Measure Theory

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<u>Text</u>

Real Analysis (4^{th} edition)

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Chapter 1

The Real Numbers

1.1 Open Sets, and σ-Algebras

Definition. We call a set U of real numbers **open** provided for any $x \in U$, there is an $r > 0$ such that $(x - r, x + r) \subseteq U$.

Lemma 1.1.1. *The set of real numbers $\mathbb{R}$, together with open sets defines a topology on $\mathbb{R}$.*

Proof. Notice that both $\mathbb{R}$ and $\emptyset$ are open sets. Moreover, if $\{U_n\}$ is a collection of open sets, then so is thier union. Now, consider the fintie collection $\{U_k\}_k = 1^n$ and let $U = \bigcap_{k=1}^n U_k$. If U is empty, we are done. Otherwise, let $x \in U$. Then $x \in U_k$ for every $1 \leq k \leq n$, and since each U_k is open, choose an $r_k > 0$ for which $(x - r_k, x + r_k) \subseteq U_k$. Then let $r = \min\{r_1, \ldots, r_n\}$. Then $r > 0$, and we have $(x - r, x + r) \subseteq U$, which makes U open in $\mathbb{R}$. ■

Lemma 1.1.2. *Every nonempty set is the disjoint union of a countable collection of open sets.*

Proof. Let U be nonempty and open in $\mathbb{R}$. LEt $x \in U$. Then there is a $y > x$ for which $(x, y) \subseteq U$ and there is a $z < x$ for which $(z, x) \subseteq U$. Now, let $a_x = \inf\{z : (z, x) \subseteq U\}$ and $b_x = \sup\{y : (x, y) \subseteq U\}$, and let $I_x = (a_x, b_x)$. Then we have

$$x \in I_x \text{ and } a_x \notin I_x \text{ and } b_x \notin I_x$$

Let $w \in I_x$ such that $x < w < b_x$. Then there is a $y > w$ such that $(x, y) \subseteq U$ so that $w \in U$. Now, if $b_x \in U$, then there is an $r > 0$ for which $(b_x - r, b_x + r) \subseteq U$, in particular, $(x, b_x + r) \subseteq U$. But b_r is the least upperbound of all such numbers, and $b_x < b_x + r$, a contradiction. Thus $b_x \notin U$, and hence $b_x \notin I_x$. A similar argument shows that $a_x \notin I_x$.

Consider now the collection $\{I_x\}_{x \in U}$. Then $U = \bigcup I_x$ and since $a_x, b_x \notin I_x$ for each x, the collection $\{I_x\}$ is a disjoint collection. Lastly, by the density of $\mathbb{Q}$ in $\mathbb{R}$ there is a 1–1 mapping between this collection and $\mathbb{Q}$, making it countable. ■

Definition. Let $E \subseteq \mathbb{R}$ a set. We call a point $x \in \mathbb{R}$ a **point of closure** of E if every open interval containing x also contains a point of E. We call the collection of all such points the **closure** of E, and denote it $\operatorname{cl} E$. If $E = \operatorname{cl} E$, then we say that E is **closed**.

Lemma 1.1.3. *For any set E of real numbers, $\mathrm{cl}\,E$ is closed; i.e. $\mathrm{cl}\,E = \mathrm{cl}\,(\mathrm{cl}\,E)$. Moreover, $\mathrm{cl}\,E$ is the smallest closed set containing E.*

Lemma 1.1.4. *Every set E of rea numbers is open if, and only if $\mathbb{R}\backslash E$ is closed.*

Definition. Let $E \subseteq \mathbb{R}$ a set. We call a collection $\{E_\lambda\}$ a **cover** of E if $E \subseteq \bigcup E_\lambda$. If each E_λ is open, then we call this collection an **open cover** of E.

Theorem 1.1.5 (Heine-Borel)**.** *For any closed and bounded set F of $\mathbb{R}$, every open cover of F has a finite subcover.*

Proof. Suppose first that $F = [a, b]$, for $a \leq b$ real numbers. Then F is closed and bounded. Let $\mathcal{F}$ be an open cover of $[a, b]$, and deifne $E = \{x \in [a, b] : [a, x] \text{ has a finite subcover by sets in } \mathcal{F}\}$. Notice that $a \in E$, so that E is nonempty. Now, since E is bounded by b, by the completeness of $\mathbb{R}$, let $c = \sup\{E\}$. Then $c \in [a, b]$ and there is a set $U \in \mathcal{F}$ with $c \in U$. Since U is open, there exists an $\varepsilon > 0$ such that $(c-\varepsilon, c+\varepsilon) \subseteq U$. Now, $c - \varepsilon$ is not an upperbound of E, so there is an $x \in E$ with $c - \varepsilon < x$, and a finite collection of open sets $\{U_i\}_{i=1}^k$ covering $[a, x]$. Then the collection $\{U_i\}_{i=1}^k \cup U$ covers $[a, x]$ so that $c = b$, and we have found a finite subcover of F.

Now, let F be closed and bounded. Then it is contained in a closed bounded interval $[a, b]$. Now, let $U = \mathbb{R}\backslash F$ open and $\mathcal{F}$ an open cover of F. Let $\mathcal{F}' = \mathcal{F} \cup U$. Since $\mathcal{F}$ covers F, $\mathcal{F}'$ covers $[a, b]$. By above, there is a finite subcover of $[a, b]$, and hence of F by sets in $\mathcal{F}'$. Removine U from $\mathcal{F}'$, we get a finite subcover of F by sets in $\mathcal{F}$. ■

Theorem 1.1.6 (The Nested Set Theorem)**.** *Let $\{F_n\}$ be a descending collection of nonempty closed sets of $\mathbb{R}$, for which F_1 is bounded. Then*

$$\bigcap F_n \neq \emptyset$$

Proof. Let $F = \bigcap F_n$, and suppose to the contrary that F is empty. Then for all $x \in \mathbb{R}$, there is an $n \in \mathbb{Z}^+$ for which $x \notin F_n$. So that $x \in U_n = \mathbb{R}\backslash F_n$. TYhen $U_n = \mathbb{R}$, and each U_n is open. So $\{U_n\}$ is an open cover of $\mathbb{R}$, and hence F_1. By the theorem of Heine-Borel, there is an $N > 0$ such that $F \subseteq \bigcup_{n=1}^N U_n$. Since $\{F_n\}$ is descending, the collection $\{U_n\}$ is ascending, and hence $\bigcup U_n = U_N = \mathbb{R}\backslash F_N$ which makes $F_1\mathbb{R}\backslash F_N$, a contradiciton. ■

Definition. Let X be a set. We call a collection $\mathcal{A}$ of subsets of X **σ-algebra** if

(1) $\emptyset \in \mathcal{A}$.

(2) For any $A \in \mathcal{A}$, $X\backslash A \in \mathcal{A}$.

(3) If $\{A_n\}$ is a countable collection of elements of $\mathcal{A}$, then their union is an element of $\mathcal{A}$.

Lemma 1.1.7. *Let $\mathcal{F}$ a collection of subsets of a set X. The intersection of all σ-algebras containing $\mathcal{F}$ is a σ-algebra. Moreover, it is the smallest such σ-algebra.*

Definition. We define the **Borel sets** of $\mathbb{R}$ to be the σ-algebra of $\mathbb{R}$ cotnaining all open sets in $\mathbb{R}$

Lemma 1.1.8. *Every closed set of $\mathbb{R}$ is a Borel set.*

Definition. We call a countable intersection of open sets of $\mathbb{R}$ a **G_δ-set** and we call a countable union of closed sets of $\mathbb{R}$ an **F_σ-set**.

