}$$

7.3 Integration and Differentiation.

Theorem 7.3.1. *Let f be Riemann integrable on an interval $[a, b]$, and for $a \leq x \leq b$, let $F(x) = \int_a^x f(x)dx$, then F is continuous on $[a, b]$. Firtheremore,, if f is continuous at $x_0 = [a, b]$, then F is differentiable at x_0 and $F'(x_0) = f(x_0)$.*

Proof. We have that f is bounded, so for some $a \leq t \leq b$, we have $|f| \leq M$. Now if $a \leq x < y \leq b$, then $|F(y) - F(x)| = |\int_x^y f(t)dt| \leq M(y - x)$.

Now given $\varepsilon > 0$, we see that $|F(y) - f(x)| < \varepsilon$, whenever $|y - x| < \frac{\varepsilon}{M}$. Thus f is continuous at x_0. Let $\varepsilon > 0$ and $\delta > 0$ such that $|f(t) - f(x_0)| < \varepsilon$ whenever $|t - x_0| < \delta$ whenever $a \leq t \leq b$. Thus if $x_0 - < s \leq x_0 \leq t < x_0 + \delta$, and $a \leq s < t \leq b$, then:

$$|\frac{F(s) - F(t)}{s - t}| = |\frac{\int_x^y f(u) - f(x_0) du}{s - t}| < \varepsilon$$

thus, we get $F'(x_0) = f(x_0)$. ■

Theorem 7.3.2 (The Fundamental Theorem of Calculus)**.** *If f is Riemann integrable on $[a, b]$, and if there is a differentiable function F on $[a, b]$ such that $F' = f$, then:*

$$\int_a^b f(x)dx = F(b) - F(a) \tag{7.25}$$

Proof. Let $\varepsilon > 0$, and construct a partition P of $[a, b]$ such that $U(f, P) - L(f, P) < \varepsilon$. Then by the mean value theorem, there is a $t_i \in [x_{i-1}, x_i]$ such that $F(x_i) - F(x_{i-1}) = f(t_i)\Delta x_i$, for $1 \leq i \leq n$. Taking sums, we have

$$\sum f(t_i)\Delta x_i = F(b) - F(a) \tag{7.26}$$

thus, by theorem 7.1.6, we have:

$$|\int_a^b f(x)dx - (F(b) - F(a))| < \varepsilon.$$

■

Theorem 7.3.3 (Integration by Parts)**.** *Let F and G be differentiable functions on $[a, b]$, and let $F' = f$ and $G' = g$ be Riemann integrable on $[a, b]$. Then:*

$$\int_a^b Fg(x)dx = FG(b) - FG(a) - \int_a^b fG(x)dx \tag{7.27}$$

Proof. By theorem 7.2.2, FG is Riemann integrable, and also notice that $(FG)' = Fg + fG$, thus by the fundamental theorem of calculus:

$$\int_a^b FGdx = \int_a^b Fgdx + \int_a^b fGdx = FG(b) - FG(a)$$

■

Remark. Integration by parts is what will allow us to find integrals of products of functions.

7.4 Integration of Vector Valued Functions.

We extend the concept of integrablility to general Euclidean spaces. Let $f_1, \ldots f_k$ be realvalued functions defined on an interval $[a, b]$, and let $f = (f_1, \ldots, f_k)$ be a vectorvalued function of $[a, b]$ onto $\mathbb{R}^k$. If α is a monotonically increasing function on $[a, b]$, we say that f is **integrable** with respect to α of $[a, b]$ if and only if f_i is integrable with respect to α for $1 \leq i \leq k$. If this integrable exists, we define it to be:

$$\int_a^b fd\alpha = (\int_a^b f_1 d\alpha, \ldots, \int_a^b f_k d\alpha). \tag{7.28}$$

The usual theorems for integrablility extend to $\mathbb{R}^k$.

Theorem 7.4.1 (The Fundamental Theorem of Calculus for Vector valued functions)**.** *If f : $[a, b] \to \mathbb{R}^k$ is a vector valued function that is Riemann integrable, and if F is a differentiable function such that $F' = f$, then:*

$$\int_a^b fdx = F(b) - f(a) \tag{7.29}$$

Proof. We apply the fundamental theorem of calculus to each of the components of f. ■

Theorem 7.4.2. *If $f : [a, b] \to \mathbb{R}^k$ is integrable with respect to α on an interval $[a, b]$, then $||f||$ is integrable, with respect to α on $[a, b]$, and:*

$$|| \int_a^b f d\alpha|| \leq \int_a^b ||f|| d\alpha$$

Proof. Let $f = (f_1, \ldots, f_k)$, then $||f|| = \sqrt{\sum f_i^2}$ for $1 \leq i \leq k$. Now f is integrable, hence so is $\sum f_i^2$, and furthermore, since $||f||$ is continuous on $[0, M]$ for $M > 0$, it is integrable by theorem 7.1.10.

Now ley $y = (y_1, \ldots, y_k)$, with $y_i = \int_a^b f_i d\alpha$ for $1 \leq i \leq k$. Then $y = \int_a^b f d\alpha$, and

$$||y||^2 = \sum y_i^2 = \sum y_i \int f_i = \int \sum y_i f_i d\alpha$$

hence, by the Cauchy-Shwarz inequality, $\sum y_i f_i \leq ||y|| ||f_i||$ for $a \leq t \leq b$, hence $||y||^2 \leq ||y|| \int ||f|| d\alpha$. ■

7.5 Rectifiable Curves.

Definition. We call a continuous mapping $\gamma : [a, b] \to \mathbb{R}^k$ a **curve** in $\mathbb{R}^k$, on an interval $[a, b]$. If γ is 1-1, we call the curve an **arc**, and we call γ a **closed curve** if $\gamma(a) = \gamma(b)$.

Now, for each partition P of $[a, b]$, and for γ a curve on $[a, b]$, and let:

$$\Lambda(\gamma, P) = \sum_{i=1}^{n} ||\gamma(x_i) - \gamma(x_{i-1})|| \tag{7.30}$$

We deefine the **length** of γ to be :

$$\Lambda(\gamma) = \sup_P \Lambda(\gamma, P) \tag{7.31}$$

If γ is of finite length (i.e. $\Lambda(\gamma) < \infty$), then we call γ a **rectifiable** curve.

Theorem 7.5.1. *Let γ be a differentiable curve on $[a, b]$, if γ' is continuous, then γ is rectifiable and*

$$\Lambda(\gamma) = \int_a^b ||\gamma'|| dt \tag{7.32}$$

Proof. If $a \leq x_{i-1} < x_i \leq b$, then $||\gamma(x_i) - \gamma(x_{i-1})|| = || \int_{x_{i-1}}^{x_i} \gamma' dt|| \leq \int_{x_{i-1}}^{x_i} ||\gamma'|| dt$, by the Fundamental theorem of calculus, thus $\Lambda(\gamma, P) \leq \int_a^b ||\gamma'|| dt$ for every partition P of $[a, b]$, then cinsequently, we have $\Lambda(\gamma) = \int ||\gamma'|| dt$. Consequently, γ is rectifiable.

Now let $\varepsilon > 0$, since γ' is uniformly continuous on $[a, b]$, there is a $\delta > 0$ such that $||\gamma'(s) - \gamma'(t)|| < \varepsilon$ whenever $|s - y| < \delta$ for $s, t \in [a, b]$. Let P be a partition of $[a, b]$, with

$\Delta x_i < \delta$ for all i, and it $x_{i-1} \leq t \leq x_i$, we have $||\gamma'(t)|| \leq ||\gamma'(x_i)|| + \varepsilon$, hence:

$$\begin{aligned}
\int_{x_{i-1}}^{x_i} ||\gamma'||\Delta x_i &\leq ||\gamma'(x_i)||\Delta x_i + \varepsilon \Delta x_i \\
&= || \int_{x_{i-1}}^{x_i} (\gamma'(t) + \gamma'(x_i) - \gamma'(t))|| + \varepsilon \Delta x_i \\
&\leq || \int_{x_{i-1}}^{x_i} \gamma' dt|| + || \int_{x_{i-1}}^{x_i} (\gamma'(x_i) - \gamma'(t))dt|| + \varepsilon \Delta x_i \\
&\leq ||\gamma(x_i) - \gamma(x_{i-1})|| + 2\varepsilon \Delta x_i
\end{aligned}$$

Thus, $\int_a^b ||\gamma'||dt \leq \Lambda(\gamma, P) + 2\varepsilon(b-a)$, so for ε small enough, we get that $\int ||\gamma'|| \leq \Lambda(\gamma)$. ■

Chapter 8

Lebesgue Theory

8.1 σ-algebras

Definition. Let X be a nonempty set. An **algebra** of sets on X is a nonempty collection $\mathcal{A}$ of subsets of X which are closed under finite unions and complements in X. We call $\mathcal{A}$ a σ**-algebra** if it is closed under countable unions.

Lemma 8.1.1. *Let X be a set and $\mathcal{A}$ an algbra on X. Then $\mathcal{A}$ is closed under finite intersections.*

Proof. Let $\{E_\lambda\}$ be a collection of sets of $\mathcal{A}$. Then by finite union $E = \bigcup E_\lambda \in \mathcal{A}$. Then by complements, $X \backslash E = \bigcap X \backslash E_\lambda \in \mathcal{A}$. ■

Corollary. *σ-algebras are closed under countable unions.*

Lemma 8.1.2. *Let X be a set, and $\mathcal{A}$ an algebra on X. Then $\emptyset \in \mathcal{A}$ and $X \in \mathcal{A}$.*

Proof. By closure of finite unions, notice that if $E \in \mathcal{A}$, then $E \cup X \backslash E = X \in \mathcal{A}$ lemma 8.1.1 gives us that $E \cap X \backslash E = \emptyset \in \mathcal{A}$. ■

Example 8.1. (1) The collections $\{\emptyset, X\}$ and 2^X are σ-algebras on any set X.

(2) Let X be an uncountable set. Then the collection

$$\mathcal{C} = \{E \subseteq X : E \text{ is countable or } X \backslash E \text{ is countable}\}$$

defines a σ-algebra of sets on X, since countable unions of countable sets are countable, and $\mathcal{C}$ is closed under complements. We call $\mathcal{C}$ the σ**-algebra of countable or cocountable sets**.

Lemma 8.1.3. *Let $\{A_\lambda\}$ be a collection of σ-algebras on a set X. Then the intersection*

$$\mathcal{A} = \bigcap \mathcal{A}_\lambda$$

is a σ-algebra on X. Moreover, if $F \subseteq X$, then there exists a unique smallest σ-algebra containing F; in particular, it is the intersection of all σ-algebras containing F.

Proof. Notice that since each $\mathcal{A}_\lambda$ is a σ-algebra, they are closed under countable unions and complements. Hence by definition, $\mathcal{A}$ must also be closed under countable unions and complements.

Now, let $F \subseteq X$ and let $\{\mathcal{A}_\lambda\}$ be the collection of all σ-algebras containing F. Then the intersection $\mathcal{A} = \bigcap \mathcal{A}_\lambda$ is also a σ-algebra containing F; by above. Now, suppose that there is a smallest σ-algebra $\mathcal{B}$ containing F. Then we have that $\mathcal{B} \subseteq \mathcal{A}$. Now, by definition of $\mathcal{A}$ as the intersection of all σ-algberas containing F, we get that $\mathcal{A} \subseteq \mathcal{B}$; so that $\mathcal{B} = \mathcal{A}$. ■

Definition. Let X be a nonempty set and $F \subseteq X$. We define the σ-algebra **generated** by F to be the smallest such σ-algebra $\mathcal{M}(F)$ containing F.

Lemma 8.1.4. *Let X be a set and let $E, F \subseteq X$. Then if $E \subseteq \mathcal{M}(F)$, then $\mathcal{M}(E) \subseteq \mathcal{M}(F)$.*

Proof. We have that since $E \subseteq \mathcal{M}(F)$, and $\mathcal{M}(E)$ is the intersection of all σ-algebras containing E, then $\mathcal{M}(E) \subseteq \mathcal{M}(F)$. ■

Definition. Let X be a topological space. We define the **Borel σ-algebra** on X to be the σ-algebra $\mathcal{B}(X)$ generated by all open sets of X; that is

$$\mathcal{B}(X) = \mathcal{M}(\mathcal{T})$$

where $\mathcal{T}$ is the topology on X. We call the elements of $\mathcal{B}(X)$ **Borel-sets**

Definition. Let X be a topological space. We call a countable intersection of open sets of X a G_δ**-set** of X. We call a countable union of closed sets of X an F_σ**-set** of X.

Theorem 8.1.5. *The Borel σ-algebra on $\mathbb{R}$, $\mathcal{B}(\mathbb{R})$, is generated by the following.*

(1) All open intervals of $\mathbb{R}$.

(2) All closed intervals of $\mathbb{R}$.

(3) All half-open intervals of $\mathbb{R}$.

(4) All open rays of $\mathbb{R}$.

(5) All closed rays of $\mathbb{R}$.

8.2 The Lebesgue Outer Measure

Definition. Let $I \subseteq \mathbb{R}$ be an interval. We define the **length** $l(I)$, of I to be ∞ if I is unbounded, and the difference of its endpoints otherwise.

Definition. Let $\{I_k\}$ be a countable collection of open bounded intervals covering a set $A \subseteq \mathbb{R}$. We define the **Lebesgue outer measure** of A to be

$$m^*(A) = \inf\left\{ \sum_{k=1}^{\infty} l(I_k) : A \subseteq \bigcup_{k=1}^{\infty} I_k \right\}$$

We now go over some basic properties of the Lebesgue outer measure.

