

# Algebraic Topology

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24.09.2022



# Chapter 1

## Categories.

### 1.1 Categories and Subcategories.

**Definition.** A **category**  $\mathcal{C}$  is a collection of a class of **objects**, denoted  $\text{obj } \mathcal{C}$  a collection of sets of **morphisms**  $\text{Hom}(A, B)$  for each  $A, B \in \text{obj } \mathcal{C}$  and a binary operation  $\circ : \text{Hom}(A, B) \times \text{Hom}(B, C) \rightarrow \text{Hom}(A, C)$ , defined by  $(f, g) \rightarrow g \circ f$ , called **composition** such that:

- (1) Each  $\text{Hom}(A, B)$  is pairwise disjoint for all  $A, B \in \text{obj } \mathcal{C}$ .
- (2)  $\circ$  is associative when defined; that is if either  $(g \circ f) \circ h$  or  $g \circ (f \circ h)$  are defined, then  $(g \circ f) \circ h = g \circ (f \circ h)$ , for morphisms  $f, g, h$ .
- (3) For each  $A \in \text{obj } \mathcal{C}$ , there exists an **identity** morphism  $1_A \in \text{Hom}(A, A)$  such that for each  $B, C \in \text{obj } \mathcal{C}$ ,  $1_A \circ f = f$  and  $g \circ 1_A = g$  for each morphism  $f \in \text{Hom}(B, A)$  and  $g \in \text{Hom}(A, C)$ .

We denote morphisms by  $f : A \rightarrow B$  instead of  $f \in (A, B)$ .

**Definition.** Let  $\mathcal{C}$  be a category and  $f : A \rightarrow B$  a morphism in  $\mathcal{C}$ . We call  $A$  and  $B$  the **domain** and **codomain** of  $f$ , respectively, and we call the set  $G_f = \{(a, f(a)) : a \in A\} \subseteq B$  the **graph** of  $f$ .

**Example 1.1.** (1) The category of all sets  $\text{Set}$  has as objects the class of all sets. The morphisms in  $\text{Set}$  are all functions  $f : A \rightarrow B$  where  $A$  and  $B$  are sets. The composition of  $\text{Set}$  is the usual composition of functions.

- (2) The category of all topological spaces  $\text{Top}$  has as objects all topological spaces, and as morphisms all continuous maps  $f : X \rightarrow Y$  from a space  $X$  to a space  $Y$ . The composition is the usual composition.
- (3) The category of all groups,  $\text{Grp}$  has as objects all groups and as morphisms all homomorphisms  $f : G \rightarrow H$ , under the usual composition.
- (4) The category of rings with unit  $\text{Rng}$  has as objects all rings with unit, along with all ring homomorphisms  $f : R \rightarrow K$  to be the morphisms under the usual composition.

**Definition.** We call a category a **subcategory** of a category  $\mathcal{C}$  if  $\text{obj } \mathcal{A} \subseteq \text{obj } \mathcal{C}$ ,  $\text{Hom}_{\mathcal{A}}(A, B) \subseteq \text{Hom}_{\mathcal{C}}(A, B)$  for all  $A, B \in \mathcal{C}$ , and  $\mathcal{A}$  inherits the composition of  $\mathcal{C}$ .

**Example 1.2.** (1) The category of all pairs of topological spaces  $(X, A)$  where  $A$  is a subspace of  $X$ , whose morphisms are pairs of continuous maps  $f = (f_1, f_2)$  such that  $f_1 i = j f_2$  where  $i : A \rightarrow X$  and  $j : B \rightarrow Y$  are inclusions, is a subcategory of  $\text{Top}$ . We denote this category  $\text{Top}^2$ .

$$\begin{array}{ccc}
 A & \xrightarrow{i} & X \\
 f_2 \downarrow & & \downarrow f_1 \\
 B & \xrightarrow{j} & Y
 \end{array}$$

- (2) The category of all **pointed spaces**,  $\text{Top}^*$  is defined with the objects being all pairs  $(X, \{x_0\})$ , where  $x_0 \in X$  with the morphisms of  $\text{Top}^2$ .  $\text{Top}^*$  is a subcategory of  $\text{Top}^2$ . We call  $x_0$  the **base point**, and we call the morphisms of  $\text{Top}^*$  **pointed maps**.
- (2) The category of all Abelian groups,  $\text{Ab}$  is a subcategory of  $\text{Grp}$ . Likewise, the category of all commutative rings with unit is a subcategory of  $\text{Rng}$ .

## 1.2 Commutative Diagrams and Congruences.

**Definition.** A **diagram** in a category is a directed graph with its vertex set a subset of objects, and whose edge set are morphisms between those objects. We call a diagram **commutative** if for any pair of objects, every pair of morphisms between those objects are equal. That is if  $A, A'$  and  $B, B'$  are pairs of objects with pairs of morphisms  $f : A \rightarrow B$ ,  $f' : A' \rightarrow B'$  and  $g : A \rightarrow A'$ ,  $g' : B \rightarrow B'$  we have that  $g' \circ f = f' \circ g$

$$\begin{array}{ccc}
 A & \xrightarrow{g} & A' \\
 f \downarrow & & \downarrow f' \\
 B & \xrightarrow{g'} & B'
 \end{array}$$

**Definition.** A **congruence** on a category  $\mathcal{C}$  is an equivalence relation  $\sim$  on morphisms in  $\mathcal{C}$  such that:

- (1) If  $f \in \text{Hom}(A, B)$ , and  $f \sim f'$ , then  $f' \in \text{Hom}(A, B)$ .
- (2) If  $f \sim g$  and  $f' \sim g'$ , then  $g \circ f \sim g' \circ f'$ .

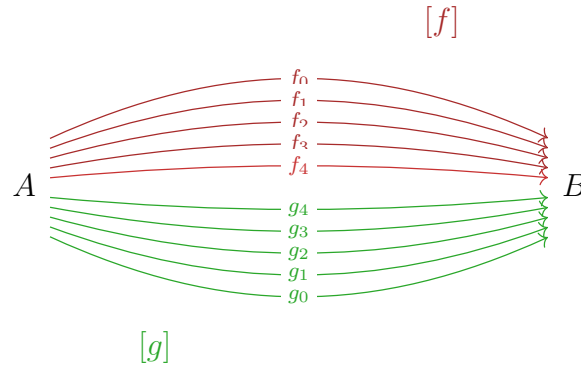


Figure 1.1: An equivalence relation between morphisms.

