Measure Theory

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Contents

Chapter 1

Measure and Measure Spaces

1.1 σ-Algebras

Definition. Let X be a nonempty set. An **algebra** of sets on X is a nonempty collection $\mathcal{A}$ of subsets of X which are closed under finite unions and complements in X. We call $\mathcal{A}$ a **σ-algebra** if it is closed under countable unions.

Lemma 1.1.1. *Let X be a set and $\mathcal{A}$ an algbra on X. Then $\mathcal{A}$ is closed under finite intersections.*

Proof. Let $\{E_\lambda\}$ be a collection of sets of $\mathcal{A}$. Then by finite union $E = \bigcup E_\lambda \in \mathcal{A}$. Then by complements, $X \backslash E = \bigcap X \backslash E_\lambda \in \mathcal{A}$. ■

Corollary. *σ-algebras are closed under countable disjoint unions.*

Proof. Let $\mathcal{A}$ a σ-algebra, and let $\{E_n\}$ a collection of (not necessarily disjoint) sets in $\mathcal{A}$. Then take

$$F_n = E_n \backslash \Big(\bigcup_{k=1}^{n-1} E_k \Big) \tag{1.1}$$

Then each F_n is a set in $\mathcal{A}$, and are pairwise disjoint. Moreover, $\bigcup E_n = \bigcup F_n$. ■

Lemma 1.1.2. *Let X be a set, and $\mathcal{A}$ an algebra on X. Then $\emptyset \in \mathcal{A}$ and $X \in \mathcal{A}$.*

Proof. By closure of finite unions, notice that if $E \in \mathcal{A}$, then $E \cup X \backslash E = X \in \mathcal{A}$ lemma **??** gives us that $E \cap X \backslash E = \emptyset \in \mathcal{A}$. ■

Example 1.1. (1) The collections $\{\emptyset, X\}$ and 2^X are σ-algebras on any set X.

(2) Let X be an uncountable set. Then the collection

$$\mathcal{C} = \{E \subseteq X : E \text{ is countable or } X \backslash E \text{ is countable}\}$$

defines a σ-algebra of sets on X, since countable unions of countable sets are countable, and $\mathcal{C}$ is closed under complements. We call $\mathcal{C}$ the **σ-algebra of countable or co-countable sets**.

Lemma 1.1.3. *Let $\{A_\lambda\}$ be a collection of σ-algebras on a set X. Then the intersection*

$$\mathcal{A} = \bigcap \mathcal{A}_\lambda$$

is a σ-algebra on X. Moreover, if $F \subseteq X$, then there exists a unique smallest σ-algebra containing F; in particular, it is the intersection of all σ-algebras containing F.

Proof. Notice that since each $\mathcal{A}_\lambda$ is a σ-algebra, they are closed under countable unions and complements. Hence by definition, $\mathcal{A}$ must also be closed under countable unions and complements.

Now, let $F \subseteq X$ and let $\{\mathcal{A}_\lambda\}$ be the collection of all σ-algebras containing F. Then the intersection $\mathcal{A} = \bigcap \mathcal{A}_\lambda$ is also a σ-algebra containing F; by above. Now, suppose that there is a smallest σ-algebra $\mathcal{B}$ containing F. Then we have that $\mathcal{B} \subseteq \mathcal{A}$. Now, by definition of $\mathcal{A}$ as the intersection of all σ-algberas containing F, we get that $\mathcal{A} \subseteq \mathcal{B}$; so that $\mathcal{B} = \mathcal{A}$. ■

Definition. Let X be a nonempty set and $F \subseteq X$. We define the σ-algebra **generated** by F to be the smallest such σ-algebra $\mathcal{M}(F)$ containing F.

Lemma 1.1.4. *Let X be a set and let $E, F \subseteq X$. Then if $E \subseteq \mathcal{M}(F)$, then $\mathcal{M}(E) \subseteq \mathcal{M}(F)$.*

Proof. We have that since $E \subseteq \mathcal{M}(F)$, and $\mathcal{M}(E)$ is the intersection of all σ-algebras containing E, then $\mathcal{M}(E) \subseteq \mathcal{M}(F)$. ■

Definition. Let X be a topological space. We define the **Borel σ-algebra** on X to be the σ-algebra $\mathcal{B}(X)$ generated by all open sets of X; that is

$$\mathcal{B}(X) = \mathcal{M}(\mathcal{T})$$

where $\mathcal{T}$ is the topology on X. We call the elements of $\mathcal{B}(X)$ **Borel-sets**

Definition. Let X be a topological space. We call a countable intersection of open sets of X a G_δ**-set** of X. We call a countable union of closed sets of X an F_σ**-set** of X.