1.2 Sequences of Real Numbers

Definition. A sequence $\{a_n\}$ of real numbers is said to **converge** to a point a, if, for every $\varepsilon > 0$, there is an $N > 0$ such that

$$|a - a_n| < \varepsilon \text{ whenever } n \geq N$$

We call a the **limit** of $\{a_n\}$ and write $\{a_n\} \to a$, or

$$\lim_{n \to \infty} \{a_n\} = a$$

Lemma 1.2.1. *Let $\{a_n\} \to a$ a sequence of real numbers converging to $a \in \mathbb{R}$. Then the limit of $\{a_n\}$ is unique, $\{a_n\}$ is bounded, and for any $c \in \mathbb{R}$, if $a_n \leq c$ for all n, then $a \leq c$.*

Theorem 1.2.2 (The Monoton CVonvergence Theorem)**.** *A monotone sequence of real numbers converges to a point if, and only if it is bounded.*

Proof. Without loss of generality, suppose that the sequence $\{a_n\}$ is increasing. If $\{a_n\} \to a$, by lemma 1.2.1, $\{a_n\}$ is bounded. On the otherhand, suppose that $\{a_n\}$ is bounded. Let $S = \{a_n : n \in \mathbb{Z}^+\}$, then by the completeness of $\mathbb{R}$, let $a = \sup S$. Let $\varepsilon > 0$. Notice that $a_n \leq a$ for all n. Now, since $a - \varepsilon$ is not an upperbound, there exists an $N > 0$ for which $a_N > a - \varepsilon$, then since $\{a_n\}$ is increasing, $a_n > a - \varepsilon$ whenever $n \geq N$. So we get

$$|a - a_n| < \varepsilon \text{ whenever } n \geq N$$

Which makes $\{a_n\} \to a$. ■

Theorem 1.2.3 (Bolzano-Weierstrass)**.** *Every bounded sequence has a convergent subsequence.*

Proof. Let $\{a_n\}$ be a bounded sequence, and let $M > 0$ such that $|a_n| \geq M$ for all $n \in \mathbb{Z}^+$. Define $E_n = \text{cl}\,\{a_j : j \geq n\}$. Then $EE \subseteq [-M, M]$. Thus $\{E_n\}$ is a decreasing sequence of closed, bounded, and nonempty sets of $\mathbb{R}$. By the nested set theorem, the intersection $E = \bigcap E_n$ is nonempty. Choose an $a \in E$. THen for every $k \in \mathbb{Z}^+$, a is a point of closure of the set $\{a_j : j \geq k\}$. SO that $a_j \in (a - \frac{1}{k}, a + \frac{1}{k})$ whenever $j \geq k$. By induction, construct a strictly increasing sequence $\{n_k\}$ of natural numbers for which $|a - a_{n_k}| < \varepsilon$. Thenb by the principle of Archimedes, $\{a_{n_k}\} \to a$, and we have a convergent subsequence. ■

Definition. We call a sequence $\{a_n\}$ **Cauchy** if for every $\varepsilon > 0$, there is an $N > 0$ for which

$$|a_m - a_n| < \varepsilon \text{ whenever } m, n \geq N$$

Theorem 1.2.4 (The Cauchy Convergence Criterion)**.** *A sequence of real numbers converges if, and only if it is Cauchy.*

Proof. Suppose that the sequence $\{a_n\} \to a$ converges to $a \in \mathbb{R}$. Then for any $m, n \in \mathbb{Z}^+$, notice that $|a_m - a_n| \leq |a_m - a| + |a - a_n|$. Let $\varepsilon > 0$ and choose $N > 0$ such that $|a - a_n| < \frac{\varepsilon}{2}$, and $|a_m - a| < \frac{\varepsilon}{2}$. Then if $n, m \geq N$, we get $|a_m - a_n| < \varepsilon$, which makes $\{a_n\}$ Cauchy.

Conversely, suppose that $\{a_n\}$ is Cauchy. Let $\varepsilon = 1$ and choose $N > 0$ such that if $m, n \geq N$, then $|a_m - a_n| < 1$. Then we get $|a_n| \leq 1 + |a_N|$ for all $n \geq N$. Define $M = 1 + \max\{|a_1|, \ldots, |a_N|\}$. Then $|a_n| \leq M$ for all n. This makes $\{a_n\}$ bounded. By the theorem of Bolzano-Weierstrass, $\{a_n\}$ has a convergent subsequence $\{a_{n_k}\} \to a$. Let $\varepsilon > 0$, since $\{a_n\}$ is Cauchy, choose an $N > 0$ such taht $|a_m - a_n| < \frac{\varepsilon}{2}$ whenever $n, m \geq N$. Likewise, we get $|a - a_{n_k}| < \frac{\varepsilon}{2}$ and $n_k \geq N$. Thus we observe that $|a_n - a| \leq |a_n - a_{n_k}| + |a - a_{n_k}| < \varepsilon$ and so $\{a_n\} \to a$. ■

Theorem 1.2.5. *Let $\{a_n\} \to a$ and $\{b_n\} \to b$ be convergent sequences. Then for any $\alpha, \beta \in \mathbb{R}$, we have that the sequence $\{\alpha a_n + \beta b_n\}$ converges and that*

$$\lim_{n \to \infty} \{\alpha a_n + \beta b_n\} = \alpha a + \beta b$$

Definition. We say a sequence $\{a_n\}$ of real numbers **converges to infinity** $\infty \in \mathbb{R}_\infty$ if for every $c \in \mathbb{R}$, there is an $N > 0$ such that $a_n \geq c$ whenever $n \geq N$. We write $\{a_n\} \to \infty$, or

$$\lim_{n \to \infty} \{a_n\} = \infty$$

Definition. Let $\{a_n\}$ be a sequence of real numbers. We define the **limit superior** of $\{a_n\}$ to be

$$\limsup \{a_n\} = \lim_{n \to \infty} (\sup \{a_k : k \geq n\})$$

Similarly, we define the **limit inferiro** of $\{a_n\}$ to be

$$\liminf \{a_n\} = \lim_{n \to \infty} (\inf \{a_k : k \geq n\})$$

Theorem 1.2.6. *For any sequences $\{a_n\}$ and $\{b_n\}$ of real numbers, the following are true:*

(1) $\limsup \{a_n\} = l \in \mathbb{R}_\infty$ *if, and only if for every $\varepsilon > 0$, there exists infintiely many $n \in \mathbb{Z}^+$ such that $a_n > l - \varepsilon$ and finitely many $n \in \mathbb{Z}^+$ for which $a_n > l + \varepsilon$.*

(2) $\limsup \{a_n\} = \infty$ *if, and only if $\{a_n\}$ is not bounded above.*

(3) $\limsup \{a_n\} = -\liminf \{-a_n\}$

(4) $\{a_n\} \to a \in \mathbb{R}_\infty$ *if, and only if* $\limsup \{a_n\} = \liminf \{a_n\}$.

(5) If $a_n \leq b_n$ for all n, then $\limsup \{a_n\} \leq \limsup \{b_n\}$.

Definition. Let $\{a_n\}$ a sequence of real numbers. We call the series $\sum_{k=1}^{\infty} a_k$ **summable** if the sequence of partial sums $\{s_n = \sum_{k=1}^{n} a_k\} \to s$ converges to a point $s \in \mathbb{R}$.

Lemma 1.2.7. *Let $\{a_n\}$ a sequence of real numbers. Then the following are true.*

(1) The series $\sum a_k$ is summable if, and only if for every $\varepsilon > 0$, there is an $N > 0$ such that

$$|\sum_{k=n}^{n+m} a_k| < \varepsilon \textit{ for all } m \in \mathbb{Z}^+ \textit{ whenever } n \geq N$$

(2) If $\sum |a_k|$ is summable, then so is $\sum a_k$.

(3) If $a_k \geq 0$, then $\sum a_k$ is summable if, and only if the sequence of partial sums $\{s_n\}$ is bounded.

