Algebraic Geometry.

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Contents

Chapter 1

Preliminaries

1.1 Affine Varieties

Definition. Let k be an algebraically closed field. We define **affine n-space** over k to be the set $\mathbb{A}^n(k)$ of n-tuples of elements of k. We write simply $\mathbb{A}^n$ when k is understood. We call the elements of $\mathbb{A}^n$ **points** and if $P = (a_1, \ldots, a_n)$ is a point of $\mathbb{A}^n$, we call each a_i the **coordinates** of P.

Example 1.1. Let k be any algebraically closed field, and consider the multivariate polynomial ring $k[x_1, \ldots, x_n]$. We can interperate the elements of $k[x_1, \ldots, x_n]$ as functions from affine space $\mathbb{A}^n(k)$ to k by taking $f(P) = f(a_1, \ldots, a_n)$, where $f \in k[x_1, \ldots, x_n]$ and $P \in \mathbb{A}^n(k)$. This leads us to be able to talk about the set of zeros of a polynomial over k.

Definition. Let k be an algebraically closed field, and $f \in k[x_1, \ldots, x_n]$ a multivariate polynomial over k. We define the **set of zeros** of f to be the set

$$Z(f) = \{P \in \mathbb{A}^n(k) : f(P) = 0\}$$

Let T be a subset of $k[x_1, \ldots, x_n]$. Then we define the **set of zeros** of T to be

$$Z(T) = \bigcap_{f \in T} Z(f)$$

Now, if $\mathfrak{a} = (f_1, \ldots, f_r)$ is an ideal of $k[x_1, \ldots, x_n]$ generated by T, then we write $Z(T) = Z(\mathfrak{a}) = Z(f_1, \ldots, f_r)$.

Definition. Let k be an algebraically closed field. We call a subset Y of $\mathbb{A}^n$ an **algebraic set** if there exists some $T \subseteq k[x_1, \ldots, x_n]$ for which Y is the set of zeros of T; i.e. $Y = Z(T)$.

Lemma 1.1.1. *Let k be an algebraically closed field. Then algebraic sets of $\mathbb{A}^n$ make $\mathbb{A}^n$ into a topology under closed sets.*

Proof. Let $\mathbb{A}^n = Z(0)$ and $\emptyset = Z(1)$. Then $\mathbb{A}^n$ and $\emptyset$ are both algebraic. Now, let X and Y be algebraic, then there are S, T such that $X = Z(S)$ and $Y = Z(T)$. Now, let $P \in X \cup Y$, then P is a zero of any polynomial $f \in ST$, conversly, suppose that $P \in Z(ST)$ where

$P \notin Y$. There exists a polynomial $f \in S$ with $f(P) \neq 0$. Now, for any $g \in T$, we have that if $fg(P) = 0$, then $g(P) = 0$, so that $P \in S$. Therefore we have $X \cup Y = Z(ST)$, making $X \cup Y$ algebraic. So that the collection of algebraic sets is closed under finite intersection.

Lastly, consider a collection $\{Y_\alpha\}$ of algebraic sets, where $Y_\alpha = Z(T_\alpha)$ for some T_α. Let

$$Y = \bigcap Y_\alpha \text{ and } T = \bigcup T_\alpha$$

and let $P \in Y$. Then P is in every Y_α making it a zero of some $f_\alpha \in T_\alpha$, thus $P \in Z(T)$. Similarly, if $P \in Z(T)$, then $P \in Y$, making $Y = Z(T)$, and making the collection of algebraic sets closed under arbitrary intersections. ■

Definition. We define the **Zariski topology** on affine n-space $\mathbb{A}^n$ to be the topology on $\mathbb{A}^n$ whose closed sets are the algebraic sets of $\mathbb{A}^n$.

Example 1.2. Consider the Zariski topology on affine 1-space $\mathbb{A}^1$. Now, since $k[x]$ is a PID, every algebraic set of $\mathbb{A}^1$ is the set of zeros of preciesly one polynomial. Moreover, by the algebraic closure of k, for any nonzero polynomial f over k, we have

$$f(x) = c(x - a_1) \dots (x - a_n)$$

where $c, a_1, \dots, a_n \in k$. Then $Z(f) = \{a_1, \dots, a_n\}$, so that the algebraic sets of $\mathbb{A}^1$ are the emptyset, itself, and finite subsets. Thus the Zariski topology on $\mathbb{A}^1$ consists of finite sets, the emptyset, and $\mathbb{A}^1$ itself. Notice that this topology is not Hausdorff.

