

# Group Theory.

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# Chapter 1

## Groups.

### 1.1 Motivations of Groups.

The notion of a “group” is perhaps the most fundamental notion in all of abstract algebra (without going into more granular structures which tend to be reserved for more advanced settings). It forms the building blocks for the study of the “ring” and “field” structures prevalent in algebra, and it even forms its (rightfully) interesting field of study.

There are a number of motivations and examples for group theory and its study. Historically, the definition of a group finds itself motivated by the study of algebraic equations; specifically polynomial equations of the form  $ax^2 + bx + c = 0$ ,  $ax^3 + bx^2 + cx + d = 0$ , and more generally,  $a_nx^n + \cdots + a_1x + a_0 = 0$ . That is the so called quadratic, cubic, and other polynomial equations. Specifically, the solutions of such equations were of interest, and finding a general form for them proved invaluable. It is known that the quadratic,  $ax^2 + bx + c = 0$  has the general solution:

$$x = -\frac{b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a} \quad (1.1)$$

dubbed the “quadratic” formula. The general solution for the cubic equation is far more convoluted. What about the solution for polynomial equations of higher degree? It was shown that the quintic equation (of degree 5) and higher has no general form for their solutions.

The study of these polynomial equations and their general solutions led mathematicians such as Niels Abel and Evariste Galois to develop the foundations for algebra. In particular, Galois, on the eve of his duel composed a manuscript concerning these equations which would form the foundations for group theory in which others would build upon.

There are other applications and motivations for group theory besides from polynomials. In number theory, one is concerned with the properties of positive integers, which form a “group” structure. For example, if the integer  $a^{40}$  has its last two digits 01 if it is not congruent 0 mod 2, nor 0 mod 5; i.e. it is not divisible by either 2 nor 5. Leonhard Euler proved this using group theoretic techniques developed by Joseph Louis Lagrange.

Group theory also plays an essential role in finding rational solutions (i.e. solutions in  $\mathbb{Q}$ ), to certain algebraic equations called Diophantine equations;  $y^2 = x^3 - x$  is an example. Solutions to these equations can be found by intersecting straight lines with these

curves. The result is that if an arithmetic is defined on the points of these curves using line intersections, the one can obtain a group structure. The most famous example of this is are elliptic curves, which have the form  $y^2 = f(x)$  where  $f$  is a polynomial. These curves form a group structure and have an intimate connection to number theory. Pierre de Fermat was interested in Diophantine equations, especially the equation  $x^n + y^n = z^n$ ; which was claimed not to have solutions for  $n > 2$ , and to which Fermat scribbled in a copy of Diophantus' book: "I have a marvelous proof to this theorem, but the margin is too small to contain it". This problem became known as "Fermat's Last Theorem" and went unproved until Andrew Wiles proved it using Elliptic curves (its much more complicated than that, but the story of Fermat's last theorem rightfully deserves its own book).

The study of the arithmetic of Elliptic curve is an area of active research and has wide applications, specifically in the field of cryptography.

## 1.2 Definitions and Examples.

**Definition.** Let  $G$  be set, we define a **binary operation**,  $*$ , on  $G$  to be a map  $*$  :  $G \times G \rightarrow G$  that takes  $(a, b) \rightarrow a * b$ . We say that a binary operation  $*$  is **associative** if for any  $a, b, c \in G$ ,  $(a * b) * c = a * (b * c)$ . We say that a binary operation,  $*$  is **commutative** if for any  $a, b \in G$ ,  $a * b = b * a$ .

We also write  $ab$  instead of  $a * b$  for convenience, and when context is clear.

**Example 1.1.** (1) The usual addition,  $+$  is an associative and commutative binary operation on the sets  $\mathbb{Z}$ ,  $\mathbb{Q}$ , and  $\mathbb{R}$  of integers, rationals, and real numbers. The addition of complex numbers,  $+$  is an associative and commutative binary operation on the complex numbers  $\mathbb{C}$ .

(2) The usual multiplication  $\cdot$  is an associative and commutative binary operation on  $\mathbb{Z}^*$ ,  $\mathbb{Q}^*$  and  $\mathbb{R}^*$ . Complex multiplication on  $\mathbb{C}^*$  is also an associative, commutative binary operation. Note we define  $F^* = F \setminus \{0\}$ , where  $F = \mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{C}$ .

(3) The usual subtraction, is a noncommutative binary operation on  $\mathbb{Z}$ ,  $\mathbb{Q}$ , and  $\mathbb{R}$ , complex subtraction is noncommutative on  $\mathbb{C}$ . The map  $a \rightarrow -a$  is not binary.

(4) The usual subtraction is not a binary operation on  $\mathbb{Z}^+$ ,  $\mathbb{Q}^+$ , and  $\mathbb{R}^+$ , notice that if  $a < b$ ,  $a - b \notin F$  where  $F = \mathbb{Z}, \mathbb{Q}, \mathbb{R}$ .

(5) The cross product,  $\times$  on two vectors in real 3-space is a nonassociative binary operation on  $\mathbb{R}^3$ .

(6) The operation  $+_n$  of addition mod  $n$  is a binary operation on the set of integers mod  $n$ ,  $\mathbb{Z}/_n\mathbb{Z}$ . Notice that for  $k, l, m \in \mathbb{Z}$ , that  $a \bmod n$  is of the form  $a + kn$ , thus  $b \bmod n$  and  $c \bmod n$ , have the forms  $b + ln$  and  $c + mn$ . Thus  $((a + kn) + (b + ln)) + (c + mn) = (a + b) + (l + k)n + (c + mn) = a + b + c + (l + k + m)n = (a + kn) + (b + c) + (l + m)n = (a + kn) + ((b + ln) + (c + mn))$ . This implies that  $(a + b) + c \bmod n = a(b + c) \bmod n$ ; additionally,  $(a + kn) + (b + ln) = (a + b) + (k + l)n = (b + a) + (l + k)n = (b + ln) + (a + kn)$ ,

so  $a + b \bmod n = b + a \bmod n$ . That is  $+_n$  is associative and commutative. We abbreviate addition  $\bmod n$  and write  $+$  instead of  $+_n$ .

- (7) Multiplication  $\bmod n$  is a binary operation on  $\mathbb{Z}/n\mathbb{Z}$  which is associative and commutative.
- (8) The operation of function composition  $\circ$  is a binary operation on any set of mappings. We have that for mappings  $f, g$ , and  $h$  that  $(f \circ g) \circ h = f \circ (g \circ h)$ , making  $\circ$  associative; but  $f \circ g \neq g \circ f$ , making  $\circ$  noncommutative.

**Definition.** Let  $G$  be a set, and  $H \subseteq G$ , and let  $*$  be a binary operation on  $G$ . We say that  $H$  is **closed** under  $*$  if  $*|_H$  is a binary operation on  $H$ .

**Definition.** Let  $G$  be a nonempty set, and let  $*$  be a binary operation on  $G$ . We call the pair  $(G, *)$  a **group** if:

- (0) For every  $a, b \in G$ ,  $ab \in G$ . That is  $G$  is closed under  $*$ .
- (1)  $(ab)c = a(bc)$  for all  $a, b, c \in G$ , i.e.  $*$  is associative.
- (2) There exists an element  $e \in G$  called the **identity** element such that  $ae = ea = a$  for all  $a \in G$ .
- (3) For each  $a \in G$ , there is an element  $b \in G$ , called the **inverse** of  $a$  such that  $ab = ba = e$ , where  $e$  is the identity element.

*Remark.* We make note that property (0) of this definition is implied by stating  $*$  as a binary operation on  $G$ , we however list it, because when verifying a given set is a group, we usually want to check for closure.

*Remark.* Instead of stating  $(G, *)$  as a group, we will often just say that  $G$  is a group under  $*$ , or simply,  $G$  is a group.

**Example 1.2.** (1) The set  $G = \{e\}$  of one element forms a group under any operation, and is called the **trivial** group. We write  $G = \langle e \rangle$ .

- (2) The sets  $\mathbb{Z}$ ,  $\mathbb{R}$ , and  $\mathbb{Q}$  are all groups under the usual addition. Here 0 is the identity, and  $-a$  is the inverse of  $a$ .  $\mathbb{C}$  is a group under complex addition with  $0 = 0 + i0$  the identity and  $-a - ib$  the inverse of  $a + ib$ .
- (3)  $\mathbb{Q}$  and  $\mathbb{R}$  are groups under the usual multiplication, with identity 1 and inverse  $a^{-1} = \frac{1}{a}$ .  $\mathbb{Z}$  is not a group under this operation, as  $\frac{1}{a} \notin \mathbb{Z}$  whenever  $a \in \mathbb{Z}$ .  $\mathbb{C}$  is a group under complex multiplication with identity  $1 = 1 + i0$  and inverse  $\frac{a}{a^2+b^2} - i\frac{b}{a^2+b^2}$  for  $a + ib$ .
- (4) Consider the set of integers  $\bmod n$ ,  $\mathbb{Z}/n\mathbb{Z}$  under addition  $\bmod n$ ,  $+$ . Since  $+$  is a binary operation on  $\mathbb{Z}/n\mathbb{Z}$ , closure is guaranteed. We also see that associativity holds. Now, notice that  $n \equiv 0 \bmod n$ , by definition, so  $a + n = n + a \equiv 0 + a \bmod n \equiv a + 0 \bmod n \equiv a$ . Moreover,  $n - a \equiv 0 - a \equiv -a \bmod n$ , and  $(n - a) + a = n(-a + a) = n \equiv 0 \bmod n$  and  $a + (n - a) = n + (a - a) = n \equiv 0 \bmod n$ . So  $(\mathbb{Z}/n\mathbb{Z}, +_n)$  is a group, with identity element  $0 \bmod n$  and inverse element  $-a \bmod n$  for each  $a \in \mathbb{Z}/n\mathbb{Z}$ .

**Example 1.3.** Suppose we removed the restriction to be nonempty in the definition of a group. We see that if  $G = \emptyset$ , then  $G$  cannot be a group, since it is not closed, trivially; furthermore, there is no identity, nor inverse to each element. Therefore the minimum number of elements a group can have is 1. This makes the trivial group minimal.

**Definition.** We call a group  $G$  under a binary operation  $*$  **Abelian**, or **commutative** if for every  $a, b \in G$ ,  $ab = ba$ .

**Example 1.4.** The above examples of groups are also examples of abelian groups.

**Example 1.5.** Consider  $(\mathbb{Z}/n\mathbb{Z}) \setminus \{0\}$  under multiplication  $\pmod n$ ,  $\cdot_n$  (abbreviated as  $\cdot$ ). This is not a group as not every element has an inverse. Specifically, take  $n = 6$ , then in  $\mathbb{Z}/6\mathbb{Z}$ ,  $2 \cdot 3 = 6 \equiv 0 \pmod 6 \notin (\mathbb{Z}/n\mathbb{Z}) \setminus \{0\}$ . However, one can still impose a group structure with modular multiplication.

Define the set  $U(\mathbb{Z}/n\mathbb{Z}) = \{a \in \mathbb{Z}/n\mathbb{Z} : (a, n) = 1\}$ , that is it is the set of all integers  $\pmod n$  coprime with  $n$ . We have that  $U(\mathbb{Z}/n\mathbb{Z})$  is closed under  $\cdot$ . Notice that if  $(a, n) = 1$ , and  $(b, n) = 1$ , then  $(ab, n) = 1$ . Also notice that  $U(\mathbb{Z}/n\mathbb{Z}) \subseteq \mathbb{Z}/n\mathbb{Z}$ , and so inherits associativity. Moreover, notice that  $(1, n) = 1$ , and  $a1 = 1a = a \in U(\mathbb{Z}/n\mathbb{Z})$ , so  $1 \pmod n$  is the identity of the set. Now for any  $a \in U(\mathbb{Z}/n\mathbb{Z})$ , since  $(a, n) = 1$ , there exist  $b, m \in \mathbb{Z}$  such that  $ab + mn = 1$ , that is  $ab \equiv 1 \pmod n$ , and moreover notice that if  $(ab, n) = 1$  and  $(a, n) = 1$ , then  $(b, n) = 1$ , thus  $b \in U(\mathbb{Z}/n\mathbb{Z})$ , and is an inverse of  $a$ . Thus we have shown that  $U(\mathbb{Z}/n\mathbb{Z})$  is a group under  $\cdot$ . We call the group the **group of units**,  $\pmod n$ , or simply the **unit group**  $\pmod n$ . Moreover, we see that this group is commutative.

**Example 1.6.** (1) The vector space axioms for some vector space  $V$  specify that under vector addition  $+$ ,  $(V, +)$  forms a group.

(2) Let  $(A, *)$ ,  $(B, \cdot)$  be groups under binary operations  $*$  and  $\cdot$ . Consider the product  $A \times B$  and take the map  $\circ : (a_1, b_1)(a_2, b_2) \rightarrow (a_1 * a_2, b_1 \cdot b_2)$ . Then  $\circ$  is a binary operation on  $A \times B$ . Then  $(A \times B, \circ)$  forms a group. We have that since  $A$  and  $B$  is closed, then so is  $A \times B$ . Furthermore, by associativity of  $*$  and  $\cdot$ ,  $((a_1, b_1) \circ (a_2, b_2)) \circ (a_3, b_3) = (a_1, b_1) \circ ((a_2, b_2) \circ (a_3, b_3))$ ; making  $\circ$  associative. Now if  $e_1$  and  $e_2$  are the identities of  $A$  and  $B$  respectively, then  $(e_1, e_2)$  is the identity for  $A \times B$ ; finding the inverse of an element  $(a, b)$  follow similarly.

**Theorem 1.2.1.** *Let  $G$  be a group under a binary operation  $*$  then the identity of  $G$  is unique, and the inverse of  $a \in G$  is unique.*

*Proof.* Suppose there exists  $e, f \in G$  such that for any  $a \in G$   $ae = ea = a$  and  $af = fa = a$ . Then we have that  $fe = e$  and  $ef = fe = f$ ; thus  $e = f$ .

Now let  $a \in G$ , and suppose  $a$  has inverses  $b, c \in G$ , then  $ab = ba = e$  and  $ac = ca = e$ , where  $e$  is the identity of  $G$ . Then we have  $b = be = b(ac) = (ba)c = ec = c$ , thus  $b = c$ . ■

*Remark.* Since the inverse of an element  $a$  is unique, we will now denote it  $a^{-1}$ .

**Corollary.**  $(a^{-1})^{-1} = a$  and  $(ab)^{-1} = b^{-1}a^{-1}$



*Proof.* Since inverses are unique, the  $a^{-1} \in G$  has the unique inverse  $a^{-1-1}$ . Then taking  $aa^{-1} = e$ , applying inverses, we get  $a(a^{-1}(a^{-1})^{-1}) = e(a^{-1})^{-1}$ ,  $a = (a^{-1})^{-1}$ .

Now Let  $a, b \in G$ , then  $ab(ab)^{-1} = e$ . Applying the inverse of  $a$  on the right to both sides, we get  $b(ab)^{-1} = a^{-1}$ ; again with the inverse of  $b$  yields  $(ab)^{-1} = b^{-1}a^{-1}$ . ■

**Theorem 1.2.2** (Generalized Associativity). *Let  $G$  be a group under a binary operation  $*$ , then for any  $a_1, a_2, \dots, a_n \in G$ , the product  $a_1 * a_1 * \dots * a_n$  is independent of the ordering of any brackets.*

*Proof.* By induction on  $n$ , for  $n = 1$ , we just have the element  $a_1$ , for  $n = 2$  we have  $a_1a_2$  has only one possible bracketing  $(a_1b_1)$ ; and for  $n = 3$ , the associativity group law guarantees  $a_1(a_2a_3) = (a_1a_2)a_3$ .

Now for any  $k < n$ , the the bracketing of  $k$  elements  $a_1, \dots, a_k$  is can be reduced to the expression

$$a_1 * (a_2 * (a_3 * (\dots * a_k))).$$

Now we see that  $a_1 * \dots * a_n$  can be bracketed into the products:

$$(a_1 * \dots * a_k) * (a_{k+1} * \dots * a_n)$$

which can be bracketed, by hypothesis as:

$$(a_1 * (a_2 * (a_3 * (\dots * a_k)))) * (a_{k+1} * (a_{k+2} * (a_{k+3} * (\dots * a_n))))$$

Therefore, applying the associative group law to this product, we get that

$$a_1 * \dots * a_n = a_1 * (a_2 * (a_3 * (\dots * a_n)))$$

This completes the proof. ■

**Theorem 1.2.3** (The Cancellation Laws). *Let  $G$  be a group under a binary operation  $*$ . Then for  $a, b, c \in G$ , we have:*

(1)  $ab = ac$  implies  $b = c$  (Left Cancellation Law).

(2)  $ba = ca$  implies  $b = c$  (Right Cancellation Law).

*Proof.* Suppose that  $ab = ac$ , then applying the inverse of  $a$  on the left, we get  $(a^{-1}a)b = (a^{-1}a)c$ , hence  $eb = ec$ , thus  $b = c$ . Similarly, we get  $b = c$  if we apply  $a^{-1}$  to the right in the equation  $ba = ca$ . ■

**Corollary.** *For  $x, y \in G$ , the equations  $ax = b$  and  $ya = b$  have unique solutions.*

*Proof.* We have  $x = ba^{-1}$  and  $y = a^{-1}b$ . Since inverses are unique, so are the solutions  $x$  and  $y$ . ■

**Definition.** Let  $G$  be a group under a binary operation  $*$ . For any  $a \in G$ , and  $n \in \mathbb{Z}^+$ , we define the  $n$ -th power of  $a$  to be:

$$a^n = \underbrace{a * \dots * a}_{n \text{ times.}}$$

We define  $a^0 = e$  and  $a^{-n} = (a^{-1})^n$ .