**Theorem 1.2.1.** Let  $\mathcal{C}$  be a category with congruence  $\sim$ . Define  $\mathcal{C}/\sim$  as follows:

- (1)  $\text{obj } \mathcal{C}/\sim = \text{obj } \mathcal{C}$ .
- (2)  $\text{Hom}_{\mathcal{C}/\sim}(A, B) = \{[f] : f \in \text{Hom}_{\mathcal{C}}(A, B)\}$ .
- (3)  $[g] \circ [f] = [g \circ f]$

Then  $\mathcal{C}/\sim$  is a category.

*Proof.* We have by equivalence that  $\text{obj } \mathcal{C}/\sim$  is a class. Moreover, since  $\sim$  partitions  $\mathcal{C}$ , it partitions all of the  $\text{Hom}(A, B)$  for each  $A, B$ . So each  $\text{Hom}(A, B)$  is a set, moreover, they are pairwise disjoint by definition of  $\sim$ . Now, notice that by hypothesis, composition in  $\mathcal{C}/\sim$  is well defined, so  $[1_A] \circ [f] = [1_A \circ f] = [f]$  and  $[g] \circ [1_A] = [g \circ 1_A] = [g]$ . This makes  $\mathcal{C}/\sim$  a category. ■

*Remark.* One can think of the category  $\mathcal{C}/\sim$  as taking all morphisms with the same domain and codomain, and collapsing them into a single morphism.

**Definition.** Let  $\mathcal{C}$  be a category and  $\sim$  a congruence of  $\mathcal{C}$ . We call the category  $\mathcal{C}/\sim$  induced by  $\sim$  the **quotient category**.

## 1.3 Functors.

**Definition.** Let  $\mathcal{A}$  and  $\mathcal{C}$  be categories. We define a **covariant functor** to be a map  $F : \mathcal{A} \rightarrow \mathcal{C}$  such that:

- (1)  $A \in \text{obj } \mathcal{A}$  implies  $F(A) \in \text{obj } \mathcal{C}$ .
- (2) If  $f : A \rightarrow B$  is a morphism in  $\mathcal{A}$ , then  $F(f) : F(A) \rightarrow F(B)$  is a morphism in  $\mathcal{C}$ .

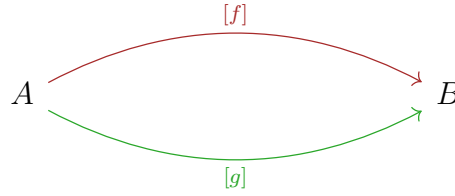


Figure 1.2: Morphisms in a category with the same domain and codomain get collapsed onto a single morphism in the corresponding quotient category.

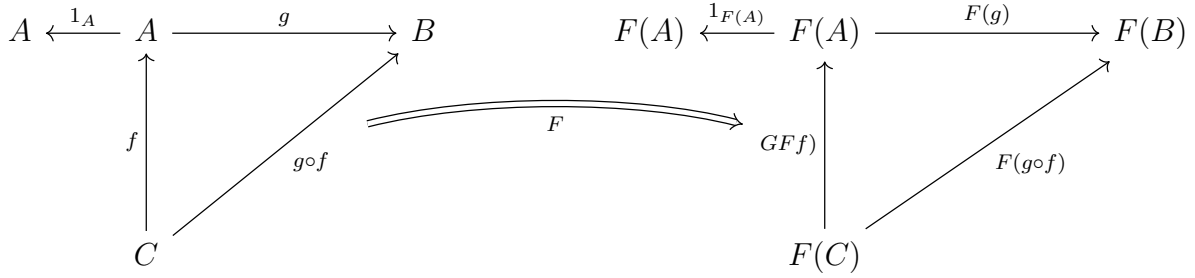


Figure 1.3: A covariant functor taking a diagram in one category to a diagram in the other.

- (3) For all morphisms  $f$  and  $g$  in  $\mathcal{A}$ , for which  $g \circ f$  is defined, we have that  $F(g \circ f) = F(g) \circ F(f)$ , and  $F(1_A) = 1_{F(A)}$ .

**Example 1.3.** (1) We define the **forgetful functor** the map  $F : \mathcal{C} \rightarrow \text{Set}$  that takes all objects in  $\mathcal{C}$  to their underlying sets, and morphisms in  $\mathcal{C}$  to themselves considered as functions under the usual composition. For example the forgetful functor  $F : \text{Top} \rightarrow \text{Set}$  takes topological spaces  $X$  to their underlying sets, and continuous maps to themselves, considered just as functions.

- (2) The **identity functor** is the functor  $I : \mathcal{C} \rightarrow \mathcal{C}$  that takes objects and morphisms in  $\mathcal{C}$  to themselves.
- (3) Let  $M$  be a topological space. Define  $F_M : \text{Top} \rightarrow \text{Top}$  by  $F_M : X \rightarrow X \times M$ , and for each continuous map  $f : X \rightarrow Y$ ,  $F(f) : X \times M \rightarrow Y \times M$  is defined by  $(x, m) \rightarrow (f(x), m)$ . Then  $F_M$  is a functor.
- (4) Let  $A \in \text{obj } \mathcal{C}$  and take the map  $\text{Hom}(A, *) : \mathcal{C} \rightarrow \text{Set}$  that takes  $A \rightarrow \text{Hom}(A, B)$  and for each morphism  $f : B \rightarrow B'$ ,  $\text{Hom}(A, f) : \text{Hom}(A, B) \rightarrow \text{Hom}(A, B')$  is given by  $g \rightarrow f \circ g$ . We call this functor the **covariant Hom functor**, and denote it  $f_*$ .

**Definition.** Let  $\mathcal{A}$  and  $\mathcal{C}$  be categories. We define a **contravariant functor** to be a map  $G : \mathcal{A} \rightarrow \mathcal{C}$  such that:

- (1)  $A \in \text{obj } \mathcal{A}$  implies  $G(A) \in \text{obj } \mathcal{C}$ .

- (2) If  $f : A \rightarrow B$  is a morphism in  $\mathcal{A}$ , then  $G(f) : G(B) \rightarrow G(A)$  is a morphism in  $\mathcal{C}$ .
- (3) For all morphisms  $f$  and  $g$  in  $\mathcal{A}$ , for which  $g \circ f$  is defined, we have that  $G(g \circ f) = G(f) \circ G(g)$ , and  $G(1_A) = 1_{G(A)}$ .

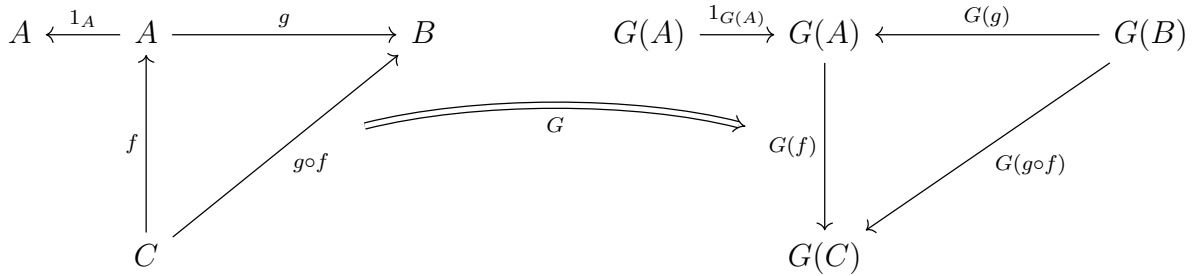


Figure 1.4: A contravariant functor taking a diagram in one category to a diagram in the other.

