Measure Theory

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Contents

Chapter 1

Measure and Measure Spaces

1.1 σ-Algebras

Definition. Let X be a nonempty set. An **algebra** of sets on X is a nonempty collection $\mathcal{A}$ of subsets of X which are closed under finite unions and complements in X. We call $\mathcal{A}$ a **σ-algebra** if it is closed under countable unions.

Lemma 1.1.1. *Let X be a set and $\mathcal{A}$ an algbra on X. Then $\mathcal{A}$ is closed under finite intersections.*

Proof. Let $\{E_\lambda\}$ be a collection of sets of $\mathcal{A}$. Then by finite union $E = \bigcup E_\lambda \in \mathcal{A}$. Then by complements, $X \backslash E = \bigcap X \backslash E_\lambda \in \mathcal{A}$. ■

Corollary. *σ-algebras are closed under countable disjoint unions.*

Proof. Let $\mathcal{A}$ a σ-algebra, and let $\{E_n\}$ a collection of (not necessarily disjoint) sets in $\mathcal{A}$. Then take

$$F_n = E_n \backslash \Big(\bigcup_{k=1}^{n-1} E_k \Big) \tag{1.1}$$

Then each F_n is a set in $\mathcal{A}$, and are pairwise disjoint. Moreover, $\bigcup E_n = \bigcup F_n$. ■

Lemma 1.1.2. *Let X be a set, and $\mathcal{A}$ an algebra on X. Then $\emptyset \in \mathcal{A}$ and $X \in \mathcal{A}$.*

Proof. By closure of finite unions, notice that if $E \in \mathcal{A}$, then $E \cup X \backslash E = X \in \mathcal{A}$ lemma **??** gives us that $E \cap X \backslash E = \emptyset \in \mathcal{A}$. ■

Example 1.1. (1) The collections $\{\emptyset, X\}$ and 2^X are σ-algebras on any set X.

(2) Let X be an uncountable set. Then the collection

$$\mathcal{C} = \{E \subseteq X : E \text{ is countable or } X \backslash E \text{ is countable}\}$$

defines a σ-algebra of sets on X, since countable unions of countable sets are countable, and $\mathcal{C}$ is closed under complements. We call $\mathcal{C}$ the **σ-algebra of countable or co-countable sets**.

Lemma 1.1.3. *Let $\{A_\lambda\}$ be a collection of σ-algebras on a set X. Then the intersection*

$$\mathcal{A} = \bigcap \mathcal{A}_\lambda$$

is a σ-algebra on X. Moreover, if $F \subseteq X$, then there exists a unique smallest σ-algebra containing F; in particular, it is the intersection of all σ-algebras containing F.

Proof. Notice that since each $\mathcal{A}_\lambda$ is a σ-algebra, they are closed under countable unions and complements. Hence by definition, $\mathcal{A}$ must also be closed under countable unions and complements.

Now, let $F \subseteq X$ and let $\{\mathcal{A}_\lambda\}$ be the collection of all σ-algebras containing F. Then the intersection $\mathcal{A} = \bigcap \mathcal{A}_\lambda$ is also a σ-algebra containing F; by above. Now, suppose that there is a smallest σ-algebra $\mathcal{B}$ containing F. Then we have that $\mathcal{B} \subseteq \mathcal{A}$. Now, by definition of $\mathcal{A}$ as the intersection of all σ-algberas containing F, we get that $\mathcal{A} \subseteq \mathcal{B}$; so that $\mathcal{B} = \mathcal{A}$. ■

Definition. Let X be a nonempty set and $F \subseteq X$. We define the σ-algebra **generated** by F to be the smallest such σ-algebra $\mathcal{M}(F)$ containing F.

Lemma 1.1.4. *Let X be a set and let $E, F \subseteq X$. Then if $E \subseteq \mathcal{M}(F)$, then $\mathcal{M}(E) \subseteq \mathcal{M}(F)$.*

Proof. We have that since $E \subseteq \mathcal{M}(F)$, and $\mathcal{M}(E)$ is the intersection of all σ-algebras containing E, then $\mathcal{M}(E) \subseteq \mathcal{M}(F)$. ■

Definition. Let X be a topological space. We define the **Borel σ-algebra** on X to be the σ-algebra $\mathcal{B}(X)$ generated by all open sets of X; that is

$$\mathcal{B}(X) = \mathcal{M}(\mathcal{T})$$

where $\mathcal{T}$ is the topology on X. We call the elements of $\mathcal{B}(X)$ **Borel-sets**

Definition. Let X be a topological space. We call a countable intersection of open sets of X a G_δ**-set** of X. We call a countable union of closed sets of X an F_σ**-set** of X.

Theorem 1.1.5. *The Borel σ-algebra on $\mathbb{R}$, $\mathcal{B}(\mathbb{R})$, is generated by the following.*

(1) All open intervals of $\mathbb{R}$.

(2) All closed intervals of $\mathbb{R}$.

(3) All half-open intervals of $\mathbb{R}$.

(4) All open rays of $\mathbb{R}$.

(5) All closed rays of $\mathbb{R}$.

Definition. Let X_α be a collection of non-empty sets, and let $X = \prod X_\alpha$. If $\mathcal{M}_\alpha$ is a σ-algebra on X_α, then we define the **product σ-algebra** on X to be the smallest σ-algebra generated by all $\pi_\alpha^{-1}(E_\alpha)$, where $E_\alpha \in \mathcal{M}_\alpha$, and $\pi_\alpha : X \to X_\alpha$ is the projection map onto the α-th coordinate. We denote the product σ-algebra by $\bigotimes \mathcal{M}_\alpha$.

