

Measure Theory

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Text

Real Analysis (4th edition)

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Chapter 1

The Real Numbers

1.1 Open Sets, and σ -Algebras

Definition. We call a set U of real numbers **open** provided for any $x \in U$, there is an $r > 0$ such that $(x - r, x + r) \subseteq U$.

Lemma 1.1.1. *The set of real numbers \mathbb{R} , together with open sets defines a topology on \mathbb{R} .*

Proof. Notice that both \mathbb{R} and \emptyset are open sets. Moreover, if $\{U_n\}$ is a collection of open sets, then so is their union. Now, consider the finite collection $\{U_k\}_{k=1}^n$ and let $U = \bigcap_{k=1}^n U_k$. If U is empty, we are done. Otherwise, let $x \in U$. Then $x \in U_k$ for every $1 \leq k \leq n$, and since each U_k is open, choose an $r_k > 0$ for which $(x - r_k, x + r_k) \subseteq U_k$. Then let $r = \min\{r_1, \dots, r_n\}$. Then $r > 0$, and we have $(x - r, x + r) \subseteq U$, which makes U open in \mathbb{R} . ■

Lemma 1.1.2. *Every nonempty set is the disjoint union of a countable collection of open sets.*

Proof. Let U be nonempty and open in \mathbb{R} . Let $x \in U$. Then there is a $y > x$ for which $(x, y) \subseteq U$ and there is a $z < x$ for which $(z, x) \subseteq U$. Now, let $a_x = \inf\{z : (z, x) \subseteq U\}$ and $b_x = \sup\{y : (x, y) \subseteq U\}$, and let $I_x = (a_x, b_x)$. Then we have

$$x \in I_x \text{ and } a_x \notin I_x \text{ and } b_x \notin I_x$$

Let $w \in I_x$ such that $x < w < b_x$. Then there is a $y > w$ such that $(x, y) \subseteq U$ so that $w \in U$. Now, if $b_x \in U$, then there is an $r > 0$ for which $(b_x - r, b_x + r) \subseteq U$, in particular, $(x, b_x + r) \subseteq U$. But b_x is the least upperbound of all such numbers, and $b_x < b_x + r$, a contradiction. Thus $b_x \notin U$, and hence $b_x \notin I_x$. A similar argument shows that $a_x \notin I_x$.

Consider now the collection $\{I_x\}_{x \in U}$. Then $U = \bigcup I_x$ and since $a_x, b_x \notin I_x$ for each x , the collection $\{I_x\}$ is a disjoint collection. Lastly, by the density of \mathbb{Q} in \mathbb{R} there is a 1-1 mapping between this collection and \mathbb{Q} , making it countable. ■

Definition. Let $E \subseteq \mathbb{R}$ a set. We call a point $x \in \mathbb{R}$ a **point of closure** of E if every open interval containing x also contains a point of E . We call the collection of all such points the **closure** of E , and denote it $\text{cl } E$. If $E = \text{cl } E$, then we say that E is **closed**.

Lemma 1.1.3. *For any set E of real numbers, $\text{cl } E$ is closed; i.e. $\text{cl } E = \text{cl}(\text{cl } E)$. Moreover, $\text{cl } E$ is the smallest closed set containing E .*

Lemma 1.1.4. *Every set E of real numbers is open if, and only if $\mathbb{R} \setminus E$ is closed.*

Definition. Let $E \subseteq \mathbb{R}$ a set. We call a collection $\{E_\lambda\}$ a **cover** of E if $E \subseteq \bigcup E_\lambda$. If each E_λ is open, then we call this collection an **open cover** of E .

Theorem 1.1.5 (Heine-Borel). *For any closed and bounded set F of \mathbb{R} , every open cover of F has a finite subcover.*

Proof. Suppose first that $F = [a, b]$, for $a \leq b$ real numbers. Then F is closed and bounded. Let \mathcal{F} be an open cover of $[a, b]$, and define $E = \{x \in [a, b] : [a, x] \text{ has a finite subcover by sets in } \mathcal{F}\}$. Notice that $a \in E$, so that E is nonempty. Now, since E is bounded by b , by the completeness of \mathbb{R} , let $c = \sup \{E\}$. Then $c \in [a, b]$ and there is a set $U \in \mathcal{F}$ with $c \in U$. Since U is open, there exists an $\varepsilon > 0$ such that $(c - \varepsilon, c + \varepsilon) \subseteq U$. Now, $c - \varepsilon$ is not an upperbound of E , so there is an $x \in E$ with $c - \varepsilon < x$, and a finite collection of open sets $\{U_i\}_{i=1}^k$ covering $[a, x]$. Then the collection $\{U_i\}_{i=1}^k \cup U$ covers $[a, x]$ so that $c = b$, and we have found a finite subcover of F .

Now, let F be closed and bounded. Then it is contained in a closed bounded interval $[a, b]$. Now, let $U = \mathbb{R} \setminus F$ open and \mathcal{F} an open cover of F . Let $\mathcal{F}' = \mathcal{F} \cup U$. Since \mathcal{F} covers F , \mathcal{F}' covers $[a, b]$. By above, there is a finite subcover of $[a, b]$, and hence of F by sets in \mathcal{F}' . Remove U from \mathcal{F}' , we get a finite subcover of F by sets in \mathcal{F} . ■