**Example 1.4.** (1) Let  $F$  be a field, and  $\text{Vec}$  the category of all finite dimensional vector spaces over  $F$ , whose morphisms are linear transformations. Define the map  $T : \text{Vec} \rightarrow \text{Vec}$  by taking  $T : V \rightarrow V^\perp$ , and  $T : f \rightarrow f^T$ . That is  $T$  takes vector spaces to their dual spaces, and linear transformation to their transpose.  $T$  is a contravariant functor called the **dual space functor**.

- (2) Define  $\text{Hom}(*, B) : \mathcal{C} \rightarrow \mathcal{C}$  by taking  $\text{Hom}(*, B) : A \rightarrow \text{Hom}(A, B)$  and for each morphism  $g : A \rightarrow A'$  in  $\mathcal{C}$ ,  $\text{Hom}(g, B) : \text{Hom}(A', B) \rightarrow \text{Hom}(A, B)$  is defined by taking  $h \rightarrow h \circ g$ . This is analogous to the covariant  $\text{Hom}$  functor, and we call it the **contravariant Hom functor**.

**Definition.** We call a morphism  $f : A \rightarrow B$  an **equivalence** if there exists a morphism  $g : B \rightarrow A$  such that  $f \circ g = 1_B$  and  $g \circ f = 1_A$ .

**Theorem 1.3.1.** Let  $\mathcal{A}$  and  $\mathcal{C}$  be categories, and  $F : \mathcal{A} \rightarrow \mathcal{C}$  be a functor. If  $f$  is an equivalence in  $\mathcal{A}$ , then  $F(f)$  is an equivalence in  $\mathcal{C}$ .

*Proof.* Suppose that  $F$  is a covariant functor. Notice that if  $f : A \rightarrow B$  is an equivalence, then there is a  $g : B \rightarrow A$  with  $f \circ g = 1_B$  and  $g \circ f = 1_A$ . Then  $F(f \circ g) = F(f) \circ F(g) = F(1_B) = 1_{F(B)}$ , and  $F(g \circ f) = F(g) \circ F(f) = F(1_A) = 1_{F(A)}$ .

Likewise, if  $F$  is contravariant, notice that  $F(f) : B \rightarrow A$  and  $F(g) : A \rightarrow B$ . Then  $F(f \circ g) = F(g) \circ F(f) = 1_{F(A)}$ , and  $F(g \circ f) = F(f) \circ F(g) = 1_{F(B)}$ . In either case, we find that  $F(f)$  is an equivalence in  $\mathcal{C}$ . ■





# Chapter 2

## Homotopy, Convexity, and Connectedness.

### 2.1 Homotopy

**Definition.** If  $X$  and  $Y$  are topological spaces, and  $f_0 : X \rightarrow Y$  and  $f_1 : X \rightarrow Y$  are continuous maps, we say that  $f_0$  is **homotopic** to  $f_1$  if there exists a continuous map  $F : X \times I \rightarrow Y$  with  $F(x, 0) = f_0(x)$  and  $F(x, 1) = f_1(x)$ . We write  $f_0 \simeq f_1$  and call  $F$  a **homotopy**. We also write  $F : f_0 \simeq f_1$  to denote a homotopy between  $f_0$  and  $f_1$ .

**Lemma 2.1.1** (The Pasting Lemma). *Let  $X$  is a topological space that is covered by open sets  $\{X_n\}$ . If  $Y$  is some topological space for which there exist unique maps  $f_n : X_n \rightarrow Y$  that coincide in the intersections of their domains, then there exists a unique map  $f : X \rightarrow Y$  such that  $f|_{X_n} = f_n$ , for all  $n$ .*

**Lemma 2.1.2.** *Homotopy between continuous maps is an equivalence relation.*

*Proof.* Let  $f : X \rightarrow Y$  be a continuous map. Define  $F : X \times I \rightarrow Y$  by  $(x, t) \rightarrow f(x)$  for all  $(x, t) \in X \times I$ . Then  $F$  is continuous by definition; moreover,  $F(x, 0) = F(x, 1) = f(x)$ , making  $f \simeq f$ .

Now suppose there exist a homotopy  $F : f \simeq g$  for maps  $f : X \rightarrow Y$  and  $g : X \rightarrow Y$ . Define the map  $G : X \times I \rightarrow Y$  by  $(x, t) \rightarrow F(x, 1 - t)$ .  $G$  is the composition of continuous maps, so  $G$  is continuous, moreover,  $G(x, 0) = F(x, 1) = g(x)$  and  $G(x, 1) = F(x, 0) = f(x)$ , so that  $g \simeq f$ .

Lastly, suppose that  $F : f \simeq g$  and  $G : g \simeq h$  for maps  $f, g, h$ . Define the map  $H : X \times I \rightarrow Y$  by:

$$H(x, t) = \begin{cases} F(x, 2t), & \text{if } 0 \leq t \leq \frac{1}{2} \\ G(x, 2t - 1), & \text{if } \frac{1}{2} \leq t \leq 1 \end{cases}$$

Notice that  $F$  and  $G$  coincide in their domains which cover  $X$ . Therefore, by the pasting lemma,  $H$  is continuous. Now notice also that  $H(x, 0) = F(x, 2 \cdot 0) = F(x, 0) = f(x)$  and  $H(x, 1) = G(x, 2 \cdot 1 - 1) = G(x, 1) = h(x)$ . This makes  $f \simeq h$ . ■

**Definition.** For any continuous map  $f : X \rightarrow Y$  we define the **homotopy class** of  $f$  to be the equivalence class of all continuous maps homotopic to  $f$ . That is:

$$[f] = \{g : X \rightarrow Y : g \text{ is continuous and } g \simeq f\}$$

**Lemma 2.1.3.** Let  $f_0 : X \rightarrow Y$ ,  $f_1 : X \rightarrow Y$  and  $g_0 : X \rightarrow Y$ ,  $g_1 : X \rightarrow Y$  be continuous maps. If  $f_0 \simeq f_1$  and  $g_0 \simeq g_1$ , then  $g_0 \circ f_0 \simeq g_1 \circ f_1$ . That is  $[g_0 \circ f_0] = [g_1 \circ f_1]$ .