Lemma 1.1.6. *Let $\{X_n\}$ be a countable collection of sets, each with a σ-algebra $\mathcal{M}_n$, and let $X = \prod X_n$. Then the product σ-algebra $\bigotimes \mathcal{M}_n$ on X is generated by all $\prod E_n$, where $E_n \in \mathcal{M}_n$.*

Proof. Let $E_n \in \mathcal{M}_n$, then by definition of the projection map, $\pi_n^{-1}(E_n) = \prod E_k$ where $E_k = X_k$ for all $k \neq n$. On the otherhand, we can see that $\prod E_n = \bigcap \pi_n^{-1}(E_n)$. ■

Lemma 1.1.7. *Let $\{X_\alpha\}$ be a collection of sets, each together with a σ-algebra $\mathcal{M}_\alpha$. If each $\mathcal{M}_\alpha$ is generated by some $\mathcal{E}_\alpha$, then$\bigotimes \mathcal{M}_\alpha$ is generated by all $\pi_\alpha^{-1}(E_\alpha)$, where $E_\alpha \in \mathcal{E}_\alpha$.*

Proof. Let $\mathcal{F} = \{\pi_\alpha^{-1}(E_\alpha) : E_\alpha \in \mathcal{E}_\alpha\}$. Then by lemma 1.1.4, $\mathcal{M}(\mathcal{F}) \subseteq \bigotimes \mathcal{M}_\alpha$. On the otherhand, for any α, the collection of all $\pi_\alpha^{-1}(E) \in \mathcal{M}(\mathcal{F})$ is a σ-algebra on X_α, containing $\mathcal{E}_\alpha$; and hence, $\mathcal{M}_\alpha$. That is, $\pi_\alpha^{-1}(E) \in \mathcal{M}(\mathcal{F})$ for all $E \in \mathcal{M}_\alpha$, which gives us the reverse inclusion. ■

Corollary. *If $\{X_\alpha\}$ is a countable collection, theb $\bigotimes \mathcal{M}_\alpha$ is generated by all $\prod E_\alpha$, where $E_\alpha \in \mathcal{E}_\alpha$.*

Lemma 1.1.8. *Let $X_1, \ldots, X_n$ be metric spaces, and $X = \prod_{i=1}^n X_i$ on the product topology. Then*

$$\bigotimes(\mathcal{B}(X_i)) \subseteq \mathcal{B}(X)$$

Moreover, if each X_i is seperable, then equality is established.

Proof. We have that $\bigotimes \mathcal{B}(X_i)$ is generated by each $\pi_i^{-1}(U_i)$, where U_i is an open set in X_i. Since these sets are open, again by lemma 1.1.4, $\bigotimes \mathcal{B}(X_i) \subseteq \mathcal{B}(X)$.

Now, suppose that each X_i is seperable, and let C_i a countable dense set in X_i, and let $\mathcal{E}_i$ be the collection of all open balls in X_i with rational radius r, and center in C_i. Then every open set in X_i is a countable union of members of $\mathcal{E}_i$. Moreover, the set of points in X whose i-th coordinate is in C_i, for all i, is countable dense in X_i. Hence, $\mathcal{B}(X_i)$ is generated by $\mathcal{E}_i$, and since (X) is generated by all $\prod_{i=1}^n E_i$, where $E_i \in \mathcal{E}_i$, we get $\mathcal{B}(X) \subseteq \mathcal{B}(X_i)$, and equality is established. ■

Corollary. $\mathcal{B}(\mathbb{R}^n) = \bigotimes_{i=1}^n \mathcal{B}(\mathbb{R})$.

Definition. We define an **elementary family** on a set X to be a collection $\mathcal{E}$ of subsets of X such that:

(1) $\emptyset \in \mathcal{E}$.

(2) If $E, F \in \mathcal{E}$, then $E \cap F \in \mathcal{E}$.

(3) If $E \in \mathcal{E}$, then $X \backslash E$ is a finite disjoint union of members of $\mathcal{E}$.

Lemma 1.1.9. *Let X be a set and $\mathcal{E}$ an elementary family on X. Let $\mathcal{A}$ be the collection of all finite disjoint unions of members of $\mathcal{E}$. Then $\mathcal{A}$ is an algebra on X.*

Proof. Let $A, B \in \mathcal{E}$, and let $X \backslash B = \bigcup_{i=1}^{n} C_i$, where each $C_i \in \mathcal{E}$ for all $1 \leq i \leq n$, and are disjoint. Then we have

$$A \cup B = (A \backslash V) \cup B \text{ and } A \backslash B = \bigcup_{i=1}^{n} (A \cap C_i)$$

so that $A \cup B \in \mathcal{A}$, and $A \backslash B \in \mathcal{A}$. Now, by induction on n, suppose that $A_1, \ldots, A_n \in \mathcal{A}$ are disjoint, then

$$\bigcup_{i=1}^{n+1} A_i = A_{n+1} \cup \bigcup_{i=1}^{n} A_i \backslash A_{n+1}$$

is also a disjoint union. Moreover, we have that if $X \backslash A_n = \bigcup_{i=1}^{N_m} B_m^i$, where the union is disjoint, then

$$X \backslash \Big(\bigcup_{m=1}^{n} A_m \Big) = \bigcap_{m=1}^{n} \Big(\bigcup_{i=1}^{N_m} B_m^i \Big)$$

is also a disjoint union. This makes $\mathcal{A}$ an algebra on X. ■

1.2 Measures

Bibliography

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