

# Measure Theory

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**Text**

Real Analysis (4<sup>th</sup> edition)

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# Chapter 1

## The Real Numbers

### 1.1 Open Sets, and $\sigma$ -Algebras

**Definition.** We call a set  $U$  of real numbers **open** provided for any  $x \in U$ , there is an  $r > 0$  such that  $(x - r, x + r) \subseteq U$ .

**Lemma 1.1.1.** *The set of real numbers  $\mathbb{R}$ , together with open sets defines a topology on  $\mathbb{R}$ .*

*Proof.* Notice that both  $\mathbb{R}$  and  $\emptyset$  are open sets. Moreover, if  $\{U_n\}$  is a collection of open sets, then so is their union. Now, consider the finite collection  $\{U_k\}_{k=1}^n$  and let  $U = \bigcap_{k=1}^n U_k$ . If  $U$  is empty, we are done. Otherwise, let  $x \in U$ . Then  $x \in U_k$  for every  $1 \leq k \leq n$ , and since each  $U_k$  is open, choose an  $r_k > 0$  for which  $(x - r_k, x + r_k) \subseteq U_k$ . Then let  $r = \min\{r_1, \dots, r_n\}$ . Then  $r > 0$ , and we have  $(x - r, x + r) \subseteq U$ , which makes  $U$  open in  $\mathbb{R}$ . ■

**Lemma 1.1.2.** *Every nonempty set is the disjoint union of a countable collection of open sets.*

*Proof.* Let  $U$  be nonempty and open in  $\mathbb{R}$ . Let  $x \in U$ . Then there is a  $y > x$  for which  $(x, y) \subseteq U$  and there is a  $z < x$  for which  $(z, x) \subseteq U$ . Now, let  $a_x = \inf\{z : (z, x) \subseteq U\}$  and  $b_x = \sup\{y : (x, y) \subseteq U\}$ , and let  $I_x = (a_x, b_x)$ . Then we have

$$x \in I_x \text{ and } a_x \notin I_x \text{ and } b_x \notin I_x$$

Let  $w \in I_x$  such that  $x < w < b_x$ . Then there is a  $y > w$  such that  $(x, y) \subseteq U$  so that  $w \in U$ . Now, if  $b_x \in U$ , then there is an  $r > 0$  for which  $(b_x - r, b_x + r) \subseteq U$ , in particular,  $(x, b_x + r) \subseteq U$ . But  $b_x$  is the least upperbound of all such numbers, and  $b_x < b_x + r$ , a contradiction. Thus  $b_x \notin U$ , and hence  $b_x \notin I_x$ . A similar argument shows that  $a_x \notin I_x$ .

Consider now the collection  $\{I_x\}_{x \in U}$ . Then  $U = \bigcup I_x$  and since  $a_x, b_x \notin I_x$  for each  $x$ , the collection  $\{I_x\}$  is a disjoint collection. Lastly, by the density of  $\mathbb{Q}$  in  $\mathbb{R}$  there is a 1-1 mapping between this collection and  $\mathbb{Q}$ , making it countable. ■

**Definition.** Let  $E \subseteq \mathbb{R}$  a set. We call a point  $x \in \mathbb{R}$  a **point of closure** of  $E$  if every open interval containing  $x$  also contains a point of  $E$ . We call the collection of all such points the **closure** of  $E$ , and denote it  $\text{cl } E$ . If  $E = \text{cl } E$ , then we say that  $E$  is **closed**.

**Lemma 1.1.3.** *For any set  $E$  of real numbers,  $\text{cl } E$  is closed; i.e.  $\text{cl } E = \text{cl}(\text{cl } E)$ . Moreover,  $\text{cl } E$  is the smallest closed set containing  $E$ .*

**Lemma 1.1.4.** *Every set  $E$  of real numbers is open if, and only if  $\mathbb{R} \setminus E$  is closed.*

**Definition.** Let  $E \subseteq \mathbb{R}$  a set. We call a collection  $\{E_\lambda\}$  a **cover** of  $E$  if  $E \subseteq \bigcup E_\lambda$ . If each  $E_\lambda$  is open, then we call this collection an **open cover** of  $E$ .

**Theorem 1.1.5** (Heine-Borel). *For any closed and bounded set  $F$  of  $\mathbb{R}$ , every open cover of  $F$  has a finite subcover.*

*Proof.* Suppose first that  $F = [a, b]$ , for  $a \leq b$  real numbers. Then  $F$  is closed and bounded. Let  $\mathcal{F}$  be an open cover of  $[a, b]$ , and define  $E = \{x \in [a, b] : [a, x] \text{ has a finite subcover by sets in } \mathcal{F}\}$ . Notice that  $a \in E$ , so that  $E$  is nonempty. Now, since  $E$  is bounded by  $b$ , by the completeness of  $\mathbb{R}$ , let  $c = \sup \{E\}$ . Then  $c \in [a, b]$  and there is a set  $U \in \mathcal{F}$  with  $c \in U$ . Since  $U$  is open, there exists an  $\varepsilon > 0$  such that  $(c - \varepsilon, c + \varepsilon) \subseteq U$ . Now,  $c - \varepsilon$  is not an upperbound of  $E$ , so there is an  $x \in E$  with  $c - \varepsilon < x$ , and a finite collection of open sets  $\{U_i\}_{i=1}^k$  covering  $[a, x]$ . Then the collection  $\{U_i\}_{i=1}^k \cup U$  covers  $[a, x]$  so that  $c = b$ , and we have found a finite subcover of  $F$ .

Now, let  $F$  be closed and bounded. Then it is contained in a closed bounded interval  $[a, b]$ . Now, let  $U = \mathbb{R} \setminus F$  open and  $\mathcal{F}$  an open cover of  $F$ . Let  $\mathcal{F}' = \mathcal{F} \cup U$ . Since  $\mathcal{F}$  covers  $F$ ,  $\mathcal{F}'$  covers  $[a, b]$ . By above, there is a finite subcover of  $[a, b]$ , and hence of  $F$  by sets in  $\mathcal{F}'$ . Remove  $U$  from  $\mathcal{F}'$ , we get a finite subcover of  $F$  by sets in  $\mathcal{F}$ . ■

**Theorem 1.1.6** (The Nested Set Theorem). *Let  $\{F_n\}$  be a descending collection of nonempty closed sets of  $\mathbb{R}$ , for which  $F_1$  is bounded. Then*

$$\bigcap F_n \neq \emptyset$$

*Proof.* Let  $F = \bigcap F_n$ , and suppose to the contrary that  $F$  is empty. Then for all  $x \in \mathbb{R}$ , there is an  $n \in \mathbb{Z}^+$  for which  $x \notin F_n$ . So that  $x \in U_n = \mathbb{R} \setminus F_n$ . Then  $U_n = \mathbb{R}$ , and each  $U_n$  is open. So  $\{U_n\}$  is an open cover of  $\mathbb{R}$ , and hence  $F_1$ . By the theorem of Heine-Borel, there is an  $N > 0$  such that  $F \subseteq \bigcup_{n=1}^N U_n$ . Since  $\{F_n\}$  is descending, the collection  $\{U_n\}$  is ascending, and hence  $\bigcup U_n = U_N = \mathbb{R} \setminus F_N$  which makes  $F_1 \subseteq \mathbb{R} \setminus F_N$ , a contradiction. ■

**Definition.** Let  $X$  be a set. We call a collection  $\mathcal{A}$  of subsets of  $X$   **$\sigma$ -algebra** if

- (1)  $\emptyset \in \mathcal{A}$ .
- (2) For any  $A \in \mathcal{A}$ ,  $X \setminus A \in \mathcal{A}$ .
- (3) If  $\{A_n\}$  is a countable collection of elements of  $\mathcal{A}$ , then their union is an element of  $\mathcal{A}$ .