*Proof.* Let  $F : f_0 \simeq f_1$  and  $G : g_0 \simeq g_1$  be the homotopies of  $f_0$  into  $f_1$  and  $g_0$  into  $g_1$ , respectively. Define the map  $H : X \times I \rightarrow Y$  by taking  $(x, t) \rightarrow G(f_0(x), t)$ . Then we have that  $H$  is continuous by composition, and that  $H(x, 0) = G(f_0(x), 0) = g_0(f_0(x))$ , and  $H(x, 1) = G(f_0(x), 1) = g_1(f_0(x))$ . Thus we see that  $g_0 \circ f_0 \simeq g_1 \circ f_0$ .

Now define the map  $K : X \times I \rightarrow Y$  by  $K = g_1 \circ F$ . We have that  $K$  is continuous by composition, and that  $K(x, 0) = g_1 \circ f_0$  and  $K(x, 1) = g_1 \circ f_1$ , making  $g_1 \circ f_0 \simeq g_1 \circ f_1$ . Therefore, by transitivity of homotopy,  $g_0 \circ f_0 \simeq g_1 \circ f_1$ . ■

**Theorem 2.1.4.** Homotopy is a congruence on the category Top.

*Proof.* The proof follows by lemmas 2.1.2 and 2.1.3. ■

**Definition.** We call the quotient category of Top induced by homotopy the **homotopy category** and denote it hTop.

**Definition.** A continuous map  $f : X \rightarrow Y$  is a **homotopy equivalence** if there exists a continuous map  $g : Y \rightarrow X$  such that  $f \circ g \simeq 1_Y$  and  $g \circ f \simeq 1_X$ . We say that the spaces  $X$  and  $Y$  have the same **homotopy type** if there exists a homotopy equivalence.

**Definition.** We call a continuous map **nullhomotopic** if it is homotopic to a constant map.

**Example 2.1.** The space of complex numbers  $\mathbb{C}$  and the unit circle  $S^1$  have the same homotopy type.

**Definition.** Let  $Y$  and  $Z$  be topological spaces, and  $X \subseteq Y$  a subspace of  $Y$ . If  $f : X \rightarrow Z$  is a continuous map, then we call the map  $g : Y \rightarrow Z$  defined by  $g \circ i = f$  an **extension** of  $f$ , where  $i : X \rightarrow Y$  is the inclusion map.

**Theorem 2.1.5.** Let  $f : S^n \rightarrow Y$  be a continuous map into a topological space  $Y$ . The following are equivalent:

- (1)  $f$  is nullhomotopic.
- (2)  $f$  can be extended to a continuous map  $B^{n+1} \rightarrow Y$ .
- (3) There exists a constant map  $k : S^n \rightarrow Y$ , taking  $x \rightarrow f(x_0)$ , for all  $x \in S^n$ , such that  $f \simeq k$ , for  $x_0 \in S^n$ .

*Proof.* Notice that (3) implies (1) immediately. Now suppose that  $f$  is nullhomotopic. Then there exists a constant map  $k : X \rightarrow Y$ , such that for some  $x_0 \in S^n$ ,  $k : x \rightarrow x_0$  for all  $x \in S^n$  implies that  $f \simeq k$ . Now, define the map  $g : B^{n+1} \rightarrow Y$  by:

$$g(x) = \begin{cases} y_0, & \text{if } 0 \leq \|x\| \leq \frac{1}{2} \\ F(\frac{x}{\|x\|}, 2 - 2\|x\|), & \text{if } \frac{1}{2} \leq \|x\| \leq 1 \end{cases}$$

Notice, that if  $\|x\| = \frac{1}{2}$ , then  $g(x) = F(2x, 1) = y_0$ . Therefore, by the pasting lemma,  $g$  is continuous. Moreover, if  $\|x\| = 1$ ,  $g(x) = F(x, 0) = f$ , which makes  $g$  an extension of  $f$ .

Now, suppose that there exists an extension  $g : B^{n+1} \rightarrow Y$  of  $f$ . Since  $S^n$  is a subspace of  $B^{n+1}$ , we have that  $g \circ i = g|_{S^n} = f$ , where  $i : Y \rightarrow S^n$  is an inclusion. Now, let  $x_0 \in S^n$  and define the constant map  $k : S^n \rightarrow Y$  by taking  $x \rightarrow f(x_0)$  for all  $x \in S^n$ . Additionally, define the map  $F : S^n \times I \rightarrow Y$  given by  $F(x, t) = g((1-t)x + x_0t)$ . We have that  $F$  is continuous by composition of continuous maps, and that  $F(x, 0) = g(x) = f(x)$ , since  $F$  has the domain  $S^n \times I$ , and that  $F(x, 1) = g(x_0) = f(x_0)$ , since  $F$  has the domain  $S^n \times I$ . This makes  $f \simeq k$  with  $F$  as the associated homotopy. ■

## 2.2 Quotient Spaces

**Definition.** Let  $X$  be a topological space, and  $X' = \{X_\alpha\}$  a partition of  $X$ . We define the **natural map**  $q : X \rightarrow X'$  by taking  $x \rightarrow X_\alpha$  where  $x \in X_\alpha$ . We define the **quotient topology** on  $X'$  to be the family:

$$\mathcal{T} = \{U' \subseteq X' : q^{-1}(U') \text{ is open in } X\}$$

We denote quotient spaces by  $X/q$ ,  $X/X'$ , or  $X/\sim$  where  $\sim$  is an equivalence relation partitioning  $X$  into  $X'$ .

**Example 2.2.** (1) Consider the space  $I = [0, 1]$  and let  $A = \{0, 1\}$ . The quotient space  $I/A$  identifies 0 to 1, and hence, under the quotient topology, is homeomorphic to  $S^1$ .

(2) Consider the space  $I \times I$  and define an equivalence relation  $(x, 0) \sim (x, 1)$  for all  $x \in I$ . Then the quotient topology formed on  $I \times I/\sim$  is homeomorphic to the cylinder  $S^1 \times I$ . Defining another equivalence  $(0, y) \sim (1, y)$  for all  $y \in I$ , we get the quotient space on  $S^1 \times I/\sim$  under this equivalence relations is homeomorphic to the torus  $S^1 \times S^1$ .

(3) Let  $h : X \rightarrow Y$  be a map, and define  $\ker h$  the equivalence relation on  $X$  such that  $x \ker h x'$  if, and only if  $h(x) = h(x')$ . The quotient space  $X/\ker h$  has the following relation to the natural map on  $X$  via the commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{h} & Y \\ \downarrow q & \searrow \phi & \\ X/\ker h & & \end{array}$$

Where  $\phi : X/\ker h \rightarrow Y$  is a 1-1 map defined by  $\phi([x]) = h(x)$ .

