

Matroid Theory

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Chapter 1

Fundamental Definitions and Examples.

The goal of matroid theory is to provide an abstract theory of independence. Matroids have their roots in algebra (especially linear algebra), graph theory, and combinatorics; and each field provides a distinct flavor to the subject. The notion of independence was first covered by Whitney in 1935, and then Van der Waerden, in 1937, in his seminal work “Moderne Algebra”. Gian Carlo Rota would also stumble upon the theory independently. Pioneering work would later be done by Tutte, Nakasawa, Birkhoff, and Mac Lane.

Perhaps the most important aspect of matroids is that they have several equivalent definitions. We begin by studying two of them.

1.1 The Independence and Circuit Axioms.

Definition. A **matroid** M , on a finite set E , called the **ground set** is a pair (E, \mathcal{I}) where $\mathcal{I} \subseteq 2^E$ is a collection of **independent sets**, such that:

(I1) $\emptyset \in \mathcal{I}$.

(I2) If $I_1 \in \mathcal{I}$, and $I_2 \subseteq I_1$, then $I_2 \in \mathcal{I}$.

(I3) If $I_1, I_2 \in \mathcal{I}$, and $|I_1| < |I_2|$, then there exists an $e \in I_2 \setminus I_1$ such that $I_1 \cup e \in \mathcal{I}$.

We call properties (I2) and (I3) the **inherence** and **augmentation** axioms, respectively.

Example 1.1. The empty set \emptyset together with $2^\emptyset = \{\emptyset\}$ forms a matroid called the **empty matroid** and the collection 2^E on a nonempty set E induces a matroid called the **trivial matroid**. It is easy to see why these two are matroids; since one encompasses only the empty set, and the other all subsets, these two matroids are not very interesting.

We provide some equivalent definitions with independent sets.

Example 1.2. (1) Let E be a finite set. Then $M = (E, \mathcal{I})$ is a matroid if, and only if:

(I'1) $\mathcal{I} \neq \emptyset$.

(I'2) Inheritance holds.

(I'3) If $I_1, I_2 \in \mathcal{I}$, with $|I_2| = |I_1| + 1$, then there is an $e \in I_2 \setminus I_1$ such that $I_1 \cup e \in \mathcal{I}$.

Notice, that if M is a matroid, then $\emptyset \in \mathcal{I}$, so (1) is satisfied. Moreover, the augmentation theorem implies (3), since if $|I_2| = |I_1| + 1$, we have $|I_1| < |I_2|$.

On the otherhand, if $\mathcal{I} \neq \emptyset$, then \mathcal{I} contains, at least, \emptyset , since $\mathcal{I} \subseteq 2^E$. (I2) is also given by (2).

Now, if $I_1, I_2 \in \mathcal{I}$ such that $|I_2| = |I_1| + 1$, then $|I_1| < |I_2|$, and it follows from there that (3) implies (I3).

This gives us an equivalent way to define the matroid M , still using independent sets, but with different rules.

(2) A finite set E together with a collection $\mathcal{I} \subseteq 2^E$ is a matroid if, and only if the following hold:

(I''1) $\emptyset \in \mathcal{I}$.

(I''2) Inheritance holds; i.e. if $I \in \mathcal{I}$ and $J \subseteq I$, then $J \in \mathcal{I}$.

(I''3) If $X \subseteq E$, and $I_1, I_2 \in \mathcal{I}$ are maximal such that $I_1, I_2 \subseteq X$ (that is, I_1 and I_2 are maximal sets of the collection $\{I \in \mathcal{I} : I \subseteq X\}$), then $|I_1| = |I_2|$.

It is rather simple to prove (1) and (2), so the nontrivial work goes to showing that property (3) and (I3) are equivalent to each other.

Definition. Let $M = (E, \mathcal{I})$ be a matroid. A subset of E that is not independent, i.e. $X \subseteq E$ with $X \notin \mathcal{I}$ is called a **dependent set**.

the following example shows why Whitney gave the name “matroid” to these structures.

Example 1.3. (1) Consider an $m \times n$ matrix $A \in F^{m \times n}$, where F is a field. Define E to be the set of all column labels of the matrix A , i.e. $A = \{1, \dots, m\}$ and define $\mathcal{I} \subseteq 2^E$ to be the collection of all multisets of E linearly independent over $F^{m \times n}$ considered as a vector space. Then $M = (E, \mathcal{I})$ is a matroid.

First notice that \emptyset is trivially linearly independent, so $\emptyset \in \mathcal{I}$. Moreover, if $I_1 \in \mathcal{I}$ is linearly independent, and $I_2 \subseteq I_1$, then I_2 is also linearly independent, so $I_2 \in \mathcal{I}$.

Now, let $X, Y \in \mathcal{I}$ be linearly independent with $|X| < |Y|$, and consider the subspace $W \subseteq F^{m \times n}$ spanned by $X \cup Y$; i.e. $\text{span } W = X \cup Y$. then $\dim W \geq |Y|$. Now, suppose tht $X \cup i$ is linearly dependent for all $i \in Y \setminus X$, then $W \subseteq \text{span } X$, thus $\dim W \leq |X| < |Y|$, which is a contradiction. Thus, there is atleast one $i \in Y \setminus X$ for which $X \cup i \in \mathcal{I}$. This makes M a matroid which we call the **vector matroid** over A , and we denote it $M[A]$.

(2) Let

$$A = \begin{pmatrix} 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 \end{pmatrix}$$

be a 2×5 matrix on \mathbb{R} . Then considering the vector matroid over M over A , with $E = \{1, 2, 3, 4, 5\}$, we get the independent sets are:

$$\begin{array}{cccc} \{1\} & \{2\} & \{4\} & \{1, 2\} \\ \{1, 5\} & \{2, 4\} & \{2, 5\} & \{4, 5\} \end{array}$$

The collection of dependent sets are:

$$\begin{array}{cccc} \{3\} & \{1, 3\} & \{1, 4\} & \{2, 3\} \\ \{3, 4\} & \{3, 5\} & \{1, 2, 5\} & \{2, 4, 5\} \end{array}$$

The minimal dependent sets of M on A is:

$$\{3\} \quad \{1, 4\} \quad \{1, 2, 5\} \quad \{2, 4, 5\}$$

Remark. Since matroids are a pair of a ground set E and a subset of 2^E we can usually just specify the matroid by E and take the collection of independent sets (or another collection) to be imposed.

Lemma 1.1.1. *Let E be a subset of a vector space V , and let \mathcal{I} be the collection of all linearly independent subsets of E . Then E , together with \mathcal{I} forms a matroid.*

Proof. The proof of example ??(1) can be repeated. In fact, notice that an $m \times n$ matrix is just a collection of n vectors of length m in F^m . We now prove this for arbitrary vector spaces.

\emptyset is trivially linearly independent, so $\emptyset \in \mathcal{I}$; moreover, if I_1 is a set of linearly independent vectors, and $I_2 \subseteq I_1$, then I_2 is also linearly independent; so inheritance holds.

Now, let $I_1 = \{v_1, \dots, v_m\}$, and $I_2 = \{u_1, \dots, u_n\}$ be linearly independent sets of vectors with $n < m$. Then $|I_2| < |I_1|$, with $u_i = v_i$ possibly equal for some $1 \leq i \leq n$. Then, there exists some $v_j \in I_1 \setminus I_2$ distinct from all other u_i . Now, suppose that $I_2 \cup v_j$ is linearly dependent. Then we have

$$\alpha_1 u_1 + \dots + \alpha_n u_n + \alpha v_j = 0$$

which implies that $\alpha \neq 0$. So we get:

$$v_j = \alpha^{-1} \alpha_1 u_1 + \dots + \alpha^{-1} \alpha_n u_n$$

This puts $v_j \in \text{span } I_2$, thus $I_1 \setminus I_2 \subseteq \text{span } I_2$, thus $(I_1 \setminus I_2) \cup I_2 = I_1 \subseteq \text{span } I_2$. This makes $|I_1| < |I_2|$; but $|I_2| < |I_1|$, a contradiction. Therefore $I_2 \cup v_j$ must be linearly independent. This makes E into a matroid. \square

Remark. Most notably, if V is a vector space, then V together with the collection of all linearly independent subsets forms a matroid.