Theorem 1.1.6 (The Nested Set Theorem). *Let $\{F_n\}$ be a descending collection of nonempty closed sets of \mathbb{R} , for which F_1 is bounded. Then*

$$\bigcap F_n \neq \emptyset$$

Proof. Let $F = \bigcap F_n$, and suppose to the contrary that F is empty. Then for all $x \in \mathbb{R}$, there is an $n \in \mathbb{Z}^+$ for which $x \notin F_n$. So that $x \in U_n = \mathbb{R} \setminus F_n$. Then $U_n = \mathbb{R}$, and each U_n is open. So $\{U_n\}$ is an open cover of \mathbb{R} , and hence F_1 . By the theorem of Heine-Borel, there is an $N > 0$ such that $F \subseteq \bigcup_{n=1}^N U_n$. Since $\{F_n\}$ is descending, the collection $\{U_n\}$ is ascending, and hence $\bigcup U_n = U_N = \mathbb{R} \setminus F_N$ which makes $F_1 \subseteq \mathbb{R} \setminus F_N$, a contradiction. ■

Definition. Let X be a set. We call a collection \mathcal{A} of subsets of X **σ -algebra** if

- (1) $\emptyset \in \mathcal{A}$.
- (2) For any $A \in \mathcal{A}$, $X \setminus A \in \mathcal{A}$.
- (3) If $\{A_n\}$ is a countable collection of elements of \mathcal{A} , then their union is an element of \mathcal{A} .

Lemma 1.1.7. *Let \mathcal{F} a collection of subsets of a set X . The intersection of all σ -algebras containing \mathcal{F} is a σ -algebra. Moreover, it is the smallest such σ -algebra.*

Definition. We define the **Borel sets** of \mathbb{R} to be the σ -algebra of \mathbb{R} containing all open sets in \mathbb{R} .

Lemma 1.1.8. *Every closed set of \mathbb{R} is a Borel set.*

Definition. We call a countable intersection of open sets of \mathbb{R} a **G_δ -set** and we call a countable union of closed sets of \mathbb{R} an **F_σ -set**.

1.2 Sequences of Real Numbers

Definition. A sequence $\{a_n\}$ of real numbers is said to **converge** to a point a , if, for every $\varepsilon > 0$, there is an $N > 0$ such that

$$|a - a_n| < \varepsilon \text{ whenever } n \geq N$$

We call a the **limit** of $\{a_n\}$ and write $\{a_n\} \rightarrow a$, or

$$\lim_{n \rightarrow \infty} \{a_n\} = a$$

Lemma 1.2.1. *Let $\{a_n\} \rightarrow a$ a sequence of real numbers converging to $a \in \mathbb{R}$. Then the limit of $\{a_n\}$ is unique, $\{a_n\}$ is bounded, and for any $c \in \mathbb{R}$, if $a_n \leq c$ for all n , then $a \leq c$.*

Theorem 1.2.2 (The Monoton C Vonvergence Theorem). *A monotone sequence of real numbers converges to a point if, and only if it is bounded.*

Proof. Without loss of generality, suppose that the sequence $\{a_n\}$ is increasing. If $\{a_n\} \rightarrow a$, by lemma 1.2.1, $\{a_n\}$ is bounded. On the otherhand, suppose that $\{a_n\}$ is bounded. Let $S = \{a_n : n \in \mathbb{Z}^+\}$, then by the completeness of \mathbb{R} , let $a = \sup S$. Let $\varepsilon > 0$. Notice that $a_n \leq a$ for all n . Now, since $a - \varepsilon$ is not an upperbound, there exists an $N > 0$ for which $a_N > a - \varepsilon$, then since $\{a_n\}$ is increasing, $a_n > a - \varepsilon$ whenever $n \geq N$. So we get

$$|a - a_n| < \varepsilon \text{ whenever } n \geq N$$

Which makes $\{a_n\} \rightarrow a$. ■

Theorem 1.2.3 (Bolzano-Weierstrass). *Every bounded sequence has a convergent subsequence.*

Proof. Let $\{a_n\}$ be a bounded sequence, and let $M > 0$ such that $|a_n| \leq M$ for all $n \in \mathbb{Z}^+$. Define $E_n = \text{cl}\{a_j : j \geq n\}$. Then $E_n \subseteq [-M, M]$. Thus $\{E_n\}$ is a decreasing sequence of closed, bounded, and nonempty sets of \mathbb{R} . By the nested set theorem, the intersection $E = \bigcap E_n$ is nonempty. Choose an $a \in E$. Then for every $k \in \mathbb{Z}^+$, a is a point of closure of the set $\{a_j : j \geq k\}$. SO that $a_j \in (a - \frac{1}{k}, a + \frac{1}{k})$ whenever $j \geq k$. By induction, construct a strictly increasing sequence $\{n_k\}$ of natural numbers for which $|a - a_{n_k}| < \frac{1}{k}$. Then by the principle of Archimedes, $\{a_{n_k}\} \rightarrow a$, and we have a convergent subsequence. ■

Definition. We call a sequence $\{a_n\}$ **Cauchy** if for every $\varepsilon > 0$, there is an $N > 0$ for which

$$|a_m - a_n| < \varepsilon \text{ whenever } m, n \geq N$$

Theorem 1.2.4 (The Cauchy Convergence Criterion). *A sequence of real numbers converges if, and only if it is Cauchy.*

Proof. Suppose that the sequence $\{a_n\} \rightarrow a$ converges to $a \in \mathbb{R}$. Then for any $m, n \in \mathbb{Z}^+$, notice that $|a_m - a_n| \leq |a_m - a| + |a - a_n|$. Let $\varepsilon > 0$ and choose $N > 0$ such that $|a - a_n| < \frac{\varepsilon}{2}$, and $|a_m - a| < \frac{\varepsilon}{2}$. Then if $n, m \geq N$, we get $|a_m - a_n| < \varepsilon$, which makes $\{a_n\}$ Cauchy.