**Lemma 1.2.4.** *Let  $G$  be a group under a binary operation  $*$ , and let  $a \in G$  and  $m, n \in \mathbb{Z}^+$ . Then:*

$$(1) \ a^m a^n = a^{m+n}.$$

$$(2) \ (a^m)^n = a^{mn}.$$

*Proof.* We have by definition that  $a^m a^n = \underbrace{a * \cdots * a}_{m \text{ times.}} \underbrace{a * \cdots * a}_{n \text{ times.}} = \underbrace{a * \cdots * a}_{m+n \text{ times.}} = a^{m+n}$ .

$$\text{Likewise, } (a^m)^n = \underbrace{a^m * \cdots * a^m}_{n \text{ times.}} = \underbrace{a * \cdots * a}_{nm \text{ times.}} = a^{mn}. \quad \blacksquare$$

We can now, unless context isn't clear enough, drop all mention to the binary operation of a group.

**Definition.** We define the **order** of a group  $G$  to be the number of elements of  $G$  and denote it  $\text{ord } G$ . That is,  $\text{ord } G = |G|$ . If  $G$  is infinite, then we say that  $G$  has **infinite order**; otherwise, we say  $G$  is of **finite order**.

**Definition.** Let  $G$  be a group and let  $a \in G$ . We define the **order** of  $a$  to be the smallest positive integer  $n \in \mathbb{Z}^+$  for which  $a^n = e$ . If there is no such integer  $n$ , then we say  $a$  has **infinite order**, otherwise, we say  $a$  has **finite order**, and write  $\text{ord } a = n$ .

We conclude the section with more examples and one last definition.

**Lemma 1.2.5.** *Let  $G$  be a group, suppose  $a, b \in G$  are elements with  $\text{ord } a = m$ ,  $\text{ord } b = n$ , for  $m, n \in \mathbb{Z}^+$ , and that  $ab = ba$ . Then  $\text{ord } ab = [m, n]$ .*

*Proof.* Let  $\text{ord } ab = k$  for  $k \in \mathbb{Z}^+$ . Since  $a$  and  $b$  commute, we get  $a^k = e$  and  $b^k = e$ . Now, by the division theorem, there are integers  $q_1, q_2, r_1, r_2 \in \mathbb{Z}^+$  such that  $k = q_1 m + r_1$  and  $k = q_2 n + r_2$ . Then we get that  $a^k = a^{q_1 m + r_1} = a^{q_1 m} a^{r_1} = a^{r_1} = e$ . Since  $\text{ord } a = m$ , this makes  $r_1 = 0$ . Similarly, we get that  $b^k = b^{r_2} = e$  implies  $r_2 = 0$ . Therefore,  $k = q_1 m = q_2 n$ ; moreover,  $\text{ord } ab = k$  is minimal by definition, thus we get  $\text{ord } ab = k = [m, n]$ .  $\blacksquare$

**Corollary.** *For  $a_1, \dots, a_k \in G$ , with  $\text{ord } a_i = n_i$  and  $a_i a_j = a_j a_i$  for all  $1 \leq i, j \leq k$ , then  $\text{ord } a_1 a_2 \dots a_k = [n_1, \dots, n_k]$ .*

*Proof.* By induction, when  $k = 1$ , the case is trivial. Now, for  $k = 2$ , the result follows by the above theorem. Now suppose that  $\text{ord } a_1 \dots a_k = [n_1, \dots, n_k]$ . Take  $a_{k+1} \in G$  with  $\text{ord } a_{k+1} = n_{k+1}$  and  $(a_1 \dots a_k) a_{k+1} = a_{k+1} (a_1 \dots a_k) = a_1 \dots a_k a_{k+1}$ . Then, by the above theorem, we get  $\text{ord } a_1 \dots a_k a_{k+1} = [[n_1, \dots, n_k], n_{k+1}] = [n_1, \dots, n_k, n_{k+1}]$ .  $\blacksquare$

**Example 1.7.** (1)  $\text{ord } \langle e \rangle = 1$ .

- (1) In any group  $G$ ,  $\text{ord } e = 1$ , and if  $a \in G$  has  $\text{ord } a = 1$ , then necessarily,  $a = e$ . That is in any group, the only element of order 1 is the identity.
- (2) The additive groups  $\mathbb{Z}$ ,  $\mathbb{Q}$ ,  $\mathbb{R}$ , and  $\mathbb{C}$  have infinite order, and their nonzero elements also have infinite order.

Figure 1.1: The group,  $D_{14}$  of symmetries on a heptagon.

- (3) The multiplicative group  $\mathbb{C}^*$  has infinite order, moreover, since  $i^4 = 1$  and  $i^3 = -i$ ,  $\text{ord } i = 4$ .
- (4)  $\text{ord } \mathbb{Z}/n\mathbb{Z} = n$ , and every element is of finite order.  $\text{ord } U(\mathbb{Z}/n\mathbb{Z}) = \phi(n)$ , where  $\phi$  is the Euler totient function. Every element of  $U(\mathbb{Z}/n\mathbb{Z})$  also has finite order.

**Definition.** Let  $G = \{g_1, \dots, g_n\}$  be a finite group of order  $n$  with  $g_1 = e$ . We define the **Cayley table**, or **multiplication table** of  $G$  to be the  $n \times n$  matrix defined by the entries  $(g_{ij})$  where

$$g_{ij} = g_i g_j$$

for  $1 \leq i, j \leq n$ .

### 1.3 Dihedral Groups and Group Generators.

Let  $\mathbb{Z}^+$  and define  $D_{2n}$  to be the set of symmetries of a regular  $n$ -gon, where a symmetry is just a permutation of the vertices. That is, if  $S$  is the set of vertices, then a symmetry on  $S$  is just a map taking  $S \rightarrow S$ . We would like to characterize the symmetries of  $D_{2n}$  (See figure 1.2).

Since the set of vertices of the  $n$ -gon is arbitrary, but finite, we can label  $S$  how we see fit. Let us label  $S = \mathbb{Z}/n\mathbb{Z}$  and define the symmetries  $t : \mathbb{Z}/n\mathbb{Z} \rightarrow \mathbb{Z}/n\mathbb{Z}$  and  $r : \mathbb{Z}/n\mathbb{Z} \rightarrow \mathbb{Z}/n\mathbb{Z}$  by  $t : i \rightarrow -i$  and  $r : i \rightarrow i + 1$ . That is  $t$  is a transposition (or reflection) of the vertices, and  $r$  is a rotation of the vertices by an angle of  $\frac{2\pi}{n}$ . Now, define the “identity” symmetry  $e : i \rightarrow i$ . Then notice that  $t^2 : i \rightarrow -i \rightarrow -(-i) = i$ , and  $r^n : i \rightarrow i + 1 \rightarrow \dots \rightarrow i + n \equiv i$ . So  $t^2 = r^n = e$  (more over, if we treat  $e$  as a rotation of the vertices by an angle of  $2\pi$ , then note that  $r^n$  applies a rotation of the vertices by an angle of  $\frac{2n\pi}{n} = 2\pi$ , which makes it the identity symmetry). It is easy to see that  $t$  and  $r$  are 1-1 and onto, so  $t^{-1} = t$  and  $r^{-1} = r^{n-1}$ .

Now that we have characterized the symmetries  $r$  and  $t$ , what about  $r \circ t$ ? Abbreviating  $r \circ t$  as  $rt$ , we see that  $rt : i \rightarrow -i \rightarrow -i + 1 = -(i + 1)$ . Now, notice that  $r^{-1} = r^{n-1} : i \rightarrow i + (n - 1) \equiv i - 1$ . Then,  $tr^{-1} : i \rightarrow i - 1 \rightarrow -(i - 1) = -i + 1$ . Thus  $rt = tr^{-1}$ . This gives us the following lemma.

**Lemma 1.3.1.** For the symmetries  $r$  and  $t$ , and for  $i \in \mathbb{Z}/n\mathbb{Z}$ ,  $r^i t = t r^{-i}$ .

*Proof.* By induction on  $i$ , we have for  $i = 1$  that  $rt = tr^{-1}$ . Now suppose for  $i$  that  $r^i t = tr^{-i}$ . Then  $r^{i+1}t = r(r^i t) = (rt)r^{-i} = tr^{-1}r^{-i} = tr^{-i-1} = tr^{-(i+1)}$ . ■

We can now characterize the set  $D_{2n}$ .

**Definition.** Let  $S$  be a set of  $n$  elements, we define the **dihedral group** to be the set of all permutations on  $S$  with the form  $D_{2n} = \langle r, t : r^n = t^2 = e, rt = tr^{-1} \rangle$

**Theorem 1.3.2.**  $D_{2n}$  forms a group under function composition  $\circ$ , and the elements of  $D_{2n}$  are of the form  $r^i t^j$  with  $i \in \mathbb{Z}/n\mathbb{Z}$  and  $j \in \mathbb{Z}/2\mathbb{Z}$ .

*Proof.* Since  $r, t \in D_{2n}$ , we have that  $rt \in D_{2n}$  since  $rt$  is also a permutation. Now by the above relations, we have that any element in  $D_{2n}$  has the form  $r^i t^j$  where  $i \in \mathbb{Z}/n\mathbb{Z}$  and  $j \in \mathbb{Z}/2\mathbb{Z}$ . Now let  $r^i t^j, r^l t^k \in D_{2n}$  with  $i, l \in \mathbb{Z}/n\mathbb{Z}$  and  $j, k \in \mathbb{Z}/2\mathbb{Z}$ . Then  $(r^i t^j)(r^l t^k) = r^i (t^j t^k) r^{-l} = r^i t^{j+k} r^{-l} = (r^i r^l) t^{j+k} = r^{i+l} t^{j+k}$ . Now, by the closure of both  $\mathbb{Z}/n\mathbb{Z}$ , and  $\mathbb{Z}/2\mathbb{Z}$ , we get that  $r^{i+l} t^{j+k} \in D_{2n}$  which establishes closure of  $D_{2n}$  under  $\circ$ . We also see that since  $D_{2n}$  is a set of permutations, it inherits the associativity of  $\circ$ .

Now consider the identity symmetry  $e : i \rightarrow i$ . We have  $e = r^n t^2$ , so for any  $r^i t^j$ ,  $(r^i t^j)(r^n t^2) = r^{i+n} t^{j+2} = r^i r^n t^j t^2 = r^i e t^j e = r^i t^j$ . Likewise  $(r^n t^2)(r^i t^j) = (r^i t^j)$ . We also get that since  $t^{-1} = t$  and  $r^{-1} = r^{n-1}$  then if  $s \in D_{2n}$ , then  $(r^i t^j)s = e$  implies, by cancellation laws, that  $s = t^j r^{-i}$ , which serves as an inverse to  $r^i t^j$ . Therefore  $D_{2n}$  is a group. ■

**Corollary.**  $\text{ord } D_{2n} = 2n$ .

*Proof.* We have that each element of  $D_{2n}$  is of the form  $r^i t^j$  where  $i \in \mathbb{Z}/n\mathbb{Z}$  and  $j = 0$  or  $j = 1$ . Thus there are two possible choices for  $j$  and  $n$  possible choices for  $i$ , therefore there are  $2n$  possible choices for  $r^i t^j$ , since this element is arbitrary, we have that this enumerates all the elements of  $D_{2n}$ . ■

**Corollary.**  $D_{2n} = \{e, t, r, r^2, \dots, r^{n-1}, rt, r^2t, \dots, r^{n-1}t\}$ .

*Proof.* Compute each element  $r^i t^j$ , iterating over  $i$  and  $j$ . ■

Thus we have entirely described the set of symmetries of a regular  $n$ -gon in group theoretic terms; and have found that they follow a certain (special case, of a more general) group structure. In fact, we have found elements  $r, t$  that “generate” the symmetries, and found relations which we can use to describe the group. This leads us to the following definition.

**Definition.** Let  $G$  be a group. We say that a subset,  $S \subseteq G$  **generates** the group  $G$  if for every  $g \in G$ ,  $g$  is the finite product of elements of  $S$ . We write  $G = \langle S \rangle$  and call  $S$  the **generator** of  $G$ . If the elements of  $S$  satisfy a set of relations  $R_1, \dots, R_n$ , then we say  $G$  is **represented** by  $S$  by  $R_1, \dots, R_n$  and write  $G = \langle S : R_1, \dots, R_n \rangle$ , and call this form the **representation** of  $G$ .

**Example 1.8.** (1)  $\mathbb{Z} = \langle 1 \rangle$ .

(2)  $\mathbb{Z}/n\mathbb{Z} = \langle 1 \rangle$ .



Figure 1.2: The transposition  $t \in D_{14}$  of vertices of a heptagon and the rotation  $r \in D_{14}$  about an angle of  $\frac{2\pi}{7}$  of the vertices.

(3)  $D_{2n} = \langle r, t \rangle$

(4) Define  $X_{2n} = \langle x, y : x^n = y^2 = e, xy = yx^2 \rangle$ . Since  $y^2 = e$ , we have  $x = xy^2 = (xy)y = y(x^2y) = y^2x^4 = x^4$ ; thus  $x^4 = x$ , hence  $x^3 = e$ . Therefore for any  $n$ ,  $\text{ord } X_{2n} = 6$ , by the same argument we made for  $D_{2n}$ .

(5) Let  $Y = \langle u, v : u^4 = v^3 = e, uv = v^2u^2 \rangle$ . We have  $u = uv^3 = v^6u^2 = (v^3)^2u^2 = e^2u^2 = u^2$ , hence  $u^2 = u$ , hence  $u = e$ ; thus we also get that  $v^2 = v$ , hence  $v = e$ . Therefore  $Y = \langle e \rangle$ , the trivial group.

The previous two examples show that not every relation may be listed in the representation of a given group. It turns out in the case of  $D_{2n}$ , that all the relations are listed, but as in the above example with  $X_{2n}$  and  $Y$ , we had the relations  $x^4 = x$  and  $u^2 = u, v^2 = v$ . Thus one may be careful when dealing with group representations and take care that no relations are left unattended. One consequence might be an erroneous arguing of group order; one can be led to believe that  $\text{ord } X_{2n} = 2n$ , where in reality  $\text{ord } X_{2n} = 6$  and that  $\text{ord } Y = 6$ , where in reality,  $\text{ord } Y = 1$ .

## 1.4 Permutation Groups and the Symmetric Group.

**Definition.** Let  $S$  be any set, we define  $S(A)$  to be the **symmetric group** on  $A$  of all permutations from  $A$  onto itself. If  $A = \mathbb{Z}/n\mathbb{Z}$  for  $n \in \mathbb{Z}^+$ , we write  $S(A) = S_n$ .

**Theorem 1.4.1.** *For any set  $S$ ,  $A(S)$  is a nonabelian group under function composition.*

*Proof.* For any  $f, g \in A(S)$ , we get that  $f \circ g$  is also a permutation, so  $f \circ g \in A(S)$ ; moreover,  $A(S)$  is associative on account of  $\circ$ .

Now, the identity map  $e : i \rightarrow i$ , for any  $i \in S$  is the identity of  $A(S)$ , and since  $f \in A(S)$  is 1-1 and onto,  $f^{-1} \in A(S)$ . This makes  $A(S)$  a group. That  $A(S)$  is nonabelian follows from the noncommutativity of  $\circ$ . ■

**Corollary.**  $\text{ord } S_n = n!$ .

*Proof.* The proof of this follows by a combinatorial argument counting the number of  $1-1$  maps, and the number of onto maps. ■

**Definition.** We define the **cycle** of a permutation  $s \in S_n$  to be a string of integers of  $\mathbb{Z}/n\mathbb{Z}$ ,  $(a_0 \dots a_{n-1})$  where  $s : a_i \rightarrow a_{i+1}$ , where  $i \in \mathbb{Z}/n\mathbb{Z}$ . We call two cycles **disjoint** if they share no entries.

**Lemma 1.4.2.** *The cycle decomposition of a given permutation  $s \in S_n$  is finite.*

*Proof.* Let  $s = (a_0 \dots a_{n-1})$  be the cycle decomposition of  $s$ . Then by definition, we get  $s : a_{n-1} \rightarrow a_{(n-1)+1} = a_n = a_0$ . ■

**Example 1.9.** The cycle  $(2 \ 0 \ 1)$  represents the permutation  $s : 2 \rightarrow 0 \rightarrow 1$  in  $S_3$ .

Now, since cycles of permutations of  $S_n$  are finite, we can define the “length” of a cycle.

**Definition.** The **length** of a cycle of a permutation  $s \in S_n$  is the number of entries in the cycle. We call a cycle of length  $k \in \mathbb{Z}^+$  a  **$k$ -cycle**.

**Lemma 1.4.3.** *For any permutation  $s \in S_n$ , the elements of  $\mathbb{Z}/n\mathbb{Z}$  can be grouped into  $k$  cycles of the form:*

$$(a_0 \ a_2 \ \dots \ a_{m_1})(a_{m_1+1} \ \dots \ a_{m_2}) \dots (a_{m_{k-1}} \ \dots \ a_{m_k}) \quad (1.2)$$

*Proof.* Since  $s$  is  $1-1$  and onto,  $s$  will permute through the entirety of  $\mathbb{Z}/n\mathbb{Z}$ ; so every integer mod  $n$  will be represented in the cycle for  $s$ .

Now, find  $x \in \mathbb{Z}/n\mathbb{Z}$  in a cycle for  $s$ . If  $x$  is not at the end of the cycle, i.e. if  $s(x)$  is not some previous element of cycle of  $x$ , then  $s(x)$  is next integer in the cycle of  $x$ . Otherwise,  $s(x)$  is the first integer of another cycle of  $s$ , i.e. if  $x = a_{m_i}$ , then  $s(x) = a_{m_i+1}$ . There are  $k$  such possible cycles for  $s$ , where  $k \in \mathbb{Z}^+$ . ■

**Definition.** For any permutation  $s \in S_n$ , we call the concatenation of all cycles of  $s$  the **cycle decomposition** of  $s$ .

We introduce a neat algorithm for finding the cycle decomposition of a permutation.