**Lemma 1.1.7.** *Let  $\mathcal{F}$  a collection of subsets of a set  $X$ . The intersection of all  $\sigma$ -algebras containing  $\mathcal{F}$  is a  $\sigma$ -algebra. Moreover, it is the smallest such  $\sigma$ -algebra.*

**Definition.** We define the **Borel sets** of  $\mathbb{R}$  to be the  $\sigma$ -algebra of  $\mathbb{R}$  containing all open sets in  $\mathbb{R}$ .

**Lemma 1.1.8.** *Every closed set of  $\mathbb{R}$  is a Borel set.*

**Definition.** We call a countable intersection of open sets of  $\mathbb{R}$  a  **$G_\delta$ -set** and we call a countable union of closed sets of  $\mathbb{R}$  an  **$F_\sigma$ -set**.

## 1.2 Sequences of Real Numbers

**Definition.** A sequence  $\{a_n\}$  of real numbers is said to **converge** to a point  $a$ , if, for every  $\varepsilon > 0$ , there is an  $N > 0$  such that

$$|a - a_n| < \varepsilon \text{ whenever } n \geq N$$

We call  $a$  the **limit** of  $\{a_n\}$  and write  $\{a_n\} \rightarrow a$ , or

$$\lim_{n \rightarrow \infty} \{a_n\} = a$$

**Lemma 1.2.1.** *Let  $\{a_n\} \rightarrow a$  a sequence of real numbers converging to  $a \in \mathbb{R}$ . Then the limit of  $\{a_n\}$  is unique,  $\{a_n\}$  is bounded, and for any  $c \in \mathbb{R}$ , if  $a_n \leq c$  for all  $n$ , then  $a \leq c$ .*

**Theorem 1.2.2** (The Monoton C Vonvergence Theorem). *A monotone sequence of real numbers converges to a point if, and only if it is bounded.*

*Proof.* Without loss of generality, suppose that the sequence  $\{a_n\}$  is increasing. If  $\{a_n\} \rightarrow a$ , by lemma 1.2.1,  $\{a_n\}$  is bounded. On the otherhand, suppose that  $\{a_n\}$  is bounded. Let  $S = \{a_n : n \in \mathbb{Z}^+\}$ , then by the completeness of  $\mathbb{R}$ , let  $a = \sup S$ . Let  $\varepsilon > 0$ . Notice that  $a_n \leq a$  for all  $n$ . Now, since  $a - \varepsilon$  is not an upperbound, there exists an  $N > 0$  for which  $a_N > a - \varepsilon$ , then since  $\{a_n\}$  is increasing,  $a_n > a - \varepsilon$  whenever  $n \geq N$ . So we get

$$|a - a_n| < \varepsilon \text{ whenever } n \geq N$$

Which makes  $\{a_n\} \rightarrow a$ . ■

**Theorem 1.2.3** (Bolzano-Weierstrass). *Every bounded sequence has a convergent subsequence.*

*Proof.* Let  $\{a_n\}$  be a bounded sequence, and let  $M > 0$  such that  $|a_n| \leq M$  for all  $n \in \mathbb{Z}^+$ . Define  $E_n = \text{cl}\{a_j : j \geq n\}$ . Then  $E_n \subseteq [-M, M]$ . Thus  $\{E_n\}$  is a decreasing sequence of closed, bounded, and nonempty sets of  $\mathbb{R}$ . By the nested set theorem, the intersection  $E = \bigcap E_n$  is nonempty. Choose an  $a \in E$ . Then for every  $k \in \mathbb{Z}^+$ ,  $a$  is a point of closure of the set  $\{a_j : j \geq k\}$ . SO that  $a_j \in (a - \frac{1}{k}, a + \frac{1}{k})$  whenever  $j \geq k$ . By induction, construct a strictly increasing sequence  $\{n_k\}$  of natural numbers for which  $|a - a_{n_k}| < \frac{1}{k}$ . Then by the principle of Archimedes,  $\{a_{n_k}\} \rightarrow a$ , and we have a convergent subsequence. ■

**Definition.** We call a sequence  $\{a_n\}$  **Cauchy** if for every  $\varepsilon > 0$ , there is an  $N > 0$  for which

$$|a_m - a_n| < \varepsilon \text{ whenever } m, n \geq N$$

**Theorem 1.2.4** (The Cauchy Convergence Criterion). *A sequence of real numbers converges if, and only if it is Cauchy.*

*Proof.* Suppose that the sequence  $\{a_n\} \rightarrow a$  converges to  $a \in \mathbb{R}$ . Then for any  $m, n \in \mathbb{Z}^+$ , notice that  $|a_m - a_n| \leq |a_m - a| + |a - a_n|$ . Let  $\varepsilon > 0$  and choose  $N > 0$  such that  $|a - a_n| < \frac{\varepsilon}{2}$ , and  $|a_m - a| < \frac{\varepsilon}{2}$ . Then if  $n, m \geq N$ , we get  $|a_m - a_n| < \varepsilon$ , which makes  $\{a_n\}$  Cauchy.

Conversely, suppose that  $\{a_n\}$  is Cauchy. Let  $\varepsilon = 1$  and choose  $N > 0$  such that if  $m, n \geq N$ , then  $|a_m - a_n| < 1$ . Then we get  $|a_n| \leq 1 + |a_N|$  for all  $n \geq N$ . Define  $M = 1 + \max\{|a_1|, \dots, |a_N|\}$ . Then  $|a_n| \leq M$  for all  $n$ . This makes  $\{a_n\}$  bounded. By the theorem of Bolzano-Weierstrass,  $\{a_n\}$  has a convergent subsequence  $\{a_{n_k}\} \rightarrow a$ . Let  $\varepsilon > 0$ , since  $\{a_n\}$  is Cauchy, choose an  $N > 0$  such that  $|a_m - a_n| < \frac{\varepsilon}{2}$  whenever  $n, m \geq N$ . Likewise, we get  $|a - a_{n_k}| < \frac{\varepsilon}{2}$  and  $n_k \geq N$ . Thus we observe that  $|a_n - a| \leq |a_n - a_{n_k}| + |a - a_{n_k}| < \varepsilon$  and so  $\{a_n\} \rightarrow a$ . ■

**Theorem 1.2.5.** *Let  $\{a_n\} \rightarrow a$  and  $\{b_n\} \rightarrow b$  be convergent sequences. Then for any  $\alpha, \beta \in \mathbb{R}$ , we have that the sequence  $\{\alpha a_n + \beta b_n\}$  converges and that*

$$\lim_{n \rightarrow \infty} \{\alpha a_n + \beta b_n\} = \alpha a + \beta b$$

**Definition.** We say a sequence  $\{a_n\}$  of real numbers **converges to infinity**  $\infty \in \mathbb{R}_\infty$  if for every  $c \in \mathbb{R}$ , there is an  $N > 0$  such that  $a_n \geq c$  whenever  $n \geq N$ . We write  $\{a_n\} \rightarrow \infty$ , or

$$\lim_{n \rightarrow \infty} \{a_n\} = \infty$$

**Definition.** Let  $\{a_n\}$  be a sequence of real numbers. We define the **limit superior** of  $\{a_n\}$  to be

$$\limsup \{a_n\} = \lim_{n \rightarrow \infty} (\sup \{a_k : k \geq n\})$$

Similarly, we define the **limit inferior** of  $\{a_n\}$  to be

$$\liminf \{a_n\} = \lim_{n \rightarrow \infty} (\inf \{a_k : k \geq n\})$$

**Theorem 1.2.6.** *For any sequences  $\{a_n\}$  and  $\{b_n\}$  of real numbers, the following are true:*

- (1)  $\limsup \{a_n\} = l \in \mathbb{R}_\infty$  if, and only if for every  $\varepsilon > 0$ , there exists infinitely many  $n \in \mathbb{Z}^+$  such that  $a_n > l - \varepsilon$  and finitely many  $n \in \mathbb{Z}^+$  for which  $a_n > l + \varepsilon$ .
- (2)  $\limsup \{a_n\} = \infty$  if, and only if  $\{a_n\}$  is not bounded above.
- (3)  $\limsup \{a_n\} = -\liminf \{-a_n\}$
- (4)  $\{a_n\} \rightarrow a \in \mathbb{R}_\infty$  if, and only if  $\limsup \{a_n\} = \liminf \{a_n\}$ .
- (5) If  $a_n \leq b_n$  for all  $n$ , then  $\limsup \{a_n\} \leq \limsup \{b_n\}$ .

