3-Manifolds

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Chapter 1

Manifolds

1.1 Topological Manifolds

Definition. A **topological n-manifold** is a second countable Hausdorff space M, together with a collection $\{(M_\alpha, \phi_\alpha)\}$ for which

(1) $\{M_\alpha\}$ is a collection of open sets of M covering M ; that is, $M_\alpha \subseteq M$ is open and $M = M_\alpha$.

(2) ϕ_a is a homeomorphism of M_α onto an open subset U of $\mathbb{R}^n$.

We call the pairs (M_α, ϕ_α) **charts** of M, and we call the collection of all such charts of M an **atlas** of M. We define the **dimension** of M to be $\dim M = n$.

Example 1.1. (1) Every subset of $\mathbb{R}^N$ is second countable and Hausdorff, so that a subset $\mathbb{R}^N$ is an n-manifold uf every point of M has a neighborhood homeomorphic to $\mathbb{R}^n$, for $n \leq N$. In particular, $\mathbb{R}^n$ is an n-manifold.

(2) The n-sphere $S^n = \{x \in \mathbb{R}^{n+1} : \|x\| = 1\}$ (see figure 1.1) is an n-manifold. It is a second countable Hausdorff space, since it is a subspace of $\mathbb{R}^n$. Moreover, the stereographic projection $h : S^n \backslash (0, \dots, 0, 1) \to \mathbb{R}^n$ is a homeomrphism. So for $x \in S^n$, $x \neq (0, \dots, 0, 1)$, x has a neighborhood homeomorphic to $\mathbb{R}^n$. Now, if we take the composition of $\mathbb{R}^n \times \{0\}$ with h to obtain the map $h' : S^n \backslash (0, \dots, 0, -1) \to \mathbb{R}^n$, then we get that $S^n \backslash (0, \dots, 0, -1)$ is a neighborhood of $(0, \dots, 0, 1)$ homeomorphic to $\mathbb{R}^n$.

(3) The n-torus $T = \underbrace{S_1 \times \cdots \times S^1}_{n \text{ times}}$ (see figure 1.2) is the quotient space obtained fron $\mathbb{R}^n$ by identfying two points $x, y \in \mathbb{R}^n$ if, and only if there is some $g \in G$ for which $g(x) = y$, where G is the group generated by all translations by distance 1 along the coordinate axes. Let $x \in T^n$ and $U = \partial B(x, \frac{1}{4})$ the sphere centered about x of radius $\frac{1}{4}$ and let $q : \mathbb{R}^n \to \mathbb{R}^n$ the quotient map of the quotient space of T^n. Then $q^{-1}|_{q(U)}$ is a homoemorphism. This makes T an n-manifold, with atlas $\{(U, q^{-1}|_{q(U)})\}$.

(4) Identify the antipodal points of S^n, then the resulting quotient space is an n-manifold called **n-dimensional real projective space** which we denote by $\mathbb{PR}^n$. Let $x \in$

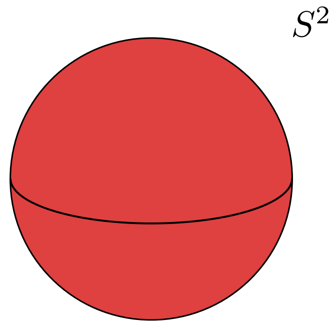

Figure 1.1: The 2-Sphere of $\mathbb{R}^3$ is a 2-manifold.

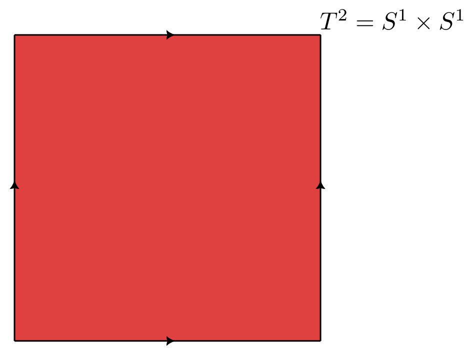

Figure 1.2: The 2-torus is a 2 manifold of $\mathbb{R}^2$.