1.3 Continuous Functions of a Real Variable.

Definition. A realvalued function f on a domain E is said to be **continuous** at a point $x \in E$ provided for any $\varepsilon > 0$ there is a $\delta > 0$ for which

$$|f(x) - f(y)| < \varepsilon \text{ whenever } |x - y| < \delta \text{ for any } y \in E$$

We call f **continuous** on E if it is continuous at every point in E. We call f **Lipschitz continuous** if there is a $c \geq 0$ for which

$$|f(x) - f(y)| \leq c|x - y| \text{ for all } x, y \in E$$

Lemma 1.3.1. *A Lipschitz continuous function on a domain is continuous on that domain.*

Lemma 1.3.2 (The Sequential Criterion)**.** *A realvalued function f defined on a domain E is continous at a point $x \in E$ if, and only if for any ssequence $\{x_n\} \to x$ of points in E, converging to x, that the sequence $\{f(x_n)\} \to f(x)$ converges to $f(x)$.*

Theorem 1.3.3 (The Extreme Value Theorem)**.** *A continuous realvalued function defined on a nonempty, closed and bounded domain takes on a maximum value, and a minimum value on that domain.*

Proof. Let f be a continuous realvalued function defined on the domain E, where E is nonempty, closed, and bounded. Let $x \in E$ and $\delta > 0$ and $\varepsilon = 1$. Define the open interval $I_x = (x - \delta, x + \delta)$. Then if $y \in E \cap I_x$, then $|f(x) - f(y)| < 1$. So that $|f(y)| \leq |f(x)| + 1$. Notice also that the collection $\{I_x\}$ is an open cover of E. By the theorem of Heine-Borel, there is a finite subcover of E, $\{I_{x_k}\}_{k=1}^n$. Define, then, $M = 1 + \max\{|f(x_1)|, \ldots, |f(x_n)|\}$. Then we get that $|f(x)| \leq M$ and f is bounded.

Now, let $m = \sup f(E)$. If f does not take the value m for any points in E, then the function $x \to \frac{1}{f(x)-m}$ is a contoinuous unbounded function on E; which is impossible. So there is an $x \in E$ with $f(x) = m$ and m is a maximum value. Now, since f is continuous, then so is $-f$, and hence $-m$ defines a minimum value on f. ∎

Theorem 1.3.4 (The Intermediate Value Theorem)**.** *If f is a continuous realvalued function on a closed bounded interval $[a, b]$, for which $f(a) < c < f(b)$, then there exists an $x_0 \in (a, b)$ for which $f(x_0) = c$.*

Proof. Define $a_1 = a$ and $b_1 = b$ and let m_1 be the midpoint of the interval $[a_1, b_1]$. If $c < f(m_1)$, define $a_2 = a_1$ and $b_2 = m_1$, otherwise define $a_2 = m_1$ and $b_2 = m_1$, so that in either case we get $f(a_2) \leq c \leq f(b_2)$ and $b_2 - a_2 = \frac{b-a}{2}$. By induction, construct the collection of closde bounded intervals $\{[a_n, b_n]\}$ such that $f(a_n) \leq c \leq f(b_n)$ and $b_n - a_n = \frac{b-a}{2^{n-1}}$. This collection is a descending collection, so by the nested set theorem, the intersection $I = \bigcap [a_n, b_n]$ is nonempty. Choose an $x_0 \in I$, and observe that

$$|a_n - x_0| \leq b_n - a_n = \frac{b-a}{2^{n-1}}$$

So the sequence $\{a_n\} \to x_0$. By the sequential criterion, since f is continuous at x_0, we get the sequence $\{f(a_n)\} \to f(x_0)$. Since $f(a_n) \leq c$, and $(-\infty, c]$ is closed, we also get $f(x_0) \leq c$.

By similar reasoning to the argument provided above, we also get that $f(x_0) \geq c$ so that equality is established. ∎

Definition. A realvalued function f on a domain E is said to be **uniformly continuous** if for everyu $\varepsilon > 0$ there is a $\delta > 0$ such that

$$|f(x) - f(y)| < \varepsilon \text{ whenever } |x - y| < \delta \text{ for all } x, y \in E$$

Lemma 1.3.5. *If f is a uniformly continuous function on a domain E, then it is continuous on E.*

Theorem 1.3.6. *A continuous realvalued function on a closed and bounded domain is unifomrly continuous.*

Proof. Let f be continuous on E, and E a closed and bounded domain. Let $\varepsilon > 0$. For every $x \in E$, there is a $\delta_x > 0$ for which $|f(x) - f(y)| < \varepsilon$ whenever $|x - y| < \delta_x$ for some $y \in E$. Define $I_x = (x - \frac{\delta_x}{2}, x + \frac{\delta_x}{2})$. Then $\{I_x\}$ is an open cover for E, so that by the theorem of Heine-Borel, there is a finite subcover $\{I_{x_k}\}_{k=1}^{n}$ of E. Define $\delta = \frac{1}{2}\min\{\frac{\delta_{x_1}}{2}, \ldots, \frac{\delta_{x_n}}{2}\}$. Then $\delta > 0$ moreover, if $x, y \in E$, with $|x - y| < \delta$, then asince $\{I_{x_k}\}$ covers E, there is a $k > 0$ such that

$$|x - x_k| < \frac{\delta_{x_k}}{2} \text{ and } |x_{x_k} - y| < \frac{\delta_{x_k}}{2}$$

Then we have $|f(x) - f(x_k)| < \frac{\varepsilon}{2}$ and $|f(x_k) - f(y)| < \frac{\varepsilon}{2}$ so that $|f(x) - f(y)| < \varepsilon$, which makes f unifomrly continuous. ∎

Chapter 2

Lebesgue Measure

2.1 Lebesgue Outermeasure

Definition. Let I be a nonempty interval of $\mathbb{R}$. We define the **lenght** of I, denoted $l(I)$, to be the difference of its endpoints, if I is bounded, and ∞ otherwise.

Definition. Let A a subset of $\mathbb{R}$. We define the **Lebesgue outer measure** of A to be

$$m^*(A) = \inf \{\sum l(I_k)\}$$

Where $\{I_k\}$ is a countable collection of bounded open sets, covering A.

Lemma 2.1.1. *The emptyset has Lebesgue outermeasure* 0*. Moreover, the Lebesgue outermeasure is monotone; that is, if* $A, B \subseteq \mathbb{R}$ *such that* $A \subseteq B$*, then* $m^*(A) \leq m^*(B)$*.*

Proof. Notice that the singleton $\{a\} = [a, a]$ covers the emptyset . Moreover $l([a, a]) = a - a = 0$, so by definition $m^*(\emptyset) = 0$.

Now, let A, B subsets of $\mathbb{R}$ such that $A \subseteq B$. Then if $\{I_k\}$ is a countable collection of bounded open sets covering B, they also cover A, hence by definition, we get $m^*(A) \leq m^*(B)$. ■

Corollory. *Lebesgue outermeasure is nonnegative. That is,* $0 \leq m^*(E)$ *for any set* $E \subseteq \mathbb{R}$*.*

Proof. Notice the length of any interval I is nonnegative. ■

Example 1. Countable sets have measure 0. Let C be a countable set with enumeration $\{c_k\}$. Let $\varepsilon > 0$ and define $I_k = (c_k - \frac{\varepsilon}{2^{k+1}}, c_k + \frac{\varepsilon}{2^{k+1}})$. Then $\{I_k\}$ is a countable collection of bounded open sets covering $C = \{c_k\}$. Hence we get that

$$0 \leq m^*(C) \leq \sum I_k \leq \sum \frac{\varepsilon}{2^k} = 0$$

So that $m^*(C) = 0$.

Lemma 2.1.2. *For any nonempty interval* I*,* $m^*(I) = l(I)$*.*

Proof. Consider first, the closed bounded interval $[a,b]$, where $a<b$. Let $\varepsilon>0$. Notice that $[a,b]\subseteq(a-\varepsilon,b+\varepsilon)$, so that $m^*([a,b])\leq l((a-\varepsilon,b+\varepsilon))=b-a+2\varepsilon$. Hence $m^*([a,b])\leq b-a$. It remains to show that $b-a\leq m^*([a,b])$.

