Algebraic Geometry.

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Contents

Chapter 1

Varieties.

1.1 Affine Varieties.

Definition. Let K be an algebraically closed set. We define **affine n-space**, $\mathbb{A}^n(K)$ over K to be the set of all n-tuples of elements of K. We call elements $P = (a_1, \ldots, a_n) \in \mathbb{A}^n(K)$ **points** and we call each of the a_i **coordinates** of P. When context is clear, we simply write $\mathbb{A}^n(K)$ as $\mathbb{A}^n$.

Definition. Let $\mathbb{A}^n(K)$ be the affine space over an algebraically closed field K. Let $f \in K[x_1, \ldots, x_n]$ and define $f(P) = f(a_1, \ldots, a_n)$. We define the set of **zeros** of f to be:

$$Z(f) = \{P \in \mathbb{A}^n(K) : f(P) = 0\}$$

For any $T \subseteq K[x_1, \ldots, x_n]$, the **zero** set of T is defined to be

$$Z(T) = \{P \in \mathbb{A}^n(K) : f(P) = 0, \text{ for all } f \in T\}$$

If $T = (a)$ the ideal of $K[x_1, \ldots, x_n]$ generated by T, then we simply write $Z(T) = Z(a)$.

Definition. We call a subset $Y \subseteq \mathbb{A}^n(K)$, of the affine space over K **algebraic** (or an **algebraic set**), if $Y = Z(T)$ for some $T \subseteq K[x_1, \ldots, x_n]$.

Lemma 1.1.1. *The collection of all algebraic sets of an affine space $\mathbb{A}^n$ forms a topology under closed sets.*

Proof. Let $\mathbb{A}^n = Z(0)$ and $\emptyset = Z(1)$. Then $\mathbb{A}^n$ and $\emptyset$ are both algebraic. Now, let X and Y be algebraic, then there are S, T such that $X = Z(S)$ and $Y = Z(T)$. Now, let $P \in X \cup Y$, then P is a zero of any polynomial $f \in ST$, conversly, suppose that $P \in Z(ST)$ where $P \notin Y$. There exists a polynomial $f \in S$ with $f(P) \neq 0$. Now, for any $g \in T$, we have that if $fg(P) = 0$, then $g(P) = 0$, so that $P \in S$. Therefore we have $X \cup Y = Z(ST)$, making $X \cup Y$ algebraic. So that the collection of algebraic sets is closed under finite intersection.

Lastly, consider a collection $\{Y_\alpha\}$ of algebraic sets, where $Y_\alpha = Z(T_\alpha)$ for some T_α. Let

$$Y = \bigcap Y_\alpha \text{ and } T = \bigcup T_\alpha$$

and let $P \in Y$. Then P is in every Y_α making it a root of some $f_\alpha \in T_\alpha$, thus $P \in Z(T)$. Similarly, if $P \in Z(T)$, then $P \in Y$, making $Y = Z(T)$, and making the collection of algebraic sets closed under arbitrary intersections. ∎

Definition. We define the **Zariski topology** on $\mathbb{A}^n$ to be the topology taking as open sets complements of algebraic sets.

Bibliography

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