**Algorithm** (The Cycle Decomposition Algorithm). *Let  $s \in S_n$  be a permutation of the elements of  $\mathbb{Z}/n\mathbb{Z}$ .*

- step 1:** Choose the smallest  $i \in \mathbb{Z}/n\mathbb{Z}$  which has not appeared in a previous cycle; if there is no previous cycle,  $i = 0$ . Start the cycle at  $i$ .
- step 2:** Compute  $s(i)$ . If  $s(i) = i$ , close the cycle, and return to **step 1**. Else, concatenate  $s(i)$  to  $i$  in the cycle.
- step 3:** Repeat **step 2** with  $s(i)$ .
- step 4:** If  $\mathbb{Z}/n\mathbb{Z}$  has been exhausted, go to **step 5**; else return to **step 3**.

**step 5:** Remove all 1-cycles and stop.

*Remark.* A neat exercise immediately introduces itself as the problem of how to program this cycle decomposition algorithm so that one can simply feed a permutation into a computer and get its cycle decomposition as an output.

**Example 1.10.** Define the permutation  $s \in S_{13}$  by:

$$\begin{array}{lll} s : 1 \rightarrow 12 & s : 2 \rightarrow 0 & s : 3 \rightarrow 3 \\ s : 4 \rightarrow 1 & s : 5 \rightarrow 11 & s : 6 \rightarrow 9 \\ s : 7 \rightarrow 5 & s : 8 \rightarrow 10 & s : 9 \rightarrow 6 \\ s : 10 \rightarrow 4 & s : 11 \rightarrow 7 & s : 12 \rightarrow 8 \\ s : 0 \rightarrow 2 & & \end{array}$$

Using the cycle decomposition algorithm we get:

$$s = (0\ 2)(1\ 12\ 8\ 10\ 4)(5\ 11\ 7)(6\ 9)$$

The cycle decomposition algorithm provides a neat way of finding the cycle decomposition  $s$ ; what about  $s^{-1}$ , when we are given  $s$ ?

**Lemma 1.4.4.** Let  $s \in S_n$  be a permutation with the cycle decomposition  $s = (a_0\ a_2\ \dots\ a_{m_1})(a_{m_1+1}\ \dots\ a_{m_2})\dots(a_{m_{k-1}}\ \dots\ a_{m_k})$ . Then  $s^{-1}$  has the cycle decomposition:

$$s^{-1} = (a_{m_1}\ \dots\ a_2\ a_0)(a_{m_2}\ \dots\ a_{m_1+1})\dots(a_{m_k}\ \dots\ a_{m_{k-1}}) \quad (1.3)$$

*Proof.* If  $s : a_{m_i} \rightarrow a_{m_i+1}$ , then  $s^{-1} : a_{m_i+1} \rightarrow a_{m_i}$ . Then by the cycle decomposition algorithm we can derive equation (1.3). ■

We finally come wish to introduce the “product” of cycles.

**Definition.** Let  $s, t \in S_n$  be a permutation with a cycle decompositions defined by the rules  $s : a_{m_i} \rightarrow a_{m_i+1}$  and  $t : b_{m_i} \rightarrow b_{m_i+1}$ . Then we define the **product** of cycle decompositions,  $\circ$ , to be:  $s \circ t$  whose cycle decomposition is defined by the rule  $s \circ t : b_{m_i} \rightarrow b_{m_i+1} \rightarrow a_{m_j}$ , where  $a_{m_j} = s(b_{m_i+1})$  and  $b_{m_i+1} = t(b_{m_i})$ . We define the concatenation of cycles to be the product of cycle decompositions.

**Example 1.11.** Consider the cycles  $(1\ 3)$  and  $(1\ 2)(3\ 0)$  in  $S_4$ . Then  $(1\ 3) \circ (1\ 2)(3\ 0) = (1\ 3\ 0)$ .

We can now rephrase theorem 1.4.1 as:

**Theorem 1.4.5.** Define  $S_n$  to be the set of all cycle decompositions of permutations of the elements of  $\mathbb{Z}/n\mathbb{Z}$ . Then  $S_n$  is a group under cycle products.

**Corollary.**  $S_n$  is nonabelian for  $n \geq 3$ .

**Corollary.** Disjoint cycles commute.

**Lemma 1.4.6.** *Let  $s \in S_n$  be a  $k$ -cycle. Then  $\text{ord } s = k$ .*

*Proof.* We have that  $s = (a_1 \ a_2 \ \dots \ a_k)$ , thus  $s$  maps  $a_i \rightarrow a_{i+1}$ , for  $i \in \mathbb{Z}/k\mathbb{Z}$ ; also notice that  $s : a_{k-1} \rightarrow a_0$ . Then  $s^k : a_i \rightarrow a_{i+k \bmod k} = a_i$ , for all  $i$ , thus  $s^k = (1)$ . Moreover, if  $m \in \mathbb{Z}^+$  such that  $s^m : a_i \rightarrow a_{i+m \bmod k}$ , then  $a_i = a_j$  for some other  $j \in \mathbb{Z}/k\mathbb{Z}$ , which implies that  $s$  is a  $k-1$ -cycle, which cannot happen. Thus  $\text{ord } s = k$ . ■

**Corollary.** *If  $s, t \in S_n$  are  $k$  and  $m$ -cycles, respectively, then  $\text{ord } st = [k, m]$ .*

**Corollary.** *Let  $s \in S_n$ , then the cycle composition of  $s$  is a product of disjoint  $m_k$  cycles where  $k \in \mathbb{Z}/n\mathbb{Z}$ .*

## 1.5 The General and Special Linear Groups of $n \times n$ Matrices.

One special class of groups are those that can be defined on matrices. We first need to define, in an elementary sense what a “field” is; though we will not go into their study here. We assume familiarity with matrix algebra such as matrix multiplication and determinants. This makes this section, in a sense, optional.

**Definition.** Let  $F$  be a set together with binary operations  $+$ , called **addition** and  $\cdot$ , called **multiplication**. We call  $(F, +, \cdot)$  a **field** if:

- (1)  $(F, +)$  forms an abelian group.
- (2)  $(F^*, \cdot)$  forms an abelian group; where  $F^* = F \setminus \{e\}$ ,  $e$  is the identity of  $F$  under  $+$ .
- (3)  $\cdot$  **distributes** over  $+$ ; that is, for  $a, b, c \in F$ ,  $a(b + c) = ab + ac$ .

**Example 1.12.** (1) The sets  $\mathbb{Q}$  and  $\mathbb{R}$  are fields under the usual addition and multiplication.

(2)  $\mathbb{C}$  is a field under complex addition and complex multiplication. So is  $\mathbb{R}$  if we take all  $a \in \mathbb{R}$  to have the form  $a + i0$ .

(3)  $\mathbb{Z}/p\mathbb{Z}$ , with  $p \in \mathbb{Z}^+$  prime forms a field under addition and multiplication  $\bmod p$ .

**Definition.** Let  $F$  be a field. We define  $F^{n \times n}$  to be the field of all  $n \times n$  matrices with entries in  $F$ . We define the **general linear group** to be  $GL(n, F) = \{A \in F^{n \times n} : \det A \neq 0\}$ . We define the **special linear group** to be  $SL(n, F) = \{A \in F^{n \times n} : \det A = 1\}$ . If  $F = \mathbb{Z}/p\mathbb{Z}$ , we write  $GL(n, \mathbb{Z}/p\mathbb{Z}) = GL(n, p)$  and  $SL(n, \mathbb{Z}/p\mathbb{Z}) = SL(n, p)$ .

**Theorem 1.5.1.** *For any field  $F$ , and  $n \in \mathbb{Z}^+$ ,  $GL(F, n)$  forms a group under matrix multiplication.*



*Proof.* Let  $A, B \in GL(F, n)$  be  $n \times n$  matrices. Then  $\det A \neq 0$  and  $\det B \neq 0$ , so  $\det AB = \det A \det B \neq 0$ , by a well known property of determinants. So  $GL(F, n)$  is closed. Now since matrix multiplication is associative, then  $GL(F, n)$  satisfies the associative law.

Now consider the  $n \times n$  identity matrix  $I$ , we have for any  $A \in GL(F, n)$ ,  $AI = IA = A$ , moreover,  $\det I = 1 \neq 0$  making  $I \in GL(F, n)$ . Likewise, since for  $A \in GL(F, n)$ ,  $\det A \neq 0$ ,  $A$  is invertible, by well known properties of matrices, so  $A^{-1}$  exists, and  $\det A^{-1} = \det A \neq 0$ . Thus  $A^{-1} \in GL(F, n)$  and since  $AA^{-1} = A^{-1}A = I$ , this makes  $A^{-1}$  the inverse of  $A$ . ■

**Corollary.**  $SL(F, n)$  forms a group under matrix multiplication.

*Proof.* Notice that  $SL(F, n) \subseteq GL(F, n)$ , so  $SL(F, n)$  inherits closure (and associativity). Now, for  $A \in SL(F, n)$ ,  $A \in GL(F, n)$ , so  $A^{-1}$  exists. Moreover,  $\det A^{-1} = \det A = 1$ , making  $A^{-1} \in SL(F, n)$ . This also implies that  $I \in SL(F, n)$ . ■

**Example 1.13.**

$$GL(2, 2) = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \right\}$$

Labeling these elements as  $I, A, B, C, D$ , and  $E$ , consecutively we find the orders to be:  $\text{ord } I = 1$ ,  $\text{ord } A = 2$ ,  $\text{ord } B = 2$ ,  $\text{ord } C = 3$ ,  $\text{ord } D = 3$ , and  $\text{ord } E = 2$ .

**Example 1.14.** Consider  $GL(2, 2)$ , and considering the labeling of the above example, we compute the Cayley table to be:

$$\begin{pmatrix} I & A & B & C & D & E \\ A & I & D & E & B & C \\ B & C & I & A & E & D \\ C & B & E & D & I & A \\ D & E & I & I & C & I \\ E & D & C & B & A & I \end{pmatrix}$$

which is not symmetric, hence  $GL(2, 2)$  is not Abelian. In general, for  $n, p \in \mathbb{Z}^+$  and  $p$  prime, for  $A, B \in GL(n, p)$  we have

$$\begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} B & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} AB & 0 \\ 0 & 1 \end{pmatrix}$$

while

$$\begin{pmatrix} B & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} BA & 0 \\ 0 & 1 \end{pmatrix}$$

then

$$\begin{pmatrix} AB & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} BA & 0 \\ 0 & 1 \end{pmatrix}$$

if, and only if  $AB = BA$ , which is in general, not true for matrices. So  $GL(n, p)$  is not necessarily Abelian.

Now, we would like to observe the order of the group  $GL(n, p)$ , the order of  $SL(n, p)$  will be derived later.

**Example 1.15.** (1) We have that if  $F$  is a field with  $\text{ord } F = p$ , then  $\text{ord } GL(n, F) < p^{n^2}$ , for, notice for any  $A \in F^{n \times n}$ , there are  $n^2$  entries, and  $p$  choices for each entry, thus  $\text{ord } F^{n \times n} = p^{n^2}$ , now, by definition,  $GL(n, F)$  excludes those with  $\det = 0$ , thus we get the result.

(2) Let  $A \in GL(2, 2)$  where:

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

where  $a, b, c, d \in \mathbb{Z}/2\mathbb{Z}$ . Then we have that if  $ad - bc \neq 0$ , then  $ad \neq bc$ , thus  $a$  is a multiple of  $c$  and  $d$  is a multiple of  $b$ , let us consider the columns. We have that if  $a = c = 0$ , then  $\det A = 0$ , thus  $a$  and  $c$  cannot be both 0, also notice that there are  $2^2$  possible choices for  $a$  and  $c$ , so the first column,  $\begin{pmatrix} a \\ c \end{pmatrix}$ , has  $2^2 - 1$  possible choices. Now, observing column  $\begin{pmatrix} b \\ d \end{pmatrix}$ , we have the  $2^2$  choices for both entries, however, since  $b$  and  $d$  are multiples of each other, we must exclude the 2 choices for the multiples  $ad$  and  $bc$ . Thus the column  $\begin{pmatrix} b \\ d \end{pmatrix}$  has  $2^2 - 2$  choices. That is,  $\text{ord } GL(2, 2) = (2^2 - 1)(2^2 - 2) = 2 \cdot 3 = 6$ .

Observing further, we can see that  $\text{ord } GL(n, 3) = (3^n - 1) \dots (3^n - 3)$ , and so on. Thus we have:

**Theorem 1.5.2.** For  $n, p \in \mathbb{Z}^+$  and  $p$  prime :

$$\text{ord } GL(n, p) = \prod_{j=1}^{n-1} (p^n - p^{n-j}) \quad (1.4)$$

*Proof.* Consider the  $n \times n$  matrix  $A = (a_{ij}) \in GL(n, p)$ , observe that there are  $p^n - 1$  choices for the first column,  $\begin{pmatrix} a_{11} \\ \vdots \\ a_{n1} \end{pmatrix}$  since  $a_{11}, \dots, a_{n1}$  cannot all be 0. Now, we have  $\det A = \sum_{j=1}^n (-1)^{i+j} a_{ij} \det A_{ij}$ , with  $A_{ij}$  the cofactor of  $A$  about the entry  $a_{ij}$ . So, for the  $j^{\text{th}}$  column  $\begin{pmatrix} a_{1j} \\ \vdots \\ a_{nj} \end{pmatrix}$ , then  $a_{ij} \det A_{ij} = \sum_{l=1}^{j-1} (-1)^{i+j} a_{il} \det A_{il} + \sum_{l=1}^{j+1} (-1)^{i+j} a_{il} \det A_{il}$ , for which there are  $p^n - p^j$  choices, given that each of the  $a_{ij}$  entries are multiples of the previous entries, for  $1 \leq i \leq n$ . Taking  $2 \leq j \leq n$  (since we already evaluated the first column), we get there are:

$$\prod_{j=1}^{n-1} (p^n - p^{n-j})$$

choices for the matrix  $A$ . Since  $A$  is arbitrary, we get the order of  $GL(n, p)$ . ■

We also need to comment on the order of  $GL(n, F)$  when the field  $F$  is infinite.

**Theorem 1.5.3.** *For any field  $F$ ,  $GL(n, F)$  is of infinite order if, and only if  $F$  is of infinite order.*

*Proof.* We show by contrapositives. Suppose that  $F$  is finite with  $\text{ord } F = k$ . Then by the same argument of theorem 1.5.2, we find there are  $\prod (k^n - k^j)$  matrices  $A \in GL(n, F)$ . Any additional elements contradict this result, and so  $GL(n, F) = \prod (k^n - k^j)$ .

On the otherhand, if  $\text{ord } GL(n, F) = k$  then there are  $k$   $n \times n$  matrices over  $F$  with  $\det \neq 0$ . Now, if  $F$  were not finite, then there exists a distinct matrix  $A \in GL(n, F)$ , making  $\text{ord } GL(n, F) = k + 1$  a contradiction. Thus,  $F$  must be finite. ■

We now introduce a separate group from the general and special linear groups.

**Definition.** Let  $F$  be a field. We define the **Heisenberg** group over  $F$  to be the set:

$$H(F) = \left\{ \begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix} : a, b, c \in F \right\} \quad (1.5)$$

That is,  $H(F)$  is the set of all upper triangular matrices over  $F$  with diagonal entries equal to 1 (the identity element of  $F$ ).

**Lemma 1.5.4.** *For any field  $F$ ,  $H(F)$  is a group under matrix multiplication.*

*Proof.* Let  $X = \begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix}$ , and  $Y = \begin{pmatrix} 1 & d & e \\ 0 & 1 & f \\ 0 & 0 & 1 \end{pmatrix}$ . Then  $XY = \begin{pmatrix} 1 & a+d & e+af+b \\ 0 & 1 & f+c \\ 0 & 0 & 1 \end{pmatrix}$ .

So  $H(F)$  is closed. Additionally,  $H(F)$  inherits the associativity of matrix multiplication.

Now, we get that  $I = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$  serves as the identity, and the matrix  $Y = \begin{pmatrix} 1 & -a & ac-b \\ 0 & 1 & -c \\ 0 & 0 & 1 \end{pmatrix}$  serves as an inverse to  $X$ . This makes  $H(F)$  into a group. ■

**Corollary.**  $H(F)$  is non-Abelian.

**Corollary.**  $\text{ord } H(F) = (\text{ord } F)^3$ .

*Proof.* Let  $\text{ord } F = k$ , then we have  $n$  choices for  $a$ ,  $b$ , and  $c$ , hence  $n^3$  choices for an arbitrary matrix in  $H(F)$ . ■

**Corollary.**  $H(F)$  is finite if, and only if  $F$  is finite.

## 1.6 Homomorphism.

In this section, we relate the structures of groups to each other. The main reason to do this is to determine which groups are “equal”, i.e. when two distinct groups share the same group structure. Doing this will often allow us to infer properties of one group from the other.

**Definition.** Let  $(G, *)$  and  $(H, \cdot)$  be groups. We call a map  $\phi : G \rightarrow H$  a group **homomorphism** if for any  $a, b \in G$ ,  $\phi(a * b) = \phi(a) \cdot \phi(b)$ . We call the homomorphism  $\phi$  a group **isomorphism** if  $\phi$  is both 1-1 and onto. If such an isomorphism exists between  $G$  and  $H$ , we call  $G$  and  $H$  **isomorphic** and write  $G \simeq H$ .

*Remark.* Frequently, we will imply the operations on  $G$  and  $H$  and write  $\phi(ab) = \phi(a)\phi(b)$ .

**Lemma 1.6.1.** *Isomorphism of groups is an equivalence relation.*

*Proof.* Let  $G$  and  $H$  be groups. First, take  $\phi : G \rightarrow G$  by  $\phi : g \rightarrow g$ , then  $\phi$  is an isomorphism, so  $G \simeq G$ .

Now suppose that  $G \simeq H$ , then there is an isomorphism  $\phi : G \rightarrow H$ . Then consider  $\phi^{-1} : H \rightarrow G$ , we have  $\phi^{-1}$  is also 1-1 and onto; moreover  $\phi^{-1}(\phi(ab)) = ab = \phi^{-1}(\phi(a))\phi^{-1}(\phi(b))$ . This makes  $\phi^{-1}$  an isomorphism and so  $H \simeq G$ .