Lemma 8.2.1. *For any set $A \subseteq \mathbb{R}$, $m^*(A) \geq 0$; in particular, $m^*(\emptyset) = 0$.*

Proof. By definition, since $l(I_k) \geq 0$, each $\sum l(I_k) \leq 0$. This makes $m^*(A) \geq 0$ for any $\{I_k\}$ a countable cover of A by bounded open intervals.

Notice that $\emptyset \subseteq (-\varepsilon, \varepsilon)$ for some $\varepsilon > 0$, and that this covers $\emptyset$, so that $m^*(\emptyset) = l((-\varepsilon, \varepsilon)) = 2\varepsilon$. Then choosing ε small enough gives us $m^*(\emptyset) = 0$. ■

Lemma 8.2.2 (Monotonicity)**.** *The Lebesgue Outer Measure is monoton; that is, if $A \subseteq B$, then*

$$m^*(A) \leq m^*(B)$$

Proof. Let $\{I_k\}$ be a countable cover of B by bounded open intervals. Then notice that $\{I_k\}$ covers A as well. Now, let

$$E = \left\{ \sum_{k=1}^{\infty} l(I_k) : A \subseteq \bigcup_{k=1}^{\infty} I_k \right\}$$
$$F = \left\{ \sum_{k=1}^{\infty} l(I_k) : B \subseteq \bigcup_{k=1}^{\infty} I_k \right\}$$

Then since $A \subseteq B$, $F \subseteq E$. Therefore, we get that

$$\inf E = m^*(A) \leq m^*(B) = \inf F$$

■

Lemma 8.2.3. *Countable sets have Lebesgue outer measure* 0*.*

Proof. Let C be a countable set, and $C = \{c_k\}_{k \in \mathbb{Z}^+}$ an enumeration for C. Let $\varepsilon > 0$, then for every $k \in \mathbb{Z}^+$, define the interval

$$I_k = (c_k - \frac{\varepsilon}{2^{k+1}}, c_k + \frac{\varepsilon}{2^{k+1}})$$

Then $\{I_k\}$ is a countable cover of C by bounded open intervals. Thus we get

$$0 \leq m^*(C) \leq \sum_{k=1}^{\infty} l(I_k) = \sum \frac{\varepsilon}{2^k} = \varepsilon$$

Therefore taking ε small enough, we get $0 \leq m^*(C) \leq 0$, and equality is established. ■

Lemma 8.2.4. *Let $I \subseteq \mathbb{R}$ be an interval. Then $m^*(I) = l(I)$.*

Proof. Consider first that $I = [a, b]$, a closed bounded interval, where $a < b$. Let $\varepsilon > 0$, then we have that $[a, b] \subseteq (a - \varepsilon, b + \varepsilon)$, so that by monotonicity

$$m^*([a, b]) \leq l((a - \varepsilon, b + \varepsilon)) = b - a + 2\varepsilon$$

Then for ε small enough, this shows that

$$m^*([a,b]) \leq b-a$$

Now, let $\{I_k\}$ be a countable cover of $[a,b]$ by bounded open intervals. Then since $[a,b]$ is compact (by Heine-Borel), there is a finite subcover $\{I_k\}_{k=1}^n$ of $[a,b]$. Now, since $a \in \bigcup_{k=1}^n I_k$, there is atleast one interval I_k containing a; denote it (a_1,b_1). Now, $a_1 < a < b_1$. If $b_1 \geq b$, then we are done as

$$\sum_{k=1}^n l(I_k) \geq b_1 - a_1 > b-a$$

Otherwise, $b_1 \in [a,b)$, and since $b_1 \notin (a_1,b_1)$, there is an interval (a_2,b_2), distinct from (a_1,b_1), containing b_1. Now, if $b_2 \geq b$, we are done. Otherwise, proceeding recursively, we obtain a subcollection $\{(a_k,b_k)\}_{k=1}^N$ of $\{I_k\}_{k=1}^n$ for which

$$a_1 < a \text{ and } a_{k+1} < b_k \text{ for all } 1 \leq k \leq N-1$$

The process of selecting such a subcollection must terminate, which leaves us with $b_N > b$, so that

$$\sum_{k=1}^n l(I_k) \geq \sum_{k=1}^N l((a_k,b_k)) = (b_N - a_N) + \cdots + (b_1 - a_1) \geq b_N - a_1 > b-a$$

INdeed, we get

$$\sum_{k=1}^n l(I_k) \geq b-a$$

so that $m^*([a,b]) = b-a$.

Now, suppose that I is any bounded interval. Then for $\varepsilon > 0$, there exist closed bounded interval J_1 and J_2 such that

$$J_1 \subseteq I \subseteq J_2$$

and

$$l(I) - \varepsilon < l(J_1) \leq l(I) \leq l(J_2) < l(I) + \varepsilon$$

By the monotonicity, and the above discussion of closed bounded intervals, we get

$$l(I) - \varepsilon < m^*(J_1) \leq m^*(I) \leq m^(J_2) < l(I) + \varepsilon$$

Therefore, for ε small enough, we get that $m^*(I) = l(I)$.

Finally, suppose that I is an unbounded interval. Then for every $n \in \mathbb{Z}^+$, there is an interval J with $l(J) = n$. So that

$$n = m^*(J) \leq m^*(I)$$

This makes $m^*(I) = l(I) = \infty$, by definition of $l(I)$. ■

Lemma 8.2.5 (Translation Invariance). *The Lebesgue outer measure is translation invariant; that is, if $A \subseteq \mathbb{R}$, and $y \in \mathbb{R}$,*

$$m^*(A+y) = m^*(A)$$

Proof. Let $\{I_k\}$ be a countable cover of A by open bounded intervals. Then the collection $\{I_k + y\}$ is a countable cover of the set $A + y$ by open bounded intervals. Moreover, notice that $l(I_k) = l(I_k + y)$. This gives us

$$\sum l(I_k) = \sum l(I_k + y)$$

and we are done. ■

Lemma 8.2.6 (Countable Subadditivity). *The Lebesgue outer measure is countable subadditive; that is, if $\{E_k\}$ is a countable collection of subsets of $\mathbb{R}$, then*

$$m^*\Big(\bigcup E_k\Big) \leq \sum m^*(E_k)$$

Proof. Let $\{E_k\}$ be a countable collection of subsets of $\mathbb{R}$, and let

$$E = E_k$$

If at least one of the E_k has $m^*(E_k) = \infty$, then we are done. Suppose then that $m^*(E_k)$ is finite for all $k \in \mathbb{Z}^+$. Then for each E_k, there is a countable cover $\{I_{k,i}\}_{i\in\mathbb{Z}^+}$ by bounded open intervals for which

$$\bigcup l(I_{k,i}) < m^*(E_k) + \frac{\varepsilon}{2^k}$$

Now, consider the countable collection

$$\{I_{k,i}\}_{k,i\in\mathbb{Z}^+} = \bigcup_{k\in\mathbb{Z}^+} \{I_{k,i}\}_{i\in\mathbb{Z}^+}$$

Then $\{I_{k,i}\}_{k,i\in\mathbb{Z}^+}$ is a countable cover of E by bounded open intervals. We get

$$m^*(E) \leq \sum_{k\in\mathbb{Z}^+} \Big(\sum_{i\in\mathbb{Z}^+} l(I_{k,i})\Big) < \sum_{k\in\mathbb{Z}^+} (m^*(E_k) + \frac{\varepsilon}{2^k}) = \Big(\sum m^*(E_k)\Big) + \varepsilon$$

Taking $\varepsilon > 0$ small enough, gives us the required subadditivity. ■

Corollary. *The Lebesgue outer measure is finitely subadditive.*

Proof. Recall that finite collections of sets are countable. ■

8.3 Lebesgue Measurable Sets

Definition. A set E of $\mathbb{R}$ is said to be **measurable** if for any set $A \subseteq \mathbb{R}$, we have

$$m^*(A) = m^*(A \cap E) + m^*(A \cap (\mathbb{R}\backslash E))$$

Lemma 8.3.1. *A set E is measurable if for any $A \subseteq \mathbb{R}$, we have*

$$m^*(A) \geq m^*(A \cap E) + m^*(A \cap (\mathbb{R}\backslash E))$$

Proof. This follows by finite subadditivity, notice that both $A \cap E \subseteq A$ and $A \cap (\mathbb{R} \backslash E) \subseteq A$, and that $A = (A \cap E) \cup (A \cap (\mathbb{R} \backslash E))$. Thus

$$m^*(A) \leq m^*(A \cap E) + m^*(A \cap (\mathbb{R} \backslash E))$$

■

Corollary. *A set E is measurable if, and only if $\mathbb{R} \backslash E$ is measurable.*

Lemma 8.3.2. *Any set of Lebesgue outer measure 0 is measurable.*

Proof. Let E be a set with $m^*(E) = 0$, and let $A \subseteq \mathbb{R}$. Then we have $A \cap E \subseteq E$ and $A \cap (\mathbb{R} \backslash E) \subseteq A$, so that by monotonicity

$$m^*(A \cap E) \leq m^*(E) = 0 \text{ and } m^*(A \cap (\mathbb{R} \backslash E)) \leq m^*(A)$$

Adding both reltions gives us the result. ■

Corollary. *Countable sets are measurable.*

Lemma 8.3.3. *The union of a finite collection of measurable sets is measurable.*

Proof. Let E_1 and E_2 be measurable and $A \subseteq \mathbb{R}$. Then

$$\begin{aligned} m^*(A) &= m^*(A \cap E_1) + m^*(A \cap (\mathbb{R} \backslash E_1)) \\ &= m^*(A \cap E_1) + m^*((A \cap E_1) \cap E_2) + m^*((A \cap (\mathbb{R} \backslash E_1)) \cap (\mathbb{R} \backslash E_2)) \end{aligned}$$

Now, we have that

$(A \cap (\mathbb{R} \backslash E_1)) \cap (\mathbb{R} \backslash E_2) = A \cap \mathbb{R} \backslash (E_1 \cup E_2)$ and $(A \cap E_1) \cup (A \cap (\mathbb{R} \backslash E_1) \cap E_2) = A \cap (E_1 \cup E_2)$

which gives us

$$m^*(A) \geq m^*(A \cap (E_1 \cup E_2)) + m^*(A \cap \mathbb{R} \backslash (E_1 \cup E_2))$$

which makes the union $E_1 \cup E_2$ measurable. Extending the argument for arbitrary $n \in \mathbb{Z}^+$ by induction gives us the result. ■

Lemma 8.3.4. *Let A be a set and $\{E_k\}_{k=1}^n$ a disjoint collection of measurable sets. Then*

$$m^*\Big(A \cap \Big(\bigcup_{k=1}^n E_k\Big)\Big) = \sum_{k=1}^n m^*(A \cap E_k)$$

Proof. By induction on n, for $n = 1$, we are done. Now, suppose that our assertion holds for all $n \geq 1$, and consider the finite disjoint collection $\{E_k\}_{k=1}^{n+1}$. The since this collection is disjoint, we have

$$A \cap \Big(\bigcup_{k=1}^{n+1} E_k\Big) \cap E_{n+1} = A \cap E_{n+1}$$

and

$$A \cap \Big(\bigcup_{k=1}^{n+1} E_k\Big) \cap (\mathbb{R} \backslash E_{n+1}) = A \cap \Big(\bigcup_{k=1}^{n} E_k\Big)$$

By the measurability of E_{n+1},, we get that

$$m^*\Big(A \cap \Big(\bigcup_{k=1}^{n+1} E_k\Big)\Big) = m^*(A \cap E_{n+1}) + \sum_{k=1}^{n} m^*(A \cap E_k) = \sum_{k=1}^{n+1} m^*(A \cap E_k)$$

■

Lemma 8.3.5. *Countable unions of measurable sets are measurable.*

Proof. Let $\{E_k\}$ be a countable collection of measurable sets, and $E = \bigcup E_k$ the countable union; suppose further without loss of generality that the collection $\{E_k\}$ is a disjoint collection. Let $A \subseteq \mathbb{R}$, and $n \in \mathbb{Z}^+$ and define

$$F_n = \bigcup_{k=1}^{n} E_k$$

Then F is measurable since it is a finite union of measurable sets and $\mathbb{R} \backslash E \subseteq \mathbb{R} \backslash F$, moreover, we have

$$m^*(A) = m^*(A \cap E_n) + m^*(A \cap (\mathbb{R} \backslash F)) \geq m^*(A \cap F_n) + m^*(A \cap (\mathbb{R} \backslash E))$$

Now, by the lemma 8.3.4, we have that

$$m^*(A \cap F) = \sum_{k=1}^{n} m^*(A \cap E_k)$$

so that

$$m^*(A) \geq \sum_{k=1}^{n} m^*(A \cap E_k) + m^*(A \cap (\mathbb{R} \backslash E))$$

Now, choosing $n \in \mathbb{Z}^+$ large enough, we get

$$m^*(A) \geq \sum m^*(A \cap E_k) + m^*(A \cap (\mathbb{R} \backslash E))$$

so that $m^*(A) \geq m^*(A \cap E) + m^*(A \cap (\mathbb{R} \backslash E))$ ■

Theorem 8.3.6. *The measurable sets of* $\mathbb{R}$ *form a* σ*-algebra on* $\mathbb{R}$.