Theorem 1.1.5. *The Borel σ-algebra on $\mathbb{R}$, $\mathcal{B}(\mathbb{R})$, is generated by the following.*

(1) All open intervals of $\mathbb{R}$.

(2) All closed intervals of $\mathbb{R}$.

(3) All half-open intervals of $\mathbb{R}$.

(4) All open rays of $\mathbb{R}$.

(5) All closed rays of $\mathbb{R}$.

Definition. Let X_α be a collection of non-empty sets, and let $X = \prod X_\alpha$. If $\mathcal{M}_\alpha$ is a σ-algebra on X_α, then we define the **product σ-algebra** on X to be the smallest σ-algebra generated by all $\pi_\alpha^{-1}(E_\alpha)$, where $E_\alpha \in \mathcal{M}_\alpha$, and $\pi_\alpha : X \to X_\alpha$ is the projection map onto the α-th coordinate. We denote the product σ-algebra by $\bigotimes \mathcal{M}_\alpha$.

Lemma 1.1.6. *Let $\{X_n\}$ be a countable collection of sets, each with a σ-algebra $\mathcal{M}_n$, and let $X = \prod X_n$. Then the product σ-algebra $\bigotimes \mathcal{M}_n$ on X is generated by all $\prod E_n$, where $E_n \in \mathcal{M}_n$.*

Proof. Let $E_n \in \mathcal{M}_n$, then by definition of the projection map, $\pi_n^{-1}(E_n) = \prod E_k$ where $E_k = X_k$ for all $k \neq n$. On the otherhand, we can see that $\prod E_n = \bigcap \pi_n^{-1}(E_n)$. ■

Lemma 1.1.7. *Let $\{X_\alpha\}$ be a collection of sets, each together with a σ-algebra $\mathcal{M}_\alpha$. If each $\mathcal{M}_\alpha$ is generated by some $\mathcal{E}_\alpha$, then$\bigotimes \mathcal{M}_\alpha$ is generated by all $\pi_\alpha^{-1}(E_\alpha)$, where $E_\alpha \in \mathcal{E}_\alpha$.*

Proof. Let $\mathcal{F} = \{\pi_\alpha^{-1}(E_\alpha) : E_\alpha \in \mathcal{E}_\alpha\}$. Then by lemma 1.1.4, $\mathcal{M}(\mathcal{F}) \subseteq \bigotimes \mathcal{M}_\alpha$. On the otherhand, for any α, the collection of all $\pi_\alpha^{-1}(E) \in \mathcal{M}(\mathcal{F})$ is a σ-algebra on X_α, containing $\mathcal{E}_\alpha$; and hence, $\mathcal{M}_\alpha$. That is, $\pi_\alpha^{-1}(E) \in \mathcal{M}(\mathcal{F})$ for all $E \in \mathcal{M}_\alpha$, which gives us the reverse inclusion. ■

Corollary. *If $\{X_\alpha\}$ is a countable collection, theb $\bigotimes \mathcal{M}_\alpha$ is generated by all $\prod E_\alpha$, where $E_\alpha \in \mathcal{E}_\alpha$.*

Lemma 1.1.8. *Let $X_1, \ldots, X_n$ be metric spaces, and $X = \prod_{i=1}^n X_i$ on the product topology. Then*

$$\bigotimes(\mathcal{B}(X_i)) \subseteq \mathcal{B}(X)$$

Moreover, if each X_i is seperable, then equality is established.

Proof. We have that $\bigotimes \mathcal{B}(X_i)$ is generated by each $\pi_i^{-1}(U_i)$, where U_i is an open set in X_i. Since these sets are open, again by lemma 1.1.4, $\bigotimes \mathcal{B}(X_i) \subseteq \mathcal{B}(X)$.