Lastly, let  $K$  be a group and suppose  $G \simeq H$  and  $H \simeq K$ . Then there are isomorphisms  $\phi : G \rightarrow H$  and  $\psi : H \rightarrow K$ . Then take  $\psi \circ \phi : G \rightarrow K$  which is 1-1 and onto by definition. Then  $\psi \circ \phi(ab) = \psi(\phi(a)\phi(b)) = \psi(\phi(a))\psi(\phi(b))$ . Thus  $G \simeq K$ . ■

**Example 1.16.** (1) The maps  $\exp : (\mathbb{R}, +) \rightarrow (\mathbb{R}^+, \cdot)$  and  $\log : (\mathbb{R}^+, \cdot) \rightarrow (\mathbb{R}, +)$  defined by  $\exp : x \rightarrow e^x$  and  $\log : y \rightarrow \log y$ . Then  $\exp$  and  $\log$  are homomorphisms. We have  $\exp x + y = \exp x \exp y$  and  $\log xy = \log x + \log y$ . Moreover,  $\exp$  and  $\log$  are group isomorphisms, infact,  $\log = \exp^{-1}$ .

(2) Let  $S$  and  $T$  be nonempty finite sets. Then  $A(S) \simeq A(T)$  if and only if  $|S| = |T|$ , i.e. the symmetric groups of  $S$  and  $T$  are isomorphic if, and only if  $S$  and  $T$  share the same cardinality.

Suppose  $S$  and  $T$  are finite, and that  $|S| = |T| = n$ . Define the map  $\phi : A(S) \rightarrow A(T)$  by  $\phi : s \rightarrow tst^{-1}$ , there  $t : S \rightarrow T$  is a bujection. Then  $\phi$  is 1-1, for  $tst^{-1} = ts't^{-1}$  implies  $s = s'$ . Moreover, we have  $\phi(A(S)) = A(T)$ , since for any  $s \in A(S)$ ,  $tst^{-1} : T \rightarrow T$  defines a bijection from  $T$  onto itself; hence  $tst^{-1} \in A(T)$ . Therefore, we get  $\phi$  is an isomorphism form  $A(S)$  to  $A(T)$ . This makes  $A(S) \simeq A(T)$ .

On the other hand, if  $A(S) \simeq A(T)$ , then  $\text{ord } A(S) = \text{ord } A(T) = n!$ , for some  $n \in \mathbb{Z}^+$ , this implies that  $|S| = |T| = n$ .

**Lemma 1.6.2.** *Let  $G$  and  $H$  be groups, and let  $\phi : G \rightarrow H$  be a homomorphism. Then the following are true:*

(1)  $\phi(e) = e'$  where  $e$  and  $e'$  are the identites of  $G$  and  $H$ , respectively.

(2)  $\phi(a^{-1}) = \phi(a)^{-1}$ .

*Proof.* We have that  $\phi(e) = \phi(aa^{-1}) = \phi(a)\phi(a^{-1})$ . Thus by cancellation, we get  $\phi(a)^{-1} = \phi(a^{-1})$ . By consequence, we also get that  $\phi(e) = \phi(a)\phi(a)^{-1} = e'$ ; for any  $a \in G$ . ■

**Corollary.** *The following are also true for any  $n \in \mathbb{Z}^+$ :*

(1)  $\phi(a^n) = \phi(a)^n$ .

(2)  $\phi(a^{-n}) = \phi(a)^{-n}$ .

*Proof.* Firstly,  $\phi(a^n) = \underbrace{\phi(a) \dots \phi(a)}_{n \text{ times}} = \phi(a)^n$ . Then we get  $\phi(a^{-n}) = \phi((a^{-1})^n) = \phi(a^{-1})^n = \phi(a)^{-n}$ . ■

**Lemma 1.6.3.** *Let  $G$  and  $H$  be groups and suppose  $G \simeq H$ . Then the following are true:*

- (1)  $\text{ord } G = \text{ord } H$ .
- (2)  $G$  is Abelian if, and only if  $H$  is abelian.
- (3) For all  $x \in G$ ,  $\text{ord } x = \text{ord } \phi(x)$ , where  $\phi : G \rightarrow H$  is the underlying isomorphism.

*Proof.* Let  $G \simeq H$ , via the isomorphism  $\phi$ . Then since  $\phi$  is 1 – 1 and onto, every element of  $G$  must get mapped to every element of  $H$ . This makes  $\text{ord } G = \text{ord } H$ .

Now, suppose  $G$  is Abelian, then for every  $a, b \in G$ ,  $ab = ba$ . Thus  $\phi(ab) = \phi(ba)$ . This makes  $\phi(a)\phi(b) = \phi(b)\phi(a)$ , since  $a, b$  are arbitrary, this makes  $H$  Abelian. The converse is an equivalent argument with  $\phi^{-1}$ .

Now suppose that  $x \in G$  has order  $\text{ord } x = n$ . Then  $x^n = e$ . Thus  $\phi(x^n) = \phi(e)$ , thus  $\phi(x)^n = e'$ . Now since  $n$  is minimal, any  $m < n$  for which  $\phi(x)^m = e'$  would imply that  $x^m = e$ , which cannot happen. Thus  $\text{ord } \phi(x) = n$ . ■

**Corollary.** *Let  $\phi : G \rightarrow H$  be a homomorphism. If  $H$  is Abelian, and  $\phi$  is 1 – 1, then  $G$  is Abelian. On the otherhand, if  $G$  is Abelian, and  $\phi$  is onto, then  $H$  is Abelian.*

*Proof.* For  $a, b \in G$ ,  $\phi(a)\phi(b) = \phi(b)\phi(a)$ , hence  $\phi(ab) = \phi(ba)$ . Since  $\phi$  is 1 – 1, this implies  $ab = ba$ .

On the otherhand, if  $ab = ba$  for any  $a, b \in G$  and  $\phi$  is onto, then we get  $\phi(ab) = \phi(ba)$  hence  $\phi(a)\phi(b) = \phi(b)\phi(a)$ . Since  $\phi(G) = H$ , this completes the proof. ■

**Example 1.17.** (1) We have  $S_3 \not\simeq \mathbb{Z}/6\mathbb{Z}$ , despite having the same order. We have  $S_3$  is nonabelian while  $\mathbb{Z}/6\mathbb{Z}$  is Abelian.

- (2)  $(\mathbb{R}, +) \not\simeq (\mathbb{R}^*, \cdot)$  since in  $\mathbb{R}^*$ ,  $\text{ord } -1 = 2$ , while there are no elements of order 2 in  $\mathbb{R}^+$ .

**Lemma 1.6.4.** *Let  $G$  and  $H$  be groups with representations; let  $G = \langle S : R_1, \dots, R_n \rangle$  and  $H = \langle T : R'_1, \dots, R'_n \rangle$ . Then if the relations  $R_i$  is satisfied by the elements of  $H$ , for each  $1 \leq i \leq n$ , then there exists a unique homomorphism  $\phi$  defined by  $\phi : R_i \rightarrow R'_i$ .*

*Remark.* We defer the proof of this lemma.

**Example 1.18.** (1) Take  $D_{2n} = \langle r, t : r^n = t^2 = e, rt = tr^{-1} \rangle$ , and take  $X_{2k} = \langle a, b : a^k = b^2 = e, ab = ba^{-1} \rangle$ . If  $n = km$  for  $m \in \mathbb{Z}^+$ , then  $a^n = (a^k)^m = e$ , so the relations of  $D_{2n}$  are satisfied by the generators of  $X_{2n}$ . Thus take homomorphism  $\phi : D_{2n} \rightarrow X_{2k}$  by  $\phi : r, t \rightarrow a, b$ . Since  $X_{2k} = \langle a, b \rangle$ ,  $\phi$  is onto.  $\phi$  is 1 – 1 if, and only if  $n = k$ . So, in general,  $D_{2n} \not\simeq X_{2k}$ .

- (2) Consider  $D_6$  and  $S_3$ . Let  $a = (1 \ 2 \ 3)$  and  $b = (1 \ 2)$ . Then  $a^3 = (1)(2)(3) = (1)$  and  $b^2 = (1)$ ; moreover,  $ab = (1 \ 2 \ 3)(1 \ 2) = (1 \ 2)(3 \ 2 \ 1) = ba^{-1}$ . Thus take  $\phi : D_6 \rightarrow S_3$  by  $\phi : r, t \rightarrow (1 \ 2 \ 3), (1 \ 2)$ . By the above reasoning,  $\phi$  is onto. Now since  $\text{ord } D_6 = \text{ord } S_3 = 6$ ,  $\phi$  is 1 – 1 and so  $\phi$  is an isomorphism and  $D_6 \simeq S_3$ .

- Example 1.19.** (1)  $\mathbb{C}^* \not\simeq \mathbb{R}^*$ , since  $i \in \mathbb{C}^*$  has  $\text{ord } i = 4$ , while  $\mathbb{R}^*$  has no elements of order 4.
- (2)  $\mathbb{R} \not\simeq \mathbb{Q}$ , for  $\mathbb{Q}$  is countable, and  $\mathbb{R}$  is not. That is if we take  $\text{ord } \mathbb{Q}$  and  $\text{ord } \mathbb{R}$  to be defined and assume results from set theory and topology, then  $\text{ord } \mathbb{Q} < \text{ord } \mathbb{R}$ .
- (3)  $\mathbb{Z} \not\simeq \mathbb{Q}$ , for suppose otherwise. If  $\phi : \mathbb{Z} \rightarrow \mathbb{Q}$  is an isomorphism, then  $\phi(1) = a$ , then  $\phi(\frac{1}{2} + \frac{1}{2}) = a$ , then  $2\phi(\frac{1}{2}) = a$ , likewise  $3\phi(\frac{1}{3}) = a$ , then  $2\phi(\frac{1}{2}) = 3\phi(\frac{1}{3})$ , implying  $\phi(\frac{1}{2}) = \phi(\frac{1}{3})$ , but  $\frac{1}{2} \neq \frac{1}{3}$ , contradicting the 1-1ness of  $\phi$ .
- (4) Notice that if  $n < m$ , then  $\text{ord } S_n = n! < \text{ord } S_m = m!$ , so  $S_m \simeq S_n$  if, and only if  $n = m$ , for  $n, m \in \mathbb{Z}^+$ .
- (5)  $D_{24} \not\simeq S_4$ . Notice  $r \in D_{24} = D_{2 \cdot 12}$  has  $\text{ord } r = 12$ . Now, any permutation in  $S_4$  is either a 4-cycle, a 3-cycle, or a product of two 2-cycles, thus every  $s \in S_4$  has  $\text{ord } s \leq 4$ , and so there are no elements of order 12 in  $S_4$ .

We finish with some results.

**Lemma 1.6.5.** *Let  $A$  and  $B$  be groups. Then  $A \times B \simeq B \times A$ .*

*Proof.* Take the map  $A \times B \rightarrow B \times A$  by taking  $(a, b) \rightarrow (b, a)$ . This map is 1-1 and onto, for  $(b, a) = (b', a')$  implies  $a = a'$  and  $b = b'$ , and  $\text{ord } A \times B = \text{ord } B \times A$ . Lastly, notice that  $(bb', aa') = (b, a)(b', a')$ , which makes it a homomorphism. Thus there is an isomorphism from  $A \times B$  onto  $B \times A$ . ■

**Lemma 1.6.6.** *Let  $A, B, C$  be groups. Then  $(A \times B) \times C \simeq A \times (B \times C)$ .*

*Proof.* Consider the map  $(A \times B) \times C \rightarrow A \times (B \times C)$  by taking  $(a, (b, c)) \rightarrow ((a, b), c)$ . This map is 1-1 and onto by the same reasoning used in the above lemma, moreover, this map is a homomorphism by closure. This completes the proof. ■

## 1.7 Group Actions.

We now present the notion of a group “acting” on a given set. The study of these “actions” will allow us to prove results for groups and also for finding underlying structures of specific sets.

**Definition.** A **left group action** of a group  $G$  on a set  $A$  is a map  $\cdot : G \times A \rightarrow A$  such that for all  $g \in G$  and  $a \in A$ :

- (1)  $g_1 \cdot (g_2 \cdot a) = (g_1 g_2) \cdot a$ .
- (2)  $e \cdot a = a$ , where  $e$  is the identity of  $G$ .

*Remark.* As before, we will drop explicit mention of the action  $\cdot$  and merely write  $ga$ . It should be taken into account that the group action  $\cdot$  is not a binary operation.

*Remark.* Similarly, one can define **right** group actions. Our study will consist of left group actions, so we drop the indication.

**Lemma 1.7.1.** *Let  $G$  be a group acting on a set  $S$ . For each  $g \in G$ , define the map  $\sigma_g : S \rightarrow S$  by  $\sigma_g : a \rightarrow ga$ . Then:*

- (1) *For each  $g \in G$ ,  $\sigma_g$  is a permutation of  $S$ .*
- (2) *The map  $G \rightarrow A(S)$  defined by  $g \rightarrow \sigma_g$  is a homomorphism.*

*Proof.* Let  $a, b \in S$  and suppose that  $\sigma_g(a) = \sigma_g(b)$ . Then  $ga = gb$ , and by the cancellation laws,  $a = b$ . This makes  $\sigma_g$  1-1. On the otherhand, since  $ga \in A$  for all  $a \in A$ , we get that  $\sigma_g$  is onto. This makes  $\sigma_g$  a permutation. Now, consider  $g^{-1} \in G$ , and notice that  $\sigma_{g^{-1}}\sigma_g(a) = \sigma_{g^{-1}}(ga) = g^{-1}(ga) = (gg^{-1})a = ea = a$ . Thus we get  $\sigma_g^{-1} = \sigma_{g^{-1}}$ .

Now consider the map  $\phi : g \rightarrow \sigma_g$ . Then  $\phi(gg')(a) = \sigma_{gg'}(a) = (gg')a = g(g'a) = \sigma_g\sigma_{g'}(a) = \phi(g)\phi(g')$ . ■

*Remark.* The main takeaway of this lemma is that group actions on a set  $S$  are merely permutations of the elements of  $S$ .

**Definition.** Let  $G$  be a group acting on a set  $S$ , and define  $\sigma_g : a \rightarrow ga$ , and define  $\phi : G \rightarrow A(S)$  by  $\phi : g \rightarrow \sigma_g$ . We call  $\phi$  the **permutation representation** of  $S$  associated with  $g$ .

**Example 1.20.** Let  $G$  be a group, and  $A \neq \emptyset$ . Then:

- (1) Define the action  $ga = a$  for all  $g \in G$ . Then  $g_1(g_2)a = g_1a = a$  and  $(g_1g_2)a = a$ , so  $g_1(g_2)a = (g_1g_2)a$ , and  $ea = a$ , so we indeed have an action. We call this the **trivial** action, and we say that  $G$  acts **trivially** on  $A$ . Define then  $\sigma_g : a \rightarrow ga = a$ , then  $\sigma_g$  is the identity map. So the permutation representation associated with  $g$  is the identity map.
- (2) In the vector space axioms, scalar multiplication  $\cdot : F^* \times V \rightarrow V$  is an action of  $F^*$  on  $V$ . We have for any  $\alpha, \beta \in F$ , and  $v \in V$ ,  $\alpha(\beta)v = (\alpha\beta)v$ , and  $1v = v$ . Here  $F$  is a field, and so  $F^*$  forms a multiplicative group under the relevant multiplication.
- (3) For any  $S \neq \emptyset$ , the symmetric group  $A(S)$  acts on  $S$  via the action  $sa = s(a)$ .

- (4) Consider again the regular  $n$ -gon. Label its vertices to be the set  $\mathbb{Z}/n\mathbb{Z}$ , then we can see that the symmetries of the  $n$ -gon act on vertices of the  $n$ -gon.

Consider the map  $D_{2n} \times \mathbb{Z}/n\mathbb{Z} \rightarrow \mathbb{Z}/n\mathbb{Z}$  of  $D_{2n} \times \mathbb{Z}/n\mathbb{Z}$  onto  $\mathbb{Z}/n\mathbb{Z}$  via the map  $(r^j t, i) \rightarrow r^j t(i)$ , where  $j \in \mathbb{Z}/n\mathbb{Z}$ . This map forms a group action of  $D_{2n}$  on  $\mathbb{Z}/n\mathbb{Z}$ . Also notice that distinct symmetries induce distinct permutations of the vertices.

- (5) Let  $G$  be any group, and let  $A = G$ . Then the binary operation on  $G$  is a group action of  $G$  onto itself. We have  $a(bc) = (ab)c$ , and so the first property is satisfied by associativity, and  $ea = a$  and the second property is satisfied by the identity law. We call the binary operation a **left regular** action. Also notice that distinct elements of  $G$  induce distinct permutations of  $G$ .

We end the section, and the chapter with two more definitions.

**Definition.** Let  $G$  be a group acting on a set  $A$ . We call the action of  $G$  on  $A$  **faithful** if distinct elements of  $G$  induce distinct permutations on  $G$ . That is if  $\phi$  is the permutation representation associated with  $G$ , then  $\phi$  is 1-1.

**Definition.** Let  $G$  be a group acting on a set  $A$ . We define the **kernel** of the group action on  $A$  to be  $\ker A = \{g \in G : ga = a \text{ for all } a \in A\}$



# Chapter 2

## Subgroups.

### 2.1 Definitions and Examples.

**Definition.** Let  $G$  be a group. We call a nonempty subset  $H \subseteq G$  a **subgroup** if  $H$  is also a group under the binary operation of  $G$ . We write  $H \leq G$ ; if  $H \neq G$ , then we write  $H < G$ .

There are two immediate results that we can develop.

**Theorem 2.1.1** (The Subgroup Criterion.). *Let  $G$  be a group. Then a subset  $H \subseteq G$  is a subgroup if, and only if:*

- (1) For every  $a, b \in H$ ,  $ab \in H$ .
- (2)  $a^{-1} \in H$  whenever  $a \in H$ .

*Proof.* Suppose first that  $H \leq G$ , then by definition, (1) and (2) are satisfied.