Conversely, suppose that $\{a_n\}$ is Cauchy. Let $\varepsilon = 1$ and choose $N > 0$ such that if $m, n \geq N$, then $|a_m - a_n| < 1$. Then we get $|a_n| \leq 1 + |a_N|$ for all $n \geq N$. Define $M = 1 + \max\{|a_1|, \dots, |a_N|\}$. Then $|a_n| \leq M$ for all n . This makes $\{a_n\}$ bounded. By the theorem of Bolzano-Weierstrass, $\{a_n\}$ has a convergent subsequence $\{a_{n_k}\} \rightarrow a$. Let $\varepsilon > 0$, since $\{a_n\}$ is Cauchy, choose an $N > 0$ such that $|a_m - a_n| < \frac{\varepsilon}{2}$ whenever $n, m \geq N$. Likewise, we get $|a - a_{n_k}| < \frac{\varepsilon}{2}$ and $n_k \geq N$. Thus we observe that $|a_n - a| \leq |a_n - a_{n_k}| + |a - a_{n_k}| < \varepsilon$ and so $\{a_n\} \rightarrow a$. ■

Theorem 1.2.5. *Let $\{a_n\} \rightarrow a$ and $\{b_n\} \rightarrow b$ be convergent sequences. Then for any $\alpha, \beta \in \mathbb{R}$, we have that the sequence $\{\alpha a_n + \beta b_n\}$ converges and that*

$$\lim_{n \rightarrow \infty} \{\alpha a_n + \beta b_n\} = \alpha a + \beta b$$

Definition. We say a sequence $\{a_n\}$ of real numbers **converges to infinity** $\infty \in \mathbb{R}_\infty$ if for every $c \in \mathbb{R}$, there is an $N > 0$ such that $a_n \geq c$ whenever $n \geq N$. We write $\{a_n\} \rightarrow \infty$, or

$$\lim_{n \rightarrow \infty} \{a_n\} = \infty$$

Definition. Let $\{a_n\}$ be a sequence of real numbers. We define the **limit superior** of $\{a_n\}$ to be

$$\limsup \{a_n\} = \lim_{n \rightarrow \infty} (\sup \{a_k : k \geq n\})$$

Similarly, we define the **limit inferior** of $\{a_n\}$ to be

$$\liminf \{a_n\} = \lim_{n \rightarrow \infty} (\inf \{a_k : k \geq n\})$$

Theorem 1.2.6. *For any sequences $\{a_n\}$ and $\{b_n\}$ of real numbers, the following are true:*

- (1) $\limsup \{a_n\} = l \in \mathbb{R}_\infty$ if, and only if for every $\varepsilon > 0$, there exists infinitely many $n \in \mathbb{Z}^+$ such that $a_n > l - \varepsilon$ and finitely many $n \in \mathbb{Z}^+$ for which $a_n > l + \varepsilon$.
- (2) $\limsup \{a_n\} = \infty$ if, and only if $\{a_n\}$ is not bounded above.
- (3) $\limsup \{a_n\} = -\liminf \{-a_n\}$
- (4) $\{a_n\} \rightarrow a \in \mathbb{R}_\infty$ if, and only if $\limsup \{a_n\} = \liminf \{a_n\}$.
- (5) If $a_n \leq b_n$ for all n , then $\limsup \{a_n\} \leq \limsup \{b_n\}$.

Definition. Let $\{a_n\}$ a sequence of real numbers. We call the series $\sum_{k=1}^{\infty} a_k$ **summable** if the sequence of partial sums $\{s_n = \sum_{k=1}^n a_k\} \rightarrow s$ converges to a point $s \in \mathbb{R}$.

Lemma 1.2.7. *Let $\{a_n\}$ a sequence of real numbers. Then the following are true.*

- (1) The series $\sum a_k$ is summable if, and only if for every $\varepsilon > 0$, there is an $N > 0$ such that

$$\left| \sum_{k=n}^{n+m} a_k \right| < \varepsilon \text{ for all } m \in \mathbb{Z}^+ \text{ whenever } n \geq N$$

- (2) If $\sum |a_k|$ is summable, then so is $\sum a_k$.
- (3) If $a_k \geq 0$, then $\sum a_k$ is summable if, and only if the sequence of partial sums $\{s_n\}$ is bounded.

1.3 Continuous Functions of a Real Variable.

Definition. A realvalued function f on a domain E is said to be **continuous** at a point $x \in E$ provided for any $\varepsilon > 0$ there is a $\delta > 0$ for which

$$|f(x) - f(y)| < \varepsilon \text{ whenever } |x - y| < \delta \text{ for any } y \in E$$

We call f **continuous** on E if it is continuous at every point in E . We call f **Lipschitz continuous** if there is a $c \geq 0$ for which

$$|f(x) - f(y)| \leq c|x - y| \text{ for all } x, y \in E$$

Lemma 1.3.1. *A Lipschitz continuous function on a domain is continuous on that domain.*

Lemma 1.3.2 (The Sequential Criterion). *A realvalued function f defined on a domain E is continuous at a point $x \in E$ if, and only if for any ssequence $\{x_n\} \rightarrow x$ of points in E , converging to x , that the sequence $\{f(x_n)\} \rightarrow f(x)$ converges to $f(x)$.*

Theorem 1.3.3 (The Extreme Value Theorem). *A continuous realvalued function defined on a nonempty, closed and bounded domain takes on a maximum value, and a minimum value on that domain.*

Proof. Let f be a continuous realvalued function defined on the domain E , where E is nonempty, closed, and bounded. Let $x \in E$ and $\delta > 0$ and $\varepsilon = 1$. Define the open interval $I_x = (x - \delta, x + \delta)$. Then if $y \in E \cap I_x$, then $|f(x) - f(y)| < 1$. So that $|f(y)| \leq |f(x)| + 1$. Notice also that the collection $\{I_x\}$ is an open cover of E . By the theorem of Heine-Borel, there is a finite subcover of E , $\{I_{x_k}\}_{k=1}^n$. Define, then, $M = 1 + \max \{|f(x_1)|, \dots, |f(x_n)|\}$. Then we get that $|f(x)| \leq M$ and f is bounded.