Let $\{I_k\}$ a countable collection of open bounded intervals covering $[a,b]$. By the theorem of Heine-Borel, there is a finite subcover $\{I_k\}_{k=1}^n$ of $[a,b]$. Notice that since $a\in\bigcup I_k$, at least one I_k contains a. Hence choose an interval (a_1,b_1) in this cover for which $a_1<a<b_1$. Now, if $b<b_1$, we are done as

$$\sum_{k=1}^{n} l(I_k)\geq b_1-a_1>b-a$$

Otherwise, $b_1\in[a,b_1)$. In this case, choose an interval (a_2,b_2), distinct from (a_1,b_1) for which $a_2<b_1<b_2$. If $b_2\geq b$, then we are done by similar reasoing as above. Otherwise, continue the process of choosing intervals. This process terminates as we eventually exhaust the endpoints of each I_k in the open cover. Thus, we get a subcollection $\{(a_k,b_k)\}_{k=1}^N$ for which $a_1<a$ and $a_{k+1}<b_k$ for all $1\leq k\leq N-1$. We also have a $b_N>b$. Then we have

$$\sum l(I_k)\geq\sum_{k=1}^{N} l((a_k,b_k))=(b_N-a_N)+\cdots+(b_1-a_1)\geq b-a$$

so that we get $b-a\leq m^*([a,b])$.

Now, let I be any bounded interval. Notice that there exist closed bounded intervals J_1 and J_2 for which

$$J_1\subseteq I\subseteq J_2$$

and for some $\varepsilon>0$,

$$l(I)-\varepsilon<l(J_1)\leq l(I)\leq l(J_1)<l(I)+\varepsilon$$

Then since J_1 and J_2 are closed and bounded intervals, and by monotonicity of m^*, we have

$$l(I)-\varepsilon<m^*(J_1)\leq m^*(I)\leq m^*(J_1)<l(I)+\varepsilon$$

so that $l(I)-\varepsilon<m^*(I)<l(I)+\varepsilon$ for all $\varepsilon>0$. This establishes equality. ■

Lemma 2.1.3. *The Lebesgue outermeasure is translation invariant. That is, if $A\subseteq\mathbb{R}$, and $y\in\mathbb{R}$, then $m^*(A)=m^*(A+y)$.*

Proof. Notice that a countable collection of open bounded intervals $\{I_k\}$ covers A if, and only if the collection $\{I_k+y\}$ of open bounded intervals covers $A+y$. Moreover, notice that $l(I_k)=l(I_k+y)$, so that we get

$$\sum l(I_k)=\sum l(I_k+y)$$

the rest follows from definition. ■

Lemma 2.1.4. *The Lebesgue outermeasure is countable subadditive; that is, if $\{E_k\}$ is a collection of subsets of $\mathbb{R}$, then*

$$m^*(\bigcup E_k)\leq\sum m^*(E_k)$$

Proof. Let $\{E_k\}$ a countable collection of sets, and let $E = \bigcup E_k$. Notice that if atleast one E_k has infinite measure, then we are done. Suppose then that for all k, $m^*(E_k)$ is finite. Let $\varepsilon > 0$. Then for all k, there exists a countable collection of open bounded intervals $\{I_{k,i}\}$ covering E_k, and $\sum_i l(I_{k,i}) < m^*(E_k) + \frac{\varepsilon}{2^k}$. By definition, we get

$$m^*(E) \leq \sum l(I_{k,i}) = \sum_k \sum_i l(I_{k,i}) < \sum_k (m^*(E_k) + \frac{\varepsilon}{2^k}) = \sum_k m^*(E_k) + \varepsilon$$

for all $\varepsilon > 0$. This inequality also holds for $\varepsilon = 0$. ■

Corollory. *The Lebesgue outermeasure is finitely subadditive.*

Proof. Recall that finite collections are also countable collectuions. ■

2.2 Lebesuge Measurable Sets

Definition. We call a set E of $\mathbb{R}$ **Lebesuge measurable**, provided for any subset A of $\mathbb{R}$,

$$m^*(A) = m^*(A \cap E) + m^*(A \cap \mathbb{R} \backslash E)$$

Lemma 2.2.1. *A set E is Lebesuge measurable if, and only if for any subset A of $\mathbb{R}$,*

$$m^*(A) \geq m^*(A \cap E) + m^*(A \cap \mathbb{R} \backslash E)$$

Proof. We have $A = (A \cap E) \cup (A \cap \mathbb{R} \backslash E)$, so by finite subadditivity, $m^*(A) \leq m^*(A \cap E) + m^*(A \cap \mathbb{R} \backslash E)$. ■

Lemma 2.2.2. *Any set of Lebesuge outer measure 0 is Lebesgue measurable.*

Proof. Let E have $m^*(E) = 0$ and let $A \subseteq \mathbb{R}$. Notice that $A \cap E \subseteq E$ and $A \cap \mathbb{R} \backslash E \subseteq E$, so that $m^*(A \cap E) \leq m^*(E) = 0$ and $m^*(A \cap \mathbb{R} \backslash E) \leq m^*(A)$. Then we have

$$m^*(A) \geq m^*(A \cap \mathbb{R} \backslash E) = 0 + m^*(A \cap \mathbb{R} \backslash E) = m^*(A \cap E) + m^*(A \cap \mathbb{R} \backslash E)$$

■

Corollory. *Countable sets are measurable.*

Lemma 2.2.3. *The union of two measurable sets is measurable.*

Proof. Let E_1 and E_2 be measurable sets and $A \subseteq \mathbb{R}$. Then $m^*(A) = m^*(A \cap E_1) + m^*(A \cap \mathbb{R} \backslash E_1) = m^*(A \cap E_1) + m^*((A \cap \mathbb{R} \backslash E_1) \cap E_2) + m^*((A \cap \mathbb{R} \backslash E_1) \cap \mathbb{R} \backslash E_2)$. Moreover, notice that

$$(A \cap \mathbb{R} \backslash E_1) \cap \mathbb{R} \backslash E_2 = A \cap \mathbb{R} \backslash (E_1 \cup E_2) \text{ and } (A \cap E_1) \cup (A \cap \mathbb{R} \backslash E_1 \cap E_2) = A \cap (E_1 \cup E_2)$$

Then we get

$$m^*(A) = m^*(A \cap E_1) + m^*((A \cap \mathbb{R} \backslash E_1) \cap E_2) + m^*(A \cap \mathbb{R} \backslash (E_1 \cup E_2)) \geq m^*(A \cap (E_1 \cup E_2)) + m^*(A \cap \mathbb{R} \backslash (E_1 \cup E_2))$$

which makes E_1 and E_2 measurable. ■

Corollory. *The union of a finite collection of measurable sets is measurable.*

Proof. Let $\{E_k\}_{k=1}^n$ a finite collection of measurable sets. By induction on n, we showed that this is true for $n = 1$ and $n = 2$. Now, consider the collections $\{E_k\}_{k=1}^{n+1}$ and suppose that the union $E = \bigcup_{k=1}^n E_k$ is measurable. Notice, then that

$$\bigcup_{k=1}^{n+1} E_k = E \cup E_{n+1}$$

both of which are measurable. Hence measurability of the union of $\{E_k\}_{k=1}^{n+1}$ follows by above. ■

Lemma 2.2.4. *Let A a subset of $\mathbb{R}$ and $\{E_k\}_{k=1}^n$ a finite, disjoint collection of measurable sets. Then*

$$m^*(A \cap \bigcup_{k=1}^n E_k) = \sum_{k=1}^n m^*(A \cap E_k)$$

Proof. By induction on n, for $n = 1$ it is true. Now, suppose that it is true for n, and consider the collection $\{E_k\}_{k=1}^{n+1}$ of disjoint measurable sets. Then we have $A \cap (\bigcup_{k=1}^n E_k) \cap E_{n+1} = A \cap E_{n+1}$ and $A \cap (\bigcup_{k=1}^n) \cap \mathbb{R} \backslash E_{n+1} = A \cap \bigcup_{k=1}^n E_k$. SInce E_{n+1} is measurable we get

$$m^*(A \cap \bigcup_{k=1}^{n+1} E_k) = m^*(A \cap E_{n+1}) + m^*(A \cap \bigcup_{k=1}^n E_k) = \sum_{k=1}^{n+1} m^*(A \cap E_k)$$

■

Definition. We call a collection of subsets of $\mathbb{R}$ an **algebra** if it contains $\mathbb{R}$ and it is closed under complements (with respect to $\mathbb{R}$) and finite unions.