$\mathbb{PR}^n$,since S^n is an n-manifold, there is a neighborhood U of x and a homeomorphism $h : U \to \mathbb{R}^n$. Let $-U = a(U)$, where $a : S^n \to S^n$ is the antipodal map. Then $-U$ is a neighborhood of $-x$, and $-h = h \circ a$ is a homeomorphism of U onto $\mathbb{R}^n$. Then the collection $\{(U, h)\}$ is an atlas for $\mathbb{PR}^n$.

(5) Consider the unit circle S^2 and the following "trefoil knot" K_3 pictured below in figure 1.3 Both of these are 1-manifolds, and are homeomorphic to each other. Consider the

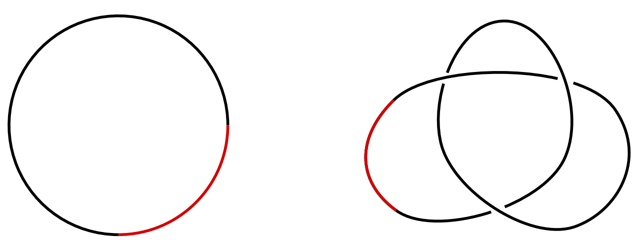

Figure 1.3: The unit circle S^2 and the trefoil knot.

neighborhoods on either space (colored in red). Notice that each of these neighborhoods is homeomorphic to an interval in $\mathbb{R}_1$, and hence homeomorphic to each other.

Definition. Let M be an n-dimensional manifold. A p**-dimensional submanifold** of M is a closed subset L of M for which there exists an atlas $\{(M_\alpha, \phi_a)\}$ of M such that for all $x \in L$, there exists a chart (M_α, ϕ_α) in where $x \in M_\alpha$ and $\phi_\alpha(L \cap M_\alpha) = \{0\} \times \mathbb{R}^p$.

Lemma 1.1.1. *Submanifolds of manifolds are manifolds.*

Lemma 1.1.2. *Let M be an m-manifold, and N an n-manifold. Then the product $M \times N$ is an $(n+m)$-manifold.*

Proof. We have that both M and N are Hausdorff, which makes $M \times N$ Hausdorff. Moreover, since M and N are second countable, they have countable bases $\mathcal{B}_M$ and $\mathcal{B}_N$. Then the product $\mathcal{B}_M \times \mathcal{B}_N$ serves as a countable basis for $M \times N$.

Now, let $\{(M_\alpha, \phi_\alpha)\}$ and $\{(N_\beta, \psi_\beta)\}$ be atlases for M and N respectively. Then since each M_α is open in M, and each N_β is open in N, $M_\alpha \times N_\beta$ is open in $M \times N$. Moreover we also have that $M = \bigcup M_\alpha$, $N = \bigcup N_\alpha$ so that

$$M \times N = (\bigcup M_\alpha) \times (\bigcup N_\beta) = \bigcup M_\alpha \times N_\beta$$

Now, we also have that ϕ_α is a homeomorphism of M_α onto an open subset of $\mathbb{R}^m$, and ψ_β is a homeomorphism of N_β onto an open subset of $\mathbb{R}^n$. Since ϕ_α and ψ_β are homeomorphisms, they are continuous with continuous inverses ϕ_α^{-1} and ψ_β^{-1}. This makes the map $\phi_\alpha \times \psi_\beta$ continuous with continuous inverse $(\phi_\alpha \times \psi_b)^{-1}$, which makes $\phi_\alpha \times \psi_b$ a homeomorphism of $M_\alpha \times N_\beta$ onto a subset of $\mathbb{R}^m \times \mathbb{R}^n \simeq \mathbb{R}^{m+n}$. Therefore $M \times N$ is an $(m+n)$-manifold. ■

Example 1.2. The equator, S^1 of S^2 is a submanifold of S^2 (see figure 1.1).

Definition. We define the **boundary** of an n-manifold M to be the set ∂M, of all points of M for which there is a neigborhood homeomorphic to a neighborhood of H^n, but no neighborhood homeomorphic to a neighborhood of $\mathbb{R}^n$; where $H^n = \{x \in \mathbb{R}^n : x_1 \geq 0\}$.