Now, let $m = \sup f(E)$. If f does not take the value m for any points in E , then the function $x \rightarrow \frac{1}{f(x)-m}$ is a continuous unbounded function on E ; which is impossible. So there is an $x \in E$ with $f(x) = m$ and m is a maximum value. Now, since f is continuous, then so is $-f$, and hence $-m$ defines a minimum value on f . ■

Theorem 1.3.4 (The Intermediate Value Theorem). *If f is a continuous realvalued function on a closed bounded interval $[a, b]$, for which $f(a) < c < f(b)$, then there exists an $x_0 \in (a, b)$ for which $f(x_0) = c$.*

Proof. Define $a_1 = a$ and $b_1 = b$ and let m_1 be the midpoint of the interval $[a_1, b_1]$. If $c < f(m_1)$, define $a_2 = a_1$ and $b_2 = m_1$, otherwise define $a_2 = m_1$ and $b_2 = b_1$, so that in either case we get $f(a_2) \leq c \leq f(b_2)$ and $b_2 - a_2 = \frac{b-a}{2}$. By induction, construct the collection of closed bounded intervals $\{[a_n, b_n]\}$ such that $f(a_n) \leq c \leq f(b_n)$ and $b_n - a_n = \frac{b-a}{2^{n-1}}$. This collection is a descending collection, so by the nested set theorem, the intersection $I = \bigcap [a_n, b_n]$ is nonempty. Choose an $x_0 \in I$, and observe that

$$|a_n - x_0| \leq b_n - a_n = \frac{b-a}{2^{n-1}}$$

So the sequence $\{a_n\} \rightarrow x_0$. By the sequential criterion, since f is continuous at x_0 , we get the sequence $\{f(a_n)\} \rightarrow f(x_0)$. Since $f(a_n) \leq c$, and $(-\infty, c]$ is closed, we also get $f(x_0) \leq c$.

By similar reasoning to the argument provided above, we also get that $f(x_0) \geq c$ so that equality is established. ■

Definition. A realvalued function f on a domain E is said to be **uniformly continuous** if for every $\varepsilon > 0$ there is a $\delta > 0$ such that

$$|f(x) - f(y)| < \varepsilon \text{ whenever } |x - y| < \delta \text{ for all } x, y \in E$$

Lemma 1.3.5. *If f is a uniformly continuous function on a domain E , then it is continuous on E .*

Theorem 1.3.6. *A continuous realvalued function on a closed and bounded domain is uniformly continuous.*

Proof. Let f be continuous on E , and E a closed and bounded domain. Let $\varepsilon > 0$. For every $x \in E$, there is a $\delta_x > 0$ for which $|f(x) - f(y)| < \varepsilon$ whenever $|x - y| < \delta_x$ for some $y \in E$. Define $I_x = (x - \frac{\delta_x}{2}, x + \frac{\delta_x}{2})$. Then $\{I_x\}$ is an open cover for E , so that by the theorem of Heine-Borel, there is a finite subcover $\{I_{x_k}\}_{k=1}^n$ of E . Define $\delta = \frac{1}{2} \min \{\frac{\delta_{x_1}}{2}, \dots, \frac{\delta_{x_n}}{2}\}$. Then $\delta > 0$ moreover, if $x, y \in E$, with $|x - y| < \delta$, then asince $\{I_{x_k}\}$ covers E , there is a $k > 0$ such that

$$|x - x_k| < \frac{\delta_{x_k}}{2} \text{ and } |x_{x_k} - y| < \frac{\delta_{x_k}}{2}$$

Then we have $|f(x) - f(x_k)| < \frac{\varepsilon}{2}$ and $|f(x_k) - f(y)| < \frac{\varepsilon}{2}$ so that $|f(x) - f(y)| < \varepsilon$, which makes f uniformly continuous. ■

Chapter 2

Lebesgue Measure

2.1 Lebesgue Outermeasure

Definition. Let I be a nonempty interval of \mathbb{R} . We define the **length** of I , denoted $l(I)$, to be the difference of its endpoints, if I is bounded, and ∞ otherwise.

Definition. Let A a subset of \mathbb{R} . We define the **Lebesgue outer measure** of A to be

$$m^*(A) = \inf \left\{ \sum l(I_k) \right\}$$

Where $\{I_k\}$ is a countable collection of bounded open sets, covering A .

Lemma 2.1.1. *The emptyset has Lebesgue outermeasure 0. Moreover, the Lebesgue outermeasure is monotone; that is, if $A, B \subseteq \mathbb{R}$ such that $A \subseteq B$, then $m^*(A) \leq m^*(B)$.*

Proof. Notice that the singleton $\{a\} = [a, a]$ covers the emptyset. Moreover $l([a, a]) = a - a = 0$, so by definition $m^*(\emptyset) = 0$.

Now, let A, B subsets of \mathbb{R} such that $A \subseteq B$. Then if $\{I_k\}$ is a countable collection of bounded open sets covering B , they also cover A , hence by definition, we get $m^*(A) \leq m^*(B)$. ■

Corollary. *Lebesgue outermeasure is nonnegative. That is, $0 \leq m^*(E)$ for any set $E \subseteq \mathbb{R}$.*

Proof. Notice the length of any interval I is nonnegative. ■

Example 1. Countable sets have measure 0. Let C be a countable set with enumeration $\{c_k\}$. Let $\varepsilon > 0$ and define $I_k = (c_k - \frac{\varepsilon}{2^{k+1}}, c_k + \frac{\varepsilon}{2^{k+1}})$. Then $\{I_k\}$ is a countable collection of bounded open sets covering $C = \{c_k\}$. Hence we get that

$$0 \leq m^*(C) \leq \sum l(I_k) \leq \sum \frac{\varepsilon}{2^k} = \varepsilon$$

So that $m^*(C) = 0$.