**Definition.** Let  $\{a_n\}$  a sequence of real numbers. We call the series  $\sum_{k=1}^{\infty} a_k$  **summable** if the sequence of partial sums  $\{s_n = \sum_{k=1}^n a_k\} \rightarrow s$  converges to a point  $s \in \mathbb{R}$ .

**Lemma 1.2.7.** *Let  $\{a_n\}$  a sequence of real numbers. Then the following are true.*

- (1) The series  $\sum a_k$  is summable if, and only if for every  $\varepsilon > 0$ , there is an  $N > 0$  such that

$$\left| \sum_{k=n}^{n+m} a_k \right| < \varepsilon \text{ for all } m \in \mathbb{Z}^+ \text{ whenever } n \geq N$$

- (2) If  $\sum |a_k|$  is summable, then so is  $\sum a_k$ .
- (3) If  $a_k \geq 0$ , then  $\sum a_k$  is summable if, and only if the sequence of partial sums  $\{s_n\}$  is bounded.



## 1.3 Continuous Functions of a Real Variable.

**Definition.** A realvalued function  $f$  on a domain  $E$  is said to be **continuous** at a point  $x \in E$  provided for any  $\varepsilon > 0$  there is a  $\delta > 0$  for which

$$|f(x) - f(y)| < \varepsilon \text{ whenever } |x - y| < \delta \text{ for any } y \in E$$

We call  $f$  **continuous** on  $E$  if it is continuous at every point in  $E$ . We call  $f$  **Lipschitz continuous** if there is a  $c \geq 0$  for which

$$|f(x) - f(y)| \leq c|x - y| \text{ for all } x, y \in E$$

**Lemma 1.3.1.** *A Lipschitz continuous function on a domain is continuous on that domain.*

**Lemma 1.3.2** (The Sequential Criterion). *A realvalued function  $f$  defined on a domain  $E$  is continuous at a point  $x \in E$  if, and only if for any ssequence  $\{x_n\} \rightarrow x$  of points in  $E$ , converging to  $x$ , that the sequence  $\{f(x_n)\} \rightarrow f(x)$  converges to  $f(x)$ .*

**Theorem 1.3.3** (The Extreme Value Theorem). *A continuous realvalued function defined on a nonempty, closed and bounded domain takes on a maximum value, and a minimum value on that domain.*

*Proof.* Let  $f$  be a continuous realvalued function defined on the domain  $E$ , where  $E$  is nonempty, closed, and bounded. Let  $x \in E$  and  $\delta > 0$  and  $\varepsilon = 1$ . Define the open interval  $I_x = (x - \delta, x + \delta)$ . Then if  $y \in E \cap I_x$ , then  $|f(x) - f(y)| < 1$ . So that  $|f(y)| \leq |f(x)| + 1$ . Notice also that the collection  $\{I_x\}$  is an open cover of  $E$ . By the theorem of Heine-Borel, there is a finite subcover of  $E$ ,  $\{I_{x_k}\}_{k=1}^n$ . Define, then,  $M = 1 + \max \{|f(x_1)|, \dots, |f(x_n)|\}$ . Then we get that  $|f(x)| \leq M$  and  $f$  is bounded.

Now, let  $m = \sup f(E)$ . If  $f$  does not take the value  $m$  for any points in  $E$ , then the function  $x \rightarrow \frac{1}{f(x)-m}$  is a continuous unbounded function on  $E$ ; which is impossible. So there is an  $x \in E$  with  $f(x) = m$  and  $m$  is a maximum value. Now, since  $f$  is continuous, then so is  $-f$ , and hence  $-m$  defines a minimum value on  $f$ . ■

**Theorem 1.3.4** (The Intermediate Value Theorem). *If  $f$  is a continuous realvalued function on a closed bounded interval  $[a, b]$ , for which  $f(a) < c < f(b)$ , then there exists an  $x_0 \in (a, b)$  for which  $f(x_0) = c$ .*

*Proof.* Define  $a_1 = a$  and  $b_1 = b$  and let  $m_1$  be the midpoint of the interval  $[a_1, b_1]$ . If  $c < f(m_1)$ , define  $a_2 = a_1$  and  $b_2 = m_1$ , otherwise define  $a_2 = m_1$  and  $b_2 = b_1$ , so that in either case we get  $f(a_2) \leq c \leq f(b_2)$  and  $b_2 - a_2 = \frac{b-a}{2}$ . By induction, construct the collection of closed bounded intervals  $\{[a_n, b_n]\}$  such that  $f(a_n) \leq c \leq f(b_n)$  and  $b_n - a_n = \frac{b-a}{2^{n-1}}$ . This collection is a descending collection, so by the nested set theorem, the intersection  $I = \bigcap [a_n, b_n]$  is nonempty. Choose an  $x_0 \in I$ , and observe that

$$|a_n - x_0| \leq b_n - a_n = \frac{b-a}{2^{n-1}}$$

So the sequence  $\{a_n\} \rightarrow x_0$ . By the sequential criterion, since  $f$  is continuous at  $x_0$ , we get the sequence  $\{f(a_n)\} \rightarrow f(x_0)$ . Since  $f(a_n) \leq c$ , and  $(-\infty, c]$  is closed, we also get  $f(x_0) \leq c$ .

By similar reasoning to the argument provided above, we also get that  $f(x_0) \geq c$  so that equality is established. ■

**Definition.** A realvalued function  $f$  on a domain  $E$  is said to be **uniformly continuous** if for every  $\varepsilon > 0$  there is a  $\delta > 0$  such that

$$|f(x) - f(y)| < \varepsilon \text{ whenever } |x - y| < \delta \text{ for all } x, y \in E$$

**Lemma 1.3.5.** *If  $f$  is a uniformly continuous function on a domain  $E$ , then it is continuous on  $E$ .*

**Theorem 1.3.6.** *A continuous realvalued function on a closed and bounded domain is uniformly continuous.*

*Proof.* Let  $f$  be continuous on  $E$ , and  $E$  a closed and bounded domain. Let  $\varepsilon > 0$ . For every  $x \in E$ , there is a  $\delta_x > 0$  for which  $|f(x) - f(y)| < \varepsilon$  whenever  $|x - y| < \delta_x$  for some  $y \in E$ . Define  $I_x = (x - \frac{\delta_x}{2}, x + \frac{\delta_x}{2})$ . Then  $\{I_x\}$  is an open cover for  $E$ , so that by the theorem of Heine-Borel, there is a finite subcover  $\{I_{x_k}\}_{k=1}^n$  of  $E$ . Define  $\delta = \frac{1}{2} \min \{\frac{\delta_{x_1}}{2}, \dots, \frac{\delta_{x_n}}{2}\}$ . Then  $\delta > 0$  moreover, if  $x, y \in E$ , with  $|x - y| < \delta$ , then asince  $\{I_{x_k}\}$  covers  $E$ , there is a  $k > 0$  such that

$$|x - x_k| < \frac{\delta_{x_k}}{2} \text{ and } |x_{x_k} - y| < \frac{\delta_{x_k}}{2}$$

Then we have  $|f(x) - f(x_k)| < \frac{\varepsilon}{2}$  and  $|f(x_k) - f(y)| < \frac{\varepsilon}{2}$  so that  $|f(x) - f(y)| < \varepsilon$ , which makes  $f$  unifomrly continuous. ■

# Chapter 2

## Lebesgue Measure

### 2.1 Lebesgue Outermeasure

**Definition.** Let  $I$  be a nonempty interval of  $\mathbb{R}$ . We define the **length** of  $I$ , denoted  $l(I)$ , to be the difference of its endpoints, if  $I$  is bounded, and  $\infty$  otherwise.