Definition. Let $H^n = \{(x_1, \ldots, x_n) \in \mathbb{R}^n : x_1 \geq 0\}$. We define an **$n$-manifold with boundary** to be a second countable Hausdorff space M with atlas $\{(M_\alpha, \phi_\alpha)\}$ such that ϕ_α is a homeomorphism from M_α to an open subset of $\mathbb{R}^n$, or H^n.

Example 1.3. (1) The unit ball $B^n = \{x \in \mathbb{R}^n : \|x\| \leq 1\}$ is an n-dimensional manifold with boundary $\partial B^n = S^{n-1}$. For interior points of B^n, this is clear. For points in S^{n-1}, extending the stererographic projection gives the required homeomorphism.

(2) The **pair of pants** (see figure 1.4) Is a 2-manifold with boundary.

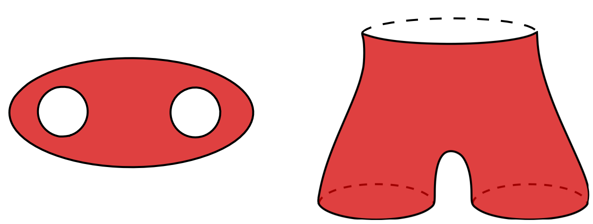

Figure 1.4:

(3) The 1-holed torus is a 2-manifold with boundary.

Definition. A p**-dimensional submanifold with boundary** of an n-dimensional manifold M is a closed subset L of M for which there is an atlas $\{(M_\alpha, \phi_a)\}$ of M and $0 \leq p \leq n$, such that for every $x \in L$ in the interior of M, there is a chart (M_α, ϕ_a) such that $x \in M_\alpha$, and $\phi_\alpha(L \cap M_\alpha) = \{0\} \times \mathbb{R}^p$, and for every $x \in L$ in the boundary of M, there is a chart (M_α, ϕ_α) such that $x \in M_\alpha$, and with $\phi_\alpha(L \cap M_\alpha) = \{0\} \times \mathbb{R}^p$, and for which $\phi_\alpha(x) \in \{0\} \times \partial H^p$.

Lemma 1.1.3. *The boundary of an n-manifold is an $(n-1)$-submanifold with boundary.*

Example 1.4. The diameter of the ball B^2 is a submanifold with boundary.

Definition. We call an n-manifold M **closed** if M is compact with nonempty boundary ∂M.

Example 1.5. The n-sphere and n-torus are closed manifolds. Additionally, the projection map $\pi_y : T^2 \to S^1$ fo $T^2 = S^1 \times S^1$ onto the second factor is a continuous map between manifolds.

Lemma 1.1.4. *If M is an n-manifold with boundary, then its boundary ∂M is an $(n-1)$-manifold without boundary.*

Proof. We first notice by our definition of boundary that $\partial M \subseteq M$. Since M is second countable and Hausdorff, ∂M inherits these properties as a subspace of M.

Now, consider a point $x \in \partial M$. Then by definition, there is a neighborhood U of x for which U is homeomorphic to a neighborhood V of H^n. Let $\phi : U \to V$ be the given homeomorphism. Notice, that $\partial H^n = \{x \in \mathbb{R}^n : x_1 = 0\}$, and that $\phi(x) \in V$ implies $\phi(x) \in V \cap \partial H^n$. Since V is open in H^n, $V \cap \partial H^n$ is open in ∂H^n as a subspace of H^n. So that $\phi(U) \simeq V \cap \partial H^n$ and U is homeomprphic to an open set in ∂H^n.

