# Field Theory and Galois Theory.

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January 13, 2023

# Contents

# Chapter 1

# Fields.

## 1.1 Field Extensions.

**Definition.** We define the **characteristic** of a field $F$ to be the smallest positive integer $p$, such that $p \cdot 1 = 0$, where 1 is the identity of $F$. We write $\operatorname{char} F = p$, and if no such $p$ exists, then we write $\operatorname{char} F = 0$.

**Lemma 1.1.1.** *Let $F$ be a field, then $\operatorname{char} F$ is either $0$, or a prime integer.*

*Proof.* Let $\Gamma F = p$. If $p = 0$, then we are done. Now suppose that $p = mn$, with $m, n \in \mathbb{Z}^+$. Then $p \cdot 1 = (mn)1 = (n \cdot 1)(m \cdot 1) = mn = 0$, which makes $m$ and $n$ 0 divisors. Since $F$ is a field, and hence an integral domain, this is impossible, and hence $p$ must be prime. ■

**Corollary.** *If $\operatorname{char} F = p$, then for all $a \in F$, $pa = \underbrace{a + \cdots + a}_{p \text{ times}}$.*

*Proof.* We have $pa = p(a \cdot 1) = (p \cdot 1)a$. ■

**Example 1.1.** (1) Both $\mathbb{Q}$ and $\mathbb{R}$ have char $= 0$. Similarly, $\operatorname{char} \mathbb{Z} = 0$, even though $\mathbb{Z}$ is just an integral domain.

(2) $\operatorname{char} \mathbb{Z}/p\mathbb{Z} = p$ and $\operatorname{char} \mathbb{Z}/p\mathbb{Z}[x] = p$ for any prime $p$.

**Definition.** We define the **prime subfield** of a field $F$ to be the subfield of $F$ generated by 1.

**Example 1.2.** (1) The prime subfields of $\mathbb{Q}$ and $\mathbb{R}$ is $\mathbb{Q}$.

(2) Let $\mathbb{Z}/p\mathbb{Z}(x)$ the field of rational functions over $\mathbb{Z}/p\mathbb{Z}$. Then the prime subfield of $\mathbb{Z}/p\mathbb{Z}(x)$ is $\mathbb{Z}/p\mathbb{Z}$. Similarly, the prime subfield for $\mathbb{Z}/p\mathbb{Z}[x]$ is also $\mathbb{Z}/p\mathbb{Z}$.

**Definition.** If $K$ is a field containing a field $F$, then we call $K$ **field extension** over $F$, and write $K/F$ (not the quotient field!) or denote it by the diagram

$$\begin{tikzcd} K \arrow[d, dash] \\ F \end{tikzcd}$$

**Lemma 1.1.2.** *Every field is a field extension of its prime subfield.*

**Lemma 1.1.3.** *Let $K$ an extension over a field $F$. Then $K$ is a vector space over $F$.*

**Definition.** Let ${}^{K}\!/_{F}$ a field extension. We define the **degree** of $K$ over $F$, $[K:F]$ to be the dimension of ${}^{K}\!/_{F}$ as a vector space.

**Definition.** Let $F$ be a field, and $f \in F[x]$ a polynomial. We call am element $\alpha \in R$ a **root** (or **zero**) of $f$ if $f(\alpha) = 0$.

**Lemma 1.1.4.** *Let $\phi : F \to L$ a field homomorphism. Then either $\phi = 0$, or $\phi$ is 1–1.*

**Lemma 1.1.5.** *Let $F$ be a field, and $p \in F[x]$ an irreducible polynomial. Then there exists a field $K$ containing an embedding of $F$, such that $p$ has a root in $K$.*

*Proof.* Consider $K = {}^{F[x]}\!/_{(p)}$. Siince $p$ is irreducible in a principle ideal domain, $(p)$ is a maximal idea, and hence $K$ is a field. Now consider the canonical map $\pi : F[x] \to K$ taking $f \to f \mod (p)$ and let $\phi = \pi|_F$. Then $\phi \neq 0$, since $\pi : 1 \to 1$. Then $\phi$ is 1–1. And so $\phi(F) \simeq F$.

Now, consider $F$ as a subfield of $K$. Then $p(x \mod (p)) \equiv p(x) \mod (p) \equiv 0 \mod (p)$, so that $x \mod (p)$ is a root of $p$ in $K$. ■

**Corollary.** *There exists a field extension of $F$ containing a root of $p$.*

**Theorem 1.1.6.** *Let $F$ be a field, and let $p \in F[x]$ an irreducible polynomial of degree $n$, and let $K = {}^{F[x]}\!/_{(p)}$, and $\theta = x \mod (p)$. Then $\{1, \theta, \dots, \theta^{n-1}\}$ forms a basis for $K$ as a vector space over $F$ and $[K:F] = n$.*

*Proof.* Let $a \in F[x]$, since $F[x]$ is Euclidean domain, there exist $q, r \in F[x]$, $q \neq 0$ for which

$$a(x) = q(x)p(x) + r(x) \text{ where } \deg r < n$$

Now, since $pq \in (p)$, $a(x) \equiv r(x) \mod (p)$, and every element of $K$ is a polynomial of degree less than $n$. Then the elements $\{1, \theta, \dots, \theta^{n-1}\}$ span $K$.