Lemma 8.3.7. *Every interval is measurable.*

Proof. It suffices to show that the intervals of the form (a, ∞) are measurable. Let $A \subseteq \mathbb{R}$, and suppose that $a \notin A$. Now, let $\{I_k\}$ be a couintable collection of open bounded intervals covering A. Now, define

$$A' = A \cap (-\infty, a) \text{ and } A'' = A \cap (a, \infty)$$

and

$$I_k' = I_k \cap (-\infty, a) \text{ and } I_k'' = I_k \cap (a, \infty)$$

Then the I_k' and I_k'' are intervals with $l(I_k) = l(I_k') + l(I_k'')$. Moreover, the countable collections $\{I_k'\}$ and $\{I_k''\}$ form open covers of A' and A'' respectively. Therefore, by the definition of outer measure, we get

$$m^*(A') \leq \sum l(I_k') \text{ and } m^*(A'') \leq \sum l(I_k'')$$

so that

$$m^*(A') + m^*(A'') \leq \sum l(I_k)$$

which makes (a, ∞) measurable. ■

Corollary. *Every Borel set of $\mathbb{R}$ is measurable.*

Corollary. *G_δ sets and F_σ sets are measurable.*

Lemma 8.3.8. *Translates of measurable sets are measurable.*

Proof. Let E be measurable, and $y \in \mathbb{R}$, and let $A \subseteq \mathbb{R}$. We have by the measurability of E, and by translation invariance of outer measure that

$$\begin{aligned} m^*(A) &= m^*(A \backslash y) \\ &= m^*((A \backslash y) \cap E) + m^*((A \backslash y) \cap (\mathbb{R} \backslash E)) \\ &= m^*(A \cap (E + y)) + m^*(A \cap \mathbb{R} \backslash (E + y)) \end{aligned}$$

■

Lemma 8.3.9 (The Excision Property)**.** *If A is a measurable set of finite outer measure contained in a set B of $\mathbb{R}$, then*

$$m^*(B \backslash A) = m^*(B) - m^*(A)$$

Proof. We have that

$$m^*(B) = m^*(B \cap A) + m^*(B \cap (\mathbb{R} \backslash A)) = m^*(A) + m^*(B \backslash A)$$

■

8.4 Outer and Inner Approximations

Theorem 8.4.1 (The Outer Approximation Theorem)**.** *LEt $E \subseteq \mathbb{R}$. Then the following are true.*

(1) E is measurable.

(2) For every $\varepsilon > 0$, there is an open set U, containing E for which $m^*(U \backslash E) < \varepsilon$

(3) There exists a G_δ set G, containing E for which $m^*(G \backslash E) = 0$.

Proof. Suppose that E is measurable, and let $\varepsilon > 0$. IF $m^*(E)$ is finite, then by definition of outer measure, there exists a countable collection of open bounded intervals $\{I_k\}$ covering E, for which

$$\sum l(I_k) < m^*(E) + \varepsilon$$

Define now, $U = \bigcup I_k$, then U is open in $\mathbb{R}$, and $E \subseteq \mathbb{R}$. Again, by the definition of outer measure, we have

$$m^*(U) \leq \sum l(I_k) < m^*(E) + \varepsilon$$

so that

$$m^*(U) - m^*(E) < \varepsilon$$

Now, since E is measurable, and $E \subseteq U$, by the excision property we get $m^*(U \backslash E) < \varepsilon$.

Now, if $m^*(E)$ is infinite, then there exists a disjoint countable collection $\{E_k\}$ of measurable sets, each of finite outer measure, for which $E = \bigcup E_k$. Then for any $k \in \mathbb{Z}^+$, there exists an open set U_k with $E_k \subseteq U_k$ for which

$$m^*(U_k \backslash E_k) < \frac{\varepsilon}{2^k}$$

Now, let $U = \bigcup U_k$, so that U is open and $E \subseteq U$. Moreover, we get $U \backslash E \subseteq \bigcup U_k \backslash E_k$; therefore, by monotonicity, we get

$$m^*(U \backslash E) \leq \sum m^*(U_k \backslash E_k) < \sum \frac{\varepsilon}{2^k} = \varepsilon$$

Now, suppose that the second assertion holds for E. Then for any $k \in \mathbb{Z}^+$, choose an open set U_k containing E for which

$$m^*(U_k \backslash E) < \frac{1}{k}$$

Define $G = \bigcap U_k$. Then G is a G_δ set and $E \subseteq G$. Moreover, for each k, we have $G \backslash E \subseteq U_k \backslash E$ so by monotonicity

$$m^*(G \backslash E) \leq m^*(U_k \backslash E) < \frac{1}{k}$$

Then as $k \to \infty$, $m^*(G \backslash E) \to 0$.

Finally, suppose that the third assertion holds for E. Now, since sets of measure 0 are measurable, and G_δ are measurable, we get

$$E = G \cap \mathbb{R} \backslash (G \backslash E)$$

is measurable, by theorem 8.3.6. ■

Theorem 8.4.2 (The Inner Approximation Theorem)**.** *LEt $E \subseteq \mathbb{R}$. Then the following are true.*

(1) E is measurable.

(2) For every $\varepsilon > 0$, there is an closed set F, contained in E for which $m^*(E \backslash F) < \varepsilon$

(3) There exists a F_σ set F, contained in E for which $m^*(E \backslash F) = 0$.

Proof. This follows by DeMorgan's laws on the outer approximation theorem. ∎

Theorem 8.4.3. *LEt E be measurable of finite outer measure. Then for every $\varepsilon > 0$, there exists a finite disjoint collection of open intervals $\{I_k\}_{k=1}^n$ for which if $U = \bigcup_{k=1}^n I_k$, then*

$$m^*(E\backslash U) + m^*(U\backslash E) < \varepsilon$$

Proof. Since E is measurable, there exists an open set V, containing E and for which

$$m^*(V\backslash E) < \frac{\varepsilon}{2}$$

Then by the excision property, we observe that since E has finite outer measure, then V must also have finite outer measure. Now, let $V = \bigcup I_k$ where $\{I_k\}$ is a countable disjoint collection of open intervals. Therefore, by lemma 8.3.3, and monotonicity we get

$$\sum_{k=1}^n l(I_k) = m^*\left(\bigcup_{k=1}^n I_k\right) \leq m^*(V)$$

Then as $n \to \infty$, we see the sum $\sum l(I_k)$ is finite. Then choosing $n \in \mathbb{Z}^+$ for which

$$\sum_{k=n+1}^{\infty} l(I_k) < \frac{\varepsilon}{2}$$

and defining

$$U = \bigcup_{k=1}^n I_k$$

we get $U\backslash E \subseteq V\backslash E$ so by monotonicity,

$$m^*(U\backslash E) \leq m^*(V\backslash E) < \frac{\varepsilon}{2}$$

on the other hand, since $E \subseteq V$, we get $E\backslash U \subseteq V\backslash U = \bigcup_{k=n+1}^{\infty} I_k$. Therefore by definition of outer measure

$$m^*(E\backslash U) \leq \sum_{k=n+1}^{\infty} l(I_k) < \frac{\varepsilon}{2}$$

This gives us

$$m^*(U\backslash E) + m^*(E\backslash U) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

∎

8.5 The Borel-Cantelli Lemma

Definition. We define the **Lebesgue measure** m to be the restriction of the Lebesgue outer measure to the σ-algebra of measurable sets of $\mathbb{R}$. That is, if $E \subseteq \mathbb{R}$ is measurable, then

$$m(E) = m^*(E)$$

Lemma 8.5.1. *The Lebesgue measure is countable additive. That is, if $\{E_k\}$ is a countable disjoint collection of measurable sets, then*

$$m\Big(\bigcup E_k\Big) = \sum m(E_k)$$

Proof. Let $\{E_k\}$ be a countable disjount collection of measurable sets, and let $E = \bigcup E_k$. Then E is also measurable, and we hve by countable subadditivity

$$m(E) \leq \sum m(E_k)$$

Now, by lemma 8.3.4, for each $n \in \mathbb{Z}^+$, we have

$$m\Big(\bigcup_{k=1}^{n} E_k\Big) = \sum_{k=1}^{n} E_k$$

and that

$$\bigcup_{k=1}^{n} E_k \subseteq E$$

Thus, by monotonicity, we have

$$\sum_{k=1}^{n} m(E_k) \leq m(E)$$

Now, since n is arbitary, choosing n large enough, we get

$$\sum m(E_k) \leq m(E)$$

and so equality is established. ∎

Theorem 8.5.2 (Continuity of Measure)**.** *The following is true for the Lebesgue measure.*

(1) IF $\{A_k\}$ is an ascending collection of measurable sets, then

$$m\Big(\bigcup A_k\Big) = \lim_{k \longrightarrow \infty} m(A_k)$$

(2) If $\{B_k\}$ is a descending collection of measurable sets, and $m(B_1)$ is finite, then

$$m\Big(\bigcap B_k\Big) = \lim_{k \longrightarrow \infty} m(B_k)$$

Proof. Let $A = \bigcup A_k$. First, suppose that there is a $k_0 \in \mathbb{Z}^+$ for which $m(A_{k_0}) = \infty$. THen $m(A) = \infty$, and so $m(A_k) = \infty$ for all $k \geq k_0$, and we are done. Now, suppose that $m(A_k)$ is finite for all $k \in \mathbb{Z}^+$. Define

$$A_0 = \emptyset \text{ and } C_{k+1} = A_{k+1} \backslash A_k \text{ for all } k \geq 1$$

Since $\{A_k\}$ is an ascending collection, the collection $\{C_k\}$ is disjoint, and we have

$$A = \bigcup A_k = \bigcup C_k$$

Then, by the countable additivity of the Lebesgue measure, we get

$$m(A) = \sum m(A_{k+1} \backslash A_k)$$

and by excision, we get

$$\begin{aligned}
\sum m(A_{k+1} \backslash A_k) &= \sum (m(A_{k+1}) - m(A_k)) \\
&= \lim_{n \longrightarrow \infty} \sum_{k=0}^{n-1} (m(A_{k+1}) - m(A_k)) \\
&= \lim_{n \longrightarrow \infty} (m(A_n) - m(A_0))
\end{aligned}$$

Now, since $m(A_0) = m(\emptyset) = 0$, we get that

$$m(A) = \lim_{k \longrightarrow 0} m(A_k)$$

Now, let $B = \bigcap B_k$, and suppose that $m(B_1)$ is finite. Define

$$D_k = B_1 \backslash B_k \text{ for all } k \in \mathbb{Z}^+$$

Since $\{B_k\}$ is a descending collection, $\{D_k\}$ is an ascending collection. By the above assertion, we get

$$m\Big(\bigcup D_k\Big) = \lim_{k \longrightarrow \infty} m(D_k)$$

Now, by DeMorgan's laws, we have

$$\bigcup D_k = B_1 \backslash \Big(\bigcap B_k\Big)$$

so that by the excision property

$$m\Big(B_1 \backslash \Big(\bigcap B_k\Big)\Big) = \lim_{k \longrightarrow \infty} (m(B_1) - m(B_k))$$

by the excision property on the left hand side of the equation, we get

$$m(B_1) - m(B) = m(B_1) - \lim_{k \longrightarrow \infty} m(B_k)$$

which gives us the result. ■

Definition. We say that a property on a Lebesgue measurable set E of $\mathbb{R}$ holds **almost everywhere** provided there exists a subsete $E_0 \subseteq E$ of $m(E_0) = 0$ for which the property holds for all $x \in E \backslash E_0$.

Lemma 8.5.3 (The Borel-Cantelli Lemma)**.** *Let $\{E_k\}$ be a countable collection of measurable sets for which $\sum m(E)_k$ is finite. Then almost all $x \in \mathbb{R}$ belong to at most finitely many E_k.*

Proof. Let $E = \bigcup_{k=n}^{\infty} E_k$, then E is measurable. By subadditivity

$$m(E) \leq \sum_{k=n}^{\infty} m(E_k) < \infty$$

Therefore, by the continuity of measure, we have

$$m\Big(\bigcap_{n=1}^{\infty} E\Big) = \lim_{n \longrightarrow \infty} m(E) \leq \lim_{n \longrightarrow \infty} \sum_{k=1}^{n} m(E_k) = 0$$

Therefore, almost all $x \in \mathbb{R}$ fail to belong to $\bigcap_{n=1}^{\infty} E$, and so must belong to at most finitely many E_k. ■

8.6 Nonmeasurable Sets

Definition. We call a set E of $\mathbb{R}$ **nonmeasurable** if it is not a measurable set.

Lemma 8.6.1. *Let E be a bnounded measurable set, and suppose there exists a bounded countably infinite set of reals Λ for which the collection of translates $\{\lambda + E\}$ is disjoint. Then $m(E) = 0$.*

Proof. SInce translates of measurable sets are measurable, by countable additivity

$$m\Big(\bigcup_{\lambda \in \Lambda} (\lambda + E)\Big) = \sum_{\lambda \in \Lambda} m(\lambda + E) = \sum_{\lambda \in \Lambda} m(E)$$

Now, since both E and Λ are bounded, then the union

$$\bigcup_{\lambda \in \Lambda} (\lambda + E)$$

is also bounded, and has finite measure. Thus, for each $l \in \Lambda$, $m(\lambda + E) = m(E)$ is finite. Additionally, by translation invariance, $m(\lambda + E) = m(E) > 0$, and since Λ is countably infinite, we must have $m(E) = 0$. ■

Definition. LEt $E \subseteq \mathbb{R}$ be an arbitrary set. We call two points $x, y \in E$ **rationally equivalent** in E provided $x - y \in \mathbb{Q}$. We write $x \sim y$.