Now, take the projection map $\pi_y : \partial H^n \to \mathbb{R}^{n-1}$ onto the second factor, defined by $\pi_y : (0, (x_2, \ldots, x_n)) \to (x_2, \ldots, x_n) = (y_1, \ldots, y_{n-1})$. Then π_y defines a homeomorphism of ∂H^n onto $\mathbb{R}^{n-1}$, so that $\pi_y \circ \phi$ is a homeomorphism of U onto an open set of $\mathbb{R}^{n-1}$. This makes ∂M an $(n-1)$-manifold. Moreover, since $x \in \partial M$ was arbitrary, there is no $x \in \partial M$ with neighborhood homeomorphic to an open set in H^{n-1}, so that $\partial(\partial M) = \emptyset$; i.e. ∂M is without boundary. ■

1.2 Smooth Manifolds

Definition. We call a map $f : \mathbb{R}^n \to \mathbb{R}^n$ q**-smooth**, or C^q, if it has continuous partial derivatives of order q. We call f **smooth**, or C^∞, if it has continuous partial derivatives of all orders.

Definition. A C**-smooth manifold**, or C^q**-manifold** with $q > 0$ is a topological manifold with an atlas that is C^q. That is, for any charts (M_α, ϕ_α) and (M_β, ϕ_β), $\phi_\beta \circ \phi_\alpha^{-1}$ is C^q wherever it is defined. We call C^∞-manifolds **smooth manifolds**, or **differentiable manifolds**. We call the maps $\phi_\beta \circ \phi_\alpha^{-1}$ **transition maps**.

Example 1.6. (1) $\mathbb{R}^n$ is a smooth manifold, as are all its open subsets.

(2) Consider the n-manifold S^n with charts

$$(S^n \backslash (0, \dots, 0, 1), h) \qquad\qquad (S^n \backslash (0, \dots, 0, -1), h')$$

where

$$h(x_1, \dots, x_{n+1}) = \frac{1}{1 - x_{n+1}}(x_1, \dots x_n) \text{ and } h'(x_1, \dots, x_{n+1}) = \frac{1}{1 + x_{n+1}}(x_1, \dots x_n)$$

The map $h' \circ h^{-1}$ is smooth. Notice that

$$h^{-1}(y_1, \dots, y_n) = \Big(\frac{2y_1}{1 + y_1^2 + \dots + y_n^2}, \dots, \frac{2y_n}{1 + y_1^2 + \dots + y_n^2}\Big)$$

So that

$$h' \circ h^{-1} = \frac{1}{y_1^2 + \dots + y_n^2}(y_1, \dots, y_n)$$

Moreover, for all $q > 0$, $\partial^q h' \circ h^{-1}$ exists, which makes $h' \circ h^{-1}$ smooth. This makes S^n a smooth manifold.

(3) The product of smooth manifolds are smooth manifolds. In particular, the torus $T^2 = S^1 \times S^1$ is a smooth manifold.

Definition. Let M and N manifolds with atlases $\{(M_\alpha, \phi_\alpha)\}$ and $\{(N_\beta, \psi_\beta)\}$. We call a map $f : M \to N$ **q-smooth**, or C^q if $\psi_\beta \circ \phi_\alpha^{-1}$ is C^q wherever it is defined. We call C^q-maps between manifolds **C^q-diffeomorphisms**. We call C^∞-diffeomorphisms **diffeomorphisms**. We call any two C^q-manifolds **diffeomorphic** if there exists a C^q-diffeomorphism between them.

Example 1.7. (1) The map $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = x^3$ is a smooth map, but it is not a diffeomorphism, since $f'(x) = 3x^2$ has a singular point at 0 (elaborate?). It is not even a C^1-diffeomorphism.

(2) The projection map of $T^2 = S^1 \times S^1$ onto the second factor is a smooth map between manifolds.

Definition. Let M a C^q-manifold, for some $q \geq 1$, and let $x \in M$ and (M_α, ϕ_α) a chart containing x. We call x a **critical point** of a map $f : M \to \mathbb{R}$ if it is a critical point of $f \circ \phi_\alpha^{-1}$. If $g : \mathbb{R}^n \to \mathbb{R}^n$ is a map, we call x a **nondegenerate** critical point if the Hessian of g is nonsingular at x, and we call x a **nondegenerate** critical point of f if it is a nondegenerate critical point of $f \circ \phi_\alpha^{-1}$.

Definition. We define a **Morse function** on a manifold M to be a smooth map $f : M \to \mathbb{R}$ such that

(1) f has nondegenerate critical points.