Now, suppose that there are $b_0, \dots, b_{n-1} \in F$ not all 0 for which

$$b_0 + b_1\theta + \cdots + b_{n-1}\theta^{n-1} = 0$$

Then

$$b_0 + b_1\theta + \cdots + b_{n-1}\theta^{n-1} \equiv 0 \mod (p)$$

so that $p|(b_0 + b_1\theta + \cdots + b_{n-1}\theta^{n-1})$ in $F$. But $\deg p = n$ and $p$ divides a polynomial of degree $n-1$, which is a contradiction. Therefore we are left with $b_0 = \cdots = b_{n-1} = 0$. ■

**Corollary.** $K = \{\alpha_0 + a_1\theta + \cdots + a_{n-1}\theta^{n-1} : a_i \in F \textit{ for all } 1 \leq i \leq n-1\}$

**Corollary.** *If $a(\theta), b(\theta) \in K$, are elements of degree less than $n$, and the operations of polynomial addition, and polynomial multiplication mod $(p)$ are defined, then $K$ forms a field.*

**Example 1.3.** (1) Consider the polynomial $x^2+1$ over $\mathbb{R}$. Then one has the field

$$\mathbb{C} = {}^{\mathbb{R}[x]}\!\diagup_{(x^2+1)}$$

an extension of $\mathbb{R}$ of degree $[\mathbb{C} : \mathbb{R}] = 2$. Let $i$ be a root of $x^2+1$ in this field, then $i^2 = -1$, and the elements of $\mathbb{C}$ are of the form $a + ib$ where $a, b \in \mathbb{R}$. Then we have described the field of complex numbers, and the addition and multiplication ( mod $x^2+1$) of these elements are the addition and multiplication of complex numbers.

One might also construct $\mathbb{C}$ differently by defining the isomorphism

$${}^{\mathbb{R}[x]}\!\diagup_{(x^2+1)} \to \mathbb{C} \text{ taking } a + xb \to a + ib$$

(2) Consider again $x^2+1$ over $\mathbb{Q}$. Then we get the field

$$\mathbb{Q}(i) = {}^{\mathbb{Q}[x]}\!\diagup_{(x^2+1)}$$

of degree $[\mathbb{Q}(i) : \mathbb{Q}] = 2$, and where $i$ is a root of $x^2+1$, so that $i^2 = -1$. Then the elements of $\mathbb{Q}(i)$ are of the form $a + ib$ where $a, b \in \mathbb{Q}$, i.e. it is isomorphic to the set of all complex numbers with rational components.

(2) Consider $x^2-2$ over $\mathbb{Q}$. by Eisenstein's criterion for $p = 2$, $x^2 - 2$ is irreducible over $\mathbb{Q}$. Let $\alpha$ a root of $x^2-2$, so that $\alpha^2 = 2$. Then we have the field

$$\mathbb{Q}(\sqrt{2}) = {}^{\mathbb{Q}[x]}\!\diagup_{(x^2-2)}$$

of degree $[Q(\sqrt{2}) : \mathbb{Q}] = 2$, and whose elements are of the form $a + b\sqrt{2}$. One can define an isomorphism between $\mathbb{Q}(i)$ and $\mathbb{Q}(\sqrt{2})$ by taking $\sqrt{2} \to i$.

(3) The polynomial $x^3-2$ over $\mathbb{Q}$ gives us the field

$$\mathbb{Q}(\sqrt[3]{2}) = {}^{\mathbb{Q}[x]}\!\diagup_{(x^3-2)}$$

of degree $[\mathbb{Q}(\sqrt[3]{2}) : \mathbb{Q}] = 3$ over 2. Here the elements are of the form $a + b\xi + c\xi^2$ where $\xi^3 = 2$.

(4) Denote $\mathbb{F}_2$ to be a finite field of 2 elements. Consider the polynomial $x^2+x+1$ over $\mathbb{F}_2$ which is irreducible. Then the field

$$\mathbb{F}_2(\alpha) = {}^{\mathbb{F}_2[x]}\!\diagup_{(x^2+x+1)}$$

is a field of degree 2 over $\mathbb{F}_2$, whose elements are of the form $a + b\alpha$, where $\alpha^2 = \alpha + 1$. In fact, one can generate this field using the fact that $\alpha^2 = \alpha + 1$.

(5) Let $F = K(t)$ the field of rational functions in $t$ over a field $K$. Let $p(x) = x^2 - t \in F[x]$, then by Eisenstien's criterion with the ideal $(t)$, $p$ is irreducible over $F[x]$. Let $\theta$ be a root for $p$, that is $\theta = \sqrt{t}$, then we get the field $K(t, \sqrt{t})$ of degree $[K(t, \sqrt{t}) : K] = 2$, whose elements are of the form $a(t) + b(t)\sqrt{t}$.

**Lemma 1.1.7.** *Let $F$ be a subfield of a field $K$, and let $\alpha \in K$. Then there exists a unique minimal subfield of $K$ containing $F$ and $\alpha$; more preciesly, it is the intersection of all subfields of $K$ containing $F$ and $\alpha$.*

**Definition.** Let $K$ be any extension of a field $F$, and let $\alpha, \beta, \dots \in K$. Then we define the subfield **generated** by $\alpha, \beta, \dots$ over $F$ to be the unique minimal subfield containing all $\alpha, \beta, \dots$ and $F$ and we denote it $F(\alpha, \beta, \dots)$. Moreover, we call $K$ a **simple extension** of $F$ if $K = F(\alpha, \beta, \dots)$. If $K = (F\alpha_1, \dots, a_n)$ for $\alpha_1, \dots, a_n \in K$, then it is a **finitely generated** simple extension.