Lemma 8.6.2. *For any set $E \subseteq \mathbb{R}$, rational equivalence defines an equivalence relation on E.*

Proof. We have that for any $x \in E$, $x - x_{=}0 \in \mathbb{Q}$, so that $x \sim x$. Now, let $x, y \in E$ with $x \sim y$. Then $x - y \in \mathbb{Q}$, and since $\mathbb{Q}$ is a field, $-(x - y) = y - x \in \mathbb{Q}$ so that $y \sim x$. Finally, let $x, y, z \in E$ with $x \sim y$ and $y \sim z$. Then $x - y, y - z \in \mathbb{Q}$, and again since $\mathbb{Q}$ is a field, $(x - y) + (y - z) = x - z \in \mathbb{Q}$ which makes $x \sim z$. ■

Definition. Let E be a nonempty set of $\mathbb{R}$. We define a **choice set** for rational equivalence on E to be a set $\mathcal{C}_E$ consisting of exactly one member of each equivalence class in ${}^{E}/_{\sim}$; such that

(1) For all $x, y \in \mathcal{C}_E$, $x \not\sim y$.

(2) For every $x \in E$, there exists a $c \in \mathcal{C}_E$ such that $x = c + q$ for some $q \in \mathbb{Q}$.

Theorem 8.6.3 (Vitali's Theorem)**.** *Any set E with positive outer measure contains a nonmeasurable set.*

Proof. By countable subadditivity, suppose that E is bounded. Now, let $\mathcal{C}_E$ be a choice set of rational equivalence on E (which exists by the axiom of choice). Then we claim that $\mathcal{C}_E$ is nonmeasurable.

Indeed, suppose that $\mathcal{C}_E$ is measurable, and let Λ be any countably infinite set of rationals. Then the collection $\{\lambda + \mathcal{C}_E\}$ is disjoint and $m(+\mathcal{C}_E) = m(\mathcal{C}_E) = 0$, by lemma 8.6.1. Then, by translatio invariance, and countable additivity, we have

$$m\Big(\bigcup_{\lambda\in\Lambda} \lambda + \mathcal{C}_E\Big) = \sum_{\lambda\in\Lambda} m(\lambda + \mathcal{C}_E) = \sum_{\lambda\in\Lambda} m(\mathcal{C}_E) = 0$$

Now, since E is bounded, $E \subseteq [-b, b]$ for some $b \in \mathbb{R}$. Choose, then a set $\Lambda_0 = [-2b, 2b] \cap \mathbb{Q}$. Then Λ_0 is bounded and countably infinite. Now, if $x \in E$, then there is a $c \in \mathcal{C}_E$ for which $x = c + q$ for some $q \in \mathbb{Q}$. But $x, c \in [-b, b]$, so that $x, c \in [-2b, 2b]$. Thus

$$E \subseteq \bigcup_{\lambda\in\Lambda_0} (\lambda + \mathcal{C}_E)$$

but $m^*(E) > 0$, and $m(\mathcal{C}_E) = 0$, which is a contradiction! Therefore, $\mathcal{C}_E$ cannot be measurable. ∎

Theorem 8.6.4. *There exist nonmeasurable disjoint sets $A, B \subseteq \mathbb{R}$ for which*

$$m^*(A \cup B) < m^*(A) \cup m^*(B).$$

8.7 The Cantor Set

Definition. We define the **Cantor set** $\mathcal{C}$ to be the intersection

$$\mathcal{C} = \bigcap C_k$$

where $\{C_k\}$ is a descending collection of closed sets in which for each $k \in \mathbb{Z}^+$, $\mathbb{C}_k$ is the disjoint union of $2^k - 1$ closed intervals each of length $\frac{1}{3^k}$.

Theorem 8.7.1. *The Cantor set is a closed and uncountable set of measure $m(\mathcal{C}) = 0$.*

Proof. Since $\mathcal{C}$ is an arbitrary intersection of closed sets in $\mathbb{R}$, $\mathcal{C}$ is closed in $\mathbb{R}$. Moreover, by the definition of each C_k, C_k is measurable and

$$m(C_k) = \left(\frac{2}{3}\right)^k \text{ for each } k \in \mathbb{Z}^+$$

Then $\mathcal{C}$ is also measurable, and we have by monotonicity of the Lebesgue measure

$$m(\mathcal{C}) \leq m(C_k) = \left(\frac{2}{3}\right)^k$$

Taking $k \to \infty$, gives us that $m(\mathcal{C}) = 0$.

Now, suppose that $\mathcal{C}$ is countable, and let $\mathcal{C} = \{c_k\}$ be an enumeration of $\mathcal{C}$. Now, one of the disjoint intervals whose union is C_1 fails to contain c_1; call it F_1. Proceeding, one of the disjoint intervals whose union is F_1 fails to contain c_2; call it F_2. Proceeding inductively, we get a descending collection $\{F_k\}$ of closed sets, for which $c_k \notin F_k$ for each $k \in \mathbb{Z}^+$. Now, by the nested set theorem, the intersection

$$F = \bigcap F_k$$

is nonempty. Moreover, $F \subseteq \mathcal{C}$. This implies that there is an element $x \in \mathcal{C}$ which is not equal to any c_k; but $\{c_k\}$ is an enumeration of $\mathcal{C}$, which is absurd! Therefore, $\mathcal{C}$ fails to be countable. ■

Definition. Let $\mathcal{C}$ be the Cantor set, and define U_k such that $C_k = [0,1] \backslash U_k$, and define $\mathcal{U} = \bigcup U_k$; so that $[0,1] = \mathcal{C} \cup \mathcal{U}$. Fix $k \in \mathbb{Z}^+$ and define

$$\phi : U_k \to \mathbb{R}$$

to be the increasing function which is constant on each of the $2^k - 1$ intervals of U_k, and whose image is

$$\phi([0,1]) = \left\{\frac{1}{2^k}, \frac{2}{2^k}, \ldots, \frac{2^k - 1}{2^k}\right\}$$

We define the **Cantor-Lebesgue function** to be the extension Φ of ϕ to $[0,1]$; defined in terms of $\mathcal{C}$ to be

$$\Phi(0) = \phi(0) = 0 \text{ and } \Phi(x) = \sup\{\phi(t) : t \in \mathcal{U} \cap [0,x]\} \text{ for all } x \in \mathcal{C}\backslash\{0\}$$

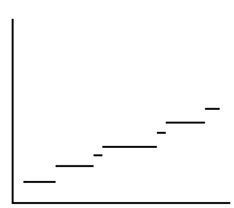

Figure 8.1: The Canotr Lebesgue Function on U_3.

Theorem 8.7.2. *The Cantor-Lebesgue function is an increasing continuous finction mapping $[0,1]$ onto $[0,1]$, and differentiable on $\mathcal{U}$ with*

$$\Phi'(x) = 0 \textit{ on } \mathcal{U} \textit{ where } m(\mathcal{U}) = 1$$

Proof. Since ϕ is an increasing fucntion, and Φ is the extension of ϕ to $[0,1]$, then Φ must also be increasing. Now, ϕ is continuous at each point of $\mathcal{U}$, since it is constant on each U_k. Now, consider $x_0 \in \mathcal{C}$ with $x_0 \neq 0, 1$. Then $x_0 \notin U_k$ for all $k \in \mathbb{Z}^+$. So for k large enought, x_0 lies in between consecutive intervals of U_k. Choose a_k to be the lowerbound of the lower interval, and b_k the upperbound of the upper interval. Then by definition of ϕ, we get

$$a_k < x_0 < b_k \text{ and } \phi(b_k) - \phi(a_k) = \frac{1}{2^k}$$

Since k is arbitrarily large, Φ fails to have a jump discontinuity at x_0, which is the only possible discontinuity it can have. Therefore Φ is continuous at x_0. A similar argument shows that Φ is continuous at $x_0 = 0, 1$.

Now, since ϕ is constant on each U_k, ϕ is differentiable on U_k and has $\phi' = 0$ on each U_k, hence $\phi' = 0$ on $\mathcal{U}$. Since $m(\mathcal{C}) = 0$ and $[0,1] = \mathcal{C} \cup \mathcal{U}$, then $m(\mathcal{U}) = 1$. Since Φ is an extension, we get that

$$\Phi' = 0 \text{ on } \mathcal{U} \text{ where } m(\mathcal{U}) = 1$$

Finally, since $\Phi(0) = 0$ and $\Phi(1) = 1$, and Φ is increasing, by the intermediate value theorem $\Phi([0,1]) = [0,1]$. ■

Lemma 8.7.3. *Let Φ be the Cantor-Lebesgue function, and define $\Psi : [0,1] \to \mathbb{R}$ by*

$$\Psi(x) = \Phi(x) + x \text{ for all } x \in [0,1]$$

Then Ψ is strictly increasing, and maps $[0,1]$ onto $[0,2]$. Moreover, Ψ maps the Cantor set onto a set of positive measure, and maps a measurable subset of the Cantor set onto a nonmeasurable set.

Proof. Since Ψ is the sum of the strictly increasing function $f(x) = x$ and the increasing function $\Phi(x)$, Ψ is strictly increasing. Moreover, Ψ is continuous since it is the sum of continuous functions, and $\Psi(0) = 0$, and $\Psi(1) = 2$, so by the intermediate value theorem, $\Psi([0,1]) = [0,2]$. Finally, notice that since Φ and $f(x) = x$ are 1–1, then Ψ is also 1–1. Therefore Ψ has a continuous inverse Ψ^{-1}. This makes $\Psi(\mathcal{C})$ closed, and $\Psi(\mathcal{U})$ open. Therefore both $\Psi(\mathcal{C})$ and $\Psi(\mathcal{U})$ are measurable.

Now, let $\{I_k\}$ be the collection of intervals removed from $[0,1]$ to form $\mathcal{C}$; i.e.

$$\mathcal{U} = \bigcup I_k$$

Since Φ is constant on each of these intervals, we get that Ψ maps I_k onto the translate $I_k + x$. Since Ψ is 1–1, we have that $\{\Psi(I_k)\}$ is a disjoint collection of measurable sets. Therefore, by countable additivity

$$m\Big(\bigcup \Psi(I_k)\Big) = \sum m(\Psi(I_k)) = \sum m(I_k + x) = \sum m(I_k) = m(\mathcal{U}) = 1$$

Since $[0,2] = \Psi(\mathcal{C}) \cup \Psi(\mathcal{U})$, we get $\Psi(\mathcal{C}) = 1$.

Now, by Vitali's theorem, there exists a nonmeasurable subset $W \subseteq \Psi(\mathcal{C})$. Notice then that $\Psi^{-1}(W) \subseteq \mathcal{C}$ is measurable of measure $m(^{-1}(W)) = 0$, since $m(\mathcal{C}) = 0$. That is, we have mapped a measurable subsete of $\mathcal{C}$ to a nonmeasurable set. This concludes the proof. ■

Theorem 8.7.4. *There exists a measurable subset of the Cantor set which is not Borel.*

Chapter 9

Lebesgue Measurable Functions

9.1 Properties of Lebesgue Measurable Functions

Lemma 9.1.1. *Let f a real-valued function on a measurable domain E. Then the following statements are equivalent for any $c \in \mathbb{R}$.*

(1) $f^{-1}((c, \infty)) = \{x \in E : f(x) > c\}$ *is measurable.*

(2) $f^{-1}([c, \infty)) = \{x \in E : f(x) \geq c\}$ *is measurable.*

(3) $f^{-1}((-\infty, c)) = \{x \in E : f(x) < c\}$ *is measurable.*

(4) $f^{-1}((-\infty, c]) = \{x \in E : f(x) \leq c\}$ *is measurable.*

Proof. It suffices to show that statements (1) and (2) are equivalent. Suppose that (1) holds. Notice that

$$f^{-1}([c, \infty)) = \bigcap_{k=1}^{\infty} \{x \in E : f(x) > c - \frac{1}{k}\}$$

which is a countable intersection of measurable sets, which makes $f^{-1}([c, \infty))$ measurable. Conversely, if (2) holds, then we have

$$f^{-1}((c, \infty)) = \bigcup_{k=1}^{\infty} \{x \in E : f(x) \geq c - \frac{1}{k}\}$$

which is a countable union of measurable sets, which makes $f^{-1}((c, \infty))$ measurable. ∎

Corollary. *If f is a real-valued function on a measurable domain E, then for any $c \in \mathbb{R}$*

$$f^{-1}(c) = \{x \in E : f(x) = c\}$$

is measurable.

Proof. Take $c \in \mathbb{R}$. Then $f^{-1}([c, \infty))$ and $f^{-1}((-\infty, c])$ are measurable, hence so is their intersection, which is $f^{-1}(c)$. ∎

Corollary. *If f is an extended real-valued function, then $f^{-1}(\infty)$ is measurable.*

Proof. Notice that

$$f^{-1}(\infty) = \{x \in E : f(x) > k\}$$

so that $f^{-1}(\infty)$ is measurable by the above lemma. ■

Definition. We call a real-valued function f on a measurable domain E **Lebesgue measurable** if for any $c \in \mathbb{R}$ and I a bounded interval of the form (c, ∞), $(-\infty, c)$, $[c, \infty)$, or $(-\infty, c]$, then $f^{-1}(I)$ is measurable.

Lemma 9.1.2. *Let f be a real-valued function defined on a measurable domain E. Then f is Lebesgue measurable if, and only if for any U open in $\mathbb{R}$, $f^{-1}(U)$ is measurable.*

Proof. Let U be open in $\mathbb{R}$, and $f^{-1}(U)$ be measurable. Since (c, ∞) is open, then $f^{-1}((c,))$ is measurable by hypothesis, which makes f measurable.