**Definition.** Let  $A$  a subset of  $\mathbb{R}$ . We define the **Lebesgue outer measure** of  $A$  to be

$$m^*(A) = \inf \left\{ \sum l(I_k) \right\}$$

Where  $\{I_k\}$  is a countable collection of bounded open sets, covering  $A$ .

**Lemma 2.1.1.** *The emptyset has Lebesgue outermeasure 0. Moreover, the Lebesgue outermeasure is monotone; that is, if  $A, B \subseteq \mathbb{R}$  such that  $A \subseteq B$ , then  $m^*(A) \leq m^*(B)$ .*

*Proof.* Notice that the singleton  $\{a\} = [a, a]$  covers the emptyset. Moreover  $l([a, a]) = a - a = 0$ , so by definition  $m^*(\emptyset) = 0$ .

Now, let  $A, B$  subsets of  $\mathbb{R}$  such that  $A \subseteq B$ . Then if  $\{I_k\}$  is a countable collection of bounded open sets covering  $B$ , they also cover  $A$ , hence by definition, we get  $m^*(A) \leq m^*(B)$ . ■

**Corollary.** *Lebesgue outermeasure is nonnegative. That is,  $0 \leq m^*(E)$  for any set  $E \subseteq \mathbb{R}$ .*

*Proof.* Notice the length of any interval  $I$  is nonnegative. ■

**Example 1.** Countable sets have measure 0. Let  $C$  be a countable set with enumeration  $\{c_k\}$ . Let  $\varepsilon > 0$  and define  $I_k = (c_k - \frac{\varepsilon}{2^{k+1}}, c_k + \frac{\varepsilon}{2^{k+1}})$ . Then  $\{I_k\}$  is a countable collection of bounded open sets covering  $C = \{c_k\}$ . Hence we get that

$$0 \leq m^*(C) \leq \sum l(I_k) \leq \sum \frac{\varepsilon}{2^k} = \varepsilon$$

So that  $m^*(C) = 0$ .

**Lemma 2.1.2.** *For any nonempty interval  $I$ ,  $m^*(I) = l(I)$ .*

*Proof.* Consider first, the closed bounded interval  $[a, b]$ , where  $a < b$ . Let  $\varepsilon > 0$ . Notice that  $[a, b] \subseteq (a - \varepsilon, b + \varepsilon)$ , so that  $m^*([a, b]) \leq l((a - \varepsilon, b + \varepsilon)) = b - a + 2\varepsilon$ . Hence  $m^*([a, b]) \leq b - a$ . It remains to show that  $b - a \leq m^*([a, b])$ .

Let  $\{I_k\}$  a countable collection of open bounded intervals covering  $[a, b]$ . By the theorem of Heine-Borel, there is a finite subcover  $\{I_k\}_{k=1}^n$  of  $[a, b]$ . Notice that since  $a \in \bigcup I_k$ , at least one  $I_k$  contains  $a$ . Hence choose an interval  $(a_1, b_1)$  in this cover for which  $a_1 < a < b_1$ . Now, if  $b < b_1$ , we are done as

$$\sum_{k=1}^n l(I_k) \geq b_1 - a_1 > b - a$$

Otherwise,  $b_1 \in [a, b]$ . In this case, choose an interval  $(a_2, b_2)$ , distinct from  $(a_1, b_1)$  for which  $a_2 < b_1 < b_2$ . If  $b_2 \geq b$ , then we are done by similar reasoning as above. Otherwise, continue the process of choosing intervals. This process terminates as we eventually exhaust the endpoints of each  $I_k$  in the open cover. Thus, we get a subcollection  $\{(a_k, b_k)\}_{k=1}^N$  for which  $a_1 < a$  and  $a_{k+1} < b_k$  for all  $1 \leq k \leq N - 1$ . We also have a  $b_N > b$ . Then we have

$$\sum_{k=1}^N l(I_k) \geq \sum_{k=1}^N l((a_k, b_k)) = (b_N - a_N) + \cdots + (b_1 - a_1) \geq b - a$$

so that we get  $b - a \leq m^*([a, b])$ .

Now, let  $I$  be any bounded interval. Notice that there exist closed bounded intervals  $J_1$  and  $J_2$  for which

$$J_1 \subseteq I \subseteq J_2$$

and for some  $\varepsilon > 0$ ,

$$l(I) - \varepsilon < l(J_1) \leq l(I) \leq l(J_2) < l(I) + \varepsilon$$

Then since  $J_1$  and  $J_2$  are closed and bounded intervals, and by monotonicity of  $m^*$ , we have

$$l(I) - \varepsilon < m^*(J_1) \leq m^*(I) \leq m^*(J_2) < l(I) + \varepsilon$$

so that  $l(I) - \varepsilon < m^*(I) < l(I) + \varepsilon$  for all  $\varepsilon > 0$ . This establishes equality. ■

**Lemma 2.1.3.** *The Lebesgue outermeasure is translation invariant. That is, if  $A \subseteq \mathbb{R}$ , and  $y \in \mathbb{R}$ , then  $m^*(A) = m^*(A + y)$ .*

*Proof.* Notice that a countable collection of open bounded intervals  $\{I_k\}$  covers  $A$  if, and only if the collection  $\{I_k + y\}$  of open bounded intervals covers  $A + y$ . Moreover, notice that  $l(I_k) = l(I_k + y)$ , so that we get

$$\sum l(I_k) = \sum l(I_k + y)$$

the rest follows from definition. ■

**Lemma 2.1.4.** *The Lebesgue outermeasure is countable subadditive; that is, if  $\{E_k\}$  is a collection of subsets of  $\mathbb{R}$ , then*

$$m^*\left(\bigcup E_k\right) \leq \sum m^*(E_k)$$

*Proof.* Let  $\{E_k\}$  a countable collection of sets, and let  $E = \bigcup E_k$ . Notice that if atleast one  $E_k$  has infinite measure, then we are done. Suppose then that for all  $k$ ,  $m^*(E_k)$  is finite. Let  $\varepsilon > 0$ . Then for all  $k$ , there exists a countable collection of open bounded intervals  $\{I_{k,i}\}$  covering  $E_k$ , and  $\sum_i l(I_{k,i}) < m^*(E_k) + \frac{\varepsilon}{2^k}$ . By definition, we get

$$m^*(E) \leq \sum_k \sum_i l(I_{k,i}) = \sum_k \sum_i l(I_{k,i}) < \sum_k (m^*(E_k) + \frac{\varepsilon}{2^k}) = \sum_k m^*(E_k) + \varepsilon$$

for all  $\varepsilon > 0$ . This inequality also holds for  $\varepsilon = 0$ . ■

**Corollary.** *The Lebesgue outermeasure is finitely subadditive.*

*Proof.* Recall that finite collections are also countable collectuions. ■

## 2.2 Lebesue Measurable Sets

**Definition.** We call a set  $E$  of  $\mathbb{R}$  **Lebesue measurable**, provided for any subset  $A$  of  $\mathbb{R}$ ,

$$m^*(A) = m^*(A \cap E) + m^*(A \cap \mathbb{R} \setminus E)$$

**Lemma 2.2.1.** *A set  $E$  is Lebesue measurable if, and only if for any subset  $A$  of  $\mathbb{R}$ ,*

$$m^*(A) \geq m^*(A \cap E) + m^*(A \cap \mathbb{R} \setminus E)$$

*Proof.* We have  $A = (A \cap E) \cup (A \cap \mathbb{R} \setminus E)$ , so by finite subadditivity,  $m^*(A) \leq m^*(A \cap E) + m^*(A \cap \mathbb{R} \setminus E)$ . ■