Now, suppose that each X_i is seperable, and let C_i a countable dense set in X_i, and let $\mathcal{E}_i$ be the collection of all open balls in X_i with rational radius r, and center in C_i. Then every open set in X_i is a countable union of members of $\mathcal{E}_i$. Moreover, the set of points in X whose i-th coordinate is in C_i, for all i, is countable dense in X_i. Hence, $\mathcal{B}(X_i)$ is generated by $\mathcal{E}_i$, and since (X) is generated by all $\prod_{i=1}^n E_i$, where $E_i \in \mathcal{E}_i$, we get $\mathcal{B}(X) \subseteq \mathcal{B}(X_i)$, and equality is established. ■

Corollary. $\mathcal{B}(\mathbb{R}^n) = \bigotimes_{i=1}^n \mathcal{B}(\mathbb{R})$.

Definition. We define an **elementary family** on a set X to be a collection $\mathcal{E}$ of subsets of X such that:

(1) $\emptyset \in \mathcal{E}$.

(2) If $E, F \in \mathcal{E}$, then $E \cap F \in \mathcal{E}$.

(3) If $E \in \mathcal{E}$, then $X \backslash E$ is a finite disjoint union of members of $\mathcal{E}$.

Lemma 1.1.9. *Let X be a set and $\mathcal{E}$ an elementary family on X. Let $\mathcal{A}$ be the collection of all finite disjoint unions of members of $\mathcal{E}$. Then $\mathcal{A}$ is an algebra on X.*

Proof. Let $A, B \in \mathcal{E}$, and let $X \backslash B = \bigcup_{i=1}^{n} C_i$, where each $C_i \in \mathcal{E}$ for all $1 \leq i \leq n$, and are disjoint. Then we have

$$A \cup B = (A \backslash V) \cup B \text{ and } A \backslash B = \bigcup_{i=1}^{n} (A \cap C_i)$$

so that $A \cup B \in \mathcal{A}$, and $A \backslash B \in \mathcal{A}$. Now, by induction on n, suppose that $A_1, \ldots, A_n \in \mathcal{A}$ are disjoint, then

$$\bigcup_{i=1}^{n+1} A_i = A_{n+1} \cup \bigcup_{i=1}^{n} A_i \backslash A_{n+1}$$

is also a disjoint union. Moreover, we have that if $X \backslash A_n = \bigcup_{i=1}^{N_m} B_m^i$, where the union is disjoint, then

$$X \backslash \Big(\bigcup_{m=1}^{n} A_m \Big) = \bigcap_{m=1}^{n} \Big(\bigcup_{i=1}^{N_m} B_m^i \Big)$$

is also a disjoint union. This makes $\mathcal{A}$ an algebra on X. ■

1.2 Measures

Definition. Let X be a set together with a σ-algebra $\mathcal{M}$. We define a **measure** on $\mathcal{M}$ to be a function $\mu : \mathcal{M} \to [0, \infty)$ for which the following hold:

(1) $m(\emptyset) = 0$.

(2) If $\{E_n\}$ is a countable disjoint collection of members of $\mathcal{M}$, then

$$m\Big(\bigcup E_n \Big) = \sum m(E_n) \tag{1.2}$$

We call m a **finitely additive measure** if instead of (2), m satisfies:

(2') If $\{E_i\}_{i=1}^{n}$ is a finite disjoint collection of members of $\mathcal{M}$, then

$$m\Big(\bigcup_{i=1}^{n} E_i \Big) = \sum_{i=1}^{n} m(E_i) \tag{1.3}$$

Definition. We call a set X together with a σ-algebra $\mathcal{M}$ a **measurable space**, and we call the members of $\mathcal{M}$ **measurable sets**. If $m : \mathcal{M} \to [0, \infty)$ is a measure on $\mathcal{M}$, then we call X together with $\mathcal{M}$ a **measure space**.

Definition. Let X together with a σ-algebra be a measure space with measure m. If $m(X) < \infty$, then we call m a **finite measure**, and if $\{E_n\}$ is a covering of X by measurable sets, each with $m(E_n) < \infty$ for all n, then we call m **σ-finite**. We also call the set $E = \bigcup E_n$ **σ-finite**. We call m **semi-finite** if for any measurable set E, of $m(E) = \infty$, there is a measurable set F contained in E such that $0 < m(F) < \infty$.