Conversely, suppose that f is measurable, and let U be open in $\mathbb{R}$. Then

$$U = \bigcup I_k$$

where $\{I_k\}$ is a countable collection of open bounded intervals. Moreover, we have each $I_k = A_k \cap B_k$ where $A_k = (a_k, \infty)$ and $B_k = (-\infty, b_k)$. Since f is measurable, then $f^{-1}(A_k)$ and $f^{-1}(B_k)$ are measurable. Hence

$$f^{-1}(U) = f^{-1}\Big(\bigcup (A_k \cap B_k)\Big) = \bigcup f^{-1}(A_k) \cap f^{-1}(B_k)$$

which is a countable union of measurable sets, which makes $f^{-1}(U)$ measurable. ■

Corollary. *If f is continuous on its domain, then it is measurable.*

Proof. Let f be continuous on E, and let U be open. Then $f^{-1}(U)$ is open and $f^{-1}(U) = E \cap V$ with some V open in $\mathbb{R}$. Now, E and V are measurable sets, which makes $f^{-1}(U)$ measurable. ■

Lemma 9.1.3. *Monotone functions defined on an interval are measurable.*

Lemma 9.1.4. *Let f and g be extended real-valued functions on a measurable domain E. Then the following are true.*

(1) If f is Lebesgue measurable, and $f = g$ almost everywhere on E, then g is Lebesgue measurable.

(2) If D is a measurable subset of E, then f is Lebesgue measurable if, and only if f is Lebesgue measurable when restricted to D and to $E \backslash D$; i.e. both $f|_D$ and $f|_{E\backslash D}$ are Lebesgue measurable.

Proof. Suppose that f is measurable and that $f = g$ almost everywhere on E. Let $A = \{x\ inE : f(x) \neq g(x)\}$. Then notice that

$$g^{-1}((c, \infty)) = \{x \in A : g(x) > c\} \cup ((\{x \in E : f(x) > c\}) \cap (E \backslash A))$$

Now, since $f = g$ almost everywhere on E, then $m(A) = 0$, so that the set $\{x \in A : g(x) > c\}$ is measurable. Now, $f^{-1}((c,\infty))$ is measurable since f is measurable, which makes $g^{-1}((c,\infty))$ measurable.

Now, let $D \subseteq E$ a measurable subset. Observe that for every $c \in \mathbb{R}$

$$f^{-1}((c,\infty)) = \{x \in D : f(x) > c\} \cup \{x \in E \backslash D : f(x) > c\}$$

hence the equivalence is proved. ■

Theorem 9.1.5. *Let f and g be Lebesgue measruable functions defined on a measurable domain E, and finite almost everywhere on E. Then for all $\alpha, \beta \in \mathbb{R}$*

(1) $\alpha f + \beta g$ is Lebesgue measurable.

(2) fg is Lebesgue measurable.

Proof. Suppose, without loss of generality, that f and g are finite on all of E. Now, for $\alpha = 0$, we have that $\alpha f = 0$ is measurable. Now, if $\alpha \neq 0$, observe that

$$(\alpha f)^{-1}((c,\infty)) = \begin{cases} \{x \in E : f(x) > \frac{c}{\alpha}\}, \alpha > 0 \\ \{x \in E : f(x) < \frac{c}{\alpha}\}, \alpha < 0 \end{cases}$$

Now, since f is measurable, both these sets are measurable. This makes αf measurable. Now, consider the function $f + g$. If $f(x) + g(x) < c$ for all $x \in E$, then $f(x) < c - g(x)$. Now, by the density of $\mathbb{Q}$ in $\mathbb{R}$, there is a rational $q \in \mathbb{Q}$ for which $f(x) < q < c - g(x)$. Thus

$$(f+g)^{-1}((c,\infty)) = \bigcup_{q \in \mathbb{Q}} (g^{-1}((-\infty, c-q)) \cap f^{-1}(-\infty, q))$$

By the countability of $\mathbb{Q}$ we have a countable union of measurable sets, which makes $(f + g)^{-1}((c,\infty))$ measurable. Hence $f + g$ is measurable.

Lastly, notice that $fg = \frac{1}{2}((f+g)^2 - f^2 - g^2)$. It suffices to show then that f^2 is measurable. For $c \geq 0$, we have that

$$(f^2)^{-1}((c,\infty)) = \{x \in E : f(x) > \sqrt{c}\} \cup \{x \in E : f(x) < -\sqrt{c}\}$$

and for $c < 0$, we have $(f^2)^{-1}((c,\infty)) = \{x \in E : f^2(x) > c\} = E$. In either case we get measurable sets, which makes f^2 measurable. ■

Example 9.1. Consider the strictly increasing function ψ on $[0,1]$ defined by $\psi(x) = \phi(x) + x$, where ϕ is the Cantor-Lebesgue function. Then there exists a measurable subset A of $[0,1]$ for which $\psi(A)$ is nonmeasurale. Now, extend ψ to the continuous function $\Psi : \mathbb{R} \to \mathbb{R}$. Then Ψ^{-1} is continuous, and hence measurable. Now, consider the function

$$\chi_A(x) = \begin{cases} 1, & \text{if } x \in A \\ 0, & \text{if } x \notin A \end{cases}$$

and take $f(x) = \chi_A \circ \Psi^{-1}(x)$. We claim that f is not a measurable function. Take I an open interval with $1 \in I$ and $0 \notin I$. Then $f^{-1}(I) = \Psi(\chi_A^{-1}(I)) = \Psi(A) = \psi(A)$ which is not measurable. Hence f fails to be a measurable function. In general, compostions of Lebesgue measurable functions need not be Lebesgue measurable.

Theorem 9.1.6. *Let g be a lebesgue measurable function on a domain E, and let f be a continuous real-valued function on all $\mathbb{R}$. Then $f \circ g$ is Lebesgue measurable on E.*

Proof. Let U be open in $\mathbb{R}$. Then $(f \circ g)^{-1}(U) = g^{-1}(f^{-1}(U))$. Now, since f is continuous, we get $f(U)$ is open in $\mathbb{R}$, and since g is measurable, this makes $g^{-1}(f^{-1}(U))$ measurable by lemma 9.1.2. Therefore $f \circ g$ is Lebesgue measurable. ■

Corollary. *$|f|^p$ is Lebesgue measurable for any $p > 0$.*

Lemma 9.1.7. *Let $\{f_k\}_{k=1}^n$ be a finite sequence of Lebesgue measurable functions on a common measurable domain E. Then the functions*

$$\overline{f}(x) = \max\{f_1(x), \dots, f_n(x)\}$$
$$\underline{f}(x) = \min\{f_1(x), \dots, f_n(x)\}$$

are Lebesgue measurable.

9.2 Simple Approximation

Definition. Let $\{f_n\}$ be a sequence of real-valued functions on a measurable domain E, and let f be a function on E, and let $A \subseteq E$. We say that $\{f_n\}$ **converges pointwise** to f on A if

$$\lim_{n \to \infty} f_n(x) = f(x) \text{ for all } x \in A$$

We say taht $\{f_n\}$ **converges uniformly** to f on A provided for all $\varepsilon > 0$, there is an $N \in \mathbb{Z}^+$ for which

$$||f(x) - f_n(x)| < \varepsilon \text{ for all } n \geq N$$

Lemma 9.2.1. *Let $\{f_n\}$ be a sequence of measurable functions on a domain E, converging pointwise almost everywhere to a real-valued function f on E. Then f is measurable.*

Proof. Let $E_0 \subseteq E$ have measure $m(E_0) = 0$, and let $\{f_n\}$ converge pointwise to f on $E \backslash E_0$. Then f is measurable if, and only if $f|_{E_0}$ and $f|_{E \backslash E_0}$ are measurable. Suppose, then, that $\{f_n\}$ converges pointwise to f on all E.

Now, fix $c \in \mathbb{R}$ and observe that for every $x \in E$, that $\lim f_n(x) = f(x)$ as $n \to \infty$. Thus $f(x) < c$ if, and only if there exists $n, k \in \mathbb{Z}^+$ for which

$$f_j(x) < c - \frac{1}{n} \text{ for all } j \geq k$$

Now, since f_j is measurable, we get $f_j^{-1}((-\infty, c - \frac{1}{n}))$ is measurable, hence the intersection

$$F_k = \bigcap_{k \neq j} f_j^{-1}((-\infty, c - \frac{1}{n}))$$

is measurable. Notice then that $f^{-1}((-\infty, c)) = \bigcup_k F_k$. ■

Definition. Let A be any set of $\mathbb{R}$. We define the **characteristic function** of A to be the real-valued function $\chi_A : \mathbb{R} \to \{0, 1\}$ defined by

$$\chi_A(x) = \begin{cases} 1, & \text{if } x \in A \\ 0, & \text{if } x \notin A \end{cases}$$

Lemma 9.2.2. *If A is a measurable set of $\mathbb{R}$, then χ_A is Lebesgue measurable.*

Definition. We call a real-valued funtion ϕ defined on a measurable set E **simple** if it takes only finitely many values. If ϕ takes the values $c_1, \dots, c_n$; i.e. $\phi(E) = \{c_1, \dots, c_n\}$, then we define

$$\phi(x) = \sum_{k=1}^{n} c_k \chi_{E_k}(x) \text{ where } E_k = \phi^{-1}(c_k)$$

the **canonical representation** of ϕ.

Lemma 9.2.3. *Simple functions are Lebesgue measurable.*

Lemma 9.2.4 (The Simple Approximation Lemma)**.** *Let f be a measurable function bounded on a measurable E. T hen for every $\varepsilon > 0$, there exists simple function ϕ_ε and ψ_ε on E such that*

$$\phi_\varepsilon(x) \leq f(x) \leq \psi_\varepsilon(x) \textit{ and } 0 \leq \psi_\varepsilon(x) - \phi_\varepsilon(x) < \varepsilon$$

Proof. Let (c, d) be an open bounded intervale containing $f(E)$, and let

$$P : \{c = y_0 < \cdots < y_n = d\}$$

a partition of the closed bounded interval $[c, d]$ such that

$$y_{k+1} - y_k < \varepsilon \text{ for all } 0 \leq k \leq n - 1$$

Define $I_k = [y_k, y_{k+1})$ and $E_k = f^{-1}(I_k)$ for all $0 \leq k \leq n - 1$. Since each I_k is an interval, and f is measurable, then each E_k is measurable. Now, define

$$\phi_\varepsilon(x) = \sum_{k=1}^{n} y_k \chi_{E_k} \text{ and } \psi_\varepsilon(x) = \sum_{k=1}^{n} y_{k+1} \chi_{E_k}$$

Then ϕ_ε and ψ_ε are simple functions on E. Now, for each $xc \in E$, we have that $f(E) \subseteq (c, d)$, so taht there exists a unique $0 \leq k \leq n - 1$ fr which $y_k \leq f(x) \leq y_{k+1}$, so that

$$y_k = \phi_\varepsilon(x) \leq f(x) \leq \psi_\varepsilon(x)$$

Moreover, we have that $y_{k+1} - y_k = \psi_\varepsilon(x) - \phi_\varepsilon(x) < \varepsilon$. ■

Theorem 9.2.5 (The Simple Approximation Theorem)**.** *An extended real-valued function f on a measurable domain E is measurable if, and only if there exists a sequence $\{\phi_n\}$ of simple functions on E, converging pointwise to f such that*

$$|\phi_n(x)| \leq |f(x)| \textit{ for all } n\iota\mathbb{Z}^+$$

Proof. Since simple functions are measurable, if $\{\phi_n\}$ converges pointwise to f on E, then f must also be measurable by lemma 9.2.1.

Now, suppose that f is measurable, and that $f(x) \geq 0$ for all $x \in E$. Let $n \in \mathbb{Z}^+$, and define $E_n = f^{-1}((-\infty, n])$. Then E_n is measurable, and $f|_{E_n}$ is a nonnegative bounded measurable function. By the simple approximation lemma, choosing $\varepsilon = \frac{1}{n}$, slect simple functions ϕ_n and ψ_n on E such that

$$\phi_n(x) \leq f(x) \leq \psi_n(x) \text{ and } 0 \leq \psi_n(x) - \phi_n(x) < \frac{1}{n} \text{ on } E_n$$

Now, extend ϕ_n to a function Φ_n on all of E such that $\Phi_n(x) = n$ if $f(x) > n$. Then Φ_n is simple, on E and

$$0 \leq \Phi_n(x) \leq f(x) \text{ on } E$$

Now, we claim that the sequence $\{\psi_n\}$ converges to f pointwise on E. Indded, let $x \in E$, if $f(x)$ is finite, choose an $N \in \mathbb{Z}^+$ such that $f(x) < N$, then

$$0 \leq f(x) - \Phi_n(x) < \frac{1}{n} \text{ for all } n \geq N$$

so that $\{\psi_n\} \to f$ as $n \to \infty$. Now, if $f(x) = \infty$, then $\Phi_n(x) = n$ for all $n \in \mathbb{Z}^+$ so that $\{\phi_n\} \to f$ as $n \to \infty$. ■

9.3 Theorems of Littlewood, Egoroff, and Lusin

Lemma 9.3.1. *Let E be a measurable set having finite measure, and let $\{f_n\}$ a sequence of measurable functions on E converging pointwise to a real-valued function f on E. Then for all $\eta > 0$ and $\delta > 0$, there exists a measurable subset A of E and an $N \in \mathbb{Z}^+$ for which*

$$|f(x) - f_n(x)| < \eta \textit{ on } A \textit{ for all } n \geq N \textit{ and } m(E \backslash A) < \delta$$

Proof. Since f is realvulued and measurable (by lemma 9.2.1), the set

$$\{x \in E : |f(x) - f_n(x)| < \eta\}$$

is measurabe. Hence, the union

$$E_n = \{x \in E : |f(x) - f_n(x)| < \eta \text{ for all } k \geq n\}$$

is measurable. Then $\{E_n\}$ is an ascending collection of measurable sets with

$$E = \bigcup E_n$$

Now, since $\{f_n\}$ converges to f pointwise, by the continuity of measure, we conclude that

$$m(E) = \lim_{n \to \infty} m(E_n)$$

Since $m(E)$ is finite, choose an $N \in \mathbb{Z}^+$ for which $m(E_N) > m(E)$-$\varepsilon, and define$A=E$_n$. Then by excision, we get $m(E \backslash A) = m(E) - m(A) < \varepsilon$. ■