**Theorem 1.1.8.** *Let $F$ be a field, and $p \in F[x]$ irreducible, and let $K$ an extension of $F$ containing a root $\alpha$ of $p$. Then*

$$F(\alpha) \simeq {}^{F[x]}\!/_{(p)}$$

*Proof.* Consider the homomorphism $F[x] \to F(\alpha)$ taking $a(x) \to a(\alpha)$. Since $p(\alpha) = 0$, $p$ is in the kernel of this homomorphism, and we get an induced homomorphsim from ${}^{F[x]}\!/_{(p)} \to F(\alpha)$. Now, since $p$ is irreducible, ${}^{F[x]}\!/_{(p)}$ is a field, and since the homomorphsim takes $1 \to 1$, it is 1–1. Then by the first isomorphsim theorem for ring homomorphsims these two fields are isomorphic. ∎

**Corollary.** *If* $\deg p = n$*, then* $F(\alpha) = \{a_0 + a_1\alpha + \dots a_{n-1}\alpha^{n-1} : a_i \in F$ *for all* $1 \le i \le n-1\}$ *and* $[F(\alpha) : F] = n$.

**Example 1.4.** (1) The polynomial $x^2 - 2$ over $\mathbb{Q}$ also has the root $-\sqrt{2}$ in $\mathbb{R}$, so that $\mathbb{Q}(-\sqrt{2})$ is of degree 2 over $\mathbb{Q}$ with elements of the form $a - b\sqrt{2}$. Notice however that $\mathbb{Q}(-\sqrt{2}) \simeq \mathbb{Q}(\sqrt{2})$ by taking $a - b\sqrt{2} \to a + b\sqrt{2}$.

(2) The polynomial $x^3 - 2$ only has the solution $\xi = \sqrt[3]{2}$ in $\mathbb{R}$. However, in $\mathbb{Q}$ it has the solutions given by

$$\sqrt[3]{2}\left(\frac{-1 \pm i\sqrt{3}}{2}\right)$$

So that the subfields generated by either of these three elements (over $\mathbb{C}$) are isomorphic.

**Theorem 1.1.9.** *Let $\phi : F \to L$ a field isomorphism and $p \in F[x]$, $q \in L[x]$ irreducible polynomials, where $q$ is obtained by applying $\phi$ to the coefficients of $p$. Let $\alpha$ a root of $p$, and $\beta$ a root of $q$. Then there exists an isomorphism $F(\alpha) \to L(\beta)$ taking $\alpha \to \beta$ and extending $\phi$. That is, we have the following diagram*

$$\begin{array}{ccc} F(\alpha) & \longrightarrow & L(\beta) \\ | & & | \\ F & \xrightarrow[\phi]{} & E \end{array}$$

*Proof.* Notice that $\phi$ induces a ring homomorphism between $F[x]$ and $L[x]$, so that $(p)$ is maximal. Since $q$ is obtained from $p$, $(q)$ is also maximal, so that $F[x]/(p)$ and $L[x]/(q)$ are fields. Then we have an isomorphsism

$$F[x]/(p) \simeq L[x]/(q)$$

Then, if $\alpha$ is a root of $p$, and $\beta$ a root of $q$, we obtain the isomorphism

$$F(\alpha) \simeq L(\beta)$$

moreover, this isomorphism takes $\alpha \to \beta$. ■

## 1.2 Algebraic Extensions.

**Definition.** Let $K/F$ be a field extension. We say that an element $\alpha \in K$ is **algebraic** over $F$, provided there exists a polynomial over $F$ having $\alpha$ as a root. Otherwise we call $\alpha$ **transcendental**. If every $\alpha \in K$ is algebraic, we call $K$ **algebraic** and $K/F$ an **algebraic extension**.

**Lemma 1.2.1.** *Let $\alpha$ be algebraic over a field $F$. Then there exista a unique monic irreducible polynomial $m \in F[x]$ having $\alpha$ as a root. Moreover, if $f \in F[x]$ is a polynomial, then $f$ has $\alpha$ as a root if, and only if $m|f$.*

*Proof.* Let $m$ a polynomial of minimal degree having $\alpha$ as a root. Suppose, also that , is monic. Now, if $m$ were reducible, then $m(x) = a(x)b(x)$ for some $a, b \in F[x]$ polynomials both of degree less than $\deg m$. Then we also have that $a(\alpha) = b(\alpha) = 0$, which contradicts that $m$ is the polynomial of minimal degree satisfying that condition. Hence, $m$ is irreducible.

Now, let $f \in F[x]$ have $\alpha$ as a root, then by the divison theorem, there exist $q, r \in F[x]$, with $q \neq 0$ for which

$$f(x) = q(x)m(x) + r(x) \text{ where } \deg r < \deg m$$

Now, since $f(\alpha) = q(\alpha)m(\alpha) + r(\alpha) = 0$, then $r(x) = 0$ for all $x$ lest we contradict the minimality of $m$. Hence $m|f$. Conversely, if $m|f$, then $f$ has $\alpha$ as a root.

Now, let $g$ a polynomial of minimal degree for which $g(\alpha) = 0$. Then by above, we have that $\deg g = \deg m$, and that moreover, $m|g$ and $g|m$. therefore $g = m$ and uniqueness is established. ■

**Corollary.** *Let $L/F$ be an extension, and $\alpha$ algebraic over $F$. Let $m_{\alpha,F}$ the unique monic irreducible polynomial over $F$ having $\alpha$ as root, and $m_{\alpha,L}$ the unique monic irreducible polynomial over $L$ having $\alpha$ as root. Then $m_{\alpha,L}|m_{\alpha,F}$ in $L[x]$.*

**Definition.** Let $F$ be a field, and $\alpha$ algebraic over $F$. We define the **minimal polynomial** $m_{\alpha,F}$, to be the polynomial over $F$ of minimal degree having $\alpha$ as a root. If the field is clear, we instead write $m_\alpha$, or even just $m$ when the root itself is also clear. We define the **degree** of $\alpha$ to be $\deg \alpha = \deg m_\alpha$.

