Real Analysis

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Chapter 1

The Real Numbers

1.1 The Field of Real Numbers

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Definition. We call a set U of $\mathbb{R}$ **open** proveded for all $x \in U$, there exists an $r > 0$ for which the open interval $(x - r, x + r) \subseteq U$.

Example 1.1. For $a < b$ the open interval (a, b) is open in $\mathbb{R}$. Let $x \in (a, b)$ and take $r = \min\{x - a, b - x\}$, then $(x - r, x + r) \subseteq (a, b)$. Similarly the intervals (a, ∞), $(-\infty, b)$, and $(-\infty, \infty)$ are also open in $\mathbb{R}$.

Lemma 1.2.1. *The set $\mathbb{R}$ of real numbers forms a topology under the open sets of $\mathbb{R}$.*

Lemma 1.2.2. *Every nonempty open set in $\mathbb{R}$ is the disjoint union of a countable collection of open sets in $\mathbb{R}$.*

Proof. Let U be a nonempty open set in $\mathbb{R}$, and take $x \in U$. There there is a $y > x$ for which $(x, y) \subseteq U$, and a $z < x$ for which $(z, x) \subseteq U$. Now, define

$$a_x = \inf\{z : (z, x) \subseteq U\}$$
$$b_x = \sup\{y : (x, y) \subseteq U\}$$

and take

$$I_x = (a_x, b_x)$$

Then I_x is an open interval containing x. Now, we claim that $I_x \subseteq U$, but that $a_x, b_x \notin U$. Indeed, take $w \in I_x$, with $x < w < b_x$, then there is a $y > w$ for which $(x, y) \subseteq U$, so that $w \in U$.

Now, suppose that $b_x \in U$, then for some $r > 0$, $(b_x - r, b_x + r) \subseteq U$, so that $(x, b_x + r) \subseteq U$, which contradicts that b_x is a least upper bound. Similar reasoning yields that $a_x \notin U$.

Now, consider the collection $\{I_x\}_{x \in U}$. Then we have that

$$U = \bigcup I_x$$

moreover, this union is disjoint since $a_x, b_x \notin U$ for each x. Now, observe that by the density of $\mathbb{Q}$ in $\mathbb{R}$, there exists a rational $q_x \in \mathbb{Q}$ for which $q_x \in I_x$. This gives us a 1–1 correspondence of the collection $\{I_x\}$ onto $\mathbb{Q}$, which makes $\{I_x\}$ countable. ■

Definition. For a set E of real numbers, we call a point $x \in \mathbb{R}$ a **limit point** of E provided every open interval containing x contains a point in E. We call the set of all limit points of E, together with E the **closure** of E and denote it $\operatorname{cl} E$. We call E **closed** if $E = \operatorname{cl} E$.

Lemma 1.2.3. *For every set E of $\mathbb{R}$, the closure of E is closed. Morevoer, $\operatorname{cl} E$ is the smallest closed set containing E.*

Proof. Let x be a linit point of $\operatorname{cl} E$, and consider an open interval I_x containing x. Then there exists an $x' \in \operatorname{cl} E \cap I_x$. Since x' is a limit point of E, and $x' \in I_x$, we get a $x \in E \cap I''_x$. Therefore every open interval that contains x also contains a point of E. This makes $x \in \operatorname{cl} E$, and hence $\operatorname{cl} E$ is closed. ■

Lemma 1.2.4. *A set of $\mathbb{R}$ is open if and only if its complement in $\mathbb{R}$ is closed.*

Proof. Suppose that $E \subseteq \mathbb{R}$ is open , and let x be a limit point of $\mathbb{R} \backslash E$. Then $x \notin E$, since otherwise there is an open interval containing x, contained in E, and hence disjoint from $\mathbb{R} \backslash E$. Therefore $x \in \mathbb{R} \backslash E$ which makes $\mathbb{R} \backslash E$ closed. ■

Corollary. *A set $\mathbb{R}$ is closed if, and only if its complement in $\mathbb{R}$ is open.*

Proof. By DeMorgan's laws. ■

Definition. We call a collection $\{E_\lambda\}$ of sets of $\mathbb{R}$ a **cover** for a set E of $\mathbb{R}$ if $E \subseteq_\lambda$. If each E_λ is open, we call the collection $\{E_\lambda\}$ an **open cover**. We call a set E of $\mathbb{R}$ **compact** if each open cover of E has a finite subcover of E.