(2) Distinct critical points map to distinct values.

Example 1.8. The projection map of the Torus $T^2 \subseteq \mathbb{R}^3$ on to the thrid coordinate is a map with critical points. It has 1 maximum value, 2 minimum values, and 2 saddle points. Moreover these critical points are nondegenerate, so that the projection is a Morse function. Notice that these critical points are nondegenerate, since they consist of minima, maxima, and saddle points. Moreover, we see that the sum of the minima and maxima, minus the saddle points is $1 + 1 - 2 = 0$.

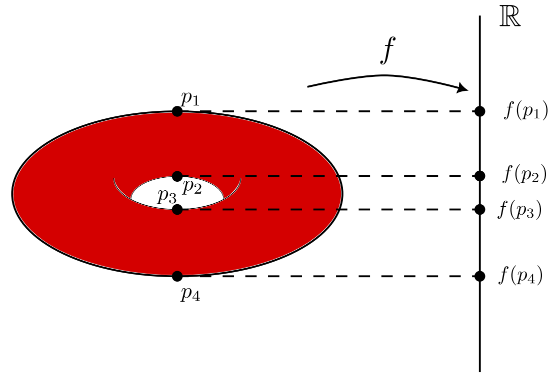

Figure 1.5: A Morse funtion on the torus T^2.

Definition. A q**-smooth submanifold**, or C^q**-submanifold** of a C^q-manifold M is a p-dimensional submanifold of M for which the transition maps restricted to L are C^q.

Definition. A q**-smooth manifold**, or C^q**-manifold** with **boundary** M is a toplogical manifold with boundary such that for any charts (M_α, ϕ_α) and (M_β, ϕ_β) in the atlas of M, the transistion map $\phi_\beta \circ \phi_\alpha^{-1}$ is C^q wherever it is defined. We call C^∞ manifolds with boundary **smooth manifolds**, or **differentiable manifolds** with **boundary**.

1.3 Orientable Manifolds

Definition. A smooth manifold M with boundary is called **orientable** if it has an atlas such that the Jacobians of all transition maps have positive determinant. Otherwise, we call M **nonorientable**. We call such atlas satisfying the above condition **orientations** of M, and we denote the orientations of M by $(M, \{\phi_\alpha\})$.

Example 1.9. (1) Let $A^1 = S^1 \times (-1, 1)$ be the 1-annulus. Then A^1 is orientable, and it is covered by the charts

$$(\{(\exp ix, t) : -\frac{\pi}{4} < x < \frac{5\pi}{4}, \text{ and } -1 < t < 1\}, \phi_1)$$

$$(\{(\exp ix, t) : -\frac{3\pi}{4} < x < \frac{9\pi}{4}, \text{ and } -1 < t < 1\}, \phi_2)$$

where $\phi_i(\exp ix, t) = (x, t)$. The transition maps are given by

$$\phi_2 \circ \phi_1^{-1}(x, t) = \begin{cases} (x + 2\pi, t), & \text{if } -\frac{\pi}{4} < x < \frac{\pi}{4} \\ (x, t), & \text{if } \frac{3\pi}{4} < x\frac{5\pi}{4} \end{cases}$$

and

$$\phi_1 \circ \phi_2^{-1}(x,t) = \begin{cases} (x-2\pi, t), & \text{if } \frac{7\pi}{4} < x < \frac{9\pi}{4} \\ (x,t), & \text{if } \frac{3\pi}{4} < x \frac{5\pi}{4} \end{cases}$$

Notice here that $\det(\operatorname{Jac}\phi_2 \circ \phi_1^{-1})$ and $\det(\operatorname{Jac}\phi_1 \circ \phi_2^{-1})$ are both positive wherever they are defined. So these charts give an orientation of A^1.