**Lemma 1.2.2.** *Let $\alpha$ algebraic over $F$. Then*

$$F(\alpha) \simeq {}^{F[x]}\!/\!_{(m_{\alpha,F})}$$

**Corollary.** $[F(\alpha) : F] = \deg m_\alpha = \deg \alpha$.

**Example 1.5.**

(1) The minimal polynomial for $\sqrt{2}$ over $\mathbb{Q}$ is $x^2 - 2$.

(3) The minimal polynomial for $\sqrt[3]{2}$ over $\mathbb{Q}$ is $x^3 - 2$.

(3) Let $n > 1$, then by the Eisenstein-Schömann criterion, $x^n - 2$ is irreducible over $\mathbb{Q}$. Moreover, $x^n - 2$ has as root in $\mathbb{R}$ $\sqrt[n]{2}$. Then $\mathbb{Q}(\sqrt[n]{2})$ is a field of degree $[\mathbb{Q}(\sqrt[n]{2}) : \mathbb{Q}] = 2$. Moreover $x^n - 2$ is the minimal polynomial of $\sqrt[n]{2}$. Notice, that over $\mathbb{R}$, $\deg [n]2 = 1$, and that $m_{\sqrt[n]{2},\mathbb{R}}(x) = x - \sqrt[n]{2}$.

(4) Consider $p(x) = x^3 - 3x - 1$ over $\mathbb{Q}$. Notice that $p$ is irreducible over $\mathbb{Q}$ and let $\alpha$ a root of $p$. Then $[\mathbb{Q}(\alpha) : \mathbb{Q}] = 3$.

**Lemma 1.2.3.** *An element $\alpha$ is algebraic over a field $F$ if, and only if the simple extension ${}^{F(\alpha)}\!/\!_{F}$ is finite.*

*Proof.* If $\alpha$ is algebraic over $F$ then $[F(\alpha) : F] = \deg \alpha \leq n$ if $\alpha$ satisfies a polynomial of degree $n$. Conversely, if $\alpha$ is an element of the finite extenson ${}^{K}\!/\!_{F}$, of degree $n$, then the set $\{1, \alpha, \ldots, \alpha^n\}$ is linearly dependent over $F$. Hence there exist $b_0, \ldots, b_n \in F$ not all 0 for which

$$b_0 + b_1\alpha + \cdots + a_n\alpha^n = 0$$

making $\alpha$ a root of a nonzero polynomial over $F$ of degree $\deg \leq n$. ■

**Corollary.** *If an extension ${}^{K}\!/\!_{F}$ is finite, then it is algebraic.*

*Proof.* If $\alpha \in K$ is algebraic, then ${}^{K}\!/\!_{F}$ implies that ${}^{F(\alpha)}\!/\!_{F}$ is finite, since $F(\alpha) \subseteq K$. ■

**Example 1.6.** Let $F$ a field of $\operatorname{char} F \neq 2$, and let $K$ an extension field of $F$ of degree $[K : F] = 2$. Let $\alpha \in K$ not in $F$, then $\alpha$ satisfies an polynomial of at most degree 2 over $F$. Now, since $\alpha \notin F$, this polynomial must have degree greater than 1. Hence it satisfies a polynomial of degree 2. Then the minimal polynomial of $\alpha$ is a quadratic

$$m_\alpha(x) = x^2 + bx + c \text{ with } b, c \in F$$

Since $F \subseteq F(\alpha) \subseteq K$, and $F(\alpha)$ is a vector space over $F$ of dimension 2, then we must have $K = F(\alpha)$; that is ${}^{K}\!/\!_{F}$ is simple.

Now, the roots of $m_\alpha$ are

$$\alpha = \frac{-b \pm \sqrt{b^2 - 4c}}{2}$$

Since $\alpha \notin F$, $b^2 - 4c$ is not a square in $F$, and $\sqrt{b^2 - 4c}$ is a root of the equation $x^2 - (b^2 - 4c) = 0$ in $K$.

Conversely, $\sqrt{b^2 - 4c} = \pm(b + 2\alpha)$ which puts $\sqrt{b^2 - 4c} \in F(\alpha)$. That is $F(\sqrt{b^2 - 4c}) = \mathbb{F}(\alpha)$. Moreover, $x^2 - (b^2 - 4c)$ does not have solutions in $K$.

We call field extensions ${}^{K}\!/\!_{F}$ of degree 2 **quadratic field extension**, where $K = F(\sqrt{D})$, and $D$ is a squarefree element of $F$.

**Theorem 1.2.4.** *Let $F \subseteq K \subseteq L$. Then $[L : F] = [L : K][K : F]$.*

*Proof.* Let $[L : K] = m$ and $[K : F] = n$. Let $\{\alpha_1, \dots, \alpha_m\}$ and $\{\beta_1, \dots, \beta_n\}$ be bases for the extensions ${}^{L}\!/_{K}$ and ${}^{K}\!/_{F}$. Now, the elements of $L$ over $K$ are of the form

$$a_1\alpha_1 + \cdots + a_m\alpha_m \text{ where } a_i \in K \text{ for all } 1 \leq i \leq m$$

Since each $a_i \in K$, which is an extension over $F$, they have the form

$$a_i = b_{i1}\beta_{i1} + \cdots + b_{in}\beta_{in} \text{ where } b_{ij} \in F \text{ for all } 1 \leq j \leq n$$

That is, every element of $L$, as a vector space over $F$ are of the form

$$\sum b_{ij}\alpha_i\beta_j$$

So the set $\{\alpha_1\beta_1, \dots \alpha_m\beta_n\}$ spans $L$. It remains to show that this set is linearly in dependent.