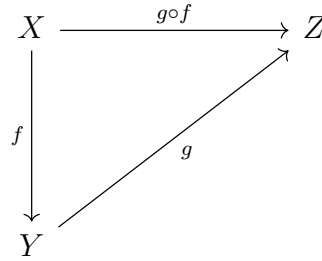
**Definition.** A continuous map  $f : X \rightarrow Y$  of a topological space  $X$  onto a topological space  $Y$  is call an **identification** if a subset  $U$  of  $Y$  is open if, and only if  $f^{-1}(U)$  is open in  $X$ . We denote the quotient space on  $X$  induced by  $f$  by  $X/f$ .

**Example 2.3.** (1) The natural map  $q : X \rightarrow X/\sim$  is an identification, where  $\sim$  is an equivalence relation on  $X$  inducing the quotient topology.

(2) If  $f : X \rightarrow Y$  takes spaces  $X$  onto  $Y$ , is open or closed, then  $f$  is an identification.

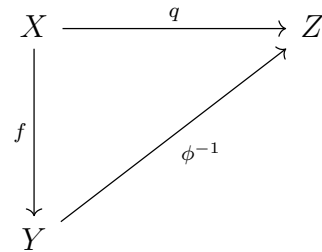
(3) If  $f : X \rightarrow Y$  is a continuous map such that there exists a map  $s : Y \rightarrow X$  such that  $f \circ s = 1_Y$ , then  $f$  is an identification. We call the map  $s$  a **section** of  $f$ .

**Theorem 2.2.1.** Let  $f : X \rightarrow Y$  be a continuous map of a topological space  $X$  onto a topological space  $Y$ .  $f$  is an identification if, and only if for any topological space  $Z$ , and all maps  $g : Y \rightarrow Z$ , then  $g$  is continuous if, and only if  $g \circ f$  is continuous.



*Proof.* Suppose that  $f$  is an identification. If  $g$  is continuous, then so is  $g \circ f$ , by continuity of  $f$ . On the other hand, if  $g \circ f$  is continuous, letting  $V$  be open in  $Z$  we have  $(g \circ f)^{-1}(V) = f^{-1}(g^{-1}(V))$  which is open in  $X$ . By hypothesis,  $g^{-1}(V)$  is open in  $Y$ , which makes  $g$  continuous.

Now, suppose that  $g$  is continuous if, and only if  $g \circ f$  is continuous. Let  $Z = X/\ker f$ , and  $q : X \rightarrow X/\ker f$  the natural map. Additionally, define the 1-1 map  $\phi : X/\ker f \rightarrow Y$  by  $\phi([x]) = f(x)$ . Since  $f$  is onto, we get that so is  $\phi$ . Consider the following commutative diagram:

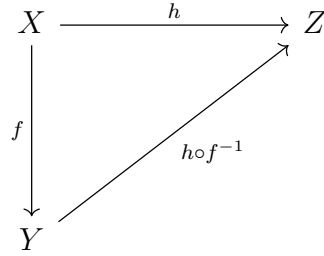


Then  $\phi^{-1} \circ f = q$  is continuous which implies that  $\phi^{-1}$  is continuous.  $\phi$  is also continuous since  $q$  is an identification. Therefore  $\phi$  is a homeomorphism between  $Y$  and  $Z$ . Notice now, that  $f = \phi \circ q$ . Then since  $q$  and  $\phi$  are continuous, this makes  $f$  continuous by composition. Moreover,  $f^{-1}(U) = q^{-1}(\phi^{-1}(U))$ . Since  $q$  is an identification,  $\phi^{-1}(U)$  is open in  $Z$ , which makes  $f^{-1}(U)$  open in  $X$ . This makes  $f$  an identification. ■

f

**Corollary.** Let  $f : X \rightarrow Y$  be an identification, and for some space  $Z$ , define  $h : X \rightarrow Z$  to

be the continuous map constant on each fiber of  $f$ . Then  $h \circ f^{-1} : Y \rightarrow Z$  is continuous.



Moreover  $h \circ f^{-1}$  is open or closed if, and only if  $h(U)$  is open or closed in  $Z$  whenever  $U = f^{-1}(f(U))$  is open or closed in  $X$ .

**Corollary.** If  $h : X \rightarrow Z$  is an identification, then the map  $\phi : X/\ker h \rightarrow Z$  defined by  $[x] \rightarrow h(x)$  is a homeomorphism.

## 2.3 Convexity and Contracibility

**Definition.** We call a subset  $X$  of  $\mathbb{R}^n$  **convex** if for every  $x, y \in X$ , the line segment joining  $x$  to  $y$  is convex. That is the line  $tx + (1 - t)y \in X$  for all  $t \in [0, 1]$ .

**Example 2.4.** The sets  $\mathbb{R}^n$ ,  $I^n$ ,  $B^n$  and  $\Delta(\mathbb{R}^n)$  are all convex. The sphere  $S^{n-1}$  is not convex.

**Definition.** We call a topological space  $X$  **contractible** if  $1_X$  is nullhomotopic.

**Example 2.5.** (1) Let  $X = \{x, y\}$  together with the topology  $\mathcal{T} = \{\emptyset, \{x\}, X\}$ . Then  $X$  is contractible under the topology  $\mathcal{T}$ . We call  $X$  together with  $\mathcal{T}$  the **Sierpinski space**.

(2) The space  $\mathbb{R}^n$  is contractible, but the sphere  $S^{n-1}$  is not contractible.

(3) Continuous images of contractible spaces need not be contractible.

**Theorem 2.3.1.** Every convex set is contractible.

*Proof.* Choose  $x_0 \in X$  and consider the constant map  $c : X \rightarrow X$  by  $x \rightarrow x_0$  for all  $x \in X$ . Define  $F : X \times I \rightarrow X$  by  $F(x, t) = tx_0 + (1 - t)x$ . This map is continuous, with  $F(x, 0) = x = 1_X(x)$  and  $F(x, 1) = x_0 = c(x)$ . Therefore  $1_X \simeq c$ . ■

**Lemma 2.3.2.** If  $X$  is a contractible space, and homeomorphic to a space  $Y$ , then  $Y$  is also contractible.

**Example 2.6.** If  $X$  and  $Y$  are subspaces of  $\mathbb{R}^n$ , with  $X$  homeomorphic to  $Y$ , and  $X$  convex, then  $Y$  is contractible by lemma 2.3.2, however,  $Y$  may not be convex. This shows that not all contractible spaces are convex spaces.