Theorem 1.2.5 (Heine-Borel)**.** *If F is a closed bounded set in $\mathbb{R}$, then F is compact.*

Proof. Consider first the case where $F = [a, b]$, for $a < b$, the closed bounded interval from a to b. Let $\mathcal{F}$ be an open cover of $[a, b]$, and define

$$E = \{x \in [a, b] : [a, x] \text{ can be covered by a finite subcollection of } \mathcal{F}\}$$

Notice then that $a \in E$, so that E is nonempty. Moreover, E is bounded above, so by the completeness of $\mathbb{R}$, $c = \sup E$ exists in $[a, b]$. Now, then, there exists a set $U\ in \mathcal{F}$ such that $c \in U$. Since U is open (well $\mathcal{F}$ is an open cover), there exists an $\varepsilon > 0$ for which the interval $(c - \varepsilon, c + \varepsilon) \subseteq U$. Now, $c - \varepsilon$ is not an upperbound of E by definition of c, so there is an $x \in E$ with $c - \varepsilon < x$. Now, threre is a finite subcollection $\{U_i\}_{i=1}^k$ of open sets in $\mathcal{F}$ covering $[a, x]$, consequently the collectuion $\{U_i\} \cup U$ covers $[a, c + \varepsilon]$, so that $c = b$. That is $[a, b]$ has a finite subcover of $\mathcal{F}$, so that $[a, b]$ is compact.

Now, let F be any closed and bounded set, and let $\mathcal{F}$ be an open cover of F. Since F is bounded, we have $F \subseteq [a, b]$ for some $a < b$, and the set $U = \mathbb{R} \backslash F$ is open. Now, let $\mathcal{F}' = \mathcal{F} \cup U$. Since $\mathcal{F}$ covers F, $\mathcal{F}'$ covers $[a, b]$. By the compactness of $[a, b]$, we obtain the compactness of F. ■

Theorem 1.2.6 (The Nested Set Theorem)**.** *Let $\{F_n\}$ a countable descending collection of closed sets of $\mathbb{R}$, for which F_1 is bounded. Then the intersection*

$$\bigcap F_n$$

is nonempty.

Proof. Suppose to the contrary that the intersection $F = \bigcap F_n$ is empty. Then for every $x \in \mathbb{R}$, there is an $n \in \mathbb{Z}^+$ for which $x \notin F_n$. That is, $x \in U_n = \mathbb{R}\backslash F_n$, and $\mathbb{R} = \bigcup U_n$. Now, since each F_n is closed, each U_n is open, making $\{U_n\}$ an open cover of $\mathbb{R}$, and hence F_1. Then by the theorem of Heine-Borel, F_1 is compact, and there is an $N \in \mathbb{Z}^+$ for which

$$F_1 \subseteq \bigcup_{n=1}^{N} U_n$$

since $\{F_n\}$ is a descending collection, the collection of open sets $\{U_n\}$ is an ascending collection. Thus we have

$$\bigcup_{n=1}^{N} U_n = U_n = \mathbb{R}\backslash F_N$$

making $F_1 \subseteq \mathbb{R}\backslash F_N$, which contradicts that $F_n \subseteq F_1$ is nonempty ■

Definition. Let X be a set. We call a collection $\mathcal{A}$ of subsetes of X a **σ-algebra** of X provided

(1) $X \in \mathcal{A}$ and $\emptyset \in \mathcal{A}$.

(2) $\mathcal{A}$ is closed under complements in X.