Definition. We call a matroid M **vectorial** if its ground set is a subset of a vector space V and the collection of independent sets consist of all linearly independent subsets of V .

Definition. We call a minimal dependent set of a matroid M a **circuit**. If C is a circuit of size $|C| = n$, we call C an n -circuit. We denote the collection of all circuits of M by \mathcal{C} .

This definition will also provide us with an alternative definition for a matroid.

Lemma 1.1.2 (The Circuit Axioms.). *The collection \mathcal{C} of circuits of a matroid satisfy the following:*

(C1) $\emptyset \notin \mathcal{C}$.

(C2) If $C_1, C_2 \in \mathcal{C}$, and $C_1 \subseteq C_2$, then $C_1 = C_2$.

(C3) If $C_1, C_2 \in \mathcal{C}$ are distinct, and $z \in C_1 \cap C_2$, then there exists a circuit $C \in \mathcal{C}$ such that $C \subseteq (C_1 \cup C_2) \setminus z$.

Proof. If M is a matroid with \mathcal{I} the collection of independent sets, then $\emptyset \in \mathcal{I}$, by definition, this makes $\emptyset \notin \mathcal{C}$. Moreover, if $C_1, C_2 \in \mathcal{C}$ are circuits, then they are minimally dependent, so if $C_1 \subseteq C_2$, it must be that $C_1 = C_2$, otherwise we would have $C_1 \in \mathcal{I}$, a contradiction.

Now, let $C_1, C_2 \in \mathcal{C}$ be distinct circuits, and let $z \in C_1 \cap C_2$. Now, suppose that $(C_1 \cup C_2) \setminus z$ does not contain a circuit; then $(C_1 \cup C_2) \setminus z \in \mathcal{I}$. Now, by the above, (C2), we have $C_2 \setminus C_1 \neq \emptyset$, so choose an $f \in C_2 \setminus C_1$. By minimality, we have $C_2 \setminus f \in \mathcal{I}$, so choose a maximally independent subset $I \subseteq C_1 \cup C_2$ such that $C_2 \setminus f \subseteq I$. Then $f \notin I$; and since C_1 is a circuit, for some $g \in C_1$, $g \notin I$, moreover, $g \neq f$. Therefore, we have:

$$|I| \leq |(C_1 \cup C_2) \setminus \{f, g\}| = |C_1 \cup C_2| - 2 \leq |(C_1 \cup C_2) \setminus z| = |C_1 \cup C_2| - 1$$

By the augmentation axiom (I3), let $I_1 = I$, $I_2 = (C_1 \cup C_2) \setminus z$, then we get $I_1 \cup g \in \mathcal{I}$ which contradicts the maximality of I . \square

Remark. This just establishes the validity of the circuit axioms for matroids. To actually show that these axioms provide an equivalent definition, we need the following theorem, and its corollary.

Theorem 1.1.3. *Let E be a finite set having $\mathcal{C} \subseteq 2^E$ satisfying (C1)-(C3). Let \mathcal{I} be the collection of all subsets of E which don't contain elements of \mathcal{C} ; i.e.*

$$\mathcal{I} = \{X \subseteq E : Y \notin \mathcal{C} \text{ given } Y \subseteq X\}$$

Then \mathcal{I} defines the collection of independent sets of a matroid on E .

Proof. Notice that \emptyset contains no subsets of E contained in \mathcal{C} , so $\emptyset \in \mathcal{I}$; furthermore, if $I_1 \in \mathcal{I}$ contains no subsets of E contained in \mathcal{C} , neither does a subset $I_2 \subseteq I_1$, so $I_2 \in \mathcal{I}$.

Now, let $I_1, I_2 \in \mathcal{I}$ with $|I_1| < |I_2|$. Suppose for some $e \in I_2 \setminus I_1$, that $I_1 \cup e$ contains a member of \mathcal{C} . Now, $I_1 \cup I_2$ contains a set $I_3 \in \mathcal{I}$ with $|I_1| < |I_3|$, moreover, choose I_3 such that $I_1 \setminus I_3$ is minimal; we have $I_1 \setminus I_3 \neq \emptyset$. Now, choose an $e' \in I_1 \setminus I_3$. Then for each $f \in I_3 \setminus I_1$, let $T_f = (I_3 \cup e') \setminus f$. Then $T_f \subseteq I_1 \cup I_3$, and $|I_1 \setminus T_f| < |I_1 \setminus I_3|$. Since we chose

$I_1 \setminus I_3$ minimal, $T_f \notin \mathcal{I}$. So there exists a $C_f \in \mathcal{C}$ such that $C_f \subseteq T_f = (I_3 \cup e') \setminus f$. Then $f \notin C_f$, moreover $e' \in C_f$, for otherwise, $C_f \subseteq I_3$ contradiction the independence of I_3 .

Suppose that $g \in I_3 \setminus I_1$. If $C_g \cap (I_3 \setminus I_1) = \emptyset$, then $C_g \subseteq ((I_1 \cap I_3) \cup e') \setminus g \subseteq I_1$, which cannot happen. So there exists an $h \in C_g \cap (I_3 \setminus I_1)$ with $C_g \neq C_h$. Now, $e' \in C_g \cap C_h$, so by (C3), there exists a $C \in \mathcal{C}$ with $C \subseteq (C_g \cap C_h) \setminus e'$, but both $C_g, C_h \subseteq I_3 \cup e'$, so $C \subseteq I_3$ another contradiction. Therefore, we find that \mathcal{I} imposes a matroid on E . \square

Corollary. *E has \mathcal{C} as its collection of circuits.*

Proof. Notice that if $I \in \mathcal{I}$ is maximal, then for any $e \in E$, $I \cup e$ is dependent. Moreover, since $I \cup e \notin \mathcal{I}$, we see that there is a $C \in \mathcal{C}$ with $C \subseteq I \cup e$. Now, since I is maximally independent, this makes $I \cup e$ minimal, and so $C = I \cup e$. This makes \mathcal{C} the set of circuits of the matroid on E . \square

Corollary. *If I is independent in a matroid M , and $e \in E$ such that $I \cup e$ is dependent, then M has a unique circuit contained in $I \cup e$, containing e .*

Proof. By above, we have that $I \cup e$ contains a circuit $C = I \cup e$, so $e \in C$. Now, if $C' \subseteq I \cup e$ is another circuit contained in $I \cup e$, containing e , such that C' is distinct from C , then by (C3), there is another circuit $C'' \in \mathcal{C}$ such that $C'' \subseteq (C \cup C') \setminus e$; a contradiction. So $C' = C$. \square

We can now provide an alternative definition.

Definition. A **matroid** M , on a finite set E , is a pair (E, \mathcal{C}) where $\mathcal{I} \subseteq 2^E$ is a collection of **circuits** such that

(C1) $\emptyset \notin \mathcal{C}$.

(C2) If $C_1, C_2 \in \mathcal{C}$, and $C_1 \subseteq C_2$, then $C_1 = C_2$.

(C3) If $C_1, C_2 \in \mathcal{C}$ are distinct, and $z \in C_1 \cap C_2$, then there exists a circuit $C \in \mathcal{C}$ such that $C \subseteq (C_1 \cup C_2) \setminus z$.

We call the property (C3) the **weak circuit elimination axiom**.

Example 1.4. (1) Let G be a graph with vertex set V and edge set E . Let \mathcal{C} be the collection of edge-set defined cycles of G (i.e. all cycles determined by their edges). The $M = (E, \mathcal{C})$ is a matroid on G , with \mathcal{C} the collection of circuits on G .

Notice that \emptyset contains no edges, and hence no cycles, so $\emptyset \notin \mathcal{C}$. Moreover, let C_1, C_2 be cycles of G , then if $C_1 \subseteq C_2$, by definition of a cycle, it must be that $C_1 = C_2$.

Now, let $C_1, C_2 \in \mathcal{C}$ be distinct cycles of G , having a common edge e with endpoints $u, v \in V$; i.e. $e = \{u, v\}$. Now, for $1 \leq i \leq 2$, let P_i be the (u, v) -path with edges in $C_i \setminus e$. Now, walk on P_1 from u to v stopping at the vertex $w \in V$ such that w is the first vertex not in P_2 . Then, walk from w to the vertex $x \neq w$ such that x is in P_2 . Since P_1 and P_2 terminate at v , such a vertex exists. Now, adjoin P_1 from w to x to P_2 from x to w , and the resultant graph is a cycle contained in $(C_1 \cup C_2) \setminus e$. We call the corresponding matroid the **cycle matroid** of G .