Lemma 2.1.2. *For any nonempty interval I , $m^*(I) = l(I)$.*

Proof. Consider first, the closed bounded interval $[a, b]$, where $a < b$. Let $\varepsilon > 0$. Notice that $[a, b] \subseteq (a - \varepsilon, b + \varepsilon)$, so that $m^*([a, b]) \leq l((a - \varepsilon, b + \varepsilon)) = b - a + 2\varepsilon$. Hence $m^*([a, b]) \leq b - a$. It remains to show that $b - a \leq m^*([a, b])$.

Let $\{I_k\}$ a countable collection of open bounded intervals covering $[a, b]$. By the theorem of Heine-Borel, there is a finite subcover $\{I_k\}_{k=1}^n$ of $[a, b]$. Notice that since $a \in \bigcup I_k$, at least one I_k contains a . Hence choose an interval (a_1, b_1) in this cover for which $a_1 < a < b_1$. Now, if $b < b_1$, we are done as

$$\sum_{k=1}^n l(I_k) \geq b_1 - a_1 > b - a$$

Otherwise, $b_1 \in [a, b]$. In this case, choose an interval (a_2, b_2) , distinct from (a_1, b_1) for which $a_2 < b_1 < b_2$. If $b_2 \geq b$, then we are done by similar reasoning as above. Otherwise, continue the process of choosing intervals. This process terminates as we eventually exhaust the endpoints of each I_k in the open cover. Thus, we get a subcollection $\{(a_k, b_k)\}_{k=1}^N$ for which $a_1 < a$ and $a_{k+1} < b_k$ for all $1 \leq k \leq N - 1$. We also have a $b_N > b$. Then we have

$$\sum_{k=1}^N l(I_k) \geq \sum_{k=1}^N l((a_k, b_k)) = (b_N - a_N) + \cdots + (b_1 - a_1) \geq b - a$$

so that we get $b - a \leq m^*([a, b])$.

Now, let I be any bounded interval. Notice that there exist closed bounded intervals J_1 and J_2 for which

$$J_1 \subseteq I \subseteq J_2$$

and for some $\varepsilon > 0$,

$$l(I) - \varepsilon < l(J_1) \leq l(I) \leq l(J_2) < l(I) + \varepsilon$$

Then since J_1 and J_2 are closed and bounded intervals, and by monotonicity of m^* , we have

$$l(I) - \varepsilon < m^*(J_1) \leq m^*(I) \leq m^*(J_2) < l(I) + \varepsilon$$

so that $l(I) - \varepsilon < m^*(I) < l(I) + \varepsilon$ for all $\varepsilon > 0$. This establishes equality. ■

Lemma 2.1.3. *The Lebesgue outermeasure is translation invariant. That is, if $A \subseteq \mathbb{R}$, and $y \in \mathbb{R}$, then $m^*(A) = m^*(A + y)$.*

Proof. Notice that a countable collection of open bounded intervals $\{I_k\}$ covers A if, and only if the collection $\{I_k + y\}$ of open bounded intervals covers $A + y$. Moreover, notice that $l(I_k) = l(I_k + y)$, so that we get

$$\sum l(I_k) = \sum l(I_k + y)$$

the rest follows from definition. ■

Lemma 2.1.4. *The Lebesgue outermeasure is countable subadditive; that is, if $\{E_k\}$ is a collection of subsets of \mathbb{R} , then*

$$m^*\left(\bigcup E_k\right) \leq \sum m^*(E_k)$$

Proof. Let $\{E_k\}$ a countable collection of sets, and let $E = \bigcup E_k$. Notice that if atleast one E_k has infinite measure, then we are done. Suppose then that for all k , $m^*(E_k)$ is finite. Let $\varepsilon > 0$. Then for all k , there exists a countable collection of open bounded intervals $\{I_{k,i}\}$ covering E_k , and $\sum_i l(I_{k,i}) < m^*(E_k) + \frac{\varepsilon}{2^k}$. By definition, we get

$$m^*(E) \leq \sum_k \sum_i l(I_{k,i}) = \sum_k \sum_i l(I_{k,i}) < \sum_k (m^*(E_k) + \frac{\varepsilon}{2^k}) = \sum_k m^*(E_k) + \varepsilon$$

for all $\varepsilon > 0$. This inequality also holds for $\varepsilon = 0$. ■

Corollary. *The Lebesgue outermeasure is finitely subadditive.*

Proof. Recall that finite collections are also countable collectuions. ■

2.2 Lebesue Measurable Sets

Definition. We call a set E of \mathbb{R} **Lebesue measurable**, provided for any subset A of \mathbb{R} ,

$$m^*(A) = m^*(A \cap E) + m^*(A \cap \mathbb{R} \setminus E)$$

Lemma 2.2.1. *A set E is Lebesue measurable if, and only if for any subset A of \mathbb{R} ,*

$$m^*(A) \geq m^*(A \cap E) + m^*(A \cap \mathbb{R} \setminus E)$$

Proof. We have $A = (A \cap E) \cup (A \cap \mathbb{R} \setminus E)$, so by finite subadditivity, $m^*(A) \leq m^*(A \cap E) + m^*(A \cap \mathbb{R} \setminus E)$. ■

Lemma 2.2.2. *Any set of Lebesue outer measure 0 is Lebesue measurable.*

Proof. Let E have $m^*(E) = 0$ and let $A \subseteq \mathbb{R}$. Notice that $A \cap E \subseteq E$ and $A \cap \mathbb{R} \setminus E \subseteq E$, so that $m^*(A \cap E) \leq m^*(E) = 0$ and $m^*(A \cap \mathbb{R} \setminus E) \leq m^*(A)$. Then we have

$$m^*(A) \geq m^*(A \cap \mathbb{R} \setminus E) = 0 + m^*(A \cap \mathbb{R} \setminus E) = m^*(A \cap E) + m^*(A \cap \mathbb{R} \setminus E)$$
■