Suppose that

$$\sum b_{ij}\alpha_i\beta_j = 0$$

for some $b_{ij} \in F$. Since $\{\alpha_1, \dots, \alpha_m\}$ are linearly indpendent in $L$ over $K$, we have that the coefficients $a_1 = \cdots = a_n = 0$ which makes

$$a_i = b_{i1}\beta_{i1} + \cdots + b_{in}\beta_{in} = 0$$

Now, since $\{\beta_1, \dots, \beta_n\}$ is linearly independent in $K$ over $F$, this implies that $b_{i1} = \cdots = b_{in} = 0$ which makes the collection $\{\alpha_1\beta_1, \dots \alpha_m\beta_n\}$ linearly independent, and hence, a basis. Moreover, notice that this basis has size $mn$. ■

**Example 1.7.** (1) The element $\sqrt{2} \notin \mathbb{Q}(\alpha)$, where $\alpha$ is the root of $x^3 - 3x - 1$; since $[\mathbb{Q}(\sqrt{2}) : \mathbb{Q}] = 2$, and $[\mathbb{Q}(\alpha) : \mathbb{Q}] = 3$.

(2) We have $[\mathbb{Q}(\sqrt[6]{2}) : \mathbb{Q}] = 6$, and since $(\sqrt[6]{2})^3 = \sqrt{2}$, we observe that $\mathbb{Q}(\sqrt{2}) \subseteq \mathbb{Q}(\sqrt[6]{2})$. Moreover, notice that by theorem 1.2.4 $[\mathbb{Q}(\sqrt[6]{2}) : Q(\sqrt{2})] = 3$. Then we have the following tower of fields for

$$\mathbb{Q} \subseteq \mathbb{Q}(\sqrt{2}) \subseteq \mathbb{Q}(\sqrt[6]{2})$$

$$\begin{array}{c} \mathbb{Q}(\sqrt[6]{2}) \\ | \\ \mathbb{Q}(\sqrt{2}) \\ | \\ \mathbb{Q} \end{array}$$

**Lemma 1.2.5.** *Let $\alpha, \beta$ be algebraic over a field $F$. Then $F(\alpha, \beta) = (F(\alpha))(\beta)$.*

*Proof.* By definition, $F(\alpha, \beta)$ contains $F$, and $\alpha$, and hence contains $F(\alpha)$. It also contains $\beta$ so that $(F(\alpha))(\beta) \subseteq F(\alpha, \beta)$. By the same argument, $(F(\alpha))(\beta)$ contains $F$, $\alpha$ and $\beta$ so that $F(\alpha, \beta) \subseteq (F(\alpha))(b)$. ■

**Corollary.** *The elements of $F(\alpha, \beta)$ are of the form $\sum a_{ij}\alpha^i b^j$, where $1 \leq i \leq \deg \alpha$ and $1 \leq j \leq \deg \beta$.*

**Example 1.8.** Consider $\mathbb{Q}(\sqrt{2}, \sqrt{3})$ generated by $\sqrt{2}$ and $\sqrt{3}$. Notice that $\deg \sqrt{3} = 2$ over $\mathbb{Q}$ so that $[\mathbb{Q}(\sqrt{2}, \sqrt{3}) : \mathbb{Q}(\sqrt{2})] \leq 2$. Now $[\mathbb{Q}(\sqrt{2}, \sqrt{3}) : \mathbb{Q}(\sqrt{2})] = 2$ if, and only if the polynomial $x^2 - 3$ is irreducible over $\mathbb{Q}(\sqrt{2})$. Then it is irreducible if, and onyl if $\sqrt{2} \in \mathbb{Q}(\sqrt{3})$. It can be shown that this is not the case by trying to find $a, b \in \mathbb{Q}$ for which $\sqrt{3} = a + b\sqrt{2}$. Moreover we have

$$[\mathbb{Q}(\sqrt{2}, \sqrt{3}) : \mathbb{Q}] = 4$$

**Theorem 1.2.6.** *An extension field $K/F$ is finite if, and only if it is generated by finitely many algebraic elements over $F$.*

*Proof.* Let $K/F$ finite of degree $n$, and $\{\alpha_1, \ldots, \alpha_n\}$ a basis. Then by theorem 1.2.4, $[F(\alpha_i) : F] | [K : F]$ for all $1 \leq i \leq n$. So each $\alpha_i$ is algebraic over $F$. Then $K$ is generated by finitely many algebraic elements.

Conversely, let $K = F(\alpha_1, \ldots, \alpha_k) = (F(\alpha_1, \ldots a_{k-1}))(\alpha_k)$ We obtain $K$ by taking the extensions $F_{i+1}/F_i$ iteratively, where $F_{i+1} = F_i(\alpha_{i+1})$, and obtain the sequence

$$F = F_0 \subseteq \cdots \subseteq F_k = K$$

Now, if the elements $\alpha_1, \ldots, \alpha_k$ are algebraic over $F$, each of $\deg \alpha_i = n_i$ for $1 \leq i \leq k$, then the extension $F_{i+1}/F_i$ is a simple extension, and $[F_{i+1} : F_i] = \deg m_{\alpha_{i+1}} \leq \deg \alpha_{i+1} = n_{i+1}$. Then we have

$$[K : F] = [F_k : F_{k-1}] \ldots [F_1, F] \leq n_1 \ldots n_k$$

which makes $K/F$ a finite extension. ∎

**Corollary.** *If $\alpha, \beta$ are algebraic over $F$, then so are $\alpha \pm \beta$, $\alpha\beta$, and $\alpha\beta^{-1}$ (for $\beta \neq 0$).*

**Corollary.** *If $L/F$ is an extension, then the collection of elements of $L$ which are algebraic over $F$ forms a subfield of $L$.*

**Example 1.9.** (1) Consider the extension $\mathbb{C}/\mathbb{Q}$, and let $\mathrm{cl}\,\mathbb{Q}$ the subfield of all elements of $\mathbb{C}$ which are algebraic over $\mathbb{Q}$. Then $\sqrt[n]{2} \in \mathrm{cl}\, Q$ for all $n \geq 1$, so that $[\mathrm{cl}\,\mathbb{Q} : \mathbb{Q}] \geq n$. This makes $\mathrm{cl}\,\mathbb{Q}$ an infinite algebraic extension, and we call $\mathrm{cl}\,\mathbb{Q}$ the **field of algebraic numbers**.