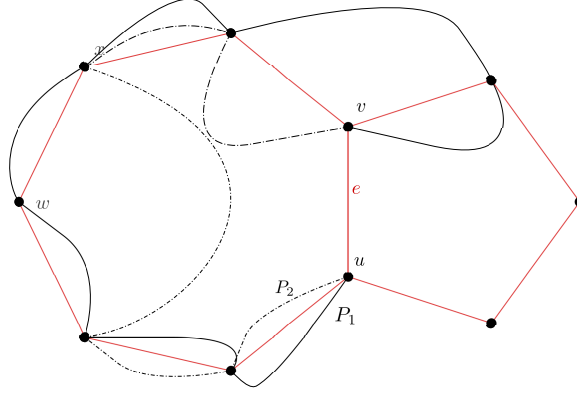


Figure 1.1: The cycle matroid corresponding to the graph G of example 1.2. Path P_1 is indicated by a solid line, and path P_2 indicated by a dotted line.

- (2) Consider the graph in figure 1.2, and the cycle matroid M on G . The set of circuits \mathcal{C} is given by the collection:

$$\{e_3\} \quad \{e_1, e_4\} \quad \{e_1, e_2, e_5\} \quad \{e_2, e_4, e_5\}$$

Comparing this with the previous matroid M' on the 2×5 matrix A from example

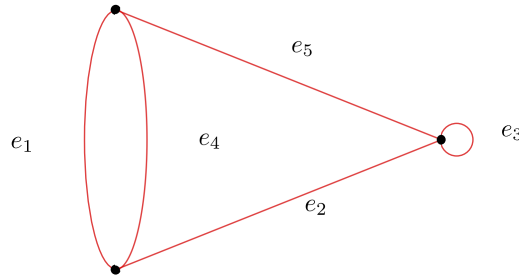


Figure 1.2:

1.1(2). We can see that they have the same structure. Take the map $\psi : i \rightarrow e_i$, which is 1-1 and onto, then a set $X \subseteq E$ is a circuit in M if, and only if it is a circuit in M' .

Example 1.5. Let

$$A = \begin{pmatrix} 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 \end{pmatrix}$$

and let $M_2(A)$ and $M_3(A)$ be the matroids on A over \mathbb{F}_2 and \mathbb{F}_3 , respectively. We can list the set of circuits of $M_2(A)$ to be:

$$\begin{array}{cccc} \{1, 2, 4\} & \{1, 3, 5\} & \{2, 3, 6\} & \{4, 5, 6\} \\ \{1, 4, 3, 6\} & \{1, 5, 2, 6\} & \{2, 4, 3, 5\} & \end{array}$$

while the set of circuits of $M_3(A)$ is:

$$\begin{array}{ccc} \{1, 2, 4\} & \{1, 3, 5\} & \{2, 3, 6\} \\ \{1, 4, 3, 6\} & \{1, 5, 2, 6\} & \{2, 4, 3, 5\} \end{array}$$

So we can clearly see that the circuits of $M_2(A)$ and $M_3(A)$ are not the same. In fact, $M_2(A)$ is graphic while $M_3(A)$ is not. We also see that $M_2(A)$ is \mathbb{F}_3 -representable, taking $A' = 2A$, while $M_3(A)$ is not \mathbb{F}_2 -representable.

The above example leads us to define what we mean by a matroid isomorphism. We present some definitions.

Definition. We call two matroids M_1 and M_2 , with ground sets E_1 and E_2 , respectively, **isomorphic** if there exists a 1–1 map $\psi : E_1 \rightarrow E_2$ of E_1 onto E_2 such that X is independent in E if, and only if $\psi(X)$ is independent in E_2 . We write $M_1 \simeq M_2$ and call ψ an **isomorphism** of the matroids.

Definition. Let G be a graph with edge set E . We call the matroid on E having \mathcal{C} the collection of all edge-set defined cycles the **cycle matroid** of G . We call a matroid M **graphic** if it is isomorphic to the cycle matroid of some graph.

Definition. We call a matroid **F -representable** if it is isomorphic to some vector matroid over a field F . We call the ground set of the vector matroid a **F -representation**.

Example 1.6. (1) The matroid on the 2×5 matrix \mathbb{R} of example 1.1(2) is \mathbb{R} -representable. It is also \mathbb{F}_2 -representable.

- (2) The above matroid is a graphic matroid, isomorphic to the cycle matroid of example 1.2(2); as a consequence, that cycle matroid is also \mathbb{F}_2 -representable.
- (3) Let M_1 and M_2 be isomorphic matroids via a map ψ . Then if \mathcal{I} is the collection of independent sets of M_1 , then $\psi(\mathcal{I})$ the collection of independent sets of M_2 .

Lemma 1.1.4. *Let M_1 and M_2 be matroids with ground sets E_1 and E_2 . If $M_1 \simeq M_2$ via the map ψ , then $C \subseteq E_1$ is a circuit of M_1 if, and only if $\psi(C) \subseteq E_2$ is a circuit of M_2 .*

Proof. Let \mathcal{I}_1 and \mathcal{I}_2 be the independent sets of M_1 and M_2 , respectively, and let C be a circuit of M_1 . Now, if $\psi(C) \in \mathcal{I}_2$, then by definition, we must have $C \in \mathcal{I}_1$, which contradicts that C is a circuit. Thus $\psi(C)$ must be a dependent set. Now, by definition, C is minimally dependent, so $C \setminus e \in \mathcal{I}_1$, thus $\psi(C \setminus e) \in \mathcal{I}_2$. Notice, then that $\psi(C) \setminus \psi(e) \subseteq \psi(C \setminus e)$, so by inheritance, $\psi(C) \setminus \psi(e) \in \mathcal{I}_2$. So we have that for $\psi(e) \in E_2$, $\psi(C) \setminus \psi(e)$ is independent, but $\psi(C)$ is dependent. This makes $\psi(C)$ minimally dependent, and thus, by definition, a circuit. \square

Corollary. *If \mathcal{C}_1 and \mathcal{C}_2 are the collection of circuits of M_1 and M_2 , respectively, then $\mathcal{C}_2 = \psi(\mathcal{C}_1)$.*

Proof. Take the above lemma together with the fact that ψ is onto. \square

Since a matroid is determined by its ground set, there is no loss of generality in referring to the ground set as the matroid itself, implying that we are imposing a collection of independent sets/circuits.

Definition. Let M be a matroid. We call an element $e \in M$ a **loop** if the singleton $\{e\}$ is a circuit of M . We call two elements $f, g \in M$ **parallel** if the doubleton $\{f, g\}$ is a circuit of M and we write $f \parallel g$. We define a **parallel class** of M to be a maximal subset $X \subseteq M$ with the property that: if $f, g \in X$ distinct, then $f \parallel g$, and no element of X is a loop. We call a parallel class of M **trivial** if it contains only one element. We call a matroid **simple** if it contains no loops, nor parallel elements.

Example 1.7. Let $A \in F^{m \times n}$ is an $m \times n$ matrix, and consider the vector matroid of A whose independent sets are linearly independent columns. Since any single nonzero column of A is linearly independent, we have the only loop in the matroid on A is the zero-column $(0 \dots 0)^T$. Likewise, the parallel elements of the matroid on A are simply the linearly dependent pairs of columns.

Theorem 1.1.5 (The Strong Circuit Elimination Axiom.). *Let M be a matroid with collection of circuits \mathcal{C} . If $C_1, C_2 \subseteq \mathcal{C}$ are distinct, such that $e \in C_1 \cap C_2$, and $f \in C_1 \setminus C_2$ then there exists a circuit $C \in \mathcal{C}$ such that $f \in C \subseteq (C_1 \cup C_2) \setminus e$.*

We now introduce one last example of a matroid for this section.