Lemma 2.2.5. *Any algebra of $\mathbb{R}$ is closed under finite intersections.*

Proof. By DeMorgan's laws. ■

Theorem 2.2.6. *The collection of all measurable sets of $\mathbb{R}$ forms an algebra.*

Lemma 2.2.7. *The union of a countable collection of measurable sets is measurable.*

Proof. Without loss of generality, let $\{E_k\}$ a countable disjoint collection of measurable sets, and let $E = \bigcup E_k$. Let A a subset of $\mathbb{R}$ and define $F_n =_{k=1}^n E_k$. Then F_n is measurable by lemma 2.2.3, and $\mathbb{R} \backslash E_n \subseteq \mathbb{R} \backslash F_n$. Then

$$m^*(A) = m^*(A \cap F_n) + m^*(A \cap \mathbb{R} \backslash F_n) \geq m^*(A \cap F_n) + m^*(A \cap \mathbb{R} \backslash E_n)$$

hence $m^*(A \cap F_n) = \sum_m^* (A \cap E_k)$ so that

$$m^*(A) \geq \sum m^*(A \cap E_k) + m^*(A \cap \mathbb{R} \backslash E)$$

By countable subadditivity of m^* we have

$$m^*(A) \geq m^*(A \cap E) + m^*(A \cap \mathbb{R} \backslash E)$$

■

Definition. We call a collection of subsets of $\mathbb{R}$ a **σ-algebra** if it forms an algebra, and it is closed under countable unions.

Lemma 2.2.8. *Any σ-algebra of $\mathbb{R}$ is closed under countable intersections.*

Theorem 2.2.9. *The collection of measurable sets of $\mathbb{R}$ forms a σ-algebra.*

Lemma 2.2.10. *Every interval of $\mathbb{R}$ is measurable.*

Proof. Consider an interval of the form (a, ∞), for any $a \in \mathbb{R}$. Let $A \subseteq \mathbb{R}$, such that $A \notin A$; otherwise, just take $A \backslash \{a\}$. Then since $m^*(A)$ is a greatest lower bound, it is sufficient to show that for any countable collection $\{I_k\}$ of open, bounded intervals covering A, that

$$m^*(A_1) + m^*(A_2) \leq \sum l(I_k)$$

where

$$A_1 = A \cap (-\infty, a) \text{ and } A_2 = A \cap (a, \infty)$$

Indeed, let $\{I_k\}$ be such a collection, and define

$$I_{k,1} = I_k \cap (-\infty, a) \text{ and } I_{k,2} = I_k \cap (a, \infty)$$

Then $\{I_{k,1}\}$ and $\{I_{k,2}\}$ are collections of open, bounded intervals which cover A_1 and A_2 respectively, Hence, by definition of m^*, we have $m^*(A_1) \leq \sum l(I_{k,1})$ and $m^*(A_2) \leq \sum l(I_{k,2})$; moreover, notice that $l(I_k) = l(I_{k,1}) + l(I_{k,2})$. Therefore, we get

$$m^*(A_1) + m^*(A_2) \leq \sum l(I_{k,1}) + \sum l(I_{k,1}) = \sum l(I_k)$$

and we are done. ■

Corollory. *Open sets, and closed sets of $\mathbb{R}$ are measurable.*

Definition. We define the intersection of all σ-algebras of $\mathbb{R}$ to be the **Borel σ-algebra**, and call its elements **Borel sets**.

Theorem 2.2.11. *The σ-algebra of all measurable sets of $\mathbb{R}$ contains the Borel σ-algebra of $\mathbb{R}$. Moreover, it contains every interval of $\mathbb{R}$, open and closed sets, as well as G_δ and F_σ sets.*

Lemma 2.2.12. *Lebesgue measurable sets are translation invariant. That is, if E is Lebesuge measurable, and $y \in \mathbb{R}$, then $E + y$ is Lebesuge measurable.*

Proof. Let E be measurable, $y \in \mathbb{R}$, and $A \subseteq \mathbb{R}$ Then

$$m^*(A) = m^*(A\backslash y) = m^*(A\backslash y \cap E) + m^*(A\backslash y \cap \mathbb{R}\backslash E) = m^*(A \cap (E+y)) + m^*(A \cap \mathbb{R}\backslash(E+y))$$

■

2.3 Inner and Outer Approximations

Lemma 2.3.1 (Excision)**.** *If A and B are sets, with A Lebesgue measurable of finite outer measure, and $A \subseteq B$, then*

$$m^*(B \backslash A) = m^*(B) - m^*(A)$$

Theorem 2.3.2 (The Outer Approximation Theorem)**.** *Let $E \subseteq \mathbb{R}$. The following are equivalent.*

(1) E is Lebesgue measurable.

(2) For all $\varepsilon > 0$ there is an opne set U of $\mathbb{R}$ containing E such that $m^*(U \backslash E) < \varepsilon$.

(3) There exists a G_δ set G containing E for which $m^*(G \backslash E) = 0$.

Proof. Suppose first that E is measurable and let $\varepsilon > 0$. Now, if $m^*(E)$ is finite, then there is a countable collection $\{I_k\}$ of open intervals covering E, for which, by definition of m^* as a greatest lower bound,

$$\sum l(I_k) < m^*(E) + \varepsilon$$

Let $U = \bigcup I_k$, then $E \subseteq U$, and U is open in $\mathbb{R}$. Thus by definition of m^* again, we have

$$m^*(U) \leq \sum l(I_k) < m^*(E) + \varepsilon$$

so that $m^*(U) - m^*(E) < \varepsilon$. Now, since E is measurable of finite outer measure, by excision, we get $m^*(U \backslash E) = m^*(U) - m^*(E) < \varepsilon$.

Now, if $m^*(E)$ is infinite, then let $\{E_k\}$ be a countable disjoint collection of measurable sets each of finite outer measure, and let $E = \bigcup E_k$. Then by above there exist open sets U_k containing E_k, for each k such that $m^*(U_k \backslash E_k) < \frac{\varepsilon}{2^k}$. Let $U = \bigcup U_k$, then U is open in $\mathbb{R}$, and $E \subseteq U$. Moreover observe that

$$U \backslash E = \bigcup U_k \backslash E_k$$

Then we get by subadditivity

$$m^*(U \backslash E) \leq \sum m^*(U_k \backslash E_k) < \sum \frac{\varepsilon}{2^k} = \varepsilon$$

Now, suppose that assertion (2) holds, and choose an open set U_k containing E for which $m^*(U_k \backslash E) < \frac{1}{k}$. Define $G = \bigcup U_k$. Then G is a G_δ set, and $E \subseteq G$. Moreovoer we have that

$$G \backslash E \subseteq U_k \backslash E \text{ for all } k$$

so by monotonicity

$$m^*(G \backslash E) \leq m^*(U_k \backslash E) < \frac{1}{k}$$

Then as $k \to \infty$, this outer measure approaches 0.

Now if (3) holds, since $m^*(G \backslash E) = 0$, the set $G \backslash E$ is measurable. Since the space of all measurable sets is a σ-algebra, then the set $E = G \cap \mathbb{R} \backslash (G \backslash E)$ is measurable. ■

Corollory (The Inner Approximation Theorem). *The following are equivalent.*

(1) E is Lebesgue measurable.

(2) For all $\varepsilon > 0$ there is a closed set V of $\mathbb{R}$ contained in E such that $m^*(E \backslash V) < \varepsilon$.

(3) There exists an F_σ set F contained in E for which $m^*(E \backslash F) = 0$.