Now suppose that  $H$  is closed under the operation on  $G$ , (1), and that  $H$  has inverses (2). Immediately, the closure and inverse laws are satisfied, moreover, since  $H \subseteq G$ ,  $H$  inherits the associativity of  $G$  under the relevant operation. Now, by (1) and (2) we have  $aa^{-1} = e \in H$ , and so the identity law is satisfied. This makes  $H \leq G$ . ■

**Corollary.** *If  $H$  is a finite subset of  $G$ , then  $H$  is a subgroup if  $H$  is closed under the operation of  $G$ .*

*Proof.* Let  $a \in H$ , by closure, we have  $a^n \in H$  for  $n \in \mathbb{Z}^+$ . So, consider the infinite collection  $\{a^i\}_{i=1} \subseteq H$ ; since  $H$  is finite, there are repetitions in the collection  $\{a^i\}$ , that is,  $a^i = a^j$  for some  $i \neq j$  and  $i, j > 0$ . Then  $a^{i-j} = e$ , so  $a^{i-j-1} = a^{-1}$ . Now, since  $i - j > 0$ , we get  $i - j - 1 \geq 0$ , which makes  $a^{-1} \in H$ . By the above theorem, we get  $H \leq G$ . ■

**Example 2.1.** (1)  $\mathbb{Z} \leq \mathbb{Q} \leq \mathbb{R}$  under the usual addition.  $\mathbb{Q}^* \leq \mathbb{R}^*$  under the usual multiplication.  $\mathbb{R} \leq \mathbb{C}$  under complex addition, and  $\mathbb{R}^* \leq \mathbb{C}^*$  under complex multiplication (here, we take  $a \in \mathbb{R}$  to have the form  $a + i0$ ).

- (2) For any group  $G$ ,  $G \leq G$  and  $\langle e \rangle \leq G$ . So the minimum number of subgroups that any group has is 2.

- (3) Let  $H = \{e, r, \dots, r^{n-1}\} \subseteq D_{2n}$ . Then  $H \leq D_{2n}$ .
- (4) Let  $n\mathbb{Z} = \{na : a \in \mathbb{Z}\}$ , for any  $n \in \mathbb{Z}$ . Then  $n\mathbb{Z} \leq \mathbb{Z}$  under the usual addition. Let  $na, nb \in n\mathbb{Z}$ , then  $na + nb = n(a + b) \in n\mathbb{Z}$ , and  $n(-a) = -na \in n\mathbb{Z}$ . We will be interested in the subgroup  $n\mathbb{Z}$  of  $\mathbb{Z}$ , in particular, for  $n \in \mathbb{Z}^+$ .
- (5) Let  $Z = \{z \in \mathbb{C} : z^n = 1\}$ . Then  $Z \leq \mathbb{C}^*$  under complex multiplication. Notice that  $z, w \in Z$  implies  $z^n w^n = (zw)^n = 1$ , and  $(z^{-1})^n = \frac{1}{z^n} = 1$ , so  $z^{-n} = 1$ . We call this group the **roots of unity** in  $\mathbb{C}$ . If we take  $n = 4$ , then we get that  $Z = \{1, i, -1, -i\}$ .
- (6) Let  $\mathbb{Z} + i\mathbb{Z} = \{a + ib \in \mathbb{C} : a, b \in \mathbb{Z}\}$ . Then  $\mathbb{Z} + i\mathbb{Z} \leq \mathbb{C}$  under complex addition. We call this group the **Gaussian integers**.
- (7) Let  $\mathbb{Q} + \sqrt{2}\mathbb{Q} = \{a + b\sqrt{2} : a, b \in \mathbb{Q}\}$ . Then  $\mathbb{Q} + \sqrt{2}\mathbb{Q} \leq \mathbb{R}$  under the usual addition.

We now give some nonexamples of subgroups.

**Example 2.2.** (1)  $\mathbb{Q}^* \not\leq \mathbb{R}$  under the usual addition (why?).

- (2)  $\mathbb{Z}^+ \not\leq \mathbb{Z}$  under the usual addition. Notice the identity of  $\mathbb{Z}$ ,  $0 \notin \mathbb{Z}^+$ .
- (3)  $D_6 \not\leq D_8$ . Notice  $D_6 = \{e, t, r, r^2, rt, r^2t\}$ , and  $D_8 = \{e, t, r, r^2, r^3, rt, r^2t, r^3t\}$ . One might be tempted to think  $D_6 \subseteq D_8$ , but notice that in  $D_6$ ,  $r^3 = e$  while in  $D_8$ ,  $r^4 = e$ . Thus  $D_6 \not\subseteq D_8$ .

## 2.2 Special Subgroups.

We introduce now, some very important examples of subgroups.

**Definition.** Let  $G$  be a group. We define the **centralizer** of an element  $a \in G$  to be the set  $C(a) = \{g \in G : gag^{-1} = a\}$ . We define the **centralizer** of a nonempty subset  $A$  of  $G$  to be the set  $C(A) = \{g \in G : gag^{-1} = a, \text{ for all } a \in A\}$ .

**Lemma 2.2.1.** Let  $G$  be a group. Then for  $a \in G$ ,  $C(a) \leq G$ . Likewise, for  $A \subseteq G$  nonempty,  $C(A) \leq G$ .

*Proof.* Notice that given  $a \in A$ ,  $C(a) \subseteq C(A)$ . Then we have that  $C(a)$ , for  $eae^{-1} = eae = a$ , so  $e \in C(a)$ , this also implies that  $e \in C(A)$ .

Now let  $x, y \in C(a)$  then  $xax^{-1} = a$ , and  $yay^{-1} = a$ . Notice then that  $a = y^{-1}ay = y^{-1}a(y^{-1})^{-1}$ , so we have  $y^{-1} \in C(a)$ . Then  $(xy)a(xy)^{-1} = xyay^{-1}x^{-1} = x(yay^{-1})x^{-1} = x(yy^{-1}ay^{-1}y)x^{-1} = xax^{-1} = a$ , so  $xy \in C(a)$ , thus  $C(a) \leq G$ , if we take  $a \in A$  arbitrary, this makes  $C(A) \leq G$  as well. ■

**Corollary.**  $C(a) \leq C(A)$  for any  $a \in A$ .

**Corollary.**  $C(A) = \bigcap_{a \in A} C(a)$ .

*Remark.* In the above corollary, notice that  $\bigcup C(a)$  is not necessarily a disjoint union.

**Example 2.3.** Let  $G$  be abelian, then  $C(G) = G$ .

**Definition.** Let  $G$  be a group. We define the **center** of  $G$  to be the set  $Z(G) = \{g \in G : gx = xg \text{ for all } x \in G\}$ .

**Lemma 2.2.2.** Let  $G$  be a group. Then  $Z(G) \leq G$ .

*Proof.* Notice that if  $g \in Z(G)$ , then for any  $x \in G$ ,  $gx = xg$  implies that  $gxx^{-1} = g$ , making  $g \in C(G)$ ; likewise,  $g \in C(G)$  implies  $gxx^{-1} = x$  which implies that  $xg = gx$ , for any  $x \in G$ ; so  $g \in Z(G)$ . That is,  $Z(G) = C(G)$  which makes  $Z(G)$  a subgroup. ■

**Definition.** Let  $G$  be a group and let  $A \subseteq G$ . Define  $gAg^{-1} = \{gag^{-1} : a \in A\}$ , where  $g \in G$ . We define the **normalizer** of  $A$  to be the set  $N(A) = \{g \in G : gAg^{-1} = A\}$ .

**Lemma 2.2.3.** Let  $G$  be a group, and let  $A \subseteq G$ . Then  $N(A) \leq G$ .

*Proof.* Let  $x, y \in N(A)$ , then  $xAx^{-1} = A$  and  $yAy^{-1} = A$ , then for any  $a \in A$ , then for some  $a, b \in A$ ,  $xaa^{-1} = b$  and  $yaa^{-1} = b$ . Then  $(xy)a(xy)^{-1} = x(yay^{-1})x^{-1} = xbx^{-1} = b$ , thus  $xy \in N(A)$ . Similarly,  $xa x^{-1} = b$  implies  $a = x^{-1}bx$ , thus  $x^{-1} \in N(A)$ . This makes  $N(A) \leq G$ . ■

**Corollary.**  $C(A) \leq N(A)$ .

**Example 2.4.** (1) If  $G$  is abelian, then  $ab = ba$  for all  $a, b \in G$ , thus  $G = Z(G)$ . Similarly, we get  $gag^{-1} = gg^{-1}a = a$  for all  $a \in A \subseteq G$  and  $g \in G$ , thus  $C(A) = N(A) = G$ .

(2) Consider the dihedral group  $D_{2n}$ . Let  $A = \{e, r, \dots, r^{n-1}\} \leq D_{2n}$ . Then  $C(A) = A$ . We have that  $A \subseteq C(A)$ , since  $r^j r^i r^{-j} = r^{j+i-j} = r^i$ . On the other hand we have  $(r^j t) r^i (r^j t)^{-1} = r^{j+i-j} t^2 = r^i$ , which makes  $C(A) \subseteq A$ . Moreover,  $N(A) = D_{2n}$ , since by the above computations we also get  $D_{2n} \subseteq N(A)$ .

(3) In  $D_{2n}$ ,  $Z(D_{2n}) = \{r^i : r^{-i} = r^i\}$ , where  $i \in \mathbb{Z}/n\mathbb{Z}$ . So in  $D_8$ ,  $Z(D_8) = \{e, r^2\}$ . Essentially, to find the center of  $D_{2n}$ , find all those powers  $i$  of  $r$  for which  $i \equiv -i \pmod{n}$ .

(4) Let  $A = \{(1), (1\ 2)\} \leq S_3$ . Then  $C(A) = N(A) = A$ . Moreover,  $Z(S_3) = \langle(1)\rangle$ . Notice that since  $S_3 \simeq D_6$ , then to preserve the group structure,  $Z(S_3) \simeq Z(D_6)$ . Notice then that  $Z(D_6) = \langle e \rangle$ .

We can treat the fact that, for a subset  $A$  of a group  $G$ , the normalizer and centralizer of  $G$ , and the center of  $G$  are all special cases of group actions.

**Definition.** If  $G$  is a group acting on a set  $A$ , then we define the **stabilizer** of  $a \in A$  in  $G$  to be the set  $\text{stab } a = \{g \in G : ga = a\}$ . We define the **stabilizer** of  $A$  to be  $\text{stab } A = \{g \in G : ga = a \text{ for all } a \in A\}$ .

**Lemma 2.2.4.** Let  $G$  be group acting on a set  $S$ . Then for any  $s \in S$ ,  $\text{stab } s \leq G$ .

*Proof.* For  $a, b \in \text{stab } s$ , we have  $(ab)s = a(bs) = as = s$ , so  $ab \in \text{stab } s$ . Likewise,  $a^{-1} \in \text{stab } s$ . ■

**Corollary.**  $\ker S \leq G$ .

*Proof.* The proof is identical to that of the above lemma. ■

**Corollary.**  $\text{stab } s \leq \ker s$ .

*Proof.* We have that both  $\text{stab } s$  and  $\ker s$  are both groups. Then notice that since  $s \in S$ , then  $\text{stab } s \subseteq \ker s$ . ■

**Corollary.**  $\ker s = \bigcap_{s \in S} \text{stab } s$ .

**Example 2.5.** (1) Consider the dihedral group  $D_8$  of symmetries of a square acting on the labeling  $\mathbb{Z}/4\mathbb{Z}$  of vertices. Then  $\text{stab } i = \{e, r^2t\}$ , where  $r^2t$  is the reflection of the square about the line crossing the vertex  $i$  and the center of the square.

(2) In general, consider the dihedral group  $D_{2n}$  of symmetries of an  $n$ -gon acting on the labeling  $\mathbb{Z}/n\mathbb{Z}$  of vertices. Notice that if  $r^jt \in \text{stab } i$ , for any  $i \in \mathbb{Z}/n\mathbb{Z}$ , then  $(r^jt)i = i$ . Then  $n + j - i = i$ , i.e.  $j \equiv 2i \pmod{n}$ , and so  $\text{stab } i = \{r^jt : j \equiv 2i \pmod{n}\} = \{e, r^2t, r^4t, \dots, r^{2n-2}t\}$ .

(3)

**Definition.** Let  $G$  be a group acting on a set  $A$ . We define the **conjugation** action of  $G$  on  $A$  to be the map  $G \times A \rightarrow A$  defined by  $(g, a) \rightarrow gag^{-1}$ . We define the **conjugation** of  $A$ ,  $gAg^{-1}$  to be the image of this action.

**Lemma 2.2.5.** *Conjugation is a group action.*

*Proof.* Let  $G$  be and  $A$  a nonempty set. Take the map  $G \times A \rightarrow A$  via  $(g, a) \rightarrow gag^{-1}$ . Then for  $e \in G$  the identity,  $(e, a) \rightarrow eaa^{-1} = a$ , now for  $x, y \in G$ ,  $(xy, a) \rightarrow (xy)a(xy)^{-1} = x(yaa^{-1})x^{-1} = (x, (y, a))$ . Thus this map is a group action, and  $G$  acts on  $A$  via conjugation. ■

**Lemma 2.2.6.** *Let  $G$  be a group acting on a set  $A \subseteq G$  via conjugation. Then  $\text{stab } A = N(A)$ .*

## 2.3 Cyclic Groups.

**Definition.** Let  $G$  be a group, and let  $H \leq G$  be a subgroup. We say that  $H$  is **cyclic** if it can be generated by a single element. That is, there is some  $x \in H$  for which  $H = \{x^n : n \in \mathbb{Z}\}$ . We write  $H = \langle x \rangle$ .

*Remark.* Notice that if  $H$  is cyclic, there is a singleton set that generates  $H$ , i.e.  $H = \langle \{x\} \rangle$ .

**Lemma 2.3.1.** *If  $G = \langle g \rangle$ , then  $G = \langle g^{-1} \rangle$ .*

*Proof.* For any  $h \in G$ , if  $h = g^n$ , then  $h^{-1} = x^{-n} = (x^{-1})^n$ . ■

**Lemma 2.3.2.** *If  $G$  is a cyclic group, then  $G$  is Abelian.*

*Proof.* Let  $G = \langle g \rangle$  and for any  $m, n \in \mathbb{Z}^+$ . Then  $g^m g^n = g^{m+n} = g^{n+m} = g^n g^m$ . ■

**Example 2.6.** (1) In  $D_{2n}$ , the set  $\{e, r, \dots, r^n\}$  is cyclic, and  $\langle r \rangle \leq D_{2n}$ .

(2)  $\mathbb{Z} = \langle 1 \rangle = \langle -1 \rangle$ .

(3) Consider the subgroup  $\mathbb{C}_n = \{z \in \mathbb{C} : z^n = 1\} \leq \mathbb{C}$  of complex primitive  $n$ -th roots of unity in  $\mathbb{C}$ , where  $n \in \mathbb{Z}^+$ . Then  $\mathbb{C} = \langle z \rangle = \langle \frac{1}{z} \rangle$ .

(4)  $\mathbb{C}_4 = \langle i \rangle = \langle -i \rangle$ .

**Lemma 2.3.3.** Let  $G = \langle g \rangle$  be a finite cyclic group, Then  $\text{ord } G = \text{ord } g$ .

*Proof.* Let  $\text{ord } g = n$ , by the division theorem, there exists  $t, q, r \in \mathbb{Z}$  such that  $t = nq + r$ , with  $0 \leq r < n$ ; i.e.  $t \equiv r \pmod{n}$ , so  $g^t = g^r$ . Thus there are at most  $n$  such elements in  $G$ . Now, suppose  $g^i = g^j$  for  $0 \leq i < j < n$ , then  $g^{j-i} = e$ , since  $j - i < n$ , this contradicts the order of  $g$ . Therefore, there are exactly  $\text{ord } g = n$  such element in  $G$ , since  $G = \langle g \rangle$ , these are the only elements, which proves the result. ■

**Corollary.** If  $G = \langle g \rangle$  is a finite cyclic group, then  $g$  is of finite order.

*Proof.* If  $g$  is of infinite order, then there are infinitely many powers of  $g$ , since  $g$  generates  $G$ , this contradicts the finiteness of  $G$ . ■

**Lemma 2.3.4.** Let  $G$  be a group and let  $g \in G$  and let  $m, n \in \mathbb{Z}^+$  distinct, such that  $g^n = e$  and  $g^m = e$ . Then  $g^{(m,n)} = e$ . Moreover,  $\text{ord } g | m$ .

*Proof.* There exist  $p, q \in \mathbb{Z}$  such that  $mp + nq = (m, n)$ . Then by hypothesis,  $g^{(m,n)} = (g^m)^p (g^n)^q = e$ . Moreover, assuming, without loss of generality, that  $m < n$ , we have by definition of the order of  $g$  that either  $\text{ord } g = m$ , or  $(\text{ord } g) | m$ . ■

**Theorem 2.3.5.** If  $G$  and  $H$  are finite cyclic groups with  $\text{ord } G = \text{ord } H$ , then  $G \simeq H$ .

*Proof.* Let  $G = \langle g \rangle$  and  $H = \langle h \rangle$ . Take the map  $\phi : \langle g \rangle \rightarrow \langle h \rangle$  via the rule  $g^k \rightarrow h^k$ , for some  $k \in \mathbb{N}$ . We have that  $\phi(g^m g^n) = \phi(g^{m+n}) = h^{m+n} = h^m h^n = \phi(g^m) \phi(g^n)$ . So  $\phi$  defines a homomorphism. Moreover, if  $\text{ord } G = \text{ord } H = n$ , and  $g^s = g^t$  for  $s, t \in \mathbb{Z}^+$ ,  $g^{s-t} = e$ , so  $n | s - t$ , that is  $s \equiv t \pmod{n}$ . Thus  $\phi(g^s) = \phi(g^t)$ . This makes  $\phi$  well defined.