Corollary. *Countable sets are measurable.*

Lemma 2.2.3. *The union of two measurable sets is measurable.*

Proof. Let E_1 and E_2 be measurable sets and $A \subseteq \mathbb{R}$. Then $m^*(A) = m^*(A \cap E_1) + m^*(A \cap \mathbb{R} \setminus E_1) = m^*(A \cap E_1) + m^*((A \cap \mathbb{R} \setminus E_1) \cap E_2) + m^*((A \cap \mathbb{R} \setminus E_1) \cap \mathbb{R} \setminus E_2)$. Moreover, notice that

$$(A \cap \mathbb{R} \setminus E_1) \cap \mathbb{R} \setminus E_2 = A \cap \mathbb{R} \setminus (E_1 \cup E_2) \text{ and } (A \cap E_1) \cup (A \cap \mathbb{R} \setminus E_1 \cap E_2) = A \cap (E_1 \cup E_2)$$

Then we get

$$m^*(A) = m^*(A \cap E_1) + m^*((A \cap \mathbb{R} \setminus E_1) \cap E_2) + m^*(A \cap \mathbb{R} \setminus (E_1 \cup E_2)) \geq m^*(A \cap (E_1 \cup E_2)) + m^*(A \cap \mathbb{R} \setminus (E_1 \cup E_2))$$

which makes E_1 and E_2 measurable. ■

Corollary. *The union of a finite collection of measurable sets is measurable.*

Proof. Let $\{E_k\}_{k=1}^n$ a finite collection of measurable sets. By induction on n , we showed that this is true for $n = 1$ and $n = 2$. Now, consider the collections $\{E_k\}_{k=1}^{n+1}$ and suppose that the union $E = \bigcup_{k=1}^n E_k$ is measurable. Notice, then that

$$\bigcup_{k=1}^{n+1} E_k = E \cup E_{n+1}$$

both of which are measurable. Hence measurability of the union of $\{E_k\}_{k=1}^{n+1}$ follows by above. ■

Lemma 2.2.4. *Let A a subset of \mathbb{R} and $\{E_k\}_{k=1}^n$ a finite, disjoint collection of measurable sets. Then*

$$m^*(A \cap \bigcup_{k=1}^n E_k) = \sum_{k=1}^n m^*(A \cap E_k)$$

Proof. By induction on n , for $n = 1$ it is true. Now, suppose that it is true for n , and consider the collection $\{E_k\}_{k=1}^{n+1}$ of disjoint measurable sets. Then we have $A \cap (\bigcup_{k=1}^n E_k) \cap E_{n+1} = A \cap E_{n+1}$ and $A \cap (\bigcup_{k=1}^n E_k) \cap \mathbb{R} \setminus E_{n+1} = A \cap \bigcup_{k=1}^n E_k$. Since E_{n+1} is measurable we get

$$m^*(A \cap \bigcup_{k=1}^{n+1} E_k) = m^*(A \cap E_{n+1}) + m^*(A \cap \bigcup_{k=1}^n E_k) = \sum_{k=1}^{n+1} m^*(A \cap E_k)$$

■

Definition. We call a collection of subsets of \mathbb{R} an **algebra** if it contains \mathbb{R} and it is closed under complements (with respect to \mathbb{R}) and finite unions.

Lemma 2.2.5. *Any algebra of \mathbb{R} is closed under finite intersections.*

Proof. By DeMorgan's laws. ■

Theorem 2.2.6. *The collection of all measurable sets of \mathbb{R} forms an algebra.*

Lemma 2.2.7. *The union of a countable collection of measurable sets is measurable.*

Proof. Without loss of generality, let $\{E_k\}$ a countable disjoint collection of measurable sets, and let $E = \bigcup E_k$. Let A a subset of \mathbb{R} and define $F_n = \bigcup_{k=1}^n E_k$. Then F_n is measurable by lemma 2.2.3, and $\mathbb{R} \setminus E_n \subseteq \mathbb{R} \setminus F_n$. Then

$$m^*(A) = m^*(A \cap F_n) + m^*(A \cap \mathbb{R} \setminus F_n) \geq m^*(A \cap F_n) + m^*(A \cap \mathbb{R} \setminus E_n)$$

hence $m^*(A \cap F_n) = \sum_{k=1}^n m^*(A \cap E_k)$ so that

$$m^*(A) \geq \sum m^*(A \cap E_k) + m^*(A \cap \mathbb{R} \setminus E)$$

By countable subadditivity of m^* we have

$$m^*(A) \geq m^*(A \cap E) + m^*(A \cap \mathbb{R} \setminus E)$$

■

Definition. We call a collection of subsets of \mathbb{R} a **σ -algebra** if it forms an algebra, and it is closed under countable unions.

Lemma 2.2.8. *Any σ -algebra of \mathbb{R} is closed under countable intersections.*

Theorem 2.2.9. *The collection of measurable sets of \mathbb{R} forms a σ -algebra.*