**Lemma 2.2.2.** *Any set of Lebesue outer measure 0 is Lebesue measurable.*

*Proof.* Let  $E$  have  $m^*(E) = 0$  and let  $A \subseteq \mathbb{R}$ . Notice that  $A \cap E \subseteq E$  and  $A \cap \mathbb{R} \setminus E \subseteq E$ , so that  $m^*(A \cap E) \leq m^*(E) = 0$  and  $m^*(A \cap \mathbb{R} \setminus E) \leq m^*(A)$ . Then we have

$$m^*(A) \geq m^*(A \cap \mathbb{R} \setminus E) = 0 + m^*(A \cap \mathbb{R} \setminus E) = m^*(A \cap E) + m^*(A \cap \mathbb{R} \setminus E)$$
■

**Corollary.** *Countable sets are measurable.*

**Lemma 2.2.3.** *The union of two measurable sets is measurable.*

*Proof.* Let  $E_1$  and  $E_2$  be measurable sets and  $A \subseteq \mathbb{R}$ . Then  $m^*(A) = m^*(A \cap E_1) + m^*(A \cap \mathbb{R} \setminus E_1) = m^*(A \cap E_1) + m^*((A \cap \mathbb{R} \setminus E_1) \cap E_2) + m^*((A \cap \mathbb{R} \setminus E_1) \cap \mathbb{R} \setminus E_2)$ . Moreover, notice that

$$(A \cap \mathbb{R} \setminus E_1) \cap \mathbb{R} \setminus E_2 = A \cap \mathbb{R} \setminus (E_1 \cup E_2) \text{ and } (A \cap E_1) \cup (A \cap \mathbb{R} \setminus E_1 \cap E_2) = A \cap (E_1 \cup E_2)$$

Then we get

$$m^*(A) = m^*(A \cap E_1) + m^*((A \cap \mathbb{R} \setminus E_1) \cap E_2) + m^*(A \cap \mathbb{R} \setminus (E_1 \cup E_2)) \geq m^*(A \cap (E_1 \cup E_2)) + m^*(A \cap \mathbb{R} \setminus (E_1 \cup E_2))$$

which makes  $E_1$  and  $E_2$  measurable. ■

**Corollary.** *The union of a finite collection of measurable sets is measurable.*

*Proof.* Let  $\{E_k\}_{k=1}^n$  a finite collection of measurable sets. By induction on  $n$ , we showed that this is true for  $n = 1$  and  $n = 2$ . Now, consider the collections  $\{E_k\}_{k=1}^{n+1}$  and suppose that the union  $E = \bigcup_{k=1}^n E_k$  is measurable. Notice, then that

$$\bigcup_{k=1}^{n+1} E_k = E \cup E_{n+1}$$

both of which are measurable. Hence measurability of the union of  $\{E_k\}_{k=1}^{n+1}$  follows by above. ■

**Lemma 2.2.4.** *Let  $A$  a subset of  $\mathbb{R}$  and  $\{E_k\}_{k=1}^n$  a finite, disjoint collection of measurable sets. Then*

$$m^*(A \cap \bigcup_{k=1}^n E_k) = \sum_{k=1}^n m^*(A \cap E_k)$$

*Proof.* By induction on  $n$ , for  $n = 1$  it is true. Now, suppose that it is true for  $n$ , and consider the collection  $\{E_k\}_{k=1}^{n+1}$  of disjoint measurable sets. Then we have  $A \cap (\bigcup_{k=1}^n E_k) \cap E_{n+1} = A \cap E_{n+1}$  and  $A \cap (\bigcup_{k=1}^n E_k) \cap \mathbb{R} \setminus E_{n+1} = A \cap \bigcup_{k=1}^n E_k$ . Since  $E_{n+1}$  is measurable we get

$$m^*(A \cap \bigcup_{k=1}^{n+1} E_k) = m^*(A \cap E_{n+1}) + m^*(A \cap \bigcup_{k=1}^n E_k) = \sum_{k=1}^{n+1} m^*(A \cap E_k)$$

■

**Definition.** We call a collection of subsets of  $\mathbb{R}$  an **algebra** if it contains  $\mathbb{R}$  and it is closed under complements (with respect to  $\mathbb{R}$ ) and finite unions.

**Lemma 2.2.5.** *Any algebra of  $\mathbb{R}$  is closed under finite intersections.*

*Proof.* By DeMorgan's laws. ■

**Theorem 2.2.6.** *The collection of all measurable sets of  $\mathbb{R}$  forms an algebra.*

**Lemma 2.2.7.** *The union of a countable collection of measurable sets is measurable.*

*Proof.* Without loss of generality, let  $\{E_k\}$  a countable disjoint collection of measurable sets, and let  $E = \bigcup E_k$ . Let  $A$  a subset of  $\mathbb{R}$  and define  $F_n = \bigcup_{k=1}^n E_k$ . Then  $F_n$  is measurable by lemma 2.2.3, and  $\mathbb{R} \setminus E_n \subseteq \mathbb{R} \setminus F_n$ . Then

$$m^*(A) = m^*(A \cap F_n) + m^*(A \cap \mathbb{R} \setminus F_n) \geq m^*(A \cap F_n) + m^*(A \cap \mathbb{R} \setminus E_n)$$

hence  $m^*(A \cap F_n) = \sum_{k=1}^n m^*(A \cap E_k)$  so that

$$m^*(A) \geq \sum m^*(A \cap E_k) + m^*(A \cap \mathbb{R} \setminus E)$$

By countable subadditivity of  $m^*$  we have

$$m^*(A) \geq m^*(A \cap E) + m^*(A \cap \mathbb{R} \setminus E)$$

■

**Definition.** We call a collection of subsets of  $\mathbb{R}$  a  **$\sigma$ -algebra** if it forms an algebra, and it is closed under countable unions.

**Lemma 2.2.8.** *Any  $\sigma$ -algebra of  $\mathbb{R}$  is closed under countable intersections.*

**Theorem 2.2.9.** *The collection of measurable sets of  $\mathbb{R}$  forms a  $\sigma$ -algebra.*

**Lemma 2.2.10.** *Every interval of  $\mathbb{R}$  is measurable.*

*Proof.* Consider an interval of the form  $(a, \infty)$ , for any  $a \in \mathbb{R}$ . Let  $A \subseteq \mathbb{R}$ , such that  $A \notin \mathcal{A}$ ; otherwise, just take  $A \setminus \{a\}$ . Then since  $m^*(A)$  is a greatest lower bound, it is sufficient to show that for any countable collection  $\{I_k\}$  of open, bounded intervals covering  $A$ , that

$$m^*(A_1) + m^*(A_2) \leq \sum l(I_k)$$

where

$$A_1 = A \cap (-\infty, a) \text{ and } A_2 = A \cap (a, \infty)$$

Indeed, let  $\{I_k\}$  be such a collection, and define

$$I_{k,1} = I_k \cap (-\infty, a) \text{ and } I_{k,2} = I_k \cap (a, \infty)$$

Then  $\{I_{k,1}\}$  and  $\{I_{k,2}\}$  are collections of open, bounded intervals which cover  $A_1$  and  $A_2$  respectively. Hence, by definition of  $m^*$ , we have  $m^*(A_1) \leq \sum l(I_{k,1})$  and  $m^*(A_2) \leq \sum l(I_{k,2})$ ; moreover, notice that  $l(I_k) = l(I_{k,1}) + l(I_{k,2})$ . Therefore, we get

$$m^*(A_1) + m^*(A_2) \leq \sum l(I_{k,1}) + \sum l(I_{k,2}) = \sum l(I_k)$$

and we are done. ■

**Corollary.** *Open sets, and closed sets of  $\mathbb{R}$  are measurable.*

**Definition.** We define the intersection of all  $\sigma$ -algebras of  $\mathbb{R}$  to be the **Borel  $\sigma$ -algebra**, and call its elements **Borel sets**.