(2) Consider $\mathrm{cl}\,\mathbb{Q} \cap \mathbb{R}$ as a subfield of $\mathbb{R}$ (i.e. the subfield of all algebraic elements of $\mathbb{Q}$). Since $\mathbb{Q}$ is countable, so is the field $\mathbb{Q}[x]$, and each polynomial in $\mathbb{Q}[x]$ has at most $n$ roots in $\mathbb{R}$, hence the number of all algebraic elements of $\mathbb{R}$ over $\mathbb{Q}$ is also countable. This means that $\mathrm{cl}\,\mathbb{Q}$ must also be countable. Now, since $\mathbb{R}$ is uncountable, then there exist uncountably transcendental numbers of $\mathbb{R}$ over $\mathbb{Q}$. Most notably the irrational numbers $\pi$ and $e$ are transcendental.

**Theorem 1.2.7.** *If $K$ is algebraic over $F$, and $L$ algebraic over $K$, then $L$ is algebraic over $F$.*

*Proof.* Let $\alpha \in L$, since $L$ is algebraic over $K$, there exists a $p \in K[x]$ having $\alpha$ as root. Let $p(x) = a_0 + a_1 x + \cdots + a_n x^n$. Consider then $F(\alpha, a_0, \ldots, a_n)$. Since ${}^{K}\!/_{F}$ is algebraic, $a_0, \ldots, a_n$ are algebraic over $F$, and so $F(\alpha, a_0, \ldots, a_n)$ is a finite extension over $F$. Then $\alpha$ generates an extension field of degree less than $n$, and we get

$$[F(\alpha, a_0, \ldots, a_n) : F] = [F(\alpha, a_0, \ldots, a_n) : F(a_0, \ldots, a_n)][F(a_0, \ldots, a_n) : F]$$

is finite, and $F(\alpha, a_0, \ldots, a_n)$ is algebraic over $F$. That is, $\alpha$ is algebraic over $F$, and so $L$ is algbraic over $F$. ■

**Definition.** Let $K_1$ and $K_2$ subfields of a field $K$. The **composite field** $K_1 K_2$ is the smallest subfield of $K$ containing both $K_1$ and $K_2$.

**Example 1.10.** The composite field of $\mathbb{Q}(\sqrt[3]{2})$ and $\mathbb{Q}(\sqrt{2})$ is $\mathbb{Q}(\sqrt[6]{2})$.

**Lemma 1.2.8.** *Let $K_1$ and $K_2$ be extensions of a field $F$ contained in a field $K$. Then $[K_1 K_2 : F] \leq [K_1 : F][K_2 : F]$ with equality holding if, and only if a basis of $F$ in the other field is linearly independent. Moreover if $\{\alpha_1, \ldots, \alpha_m\}$ and $\{\beta_1, \ldots, \beta_n\}$ are bases for $K_1$ and $K_2$, then $\{\alpha_1, \beta_1, \ldots, \alpha_m \beta_n\}$ span $K_1$ and $K_2$.*

$$\begin{tikzcd}
 & K_1K_2 \arrow[dl, dash, "\leq m"'] \arrow[dr, dash, "\leq n"] & \\
K_1 \arrow[dr, dash, "n"'] & & K_2 \arrow[dl, dash, "m"] \\
 & F &
\end{tikzcd}$$

**Corollary.** *If $[K_1 : F] = m$, and $[K_2 : F] = n$ with $m$ and $n$ coprime, then $[K_1 K_2 : F] = [K_1 : F][K_2 : F]$.*

*Proof.* We have that $m, n | [K_1 K_2 : F]$ and since $K_1, K_2 \subseteq K_1 K_2$ are subfields of $K_1 K_2$, we get the least common multiple $[m, n] | [K_1 K_2 : F]$. Now, since $(m, n) = 1$, we get $[m, n] = mn$ so that $mn \leq= [K_1 K_2 : F]$. ■

## 1.3 Ruler and Compass Constructions.

## 1.4 Splitting Fields and Algebraic Closures.

**Definition.** Let $K$ be an extension of a field $F$. We say a polynomial $f$ over $F$ **splits completely** over $K$ if $f$ factors into linear factors over $K$. If $f$ splits completely over $K$, and in no other proper subfield, then we say $K$ is the **splitting field** of $f$ over $F$.

**Theorem 1.4.1.** *If $f$ is a polynomial over a field $F$, then there exists a splitting field $K$ of $f$ over $F$.*

*Proof.* Let $E$ an extension of $F$ with $[E : F] = n$. By induction on $n$, for $n = 1$, we take $E = F$ and we are done. Now, for $n \geq 1$, suppose the irreducible factors of $f$ are of deg $= 1$. Then $f$ has all its roots in $F$, and hence splits completely over $F$. Then take $E = F$. On the other hand, if $f$ has at least one irreducible factor of deg $\geq 2$, then there is an extension $E_1$ of $F$ for which $f$ has the factor $(x - \alpha)$ for some root $\alpha$. Then $f(x) = (x - \alpha)f_1(x)$ where deg $f_1 = n - 1$. Therefore by the induction hypothesis, there is an extension $E$ of $E_1$ containing all the roots of $f_1$. Hence, it contains all the roots of $f$ and $f$ splits completely over $E$.