Example 1.8. Consider the following bipartition of the sets $X = \{1, 2, 3\}$ and $Y = \{a, b, c, d, e, f\}$, together with the relation $E = \{(1, a), (2, a), (2, b), (2, c), (3, a), (3, b), (3, c), (3, d), (3, e), (3, f)\}$. We can represent this as the following bipartite graph, $G = (X \cup Y, E)$ in figure 1.3 Then if

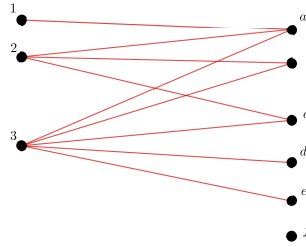


Figure 1.3:

we define those “independent sets” \mathcal{I} consisting of all subsets of Y that can be “matched” to elements of X . i.e. those subsets of Y whose edges between X contain no incident edges. Then Y together with \mathcal{I} forms a matroid.

Definition. Let G be a bipartite graph with bipartition (X, Y) . For a subset U of Y , if E is the set of edges on (X, U) , then we call E a **matching** between X and U if no two edges in E have a common endpoint. We say that U can be **matched** to X . We say that E is a complete matching if $U = Y$.

Lemma 1.1.6. *Let G be a bipartite graph, with bi-partition (X, Y) . Let \mathcal{I} be the collection of all subsets of Y that can be matched to X . Then $M = (Y, \mathcal{I})$ forms a matroid.*

Proof. We have that for $\emptyset \subseteq Y$, there are no two edges on the bipartition (X, \emptyset) with a common endpoint; infact, there are no edges in this bipartition. now suppose that $I \subseteq Y$ can be matched to X , that means no two edges between the bipartition (X, I) has a common endpoint. Now, if $I' \subseteq I$, then it follows that no two edges between (X, I') has a common endpoint. Therefore (I1), and (I2) are satisfied.

Now suppose subsets I and I' of Y can be matched to X , and that $|I| < |I'|$. Then the bipartitions (X, I) and (X, I') have no two edges incident to each other. Then the bipartition $(X, I \cup I')$ have no two edges incident with each other. Now, choose an $e \in I' \setminus I$, then the edge from e to some point in X has no other incident edge, therefore the edges between X and $I \cup e$ have no two edges incident. That is no two edges are incident between $(X, I \cup e)$, therefor (I3) is satisfied. This makes $M = (Y, \mathcal{I})$ into a matroid. \square

Remark. Notice that we can make G into a bi-partite graph by taking the set of edges to be all matchings from the set X to subsets of Y . Matroids of this type are called “transversal” and will be studied in more rigor later.

1.2 The Base Axioms.

Definition. We call a maximally independent set of a matroid M a **basis** of M . We denote the collection of all bases of M to be \mathcal{B} .

Lemma 1.2.1. *For any two bases B_1 and B_2 of a matroid, we have $|B_1| = |B_2|$.*

Proof. Suppose not, that $|B_1| < |B_2|$. Then, since $B_1, B_2 \in \mathcal{I}$, by augmentation, we can choose $e \in B_2 \setminus B_1$ such that $B_1 \cup e \in \mathcal{I}$. But B_1 is maximal, a contradiction! Therefore, $|B_1| \geq |B_2|$. Similarly, we get $|B_2| \geq |B_1|$. \square

Lemma 1.2.2 (The Base Axioms). *The collection \mathcal{B} of bases of a matroid has the following properties:*

(B1) $\mathcal{B} \neq \emptyset$.

(B2) If $B_1, B_2 \in \mathcal{B}$, and $x \in B_1 \setminus B_2$, then there exists $y \in B_2 \setminus B_1$ such that $(B_1 \setminus x) \cup y \in \mathcal{B}$.

Proof. For (B1), if $\mathcal{B} = \emptyset$, then necesarrily, $\mathcal{I} = \emptyset$, which cannot happen by (I1).

Now, notice that both $B_1 \setminus x$ and B_2 are independent, and that $|B_1 \setminus x| < |B_2|$ by lemma 1.2.1. Therefore, by augmentation, take $y \in B_2 \setminus (B_1 \setminus x)$, that is, $y \in B_2 \setminus B_1$, such that $(B_1 \setminus x) \cup y \in \mathcal{I}$. Then there is a basis $B' \in \mathcal{B}$ such that $(B_1 \setminus x) \cup y \subseteq B'$. Now, notice that $|(B_1 \setminus x) \cup y| = |B_2| = |B'|$, thus $(B_1 \setminus x) \cup y = B'$, makinng $(B_1 \setminus x) \cup y \in \mathcal{B}$ a basis. \square

With this lemma, we have proved that the independence axioms imply the base axiom. We now show that the base axioms imply independence.

Theorem 1.2.3. *Let E be a finite set and $\mathcal{B} \subseteq 2^E$ a collection of subsets of E satisfying (B1) and (B2). let $\mathcal{I} = \{I \subseteq E : I \subseteq B, \text{ where } B \in \mathcal{B}\}$. Then \mathcal{I} induces a matroid on E .*

Proof. If $\mathcal{B} \neq \emptyset$, then we have at least $\emptyset \in \mathcal{I}$. Moreover, if $I_1 \in \mathcal{I}$, then $I_1 \subseteq B$, for some $B \in \mathcal{B}$. Then if $I_2 \subseteq I_1$, $I_2 \subseteq B$ so that $I_2 \in \mathcal{I}$.

Now suppose that $B_1, B_2 \in \mathcal{B}$ with $|B_1| > |B_2|$, such that $|B_1 \setminus B_2|$ is minimal. Notice that $B_1 \setminus B_2 \neq \emptyset$, so choose $x \in B_1 \setminus B_2$. Then by (B2), there exists a $y \in B_2 \setminus B_1$ such that $(B_1 \setminus x) \cup y \in \mathcal{B}$. Notice then that $|(B_1 \setminus x) \cup y| = |B_1| > |B_2|$, so $|((B_1 \setminus x) \cup y \setminus B_2)| < |B_1 \setminus B_2|$ which contradicts minimality. So we have $|B_1| = |B_2|$.

Now suppose that the augmentation axiom, (I3), fails. Then for $I_1, I_2 \in \mathcal{I}$ with $|I_1| < |I_2|$, there is an $e \in I_2 \setminus I_1$ such that $I_1 \cup e \notin \mathcal{I}$. Now, by definition, there exists $B_1, B_2 \in \mathcal{B}$ with $I_1 \subseteq B_1$ and $I_2 \subseteq B_2$. Choose, then, B_2 such that $|B_2 \setminus (B_1 \cup I_2)|$ is minimal. Then $I_2 \setminus B_1 = I_2 \setminus I_1$. Now, supposing that $B_2 \setminus (B_1 \cup I_2) \neq \emptyset$, choose $x \in B_2 \setminus (B_1 \cup I_2)$. Then by (B2), there exists a $y \in B_1 \setminus B_2$ such that $(B_2 \setminus x) \cup y \in \mathcal{B}$; but then $((B_2 \setminus x) \cup y) \setminus (B_1 \cup I_2) < |B_2 \setminus (B_1 \cup I_2)|$, which contradicts minimality. So $B_2 \setminus (B_1 \cup I_2) = \emptyset$, and so $B_2 \setminus B_1 = I_2 \setminus B_1$; that is:

$$B_2 \setminus B_1 = I_2 \setminus I_1$$

Now suppose that $B_1 \setminus (B_2 \cup I_1) \neq \emptyset$. Then cor $x \in B_1 \setminus B_2 \cup I_1$, there exists $y \in B_2 \setminus B_1$ such that $(B_1 \setminus x) \cup y \in \mathcal{B}$. Now, we have $I_1 \cup y \subseteq (B_1 \setminus x) \cup y$, putting $I_1 \cup e \in \mathcal{I}$. Since $y \in B_2 \setminus B_1$, we have $y \in I_2 \setminus I_1$, which contradicts the hypothesis. So $B_1 \setminus (B_2 \cup I_1) = \emptyset$. Thus, $B_1 \setminus B_2 = I_1 \setminus B_2$. It follows then that $B_1 \setminus B_2 \subseteq I_2 \setminus I_1$. Now, $|B_1| = |B_2|$, so $|B_1 \setminus B_2| = |B_2 \setminus B_1|$, thus $|I_1 \setminus I_2| = |I_2 \setminus I_1|$, so that $|I_1| \geq |I_2|$. but $|I_1| < |I_2|$, a contradiction. Therefore, (I3) must be satisfied, making (E, \mathcal{I}) into a matroid. \square

Corollary. *The matroid on E induced by \mathcal{I} has \mathcal{B} as its collection of bases.*

We now come to our next equivalent definition of a matroid.