**Theorem 2.2.11.** *The  $\sigma$ -algebra of all measurable sets of  $\mathbb{R}$  contains the Borel  $\sigma$ -algebra of  $\mathbb{R}$ . Moreover, it contains every interval of  $\mathbb{R}$ , open and closed sets, as well as  $G_\delta$  and  $F_\sigma$  sets.*

**Lemma 2.2.12.** *Lebesgue measurable sets are translation invariant. That is, if  $E$  is Lebesgue measurable, and  $y \in \mathbb{R}$ , then  $E + y$  is Lebesgue measurable.*

*Proof.* Let  $E$  be measurable,  $y \in \mathbb{R}$ , and  $A \subseteq \mathbb{R}$ . Then

$$m^*(A) = m^*(A \setminus y) = m^*(A \setminus y \cap E) + m^*(A \setminus y \cap \mathbb{R} \setminus E) = m^*(A \cap (E + y)) + m^*(A \cap \mathbb{R} \setminus (E + y))$$

■

## 2.3 Inner and Outer Approximations

**Lemma 2.3.1** (Excision). *If  $A$  and  $B$  are sets, with  $A$  Lebesgue measurable of finite outer measure, and  $A \subseteq B$ , then*

$$m^*(B \setminus A) = m^*(B) - m^*(A)$$

**Theorem 2.3.2** (The Outer Approximation Theorem). *Let  $E \subseteq \mathbb{R}$ . The following are equivalent.*

- (1)  $E$  is Lebesgue measurable.
- (2) For all  $\varepsilon > 0$  there is an open set  $U$  of  $\mathbb{R}$  containing  $E$  such that  $m^*(U \setminus E) < \varepsilon$ .
- (3) There exists a  $G_\delta$  set  $G$  containing  $E$  for which  $m^*(G \setminus E) = 0$ .

*Proof.* Suppose first that  $E$  is measurable and let  $\varepsilon > 0$ . Now, if  $m^*(E)$  is finite, then there is a countable collection  $\{I_k\}$  of open intervals covering  $E$ , for which, by definition of  $m^*$  as a greatest lower bound,

$$\sum l(I_k) < m^*(E) + \varepsilon$$

Let  $U = \bigcup I_k$ , then  $E \subseteq U$ , and  $U$  is open in  $\mathbb{R}$ . Thus by definition of  $m^*$  again, we have

$$m^*(U) \leq \sum l(I_k) < m^*(E) + \varepsilon$$

so that  $m^*(U) - m^*(E) < \varepsilon$ . Now, since  $E$  is measurable of finite outer measure, by excision, we get  $m^*(U \setminus E) = m^*(U) - m^*(E) < \varepsilon$ .

Now, if  $m^*(E)$  is infinite, then let  $\{E_k\}$  be a countable disjoint collection of measurable sets each of finite outer measure, and let  $E = \bigcup E_k$ . Then by above there exist open sets  $U_k$  containing  $E_k$ , for each  $k$  such that  $m^*(U_k \setminus E_k) < \frac{\varepsilon}{2^k}$ . Let  $U = \bigcup U_k$ , then  $U$  is open in  $\mathbb{R}$ , and  $E \subseteq U$ . Moreover observe that

$$U \setminus E = \bigcup U_k \setminus E_k$$

Then we get by subadditivity

$$m^*(U \setminus E) \leq \sum m^*(U_k \setminus E_k) < \sum \frac{\varepsilon}{2^k} = \varepsilon$$

Now, suppose that assertion (2) holds, and choose an open set  $U_k$  containing  $E$  for which  $m^*(U_k \setminus E) < \frac{1}{k}$ . Define  $G = \bigcup U_k$ . Then  $G$  is a  $G_\delta$  set, and  $E \subseteq G$ . Moreover we have that

$$G \setminus E \subseteq U_k \setminus E \text{ for all } k$$

so by monotonicity

$$m^*(G \setminus E) \leq m^*(U_k \setminus E) < \frac{1}{k}$$

Then as  $k \rightarrow \infty$ , this outer measure approaches 0.

Now if (3) holds, since  $m^*(G \setminus E) = 0$ , the set  $G \setminus E$  is measurable. Since the space of all measurable sets is a  $\sigma$ -algebra, then the set  $E = G \cap \mathbb{R} \setminus (G \setminus E)$  is measurable. ■



**Corollary** (The Inner Approximation Theorem). *The following are equivalent.*

- (1)  $E$  is Lebesgue measurable.
- (2) For all  $\varepsilon > 0$  there is a closed set  $V$  of  $\mathbb{R}$  contained in  $E$  such that  $m^*(E \setminus V) < \varepsilon$ .
- (3) There exists an  $F_\sigma$  set  $F$  contained in  $E$  for which  $m^*(E \setminus F) = 0$ .

*Proof.* One can apply DeMorgan's laws. ■

**Theorem 2.3.3.** *Let  $E$  a Lebesgue measurable set of finite outer measure. then for every  $\varepsilon > 0$  there is a finite disjoint collection  $\{I_k\}$  of open intervals for which if  $U = \bigcup I_k$ , then*

$$m^*(E \setminus U) + m^*(U \setminus E) < \varepsilon$$

*Proof.* By the outer approximation theorem, there is an open set  $V$  containing  $E$  for which  $m^*(V \setminus E) < \frac{\varepsilon}{2}$ . Now, since  $E$  is measurable of finite outer measure, by excision we have

$$m^*(V) - m^*(E) < \frac{\varepsilon}{2}$$

so that  $m^*(V)$  is also finite. Now, recall that every open set of real numbers is the disjoint collection of open intervals, hence let  $V = \bigcup I_k$ . Each  $I_k$  is measurable with  $m^*(I_k) = l(I_k)$ . Thereofre, by lemma 2.2.4 and monotonicity,

$$\sum_{k=1}^n l(I_k) \leq m^*(V) \text{ is finite}$$

So  $\sum I_k$  is finite. Now, choose an  $n \in \mathbb{Z}^+$  for which  $\sum_{k=n+1}^\infty l(I_k) < \frac{\varepsilon}{2}$  and define  $U = \bigcup_{k=1}^n I_k$ . Then  $U \setminus E \subseteq V \setminus E$  so by monotonicity,  $m^*(U \setminus E) < \frac{\varepsilon}{2}$ . Moreover, we have  $E \setminus U \subseteq V \setminus U = \bigcup_{k=n+1}^\infty I_k$  so that  $m^*(E \setminus U) < \frac{\varepsilon}{2}$ . Therefore, we see that

$$m^*(U \setminus E) + m^*(E \setminus U) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$
■

## 2.4 The Borel-Cantelli Lemma

**Definition.** We define the **Lebesgue measure**  $m$  to be the restriction of the Lebesgue outer measure,  $m^*$  to the space of all Lebesgue measurable sets. That is, if  $E$  is Lebesgue measurable, the

$$m(E) = m^*(E)$$

**Lemma 2.4.1** (Countable additivity). *The Lebesgue measure is countable additive. That is, if  $\{E_k\}$  is a countable collection of disjoint measurable sets, then*

$$m\left(\bigcup E_k\right) = \sum m(E_k)$$

*Proof.* Since the space of Lebesgue measurable sets forms a  $\sigma$ -algebra, and are closed under countable unions, the set  $E = \bigcup E_k$  is Lebesgue measurable. Moreover, by subadditivity of  $m^*$ , and definition of  $m$ ,

$$m(E) \leq \sum_k = 1^\infty m(E_k)$$

Notice, however, that  $\bigcup_{k=1}^n E_k \subseteq E$ , so that by monotonicity,  $\sum_{k=1}^n m(E_k) \leq m(E)$ . Then as  $n \rightarrow \infty$ , this sum converges to  $\sum_{k=1}^\infty E_k$  so

$$\sum_{k=1}^\infty E_k \leq m(E)$$

and equality is established. ■

**Corollary.** *The Lebesgue measure is finitely additive.*

**Theorem 2.4.2.** *The Lebesgue measure assigns to intervals their lengths, is translation invariant, and countable additive.*