Theorem 9.3.2 (Egoroff's Theorem)**.** *Let E be a measurable set with finite measure, and let $\{f_n\}$ a sequence of measurable functions on E converging pointwise to a real-valued function f on E. Then for each $\varepsilon > 0$, there exists a closed subset F of E for which $\{f_n\}$ converges uniformly to f on F, and $m(E\backslash F) < \varepsilon$.*

Proof. For every $n \in \mathbb{Z}^+$, let A_n be a measurable subset of E, and let $N(n) \in \mathbb{Z}^+$ such that

$$|f(x) - f_k(x)| < \frac{1}{n} \text{ on } A_n \text{ for all } k \geq N(n) \text{ and where } m(E\backslash A_n) < \frac{\varepsilon}{2^{n+1}}$$

Define

$$A = \bigcap_{n=1}^{\infty} A_n$$

Then by DeMorgan's laws and countable subadditivity

$$m(E\backslash A) = \sum_{n=1}^{\infty} m(E\backslash A_n) < \sum_{n=0}^{\infty} \frac{\varepsilon}{2^{n+1}} < \frac{\varepsilon}{2}$$

Now, let $\varepsilon > 0$ and choose an $n_0 \in \mathbb{Z}^+$ such that $\frac{1}{n_0} < \varepsilon$. Then

$$|f(x) - f_k(x)| < \frac{1}{n_0} \text{ on } A \text{ for all } k \geq N(n_0)$$

Now, since $A \subseteq A_{n_0}$, and our hypothesis on n_0, we get

$$|f(x) - f_k(x)| < \varepsilon \text{ on } A \text{ for all } k \geq N(n_0)$$

so that $\{f_n\}$ converges uniformly to f on A with $m(E\backslash A) < \frac{\varepsilon}{2}$. Now, by the inner approximation theorem, choose an FA, closed, for which $m(A\backslash F) < \frac{\varepsilon}{2}$. Then $m(E\backslash F) < m(E\backslash A) + m(A\backslash F) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$. So that $\{f_n\}$ converges uniformly to f on F. ■

Theorem 9.3.3 (Littlewood's Theorem)**.** *Let f be a simple function on a measurable set E. Then for every $\varepsilon > 0$, there eixsts a continuous function g on $\mathbb{R}$ and a closed subset F of E such that $f = g$ on F and $m(E\backslash F) < \varepsilon$.*

Proof. Let $\{a_1, \ldots a_n\}$ the distinct values taken on by f, taken on the sets $E_1, \ldots, E_n$, respectively. Then $\{E_k\}_{k=1}^n$ is a finite disjoint collection of measurable sets whose union is E, since each a_i is disjoint. Now, choose closed sets $F_1, \ldots, F_n$ such that for all $1 \leq k \leq n$, $F_k \subseteq E_k$ and $m(E_k\backslash F_k) < \frac{\varepsilon}{n}$. Then

$$F = \bigcup_{k=1^n} F_n$$

is closed and since $\{E_k\}_{k=1}^n$ is a disjoint collection, we get

$$m(E\backslash F) = \sum_{k=1}^{n} m(E_k\backslash F_k)$$

Now, define $g : F \to \mathbb{R}$ to take $g(x) = a_k$ for all $x \in F_k$, for all $1 \leq k \leq n$. Since $\{F_k\}$ is a disjoint collection, g is well defined. Moreover, g is continuous on F. Then for every $x \in F_i$, there exists an interval I, containing x, and disjoint from the union $\bigcup_{k \neq i} F_k$. Thus g is constant on $F \cap I$. Now, extend g to a continuous function $G : \mathbb{R} \to \mathbb{R}$. Since $G(x) = a_k$ on each F_k, and $m(E_k\backslash F_k) < \frac{\varepsilon}{n}$, we get $f = G$ on F where $m(E\backslash F) < \varepsilon$. ■

Theorem 9.3.4 (Lusin's Theorem). *Let f be a measurable function on a measurable set E. Then for every $\varepsilon > 0$, there eixsts a continuous function g on $\mathbb{R}$ and a closed subset F of E such that $f = g$ on F and $m(E \backslash F) < \varepsilon$.*

Proof. Consider first when E has finite measure. By the simple approximation theorem (theorem 9.2.5), there exists a sequence $\{f_n\}$ of simple functions on E, converging pointwise to f on E. Let $n \in \mathbb{Z}^+$, then byt Littlewood's theorem, choose a g_n continuous on $\mathbb{R}$ and a closed subset F_n of E for which $f = g_n$ on F_n and $m(E \backslash F_n) < \frac{\varepsilon}{2^{n+1}}$. By Egoroff's theorem, there exists a closed set $F_0 \subseteq E$ such that $\{f_n\}$ converges uniformly to f on F_0, with $m(E \backslash F_0) < \frac{\varepsilon}{22}$. Define then

$$F = \bigcap_{n=0}^{\infty} F_n$$

Then F is closed and by DeMorgan's laws,

$$m(E \backslash F) \leq \frac{\varepsilon}{2} + \sum_{n=0}^{\infty} \frac{\varepsilon}{2^{n+1}} = \varepsilon$$

Now, each f_n is continuous on F, and since $F \subseteq F_n$, and $f = g_n$ on f_n, we get $\{f_n\}$ converges uniformly to f on F, since $F \subseteq F_0$. Moreover, $f|_F$ is continuous on F, thus choose a continuous function $g : \mathbb{R} \to \mathbb{R}$ such that $f = g$ on F, and we are done. ■

Chapter 10

Lebesgue Integration

10.1 The Lebesgue Integral of Bounded Measurable Functions over Sets of Finite Measure

We begin this chapter with an example of the deficiencies of the Riemann integral. For the who chapter, we denot the Riemann integral of a real-valued function f to be

$$(R)\int f$$

Example 10.1. (1) Define f on $[0,1]$ to be $f(x) = 1$ if $x \in \mathbb{Q}$, and $f(x) = 0$ if $x \in \mathbb{R}\backslash\mathbb{Q}$. That is, $f = \chi_{\mathbb{Q}}$ on $[0,1]$. Let P a partition of $[0,1]$, then by the density of $\mathbb{Q}$ and $\mathbb{R}\backslash\mathbb{Q}$ in $\mathbb{R}$ we have

$$L(f,P) = 0 \text{ and } U(f,P) = 1$$

so that

$$\underline{(R)\int_0^1} f = 0 \text{ and } \overline{(R)\int_0^1} f = 1$$

Thus, f is not Riemann integrable. We call f **Dirichlet's function**.

(2) Let $\{q_k\}$ an enumeration of $\mathbb{Q}$ in $[0,1]$, and define f_n on $[0,1]$ to be $f_n(x) = 1$ if $x = q_k$ for any $1 \leq k \leq n$, and $f(x) = 0$ otherwise. Then each f_n is a step function, and hence Riemann integrable. However, notice that $\{f_n\} \to f$, the Dirichlet function, which is not Riemann integrable. That is, we have a sequence of Riemann integrable functions (with $|f_n| \leq 1$ for each n) converging to a non-Riemann integrable function.

This example illustrates the short comings of the Riemann-Stieltjes integral, in which not every function is integrable. Moreover, givena sequence of Reimann integrable functions, we cannot justify the passage of a limit under the integal.

It becomes now necessary to define an alternative integral, using measure, for which we can get the above properties.

Definition. For a simple function f on a set E of finite measure, and canonical representation $f(x) = \sum_{i=1}^n a_i\chi_{E_i}$, we define the **integral** of f over E to be

$$\int_E f\ dm = \sum_{i=1}^n a_i m(E_i) \text{ where } f^{-1}(a_i) = E_i$$

Lemma 10.1.1. *Let $\{E_i\}_{i=1}^n$ be a finite disjoint collection of measurable subsets of a set E of finite measure, and let f be a simple function with*

$$f(x) = \sum_{i=1}^{n} a_i \chi_{E_i} \text{ where } f^{-1}(a_i) = E_i$$

where each a_i is not necessarily distinct. Then

$$\int_E f = \sum_{i=1}^{n} a_i m(E_i)$$

Proof. If each a_i is distinct, then we have the canonical representation of f, and the result follows directly under definition. Hence, suppose not all the a_i are distinct, and let f have the canonical representation

$$f(x) = \sum_{j=1}^{m} \lambda_j \chi_{A_j} \text{ where } f^{-1}(\lambda_j) = A_j$$

Then by definition, we have

$$\int_E f = \sum_{j=1}^{m} \lambda_j m(A_j)$$

Now, let I_j be the index set of i in which $a_i = \lambda_j$. Then notice that $\bigcup I_j = \{1, \ldots, n\}$, and is a disjoint union. Then by finite additivity of measure, we have

$$m(A_j) = \sum_{i \in I_j} m(E_i) \text{ for all } 1 \le j \le m$$

Therefore

$$\int_E f = \sum_{j=1}^{m} \lambda_j m(A_j) = \sum_{j=1}^{m} \lambda_j \sum_{i \in I_j} m(E_i) = \sum_{i=1}^{n} a_i m(E_i)$$

and we are done. ■

Lemma 10.1.2. *Let f and g be simple functions defined on a set E of finite measure. Then for any $\alpha, \beta \in \mathbb{R}$*

$$\int_E \alpha f + \beta g \; dm = \alpha \int_E f \; dm + \beta \int_E g \; dm$$

Moreover, if $f \le g$ on E, then

$$\int_E f \le \int_E g$$

Proof. Choose a finite disjoint measurable cover $\{E_i\}_{i=1}^n$ of E such that f and g are constant on each E_i. Let f and g take on the values a_i and b_i, respectively, for all $1 \le i \le n$. Then by lemma 10.1.1

$$\int_E f = \sum_{i=1}^{n} a_i m(E_i) \text{ and } \int_E g = \sum_{i=1}^{n} b_i m(E_i)$$

Now, notice that since f and g are simple, then for $\alpha, \beta \in \mathbb{R}$, the function $\alpha f + \beta g$ is simple, and takes on the values $\alpha a_i + \beta b_i$ for all $1 \leq i \leq n$. Therefore, by definition of the integral of a simple function, we have

$$\begin{aligned}\int_E \alpha f + \beta g \; dm &= \sum_{i=1}^{n} (\alpha a_i + \beta b_i) m(E_i) \\ &= \alpha \sum_{i=1}^{k} a_i m(E_i) + \beta \sum_{i=1}^{k} b_i m(E_i) \\ &= \alpha \int_E f \; dm + \beta \int_E g \; dm\end{aligned}$$

Now, let $h = g - f$. Then h is a simple function, and hence by linearity, we have

$$\int_E f - \int_E g = \int_E f - g = \int_E h \geq 0$$

■

Definition. Let f be a bounded measurable real-valued function on a set E of finite measure. We define the **upper Lebesgue integral** of f over E to be

$$\overline{\int_E} f = \sup \left\{ \int_E \phi : \phi \text{ is a simple function and } \phi \leq f \right\}$$

We define the **lower Lebesgue integral** of f over E to be

$$\underline{\int_E} f = \inf \left\{ \int_E \phi : \phi \text{ is a simple function and } f \leq \phi \right\}$$

We call f **Lebesgue integrable** if

$$\overline{\int_E} f = \underline{\int_E} f = \int_E f$$

and call the value in which the upper and lower Lebesgue integrals coincide the **Lebesgue integral**.

Theorem 10.1.3. *Let f be a bounded real-valued function on a closed bounded interval $[a, b]$. Then if f is Riemann integrable, it is Lebesgue integrable. Moreover, the two integrals coincide.*

Proof. If f is Riemann integrable, then we have

$$\overline{(R) \int_a^b} f = \underline{(R) \int_a^b} f$$

Notice now that

$$\overline{(R)\int_E} f = \sup\left\{ \int_E \phi : \phi \text{ is a step function and } \phi \leq f \right\}$$
$$\underline{(R)\int_E} f = \sup\left\{ \int_E \phi : \phi \text{ is a step function and } f \leq \phi \right\}$$

Now, step functions are simple functions, so these definitions for the upper and lower Riemann-Stieltjes integrals coincides with the definitions for the upper and lower Lebesgue integrals. ■

Example 10.2. Lebesgue integrable functions need not be Riemann integrable. Consider again Dirichlet's function f on $[0,1]$. We have that f is Lebesgue integrable with

$$\int_{[0,1]} f = \int 1 \times \chi_{\mathbb{Q}}\ dm = 1 \times m(\mathbb{Q}) = 0$$

However, it was shown in example 10.1 that f is not Riemann integrable.