(2) The **Möbius band** given by identifying two sides of a rectangle given by figure 1.6 is nonorientable. Consider the "core" of the Möbius band, signified by a dotted line in

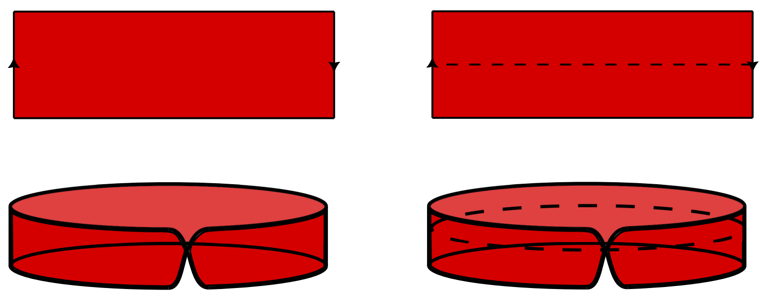

Figure 1.6: The Möbius band obtained by identifying two sides of a rectangle.

1.6. One cannot draw a curve above the core of the band, so this makes the surface nonorientable. The notion of Jacobians helps make sense of the intuitive notion of something being "above" an oriented surface.

Definition. Let M be a smooth manifold with orientations $(M, \{\phi_\alpha\})$ and $(M, \{\psi_\beta\})$. We say the the orientations **coincide** on a subset if the transition map $\phi_\alpha \circ \psi_\beta^{-1}$ is defined on that subset, and has Jacobian with positive determinant. We say that the orientations **differ** on a subset if the oreintation $\phi_\alpha \circ \psi_\beta^{-1}$ is defined on that subset, and has Jacobian with negative determinant.

Definition. Let M be an orientable manifold with orientation $(M, \{\phi_\alpha\})$. Where $\phi_\alpha : U_\alpha \to \mathbb{R}^n$ is defined by $\phi_\alpha(x) = (x_1, \ldots, x_n)$. We define the **opposite orientation** of M to be the orientation $(M, \{\psi_\alpha\})$ where $\psi_\alpha : U_\alpha \to \mathbb{R}^n$ is defined by $\psi_\alpha(x) = (-x_1, \ldots, x_n)$. We denote M with the orientation $(M, \{\psi_\alpha\})$ by $-M$.

Definition. If M is an orientable manifold with boundary, we define the **induced orientation** on ∂M to be the natural orientation on ∂M.

Definition. Let M and N be smooth orientable manifolds of the same dimension with orientations $(M, \{\phi_\alpha\})$ and $(N, \{\psi_\beta\})$. We call a smooth map $h : M \to N$ **orientation preserving** if the Jacobians on all the maps $\psi_\beta \circ h \circ \phi_a^{-1}$ have positive determinant wherever defined. If the Jacobians of all $\psi_\beta \circ h \circ \phi_a^{-1}$ have negative determinant, we call h **orientation reversing**.

Example 1.10. (1) The map $f : S^1 \to S^1$ given by $f(\exp 2i\pi x) = \exp 4i\pi x$ is orientation preserving, while the map $g : S^1 \to S^1$ given by $g(\exp 2i\pi x) = \exp -2i\pi x$ is orientation reversing.

Definition. Let M be a smooth manifold, and c a 1-dimensional submanifold of M. If there exists an atlas of M for which the charts that meet c have Jacobians with positive determinant, then we call c an **orientation preserving** closed 1-dimensional submanifold. Otherwise, if the Jacobians have negatice determinant, we call c an **orientation reversing** closed 1-dimensional submanifold.

Lemma 1.3.1. *A manifold M is nonorientable if, and only if M contains an orientation reversing closed 1-dimensional submanifold.*

1.4 Triangulated Manifolds

Definition. We define the **standard closed k-simplex** in $\mathbb{R}^{k+1}$ to be the set $[e_0, \ldots, e_k]$ of all affine combinations of the standard basis $\{e_0, \ldots, e_k\}$ of R^{k+1}. That is,

$$[e_0, \ldots, e_k] = \{a_0 e_0 + \cdots + a_k e_k : a_i \geq 0 \text{ for all } 0 \leq i \leq k \text{ and } \sum_{i=0}^{k} a_i = 0\}$$