**Theorem 2.4.3** (Continuity). *The following are true for the Lebesgue measure.*

(1) *If  $\{A_k\}$  is an increasing sequence of Lebesgue measurable sets, then*

$$m\left(\bigcup A_k\right) = \lim_{k \rightarrow \infty} m(A_k)$$

(2) *If  $\{B_k\}$  is an decreasing sequence of Lebesgue measurable sets for which  $m(B_1)$  is finite, then*

$$m\left(\bigcap B_k\right) = \lim_{k \rightarrow \infty} m(B_k)$$

*Proof.* If  $k_0 \in \mathbb{Z}^+$  is such that  $m(A_{k_0})$  is infinite, then by monotonicity,  $m(\bigcup A_k)$  is infinite so that  $m(A_k)$  is infinite for all  $k \geq k_0$ . Suppose then, that  $m(A_k)$  is finite for all  $k$  and define  $A_0 = \emptyset$ . Furthermore, define  $C_k = A_k \setminus A_{k-1}$  for all  $k \geq 1$ . then since  $\{A_k\}$  is a disjoint collection of measurable sets, then so is  $C_k$ , and  $\bigcup A_k = \bigcup C_k$ . By countable additivity, we have

$$m\left(\bigcup A_k\right) = m\left(\bigcup C_k\right) = \sum m(A_k \setminus A_{k-1})$$

By excision, we get

$$\sum m(A_k) - m(A_{k-1}) = \lim_{n \rightarrow \infty} \sum_{k=1}^n m(A_k) - m(A_{k-1}) = \lim (m(A_n) - m(A_0)) = \lim_{n \rightarrow \infty} m(A_n)$$

since  $m(A_0) = 0$ .

Now, define  $D_k = B_1 \setminus B_k$ . Since  $\{B_k\}$  is decreasing, the sequence  $\{D_k\}$  of measurable sets is increasing. Then by above,

$$m\left(\bigcup D_k\right) = \lim_{k \rightarrow \infty} m(D_k)$$

By DeMorgan's laws,  $\bigcup D_k = B_1 \setminus \bigcap B_k$ . On the otherhand, by excision, since  $m(B_1)$  is finite, we get

$$m(D_k) = m(B_1) - m(B_k)$$

so that

$$m(B_1 \setminus \bigcap B_k) = \lim_{n \rightarrow \infty} (m(B_1) - m(B_n))$$

By excision again, we are done. ■

**Definition.** We say a property holds **almost everywhere** on a measurable set  $E$  if there exists a measurable set  $E_0 \subseteq E$  with  $m(E_0) = 0$  for which the property holds for all  $x \in E \setminus E_0$ .

**Lemma 2.4.4** (Borel-Cantelli). *Let  $\{E_k\}$  a countable collection of measurable sets such that the sum  $\sum m(E_k)$  is finite. Then almost all  $x \in \mathbb{R}$  belong to at most finitely many of the  $E_k$ .*

*Proof.* By countable subadditivity, we have  $m(\bigcup E_k) \leq \sum_{k=n} m(E_k)$  is finite. Thus, by continuity, we have

$$m\left(\bigcap_{n=1} \left(\bigcup_{k=n} E_k\right)\right) = \lim_{n \rightarrow \infty} m\left(\bigcup_{k=n} E_k\right) \leq \lim_{n \rightarrow \infty} \sum_{k=n} m(E_k) = 0$$

so that almost all  $x$  does not belong to the intersection  $\bigcap_{n=1} \bigcup_{k=n} E_k$  and hence belongs to at most finitely many of the  $E_k$ . ■

## 2.5 Nonmeasurable Sets, The Cantor Set, and The Cantor-Lebesgue Function

**Definition.** We call a set  $E$  of real numbers **nonmeasurable** if it is not measurable.

**Lemma 2.5.1.** *If  $E$  is a bounded measurable set of real numbers, and there is a countably infinite disjoint collection of translates  $\{E + \lambda\}$ , then  $m(E) = 0$ .*

*Proof.* Since  $E$  is measurable, so is  $E + \lambda$  for every  $\lambda$ . Then by countable additivity, we have

$$m\left(\bigcup E + \lambda\right) = \sum m(E + \lambda) = \sum m(E)$$

Now, since  $E$  is bounded, so is each  $E + \lambda$ , and hence, so is  $\bigcup E + \lambda$  so that  $m(\bigcup E + \lambda)$  is finite. Therefore,  $m(E)$  is finite. Moreover, since the collection  $\{E + \lambda\}$  is countably infinite, and  $m(E)$  is finite, this forces  $m(E) = 0$ . ■

**Definition.** We call two real numbers  $x, y \in \mathbb{R}$  **rationally equivalent** if  $x - y \in \mathbb{Q}$ .

**Lemma 2.5.2.** *Rational equivalence is an equivalence relation on  $\mathbb{R}$ .*

**Theorem 2.5.3** (Vitali's Theorem). *Any set  $E$  of real numbers with positive outer measure contains a nonmeasurable set.*

*Proof.* Consider rational equivalence on  $E$ , which partitions  $E$  into equivalence classes. Define  $\mathcal{C}_E$  a choice set of the equivalence classes on  $E$  consisting of exactly one member from each class, such that

- (1) For all  $x, y \in \mathcal{C}_E$ ,  $x - y \notin \mathbb{Q}$ .
- (2) For all  $x \in E$ , there exists a  $c \in \mathcal{C}_E$  for which  $x = c + q$  for some  $q \in \mathbb{Q}$ .

Now, by countable subadditivity, suppose that  $E$  is bounded, and consider the choice set  $\mathcal{C}_E$  (defined above) of  $E$ . Then  $\mathcal{C}_E$  is nonmeasurable.

Suppose otherwise. Let  $\Lambda_0$  a bounded countably infinite set of rational numbers. Then each  $\{\mathcal{C}_E + \lambda\}$  is measurable for each  $\lambda \in \Lambda_0$ . Then we have a countably infinite disjoint collection of bounded translates, hence by lemma 2.5.1,  $m(\mathcal{C}_E) = 0$ . That is,

$$m(\bigcup \mathcal{C}_E + \lambda) = \sum m(\mathcal{C}_E + \lambda) = 0$$

Since  $E$  is bounded, choose  $\Lambda_0 = [-2b, 2b] \cap \mathbb{Q}$ , for some  $b \in \mathbb{R}$ . If  $x \in E$ , there exists a  $c \in \mathcal{C}_E$  and a  $q \in \mathbb{Q}$  such that  $x = c + q$ . That is  $x, c \in [-b, b]$  and  $q \in [-2b, 2b]$  so that  $E \subseteq \bigcup \mathcal{C}_E + \lambda$ . But  $m(E)$  is positive, which yields a contradiction as  $m(\mathcal{C}_E) = 0$ . Therefore  $\mathcal{C}_E$  can't possibly be measurable. ■

**Theorem 2.5.4.** *There exist disjoint sets  $A$  and  $B$  of real numbers such that*

$$m^*(A \cup B) < m^*(A) + m^*(B)$$

**Definition.** We define the **Cantor set** to be the intersection

$$\mathcal{C} = \bigcap C_k$$

where  $\{C_k\}$  is a decreasing sequence of closed sets such that for every  $k$ ,  $C_k$  is the disjoint union of  $2^k$  closed intervals of length  $\frac{1}{3^k}$

**Theorem 2.5.5.** *The Cantor set is a closed uncountable set of measure 0.*

*Proof.* Since  $\mathcal{C}$  is an arbitrary intersection of closed sets, it is closed in  $\mathbb{R}$ . Moreover, since each  $C_k$  is the disjoint union of closed intervals, which are measurable, and since measurable sets form a  $\sigma$ -algebra, then each  $C_k$  is measurable, which makes  $\mathcal{C}$  measurable.