Now, let $K$ be the intersection of all subfields of $E$ for which $f$ splite; i.e. all subfields containing the roots of $f$. Then by definition, $K$ is the splitting field of $f$ over $F$. ∎

**Definition.** If $K$ is an algebraic extesnion of $F$ such that it is the splitting field for a collection of polynomials over $F$, then we say that $K$ is a **normal extension** of $F$.

**Example 1.11.** (1) The splitting field of $x^2 - 2$ over $\mathbb{Q}$ is $\mathbb{Q}(\sqrt{2})$, since $x^2 - 2 = (x + \sqrt{2})(x - \sqrt{2})$ and $\pm\sqrt{2} \in \mathbb{Q}(\sqrt{2})$ and $[\mathbb{Q}(\sqrt{2}) : \mathbb{Q}] = 2$, so there is no other subfield in between.

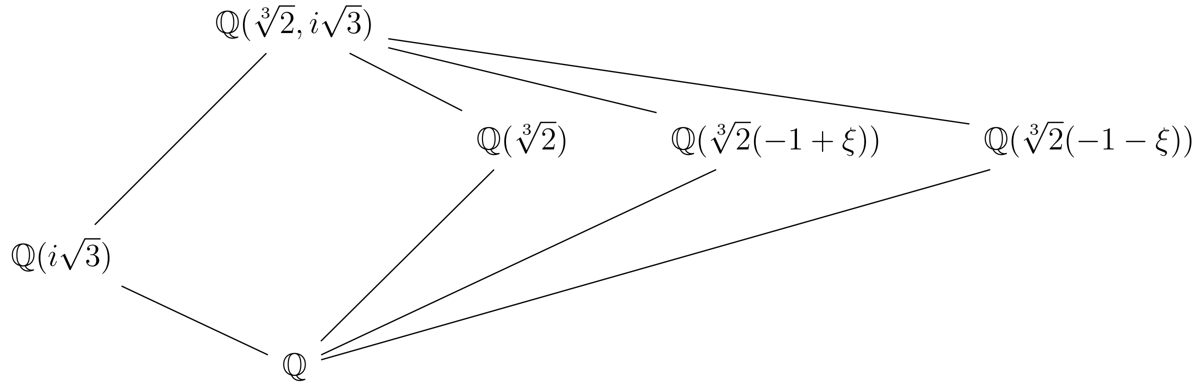
(2) The splitting field for $(x^2-2)(x^2-3) = (x+\sqrt{2})(x-\sqrt{2})(x+\sqrt{3})(x-\sqrt{3})$ is $\mathbb{Q}(\sqrt{2}, \sqrt{3})$. Now, $[\mathbb{Q}(\sqrt{2}, \sqrt{3}) : Q] = 4$ and the lattice of fields is

$$\begin{tikzcd}
& \mathbb{Q}(\sqrt{2},\sqrt{3}) \arrow[dl, dash, "2"'] \arrow[d, dash, "2"] \arrow[dr, dash, "2"] & \\
\mathbb{Q}(\sqrt{2}) \arrow[dr, dash, "2"'] & \mathbb{Q}(\sqrt{6}) \arrow[d, dash, "2"] & \mathbb{Q}(\sqrt{3}) \arrow[dl, dash, "2"] \\
& \mathbb{Q} &
\end{tikzcd}$$

(3) Let $\xi = i\frac{\sqrt{3}}{2}$. Notice that $x^3 - 2$ factors into $x^3 - 2 = (x - \sqrt[3]{2})(x + \sqrt[3]{2}(-1 + \xi))(x + \sqrt[3]{2}(-1 - \xi))$. Now, $-1 + \xi, -1 - \xi \notin \mathbb{Q}(\sqrt[3]{2})$, so $\mathbb{Q}(\sqrt[3]{2})$ is not the splitting field for $x^3 - 2$. Let $K$ be the splitting field of $x^3 - 2$. Then $K$ conmtains $-1 \pm \xi$, so that $i\sqrt{3} \in K$. Thus

$$K = \mathbb{Q}(\sqrt[3]{2}, i\sqrt{3})$$

Moreover, $[\mathbb{Q}(\sqrt[3]{2}, i\sqrt{3}) : \mathbb{Q}(\sqrt[3]{2})] \geq 2$ and since $\mathbb{Q}(\sqrt[3]{2})$ is not the splitting field, $[\mathbb{Q}(\sqrt[3]{2}, i\sqrt{3}) : \mathbb{Q}(\sqrt[3]{2})] = 2$. Hence $[\mathbb{Q}(\sqrt[3]{2}, i\sqrt{3}) : \mathbb{Q}] = 6$. We have the following lattice.




(4) Notice that $x^4+4 = (x^2+2x+2)(x^2-2x+2)$ over $\mathbb{Q}$ which is irreducible by Eisenstein's criterion. Using the quadratic formula, we get $\pm 1$ and $\pm i$ as the roots, moreover, notice that $\pm 1, \pm i \in \mathbb{Q}(i)$ and since $[\mathbb{Q}(i) : \mathbb{Q}] = 2$ there are no subfields between $\mathbb{Q}$ and $\mathbb{Q}(i)$ so that $\mathbb{Q}(i)$ is the splitting field of $x^4+4$ over $\mathbb{Q}$.