When the basis is understood, we simply write $[s]$. We define the **standard open k-simplex** in $\mathbb{R}^{k+1}$, $(e_0, \ldots, e_k)$ to be the interior of $[e_0, \ldots, e_k]$; that is

$$(e_0, \ldots, e_k) = \{a_0 e_0 + \cdots + a_k e_k : a_i > 0 \text{ for all } 0 \leq i \leq k \text{ and } \sum_{i=0}^{k} a_i = 0\}$$

again, when the basis is clear, we write (s). We denote the **dimension** of $[e_0, \ldots, e_k]$ and $(e_0, \ldots, e_k)$ to be

$$\dim [e_0, \ldots, e_k] = \dim (e_0, \ldots, e_k) = k$$

Example 1.11. The standard 0-simplex is the point set $\{1\}$ of $\mathbb{R}$.

Definition. We define a **k-simplex** in a topological space X to be a continuous map $f : [s] \to X$ such that $[s]$ is the standard k-simplex, and $f|_{(s)}$ is homeomorphic onto its image $f|_{(s)}((s))$. We define an **open k-simplex** to be the restriction of a k-simplex from $[s]$ to its interior (s). We denote the **dimension** of a k-simplex f to be

$$\dim f = \dim [s]$$

Example 1.12. (1) The standard 1-simplex, shown in figure 1.7 consists of a line segment in $\mathbb{R}^2$ with vertices $(1, 0)$ and $(0, 1)$. It can be shown to be homeomorphic to an interval. Figure 1.8 also shows a 1-simplex $f : [e_0, e_1] \to X$.

(2) The standarde 2 simplex is a triangle in $\mathbb{R}^3$ with vertices $(1, 0, 0)$, $(0, 1, 0)$, and $(0, 0, 1)$. It can be visualized in figure 1.8 to be a 2-simplex, which is just a map $f : [e_0, e_1, e_2] \to X$.

(3) The standard 3-simplex is a tetrahedron (in $\mathbb{R}^4$) with vertices at $(1, 0, 0, 0)$, $(0, 1, 0, 0)$, $(0, 0, 1, 0)$, and $(0, 0, 0, 1)$. Figure 1.8 again shows a 3-simplex.

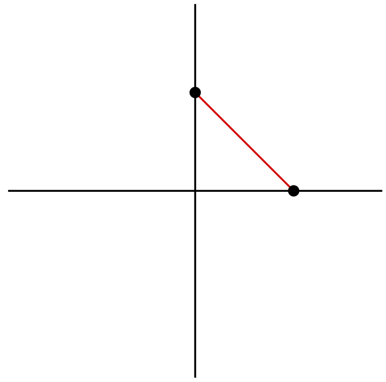

Figure 1.7: The standard 1-simplex $[e_0, e_1]$.

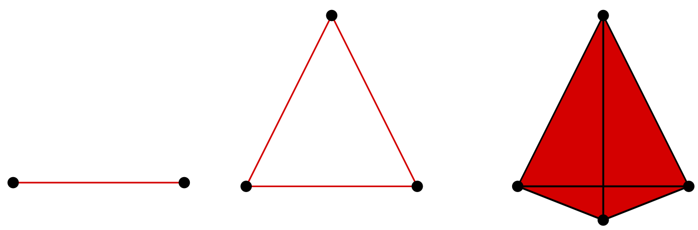

Figure 1.8: A 1-simplex, a 2-simplex, and a 3-simplex

Definition. For $0 \leq i \leq j$, we define a j**-face** of a k-simplex $f : [s] \to X$ to be a subset of the form

$$\{a_0 e_0 + \cdots + a_k e_k : a_{i_1} = \cdots = a_{i_j} = 0\}$$

We denote the **dimension** of a face to be $k - j$. We call a 0-face a **vertex**, and a 1-face an **edge**.

Lemma 1.4.1. *Let $f : [s] \to X$ be a k simplex, for any topological space X. Then a j face of f is a $k - j$ simplex.*

Example 1.13. (1) 0-simpleces have only themselves as faces.

(2) The standard 1-simplex $[e_0, e_1]$ has only the set $\{e_0, e_1\}$ as a 1-face, and has the point sets $\{e_0\}$ and $\{e_1\}$ as its 0-faces.