Now, by definition, every element of  $\langle h \rangle$  is of the form  $h^k = \phi(g^k)$  for some  $k \in \mathbb{N}$ , this makes  $\phi$  onto. Since  $\phi$  is onto, and  $\text{ord } G = \text{ord } H$  we get that  $\phi$  is 1-1, and so an isomorphism. ■

**Corollary.** If  $\langle g \rangle$  is an infinite cyclic group, then  $\mathbb{Z} \simeq \langle g \rangle$ .

*Proof.* Define  $\phi : \mathbb{Z} \rightarrow \langle g \rangle$  by  $m \rightarrow g^m$ . Then  $\phi(m+n) = g^{m+n} = g^m g^n = \phi(m) \phi(n)$ , moreover, if  $m = n$ , then  $\phi(m) = g^m = g^n = \phi(n)$ ; so  $\phi$  is a well defined homomorphism.

Now if  $\phi(m) = \phi(n)$ , then  $g^m = g^n$ , that is  $g^{m-n} = e$ , so  $m - n = 0$ , hence  $m = n$ .  $\phi$  is 1-1.  $\phi$  is also onto by definition therefore  $\phi$  is an isomorphism of  $\mathbb{Z}$  onto  $\langle g \rangle$ . ■

**Example 2.7.** (1) In  $D_{2n}$ ,  $\langle r \rangle \simeq \mathbb{Z}/n\mathbb{Z}$ , and  $\langle t \rangle = \{e, t\} \simeq \mathbb{Z}/2\mathbb{Z}$ .

(2) If  $G$  is any finite cyclic group of  $\text{ord } G = n$ , then  $G \simeq \mathbb{Z}/n\mathbb{Z}$ .

**Lemma 2.3.6.** *If  $G$  is a group, and  $g \in G$ , and  $k \in \mathbb{Z}^*$ , then the following are true:*

(1) *If  $g$  is of infinite order, then so is  $g^k$ .*

(2) *If  $\text{ord } g = n$ , then  $\text{ord } g^k = \frac{n}{(n,k)}$ .*

*Proof.* Suppose that  $g$  is of infinite order, but that  $\text{ord } g^k = m$  for some  $m \in \mathbb{Z}^+$ . Then  $(g^k)^m = g^{km} = e$ . Now, either  $km > 0$ , or  $-km > 0$ , thus, we get  $\text{ord } g \leq km$  or  $\text{ord } g \leq -km$ . Both contradict the infinite order of  $g$ .

Now let  $\text{ord } g = n$ , and let  $h = g^k$  and define  $d = (n, k)$ . Then  $n = dm$  and  $k = dl$  for  $m, l \in \mathbb{Z}$ ,  $m > 0$ . Then we have  $na + kb = d$ , for  $a, b \in \mathbb{Z}$ ; this implies that  $ma + lb = 1$ , so  $(m, l) = 1$ . Now, let  $\text{ord } h = p$ , then  $h^m = g^{km} = g^{dlm} = (g^{dm})^l = (g^n)^l = e$ , so  $p|m$ . On the other hand, since  $(m, l) = 1$ , we get  $m|p$ , thus  $p = \text{ord } h = m$ . Since  $m = \frac{n}{d}$ , we get the result. ■

**Corollary.** *Of  $k|n$ , then  $\text{ord } g^k = \frac{n}{k}$ .*

*Proof.* If  $k|n$ , then  $(n, k) = k$ . ■

**Lemma 2.3.7.** *Let  $\langle g \rangle$  be a cyclic group, then:*

(1) *If  $g$  is of infinite order, then  $\langle g \rangle = \langle g^k \rangle$  if, and only if  $k = \pm 1$ .*

(2) *If  $\text{ord } g = n$ , then  $\langle g \rangle = \langle g^k \rangle$  if, and only if  $(n, k) = 1$ .*

*Proof.* First, if  $k = \pm 1$ , then  $g^k = g$  or  $g^k = g^{-1}$ . By lemma 2.3.1, we get the result. Now suppose  $\langle g \rangle = \langle g^k \rangle$  for  $k > 1$ . Then  $g = g^k$  for some  $k > 1$ . If  $k$  is odd, then  $k = 2l + 1$  and if  $k$  is even,  $k = 2l$  for  $l \in \mathbb{Z}^+$ . Then  $g = g^{2l+1} = g^{2l}g$ , making  $g^{2l} = e$ . On the otherhand, if  $g = g^k = g^{2l}$ , then  $g^{2l}g^{-1} = g^{2l-1} = e$ . Both these cases contradict the infinite order of  $g$ . So  $k \leq 1$ . Now, since  $\langle g \rangle = \langle e \rangle$  cannot happen,  $k \neq 0$ . Now if  $k < -1$ , then we get the same result using  $-k$ . Thus either  $k = 1$  or  $k = -1$ .

Now suppose that  $\text{ord } g = n$ . Then  $g^k$  generates a subgroup of  $\text{ord } g^k = \frac{n}{(n,k)}$ . Now  $\langle g \rangle = \langle g^k \rangle$  if, and only if  $\text{ord } g = \text{ord } g^k$ . That is, if and only if  $n(n, k) = n$ , i.e. if, and only if  $(n, k) = 1$ . ■

**Corollary.** *The number of generators of  $\langle g \rangle$  is  $\phi(n)$ , the Euler- $\phi$  function.*

**Example 2.8.** (1) Any  $k \in \mathbb{Z}/n\mathbb{Z}$ , coprime with  $n$  generates  $\mathbb{Z}/n\mathbb{Z}$ . So the generators of  $\mathbb{Z}/n\mathbb{Z}$  are the elements of  $U(\mathbb{Z}/n\mathbb{Z})$ .

(2)  $\mathbb{Z}/6\mathbb{Z} = \langle 1 \rangle = \langle 5 \rangle$ .

(3)  $\mathbb{Z}/12\mathbb{Z} = \langle 1 \rangle = \langle 5 \rangle = \langle 7 \rangle = \langle 11 \rangle$ .

**Theorem 2.3.8.** *Let  $G = \langle g \rangle$  be a cyclic group. The following are true:*

(1) *Every subgroup of  $\langle g \rangle$  is cyclic, and has the form  $\langle g^d \rangle$ , with  $0 \leq d < \text{ord } g$ .*

- (2) If  $\langle g \rangle$  is infinite, then for any  $m, n \in \mathbb{Z}^+$  distinct,  $\langle g^m \rangle \neq \langle g^n \rangle$ . Moreover,  $\langle g^m \rangle = \langle g^{|m|} \rangle$ , and there is a 1-1 map of subgroups of  $\langle g \rangle$  onto  $\mathbb{N}$ .
- (3) If  $\text{ord } \langle g \rangle = n$ , then for each divisor  $k$  of  $n$ , there is a unique subgroup of order  $k$ , which is  $\langle g^d \rangle$  where  $d = \frac{n}{k}$ .

*Proof.* First, let  $H \leq \langle g \rangle$ , if  $K = \langle e \rangle$ , we are done. Otherwise, there is some  $k \neq 0$  with  $g^k \in K$ . If  $k < 0$ , then  $g^{-k} \in K$ . Now, define  $P = \{l \in \mathbb{Z}^+ : g^l \in K\}$ . We have by above that  $P$  is nonempty, thus by the Well Ordering Principle,  $P$  has a least element,  $d$ . Now,  $g^d \in K$  and  $K \leq \langle g \rangle$ , so  $\langle g^d \rangle \leq K$ . Now, for any  $g^k \in K$ , by the division theorem, we have  $k = qd + r$ ,  $0 \leq r < d$ , with  $q, r \in \mathbb{Z}$ . Then  $g^r = g^{k-qd} = g^k(g^d)^{-q}$ . Since  $g^k, g^d \in K$ , by the minimality of  $d$ , we must have  $r = 0$ . So  $k = qd$ , this makes  $g^k = (g^d)^q \in \langle g^d \rangle$  and hence  $K \leq \langle g^d \rangle$ . Thus  $K$  is cyclic. Moreover, if  $\text{ord } g = n$ , and if  $d > n$ , then by the division theorem  $d \equiv r \pmod{n}$ , hence there are  $n$  subgroups of  $\langle g \rangle$ .

Now, if  $\langle g \rangle$  is infinite, and if  $\langle g^m \rangle = \langle g^n \rangle$ , then  $g^m = g^n$ , so  $g^{m-n} = e$ , implying that  $g$  has finite order; hence  $\langle g \rangle$  has finite order. This cannot happen, so  $\langle g^m \rangle \neq \langle g^n \rangle$ . Moreover, we get  $\langle g^m \rangle = \langle g^{-m} \rangle$ , so  $\langle g^m \rangle = \langle g^{|m|} \rangle$ .

Now, define  $\phi : m \rightarrow \langle g^m \rangle$ . By above, we get  $\phi$  is 1-1 and onto, so we have established a 1-1 correspondence between the subgroups of  $\langle g \rangle$  onto  $\mathbb{N}$ .

Finally, let  $\text{ord } \langle g \rangle = n$  and let  $k|n$ . Then letting  $d = \frac{n}{(n,k)} = \frac{n}{k}$  we have  $\langle g^d \rangle$  is a subgroup of order  $\text{ord } \langle g^d \rangle = k$ . Now, suppose  $K$  is any other subgroup of  $\text{ord } K = k$ . Then  $K = \langle g^l \rangle$ , where  $l \in \mathbb{Z}^+$  is the smallest integer of the set  $P$  in the above arguments. Then  $\text{ord } K = \frac{n}{(n,l)} = \frac{n}{d}$ , so  $\frac{n}{d} = \frac{n}{(n,l)}$  so  $d = (n, l)$ . In particular,  $d|l$ , so  $g^l \in \langle g^d \rangle$  and since  $\text{ord } K = k$ , this makes  $K = \langle g^d \rangle$ . ■

**Example 2.9.** (1) The subgroups of  $\mathbb{Z}/12\mathbb{Z}$  are:

$$\begin{aligned} \mathbb{Z}/12\mathbb{Z} &= \langle 1 \rangle = \langle 5 \rangle = \langle 7 \rangle = \langle 11 \rangle \\ \langle 2 \rangle &= \langle 10 \rangle \\ \langle 3 \rangle &= \langle 9 \rangle \\ \langle 4 \rangle &= \langle 8 \rangle \\ \langle 6 \rangle & \\ \langle 0 \rangle & \end{aligned}$$

- (2) Let  $G$  be any group, and let  $g \in G$ . We have that  $C(g) = C(\langle g \rangle)$  and  $\langle g \rangle \leq N(\langle g \rangle)$ .

## 2.4 Subgroups of Generating Sets.

We now make precise the notion of when a subgroup of a group  $G$  is generated by a subset  $A \subseteq G$ . Namely,  $A$  will be a set of “generators”.

**Lemma 2.4.1.** *If  $\mathcal{H}$  is a nonempty collection of subgroups of  $G$ , then the intersection of all elements of  $\mathcal{H}$  forms a subgroup of  $G$ .*

*Proof.* Let  $\mathcal{H} = \{H_i\}$  where  $H_i \leq G$  for all  $i \in \mathbb{Z}^+$ , and let  $K = \bigcap H_i$ . We have  $e \in H_i$ , for all  $i$ , so  $e \in K$ . Moreover, if  $a, b \in K$ , then  $a, b \in H_i$  for all  $i$ , making  $ab^{-1} \in H_i$  for all  $i$ . Thus  $ab^{-1} \in K$ , and so by the subgroup criterion,  $K \leq G$ . ■

**Definition.** Let  $G$  be a group, and  $\{H_i\}$  the collection of all subgroups of  $G$ . For any subset  $A \subseteq G$ , we define the subgroup  $\langle A \rangle$  **generated** by  $A$  to be the intersection of all subgroups  $H_i$  containing  $A$ ; i.e.

$$\langle A \rangle = \bigcap_{A \subseteq H_i, H_i \leq G} H_i \quad (2.1)$$

**Lemma 2.4.2.** *Let  $A$  be a nonempty subset of a group  $G$ . Then  $\langle A \rangle \leq G$ .*

*Proof.* Let  $\mathcal{H} = \{H \leq G : A \subseteq H\}$ . Then the result follows from lemma 2.4.1. ■

Having  $\langle A \rangle$  defined as the intersection of all subgroups containing  $A$ , while a useful definition, will not help us much in enumerating the elements of  $\langle A \rangle$ . For this, we turn to another subgroup, which can allow us to enumerate its elements effectively, and prove that it is equal to  $\langle A \rangle$ .

**Definition.** Let  $G$  be a group and let  $A \subseteq G$ . We define the **closure** of  $A$  to be the set  $\text{cl } A$  of all finite products of elements of  $A$ . That is:

$$\text{cl } A = \{a_1^{e_1} \dots a_n^{e_n} : n \in \mathbb{Z}^+, a_i \in A, e_i = \pm 1 \text{ for all } i\} \quad (2.2)$$

We call the elements of  $\text{cl } A$  **words**, and we define  $\text{cl } A = \langle e \rangle$ , if  $A = \emptyset$ .

**Lemma 2.4.3.** *Let  $G$  be a group and  $A \subseteq G$ . Then  $\text{cl } A = \langle A \rangle$ .*

*Proof.* Notice, that by definition that atleast  $e \in \text{cl } A$ , for if  $A$  is empty,  $\text{cl } A = \langle e \rangle$ , and if  $A$  is not, then  $a_i a_i^{-1} = e \in \text{cl } A$ . Now, if  $a, b \in \text{cl } A$  are words with  $a = a_1^{e_1} \dots a_n^{e_n}$ , and  $b = b_1^{f_1} \dots b_m^{f_m}$ , then notice that  $b^{-1} = b_m^{-f_m} \dots b_1^{-f_1}$ . So  $b^{-1} \in \text{cl } A$ , so  $ab^{-1} \in \text{cl } A$ . This makes  $\text{cl } A \leq G$ .

Furthermore, notice that  $A \subseteq \text{cl } A$ , hence  $\langle A \rangle \subseteq \text{cl } A$ . However, also notice that  $A \subseteq \langle A \rangle$ , and  $\langle A \rangle$  is closed under the operation of  $G$ , so  $\langle A \rangle$  contains all finite products of elements of  $A$ ; i.e.  $\text{cl } A \subseteq \langle A \rangle$ . This establishes equivalence. ■

**Corollary.**  $\langle A \rangle = \{a_1^{\alpha_1} \dots a_n^{\alpha_n} : a_i \neq a_{i+1}, n \in \mathbb{Z}^+, \text{ and } \alpha_i \in \mathbb{Z}\}$ .

**Corollary.** *If  $G$  is Abelian, then  $\langle A \rangle = \{a_1^{\alpha_1} \dots a_n^{\alpha_n} : \alpha_i \in \mathbb{Z}\}$*

**Corollary.** *If  $G$  is Abelian, and if each  $a_i$  has order  $\text{ord } a_i = d_i$ , then  $\text{ord } \langle A \rangle \leq d_1 \dots d_n$ .*

The above corollary is not always true, and only holds for Abelian groups. For example:

**Example 2.10.** (1) Consider  $D_8 = \langle r, t \rangle$ . let  $a = t$ , and  $b = rt$  and let  $A = \{a, b\}$ . Then  $D_8 = \langle a, b \rangle$ . Notice however, that  $\text{ord } a = \text{ord } b = 2$ , but that  $\text{ord } D_8 = 8 \geq 4 = 2 \cdot 2$ . Moreover notice that no element of  $D_8$  can be represented in the form  $a^\alpha b^\beta$ . For example,  $aba = trtt = tr$  which cannot be written in the form  $a^\alpha b^\beta$ .



- (2) Consider  $S_n = \langle (1\ 2), (1\ 2\ \dots\ n) \rangle$ . We have  $\text{ord}(1\ 2) = 2$  and  $\text{ord}(1\ 2\ \dots\ n) = n$ , but  $\text{ord } S_n = n!$ .
- (3) In  $GL(2, \mathbb{R})$ , let  $A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$  and  $B = \begin{pmatrix} 0 & 2 \\ \frac{1}{2} & 0 \end{pmatrix}$ . Then  $A^2 = B^2 = I$ , but  $AB = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & 2 \end{pmatrix}$ , which has infinite order. So  $\langle A, B \rangle$  is a subgroup of infinite order, generated by elements of finite order.

## 2.5 Lattices of Groups.

**Definition.** Let  $G$  be a group. We define the **lattice** of subgroups of  $G$  to be a directed graph whose set of vertices are subgroups of  $G$ , and whose edges are given by the relation  $(H, K)$  if  $H \leq K$ , for  $H, K$  subgroups of  $G$ .

We detail an algorithm for generating the lattice of subgroups of a given group  $G$ .

**Algorithm.** For any group  $G$ :

**step 1:** Plot  $G$  and  $\langle e \rangle$  opposite each other.

**step 2:** Plot a subgroup  $H$  of  $G$ . If there are no subgroups  $A$  such that  $H \leq A \leq G$ , connect  $H$  to  $G$ . If there are no subgroups  $A$  such that  $\langle e \rangle \leq A \leq H$ , connect  $H$  to  $\langle e \rangle$ . If there are no subgroups left to plot, go to **step 5**.

**step 3:** For any other subgroup  $K$  plotted on the graph, if there is no subgroup  $A$  such that  $H \leq A \leq K$ , or  $K \leq A \leq H$ , then connect  $K$  to  $H$ .

**step 4:** return to **step 2**.

**step 5** return.

The lattice of subgroups of a group, can also be viewed as a lattice of partially ordered sets, instead of a graph. We focus however on the graph theoretic approach. We now list numerous examples.