**Lemma 1.4.2.** *A splitting field of a polynomial of degree $n$ over a field $F$ is of degree at most $n!$ over $F$.*

*Proof.* Let $f \in F[x]$ a polynomial of $\deg f = n$. Adjoining one root of $f$ to $F$, we have an extension ${}^{F_1}\!/_{F}$ of degree $[F_1 : F] = n$. Now,$f$ over $F_1$ has at leas one linear factor, and so any root of $f$ satisfies a polynomial of degree $n-1$. Hence proceeding inductively gives the result. ■

**Example 1.12.** Consider the polynomial $x^n - 1$ over $\mathbb{Q}$. Then the roots of $x^n - 1$ are of the form $\xi$ where $\xi^n = 1$. Notice, that in $\mathbb{C}$, $\xi = e^{\frac{2i\pi}{n}}$, so that $\mathbb{C}$ contains a splitting field of $x^n - 1$. Hence $\mathbb{Q}(\xi) \subseteq \mathbb{C}$ is a splitting field of $x^n - 1$ over $\mathbb{Q}$. Notice that the set of all roots $\xi$ of $x^n - 1$ forms a cyclic group generated by $\xi$.

**Definition.** Consider a field $F$ and the polynomial $x^n - 1$ over $F$. We call the roots $\xi$ of $x^n - 1$, where $\xi^n = 1$ the **primitive $n$-th roots of unity** over $F$. We call $F(\xi)$ the **cyclotomic field** over $F$.

**Example 1.13.** Let $p$ be a prime, and consider the splitting field $x^p - 2$ over $\mathbb{Q}$. If $\alpha$ is a root, then $\alpha^p = 2$ so that $(\xi\alpha)^p = 2$ where $\xi$ is a primitive $p$-th root of unity over $\mathbb{Q}$. So the roots of $x^2 - 2$ are

$$\sqrt[p]{2} \text{ and } \xi\sqrt[p]{2}$$

Notice that $\frac{\xi\sqrt[p]{2}}{\sqrt[p]{2}} = \xi$ so the splitting field contains $\mathbb{Q}(\xi, \sqrt[p]{2})$, Moreover, $\mathbb{Q}(\xi, \sqrt[p]{2})$ contains all the roots of $x^p - 2$ so that $\mathbb{Q}(\xi, \sqrt[p]{2})$ is the splitting field of $x^p - 2$ over $\mathbb{Q}$.

Notice, that $\mathbb{Q}(\xi) \subseteq \mathbb{Q}(\xi, \sqrt[p]{2})$ so that $[\mathbb{Q}(\xi, \sqrt[p]{2}) : \mathbb{Q}(\xi)] \leq p$. not, since $\mathbb{Q}(\sqrt[p]{2})$ is also a subfield, we get $[\mathbb{Q}(\xi, \sqrt[p]{2}) : Q] \leq p(p-1)$. Since $(p, p-1) = 1$ (i.e. they are coprime), we have $p(p-1)|[\mathbb{Q}(\xi, \sqrt[p]{2}) : \mathbb{Q}]$ so that $[p]2) : \mathbb{Q}] = p(p-1)$. We have the follwing lattice.

$$\begin{tikzcd}
 & \mathbb{Q}(\xi, \sqrt[p]{2}) \arrow[dl, dash] \arrow[dr, dash] & \\
\mathbb{Q}(\xi) \arrow[dr, dash] & & \mathbb{Q}(\sqrt[p]{2}) \arrow[dl, dash] \\
 & \mathbb{Q} &
\end{tikzcd}$$

**Theorem 1.4.3.** *Let $\phi : F \to F'$ a field isomorphism. Let $f$ and $f'$ polynomials over $F$ and $F'$, where $f'$ is obtained by applying $\phi$ to the coefficients of $f$. Let $E$ and $E'$ be splitting fields of $f$ and $f'$ over $F$ and $F'$, respectively. Then $\phi$ extends to an isomorphism betweenn $E$ and $E'$; i.e. $E \simeq E'$.*

$$\begin{tikzcd}
E \arrow[r] \arrow[d, dash] & E' \arrow[d, dash] \\
F \arrow[r, "\phi"] & F'
\end{tikzcd}$$

*Proof.* Let $\deg f = n$. By induction on $n$. If $f$ has all its roots in $F$, $f$ splits completely over $F$, and $f'$ over $F'$. Then take $E = F$ and $E' = F'$ and we are done for $n = 1$.

Now, for $n \geq 1$, suppose the theorem is true. Let $p$ an irreducible factor of $f$, and $p'$ an irreducible factor of $f'$. If $\alpha$ and $\alpha'$ are roots of $p$ and $p'$, respectively, then extend $\phi$ to $F(\alpha)$ and $F'(\alpha')$. Then $f(x) = (x-\alpha)f_1(x)$ and $f'(x) = (x-\alpha')f_1'(x)$; with $\deg f_1 = \deg f_1' = n-1$. Then let $E$ the splitting field of $f_1$ over $F(\alpha)$, and $E'$ the splitting field of $f_1'$ over $F'(\alpha')$

$$\begin{tikzcd}
E \arrow[r] \arrow[d, dash] & E' \arrow[d, dash] \\
F(\alpha') \arrow[r] \arrow[d, dash] & F'(\alpha') \arrow[d, dash] \\
F \arrow[r, "\phi"'] & F'
\end{tikzcd}$$

The the roots of $f_1$ and $f_1'$ are in $E$ and $E'$, respectively, and hence so are the roots of $f$ and $f'$. Then by the induction hypothesis, we can extend $\phi$ to $E$ and $E'$ so that $E \simeq E'$. ∎

**Corollary.** *Any two splitting fields of a given polynomial over a field are isomorphic.*

*Proof.* Take $\phi$ to be the identity map. ∎

# Bibliography

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