Now, by definition of  $C_k$ , by finite additivity, we have

$$m(C_k) = \left(\frac{2}{3}\right)^k$$

so that by monotonicity of measure,

$$m(\mathcal{C}) \leq m(C_k) = \left(\frac{2}{3}\right)^k$$

now, as  $k \rightarrow \infty$ ,  $m(C_k) \rightarrow 0$  so that  $m(\mathcal{C}) = 0$ . It remains to show that  $\mathcal{C}$  is uncountable.

Suppose  $\mathcal{C}$  is countable, and let  $\{c_k\}$  be an enumeration of  $\mathcal{C}$ . Now, there is a disjoint interval  $F_1$  in  $C_1$  which fails to contain the point  $c_1$ ; similarly, there is a disjoint interval  $F_2$  in  $C_2$ , whose union is  $F_1$ , that fails to contain  $c_2$ . Proceeding inductively, we obtain a countable collection  $\{F_k\}$  such that

- (1) Each  $F_k$  is closed.
- (2)  $F_k \subseteq C_k$ .
- (3)  $c_k \notin F_k$ .

by the nested set theorem, the intersection  $F = \bigcap F_k$  is nonempty. Now, let  $x \in F$ , then we get that  $x \in C_k$  for some  $k$ . But since  $C_k$  is countable, and enumerated by  $\{c_k\}$ , then  $x = c_n$  for some  $n$ . That is,  $c_n \in F$  which contradicts that  $c_n \notin F_n$ . Therefore  $\mathcal{C}$  is uncountable. ■

**Definition.** Define  $U_k = [0, 1] \setminus C_k$  and  $\mathcal{U} = \bigcup U_k$ , so that  $\mathcal{C} = [0, 1] \setminus \mathcal{U}$ . Define the function  $\phi : U_k \rightarrow \mathbb{R}$  to be the increasing function, which is constant on each of the  $2^k - 1$  open intervals, and which takes the values of the form  $\frac{2^k - 1}{2^k}$  in each of the intervals. We define the **Cantor-Lebesgue function** to be the extension of  $\phi$  to  $[0, 1]$  by defining it on  $\mathcal{C}$  as follows

$$\phi(0) = 0 \text{ for all } x \in \mathcal{U} \text{ and } \phi(x) = \sup \{ \phi(t) : t \in U \cap [0, x] \text{ if } x \in \mathcal{C} \setminus \{0\} \}$$

**Lemma 2.5.6.** *The Cantor-Lebesgue function is increasing continuous whose image is the interval  $[0, 1]$ . Moreover,  $\phi$  is differentiable on  $\mathcal{U}$ , with  $\phi' = 0$  on  $\mathcal{U}$ , where  $m(\mathcal{U}) = 1$ .*

*Proof.* By definition,  $\phi|_{U_k}$  is increasing so the extension  $\phi$  is increasing as well. Likewise,  $\phi|_{U_k}$  is continuous, hence so is the extension  $\phi$ .

Now, consider  $x_0 \in \mathcal{C}$  such that  $x_0 \neq 0, 1$ . Then  $x_0 \notin U_k$ , and for  $k$  large enough,  $x_0$  is between two consecutive intervals of  $U_k$ . Let  $a_k$  be in the lower of these two intervals, and  $b_k$  in the upper. Since  $\phi$  is increasing, in particular, by  $\frac{1}{2^k}$ , we get  $a_k < x_{b_k}$  and  $\phi(b_k) - \phi(a_k) = \frac{1}{2^k}$ . Then as  $k \rightarrow \infty$   $\phi(b_k) - \phi(a_k) \rightarrow 0$  so that  $\phi$  has no jump discontinuities at  $x_0$ . This makes  $\phi$  continuous at  $x_0$ . Now, if  $x_0 = 0$  or  $x_0 = 1$ , a similar argument follows. Now, since  $\phi$  is constant on  $\mathcal{U}$ , and continuous on  $\mathcal{U}$ , it is differentiable on  $\mathcal{U}$ , with derivative  $\phi'(x) = 0$  for all  $x \in \mathcal{U}$ . Moreover, since  $\mathcal{C}$  is measurable with  $m(\mathcal{C}) = 0$ , and  $\mathcal{U} = [0, 1] \setminus \mathcal{C}$ , by excision, we get  $m(\mathcal{U}) = 1$ . Finally, notice that since  $\phi(0) = 0$ , and  $\phi(1) = 1$ , and by continuity, by the intermediate value theorem,  $\phi([0, 1]) = [0, 1]$ . ■

**Lemma 2.5.7.** *Let  $\phi$  be the Cantor-Lebesgue function and define  $\psi : [0, 1] \rightarrow \mathbb{R}$  by  $\psi(x) = \phi(x) + x$  for all  $x \in [0, 1]$ . Then  $\psi$  is strictly increasing, and takes  $[0, 1]$  onto  $[0, 2]$ . Moreover*

- (1)  $\psi$  maps  $\mathcal{C}$  onto a measurable set of positive measure.
- (2)  $\psi$  maps a measurable subset of  $\mathcal{C}$  onto a nonmeasurable set.

*Proof.*  $\psi$  is continuous since it is the sum of two continuous functions. Moreover, since  $\phi$  is increasing and the function  $f(x) = x$  is strictly increasing then so is  $\psi$ . Notice, also, that  $\psi(0) = 0$  and  $\psi(1) = 2$  so by the intermediate value theorem,  $\psi([0, 1]) = [0, 2]$ .

Now, since  $[0, 1] = \mathcal{U} \cup \mathcal{C}$  (where  $\mathcal{U}$  is defined in the definition of the Cantor-Lebesgue function), we have  $[0, 2] = \psi(\mathcal{U}) \cup \psi(\mathcal{C})$ . Since  $[0, 2]$  is measurable, and measurable sets are closed under unions, then  $\psi(\mathcal{C})$  is measurable; moreover, since  $\psi$  is continuous and increasing, it has continuous inverse, and hence maps  $\mathcal{C}$  to a measurable set  $\psi(\mathcal{C})$ . Moreover,  $\psi(\mathcal{C})$  is closed, and  $\psi(\mathcal{U})$  is open.

Now, let  $\{I_k\}$  a collection of intervals of  $\mathcal{U}$ , i.e.  $\mathcal{U} = \bigcup I_k$ . Since  $\phi$  is continuous on each  $I_k$ ,  $\psi$  takes  $I_k$  onto translates of  $I_k$ , and since  $\psi$  is 1-1, the collection  $\{\psi(I_k)\}$  is disjoint. Therefore, by countable additivity

$$m * (\psi(\mathcal{U})) = \sum l(\psi(I_k)) = \sum l(I_k + \lambda) = \sum l(I_k) = m(\mathcal{U})$$

since  $m(\mathcal{C}) = 0$  and  $m(\mathcal{U}) = 1$ ,  $m(\psi(\mathcal{U})) = 1$  and  $m(\psi(\mathcal{C})) = 1$  as well.

Finally, by Vitali's theorem, there exists a nonmeasurable set  $W \subseteq \psi(\mathcal{C})$ , with  $\psi^{-1}(W)$  measurable with  $m(\psi^{-1}(W)) = 0$ . ■

**Theorem 2.5.8.** *There exists a measurable subset of  $\mathcal{C}$  which is not Borel.*

# Bibliography

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