(3) $\mathcal{A}$ is closed under countable unions.

Example 1.2. The collections $\{\emptyset, X\}$ and 2^X are σ-algebras on X.

Lemma 1.2.7. *Let $\mathcal{F}$ be a collection of subsets of a set X. Then the intersection $\mathcal{A}$ of all σ-algebras of X containing $\mathcal{F}$ is a σ-algebra containing $\mathcal{F}$. Moreover, it is the smallest such σ-algebra of X containing $\mathcal{F}$.*

Definition. We define the collection $\mathcal{B}$ of **Borel sets** of $\mathbb{R}$ to be the smallest σ-algebra of $\mathbb{R}$ containing all open sets of $\mathbb{R}$.

1.3 Sequences in $\mathbb{R}$

Definition. We define a **sequence** of real numbers to be a real-valued function $f : \mathbb{Z}^+ \to \mathbb{R}$ where $f(n) = a_n$, for some $a_n \in \mathbb{R}$. We denote sequences by $\{a_n\}$. We call a sequence $\{a_n\}$ of real numbers **bounded** provided there exists an $M \geq 0$ for which $|a_n| \leq M$ for all $n \in \mathbb{Z}^+$. We say the sequence $\{a_n\}$ is **monotone increasing** if $a_n \leq a_{n+1}$ for all $n \in \mathbb{Z}^+$, and we call it **monotone decreasing** if the sequence $\{-a_n\}$ is monotone increasing.

Definition. We say a sequence $\{a_n\}$ of real numbers **converges** to a real numbers $a\ in \mathbb{R}$ if for every $\varepsilon > 0$, there is an $N \in \mathbb{Z}^+$ for which

$$|a_n - a| < \varepsilon \text{ whenever } n \geq N$$

and we write $\{a_n\} \to a$ as $n \to \infty$, or

$$\lim_{n \to \infty} a_n = a$$

We call a the **limit** of the sequence.

Lemma 1.3.1. *Suppose a sequence $\{a_n\}$ of real numbers converges to some $a \in]\mathbb{R}$. Then this limit is unique and $\{a_n\}$ is bounded. Moreover, for every $M \in \mathbb{R}$, if $a_n \geq M$, then $a \geq M$.*

Theorem 1.3.2 (The Monotone Convergence Theorem)**.** *A monotone sequence of real numbers converges if, and only if it is bounded.*

Proof. Suppoe, without loss of generality, that $\{a_n\}$ is a monotone increasing function; and that $\{a_n\} \to a$ as $n \to \infty$. Then by lemma 1.3.1, $\{a_n\}$ must be bounded.

Conversely, suppose that $\{a_n\}$ is monotone increasing and bounded. By the completeness of $\mathbb{R}$, we have the set

$$\mathcal{S} = \{a_n : n \in \mathbb{Z}^+\}$$

has a least upper bounde $a = \sup \mathcal{S}$. Now, let $\varepsilon > 0$, then since $\mathcal{S}$ has a least upper bound, $a_n \leq a$ for all $n \in \mathbb{Z}^+$, and $a - \varepsilon$ is not an upper bound of $\mathcal{S}$. Hence, there is an $N \in \mathbb{Z}^+$ for whcih $A_N > a - \varepsilon$. Since $\{a_n\}$ is monotone increasingm we gave $a_n > a - \varepsilon$ for all $n \geq N$, so that

$$|a - a_n| < \varepsilon$$

this makes $\{a_n\} \to a$. ■

Theorem 1.3.3 (Bolzano-Weierstrass)**.** *Every bounded sequence of real numbers has a convergent subsequence.*

