Complex Analysis

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Text

Complex Analysis (4^{th} edition)

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Contents

Chapter 1

Complex Numbers and Funtions

1.1 Complex Numbers

1.2 Complex Valued Functions

Definition. We define a **complex valued function** to be a function $f : S \to \mathbb{C}$, where $S \subseteq \mathbb{C}$. Writing $f(z) = f(x+iy) = u(x,y) + iv(x,y)$, where $u : U_1 \times U_1 \to \mathbb{R}$ and $v : V_1 \times V_2 \to \mathbb{R}$ are real valued functions (with U_1, U_2, V_1, V_2 open in $\mathbb{R}$), we define the **real part** of f to be $\operatorname{Re} f = u(x,y)$, and the **imaginary part** of f to be $\operatorname{Im} f = v(x,y)$.

Remark. It should be noted that the domain of a complex valued function f depends on the domain of its real and imaginary parts, and vice versa.

Example 1. (1) The real and imaginary parts of the complex valued function $f(z) = x^3y + i\sin(x+y)$ to be $u(x,y) = x^3y$ and $v(x,y) = \sin(x+y)$, respectively.

(2) Consider the complex valued function $f(z) = z^n$, for $n \in \mathbb{Z}^+$. Writing $z = re^{i\theta}$, we get $f(z) = r^n \cos n\theta + ir^n \sin n\theta$. The real part of f is then $u(x,y) = r^n \cos n\theta$, and the imaginary part of f to be $v(x,y) = r^n \sin n\theta$.

Lettinh $\overline{B^1} = \{z \in \mathbb{C} : |z| \leq 1\}$ the closed unit ballm notice if $z \in \overline{B^1}$, then $|z^n| = |z|^n \leq 1^n = 1$, so that $z^n \in \overline{B^1}$, and hence $f(\overline{B^1}) = \overline{B^1}$.

Definition. We call the solutions to the polynomial $z^n - 1$ over $\mathbb{C}$ the complex n**-th roots of unity**.

Theorem 1.2.1. *Let ξ be a complex n-th root of unity. Then $\xi = e^{\frac{2i\pi}{n}}$.*

Corollory. *If ξ is an n-th root of unity, then so is ξ^k for all $k \in {}^{\mathbb{Z}}\!/_{n\mathbb{Z}}$.*

1.3 Complex Differentiation and Holomorphic Functions

Definition. Let U be an open set of $\mathbb{C}$, and let $w \in U$. We call a complex valued function $f : U \to \mathbb{C}$ **complex differentiable** at w if the limit

$$f'(w) = \lim_{h \longrightarrow 0} \frac{f(w+h) - f(w)}{h} = \lim_{z \longrightarrow w} \frac{f(z) - f(w)}{z - w}$$

exists. We call $f'(w)$ the **complex derivative** of f at w.

Theorem 1.3.1. *Let $f : U \to \mathbb{C}$ and $g : U \to \mathbb{C}$ be complex valued functions. If f and g are complex differentiable at a point $z \in U$, then following are true*

(1) $f + g$ is complex differentiable at z, with

$$(f+g)'(z) = f'(z) + g'(z)$$

(2) $(fg)'$ is complex differentiable at z, with

$$(fg)'(z) = f'(z)g(z) + f(z)g'(z)$$

Corollory. *The function $\frac{f}{g}$ is complex differentiable at z, provided $g(z) \neq 0$, with*

$$\left(\frac{f}{g}\right)'(z) = \frac{g(z)f'(z) - f(z) - g'(z)}{(g(z))^2}$$

Example 2. For all $n \in \mathbb{Z}^+$, the function $f(z) = z^n$ is complex differentiable on all of $\mathbb{C}$, with $f'(z) = nz^{n-1}$. In fact, z^n is what we call a "holomorphic" function.

Theorem 1.3.2 (The Chain Rule)**.** *Let U and V be open sets of $\mathbb{C}$, and let $f : U \to \mathbb{C}$, and $g : V \to \mathbb{C}$ be complex valued functions, with $f(U) \subseteq V$. If f is complex differentiable at a point $z \in Z$, and g is complex differentiable at the point $f(z) \in f(U)$, then $g \circ f$ is complex differentiable at z with*

$$(g \circ f)'(z) = (g' \circ f)(z)f'(z) = g'(f(z))f'(z)$$

Definition. We call a complex valued function $f : U \to \mathbb{C}$ **holomorphic** on U if it is complex differentiable at every point of U.

Remark. It is convention to simply say that f is "holomorphic" when it is holomorphic on all of $\mathbb{C}$.

Definition. Let $f : U \to \mathbb{C}$ a complex valued function with $f(z) = u(x, y) + iv(x, y)$. We define the **vector field** of f to be the map $F : U \to V \to \mathbb{R} \times \mathbb{R}$ defined by

$$F(x, y) = (u(x, y), v(x, y))$$

Where U and V are open in $\mathbb{R}$.

Theorem 1.3.3. *If f is holomorphic on its domain, then F is real differentiable on its domain (resepctively to the domain of f) and has derivative*

$$\operatorname{Jac} F = \begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{pmatrix}$$

Where $\operatorname{Jac} F$ is the Jacobian of F.

Corollory. $f'(z) = \frac{\partial u}{\partial x} - i\frac{\partial v}{\partial y}$*, and the we have the following of equations*

$$\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} = 0$$
$$\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = 0$$

Theorem 1.3.4. *If $u : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, and $v : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are continuously real differentiable realvalued functions satisfying the equations*

$$\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} = 0$$
$$\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = 0$$

Then the function $f(z)u(x, y) + iv(x, y)$ is holomorphic on its domain.

Definition. Let $u : U_1 \times U_2 \to \mathbb{R}$ and $v : V_1 \times V_2 \to \mathbb{R}$ be continuously real differentiable real valued functions. We define the **Cauchy-Riemann equations** to be the set of equations

$$\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} = 0$$
$$\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = 0$$

Chapter 2

Power Series

2.1 Formal Power Series

Bibliography

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