Definition. A **matroid** on a finite set E is a pair (E, \mathcal{B}) , where $\mathcal{B} \subseteq 2^E$, such that:

(B1) $\mathcal{B} \neq \emptyset$.

(B2) If $B_1, B_2 \in \mathcal{B}$, and $x \in B_1 \setminus B_2$, then there exists $y \in B_2 \setminus B_1$ such that $(B_1 \setminus x) \cup y \in \mathcal{B}$.

We call \mathcal{B} the collection of **bases** of the matroid.

Corollary. *For any $B \in \mathcal{B}$ and $e \in E \setminus B$, $B \cup e$ contains a unique circuit $C(e, B)$ which contains e .*

Proof. This follows immediately from the analogous corollary to theorem 1.1.3. \square

Definition. Let M be a matroid with ground set E and collection of bases \mathcal{B} . For $e \in E \setminus B$, we define the **fundamental circuit** of e with respect to B to be the circuit $C(e, B)$ with the property that $e \in C(e, B) \subseteq B$.

Lemma 1.2.4 (Fundamental Circuit Theorem.). *Every circuit of a matroid is the fundamental circuit of some element with respect to some basis.*

Proof. Let C be a circuit of some matroid M , and choose $e \in C$. Then choose some basis B of M in which $e \in E \setminus B$ (E the ground set of M). Then by the corollary to theorem 1.2.3, we have that there is a circuit $C(e, B)$ such that $e \in C(e, B) \subseteq B \cup e$. We then have that $e \in C \subseteq B \cup e$, and that $e \in C \cap C(e, B)$. Then by weak circuit elimination, there is a circuit $C' \subseteq (C \cup C(e, B)) \setminus e$. Then $C' \subseteq C \setminus e$ and $C' \subseteq C(e, B) \setminus e$. That is $C' \cup e \subseteq C$ and $C' \cup e \subseteq C(e, B)$. Thus, either $C \subseteq C(e, B)$ or $C(e, B) \subseteq C$; in either case we get $C = C(e, B)$. \square

Example 1.9. (1) let $m, n \in \mathbb{Z}^+$ with $m \leq n$. Let E be an n -element set, and \mathcal{B} the collection of all m -element subsets of E . Then \mathcal{B} is the collection of bases for a matroid on E .

Notice that since $|E| = n$, and $m \leq n$, then there exists at least one m -element subset of E , so that $\mathcal{B} \neq \emptyset$. Now, let $B_1, B_2 \in \mathcal{B}$ be distinct with $|B_1| = |B_2| = m$. Then for $x \in B_1 \setminus B_2$, notice that $|B_1 \setminus x| = m - 1$. Thus, choose $y \in B_2 \setminus B_1$ (which exists), then $|(B_1 \setminus x) \cup y| = m$, making $B_1 \setminus x \cup y \in \mathcal{B}$. Then (E, \mathcal{B}) forms a matroid which we call the **uniform matroid** of rank m on an n -element set. We denote the uniform matroid by $U_{m,n}$.

(2) In the uniform matroid, $U_{m,n}$, we have the collection of independent sets is given by:

$$\mathcal{I} = \{X \subset E : |X| \leq m\}$$

and the collection of circuits is given by $\mathcal{C} = \emptyset$ if $m = n$, or

$$\mathcal{C} = \{X \subseteq E : |X| = m + 1\}.$$

if $m < n$.

(3) If $m = 0$, then the only independent sets of $U_{0,n}$ are emptysets, i.e. $\mathcal{I} = \mathcal{B} = \{\emptyset\}$. This makes all n -elements of the ground set loops. Likewise, if $m = 1$, then the only independent sets of $U_{1,n}$ are singletons, hence $U_{1,n}$ consists of n parallel elements. If $m \geq 2$, then $U_{m,n}$ is simple.

(4) $U_{n,n}$ has no circuits, since by above, $\mathcal{C} = \emptyset$, and $U_{0,0} = \emptyset$ the empty matroid. We call $U_{n,n}$ the **Boolean algebra**.

Definition. We call a matroid M **free** if its collection of circuits is empty.

Example 1.10. (1) Let E be a nonempty set and (E_1, \dots, E_r) be a partition π of E into nonempty subsets. Let \mathcal{B} be the collection of all subsets of E containing exactly one element of E_i for each $1 \leq i \leq r$. Then \mathcal{B} is the collection of bases of a matroid on E called the **partition matroid**, M_π .

(2) M_π has as its collection of independent sets, bases, and circuits given by:

$$\begin{aligned} \mathcal{I} &= \{X \subseteq E : |X \cap E_i| \leq 1, \text{ for all } 1 \leq i \leq r\} \\ \mathcal{B} &= \{X \subseteq E : |X \cap E_i| = 1, \text{ for all } 1 \leq i \leq r\} \\ \mathcal{C} &= \{\{a, b\} \subseteq E : \{a, b\} \subseteq E_i, \text{ for all } 1 \leq i \leq r\} \end{aligned}$$

(3) If $|E_i| = 1$ for all i , then $M_\pi \simeq U_{r,r}$. In general, we also have that M_π is graphic, as it is isomorphic to the cycle matroid of a vertex-disjoint union of graphs G_1, \dots, G_r where G_i consists of 2 vertices joined by $|E_i|$ distinct edges.

- Example 1.11.** (1) Let G be a graph. Then a subset $X \subseteq E$ of edges of G is independent if, and only if they form a forest of the graph G . Then X is a basis if X is a forest such that for $e \in E \setminus X$, $X \cup e$ contains a cycle. That is, X forms a spanning tree in all connected components of G . If G is connected, then X is a spanning tree. Likewise, it can be shown that spanning trees are bases, so that we have that X is the basis of matroid on a graph G if, and only if it forms spanning trees across all connected components of G .
- (2) Let A be an $m \times n$ matrix over a field F . Then the columns of A span a subspace W of dimension r . We also see that independent sets of columns of A are linearly independent. Thus, if we have a subset of columns of A , linearly independent, also spanning the subspace W , they form a basis of W . That is, a set B of columns of A forms a basis for W if, and only if B is linearly independent, and $|B| = r$; so that B is a set of basis vectors for the subspace W .
- (3) If $A = \begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 2 \end{pmatrix}$ is a 2×4 matrix over \mathbb{F}_3 , then $\dim W = 2$, and so the bases are all 2 element subsets of linearly independent columns. Thus $M(A) \simeq U_{2,4}$.
- (4) Consider the graph G in figure 1.3, we would like to know how many spanning trees this graph has.

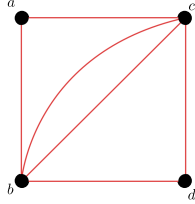


Figure 1.4:

We first form the $|V| \times |V| = 4 \times 4$ matrix L defined as follows:

$$(L_{u,v}) = \begin{cases} \deg v, & \text{if } u = v \\ -|E_{u,v}|, & \text{otherwise} \end{cases}$$

Where $E_{u,v}$ is the set of edges between u and v . We call L the **Laplacian** of G . By Kirchoff's matrix tree theorem combinatorics, the number of spanning trees of G is equal to any cofactor of L , in this case, we have:

$$L = \begin{pmatrix} 2 & -1 & -1 & 0 \\ -1 & 4 & -2 & -1 \\ -1 & -2 & 4 & -1 \\ 0 & -1 & -1 & 2 \end{pmatrix}$$

Which has a cofactor of 12, and hence there are 12 spanning trees on G . This also means that the cycle matroid on G has 12 bases.

Theorem 1.2.5. *Let M be a graphic matroid. Then M is isomorphic to the cycle matroid of some connected graph G .*

Proof. Let M be a matroid, and denote the cycle matroid of any graph G by $M(G)$. Then $M \simeq M(H)$ for some graph H . If H is connected, we are done.