Proof. Let $\{a_n\}$ be a bounded sequence of real numbers. Choose an $M > 0$ for which $|a_n| \leq M$, for all $n \in \mathbb{Z}^+$. Now, define

$$E_n = \operatorname{cl} \{a_j\}_{j \geq n}$$

Then $E_n \subseteq [-M, M]$, moreover, each E_n is closed and $\{E_n\}$ is a descending collection of closed sets in which E_1 is bounded. Therefore by the nested set theorem, the intersection

$$E = \bigcap E_n$$

is nonempty. Choose then a point $a \in E$. Then for every $k \in \mathbb{Z}^+$, a is a limiti point of E_k, and hence for infinitely many $j \geq n$, we have $A_j \in (a - \frac{1}{k}, a + \frac{1}{k})$. Hence, proceeding inductively, choose a strictly increasing sequence $\{n_k\}$ such that

$$|a - a_{n_k}| < \frac{1}{k}$$

By the Archimedean principle of $\mathbb{R}$, $\{a_{n_k}\} \to a$ as $k \to \infty$. ■

Definition. We call a sequence $\{a_n\}$ of real numbers **Cauchy** if for every $\varepsilon > 0$, there is an $N \in \mathbb{Z}^+$ for which

$$|a_m - a_n| < \varepsilon \text{ whenever } m, n \geq N$$

Theorem 1.3.4 (Cauchy's Convergence Criterion)**.** *A sequence of real numbers converges if, and only if it is Cauchy.*

Proof. SUppose that $\{a_n\} \to a$. Then observe that for all $m, n \in \mathbb{Z}^+$, that

$$|a_m - a_n| \leq |a_m - a| + |a - a_n|$$

now, let $\varepsilon > 0$ and choose $N \in \mathbb{Z}^+$ for whic $|a_n - a|M < \frac{\varepsilon}{2}$ whenever $n \geq N$. Then observe that whenever $m, n \geq N$, we get

$$|a_m - a_n| \leq |a_m - a| + |a - a_n| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

which makes $\{a_n\}$ Cauchy.

Conversely, suppose the sequence $\{a_n\}$ is Cauchy. We claim that $\{a_n\}$ is bounded. Let $\varepsilon = 1$ and choose $N \in \mathbb{Z}^+$ such that if $m, n \geq N$, then $|a_m - a_n| < 1$. Notice then that

$$|a_n| \leq |a_n - a_N| + |a_N| \leq 1 + |a_N| \text{ for all } n \geq N$$

Now, define $M = \max\{|a_1|, \dots, |a_N|\}$. Then $|a_n| \leq M$ for all $n \in \mathbb{Z}^+$ and $\{a_n\}$ is bounded. By the theorem of Bolzano-Weierstrass, $\{a_n\}$ has a convergent subsequence $\{a_{n_k}\} \to a$. Now, let $\varepsilon > 0$; since $\{a_n\}$ is Cauchy, choose an $N \in \mathbb{Z}^+$ such that $|a_m - a_| < \frac{\varepsilon}{2}$ whenever $m, n \geq N$. In particular, we have whenever $n_k \geq N$,

$$|a_n - a_{n_k}| < \frac{\varepsilon}{2}$$

so that

$$|a_n - a| \leq |a_n - a_{n_k}| + |a_{n_k} - a| < \varepsilon$$

which makes $\{a_n\} \to a$. ■

Theorem 1.3.5. *Let $\{a_n\}$ and $\{b_n\}$ be convergent sequences of real numbers, with $\{a_n\} \to a$ and $\{b_n\} \to b$. The for all $\alpha, \beta \in \mathbb{R}$, we have the sequence $\{\alpha a_n + \beta b_n\}$ is convergent, and*

$$\lim_{n \longrightarrow \infty} \alpha a_n + \beta b_n = \alpha a + \beta b$$

Moreover, if $a_n \leq b_n$ for all n, then $a \leq b$.