### Example 2.11. 2.11

- (1) For  $\mathbb{Z}/m\mathbb{Z} = \langle 1 \rangle$ , we plot the lattice of subgroups for  $m = 2, 4, 12, p^n$  where  $p$  prime and  $n \in \mathbb{Z}^+$ : Notice that only the lattice of  $\mathbb{Z}/12\mathbb{Z}$  contains any cycles.
- (2) The lattice of subgroups of  $D_8 = \langle r, t \rangle$  is that of figure 2.2 This lattice can be drawn as a planar graph, however it is not true that all lattices of  $D_8$  are planar. The lattice of  $D_{16}$  for example, cannot be drawn as a planar graph.
- 3 The lattice of  $S_3$  is the lattice in figure 2.3
- (4) The lattice of  $U(\mathbb{Z}/12\mathbb{Z})$  is that of figure 2.4

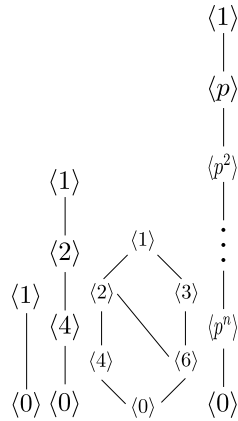


Figure 2.1: The lattices of  $\mathbb{Z}/2\mathbb{Z}$ ,  $\mathbb{Z}/8\mathbb{Z}$ ,  $\mathbb{Z}/12\mathbb{Z}$ , and  $\mathbb{Z}/p^n\mathbb{Z}$  respectively.

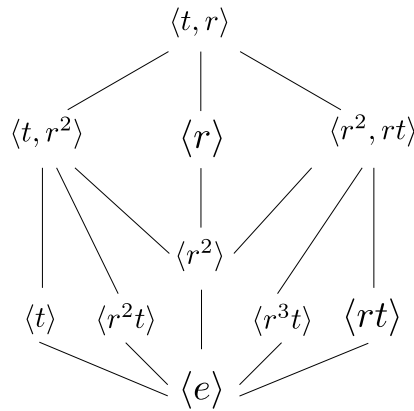


Figure 2.2: The lattices of  $D_8$ .

(5) The **Klein-4 group** is the Abelian group  $V_4$  defined by the following Cayley table:

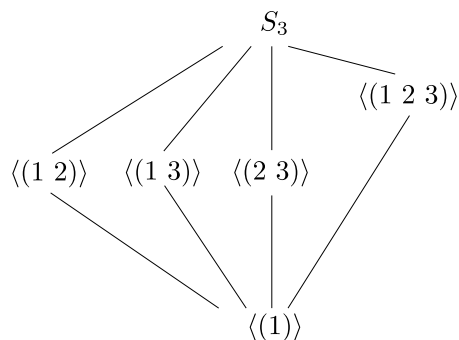
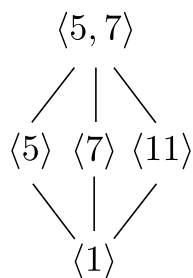
1	$a$	$b$	$c$
$a$	1	$c$	$b$
$b$	$c$	1	$a$
$c$	$b$	$a$	1

and has the lattice of figure 2.5. Notice that as a graph, the lattice of subgroups of  $V_4$  is isomorphic to the lattice of subgroups of  $U(\mathbb{Z}/12\mathbb{Z})$ .

These examples provide interesting notions. The first is the notion of the “join” of two subgroups.

**Definition.** For any two distinct subgroups  $H$  and  $K$  of a group, the **join** of  $H$  and  $K$  is the smallest subgroup,  $\langle H, K \rangle$ , containing both  $H$  and  $K$ .

**Lemma 2.5.1.** *Let  $H$  and  $K$  be distinct subgroups of a group. Then the join  $\langle H, K \rangle$  is unique.*

Figure 2.3: The lattices of  $S_3$ .Figure 2.4: The lattices of  $U(\mathbb{Z}/12\mathbb{Z})$ .

*Proof.* Suppose there exists another smallest subgroup containing  $H$  and  $K$ , and call it  $A$ . Then  $A \subseteq \langle H, K \rangle$ . But by definition,  $\langle H, K \rangle$  is minimal, thus it must be contained in  $A$  as well, so  $\langle H, K \rangle \subseteq A$ . This establishes equality, and hence uniqueness. ■

**Lemma 2.5.2.** *If  $G$  and  $H$  are isomorphic groups, then they have the same lattice of subgroups.*

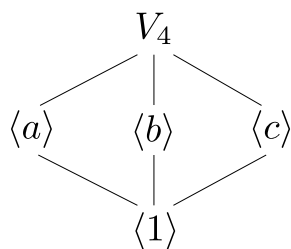


Figure 2.5: The lattices of  $V_4$ .

# Chapter 3

## Quotient Groups, and Homomorphisms.

### 3.1 Quotient Groups.

**Definition.** Let  $G$  and  $H$  be groups and let  $\phi : G \rightarrow H$  be a homomorphism of  $G$  into  $H$ . We define the **fiber** of  $\phi$  over an element  $h \in H$  to be the preimage of  $h$  under  $\phi$ , i.e.  $\phi^{-1}(h) = \{g \in G : \phi(g) = h\}$ .

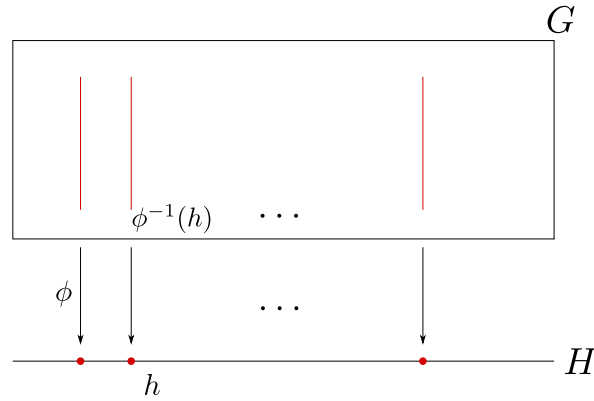


Figure 3.1: The fibers of the elements of  $H$  represented as vertical lines.

We can immediately define a special fiber for any homomorphisms.

**Definition.** Let  $G$  and  $H$  be groups and let  $\phi : G \rightarrow H$  be a homomorphism of  $G$  into  $H$ . We define the **kernel** of  $\phi$  to be the fiber  $\phi^{-1}(e')$  where  $e'$  is the identity of  $H$ . That is:

$$\ker \phi = \{g \in G : \phi(g) = e'\} \quad (3.1)$$

**Lemma 3.1.1.** *If  $\phi : G \rightarrow H$  is a 1 – 1 homomorphism then  $\ker \phi = \langle e \rangle$ .*

*Proof.* We have that  $\phi(e) = e'$ , thus since  $\phi$  is 1 – 1, we have  $\phi(a) = e'$  implies  $a = e$ . ■

**Lemma 3.1.2.** *Let  $G$  and  $H$  be groups and let  $\phi : G \rightarrow H$  be a homomorphism of  $G$  into  $H$ . Then  $\ker \phi \leq G$ , and  $\phi(G) \leq H$ .*

*Proof.* We have that  $\phi(e) = e'$  and for any  $a, b \in \ker \phi$ , that  $\phi(ab) = \phi(a)\phi(b) = e'$ , and that  $\phi(a^{-1}) = \phi(a)^{-1} = e'$ . This makes  $e', ab, a^{-1} \in \ker \phi$ . ■

**Example 3.1.** Let  $Z_n = \langle x \rangle$  be the cyclic group of order  $n$  and define the homomorphism  $\phi : \mathbb{Z} \rightarrow Z_n$  by  $a \rightarrow x^a$ . Then  $\phi(a+b) = x^{a+b} = x^a x^b = \phi(a)\phi(b)$ , so  $\phi$  is a homomorphism. Moreover,  $\phi^{-1}(x^a) = \{m \in \mathbb{Z} : m \equiv a \pmod{n}\} = n\mathbb{Z}$ . So the set of fibers is  $\mathbb{Z}/n\mathbb{Z}$ , moreover, we get the kernel of  $\phi$  to be  $\ker \phi = \{m \in \mathbb{Z} : m \equiv 0 \pmod{n}\} = n\mathbb{Z}$  the set of all multiples of  $n$ .

This motivates the following results.

**Theorem 3.1.3.** Let  $\phi^{-1}(H)$  be the set of all fibers of  $\phi$  under elements of  $H$ , and define the operation  $\phi^{-1}(a)\phi^{-1}(b) = \phi^{-1}(ab)$ . Then  $\phi^{-1}(H)$  forms a group under this operation.

*Proof.* Let  $\phi^{-1}(a)$  and  $\phi^{-1}(b)$  be fibers of  $\phi$  under  $a$  and  $b$  respectively. Then for  $g \in \phi^{-1}(a)$  and  $g' \in \phi^{-1}(b)$ , we have  $\phi(gg') = \phi(g)\phi(g') = ab$ , so that  $gg' \in \phi^{-1}(ab)$ , so this ensures that the operation is well defined. Moreover, we have that  $\phi^{-1}(H)$  satisfies associativity because of the associativity of  $H$ .

Notice, then that  $\phi^{-1}(e)$  is the identity, as, for  $g_e \in \phi^{-1}(e)$ ,  $\phi(gg_e) = \phi(g)\phi(g_e) = ae = a = ea = \phi(g_e)\phi(g) = \phi(g_e g)$ , where  $g \in \phi^{-1}(a)$  and  $g_e \in \phi^{-1}(e)$ . Similarly, we find that  $\phi^{-1}(a^{-1})$  is the inverse of  $\phi^{-1}(a)$  of  $\phi^{-1}(H)$ . ■

**Definition.** Let  $G$  and  $H$  be groups, and let  $\phi : G \rightarrow H$  be a homomorphism with  $\ker \phi = K$ . We define the **quotient group** of  $G$  **mod**  $K$  to be the group  $G/K$  of all fibers of  $\phi$  under elements of  $K$ .

**Lemma 3.1.4.** Let  $\phi$  be a homomorphism defined on a group  $G$  with  $\ker \phi = K$ . Then for any fiber  $\phi(a)^{-1} \in G/K$ ,  $a \in K$ . we have for any  $u \in \phi^{-1}(a)$ , that  $\phi^{-1}(a) = \{uk : k \in K\}$  and  $\phi^{-1}(a) = \{ku : k \in K\}$ .

*Proof.* Let  $uK = \{uk : k \in K\}$ . Then since  $\phi(u) = a$ , for any  $k \in K$ , we have  $\phi(uk) = \phi(u)\phi(k) = ae = a$ , making  $uk \in \phi^{-1}(a)$ . On the other hand, for some other  $x \in \phi^{-1}(a)$ , let  $k = u^{-1}x$ . Then  $\phi(k) = \phi(u^{-1})\phi(x) = \phi(u)^{-1}\phi(x) = a^{-1}a = e'$ , so  $k \in K$ . Then  $x = uk$ , this  $x \in uK$ . This makes  $\phi^{-1}(a) = uK$ . The second assertion follows similarly. ■

**Definition.** Let  $G$  be a group, and let  $H \leq G$ . For any  $g \in G$ , we define the **left** and **right cosets** of  $H$  in  $G$  to be the sets  $gH = \{gh : h \in H\}$  and  $Hg = \{hg : h \in H\}$ , respectively.

**Lemma 3.1.5.** Let  $G$  and  $H$  be groups and  $\phi : G \rightarrow H$  a homomorphism with kernel  $K$ . Then for any  $g \in G$ ,  $gK = Kg$ .

*Proof.* By the previous lemma 3.1.4, if  $a \in gK$  then  $a \in \phi^{-1}(g)$ , which makes  $a = kg$  for some  $k \in K$ , thus  $a \in Kg$ . Similarly, if  $b \in Kg$ , then  $b \in gK$ . ■

It is much easier to think in terms of cosets, rather than fibers, so we develop the theory of cosets further. In particular, if  $G$  is a group, and  $K$  is the kernel of some homomorphism  $\phi$  defined on  $G$ , then define the operation of **coset multiplication** by the rule  $aKbK = abK$ . Then we get the following result.

**Theorem 3.1.6.** *Let  $G$  be a group and  $\phi$  a homomorphism on  $G$  with kernel  $K$ . Then the set of all cosets of  $K$  in  $G$  forms a group under coset multiplication.*

*Proof.* Let  $a' \in aK$  and  $b' \in bK$ . Then  $a' = ak_1$  and  $b' = bk_2$ , thus  $a'b' = (ak_1)(bk_2) = a(k_1b)k_2 = a(bk')k_2 = ab(k'k_2) = abk''$ . This makes  $a'b' \in abK$ , and by definition  $aKbK = abK$ , we get  $a'b'K = abK$ . Therefore coset multiplication is well defined.

Now, notice that for the identity element  $e \in G$ , that  $eK = K$  satisfies  $aKK = KaK = aeK = aK$ . Moreover, for the inverse  $a^{-1} \in G$  of  $a$ , we have  $a^{-1}KaK = aKa^{-1}K = aa^{-1}K = eK = K$ . So  $K$  is the identity element and  $a^{-1}K$  is the inverse of  $aK$ . Lastly, associativity is inherited from the associativity of  $G$ ; i.e.  $(aKbK)cK = abKcK = (ab)cK = a(bc)K = aKbcK = aK(bKcK)$  ■

**Corollary.** *The group of cosets of  $K$  in  $G$  is precisely the quotient group  $G/K$ .*

*Proof.* This follows directly from lemma 3.1.4. Since  $G/K$  is the set of all fiber  $\phi^{-1}(a)$  Then notice that  $\phi^{-1}(a) = aK = Ka$ . ■

**Example 3.2.** (1) For  $\mathbb{Z}$  and  $Z_n = \langle x \rangle$  in example 3.2, the homomorphism  $\phi : a \rightarrow x^a$  has kernel  $n\mathbb{Z}$ , so the cosets of  $n\mathbb{Z}$  in  $\mathbb{Z}$  are of the form  $a + n\mathbb{Z}$  which are the elements mod  $n$ . Thus the quotient group is  $\mathbb{Z}/n\mathbb{Z}$ .

(2) Let  $G$  and  $H$  be isomorphic to each other via the isomorphism  $\phi : G \rightarrow H$ . Then  $\ker \phi = \langle e \rangle$ , and so the cosets are all  $a\langle e \rangle = a$  so we get  $G/\langle e \rangle = G$ .

(3) Let  $G$  be a group and define  $\phi : G \rightarrow \langle e \rangle$  by  $g \rightarrow e$ . Then  $\phi$  is a homomorphism with kernel  $\ker \phi = G$ . The cosets are precisely all  $gG = G$ . Thus the quotient group is  $G/G \simeq \langle e \rangle$

(4) Define the map  $\phi : \mathbb{R}^2 \rightarrow \mathbb{R}$  to be the pojection onto the  $x$ -axis  $(x, y) \rightarrow x$ . Then  $\phi((x_1, y_1) + (x_2, y_2)) = x_1 + x_2 = \phi((x_1, y) + (x_2, y'))$ . So  $\phi$  is a homomorphism. Then  $\ker \phi = \{x \in \mathbb{R} : \phi(x, y) = 0\} = (0, \mathbb{R})$ ; i.e. the  $y$ -axis. Then the cosets are all the vertical lines  $(x, \mathbb{R})$ , of the quotient group  $\mathbb{R}^2/(0, \mathbb{R})$ . Taking  $(x, \mathbb{R}) \rightarrow x$ , we get  $\mathbb{R}^2/(0, \mathbb{R}) \simeq \mathbb{R}$ .

(5) Consider the quaternions  $\mathbb{H}$  and the Keln-4 group  $V_4$ . Take  $\phi : \mathbb{H} \rightarrow V_4$  defined by the rule  $\pm 1 \rightarrow 1$ ,  $\pm i \rightarrow a$ ,  $\pm j \rightarrow b$ , and  $\pm k \rightarrow c$ . Then  $\phi$  is a homomorphism with kernel  $\ker \phi = \langle -1, 1 \rangle$ . Then the cosets of the quotient group  $\mathbb{H}/\langle -1, 1 \rangle$  are all  $\langle -1, 1 \rangle$ ,  $\langle -i, i \rangle$ ,  $\langle -j, j \rangle$ , and  $\langle -k, k \rangle$ , all collaped to  $1, a, b, c \in V_4$ .

## 3.2 Alternative Definitions.

There is another way for driving the theory for quotient groups that does not involve the use of homomorphisms at all. Instead, it relies on the notion of equivalence classes.

**Definition.** Let  $G$  be a group and  $H \leq G$  a subgroup. For  $a, b \in G$ , we say that  $a$  is **congruent** to  $b \pmod H$  if  $b^{-1}a \in H$ . We write  $a \equiv b \pmod H$ .

**Lemma 3.2.1.** *For any subgroup  $H$  of any group, the relation  $\equiv$  of congruence mod  $H$  is an equivalence relation.*

*Proof.* Let  $G$  be group and  $H \leq G$ . We have that  $a^{-1}a = e' \in H$  so that  $a \equiv a \pmod{H}$ .

Now, suppose that  $a \equiv b \pmod{H}$ . Then  $b^{-1}a \in H$ , so  $b^{-1}a = h$ . Then  $h^{-1} = a^{-1}b \in H$ ; so  $b \equiv a \pmod{H}$ . Lastly, suppose that  $a \equiv b \pmod{H}$  and that  $b \equiv c \pmod{H}$ . Then  $b^{-1}a \in H$  and  $c^{-1}b \in H$ . By closure then,  $(c^{-1}b)(b^{-1}a) = c^{-1}(bb^{-1})a = c^{-1}a \in H$ . So  $c^{-1}a \in H$  making  $a \equiv c \pmod{H}$ . ■

**Corollary.** *The equivalence classes of  $G/\equiv$  are precisely the left cosets  $aH$ , of  $H$  in  $G$ .*

*Proof.* We have that the equivalence classes of  $\equiv$  in  $G$  are of the form  $[a] = \{b \in G : b \equiv a \pmod{H}\}$ . Then for  $b \in [a]$ , we have  $a^{-1}b = h$  for some  $h \in H$ , so  $b = ah$ . Likewise, for  $b \in aH$ ,  $b = ah'$  for some  $h' \in H$ ; hence  $a^{-1}b = h' \in H$ . This makes  $[a] = aH$ . ■

**Example 3.3.** Consider the subgroup  $n\mathbb{Z} \leq \mathbb{Z}$  and define  $\sim$  to be  $a \sim b \pmod{n\mathbb{Z}}$  if  $a - b \in n\mathbb{Z}$ . Then if  $a \sim b \pmod{n\mathbb{Z}}$  we have  $a - b = nm$ , so that  $n|(a - b)$ , making  $a \equiv b \pmod{n}$ . Conversely, if  $a \equiv b \pmod{n}$ , then  $n|(a - b)$ , making  $a - b = nk$ , hence  $a \sim b \pmod{n\mathbb{Z}}$ . Therefore, the equivalence relation  $\sim$  defined is exactly that of  $\equiv_n$  of congruence mod  $n$ . Thus  $\mathbb{Z}/\sim = \mathbb{Z}/\equiv_n = \mathbb{Z}/n\mathbb{Z}$ .