**Lemma 2.3.3.** Contractible spaces are connected.

**Corollary.** *Convex sets are connected.*

*Proof.* This follows from theorem 2.3.1. ■

**Definition.** If  $X$  is a topological space, define the equivalence relation  $\sim$  on  $X \times I$  by  $(x, t) \sim (x', t')$  if, and only if  $t = t' = 1$ . Denote the equivalence classes of  $(x, t)$  as  $[x, t]$ . We call the quotient space  $X \times I / \sim$  the **cone** over  $X$ , and denote it  $CX$ . We call the equivalence class  $[x, 1]$  the **vertex** of  $CX$ .

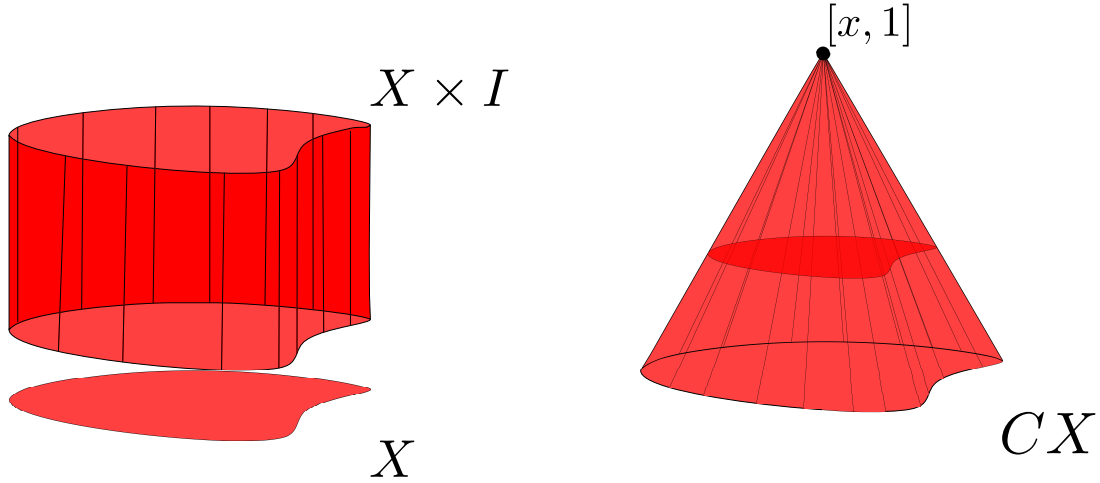


Figure 2.1: The space  $X$  and the cone  $CX$  formed by identifying all  $t = 1$  of  $X \times I$  to a point.

**Example 2.7.** (1) For topological spaces  $X$  and  $Y$ , every continuous map  $f : X \times I \rightarrow Y$  with  $f(x, 1) = y_0$  for some  $y_0 \in Y$  induces a continuous map  $Cf : CX \rightarrow Y$  by taking  $[x, t] \rightarrow f(x, t)$ .

(2) The cone over  $S^{n-1}$  is  $CS^{n-1} = D^n$  and has the vertex 0.

**Theorem 2.3.4.** *For any topological space  $X$ , the cone over  $X$  is contractible.*

*Proof.* Define the map  $F : CX \times I \rightarrow CX$  by taking  $([x, t], s) \rightarrow [x, (1-s)t + s]$ . This map is continuous by composition, moreover  $F([x, t], 0) = [x, t]$  and  $F([x, t], 1) = [x, 1]$  which makes  $1_{CX} \simeq c$  where  $c : CX \rightarrow CX$  is the constant map taking  $[x, t] \rightarrow [x, 1]$  for all  $x \in X$ . ■

**Theorem 2.3.5.** *A topological space has the same homotopy type as a point if, and only if  $X$  is contractible.*

*Proof.* Let  $\{a\}$  be a point space, and suppose that  $X \simeq \{a\}$  have the same homotopy type. Then there are maps  $f : X \rightarrow \{a\}$  and  $g : \{a\} \rightarrow X$  with  $a \xrightarrow{g} x_0$  such that  $g \circ f \simeq 1_X$  and  $f \circ g \simeq 1_{\{a\}}$ . Notice that  $g \circ f(x) = g(a) = x_0$ , for all  $x \in X$ , so  $g \circ f$  is constant. This makes  $1_X$  (and  $1_Y$ ) nullhomotopic. Therefore  $X$  is contractible.

On the otherhand, supposing that  $X$  is contractible, let  $1_X \simeq c$  where  $c : X \rightarrow X$  is the constant map defined by  $x \rightarrow x_0$  for all  $x \in X$ . Define the maps  $f : X \rightarrow \{x_0\}$  and  $g : \{x_0\} \rightarrow X$  by  $x \xrightarrow{f} x_0$  and  $x_0 \xrightarrow{g} x_0$ . Observe that  $g \circ f = 1_X$ , and that  $f \circ g \simeq 1_{\{x_0\}}$ . Therefore  $X$  is of the same homotopy type as  $\{x_0\}$ . ■

*Remark.* This theorem shows that the simplest objects in  $\mathbf{hTop}$  are the contractible spaces.

**Theorem 2.3.6.** *If  $Y$  is a contractible space, then any two maps  $X \rightarrow Y$  are homotopic.*

*Proof.* Suppose that  $1_Y \simeq c$  where  $c : Y \rightarrow Y$  takes  $y \rightarrow y_0$  for all  $y \in Y$ . Define  $g : X \rightarrow Y$  by taking  $x \rightarrow y_0$  for all  $x \in X$ . If  $f : X \rightarrow Y$  is any continuous map, then  $f \simeq g$ . Consider the diagram

$$X \longrightarrow Y \begin{array}{c} \xrightarrow{k} \\ \xleftarrow{1_Y} \end{array} Y$$

Since  $1_Y \simeq k$ , we get that  $f = 1_Y \circ f \simeq k \circ f = g$ . ■

**Corollary.** *Any two maps  $X \rightarrow Y$  are nullhomotopic.*

## 2.4 Path Connectedness.

**Definition.** A **path** in a topological space  $X$  is a continuous map  $f : [0, 1] \rightarrow X$  such that  $f(0) = a$  and  $f(1) = b$  for some  $a, b \in X$ . We call  $a$  and  $b$  the **endpoints** of  $f$ , we say  $f$  goes **from  $a$  to  $b$** .

**Definition.** We call a topological space  $X$  **path connected** if there exists a path from  $a$  to  $b$  for all  $a, b \in X$ .