Lemma 1.2.1. *σ-finite measures are semi-finite.*

Example 1.2. (1) LEt X be a non-empty set, and let $f : X \to [0, \infty)$ be any function on X. Then f defines a measure m on 2^X by the rule

$$m(E) = \sum_{x \in E} f(x)$$

Now, m is semi-finite if, and only if $f(x) < \infty$ for all $x \in X$, and m is σ-finite if, and only if m is semi-finite, and the pre-image $f^{-1}((0, \infty))$ is countable.

(2) Consider the measure m of example (1) above, where $f(x) = 1$ for all $x \in X$. Then we call m the **counting measure** on 2^X. Indeed, observe that

$$m(E) = \sum_{x \in E} 1 = |E|$$

which counts the elements of E.

(3) Consider the measure m of example (1) above, where f is defined for any $x_0 \in X$ to be:

$$f(x) = \begin{cases} 1, & \text{if } x = x_0 \\ 0, & \text{if } x \neq x_0 \end{cases}$$

We call this measure the **Dirichlet measure**.

(4) Let X be an uncountable set, and let $\mathcal{M}$ the σ-algebra of all countable or co-countable sets. Define m on $\mathcal{M}$ by $m(E) = 0$ if E is countable, and $m(E) = 1$ if E is co-countable. Then m defines a measure on $\mathcal{M}$.

(5) Let X be an infinite set, and define m on 2^X by $m(E) = 0$ if E is finite, and $m(E) = \infty$ if E is infinite. Then m is a finitely subadditive measure on 2^X, but not a measure on 2^X.

Theorem 1.2.2. *Let X be a measure space with measure m. The following are true.*

(1) If E and F are measurable with $E \subseteq F$, then

$$m(E) \leq m(F)$$

(2) If $\{E_n\}$ is a countable collection of measurable sets, then

$$m\Big(\bigcup E_n\Big) \leq \sum m(E_n)$$

(3) If $\{E_n\}$ is a countable collection of measurable sets, in which $E_1 \subseteq E_2 \subseteq \dots$, then

$$m\Big(\bigcup E_n\Big) = \lim_{n \longrightarrow \infty} m(E_n)$$

(4) *If $\{E_n\}$ is a countable collection of measurable sets, in which $\ldots, \subseteq E_2 \subseteq E_1$ and $m(E_1) < \infty$, then*

$$m\Big(\bigcap E_n\Big) = \lim_{n\longrightarrow\infty} m(E_n)$$

Proof. For the first statement, let $E \subseteq F$ be measurable sets, then observe that

$$m(E) \leq m(E) + m(F\backslash E) = m(E \cup F\backslash E) = m(F)$$

For the second statement, define $F_1 = E_1$, and $F_i = E_i\backslash\bigcup_{i=1}^{i-1} E_i$ for all $i > 1$. Then $\{F_n\}$ is a finite disjoint collection of measurable sets, with $\bigcup_{i=1}^{n} F_i = \bigcup_{i=1}^{n} E_i$. By the above argument, we get

$$m\Big(\bigcup_{i=1}^{n} E_i\Big) = m\Big(\bigcup_{i=1}^{n} F_i\Big) = \sum_{i=1}^{n} m(F_i) \leq \sum_{i=1}^{n} m(E_i)$$

Now, for (3), let $E_0 = \emptyset$, then

$$m(\bigcup E_n) = \sum m(E_i\backslash E_{i-1}) = \lim_{n\longrightarrow\infty} \sum_{i=1}^{n} m(E_i\backslash E_{i-1}) = \lim_{n\longrightarrow\infty} m(E_n)$$

Additionally, consider when the collection $\{E_n\}$ is decreasing with $m(E_1) < \infty$. Take $F_i = E_1\backslash E_i$, then $\{F_n\}$ is an increasing collection of measurable sets, and hence we apply the above argument. We get that $m(E_1) = m(F_n) + m(E_n)$, and

$$\bigcup F_n = E_1\backslash\bigcap E_n$$

therefore, we get

$$m(E_1) = m\Big(\bigcap E_n\Big) + \lim_{n\longrightarrow\infty} m(F_i) = m\Big(\bigcap E_n\Big) + \lim_{n\longrightarrow\infty} (m(E_1) - m(E_n))$$

Subtracting $m(E_1)$ from both sides of the equation yields the result. ■

Definition. Let X be a measure space with measure m. We say that a statement about points in X holds **almost everywhere** (with respect to m) if it holds for all $x \in X\backslash E$, where $m(E) = 0$. We call the measure m **complete** if its domain contains all subsets of sets with measure 0.