Now suppose that H is not connected. Consider then the connected components H_1, \dots, H_n of H . Choose a vertex $v_i \in H_i$ and form the graph G with vertices v_1, \dots, v_n all labeled as one vertex. Then the edge set $E(H) = E(G)$ the edge set of G ; moreover, $X \subseteq E(G)$ is a cycle if, and only if it is a cycle in $E(H)$. Thus we have $M \simeq M(H)$. \square

To finish the section, we make the following observation. The independence, base, and circuit axioms constitute the three main equivalent ways of defining a matroid. We can denote these definitions as (\mathcal{I}) , (\mathcal{B}) , and (\mathcal{C}) . Now, by theorems 1.1.3 and 1.2.3, we have that (\mathcal{I}) is equivalent to (\mathcal{B}) and (\mathcal{C}) . Moreover, notice that (\mathcal{B}) implies (\mathcal{I}) , and (\mathcal{I}) implies (\mathcal{C}) . Thus, by transitivity of implication, (\mathcal{B}) implies (\mathcal{C}) . Conversely, we have that (\mathcal{C}) implies (\mathcal{I}) , which implies (\mathcal{B}) . So by transitivity again, we get that (\mathcal{C}) implies (\mathcal{B}) . Therefore (\mathcal{B}) and (\mathcal{C}) are equivalent definitions of a matroid. Collecting our definitions into a digraph, whose edge set is given by implication, we get the following figure:

Remark. That is, two vertices a, b in the digraph form an edge (a, b) if a implies b .

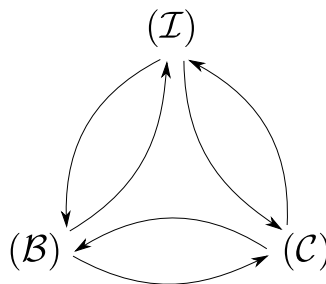


Figure 1.5: The implication digraph of (\mathcal{I}) , (\mathcal{B}) , and (\mathcal{C}) .

1.3 The Rank of a Matroid.

Let $M = (E, \mathcal{I})$ be a matroid, and let $X \subseteq E$, and $\mathcal{I}|X = \{I \subseteq X : I \in \mathcal{I}\}$. Then the pair $(X, \mathcal{I}|X)$ forms a matroid which we can define as follows.

Definition. If $M = (E, \mathcal{I})$ is a matroid, and $X \subseteq E$, then we call the matroid $M|X = (X, \mathcal{I}|X)$ the **restriction** of M to X , or alternatively, the **deletion** of $E \setminus X$ from M .

This also motivates the following definition for the “rank” of a matroid.

Definition. Let $M = (E, \mathcal{I})$ be a matroid, and $X \subseteq E$. We define the **rank** of X to be the cardinality of some basis B of $M|X$, and denote it $\text{rank } X$; we also call B a **basis** of X . We define the **rank function** of the matroid M to be the mapping $\text{rank} : 2^E \rightarrow \mathbb{N}$, and we write $\text{rank } M = \text{rank } E$. That is the rank of the matroid M is the rank of its ground set.

Lemma 1.3.1. *Let M be a matroid on a set E , and let $X \subseteq E$. If rank is the rank function of M , then the following are true:*

(R1) $0 \leq \text{rank } X \leq |X|$.

(R2) If $Y \subseteq E$ with $X \subseteq Y$, then $\text{rank } X \leq \text{rank } Y$.

(R3) If $X \subseteq E$ and $Y \subseteq E$, then $\text{rank}(X \cup Y) + \text{rank}(X \cap Y) \leq \text{rank } X + \text{rank } Y$.

Proof. Since rank maps a subset of E to a nonnegative integer in \mathbb{N} , we see that $\text{rank } X \geq 0$. Now, if B is a basis of X , then $B \subseteq X$, so that $|B| \leq |X|$, so $\text{rank } X \leq |X|$.

Moreover, if $X \subseteq Y$, and B is a basis of X , then it is a basis of the matroid $M|X$, as well as being an independent set of the matroid $M|Y$. So if B' is the basis for Y ; i.e. a basis of $M|Y$, then we have $|B| \leq |B'| \leq |Y|$, so that $\text{rank } X \leq \text{rank } Y$.

Now, let $B_{X \cap Y}$ be a basis of $X \cap Y$, then it is an independent set in $M|(X \cup Y)$, and hence contained in a basis $B_{X \cup Y}$ of $X \cup Y$. Now, we also have that $B_{X \cup Y} \cap X$ and $B_{X \cup Y} \cap Y$ are independent in $M|X$ and $M|Y$, respectively; So we get that $\text{rank } X \geq |B_{X \cup Y} \cap X|$ and $\text{rank } Y \geq |B_{X \cup Y} \cap Y|$, thus $\text{rank } X + \text{rank } Y \geq |B_{X \cup Y} \cap X| + |B_{X \cup Y} \cap Y| = |B_{X \cup Y}| + |B_{X \cap Y}| = \text{rank}(X \cup Y) + \text{rank}(X \cap Y)$. \square

Corollary. *For any matroid M on E with $X \subseteq E$ and $x \in E$:*

$$\text{rank } X \leq \text{rank}(X \cup x) \leq \text{rank } X + 1$$

Example 1.12. (1) Let V be a vector space and consider the matroid $M[V]$ on V , the dimension of V is a rank function of $M[V]$ by definition. Moreover, if U and W are subspaces of V , then we can define matroids $M[U]$ and $M[W]$ on U and W , respectively by taking $M[U] = M[V]|U$ and $M[W] = M[V]|W$, moreover, we also get by a well known theorem of linear algebra that

$$\dim(U + W) + \dim(U \cap W) = \dim U + \dim W$$

(2) Infact, for any representable matroid $M[V]$ over a vector space V , by above we get that (R3) attains equality, so that any matroid isomorphic to $M[V]$ attains equality on (R3).

(3) Let M be a matroid, and let B_X be a basis of X in M , now suppose that $I \subseteq E \setminus X$ such that $I \cup B$ is independent. Then by definition of independence via the rank, we have $\text{rank}(I \cup B) = |I \cup B| = |I| + |B|$. Now suppose $X' \subseteq X$ is also independent, then $\text{rank } X' = |X'|$. Then B is a basis of X , so $|X'| \leq |B|$, so we can take the complement of B with respect to X' . Now since B is independent, we must also have that $B \setminus X' \subseteq B$ must also be independent, so we get that $\text{rank}(B \setminus X') = |B| - |X'|$, then we have that $\text{rank}(I \cup B) - \text{rank}(B \setminus X') = |I| + |B| - |B| + |X'| = |I| + |X'| = |I \cup X'|$. Moreover, $\text{rank}((I \cup B) \setminus (B \setminus X')) = \text{rank}(I \cup X') = |I \cup X'|$. This makes $I \cup X'$ independent as well.

Lemma 1.3.2. *Let E be a set and $r : 2^E \rightarrow \mathbb{N}$ be a mapping satisfying (R1) and (R2). if X and Y are subsets of E such that $r(X \cup y) = r(X)$ for all $y \in Y \setminus X$, then $r(X \cup Y) = r(X)$.*

Proof. Let $X \setminus Y = \{y_1, \dots, y_n\}$. Then by induction on n , when $n = 1$, we have by hypothesis that $r(X \cup y_1) = r(X)$. Now suppose that $r(X \cup \{y_1, \dots, y_n\}) = r(X)$ for all $n \geq 1$, and consider the set $\{y_1, \dots, y_{n+1}\}$. Then we get that

$$\begin{aligned} r(X) + r(X) &= r(X \cup \{y_1, \dots, y_n\}) + r(X \cup y_{n+1}) \\ &\geq r(X \cup \{y_1, \dots, y_{n+1}\}) + r(X \cap \{y_1, \dots, y_{n+1}\}) \\ &= r(X \cup \{y_1, \dots, y_{n+1}\}) + r(X) \\ &\geq r(X) + r(X) \end{aligned}$$

So by equality, we get that $r(X \cup \{y_1, \dots, y_{n+1}\}) = r(X)$. □

Theorem 1.3.3. *Let E be a set and $r : 2^E \rightarrow \mathbb{N}$ be a mapping of 2^E into \mathbb{N} such that r satisfies (R1)-(R3). If $\mathcal{I} = \{X \subseteq E : r(X) = |X|\}$ Then r is the rank function of a matroid M on E*

Proof. By (R1), we have that $0 \leq r(\emptyset) \leq |\emptyset| = 0$, so $r(\emptyset) = |\emptyset| = 0$; this makes $\emptyset \in \mathcal{I}$.