Since  $aH$  describes the equivalence classes of the relation  $a \equiv b \pmod{H}$ , they partition the group  $G$  into disjoint subsets. That is:

**Theorem 3.2.2.** *Let  $G$  be a group, and  $H \leq G$ . Then the left cosets of  $H$  in  $G$  partition  $G$  into disjoint subsets. That is:*

$$G = \bigcup_{g \in G} gH \quad (3.2)$$

Where  $aH \cap bH = \emptyset$  for  $a, b \in G$  distinct.

*Proof.* We have that  $g \in gH$  for all  $g \in G$ , so that  $G$  is the union of left cosets of  $H$  in  $G$ . Now suppose that  $g \in aH \cap bH$ . Then  $g = ah$  and  $g = bh'$ . So  $ah = bh'$ , so  $b^{-1}a = h'h^{-1} = h''$ . So  $b \in aH$ . This makes  $aH = bH$ . Therefore if  $aH$  and  $bH$  are distinct, they are disjoint. ■

Now to define the quotient group, it is not enough to define left cosets as equivalence classes. In general, coset multiplication is not well defined for arbitrary cosets. So we wish to make an assertion.

**Theorem 3.2.3.** *Let  $G$  be a group and  $H \leq G$ , and define the operation of coset multiplication by  $aHbH = abH$ . Then this operation is well defined if, and only if for any  $h \in H$ ,  $ghg^{-1} \in H$  for all  $g \in G$ .*

*Proof.* Suppose that  $aHbH = abH$  is well defined. That is, if  $a' \in aH$  and  $b' \in bH$  then  $a'Hb'H = a'b'H = abH$ . Now, let  $g \in G$  and  $h \in H$ . Then take  $g^{-1}H = hg^{-1}H$ . Then, since coset multiplication is well defined, we get  $gh^{-1}g^{-1}H = H$ , so that  $gh^{-1}g^{-1} \in H$  for  $h^{-1} \in H$  arbitrary.

Conversely, suppose that  $ghg^{-1} \in H$  then  $ghg^{-1} = h'$ , hence  $gh = h'g$ . That is  $gh \in Hg$  and  $h'g \in gH$  making  $gH = Hg$ . Thus if  $a' \in aH$  and  $b' \in bH$  then  $a' = ah_1$ ,  $b' = bh_2$  so that  $a'b' = (ah_1)(bh_2) = a(h_1b)h_2 = ab(bh'h_2) = ab(h'h_2)$ , making  $a'b' \in abH$ . Therefore,  $a'Hb'H = a'b'H = abH$  making coset multiplication well defined. ■



*Proof.* If  $H$  is such that  $ghg^{-1} \in H$  for all  $g \in G$ , then  $gH = Hg$ . That is, every left coset is also a right coset. ■

**Corollary.** If  $ghg^{-1} \in H$  for all  $g \in G$ , then the set of all cosets of  $H$  in  $G$  form a group under coset multiplication.

*Proof.* This proof is identical to that of theorem 3.1.6. ■

**Definition.** Let  $G$  be a group and  $N$  a subgroup of  $H$ . For  $n \in N$  we  $gng^{-1}$  the **conjugate** of  $n$  by  $g$ . We call the set  $gNg^{-1} = \{gng^{-1} : n \in N\}$  the **conjugate** of  $N$  by  $g$ . We say that the subgroup  $N$  is **normal** in  $G$  if  $gNg^{-1} = N$ . That is, every every element of  $G$  normalizes  $N$ . If  $N$  is a normal subgroup of  $G$ , we write  $N \trianglelefteq G$ .

**Definition.** Let  $G$  be a group, and let  $N \trianglelefteq G$  be a normal subgroup of  $G$ . Then the **quotient group**  $G/N$  is the group of all cosets of  $N$  in  $G$ .

**Lemma 3.2.4.** Let  $G$  be a group, and  $N \leq G$ . Then the following are equivalent:

- (1)  $N \trianglelefteq G$ .
- (2) The normalizer of  $N$  is precisely  $G$ .
- (3)  $gN = Ng$  for all  $g \in G$ .
- (4)  $gNg^{-1} \subseteq N$ .

**Theorem 3.2.5.** Let  $G$  be a group and  $N \leq G$ . Then  $N$  is normal if, and only if it is the kernel of some homomorphism defined on  $G$ .

*Proof.* We have shown that if  $\phi$  is a homomorphism on  $G$ , then  $\ker \phi \trianglelefteq G$ ; see lemmas 3.1.4 and 3.2.4.

Now, suppose that  $N \trianglelefteq G$ . Now, let  $\phi : G \rightarrow G/N$  be defined by  $\phi : g \rightarrow gN$ . Since  $N$  is normal, coset multiplication is well defined and, we have that  $abN = aNbN$ , so that  $\phi$  is a homomorphism. Now, notice then that  $\ker \phi = \{g \in G : \phi(g) = N\}$ . So if  $g \in \ker \phi$ , then  $gN = N$ , so that  $g \in N$ . Moreover, if  $g \in N$ , then  $\phi(g) = gN = N$ , so that  $g \in \ker \phi$ . This makes  $\phi = N$ . ■

**Corollary.**  $\phi$  is onto.

*Proof.* Notice that the cosets of  $N$  in  $G$  partition  $G$ , so that each distinct  $g$  gets mapped to a distinct  $gN$ . ■

*Remark.* With this theorem, we have tied together the equivalence of the quotient group as defined with fibers, to the quotient group as defined with equivalence classes of cosets of normal subgroups.

The homomorphism in theorem 3.2.5 warrants its own definition.

**Definition.** Let  $G$  be a group and  $N \trianglelefteq G$ . The homomorphism from  $G$  onto  $G/N$  defined by  $g \rightarrow gN$  is called the **natural projection** of  $G$  onto  $G/N$ . If  $H \leq G/N$ , then the **complete fiber** of  $H$  in  $G$  is the preimage of  $H$  under the natural projection.

We conclude with some examples.

**Example 3.4.** (1) let  $G$  be any group. Then  $G$  and  $\langle e \rangle$  are normal in  $G$ , with quotient groups  $G/\langle e \rangle \simeq G$  and  $G/G \simeq \langle e \rangle$ .

(2) If  $G$  is Abelian, then any subgroup  $H \leq G$  is normal. We have  $gh = hg$  for all  $g \in G$ ,  $h \in H$ , so that  $gH = Hg$  for all  $g \in G$ . Thus quotient groups can be described for every subgroup.

(3) Every subgroup of  $\mathbb{Z}$  is normal, and cyclic, so  $N = \langle n \rangle = N = \langle -n \rangle = n\mathbb{Z}$ . The quotient group  $\mathbb{Z}/n\mathbb{Z} = \langle 1 \rangle$  is also cyclic.

(4) If  $N \leq Z(G)$  for a group  $G$ , then  $N \trianglelefteq G$ .

### 3.3 Lagrange's Theorem.

**Theorem 3.3.1** (Lagrange's Theorem.). *Let  $G$  and  $H$  be groups, then the order of  $H$  divides the order of  $G$ , i.e.  $\text{ord } H \mid \text{ord } G$ .*

*Proof.* Let  $H \leq G$  and consider the coset  $gH$ . We claim that  $\text{ord } H = |gH|$ . If  $g' \in gH$ , then  $g' = gh$ , for some  $h \in H$ . Now, define the map  $\phi : H \rightarrow gH$  by  $h \rightarrow gh$ . First, we have that if  $gh = gh'$  for  $h, h' \in H$ , then by cancellation,  $h = h'$ . Moreover, since  $gH$  is defined for all  $h \in H$ ,  $\phi$  is onto. Therefore  $\phi$  is a 1-1 mapping of  $H$  onto  $gH$ , thus  $|gH| = \text{ord } H$ .

Now, the group  $G$  is partitioned into disjoint subsets by the cosets of  $H$  in  $G$ . Assume that there are  $k$  such cosets. Then:

$$\text{ord } G = \sum_{g \in G} |gH| = \sum_{i=1}^k \text{ord } H = k \text{ord } H.$$

Therefore,  $\text{ord } H \mid \text{ord } G$ . ■

**Corollary.** *There are*

$$\frac{\text{ord } G}{\text{ord } H}$$

*cosets of  $H$  in  $G$ .*

*Proof.* We have that  $\text{ord } G = k \text{ord } H$ , where  $k$  is the number of cosets of  $H$  in  $G$ . ■

**Corollary.** *If  $N \trianglelefteq G$ , then*

$$\text{ord } G/N = \frac{\text{ord } G}{\text{ord } N} \tag{3.3}$$

**Definition.** Let  $G$  be a (not necessarily finite) group, and  $H \leq G$ . We define the **index** of  $H$  in  $G$  to be the number of left (or right) cosets of  $H$  in  $G$ , and write  $[G : H]$ .

*Remark.* By the above corollaries, we have that if  $G$  is finite, then  $[G : H] = \frac{\text{ord } G}{\text{ord } H}$ .

**Lemma 3.3.2.** *If  $G$  is a finite group, and  $g \in G$ , then  $\text{ord } g \mid \text{ord } G$ .*

*Proof.* Consider the cyclic subgroup generated by  $g$ ,  $\langle g \rangle \leq G$ . By Lagrange's theorem, we get  $\text{ord } \langle g \rangle = \text{ord } g \mid \text{ord } G$ . ■

**Corollary.** For any  $g \in G$ ,  $g^{\text{ord } G} = e$ .

*Proof.*  $g^{\text{ord } G} = g^{k \text{ord } g} = (g^{\text{ord } g})^k = e^k = e$ , for some  $k \in \mathbb{Z}^+$ . ■

**Lemma 3.3.3.** If  $G$  is a finite group of prime order  $p$ , then  $G$  is cyclic.

*Proof.* Let  $\text{ord } G = p$  a prime, and let  $g \in G$  be such that  $g \neq e$ . Then  $\text{ord } g > 1$ ; moreover, we have  $\text{ord } g \mid p$ , by lemma 3.3.2. Then either  $\text{ord } g = 1$  or  $\text{ord } g = p$ . By our choice of  $g$ , its order cannot be 1, thus  $\text{ord } g = p = \text{ord } G$ . Then  $\text{ord } \langle g \rangle = \text{ord } G$ , and since  $\langle g \rangle \leq G$ , we can conclude that  $G = \langle g \rangle$ . ■

**Corollary.** If  $G$  is of prime order, then  $G \simeq \mathbb{Z}/p\mathbb{Z}$ .

*Proof.* Take the homomorphism  $g^i \rightarrow i$ , for some generator  $g \in G$ , and  $1 \leq i \leq \text{ord } g$  ■

**Example 3.5.** (1) Let  $H = \langle (1 \ 2 \ 3) \rangle \leq S_3$ . Now, we have that  $H \leq N(H) \leq S_3$ , so by Lagrange's theorem,  $\text{ord } H \mid \text{ord } N(H) \mid \text{ord } S_3$ . Now,  $\text{ord } H = 3$  and  $\text{ord } S_3 = 6$ . This makes  $N(H) = H$  or  $N(H) = S_3$ . Notice, then that  $(1 \ 2)(1 \ 2 \ 3)(1 \ 2) = (1 \ 2 \ 3)$ , and  $(1 \ 2) = (1 \ 2)^{-1}$ , so  $(1 \ 2)$  conjugates  $(1 \ 2 \ 3)$ , so  $(1 \ 2) \in N(H)$ . This makes  $N(H) \neq H$ , so  $N(H) = S_3$ ; by lemma 3.2.4,  $H \trianglelefteq S_3$ .

(2) Let  $G$  be any group, and let  $H \leq G$ , such that  $[G : H] = 2$ . Then the total number of left cosets in  $G$  is  $H$  and  $gH$ . Now, suppose that  $Hg = H$ , since  $g \in Hg$ , we get  $g \in H$ , this makes  $Hg = H$ , but this also makes  $gH = H$ , implying  $[G : H] = 1$ , which cannot happen. Thus it must be that  $Hg = gH$ . By lemma 3.2.4, this makes  $H \trianglelefteq G$ .

(3) The relation  $\trianglelefteq$  defined by  $HG$  if  $H$  is a normal subgroup of  $G$ , is not in general transitive. Take  $D_8$ . We have

$$\langle t \rangle \trianglelefteq \langle t, r^2 \rangle \trianglelefteq D_8$$

but  $\langle t \rangle$  is not normal in  $D_8$ , for  $(rt)t(rt) = r^2t \neq t$ , so  $N(\langle t \rangle) \neq D_8$ .

For some nonexamples, we have:

**Example 3.6.** (1) Let  $H = \langle (1 \ 2) \rangle \leq S_3$ . Notice that  $(1 \ 3)(1 \ 2)(1 \ 3) = (2 \ 3) \neq (1 \ 2)$ , since  $(1 \ 3) = (1 \ 3)^{-1}$ , we get that  $(1 \ 3) \notin H$ . Thus  $N(H) \neq S_3$  meaning that  $H$  is not normal in  $S_3$ . Moreover, notice that  $[S_3 : H] = 3$ , then there are 3 left cosets of  $H$  in  $S_3$ , namely,  $H$ ,  $(1 \ 3)H$ , and  $H(1 \ 3)$ .

(2) For general  $S_n$ ,  $n \in \mathbb{Z}^+$  let  $G_i = \{s \in S_n : s(i) = i\}$  for  $1 \leq i \leq n$ . If  $t \in S_n$  such that  $t(i) = j$  then  $ts(i) = j = t(i)$ , so that  $tst^{-1} = e$ . Moreover, if  $m(i) = j$ , then  $t^{-1}m(i) = i$ , so that  $m \in tG_i$ . Thus

$$tG_i = \{m \in S_n : m(i) = j\}$$

Similarly, taking  $k = t^{-1}(j)$ , we get

$$G_it = \{l \in S_n : l(k) = i\}$$

Now, if  $tG_i$  and  $mG_i$  are distinct cosets, we have  $tG_i \cap mG_i = \emptyset$  by theorem 3.2.2. Moreover, notice that there are  $n$  such cosets, for each distinct image of  $i$ . Therefore,  $[S_n : G_i] = n$ . Notice that, in general  $tG_i \neq G_it$ , and that  $N(G_i) = G_i$ .

(3)  $\langle r \rangle$  is the only normal subgroup of order 2 in  $D_8$ .

**Theorem 3.3.4** (Cauchy). *If  $G$  is a finite group, and  $p \mid \text{ord } G$  for some prime  $p$ , then  $G$  contains an element of order  $p$ .*

**Theorem 3.3.5** (Sylow). *If  $G$  is a finite group of order  $p^a m$ ,  $p$  prime and  $p \nmid m$ , then  $G$  has an element of order  $p^a$ .*

**Theorem 3.3.6.** *Let  $H$  and  $K$  be subgroups of a group, and let  $HK = \{hk : h \in H, k \in K\}$ . If  $H$  and  $K$  are finite, then*

$$\text{ord } HK = \frac{\text{ord } H \text{ ord } K}{\text{ord } H \cap K} \quad (3.4)$$

*Proof.* Notice that  $HK = \bigcup_{h \in H} hK$  is a union of left cosets of  $K$  in  $H$ . Now, for  $h, h' \in H$ ,  $hK = h'K$  if, and only if  $h^{-1}h' \in K$ , i.e. if, and only if  $h^{-1}h' \in H \cap K$ . So that  $hK = h'K$  if and only if  $h(H \cap K) = h'(H \cap K)$ . Thus the number of left cosets of  $H \cap K$  in  $H$  is the same as the number of left cosets of  $K$  in  $H$ . By Lagrange's theorem, we then have:

$$[H : K] = [H : H \cap K] = \frac{\text{ord } H}{\text{ord } H \cap K}$$

Then, noticing that there are  $\text{ord } K$  elements in each coset  $hK$ , gives the result. ■

**Lemma 3.3.7.** *If  $H$  and  $K$  are subgroups of a group, then  $HK$  is a subgroup if, and only if  $HK = KH$ .*

*Proof.* Suppose first, that  $HK = KH$ . Let  $a, b \in HK$ , then  $a = hk$  and  $b = h'k'$  so  $ab = (hk)(h'k') = h(kh')k' = h(h''k'')k = (hh'')(k''k) \in HK$ . Additionally,  $a^{-1} = k^{-1}h^{-1} \in KH = HK$ . Thus  $HK$  is a subgroup.

Conversely, suppose that  $HK$  is a subgroup. Then we have  $K \leq HK$  and  $H \leq HK$ , then by closure, we have that  $KH \subseteq HK$ . On the other hand, since  $HK$  is a subgroup, then for every  $a \in HK$ ,  $a^{-1} \in HK$ . Then if  $a = hk$ ,  $a^{-1} = k^{-1}h^{-1}$ . That is,  $HK \subseteq KH$ . The equality is established. ■

**Corollary.** *If  $H \leq N(K)$  then  $HK$  is a subgroup.*

*Proof.* If  $H \leq N(K)$ , then for any  $h \in H$ , and  $k \in K$ ,  $hkh^{-1} \in K$ , so  $hk = k'h$  for some  $k' \in K$ . So  $hk \in KH$ . Conversely, we get  $k'h \in HK$  so that  $HK = KH$ . ■

**Corollary.** *if  $K$  is normal, then  $HK$  is a subgroup for any  $H$ .*

**Definition.** Let  $H$  be a subgroup of any group. We say that a subset  $A$  of the group **normalizes**  $H$  if  $A \subseteq N(H)$ . Similarly, we say  $A$  **centralizes**  $H$  if  $A \subseteq C(H)$ .

## 3.4 The Isomorphism Theorems.

# Bibliography

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