Proof. One can apply DeMorgan's laws. ■

Theorem 2.3.3. *Let E a Lebesgue measurable set of finite outer measure. then for every $\varepsilon > 0$ there is a finite disjoint collection $\{I_k\}$ of open intervals for which if $U = \bigcup I_k$, then*

$$m^*(E \backslash U) + m^*(U \backslash E) < \varepsilon$$

Proof. By the outer approximation theorem, there is an open set V containing E for which $m^*(V \backslash E) < \frac{\varepsilon}{2}$. Now, since E is measurable of finite outer measure, by excision we have

$$m^*(V) - m^*(E) < \frac{\varepsilon}{2}$$

so that $m^*(V)$ is also finite. Now, recall that every open set of real numbers is the disjoint collection of open intervals, hence let $V = \bigcup I_k$. Each I_k is measurable with $m^*(I_k) = l(I_k)$. Thereofre, by lemma 2.2.4 and monotonicity,

$$\sum_{k=1}^{n} l(I_k) \leq m * (V) \text{ is finite}$$

So $\sum I_k$ is finite. Now, choose an $n \in \mathbb{Z}^+$ for which $\sum_{k=n+1} I_k < \frac{\varepsilon}{2}$ and define $U = \bigcup_{k=1}^{n} I_k$. Then $U \backslash E \subseteq V \backslash E$ so by monotonicicty, $m^*(U \backslash E) < \frac{\varepsilon}{2}$. Moreover, we have $E \backslash U \subseteq V \backslash U = \bigcup_{k=n+1} I_k$ so that $m^*(E \backslash U) < \frac{\varepsilon}{2}$. Therefore, we see that

$$m^*(U \backslash E) + m^*(E \backslash U) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

■

2.4 The Borel-Cantelli Lemma

Definition. We define the **Lebesgue measure** m to be the restriction of the Lebesgue outer measure, m^* to the space of all Lebesgue measurable sets. That is, if E is Lebesgue measurable, the

$$m(E) = m^*(E)$$

Lemma 2.4.1 (Countable additivity). *The Lebesgue measure is countable additive. That is, if $\{E_k\}$ is a countable collection of disjoint measurable sets, then*

$$m(\bigcup E_k) = \sum m(E_k)$$

Proof. Since the space of Lebesgue measurable sets forms a σ-algebra, and are closed under countable unions, the set $E = \bigcup E_k$ is Lebesgue measurable. Moreover, by subadditivity of m^*, and definition of m,

$$m(E) \leq \sum_k = 1^{\infty} m(E_k)$$

Notice, however, that $\bigcup_{k=1}^n E_k \subseteq E$, so that by monotonicity, $\sum_{k=1}^n m(E_k) \leq m(E)$. Then as $n \to \infty$, this sum converges to $\sum_{k=1}^{\infty} E_k$ so

$$\sum_{k=1}^{\infty} E_k \leq m(E)$$

and equality is established. ∎

Corollory. *The Lebesgue measure is finitely additive.*

Theorem 2.4.2. *The Lebesgue measure assignes to intervals thier lengths, is translation invariant, and countable additive.*

Theorem 2.4.3 (Continuity)**.** *The following are true for the Lebesgue measure.*

(1) If $\{A_k\}$ is an increasing sequence of Lebesgue measurable sets, then

$$m(\bigcup A_k) = \lim_{k \longrightarrow \infty} m(A_k)$$

(2) If $\{B_k\}$ is an decreasing sequence of Lebesgue measurable sets for which $m(B_1)$ is finite, then

$$m(\bigcap B_k) = \lim_{k \longrightarrow \infty} m(B_k)$$

Proof. If $k_0 \in \mathbb{Z}^+$ is such that $m(A_{k_0})$ is infinite, then by monotonicity, $m(\bigcup A_k)$ is infinite so that $m(A_k)$ is infinite for all $k \geq k_0$. Suppose then, that $m(A_k)$ is finite for all k and define $A_0 = \emptyset$. Furthermore, define $C_k = A_k \backslash A_{k-1}$ for all $k \geq 1$. then since $\{A_k\}$ is a disjoint collection of measurable sets, then so is C_k, and $\bigcup A_k = \bigcup C_k$. By countable additivity, we have

$$m(\bigcup A_k) = m(\bigcup C_k) = \sum m(A_k \backslash A_{k-1})$$

By excision, we get

$$\sum m(A_k) - m(A_{k-1}) = \lim_{n \longrightarrow \infty} \sum_{k=1}^{n} m(A_k) - m(A_{k-1}) = \lim \left(m(A_n) - m(A_0)\right) = \lim_{n \longrightarrow \infty} m(A_n)$$

since $m(A_0) = 0$.

Now, define $D_k = B_1 \backslash B_k$. Since $\{B_k\}$ is decreasing, the sequence $\{D_k\}$ of measurable sets is increasing. Then by above,

$$m(\bigcup D_k) = \lim_{k \longrightarrow \infty} m(D_k)$$

By DeMorgan's laws, $\bigcup D_k = B_1 \backslash \bigcap B_k$. On the otherhand, by excision, since $m(B_1)$ is finite, we get

$$m(D_k) = m(B_1) - m(B_k)$$

so that

$$m(B_1 \backslash \bigcap B_k) = \lim_{n \to \infty} (m(B_1) - m(B_n))$$

By excision again, we are done. ■

Definition. We say a property holds **almost everywhere** on a measurable set E if there exists a measurable set $E_0 \subseteq E$ with $m(E_0) = 0$ for which the property holds for all $x \in E \backslash E_0$.

Lemma 2.4.4 (Borel-Cantelli)**.** *Let $\{E_k\}$ a countable collection of measurable sets such that the sum $\sum m(E_k)$ is finite. Then almost all $x \in \mathbb{R}$ belong to at most finitely many of the E_k.*

Proof. By countable subadditivity, we have $m(\bigcup E_k) \leq \sum_{k=n} m(E_k)$ is finite. Thus, by continuity, we have

$$m(\bigcap_{n=1} (\bigcup_{k=n} E_k)) = \lim_{n \to \infty} m(\bigcup_{k=n} E_k) \leq \lim_{n \to \infty} \sum_{k=n} m(E_k) = 0$$

so that almost all x does not belong to the intersection $\bigcap_{n=1} \bigcup_{k=n} E_k$ and hence belongs to at most finitely many of the E_k. ■

2.5 Nonmeasurable Sets, The Cantor Set, and The Cantor-Lebesgue Function

Definition. We call a set E of real numbers **nonmeasurable** if it is not measurable.

Lemma 2.5.1. *If E is a bounded measurable set of real numbers, and there is a countably infinite disjoint collection of translates $\{E + \lambda\}$, then $m(E) = 0$.*

Proof. Since E is measurable, so is $\mathbb{E} + \lambda$ for every λ. Then by countable additivity, we have

$$m(\bigcup E + \lambda) = \sum m(E + \lambda) = \sum m(E)$$

Now, since E is bounded, so is each $E + \lambda$, and hence, so is $\bigcup E + \lambda$ so that $m(\bigcup E + \lambda)$ is finite Therefore, $m(E)$ is finite. Moreover, since the collection $\{E + \lambda\}$ is countably infinite, and $m(E)$ is finite, this forces $m(E) = 0$. ■

Definition. We call two real numbers $x, y \in \mathbb{R}$ **rationally equivalent** if $x - y \in \mathbb{Q}$.

Lemma 2.5.2. *Rational equivalence is an equivalence relation on $\mathbb{R}$.*

Theorem 2.5.3 (Vitali's Theorem)**.** *Any set E of real numbers with positive outer measure contains a nonmeasurable set.*

Proof. Consider rational equivalence on E, which partitions E into equivalence classes. Define $\mathcal{C}_E$ a choice set of the equivalence classes on E consisting of exactly one member from each class, such that

(1) For all $x, y \in \mathcal{C}_E$, $x - y \notin \mathbb{Q}$.

(2) For all $x \in E$, there exists a $c \in \mathcal{C}_E$ for which $x = c + q$ for some $q \in \mathbb{Q}$.

Now, by countable subadditivity, suppose that E is bounded, and consider the choice set $\mathcal{C}_E$ (defined above) of E. Then $\mathcal{C}_E$ is nonmeasurable.