Definition. Let X be a topological space, and Y a subspace of X. We call Y **irreducible** if it cannot be written as the union $Y = Y_1 \cup Y_2$ of two sets Y_1 and Y_2 closed in Y. We make the convention that the emptyset is not irreducible.

Example 1.3. (1) Notice that the affine space $\mathbb{A}^1$ is irreducible. We have the only closed sets are finite sets, and since k is algebraically closed, and hence infinite, then $\mathbb{A}^1$ must be infinite.

(2) Subspaces of irreducible spaces are irreducible and dense.

(3) If Y is an irreducible space of a topological space X, then the closure $\operatorname{cl} Y$ is also irreducible in X.

Definition. We define an **algebraic affine variety** to be an irreducible closed subset of $\mathbb{A}^1$ under the Zariski topology. We define an open set of an affine variety to be a **quasi-affine variety**.

Definition. We define the **ideal** of a subset Y in $\mathbb{A}^n$, to be the set

$$I(Y) = \{f \in k[x_1, \dots, x_n] : f(P) = 0 \text{ for all } P \in Y\}$$

where k is an algebraically closed field.

Theorem 1.1.2 (Hilbert's Nullstellensatz)**.** *Let k be an algebraically closed field and $\mathfrak{a}$ an ideal of $k[x_1, \dots, x_n]$, and let $f \in k[x_1, \dots, x_n]$ be a polynomial vanishing at all points of $Z(\mathfrak{a})$. Then there exists an $r \in \mathbb{Z}^+$ for which $f^r \in \mathfrak{a}$.*

Lemma 1.1.3. *The followin are true for any algebraically closed field k.*

(1) If $T_1, T_2 \subseteq k[x_1, \ldots, x_n]$ with $T_1 \subseteq T_2$, then $Z(T_2)Z(T_2)$.

(2) If $Y_1, Y_2 \subseteq \mathbb{A}^n$ with $Y_1 \subseteq Y_2$, then $I(Y_2) \subseteq I(Y_1)$.

(3) For any $Y_1, Y_2 \subseteq \mathbb{A}^n$, $I(Y_1 \cup Y_2) = I(Y_1) \cap I(Y_2)$.

(4) For any ideal $\mathfrak{a}$ of $k[x_1, \ldots, x_n]$, $I(Z(J)) = \sqrt{\mathfrak{a}}$, where

$$\sqrt{\mathfrak{a}} = \{f \in k[x_1, \ldots, x_n] : f^r \in J \text{ for some } r \in \mathbb{Z}^+\}$$

(5) For every $Y \subseteq \mathbb{A}^n$, $Z(I(Y)) = \operatorname{cl} Y$ in the Zariski topology.

Proof. Let $T_1 \subseteq T_2$ be subsets of $k[x_1, \ldots, x_n]$, and choose a polynomial $f \in T_1$, and a point $P \in Z(T_2)$. We have by inclusion that $f \in T_2$, and that $f(P) = 0$. Now since $f \in T_1$, this puts $P \in Z(T_1)$ and we get the required inclusion. The proof for statement (2) is identical.

Now, let $P \in Y_1 \cup Y_2$, and choose a polynomial $f \in I(Y_1) \cap I(Y_2)$. Then we have that the point P is either contained in Y_1 or Y_2 (or both), so that $f(P) = 0$, which makes $I(Y_1 \cup Y_2) \subseteq I(Y_1) \cap I(Y_2)$. Conversely, if $f \in I(Y_1) \cap I(Y_2)$, then for any points $P \in Y_1 \cup Y_2$, $f(P) = 0$, which puts $f \in I(Y_1 \cup Y_2)$.

For part (4), notice this is a direct consequence of Hilbert's Nullstellensatz. Now, for part (5), notice that $Y \subseteq Z(I(Y))$, which is a closed set in the Zariski topology, so that $\operatorname{cl} Y \subseteq Z(I(Y))$. Now, let W be a closed set in $\mathbb{A}^n$, containing Y. Then we have $W = Z(\mathfrak{a})$, for some ideal af of $k[x_1, \ldots, x_n]$, so that $Y \subseteq Z(\mathfrak{a})$. Then by part (2), observe that $I(Y) \subseteq I(Z(\mathfrak{a}))$, but $\mathfrak{a} \subseteq I(Z(\mathfrak{a}))$, so by part (1), we have $Z(I(Y)) \subseteq Z(\mathfrak{a})$. This makes $Z(I(Y)) = \operatorname{cl} Y$. ■