Theorem 1.2.3. *Let X be a measure space with s-algebra $\mathcal{M}$, and measure m. Let $\mathcal{N} = \{N \in \mathcal{M} : m(N) = 0\}$, and define*

$$\overline{\mathcal{M}} = \{E \cup F : E \in \mathcal{M} \textit{ and } F \subseteq N \textit{ for some } N \in \mathcal{N}\}$$

Then $\overline{\mathcal{M}}$ is a σ-algebra, and there exists a unique complete measure $\overline{m}$ on $\overline{\mathcal{M}}$.

Proof. Since $\mathcal{M}$ is a σ-algebra, then so is $\mathcal{N}$, moreover, since both are closed under countable unions, so is $\overline{\mathcal{M}}$. Additionally, let $E \cup F \in \overline{\mathcal{M}}$, then we get $E \cup F = (E \cup N) \cap ((X\backslash N) \cup F)$, so that $X\backslash(E \cup F) = X\backslash(E \cup N) \cup N\backslash F$. Since $X\backslash(E \cup N) \in \overline{\mathcal{M}}$, and $N\backslash F \subseteq F$, then we get $X\backslash(E \cup F) \subseteq \overline{\mathcal{M}}$. This makes $\mathcal{M}$ a σ-algebra.

Now, for $E \cup F \in \overline{\mathcal{M}}$, define $\overline{m}$ on $\overline{\mathcal{M}}$ by $\overline{m}(E \cup F) = m(E)$. Then $\overline{m}$ is well defined. Let $E_1 \cup F_1 = E_2 \cup F_2$, where $F_i \subseteq N_i$, with $N_i \in \mathcal{N}$, for $i = 1, 2$. Then $E_1 \subseteq E_2 \cup N_2$, so that $m(E_1) \leq m(E_2) + m(N_1) = m(E_2)$. Similarly, we also get $m(E_2) \leq m(E_1)$.

Now, let $E \in \overline{\mathcal{M}}$, such that $\overline{m}(E) = 0$. Now, we have $E = A \cup B$, where $A \in \mathcal{M}$ and $B \subseteq N$, for some $N \in \mathcal{N}$. Moreover, $\overline{m}(E) = m(A) = 0$. Now, we get $E \subseteq A \cup N \in \mathcal{N}$, since $m(A) = 0$. Now, let $F \subseteq E$. Then observe that $F \subseteq A \cup N$, so that $F \in \mathcal{N}$. Then $F = \emptyset \cup F$, so that $F \in \overline{\mathcal{M}}$. Moreover, $\overline{m}(F) = m(\emptyset) = 0$.

Lastly, suppose there is another complete meaure $\overline{n}$ on $\overline{\mathcal{M}}$ for which $\overline{n}(E \cup F) = m(E)$. Let $E \in \overline{\mathcal{M}}$. Then $E = A \cup B$ where $A \in \mathcal{M}$, and $B \subseteq N$, $N \mathcal{N}$. Then $\overline{n}(E) = \overline{n}(A \cup B) = m(A) \leq m(A) + m(B) = m(A \cup B) = \overline{m}(E)$. By similar reasoning, we get $\overline{m}(E) \leq \overline{n}(E)$, which establishes uniqueness. ■

Definition. Let X be a measure space with s-algebra $\mathcal{M}$, and measure m. Let $\mathcal{N} = \{N \in \mathcal{M} : m(N) = 0\}$, and define

$$\overline{\mathcal{M}} = \{E \cup F : E \in \mathcal{M} \text{ and } F \subseteq N \text{ for some } N \in \mathcal{N}\}$$

We call $\overline{\mathcal{M}} the$**completion**$of$M$with respect to$m$, and we call the unique complete measure, \overline{m}$ on $\overline{\mathcal{M}}$ the **completion** of m.

1.3 Outer Measures

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