Suppose otherwis e. Let Λ_0 a bounded countably infinite set of rational numbers. Then each $\{\mathcal{C}_E + \lambda\}$ is measurable for each $\lambda \in \Lambda_0$. Then we have a countably infinite disjoint collection of bounded translates, hence by lemma 2.5.1, $m(\mathcal{C}_E) = 0$. That is,

$$m(\bigcup \mathcal{C}_E + \lambda) = \sum m(\mathcal{C}_E + \lambda) = 0$$

Since E is bounded, choose $\Lambda_0 = [-2b, 2b] \cap \mathbb{Q}$, for some $b \in \mathbb{R}$. If $x \in E$, there exists a $c \in \mathcal{C}_E$ and a $q \in \mathbb{Q}$ such that $x = c + q$. That is $x, c \in [-b, b]$ and $q \in [-2b, 2b]$ so that $E \subseteq \bigcup \mathcal{C}_E + \lambda$. But $m(E)$ is positive, which yields a contradiction as $m(\mathcal{C}_E) = 0$. Therefore $\mathcal{C}_E$ can't possibly be measurable. ■

Theorem 2.5.4. *There exist disjoint sets A and B of real numbers such that*

$$m^*(A \cup B) < m^*(A) + m^*(B)$$

Definition. We define the **Cantor set** to be the intersection

$$\mathcal{C} = \bigcap C_k$$

where $\{C_k\}$ is a decreasing sequence of closed sets such that for every k, C_k is the disjoint union of 2^k closed intervals of length $\frac{1}{3^k}$

Theorem 2.5.5. *The Cantor set is a closed uncountable set of measure* 0*.*

Proof. Since $\mathcal{C}$ is an arbitrary intersection of closed sets, it is closed in $\mathbb{R}$. Moreover, since each C_k is the disjoint union of closed intervals, which are measurable, and since measurable sets form a σ-algebra, then each C_k is measurable, which makes $\mathcal{C}$ measurable.

Now, by definition of C_k, by finite additivity, we have

$$m(C_k) = (\frac{2}{3})^k$$

so that by monotonicity of measure,

$$m(\mathcal{C}) \leq m(C_k) = (\frac{2}{3})^k$$

now, as $k \to \infty$, $m(C_k) \to 0$ so that $m(\mathcal{C}) = 0$. It remains to show that $\mathcal{C}$ is uncountable.

Suppose $\mathcal{C}$ is countable, and let $\{c_k\}$ be an enumeration of $\mathcal{C}$. Now, there is a disjoint interval F_1 in C_1 which fails to contain the point c_1; similarly, there is a disjoint interval F_2 in C_2, whose union is F_1, that fails to contain c_2. Proceeding inductively, we obtain a countable collection $\{F_k\}$ such that

(1) Each F_k is closed.

(2) $F_k \subseteq C_k$.

(3) $c_k \notin F_k$.

by the nested set theorem, the intersection $F = \bigcap F_k$ is nonempty. Now, let $x \in F$, then we get that $x \in C_k$ for some k. But since $\mathcal{C}_k$ is countable, and enumerated by $\{c_k\}$, then $x = c_n$ for some n. That is, $c_n \in F$ which contradicts that $c_n \notin F_n$. Therefore $\mathcal{C}$ is uncountable. ■

Definition. Define $U_k = [0,1]\backslash C_k$ and $\mathcal{U} = \bigcup U_k$, so that $\mathcal{C} = [0,1]\backslash\mathcal{U}$. Define the functin $\phi : U_k \to \mathbb{R}$ to be the increasing function, which is constant on each of the $2^k - 1$ open intervals, and which takes the values of the form $\frac{2^k-1}{2^k}$ in each of the intervals. We define the **Cantor-Lebesgue function** to be the extension of ϕ to $[0,1]$ by defining it on $\mathcal{C}$ as follows

$$\phi(0) = 0 \text{ for all } x \in \mathcal{U} \text{ and } \phi(x) = \sup\{\phi(t) : t \in U \cap [0,x) \text{ if } x \in \mathcal{C}\backslash 0\}$$

Lemma 2.5.6. *The Cantor-Lebesgue function is increasing continuous whos image is the interval* $[0,1]$. *Moreover,* ϕ *is differentiable on* $\mathcal{U}$, *with* $\phi' = 0$ *on* $\mathcal{U}$, *where* $m(\mathcal{U}) = 1$.

Proof. By definition, $\phi|_{U_k}$ is increasing so the extension ϕ is increasing as well. Likewise, $\phi|_{U_k}$ is continuous, hence so is the extension ϕ.

Now, consider $x_0 \in \mathcal{C}$ such that $x_0 \neq 0, 1$. Then $x \notin U_k$, and for k large enough, x_0 is between two consecutive intervals of U_k. Let a_k be in the lower of these two intervals, and b_k in the upper. Since ϕ is increasing, inparticular, by $\frac{1}{2^k}$, we get $a_k < x_{bk}$ and $\phi(b_k) - \phi(a_k) = \frac{1}{2^k}$.. Then as $k \to \infty$ $\phi(b_k) - \phi(a_k) \to 0$ so that ϕ has no jump discontinuities at x_0. This makes ϕ continuous at x_0. Now, if $x_0 = 0$ or $x_0 = 1$, a similar argument follows. Now, since ϕ is constant on $\mathcal{U}$, and continuous on $\mathcal{U}$, it is differentiable on $\mathcal{U}$, whith derivative $\phi'(x) = 0$ for all $x \in U$. Moreover, since $\mathcal{C}$ is measurable with $m(\mathcal{C} = 0)$, and $\mathcal{U} = [0,1]\backslash\mathcal{C}$, by excision, we get $m(\mathcal{U}) = 1$. Finally, notice that since $\phi(0) = 0$, and $\phi(1) = 1$, and by continuity, by the intermediate value theorem, $\phi([0,1]) = [0,1]$. ■

Lemma 2.5.7. *Let* ϕ *be the Cantor-Lebesgue function and define* $\psi : [0,1] \to \mathbb{R}$ *by* $\psi(x) = \phi(x) + x$ *for all* $x \in [0,1]$. *Then* ψ *is strictly increasing, and takes* $[0,1]$ *onto* $[0,2]$. *Moreover*

(1) ψ *maps* $\mathcal{C}$ *onto a measurable set of positive measure.*

(2) ψ *maps a measurable subset of* $\mathcal{C}$ *onto a nonmeasurable set.*

Proof. ψ is continuous since it is the sum of two continuous functions. Moreover, since ϕ is increasing and the function $f(x) = x$ is strictly increasing then so is ψ. Notice, also, that $\psi(0) = 0$ and $\psi(1) = 2$ so by the intermediate value theorem, $\phi([0,1]) = [0,2]$.

Now, since $[0,1] = \mathcal{U} \cup \mathcal{C}$ (where $\mathcal{U}$ is defined in the definiton of the Cantor-Lebesgue function), we have $[0,2] = \psi(\mathcal{U}) \cup \psi(\mathcal{C})$. Since $[0,2]$ is measurable, and measurable sets are closed under unions, then $\psi(\mathcal{C})$ is measurable; moreover, since ψ is continuous and increasing, it has continuous inverse, and hence maps $\mathcal{C}$ to a measurable set $\psi(\mathcal{C})$. Moreover, $\psi(\mathcal{C})$ is closed, and $\psi(\mathcal{U})$ is open.

Now, let $\{I_k\}$ a collection of intervals of $\mathcal{U}$, i.e. $\mathcal{U} = \bigcup I_k$. Since ϕ is continuous on each I_k, ψ takes I_k onto translates of I_k, and since ψ is 1–1, the collection $\{\psi(I_k)\}$ is disjoint. Therefore, by countable additivity

$$m * (\psi(\mathcal{U})) = \sum l(\psi(I_k)) = \sum l(I_k + \lambda) = \sum l(I_k) = m(\mathcal{U})$$

since $m(\mathcal{C}) = 0$ and $m(\mathcal{U}) = 1$, $m(\psi(\mathcal{U})) = 1$ and $m(\psi(\mathcal{C})) = 1$ as well.

Finally, by Vitali's theorem, there exists a nonmeasurable set $W \subseteq \psi(\mathcal{C})$, with $\psi^{-1}(W)$ measurable with $m(\psi^{-1}(W)) = 0$. ■

Theorem 2.5.8. *There exists a measurable subset of $\mathcal{C}$ which is not Borel.*

Chapter 3

Lebesgue Measurable Functions

3.1 Properties of Lebesgue Measurable Functions

Lemma 3.1.1. *Let f be an extended realvalued function on a measurable domain E. Then the following are equivalent.*

(1) for some $c \in \mathbb{R}$, the set $\{x \in E : f(x) > c\}$ is measurable.

(2) for some $c \in \mathbb{R}$, the set $\{x \in E : f(x) \geq c\}$ is measurable.