Now, suppose that $I \in \mathcal{I}$ and that $I' \subseteq I$. Then $r(I) = |I|$, and $r(I' \cup (I \setminus I')) + r(I' \cap (I \setminus I')) \leq r(I') + r(I \setminus I')$, so $r(I) \leq r(I') + r(I \setminus I')$, i.e. $|I| \leq r(I') + r(I \setminus I') \leq |I'| + |I \setminus I'| = |I|$. This makes $r(I') = |I'|$, so that $I' \in \mathcal{I}$.

Now suppose that $I, I' \in \mathcal{I}$, with $|I| < |I'|$, and suppose for all $e \in I' \setminus I$ that $I \cup e \notin \mathcal{I}$, that is $r(I \cup e) \neq |I \cup e| = |I| + 1$. Now, we have that $|I| = r(I) \leq r(I \cup e) < |I| + 1$, so $r(I \cup e) = r(I) = |I|$; then by lemma 1.3.2, we have $r(I \cup I') = r(I)$, but we also have that $r(I \cup I') \geq r(I') = |I'|$ making $|I| \geq |I'|$, a contradiction, therefore $I \cup e \in \mathcal{I}$ for all $e \in I' \setminus I$. This makes M into a matroid on E .

Now suppose that $X \in \mathcal{I}$ is independent. Then $r(X) = |X|$, moreover, X is a basis for the matroid $M|_X$, so that $r(X) = \text{rank } X$. Now if X isn't independent, and B is a basis for $M|_X$, then we have $\text{rank } B = |B|$, moreover, $B \cup x \notin \mathcal{I}$ for any $x \in X \setminus B$, so that $|B| = r(B) \leq r(B \cup x) < |B \cup x|$, making $r(B \cup x) = r(B)$. This makes $r(X) = r(B) = |B| = \text{rank } X$, therefore $r = \text{rank}$, and r is the rank function on M . □

Corollary. *r is the rank function of a matroid on E if, and only if it satisfies (R1)-(R3).*

This gives the next definition of a matroid.

Definition. A **matroid** on a finite nonempty set E is a pair $M = (E, \text{rank})$ where $\text{rank} : 2^E \rightarrow \mathbb{N}$ is a mapping taking subsets of E to \mathbb{N} such that:

(R1) $0 \leq \text{rank } X \leq |X|$ for all $X \subseteq E$

(R2) If $X \subseteq Y \subseteq E$, then $\text{rank } X \leq \text{rank } Y$.

(R3) If $X, Y \subseteq E$, then $\text{rank } X \cup Y + \text{rank } X \cap Y \leq \text{rank } X + \text{rank } Y$.

We call $\text{rank } X$ the **rank** of X , and we call rank the **rank function** of the matroid M .

Theorem 1.3.4. *Let M be a matroid on a set E together with a rank function rank . Then the following are true for any $X \subseteq E$:*

(1) X is independent if, and only if $\text{rank } X = |X|$.

- (2) X is a basis if, and only if $\text{rank } X = \text{rank } M$.
- (3) X is a circuit if, and only if X is nonempty and for all $x \in X$, $\text{rank}(X \setminus x) = |X| - 1 = \text{rank } X$.

Proof. The first statement follows immediately from theorem 1.3.3. Now, suppose that X is a basis of the matroid M , then X is independent, so $\text{rank } X = |X|$, and by definition $\text{rank } M = |X|$. Conversely, if $X \subseteq E$ such that $\text{rank } M = \text{rank } X$, then by definition $\text{rank } M = |B|$ for some basis B of M , and $|B| = \text{rank } X$, this makes X an independent set, then by the first statement, $\text{rank } X = |X| = |B|$, making X a basis.

Now, suppose that X is a circuit, then by minimality, $X \setminus x$ is independent whenever $x \in X$, so that $\text{rank}(X \setminus x) = |X \setminus x| = |X| - 1 = \text{rank } X$. Conversely, suppose for X nonempty that $\text{rank}(X \setminus x) = |X| - 1 = \text{rank } X$ whenever $x \in X$. Then we have that since $\text{rank}(X \setminus x) = |X| - 1 = |X \setminus x|$, $X \setminus x$ is independent, but since $\text{rank } X = |X| - 1$, X is not independent. That makes X a minimally dependent set of E , thus X is a circuit of M . \square

Example 1.13. (1) Let E be any set, and consider the uniform matroid $U_{m,n}$ for $m, n \in \mathbb{Z}^+$. Then by definition, the rank function of $U_{m,n}$ is defined to be:

$$\text{rank } X = \begin{cases} |X|, & \text{if } |X| < m \\ m, & \text{if } |X| \geq m \end{cases}$$

In fact, any matroid is isomorphic to a uniform matroid if and only if there are no circuits of size less than $\text{rank } M + 1$. By definition, we have that if $U_{m,n}$ is the uniform matroid of rank m , then by definition the circuits are those sets C for which $|C| = m + 1$; so any matroid isomorphic to $U_{m,n}$ must also satisfy this. Conversely, if M is a matroid of rank m , with all circuits C of size $m + 1$, then by minimality of C , $I = C \setminus c$ is maximally independent for appending any element makes it of size $m + 1$, which necessarily makes it a circuit. Thus I is a base, so that $\text{rank } I = m$, since C is arbitrary, so is I , this then makes M isomorphic to $U_{m,n}$.

It happens for $U_{1,n}$, (R3) does not attain equality. Notice any basis B has $\text{rank } B = 1$, and any circuit C has $\text{rank } C = 1$, so $\text{rank } B + \text{rank } C = 2$. However, if B and C are disjoint, then $\text{rank}(B \cup C) = 1$ and $\text{rank}(B \cap C) = 0$, so that $\text{rank}(B \cup C) + \text{rank}(B \cap C) = 1 < 2$.

- (2) If A is an $m \times n$ matrix over a field F , then the rank function of $M[A]$ is the rank of the $m \times |X|$ submatrix $A|X$ consisting of columns of A labeled by members of X . Moreover, the rank of X is the dimension of the subspace of the vector space over F spanned by the columns of $A|X$.
- (3) Let G be a graph and $M(G)$ be the graphic matroid on G . If G is connected, then a basis of G is a spanning tree T of G , for which $|V(T)| = |E(T)| + 1$, so the rank of $M(G)$ in this case is:

$$\text{rank } M(G) = |V(G)| - 1$$

Now, if G has ω connected components, then by similar reasoning, the rank of $M(G)$ is:

$$\text{rank } M(G) = |V(G)| - \omega$$

Now, if X is a subset of edges of G , then

$$\text{rank } X = |V(X)| - \omega_X$$

where ω_X is the number of connected components of X .

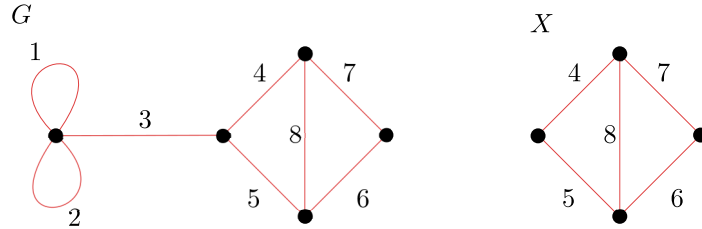


Figure 1.6:

- (4) Let G be the graph in figure 1.6, then G is a connected graph so $\text{rank } M(G) = |V(G)| - 1 = 4$. Now, if $X = \{4, 5, 6, 7, 8\}$, then $M(X) = M(G)|_X$ has as a basis the set $\{4, 7, 8\}$, so that $\text{rank } X = 3$. If $Z = \{1, 2\}$, then $M(Z)$ only has \emptyset as a basis, so that $\text{rank } Z = 0$.
- (5) In any matroid M , we have an element x is a loop if, and only if $\text{rank } \{x\} = 0$, and two elements x and y are parallel if and only if $\text{rank } \{x, y\} = 1$.