Proof. Observe that

$$|(\alpha a_n + \beta b_n) - (\alpha a + \beta b)| \leq |\alpha||a_n - a| + |\beta||b_n - b|$$

for all $n \in \mathbb{Z}^+$. Let $\varepsilon > 0$ and choose $N \in \mathbb{Z}^+$ such that

$$|a - a_n| < \frac{\varepsilon}{2 + 2|\alpha|} \text{ and } |b - b_n| < \frac{\varepsilon}{2 + 2|\beta|} \text{ for all } n \geq N$$

Then

$$|(\alpha a_n + \beta b_n) - (\alpha a + \beta b)| < \varepsilon$$

Now, suppose that $a_n \leq b_n$ for all $n \in \mathbb{Z}^+$, and consider the sequence $c_n\}$ where $c_n = b_n - a_n$, and let $c = b - a$. Then $c_n \geq 0$, and by the inearity proved above, $\{c_n\} \to c$. Now, let $\varepsilon > 0$, then there is an $N \in \mathbb{Z}^+$ such that $-\varepsilon < c - c_n < \varepsilon$ for all $n \geq N$. In particular, $0 \leq c_N < c + \varepsilon$, and since $c > -\varepsilon$, we get that $c \geq 0$. ■

Definition. We say a sequence $\{a_n\}$ **converges to infinity** if for every $M \in \mathbb{R}$, there is an $N \in \mathbb{Z}^+$ for which

$$a_n \geq M \text{ for all } n \geq N$$

and we write $\{a_n\} \to \infty$. We say that $\{a_n\}$ **converges to minus infinity** if the sequence $\{-a_n\}$ converges to infinity, and we write $\{a_n\} \to -\infty$.

Definition. Let $\{a_n\}$ be a sequence of real numbers. We define the **limit superior** of $\{a_n\}$ to be

$$\limsup \{a_n\} = \lim_{n \longrightarrow \infty} (\sup \{a_k : k \geq n\})$$

Similarly, we define the **limit inferior** of $\{a_n\}$ to be

$$\liminf \{a_n\} = \lim_{n \longrightarrow \infty} (\inf \{a_k : k \geq n\})$$

Lemma 1.3.6. *Let $\{a_n\}$ and $\{b_n\}$ be sequences of real numbers. Then the following are true.*

(1) $\limsup \{a_n\} = l \in \mathbb{R}$ *if, and only if there is infinitely many $n \in \mathbb{Z}^+$ for which $a_n > l - \varepsilon$ and only finitely many $n \in \mathbb{Z}^+$ for which $a_n > l + \varepsilon$.*

(2) $\limsup \{a_n\} = \infty$ *if and only if $\{a_n\}$ is not bounded above.*

(3) $\limsup \{a_n\} = -\liminf \{-a_n\}$

(4) $\{a_n\} \to a \in \mathbb{R}_\infty$ *if and only if*

$$\limsup \{a_n\} = \liminf \{a_n\} = a$$

(5) If $a_n \leq b_n$ for all $n \in \zeta^+$, then

$$\limsup \{a_n\} \leq \liminf \{b_n\}$$

Definition. Let $\{a_n\}$ be a sequence of real numbers. We define the **n-th partial sum** of $\{a_n\}$ to be

$$s_n = \sum_{k=1}^{n} a_k$$

We call a sum $\sum a_n$ **summable** to a **sum** $s \in \mathbb{R}$ if the sequence $\{s_n\} \to s$; that is the sequence of n-th partial sums of the sequence $\{a_n\}$ converges to s as $n \to \infty$.

Lemma 1.3.7. *Let $\{a_n\}$ be a sequence of areal numbers. The following are true.*

(1) The sum $\sum a_n$ is summable if, and only if for every $\varepsilon > 0$, there is an $N \in \mathbb{Z}^+$ such that

$$\Big|\sum_{n=1}^{m+n} a_k\Big| < \varepsilon \text{ for all } n \geq N \text{ and some } m \in \mathbb{Z}^+$$

(2) If the sum $\sum |a_n|$ is summable, then so is $\sum a_n$.

(3) If $a_n \geq 0$, then $\sum a_n$ is summable if, and only if the sequence of n-th partial sums of $\{a_n\}$ is bounded.

1.4 Contiunous Real-valued Functions

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