Corollary. *There exists a 1–1, inclusion reversing, corresponedece of algebraic sets of $\mathbb{A}^n$ onto radical ideals in $k[x_1, \ldots, x_n]$; given by the maps*

$$\begin{array}{ccc} Y & \to & I(Y) \\ \mathfrak{a} & \to & Z(\mathfrak{a}) \end{array}$$

Moreover, an algebraic set in $\mathbb{A}^n$ is irreducible if, and only if its ideal in $k[x_1, \ldots, x_n]$ is prime.

Proof. Notice that parts (1), (2), and (3) of the above lemma provide the required correspondence.

Now, suppose that Y is irreducible in $\mathbb{A}^n$, and take $fg \in I(Y)$. Then $Y \subseteq Z(fg) = Z(f) \cup Z(g)$, so that

$$Y = (Y \cap Z(f)) \cup (Y \cap Z(g))$$

which is the union of two closed sets in $\mathbb{A}^n$. Now, since Y is irreducible, we get that either $Y = Y \cap Z(f)$, or $Y = Y \cap Z(g)$; in either case, $Y \subseteq Z(f)$, or $Y \subseteq Z(g)$. This puts either $f \in I(Y)$, or $g \in I(Y)$, which makes the ideal $I(Y)$ prime.

Conversely if $\mathfrak{p}$ is a prime ideal, let $Z(\mathfrak{p}) = Y_1 \cup Y_2$, then $\mathfrak{p} = I(Y_1) \cap I(Y_2)$, so that either $\mathfrak{p} = I(Y_1)$ or $\mathfrak{p} = I(Y_2)$, since $\mathfrak{p}$ is prime. This makes $Z(\mathfrak{p}) = Y_1$ or $Z(\mathfrak{p}) = Y_2$, which makes $Z(\mathfrak{p})$ irreducible in $\mathbb{A}^n$. ■

Example 1.4. Consider k to be an algebraically closed field.

(1) Notice that $\mathbb{A}^n$ maps to the ideal (0) in $k[x_1, \ldots, x_n]$, which is a prime ideal. This makes $\mathbb{A}^n$ irreducible by the corollary to lemma 1.1.3.

(2) Let $f \in k[x, y]$ be an irreducible polynomial. Then f generates the prime ideal (f) in $k[x, y]$, since $k[x, y]$ is a UFD. Thus the zero set $Y = Z(f)$ is irreducible, and closed in $\mathbb{A}^n$; hence it is an affine variety. We call Y an **affine curve** in $\mathbb{A}^n$ defined by the equation $f(x, y) = 0$ of **degree** $\deg f = d$. Now, moregenerally, if f is an irreducible polynomial in $k[x_1, \ldots, x_n]$, then we call the affine variety $Y = Z(f)$ a **surface** in $n = 3$ and a **hypersurface** in $n > 3$.

(3) A maximal ideal $\mathfrak{m}$ of $k[x_1, \ldots, x_n]$ correspondes to a minimal affine variety of $\mathbb{A}^n$, which are the point-sets $\{P\}$ of $\mathbb{A}^n$; where $P = (a_1, \ldots, a_n)$. Thus every maximal ideal of $k[x_1, \ldots, x_n]$ is of the form $M = (x_1 - a_1, \ldots, x_n - a_n)$ for some $a_1, \ldots, a_n \in k$.

(4) Consider the field $\mathbb{R}$ which is algebraically closed, and the curve defined by $x^2 + y^2 + 1 = 0$ in $\mathbb{A}^2(\mathbb{R})$. Notice that this curve is irreducible, but has no points in $\mathbb{A}^2$ (i.e. no roots in $\mathbb{R}$). This shows that if the field k is no algebraically closed, then the above results do not hold in general.

Definition. Let k be an algebraically closed field, and Y an affine algebraic set of $\mathbb{A}^n$. We definen the **affine coordinate ring** of Y to be the factor ring

$$A(Y) = {}^{k[x_1, \ldots, x_n]}\!\big/_{I(Y)}$$

Lemma 1.1.4. *If Y is an affine variety, then $A(Y)$ is an integral domain. Moreover, there exists a 1–1 correspondence of finitely generated k-algebras onto affine coordinate rings of affine varieties.*

Bibliography

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