Lemma 2.2.10. *Every interval of \mathbb{R} is measurable.*

Proof. Consider an interval of the form (a, ∞) , for any $a \in \mathbb{R}$. Let $A \subseteq \mathbb{R}$, such that $A \notin \mathcal{A}$; otherwise, just take $A \setminus \{a\}$. Then since $m^*(A)$ is a greatest lower bound, it is sufficient to show that for any countable collection $\{I_k\}$ of open, bounded intervals covering A , that

$$m^*(A_1) + m^*(A_2) \leq \sum l(I_k)$$

where

$$A_1 = A \cap (-\infty, a) \text{ and } A_2 = A \cap (a, \infty)$$

Indeed, let $\{I_k\}$ be such a collection, and define

$$I_{k,1} = I_k \cap (-\infty, a) \text{ and } I_{k,2} = I_k \cap (a, \infty)$$

Then $\{I_{k,1}\}$ and $\{I_{k,2}\}$ are collections of open, bounded intervals which cover A_1 and A_2 respectively. Hence, by definition of m^* , we have $m^*(A_1) \leq \sum l(I_{k,1})$ and $m^*(A_2) \leq \sum l(I_{k,2})$; moreover, notice that $l(I_k) = l(I_{k,1}) + l(I_{k,2})$. Therefore, we get

$$m^*(A_1) + m^*(A_2) \leq \sum l(I_{k,1}) + \sum l(I_{k,2}) = \sum l(I_k)$$

and we are done. ■

Corollary. *Open sets, and closed sets of \mathbb{R} are measurable.*

Definition. We define the intersection of all σ -algebras of \mathbb{R} to be the **Borel σ -algebra**, and call its elements **Borel sets**.

Theorem 2.2.11. *The σ -algebra of all measurable sets of \mathbb{R} contains the Borel σ -algebra of \mathbb{R} . Moreover, it contains every interval of \mathbb{R} , open and closed sets, as well as G_δ and F_σ sets.*

Lemma 2.2.12. *Lebesgue measurable sets are translation invariant. That is, if E is Lebesgue measurable, and $y \in \mathbb{R}$, then $E + y$ is Lebesgue measurable.*

Proof. Let E be measurable, $y \in \mathbb{R}$, and $A \subseteq \mathbb{R}$. Then

$$m^*(A) = m^*(A \setminus y) = m^*(A \setminus y \cap E) + m^*(A \setminus y \cap \mathbb{R} \setminus E) = m^*(A \cap (E + y)) + m^*(A \cap \mathbb{R} \setminus (E + y))$$

■

2.3 Inner and Outer Approximations

Lemma 2.3.1 (Excision). *If A and B are sets, with A Lebesgue measurable of finite outer measure, and $A \subseteq B$, then*

$$m^*(B \setminus A) = m^*(B) - m^*(A)$$

Theorem 2.3.2 (The Outer Approximation Theorem). *Let $E \subseteq \mathbb{R}$. The following are equivalent.*

- (1) E is Lebesgue measurable.
- (2) For all $\varepsilon > 0$ there is an open set U of \mathbb{R} containing E such that $m^*(U \setminus E) < \varepsilon$.
- (3) There exists a G_δ set G containing E for which $m^*(G \setminus E) = 0$.

Proof. Suppose first that E is measurable and let $\varepsilon > 0$. Now, if $m^*(E)$ is finite, then there is a countable collection $\{I_k\}$ of open intervals covering E , for which, by definition of m^* as a greatest lower bound,

$$\sum l(I_k) < m^*(E) + \varepsilon$$

Let $U = \bigcup I_k$, then $E \subseteq U$, and U is open in \mathbb{R} . Thus by definition of m^* again, we have

$$m^*(U) \leq \sum l(I_k) < m^*(E) + \varepsilon$$

so that $m^*(U) - m^*(E) < \varepsilon$. Now, since E is measurable of finite outer measure, by excision, we get $m^*(U \setminus E) = m^*(U) - m^*(E) < \varepsilon$.

Now, if $m^*(E)$ is infinite, then let $\{E_k\}$ be a countable disjoint collection of measurable sets each of finite outer measure, and let $E = \bigcup E_k$. Then by above there exist open sets U_k containing E_k , for each k such that $m^*(U_k \setminus E_k) < \frac{\varepsilon}{2^k}$. Let $U = \bigcup U_k$, then U is open in \mathbb{R} , and $E \subseteq U$. Moreover observe that

$$U \setminus E = \bigcup U_k \setminus E_k$$

Then we get by subadditivity

$$m^*(U \setminus E) \leq \sum m^*(U_k \setminus E_k) < \sum \frac{\varepsilon}{2^k} = \varepsilon$$

Now, suppose that assertion (2) holds, and choose an open set U_k containing E for which $m^*(U_k \setminus E) < \frac{1}{k}$. Define $G = \bigcup U_k$. Then G is a G_δ set, and $E \subseteq G$. Moreover we have that

$$G \setminus E \subseteq U_k \setminus E \text{ for all } k$$

so by monotonicity

$$m^*(G \setminus E) \leq m^*(U_k \setminus E) < \frac{1}{k}$$

Then as $k \rightarrow \infty$, this outer measure approaches 0.

Now if (3) holds, since $m^*(G \setminus E) = 0$, the set $G \setminus E$ is measurable. Since the space of all measurable sets is a σ -algebra, then the set $E = G \cap \mathbb{R} \setminus (G \setminus E)$ is measurable. ■

Corollary (The Inner Approximation Theorem). *The following are equivalent.*

- (1) E is Lebesgue measurable.
- (2) For all $\varepsilon > 0$ there is a closed set V of \mathbb{R} contained in E such that $m^*(E \setminus V) < \varepsilon$.
- (3) There exists an F_σ set F contained in E for which $m^*(E \setminus F) = 0$.