Theorem 10.1.4. *Let f be a bounded measurable function on a set E of finite measure. Then f is Lebesgue integrable over E.*

Proof. Let $n \in \mathbb{Z}^+$, by the simple approximation theorem, and choosing $\varepsilon = \frac{1}{n}$, there exist simple functions ϕ_n and ψ_n for which

$$\phi_n(x) \leq f(x) \leq \psi_n(x) \text{ and } 0 \leq \psi_n(x) - \phi_n(x) \leq \frac{1}{n} \text{ on } E$$

Then we get

$$0 \leq \int_E \psi_n - \int_E \phi_n = \int_E \psi_n - \phi_n \leq \frac{1}{n} m(E)$$

Now, notice also that

$$0 \leq \underline{\int_E} f - \overline{\int_E} f \leq \int_E \psi_n - \int_E \psi_n \leq \frac{1}{n} m(E)$$

Since n is arbitrary, this establishes equlity of the integrals. ■

Theorem 10.1.5. *Let f and g be bounded measurable functions on a set E of finite measure. If f and g are Lebesgue integrable then for any $\alpha, \beta \in \mathbb{R}$, $\alpha f + \beta g$ is Lebesgue integrable, and*

$$\int_E \alpha f + \beta g\ dm = \alpha \int_E f\ dm + \beta \int_E g\ dm$$

Moreover, if $f \leq g$ on E, then

$$\int_E f \leq \int_E g$$

Proof. Notice that since f and g are bounded and measurable on E, then so is $\alpha f + \beta g$. This makes $\alpha f + \beta g$ Lebesgue integrable on E. Now, let ψ a simple function; then so is $\alpha\psi$ for $\alpha \neq 0$. So, for $\alpha > 0$ we have

$$\int_E \alpha f = \inf_{\psi \geq \alpha f} \left\{ \int_E \psi \right\} = \alpha \inf_{\frac{\psi}{\alpha} \geq f} \left\{ \int_E \frac{\psi}{\alpha} \right\} = \alpha \int_E f$$

Similarly, for $\alpha < 0$, we have

$$\int_E \alpha f = \inf_{\psi \leq \alpha f} \left\{ \int_E \psi \right\} = \alpha \inf_{\frac{\psi}{\alpha} \leq f} \left\{ \int_E \frac{\psi}{\alpha} \right\} = \alpha \int_E f$$

Thus $\int_E \alpha f = \alpha \int_E f$. Now, let ψ_1 and ψ_2 be simple functions for which $f \leq \psi_1$ and $g \leq_2$. Then $\psi_1 + \psi_2$ is a simple function with $f + g \leq \psi_1 + \psi_2$. Then we get

$$\int_E f + g \; dm \leq \int_E \psi_1 + \psi_2 \; dm \int_E \psi_1 \; dm + \int_E \psi_2 \; dm$$

which makes

$$\int_E f + g \; dm \leq \int_E f \; dm + \int_E g \; dm$$

Conversely, if ϕ_1 and ϕ_2 are simple functions for which $\psi_1 \leq f$ and $_2 \leq g$. Then $\phi_1 + \phi_2$ is a simple function with $\phi_1 + \phi_2 \leq f + g$. Therefore,

$$\int_E \phi_1 + \phi_2 \; dm \int_E \phi_1 \; dm + \int_E \phi_2 \; dm \leq \int_E f + g \; dm$$

so that

$$\int_E f \; dm + \int_E g \; dm \leq \int_E f + g \; dm$$

and linearity is established.

Finally, letting $h = g - f$, we obtain monotonicity from lineaerity. ■

Corollary. *If A and B are disjoint measurable subsets of E, then*

$$\int_{A \cup B} f = \int_A f + \int_B f$$

Proof. Botth $f\chi_A$ and $f\chi_B$ are bounded measurable functions, hence so is $f\chi_A + f\chi_B$. Moreover, since A and B are disjoint, we have $f\chi_{A\cup B} = f\chi_A + f\chi_B$. Now, observe that if E_1 is a measurble subset of E, then

$$\int_{E_1} f = \int_E f\chi_{E_1}$$

Therefore, we get

$$\int_{A\cup B} f = \int_E f\chi_{A\cup B} = \int_E f\chi_A + \int_E f\chi_B = \int_A f + \int_B f$$

■

Corollary.

$$\left|\int_E f\right| \leq \int_E |f|$$

Proof. We have that $|f|$ is a bounded measurable function, and hence is Lebesgue integrable. Notice also that $-|f| \leq f \leq |f|$, so that

$$-\int_E |f| \leq \int_E f \leq \int_E |f|$$

which gives us the result. ∎

Lemma 10.1.6. *If $\{f_n\}$ is a sequence of bounded measurable functions, converging uniformly to a real-valued function f, on a set E of finite measure, then*

$$\lim_{n\to\infty}\int_E f_n = \int_E f$$

Proof. Since $\{f_n\} \to f$ uniformly on E, then $\{f_n\} \to f$ pointwise on f, which makes f measurable. Moreover, since each f_n is bounded as well, f must also be bounded. Now, let $\varepsilon > 0$ and choose $N \in \mathbb{Z}^+$ for which

$$|f - f_n| < \frac{\varepsilon}{m(E)} \text{ on } E \text{ whenever } n \geq N$$

Then

$$\left|\int_E f - \int_E f + n\right| = \left|\int_E f - f_n\right| \leq \int_E |f - f_n| \leq \frac{\varepsilon}{m(E)} m(E) = \varepsilon$$

and we are done. ∎

Example 10.3. Let $n \in \mathbb{Z}^+$, and define f_n on $0, 1$ by

$$f_n(x) = \begin{cases} 0, & \text{if } x \geq \frac{2}{n} \\ n, & \text{if } x = \frac{1}{n} \\ 0, & \text{if } x = 0 \end{cases}$$

and let f be linear on $[0, \frac{1}{n}]$ and $[\frac{1}{n}, \frac{2}{n}]$. Observe then that $\int_0^1 f_n = 1$ for every $n \in \mathbb{Z}^+$. Now, define $f(x) = 0$ for all $x \in [0, 1]$, then observe that $\{f_n\} \to f$ pointwise as $n \to \infty$, however

$$\lim_{n\to\infty}\int_0^1 f_n \neq \int_0^1 f$$

That is, pointwise convergence of a sequence of functions is not enough for passage of a limit under the Lebesgue integral. In fact, lemma 10.1.6 tells us that the sequence needs to at least be unifomrly convergent.

Theorem 10.1.7 (The Bounded Convergence Theorem)**.** *Let $\{f_n\}$ be a uniformly pointwise bounded sequence of measurable functions on a set E of finite measure. If $\{f_n\}$ converges pointwise to a measurable function f on E, then*

$$\lim_{n\to\infty}\int_E f_n = \int_E f$$

Proof. Since $\{f_n\}$ is uniformly pointwise bounded, there is an $M \geq 0$ for which $|f_n| \leq M$ for all $n \in \mathbb{Z}^+$. Now, let A be a measurable subset of E, and choose an index $n \in \mathbb{Z}^+$. Then we habe

$$\int_E f_n - \int_E f = \int_E (f_n - f) = \int_E (f_n - f) + \int_{E \setminus A} f_n + \int_{E \setminus A} f$$

Thus

$$\Big| \int_E f_n - f \Big| \leq \int_A |f_n - f| + 2M + m(E \setminus A)$$

Now, let $\varepsilon > 0$. By Egoroff's theorem, $\{f_n\} \to f$ uniformly, on A and $m(E\setminus A) < \frac{\varepsilon}{4M}$, by uniform convergence, there is an $N \in \mathbb{Z}^+$ for which

$$|f_n - f| < \frac{\varepsilon}{2m(E)} \text{ on } A \text{ whenever } n \geq N$$

Thus,

$$\Big| \int_E f_n - f \Big| \leq \frac{\varepsilon}{2M(E)} m(A) + 2M + m(E \setminus A) < \varepsilon$$

and we are done. ■

10.2 The Lebesgue Integral of Measurable Nonnegative Functions

Definition. We say a measurable function f on a set E **vanishes** outside a set of finite measure provided there is a subset E_0 of E of finite measure for which $f(x) = 0$ for all $x \in E \setminus E_0$. We define the **support** of f to be the set

$$\operatorname{supp} f = \{x \in E : f(x) \neq 0\}$$

We say that f is of **finite support** if $m(\operatorname{supp} f)$ is finite.

Lemma 10.2.1. *A measurable function f on a set E vanishes outside a set of finite measure if, and only if it is of finite support.*

Proof. exercise ■

Definition. We define the **Lebesgue integral** of a nonnegative measurable function f on E to be

$$\int_E f \, dm = \sup \Big\{ \int_E h \, dm : h \text{ is bounded, of finite support, and } 0 \leq h \leq f \Big\}$$

Theorem 10.2.2 (Chebychev's Inequality)**.** *Let f be a nonnegative measurable function on a set E. Then for all $\lambda > 0$,*

$$\lambda m(E_\lambda) \leq \int_E f \, dm$$

where $E_\lambda = \{x \in E : f(x) \geq \lambda\}$.

Proof. Suppose first that E has infinite measure, and for $n \in \mathbb{Z}^+$, define $E_{\lambda,n} = E_\lambda \cap [-n, n]$. Define also $\psi_n = \lambda \chi_{E_{\lambda,n}}$. Then ψ is a bounded measurable function of finite support and for which

$$\lambda m(E_{\lambda,n}) = \int_E \psi$$

and $0 \leq \psi_n \leq f$. Then by the continuity of measure, we get

$$\lambda m(E_\lambda) = \lambda \lim_{n \longrightarrow \infty} m(E_{\lambda,n}) = \lim_{n \longrightarrow \infty} \int_E \psi \leq \int_E f$$

Now, suppose that $m(E)$ is finite, and define $h = \lambda \chi_{E_\lambda}$. Then h is a bounded measurable function of finite support, with $0 \leq h \leq f$, thus, by definition

$$\lambda m(E_\lambda) = \int_E h \leq \int_E f$$

■

Lemma 10.2.3. *Let f be a nonnegative measurable function on E. Then*

$$\int_E f = 0 \textit{ if, and only if } f = 0 \textit{ almost everywhere on } E$$

Proof. Suppose first that $\int_E f = 0$. By Chebychev's inequality, for any $n \in \mathbb{Z}^+$, define $E_{\frac{1}{n}} = \{x \in E : f(x) \geq \frac{1}{n}\}$ and $E_0 = \lim \{E_n\}$ as $n \to \infty$. By countable additivity, we get that $m(E_0) = 0$, which makes $f = 0$ on $E \backslash E_0$.

Conversely, suppose that $f = 0$ almost everywhere on E. Let ϕ be a simple function, and h a bounded measurable function of finite support. Then $0 \leq \phi \leq h \leq f$. Then by hypothesis, $\phi = 0$ almost everywhere on E, so that $\int_E \phi = 0$. This implies that $h = 0$ almost everywhere on E as well, so that $\int_E h = 0$, which implies that $\int_E f = 0$. ■

Theorem 10.2.4. *Let f and g be nonnegative measurable functions on a set E. Then for all $\alpha > 0$, $\beta > 0$*

$$\int_E \alpha f + \beta g \; dm = \alpha \int_E f \; dm + \beta \int_E g \; dm$$

Moreover, if $f \leq g$ on E, then

$$\int_E f \leq \int_E g$$

Proof. For $\alpha > 0$, we have that $0 \leq h \leq f$ if, and only if $0 \leq \alpha h \leq \alpha f$. Therefore, by the linearity of the Lebesgue integral for bounded functions, we have

$$\int_E \alpha f = \alpha \int_E f$$

Now, let h and k be bounded measurable functions of finite support, for which $0 \leq h \leq f$ and $0 \leq k \leq g$. Then $h + k$ is a bounded measurable function of finite support with $0 \leq h + k \leq f + g$. Hence

$$\int_E (h + k) = \int_E h + \int_E k \leq \int_E f + g$$

That is, by definition,

$$\int_E f + \int_E g \leq \int_E (f+g)$$

Now, by definition of $\int_E (f+g)$ as a least upper bound on integrals of nonnegative bounded measurable functions of finite support, less than or equal to $f+g$, let l be a bounded measurable function of finite support, for which $0 \leq l \leq f+g$. Now, define

$$\begin{aligned} h &= \min\{f, l\} \\ k &= l - f \end{aligned}$$

on E. Then for $x \in E$, if $l(x) \leq f(x)$, $k(x) = 0 \leq g(x)$. If $l(x) > f(x)$, then $k(x) = l(x) - f(x) \leq g(x)$, in either case, $0 \leq k \leq g$. Now, we have that both h and k are bounded measurable functions of finite support, in which $0 \leq h \leq f$ and $0 \leq k \leq g$. so that

$$\int_E l = \int_E h + \int_E k \leq \int_E f + \int_E g$$

which establishes the equality.