**Example 2.8.** The sphere  $S^n$  is path connected.

**Lemma 2.4.1.** *If  $f : X \rightarrow Y$  is a continuous map and  $X$  is a path connected space, then  $f(X)$  is also path connected.*

**Theorem 2.4.2.** *If  $X$  is a path connected space, then  $X$  is a connected space.*

*Proof.* Suppose that  $X$  is disconnected. Then there exists a separation of  $X$  into disjoint open sets  $U$  and  $V$ . That is  $X = U \cup V$ . Suppose however that  $X$  is path connected. Then for points  $a \in U$  and  $b \in V$ , there is a path  $f : [0, 1] \rightarrow X$  from  $a$  to  $b$ . Since  $[0, 1]$  is a connected space, so is  $f([0, 1])$ ; however notice that  $f([0, 1]) = (U \cap f([0, 1])) \cup (f([0, 1]) \cap V)$ , which is a separation of  $f([0, 1])$ , since  $U$  and  $V$  form a separation. ■

**Example 2.9.** The converse of theorem 2.4.1 is not true in general. Consider the following two examples:

- (1) Consider the subspace  $X = (0 \times [0, 1]) \cup G$  where  $G$  is the graph of  $\sin \frac{1}{x}$  on the interval  $(0, 2\pi]$ . We have that  $X$  is connected, since the component containing  $G$  is closed, and  $0 \times [0, 1] \subseteq \text{cl } G$ . However,  $X$  is not path connected. We call the space  $X$  the **topologists sine curve**.
- (2) Another example of a connected space in  $\mathbb{R}^2$  that is not path connected is the **topologist's whirlpool**.

**Lemma 2.4.3.** *Every contractible space is path connected.*

**Lemma 2.4.4.** *A topological space  $X$  is path connected if, and only if any two constant maps  $X \rightarrow X$  are homotopic.*

**Lemma 2.4.5.** *If  $X$  is a contractible space and  $Y$  a path connected space, then any two continuous maps  $X \rightarrow Y$  are homotopic.*

**Corollary.** *The continuous maps are nullhomotopic.*

**Lemma 2.4.6.** *If  $X$  and  $Y$  are path connected spaces, then so is  $X \times Y$ .*

**Lemma 2.4.7.** *If  $f : X \rightarrow Y$  is a continuous map and  $X$  is a path connected space, then  $f(X)$  is also path connected.*

**Theorem 2.4.8.** *If  $X$  is a topological space, then the relation  $\sim$  defined on  $X$  by  $a \sim b$  if, and only if there is a path from  $a$  to  $b$ , is an equivalence relation.*

*Proof.* Consider the constant path  $c : [0, 1] \rightarrow X$  where  $c(x) = a$  for all  $x \in A$ .  $c$  is continuous, and  $c(0) = c(1) = a$ . So  $a \sim a$ .

Now suppose that for  $a, b \in X$ , that  $a \sim b$ . Then there is a path  $f : [0, 1] \rightarrow X$  with  $f(0) = a$  and  $f(1) = b$ . Consider the map  $g : [0, 1] \rightarrow X$  defined by  $g(t) = f(1 - t)$ .  $g$  is continuous by composition, and  $g(0) = f(1) = b$  and  $g(1) = f(0) = a$ , which makes  $b \sim a$ .

Lastly, suppose that  $a \sim b$  and  $b \sim c$  for some  $a, b, c \in X$ . Then there exist paths  $f : [0, 1] \rightarrow X$  and  $g : [0, 1] \rightarrow X$  with  $f(0) = a$ ,  $f(1) = b$ , and  $g(0) = b$ ,  $g(1) = a$ . Now, consider the map  $h : [0, 1] \rightarrow X$  defined by:

$$h(t) = \begin{cases} f(2t), & \text{if } 0 \leq t \leq \frac{1}{2} \\ g(2t - 1), & \text{if } \frac{1}{2} \leq t \leq 1 \end{cases}$$

Notice that  $f(\frac{1}{2}) = g(\frac{1}{2}) = f(1) = g(0) = b$ , so the domains of  $f$  and  $g$  coincide. Therefore by the pasting lemma,  $h$  is continuous. Now, observe that  $h(0) = f(0) = a$ , and that  $h(1) = g(1) = c$ . This makes  $a \sim c$ . ■

**Definition.** We define the equivalence classes of  $X$  under path connectedness to be called **path components** of  $X$ .

**Definition.** We denote the collection of all path components of a topological space  $X$  to be  $\pi_0(X)$ ; that is  $\pi_0(X) = X/\sim$  (not necessarily as a quotient space). Moreover, we define the map  $\pi_0(f) : \pi_0(X) \rightarrow \pi_0(Y)$  to be the map taking the path component  $C$  to the unique path component of  $Y$  containing  $f(C)$ .

**Theorem 2.4.9.**  $\pi_0 : \text{Top} \rightarrow \text{Set}$  is a functor.

*Proof.* Consider  $1_X : X \rightarrow X$  the identity on  $X$ . Let  $\pi_0(X) = \{X_\alpha\}$  where  $X_\alpha$  is a path component of  $X$ . We have that  $\pi_0(1_X) : \pi_0(X) \rightarrow \pi_0(X)$  sends  $X_\alpha \rightarrow X_\beta$  where  $X_\beta$  is the unique path component of  $X$  containing  $1_X(X_\alpha) = X_\alpha$ . However, since  $X_\alpha$  and  $X_\beta$  are equivalence classes, we have  $X_\alpha \subseteq X_\beta$  if and only if  $\alpha = \beta$ , i.e.  $X_\alpha = X_\beta$ . This makes  $\pi_0(1_X) = 1_{\pi_0(X)}$ .

Now let  $f : X \rightarrow Y$  and  $g : Y \rightarrow Z$  be continuous maps. Let  $\pi_0(X) = \{X_\alpha\}$ ,  $\pi_0(Y) = \{Y_\beta\}$ ,  $\pi_0(Z) = \{Z_\gamma\}$  the collection of path components of  $X$ ,  $Y$ , and  $Z$ , respectively. Now



consider  $X_\alpha$  and  $Z_\gamma$  such that  $\pi_0(g \circ f)(X_\alpha) = Z_\gamma$ . Then  $Z_\gamma$  is the unique path component of  $Z$  containing  $g(f(X_\alpha))$ . Now, if  $Y_\beta$  is the unique path component of  $Y$  containing  $X_\alpha$ , then  $\pi_0(f)(X_\alpha) = Y_\beta$  and we see that  $g(f(X_\alpha)) \subseteq g(Y_\beta)$ . Moreover, if  $Z_{\gamma'}$  is the unique path component of  $Z$  containing  $g(Y_\beta)$ , then  $\pi_0(g)(Y_\beta) = Z_{\gamma'}$ , and  $g(Y_\beta) \subseteq Z_{\gamma'}$ . But  $g(f(X_\alpha)) \subseteq g(Y_\beta) \subseteq Z_{\gamma'}$ ; by above, and since path components partition their spaces, this makes  $\gamma = \gamma'$ . Thus  $Z_\gamma = Z_{\gamma'}$  and we have that  $g(f(X_\alpha)) \subseteq g(Y_\beta) \subseteq Z_\gamma$ . Therefore  $Z_\gamma$  is the unique path component of  $Z$  containing both  $g(f(X_\alpha))$  and  $g(Y_\gamma)$ ; that is  $\pi_0(g)(Y_\beta) = Z_\gamma$ , where  $\pi_0(f)(X_\alpha) = Y_\beta$ . This implies that  $\pi_0(g \circ f) = \pi_0(g) \circ \pi_0(f)$ , which makes  $\pi_0$  a functor. ■