Proof. One can apply DeMorgan's laws. ■

Theorem 2.3.3. *Let E a Lebesgue measurable set of finite outer measure. then for every $\varepsilon > 0$ there is a finite disjoint collection $\{I_k\}$ of open intervals for which if $U = \bigcup I_k$, then*

$$m^*(E \setminus U) + m^*(U \setminus E) < \varepsilon$$

Proof. By the outer approximation theorem, there is an open set V containing E for which $m^*(V \setminus E) < \frac{\varepsilon}{2}$. Now, since E is measurable of finite outer measure, by excision we have

$$m^*(V) - m^*(E) < \frac{\varepsilon}{2}$$

so that $m^*(V)$ is also finite. Now, recall that every open set of real numbers is the disjoint collection of open intervals, hence let $V = \bigcup I_k$. Each I_k is measurable with $m^*(I_k) = l(I_k)$. Thereofre, by lemma 2.2.4 and monotonicity,

$$\sum_{k=1}^n l(I_k) \leq m^*(V) \text{ is finite}$$

So $\sum I_k$ is finite. Now, choose an $n \in \mathbb{Z}^+$ for which $\sum_{k=n+1}^\infty l(I_k) < \frac{\varepsilon}{2}$ and define $U = \bigcup_{k=1}^n I_k$. Then $U \setminus E \subseteq V \setminus E$ so by monotonicity, $m^*(U \setminus E) < \frac{\varepsilon}{2}$. Moreover, we have $E \setminus U \subseteq V \setminus U = \bigcup_{k=n+1}^\infty I_k$ so that $m^*(E \setminus U) < \frac{\varepsilon}{2}$. Therefore, we see that

$$m^*(U \setminus E) + m^*(E \setminus U) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$
■

2.4 The Borel-Cantelli Lemma

Definition. We define the **Lebesgue measure** m to be the restriction of the Lebesgue outer measure, m^* to the space of all Lebesgue measurable sets. That is, if E is Lebesgue measurable, the

$$m(E) = m^*(E)$$

Lemma 2.4.1 (Countable additivity). *The Lebesgue measure is countable additive. That is, if $\{E_k\}$ is a countable collection of disjoint measurable sets, then*

$$m\left(\bigcup E_k\right) = \sum m(E_k)$$

Proof. Since the space of Lebesgue measurable sets forms a σ -algebra, and are closed under countable unions, the set $E = \bigcup E_k$ is Lebesgue measurable. Moreover, by subadditivity of m^* , and definition of m ,

$$m(E) \leq \sum_k = 1^\infty m(E_k)$$

Notice, however, that $\bigcup_{k=1}^n E_k \subseteq E$, so that by monotonicity, $\sum_{k=1}^n m(E_k) \leq m(E)$. Then as $n \rightarrow \infty$, this sum converges to $\sum_{k=1}^\infty E_k$ so

$$\sum_{k=1}^\infty E_k \leq m(E)$$

and equality is established. ■

Corollary. *The Lebesgue measure is finitely additive.*

Theorem 2.4.2. *The Lebesgue measure assigns to intervals their lengths, is translation invariant, and countable additive.*

Theorem 2.4.3 (Continuity). *The following are true for the Lebesgue measure.*

(1) *If $\{A_k\}$ is an increasing sequence of Lebesgue measurable sets, then*

$$m\left(\bigcup A_k\right) = \lim_{k \rightarrow \infty} m(A_k)$$

(2) *If $\{B_k\}$ is an decreasing sequence of Lebesgue measurable sets for which $m(B_1)$ is finite, then*

$$m\left(\bigcap B_k\right) = \lim_{k \rightarrow \infty} m(B_k)$$

Proof. If $k_0 \in \mathbb{Z}^+$ is such that $m(A_{k_0})$ is infinite, then by monotonicity, $m(\bigcup A_k)$ is infinite so that $m(A_k)$ is infinite for all $k \geq k_0$. Suppose then, that $m(A_k)$ is finite for all k and define $A_0 = \emptyset$. Furthermore, define $C_k = A_k \setminus A_{k-1}$ for all $k \geq 1$. then since $\{A_k\}$ is a disjoint collection of measurable sets, then so is C_k , and $\bigcup A_k = \bigcup C_k$. By countable additivity, we have

$$m\left(\bigcup A_k\right) = m\left(\bigcup C_k\right) = \sum m(A_k \setminus A_{k-1})$$

By excision, we get

$$\sum m(A_k) - m(A_{k-1}) = \lim_{n \rightarrow \infty} \sum_{k=1}^n m(A_k) - m(A_{k-1}) = \lim (m(A_n) - m(A_0)) = \lim_{n \rightarrow \infty} m(A_n)$$

since $m(A_0) = 0$.

Now, define $D_k = B_1 \setminus B_k$. Since $\{B_k\}$ is decreasing, the sequence $\{D_k\}$ of measurable sets is increasing. Then by above,

$$m\left(\bigcup D_k\right) = \lim_{k \rightarrow \infty} m(D_k)$$

By DeMorgan's laws, $\bigcup D_k = B_1 \setminus \bigcap B_k$. On the otherhand, by excision, since $m(B_1)$ is finite, we get

$$m(D_k) = m(B_1) - m(B_k)$$

so that

$$m(B_1 \setminus \bigcap B_k) = \lim_{n \rightarrow \infty} (m(B_1) - m(B_n))$$

By excision again, we are done. ■

Definition. We say a property holds **almost everywhere** on a measurable set E if there exists a measurable set $E_0 \subseteq E$ with $m(E_0) = 0$ for which the property holds for all $x \in E \setminus E_0$.

Lemma 2.4.4 (Borel-Cantelli). *Let $\{E_k\}$ a countable collection of measurable sets such that the sum $\sum m(E_k)$ is finite. Then almost all $x \in \mathbb{R}$ belong to at most finitely many of the E_k .*

Proof. By countable subadditivity, we have $m(\bigcup E_k) \leq \sum_{k=n} m(E_k)$ is finite. Thus, by continuity, we have

$$m\left(\bigcap_{n=1} \left(\bigcup_{k=n} E_k\right)\right) = \lim_{n \rightarrow \infty} m\left(\bigcup_{k=n} E_k\right) \leq \lim_{n \rightarrow \infty} \sum_{k=n} m(E_k) = 0$$

so that almost all x does not belong to the intersection $\bigcap_{n=1} \bigcup_{k=n} E_k$ and hence belongs to at most finitely many of the E_k . ■

2.5 Nonmeasurable Sets, The Cantor Set, and The Cantor-Lebesgue Function

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