Finally, if h is a bounded measurable function of finite support, for which $00 \leq h \leq f$, then if $f \leq g$, we get $0 \leq h \leq g$ so that

$$\int_E h \leq \int_E g$$

■

Theorem 10.2.5. *Let f be a nonnegative measurable function on a set E and let A and B be disjoint measurable subsets of E. Then*

$$\int_{A\cup B} f\ dm = \int_A f\ dm + \int_B f\ dm$$

Moreover, if E_0 is a subset of E of measure $m(E_0) = 0$, then

$$\int_E f = \int_{E\setminus E_0} f$$

Lemma 10.2.6 (Fatou's Lemma)**.** *Let $\{f_n\}$ be a sequence of nonnegative measurable functions on a set E, converging pointwise, almost eveyrhwere to a nonnegative measurable function f on E. Then*

$$\int_E f \leq \liminf \int_E f_n$$

Proof. Suppose, without loss of generality that $\{f_n\} \to f$ pointwise on all of E. Then f is nonnegative and measurable since each f_n is nonnegative and measurable. Now, let h be a bounded measurable function of finite support for which $0 \leq h \leq f$ on E; and choose an $M \geq 0$ for which $|h| \leq M$ on E. Since h is of finite support we have $m(E_0) = 0$, where $E_0 = \operatorname{supp} h$. Let $n \in \mathbb{Z}^+$, and define h_n on E by $h_n = 0$ on $E\setminus E_0$. Then for any $x \in E$, since $h \leq f$, and $\{f_n\} \to f$, then $\{h_n\} \to h$ pointwise on E. By the bounded convergence theorem, applied to the uniformly bounded sequence $\{h_n|_{E_0}\}$ we observe that

$$\lim_{n\to\infty} \int_E h_n = \lim_{n\to\infty} \int_{E_0} h_n = \int_{E_0} h = \int_E h$$

Now, observe that $h_n \leq f_n$, and by definition,

$$\int_E h_n \leq \int_E f$$

thus

$$\int_E h = \lim_{n\to\infty} \int_E h_n \leq \liminf \int_E f_n$$

■

Example 10.4. (1) Define $f_n = n\chi_{(0,\frac{1}{n})}$. Then $\{f_n\} \to f = 0$ pointwise on $(0, 1]$, but

$$\int_E f = 0 < 1 = \lim_{n\to\infty} \int_E f_n$$

(2) Define $g_n = \chi_{(n,n+1)}$. Then $\{g_n\} \to 0$ pointwise on all of $\mathbb{R}$, however

$$\int_E g = 0 < 1 = \lim_{n\to\infty} \int_E g_n$$

Theorem 10.2.7 (The Monotone Convergence Theorem)**.** *Let $\{f_n\}$ be an increasing sequence of nonnegative measurable functions on a set E converging pointwise almost everywhere to a nonnegative measurable function f on E. Then*

$$\lim_{n\to\infty} \int_E f = \int_E f$$

Proof. By Fatou's lemma, we have $\int_E f \leq \liminf \int_E f$. Now, for any $n \in \mathbb{Z}^+$, $f_n \leq f$ almost everywhere on E, so by monotonicity

$$\int_E f_n \leq \int_E f$$

Therefore

$$\limsup \int_E f_n \leq \int_E f$$

which establishes the equality. ■

Corollary. *Let $\{u_n\}$ be a sequence of nonnegative functions on a set E such that*

$$f = \sum u_n \text{ almost everywhere on } E$$

Then

$$\int_E f = \sum \int_E u_n$$

Definition. We call a nonnegative measurable function f on a set E **Lebesgue integrable** on E provided that

$$\int_E f \text{ is finite}$$

Lemma 10.2.8. *If f is a nonnegative measurable function on a set E, Lebesgue integrable on E, then f is finite almost everywhere on E.*

Proof. Let $n \in \mathbb{Z}^+$, and define $E_n = \{x \in E : f(x) \geq n\}$, and $E_\infty = \lim \{E_n\}$ as $n \to \infty$. Then, by Chebychev's inequality, we have

$$nm(E_\infty) \leq nm(E_n) \leq \int_E f$$

and since $\int_E f$ is finite, we get $m(E_\infty) = 0$. ∎

Lemma 10.2.9 (Beppo Levi's Lemma)**.** *Let $\{f_n\}$ be an increasing sequence of nonnegative meaurable functions on a set E. If the sequence of integrals $\{\int_E f_n\}$ is bounded, then $\{f_n\}$ converges pointwise to a nonnegative measurable function f, finite almost everywhere on E, and*

$$\lim_{n \to \infty} \int_E f_n = \int_E f$$

Proof. Since $\{f_n\}$ is a sequence of extended real-valued functions on E, define the extended real-valued function f, pointwise on E, by

$$f(x) = \lim_{x \to \infty} f(x)$$

By the monotone convergence theorem, we get the $\{\int_E f_n\} \to \int_E f$, and since the sequence $\{\int_E f_n\}$ is bounded, $\int_E f$ is finite. By lemma 10.2.8, this makes f Lebesgue integrable on E, and hence finite almost everywhere on E. ∎

10.3 The General Lebesgue Integral

Definition. Let f be an extended real-valued function on a set E. We define the **postive part** of f to be the function

$$f^+(x) = \max \{f(x), 0\}$$

and we define the **negative part** of f to be the function

$$f^-(x) = \max \{-f(x), 0\}$$

Lemma 10.3.1. *An extended real-valued function f on a set E is measurable if, and only if f^+ and f^- are measurable. Moreover f^+ and f^- are nonnegative.*

Lemma 10.3.2. *Let f be a measurable function on a set E. Then f^+ and f^- are Lebesgue integrable over E if, and only if $|f|$ is Lebesgue integrable over E.*

Proof. Notice that $|f|- = f^+ + f^-$. Hence if f^+ and f^- are Lebesgue integrable, then so is $|f|$. Moreover, if $|f|$ is Lebesgue integrable, then observe that $0 \leq f^+ \leq |f|$ and $0 \leq f^- \leq |f|$. By the monotonicity of the Lebesgue integral for nonnegative measurable functions, f^+ and f^- are integrable. ∎

Definition. We call a measurable function f on a set E **Lebesgue integrable** on E provided that $|f|$ is Lebesgue integrable on E. We define the **Lebesgue integral** of f on E to be

$$\int_E f\ dm = \int_E f^+\ dm - \int_E f^-\ dm$$

Lemma 10.3.3. *Let f be a measurable function on a set E. If f is Lebesgue integrable on E, then f is finite almost everywhere on E, and for any $E_0 \subseteq E$, with $m(E_0) = 0$*

$$\int_E f = \int_{E \setminus E_0} f$$

Proof. If f is integrable on E, then so is $|f|$ by definition. That makes $|f|$ finite alomst everywhere on E, and hence f must also be finite almost everywhere on E. ∎

Definition. We say that a real-valued function f on a set E is **dominated** by a real-valued function g on E if $|f| \leq g$ on E.

Theorem 10.3.4 (The Integral Comparison Test)**.** *Let f be a measurable function on a set E, and let g be a nonnegative Lebesgue integrable function dominating f. Then f is integrable and*

$$\left| \int_E f \right| \leq \int_E |f|$$

Proof. By the monotonicity of the Lebesgue integral for nonnegative measurable functions, if $|f| \leq g$, and g is integrable, then $|f|$ is integrable, which makes f integrable. Now, we also have that

$$\left| \int_E f \right| = \left| \int_E f^+ - \int_E f^- \right| \leq \int_E f^+ + \int_E f^- = \int_E |f|$$

∎

Theorem 10.3.5. *Let f and g be measurable functions on a set E. If f and g are integrable, then for any $\alpha, \beta \in \mathbb{R}$, $\alpha f + \beta g$ is Lebesgue integrable, and*

$$\int_E \alpha f + \beta g\ dm = \alpha \int_E f\ dm + \beta \int_E g\ dm$$

Moreover, if $f \leq g$ on E, then

$$\int_E f \leq \int_E g$$

Proof. If $\alpha > 0$, then $(\alpha f)^+ = \alpha f^+$ and $(\alpha f)^- = \alpha f^-$. If $\alpha < 0$, then $(\alpha f)^+ = -\alpha f^-$ and $(\alpha f)^- = -\alpha f^+$. In either case, since f is integrable, we observe that $\int_E \alpha f = \alpha \int_E f$.

Now, we also have that $|f|$ and $|g|$ are integrable so that $|f| + |g|$ is integrable, and since $|f + g| \leq |f| + |g|$, by the monotonicity of the Lebesgue integral for nonnegative functions,

$|f+g|$ is integrable, which makes $f+g$ integrable. Moreover, f and g are finite almost everywhere on E, hence so is $f+g$. Excise a set of measure zero from E and suppose that f and g are finite on all of E. Notice that

$$(f+g)^+ - (f+g)^- = f+g = (f^+ - f^-) + (g^+ - g^-)$$

so that

$$(f+g)^+ + f^- + g^- = (f+g)^- + f^+ + g^+$$

Now, since f and g, and $f+g$ are integrable, the integrals of these functions are finite, so that we obtain

$$\int_E f+g\ dm = \int_E f\ dm + \int_E g\ dm$$

Now, suppose that $f \leq g$ on E, and define $h = g-f$. Then h is a nonnegative measurable function, so that

$$\int_E g - \int_E f = \int_E (g-f) = \int_E h \geq 0$$

■

Corollary. *Let f be a Lebesgue integrable function on a set E, and let A and B disjoint measurable subsets of E. Then*

$$\int_{A\cup B} f\ dm = \int_A f\ dm + \int_B f\ dm$$

Theorem 10.3.6 (The Lebesgue Dominated Convergence Theorem). *Let $\{f_n\}$ a sequence of measurable functions on a set E, converging pointwise to a measurable function f on E. If g is a Lebesgue integrable function on E, dominating each f_n, then f is Lebesgue integrable on E, and*

$$\lim_{n\to\infty} \int_E f_n = \int_E f$$

Proof. Since $|f_n| \leq g$ for all $n \in \mathbb{Z}^+$, we have that $|f| \leq g$, and since g is integrable, we get $|f|$ integrable, and hence f must be integrable. Now, excise a countable collection of sets of measure zero from E, and by countable additivity, suppose that f is finite on E and f_n is finite on E for each $n \in \mathbb{Z}^+$. Then $g-f$ is a well defined function, and so is $g-f_n$, for all $n \in \mathbb{Z}^+$. Moreover, they are nonnegative measurable functions, and $\{g-f_n\} \to g-f$ pointwise almost everywhere on E. By Fatou's lemma

$$\int_E f+g\ dm \leq \liminf \int_E g - f_n\ dm$$

and by linearity

$$\int_E g\ dm - \int_E f\ dm \leq \int_E g\ dm - \limsup \int_E f_n\ dm$$

So that

$$\limsup \int_E f_n \leq \int_E f$$

Now, making the same argument with the sequence $\{g + f_n\}$ yields

$$\int_E f \leq \liminf \int_E f_n$$

which gives the equality. ■

Theorem 10.3.7 (The General Dominated Convergence Theorem). *Let $\{f_n\}$ a sequence of measurable functions on a set E converging pointwise almost everywhere on E. Suppose there is a sequence $\{g_n\}$ of nonnegative measurable functions on E, such that each g_n dominates each f_n, and converging pointwise almost everywhere to g on E. Then if*

$$\lim_{n\to\infty} \int_E g_n = \int_E g \text{ is finite}$$

then

$$\lim_{n\to\infty} \int_E f_n = \int_E f$$

10.4 Countable Additivity and Continuity of the Lebesgue Integral

Theorem 10.4.1 (Countable Additivity of The Lebesgue Integral). *Let f be a Lebesgue integrable function on a set E, and let $\{E_n\}$ be a disjoint cover of E by countable many measurable subsets of E. Then*

$$\int_E f \, dm = \sum_{n=0}^{\infty} \int_{E_n} f \, dm$$

Proof. For $n \in \mathbb{Z}^+$, define $f_n = f\chi_n$, where χ_n is the characteristic function on the measurable union $\bigcup_{k=1}^n E_k$. Then f_n is a measurable function on E, and $|f_n| \leq |f|$. Moreover, notice that $\{f_n\} \to f$, so by the Lebesgue dominated convergence theorem

$$\lim_{n\to\infty} \int_E f_n = \int_E f$$

Moreover, since the collection $\{E_n\}$ is disjoint

$$\int_E f_n = \sum_{k=1}^{n} \int_{E_k} f$$

so taking limits on boths sides gives us

$$\int_E f = \sum_{n=1}^{\infty} \int_{E_n} f$$

■

Before we prove the continuity of integration of the Lebesgue integral, we prove a prelimanry result.

Lemma 10.4.2. *Let f be a Lebesgue integrable function on a set E. If A is a measurable subset of E, then*

$$\int_{E\backslash A} f\ dm = \int_E f\ dm - \int_A f\ dm$$

Proof. Notice that $E = A \cup E\backslash A$, so that

$$\int_E f = \int_A f + \int_{E\backslash A} f$$

■

Theorem 10.4.3 (Continuity of The Lebesgue Integral)**.** *If f is a Lebesgue integrable function on a set E, then the following are true*

(1) If $\{E_n\}$ is an ascending collection of measurable subsets of E, then

$$\int_{\bigcup E_n} f\ dm = \lim_{n\to\infty} \int_{E_n} f\ dm$$

(2) If $\{E_n\}$ is a descending collection of measurable subsets of E, then

$$\int_{\bigcap E_n} f\ dm = \lim_{n\to\infty} \int_{E_n} f\ dm$$

Proof. The proof proceeds identically as in theorem 8.5.2. Define $E_0 = \emptyset$, and $A_{k+1} = E_{k+1}\backslash E_k$. Since $\{E_n\}$ is an ascending collection of measurabel sets, then $\{A_n\}$ is a disjoint countable collection of measurable sets, and $E = \bigcup A_n$, then by the countable additivity of the Lebesgue measure,

$$\int_E f = \sum \int_{A_k} f = \sum \int_{E_{k+1}\backslash E_k} f = \sum \Big(\int_{E_{k+1}} f - \int_{E_k} f \Big)$$

Taking limits

$$\int_E f = \lim_{n\to\infty} \sum_{k=0}^{n-1} \Big(\int_{E_{k+1}} f - \int_{E_k} f \Big) = \lim_{n\to\infty} \Big(\int_{E_n} f - \int_{E_0} f \Big) = \lim_{n\to\infty} \int_{E_n} f$$

since $\int_{E_0} f = 0$.

Now, suppose that $\{E_n\}$ is a descending collection. Then define the countable collection of measurable sets $\{B_k\}$ by $B_k = E_1\backslash E_k$ for all $k \in \mathbb{Z}^+$. Then $\{B_k\}$ is an ascending collection of measurable sets, and by DeMorgan's laws, we have

$$\bigcup B_k = E_1 \backslash \bigcap E_k$$

Therefore,

$$\int_{\bigcup B_k} = \int_{E_1} - \int_{\bigcap E_k} f$$

On the otherhand, by the above argument for ascending collections of measurable subset of E,

$$\int_{\bigcup B_k} f = \int_{E_1} f - \lim_{k \longrightarrow \infty} \int_{E_k} f$$

Equating the two gives us the result. ■

Bibliography

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