**Corollary.** *If  $f \simeq g$ , then  $\pi_0(f) = \pi_0(g)$ .*

*Proof.* Suppose that  $F : f \simeq g$  is a homotopy between the maps  $f : X \rightarrow Y$  and  $g : X \rightarrow Y$ . Let  $C$  be a path component of  $X$ , then  $C \times I$  is path connected by lemma 2.4.6. Thus by lemma 2.4.1,  $F(C \times I)$  is also path connected. Notice then that:

$$f(C) = F(C \times 0) \subseteq F(C \times I)$$

and

$$g(C) = F(C \times 1) \subseteq F(C \times I)$$

So the unique path connected component of  $Y$  containing  $F(C \times I)$  contains both  $f(C)$  and  $g(C)$ . Therefore  $\pi_0(f) = \pi_0(g)$ . ■

**Corollary.** *If  $X$  and  $Y$  are topological spaces with the same homotopy type, then they have the same number of path components.*

*Proof.* Suppose that  $f : X \rightarrow Y$  and  $g : Y \rightarrow X$  are continuous maps with  $g \circ f = 1_X$  and  $f \circ g = 1_Y$ . Since  $f$  is a homotopy equivalence, then  $[f]$  is an equivalence in  $\mathbf{hTop}$ . Restricting  $\pi_0$  to  $\mathbf{hTop}$ , this also gives use that  $\pi_0([f])$  is an equivalence in  $\mathbf{Set}$ . That is  $f$  is 1-1 and onto. ■

**Definition.** A topological space  $X$  is **locally path connected** if, for each  $x \in X$ , and every open neighborhood  $U$  of  $x$  there is an open set  $V$  with  $x \in V \subseteq U$  such that any two points in  $V$  can be joined by a path in  $U$ .

**Example 2.10.** Form the subspace  $X$  of  $\mathbb{R}^2$  by adjoining a curve from  $(0, 1)$  to  $(\frac{1}{2\pi}, 0)$  on the topologist's sine curve. Then  $X$  is path connected, but not locally path connected.

**Theorem 2.4.10.** *A topological space is locally path connected if, and only if path components of open sets are open.*

*Proof.* Suppose that  $X$  is locally path connected, and let  $U$  be open in  $X$ . Let  $x \in C$ , where  $C$  is a path component of  $U$ . Then there is an open  $V$  with  $x \in V \subseteq U$  such that every point of  $V$  can be joined to  $x$  by a path in  $U$ . Thus each point of  $V$  lies in the path component of  $x$ , which is  $C$ . Thus  $V \subseteq C$ , which makes  $C$  open.

Conversely, suppose that path components of open sets in  $X$  are open. Let  $U$  be an open set of  $X$ , and for some  $x \in U$ , let  $C$  be the path component of  $x$  in  $U$ . Then we have  $x \in C \subseteq U$ . Since  $C$  is open, this makes  $X$  locally path connected. ■

**Corollary.** *If  $X$  is locally path connected, then its path components are open.*

**Corollary.**  *$X$  is locally path connected if, and only if for every  $x \in X$ , and each open neighborhood  $U$  of  $x$ , there is an open path connected set  $V$  with  $x \in V \subseteq U$ .*

**Corollary.** *If  $X$  is locally path connected, then the connected components of every open set coincide with its path components. In particular the connected components of  $X$  coincide with the path components of  $X$ .*

**Corollary.** *If  $X$  is connected, and locally path connected, then  $X$  is connected.*

**Definition.** Let  $A$  be a subspace of a topological space  $X$ , and let  $i : A \rightarrow X$  be the inclusion. Then  $A$  is a **deformation retract** of  $X$  if there is a continuous map  $r : X \rightarrow A$  such that  $r$  is a retraction of  $X$ ; i.e.  $r \circ i = 1_A$  and  $i \circ r = 1_X$ .

**Lemma 2.4.11.** *Every deformation retract is a retract.*

**Theorem 2.4.12.** *If  $A$  is a deformation retract of a topological space  $X$ , then  $X$  and  $A$  have the same homotopy type.*

**Corollary.**  $S^1$  is a deformation retract of  $\mathbb{C} \setminus 0$ .

*Proof.* For every  $z \in \mathbb{C} \setminus 0$ , we can write  $z$  as  $z = \rho e^{i\theta}$ , where  $\rho > 0$ , and  $0 \leq \theta \leq 2\pi$ . Now, define  $F : (\mathbb{C} \setminus 0) \times I \rightarrow \mathbb{C} \setminus 0$  by taking  $(\rho e^{i\theta}, t) \rightarrow ((1-t)\rho + t)e^{i\theta}$ . Notice that  $F$  is never 0, and that  $F$  is continuous, with  $F(\rho e^{i\theta}, 0) = \rho e^{i\theta}$ ,  $F(e^{i\theta}, 1) = e^{i\theta}$ . Moreover  $F(\rho e^{i\theta}, 1) = F(e^{i\theta}) = e^{i\theta}$ . Writing  $S^1$  as  $S^1 = \{e^{i\theta} : 0 \leq \theta \leq 2\pi\}$ . We see that  $F$  makes  $S^1$  into a deformation retract of  $\mathbb{C} \setminus 0$ . ■

**Corollary.**  $S^1$  has the same homotopy type as  $\mathbb{C} \setminus 0$ .

**Definition.** Let  $f : X \rightarrow Y$  be a continuous map from a topological space  $X$  to a topological space  $Y$ . Define

$$M_f = (X \times I) \cup Y / \sim$$

Where  $(X \times I) \cup Y$  is a disjoint union, and  $\sim$  is an equivalence relation defined by  $(x, t) \sim y$  if  $y = f(x)$  and  $t = 1$ . Denote the equivalence classes of  $(x, t)$  by  $[x, t]$ . We call the quotient space  $M_f$  the **mapping cylinder** of  $f$ .

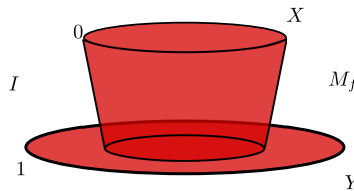


Figure 2.2: The mapping cylinder of a continuous map  $f : X \rightarrow Y$ .

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