Certainly, if x is a loop, then $\{x\}$ is a circuit, so that the only basis of $M|\{x\}$ is \emptyset , this makes $\text{rank } \{x\} = 0$. Likewise, if x and y are parallel, then $\{x, y\}$ is a circuit, making $\{x\}$ and $\{y\}$ maximally independent, and hence bases of $M|\{x, y\}$, thus $\text{rank } \{x, y\} = 1$.

Conversely, if $\text{rank } \{x\} = 0$, then $\{x\}$ only has \emptyset as a basis, hence $\{x\}$ is minimally dependent, hence a circuit, hence it is a loop. By similar reasoning for parallel elements, if $\text{rank } \{x, y\} = 1$, then $x \parallel y$.

Lemma 1.3.5. *Let M be a matroid and $k \in \mathbb{N}$ such that $k \leq \text{rank } M$. Define the map $r_{(k)} : 2^E \rightarrow \mathbb{N}$ by $r_{(k)}(X) = \min\{k, \text{rank } X\}$. Then $r_{(k)}$ is the rank function for a matroid.*

Proof. Certainly by definition, we have $0 \leq r_{(k)}(X) \leq |X|$. Moreover, if $X \subseteq Y$, then $\min\{k, \text{rank } X\} \leq \min\{k, \text{rank } Y\}$, so we get $r_{(k)}(X) \leq r_{(k)}(Y)$.

Now, if X and Y , then we get $r_{(k)}(X \cup Y) = \min\{k, \text{rank}(X \cup Y)\}$, and $r_{(k)}(X \cap Y) = \min\{k, \text{rank}(X \cap Y)\}$, then $r_{(k)}(X \cup Y) + r_{(k)}(X \cap Y) = \min\{k, \text{rank}(X \cup Y) + \text{rank } X \cap Y\} \leq \min\{k, \text{rank } X + \text{rank } Y\} = r_{(k)}(X) + r_{(k)}(Y)$. This makes $r_{(k)}$ the rank function of the matroid. \square

Corollary. $r_{(k)}(M) = k$

Proof. We have that $k \leq M$, so $r_{(k)}(M) = \min\{k, \text{rank } M\} = k$ in all cases. \square

Remark. We call this matroid obtained through $r_{(k)}$ a matroid obtained by a sequence of “truncations”. The collection of independent sets on these matroids are the collection of all independent sets of M of size less than or equal to k . That is:

$$\mathcal{I} = \{I \subseteq M : |I| \leq k\}$$

Definition. We call a matroid **paving** if it has no circuits of size less than rank M .

Example 1.14. For any $m, n \in \mathbb{Z}^+$, the uniform matroid $U_{m,n}$ only has circuits of size $m + 1$, thus $U_{m,n}$ is paving for all $m, n \in \mathbb{Z}^+$. In general, a matroid has circuits of size at most rank $M + 1$, so that M is uniform if, and only if it has no circuits of size less than rank $M + 1$. So a useful check for determining whether a matroid is paving is to show that it is uniform. However, not every paving matroid is uniform. For example, the matroid $M(K_5)$ on the complete graph on 5 vertices is paving, but not uniform.

Theorem 1.3.6. *Let \mathcal{D} be a collection of nonempty subsets of a set E . Then \mathcal{D} is the collection of circuits of a paving matroid on E if, and only if there is a $k \in \mathbb{Z}^+$ with $k < |E|$, and a subcollection $\mathcal{D}' \subseteq \mathcal{D}$ such that:*

- (1) Every $D' \in \mathcal{D}'$ has k elements and if $D', D'' \in \mathcal{D}'$ are distinct, then $\binom{D' \cup D''}{k} \in \mathcal{D}'$.
- (2) $\mathcal{D} \setminus \mathcal{D}'$ consists of all $k + 1$ -element subsets containing no member of \mathcal{D}' .

Lemma 1.3.7 (Brylawski). *Let M be a matroid on a set E , and let $k \in \mathbb{N}$ such that $\text{rank } M \leq k \leq |E|$. Then if S_k is the collection of all k -element subsets of E with rank $\text{rank } M$, then S_k is the collection of basis of a matroid on E .*

Proof. Notice by definition that if $S, S' \in S_k$, then $|S| = |S'|$. Now, if S_k is empty that means no k -element subset of E has rank $\text{rank } M$, then if $k = |E|$, $\text{rank } E \neq \text{rank } M$, which cannot happen. So we have that S_k is nonempty.

Now let $S, S' \in S_k$, then $\text{rank } S = \text{rank } S' = \text{rank } M$. Now choose $s \in S \setminus S'$, then $S \setminus s$ is a $k - 1$ -element set. Taking $s' \in S' \setminus S$, we have $(S \setminus s) \cup s'$ is a k -element subset of E , hence $|(S \setminus s) \cup s'| \leq |E|$ so we get $\text{rank } (S \setminus s) \cup s' \leq \text{rank } E = \text{rank } M$, moreover, we have that $\text{rank } S \leq \text{rank } ((S \setminus s) \cup s')$ by the corollary to lemma 1.3.1, so we get that $\text{rank } (S \setminus s) \cup s' = \text{rank } S = \text{rank } M$, which makes $(S \setminus s) \cup s' \in S_k$. This makes S_k the collection of bases of a matroid on E . \square

Definition. Let M be a matroid on E , we define the **elongation** of M to rank k , where $\text{rank } M \leq k \leq |E|$ to be the matroid on E having collection of bases S_k the collection of all k -element subsets of E of rank M . If $k = \text{rank } M + 1$, we call the elongation of M the **Higgs lift** of M .

Lemma 1.3.8 (Edmonds and Rota). *Let $f : 2^E \rightarrow \mathbb{N}$ be a mapping of subsets of a set E satisfying the following:*

- (1) $f(\emptyset) = 0$.
- (2) If $X \subseteq Y$, then $f(X) \leq f(Y)$.
- (3) $f(X \cup Y) + f(X \cap Y) \leq f(X) + f(Y)$.

Then f defines a matroid M on E having the collection of independent sets $\mathcal{I}(f) = \{X \subseteq E : f(X) \geq |X|\}$.

Proof. By definition, we have $f(\emptyset) = 0 = |\emptyset|$, so $\emptyset \in \mathcal{I}(f)$. Now, suppose that $Y \in \mathcal{I}(f)$, then $f(Y) \geq |Y|$, now if $X \subseteq Y$, we either have $|X| \leq f(X) \leq |Y| \leq f(Y)$, or $|X| \leq |Y| \leq f(X) \leq f(Y)$; either case inheretence is satisfied, and $X \in \mathcal{I}(f)$.

Now suppose that $X, Y \in \mathcal{I}(f)$, and that $|X| < |Y|$. Now, let $y \in Y \setminus X$, and suppose that $f(X \cup y) < |X \cup y| = |X| + 1$, then we have $f(X \cup y) \leq f(X) + f(y) - f(X \cap y) = f(X) + f(y) \leq |X| + 1 \leq f(X) + f(y)$, which makes $f(X) = |X|$ and $f(y) = 1$, thus we get $f(X \cup y) = f(X) + f(y) = |X| + 1$, which contradicts the assumption. Thus we must have that $|X \cup e| \leq f(X \cup e)$. This makes $X \cup e \in \mathcal{I}(f)$, and makes $\mathcal{I}(f)$ the collection of independent sets of a matroid on E . \square

Corollary. *The rank function of the matroid $M = (E, \mathcal{I}(f))$ is the map $r : 2^E \rightarrow \mathbb{N}$ defined by:*

$$r(X) = \begin{cases} |X|, & \text{if } f(X) = |X| \\ |B|, & \text{for some } B \subseteq X \text{ such that } f(B) = |B| \end{cases}$$

Proof. Let $X \in \mathcal{I}(f)$ be independent. Then we see that $f(X) \geq |X|$. Then X is dependent if $f(X) < |X|$. Thus, X is a basis if, and only if $f(X) = |X|$. \square

Remark. We call mappings satisfying properties (1)-(3) of lemma 1.3.8 **submodular functions**, and we will study them and their matroids later on. It is no surprise that the rank function is also a submodular function.

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