Proof. Let $E_1 = \{x \in E : f(x) > 0\}$ and $E_2 = \{x \in E : f(x) \geq c\}$. Suppose that S is measurable, then notice that

$$T = \bigcap \{x \in E : f(x) > c - \frac{1}{k}\}$$

Now, each of the sets in this intersection is measurable, and since measurable sets form a σ-algebra, T must also be measurable. Likewise, if T is measurable, notice that

$$S = \bigcup \{x \in E : f(x) > c + \frac{1}{k}\}$$

is measurable by the same argument. ■

Corollory. *The followingh are equivalent*

(1) for some $c \in \mathbb{R}$, the set $\{x \in E : f(x) < c\}$ is measurable.

(2) for some $c \in \mathbb{R}$, the set $\{x \in E : f(x) \leq c\}$ is measurable.

Proof. Notice that these statements are the contrapostives of the statements above. ■

Corollory. *For some $c \in \mathbb{R}$, the set $\{x \in E : f(x) = x\}$ is measurable.*

Proof. Let $E_3 = \{x \in E : f(x) = c\}$. If c is finite, notice that $E_3 = \{x \in E : f(x) \geq c\} \cap \{x \in E : f(x) \leq x\}$, which makes E_3 measurable. Now, if $c = \infty$, then $\{x \in E : f(x) = \infty\} = \{x \in E : f(x) > k\}$ for some k, which is again, measurable. ■

Definition. Let f be an extended realvalued function on a measurable domain. We say f is **Lebesgue measurable** if it statisfies one of the conditions of lemma 3.1.1 (or its corollories).

Lemma 3.1.2. *Let f be an extended realvalued functoon on a measurable domain E. Then f is measurable if, and only if, there exists an open set U, such that $f^{-1}(U)$ is measurable.*

Proof. Suppose that U is open in $\mathbb{R}$ such that $f^{-1}(U)$ is measurable. Then the interval (c, ∞) is open, which makes $f^{-1}((c, \infty))$ measurable. Notice that $f^{-1}((c, \infty)) = \{x \in E : f(x) > c\}$. This makes f measurable.

Conversely, suppose that f is measurable, and let U be open in $\mathbb{R}$. Then $U = \bigcup I_k$ for some countable collection of bounded open intervals $\{I_k\}$. Let $I_+ = B_k \cap A_k$ where

$$B_k = (-\infty, b_k) \text{ and } A_k = (a_k, \infty) \text{ for some } a_k, b_k \in \mathbb{R}$$

Since f is measurable, then the preimages $f^{-1}(A_k)$ and $f^{-1}(B_k)$ are measurable. Hence, so is the union

$$\bigcup (f^{-1}(B_k) \cap f^{-1}(A_k)) = f^{-1}(I_l) = f^{-1}(\bigcup I_k) = f^{-1}(U)$$

■

Corollory. *A realvalued function continuous on a measurable domain is measurable.*

Lemma 3.1.3. *Monotone functions defined on an interval are measurable.*

Lemma 3.1.4. *Let f be an extended realvalued function on a measurable domain E. The following are true*

(1) If f is measurable on E, and $f = g$ almost everywhere on E, for some extended realvalued function g on E, then g is measurable on E.

(2) If $D \subseteq E$ is measurable, then f is measurable if, and only if the restrictions $f|_D$ and $f|_{E \backslash D}$ are measurable.

Proof. Suppose that f is measurable and that g is an extended realvalued function on E for which $f = g$ a.e. on E. Let $A = \{x \in E : f \neq g\}$. Observe that

$$E_1 = \{x \in E : g(x) > c\} = \{x \in A : g > c\}z \cup \{x \in E : f > c\} \cap E \backslash A$$

Since $f = g$ a.e. on E, then $m(A) = 0$, so that $\{x \in A : g > c\}$ is measurable. Then since measurable sets are a σ-algebra, E_1 is measurable. This makes g measurable.

Now, observe, also, that for every $c \in \mathbb{R}$, and $D \subseteq E$ measurable, that

$$\{x \in E : f > c\} = \{x \in D : f > c\} \cup \{x \in E \backslash D : f > c\}$$

So that if f is measurable, so are its restirctions $f|_D$ and $f|_{E \backslash D}$, and vice versa. ■

Theorem 3.1.5. *Let f and g be measurable functions on a measurable domain, for which f and g are finite almost everywhere on E. Then*

(1) *For all* $\alpha, \beta \in \mathbb{R}$, *the function* $\alpha f + \beta g$ *is measurable.*

(2) fg *is measurable.*

Proof. Suppose, without loss of generality, that f and g are finite on all E. If $\alpha = 0$ and $\beta = 0$, the $\alpha f = 0$ and we are done. Now, take $\alpha \neq 0$ and $\beta = 0$. Then observe that if $\alpha > 0$ then $\{x \in E : \alpha f > c\} = \{x \in E : f > \frac{c}{\alpha}\}$, where as if $\alpha < 0$ then $\{x \in E : \alpha f > c\} = \{x \in E : f < \frac{c}{\alpha}\}$. Since f is measurable, both these sets are measurable, which makes αf measurable.

Now, take $\alpha = \beta = 1$ and observe the function $f + g$. If $f + g < c$ for all $x \in E$, then $f < c - g$, and by the density of $\mathbb{Q}$ in $\mathbb{R}$, there is a rational number q for which $f < q < c - g$. Then notice that

$$\{x \in E : f + g < c\} = \bigcup_{q \in \mathbb{Q}} \{x \in E : g < c - q\} \cap \{x \in E : f < q\}.$$

then since f and g are both measurable, this countable union is measurable.

Lastly, notice that $fg = \frac{1}{2}((f+g)^2 - f^2 - g^2)$ so that it suffices to show that f^2 is measurable. Indeed, for $c \geq 0$ $\{x \in E : f^2 > c\} = \{x \in E : f > \sqrt{c}\}$ and for $c < 0$, $\{x \in E : f^2 > c\} = \{x \in E : f < -\sqrt{c}\}$. In either case, f^2 is measurable. Hence, by linearity, so is fg. ■

Example 2. Consider the function $\psi : [0,1] \to \mathbb{R}$ given by $\psi(x) = \phi(x) + x$, where ϕ is the Cantor-Lebesgue function. Then ψ is strictly increasing and maps a measurable subset $A \subseteq [0,1]$ to a nonmeasurable set $\psi(A)$. Extending ψ to the function $\Psi : \mathbb{R} \to \mathbb{R}$, Ψ^{-1} is continuous, and hence, measurable. Now, since A is also measurable, so is the characteristic function for A, χ_A. However, let I be an open interval with $1 \in I$ but $0 \notin I$. Then $(\chi_A \circ \Phi^{-1})^{-1}(A) = \Phi(\chi_A^{-1}(I)) = \Psi(A)$. Since Ψ is an extension of ψ, $\Psi(A)$ is nonmeasurable, so that the function $\chi_A \circ \Psi^{-1}$ is nonmeasurable; despite being the composition of two measurable functions.

Lemma 3.1.6. *Let* g *a measurable function on a measurable* E *and* f *a continuous function on* $\mathbb{R}$. *Then* $f \circ g$ *is measurable in* E.

Proof. Let U be open in $\mathbb{R}$, by continuity, $f^{-1}(U) = V$ is open, and since g is measurable, $g^{-1}(V) = g^{-1}(f^{-1}(U)) = (f \circ g)^{-1}(U)$ is measurable, which makes $f \circ g$ measurable. ■

Corollory. *If* f *is measurable, then so is the function* $|f|^p$ *on* E, *for all* $p > 0$.

Lemma 3.1.7. *For a finite collection* $\{f_k\}_{k=1}^n$ *of measurable functions with common measurable domain* E, *the functions* $\overline{f} = \max\{f_1, \ldots, f_n\}$ *and* $f = \min\{f_1, \ldots, f_n\}$ *are measurable.*

Proof. For all $c \in E$, notice that $\{x \in E : \overline{f} > c\} = \bigcup_{k=1}^n \{x \in E : f_k > c\}$ and $\{x \in E : f > c\} = \bigcup_{k=1}^n \{x \in E : f_k < c\}$. ■

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