(3) A 2-simplex has one 2-face, three 1-faces, and three 0-faces. A 3-simplex has one 3-face, four 2-faces, six 1-faces, and four 0-faces.

Definition. A **simplicial complex** based on a topologica space X is a set K of simplices $f : [s] \to X$ such that

(1) For any simplex $f \in K$, the faces of f are in K.

(2) For any pair of simplices $f_1, f_2 \in K$, if the intersection of the images of $f|_{(s_1)}$ and $f_2|_{(s_2)}$ is nonempty, then the images are the same.

We define the **dimension** of K to be $\dim K = \sup\{\dim f_i\}$, for all $f_i \in K$, and we call the union of the images of simplices in K the **underlying space**, and denote it $\|K\|$.

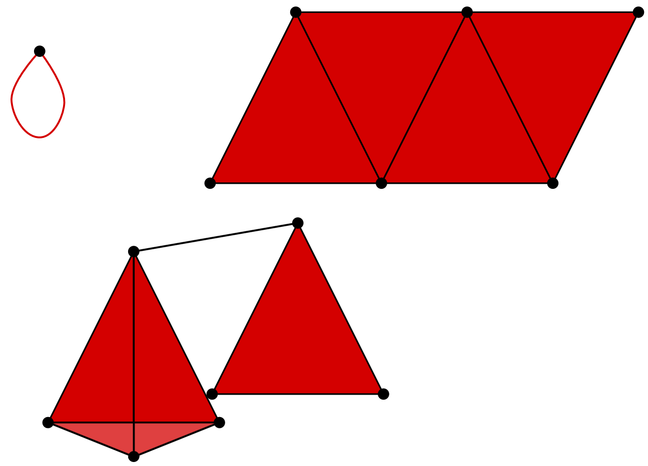

Figure 1.9: A collection of simplicial complexes.

Definition. A **triangulated n-manifold** is a topological n-manifold M, together with a simplicial comlex K based in M such that

(1) The underlying space of K is M; i.e. $\|K\| = M$.

(2) K is locally finite; i.e. if C is a compact subset of M, then the set of all nonempty intersections $C \cap f([s])$ is finite, where $f \in K$.

(3) For complexes $f, g \in K$, restricted to open simplices, the composition $g^{-1} \circ f$ is affine on its domain.

We call K a **triangulation** of M, and for any simplex $f : [s] \to M$, we call the pair $(f([s]), f^{-1})$ a **simplicial chart** of K. Writing $K = \{f_\alpha\}$, we also call f_α a **simplex** of M, for each α.

Example 1.14. (1) We have the triangulation of the sphere S^2 and the torus T^2 as show in figure 1.9.

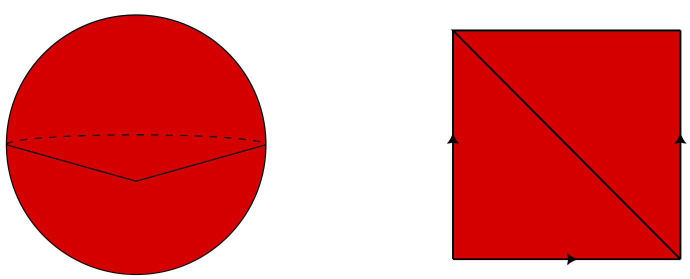

Figure 1.10:

(2) The following figure 1.11 shows a triangulated cube. This is a portion of the triangulation of the 3-torus T^3. The triangulation of T^3 is taken by indentifying 8 appropriately selected reflections of the given cube. The resulting triangulation has forty 3-simplices.

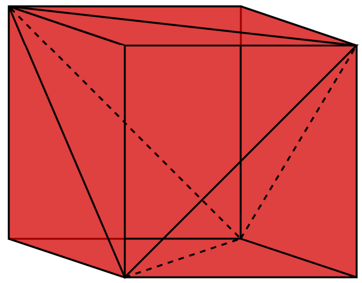

Figure 1.11:

Theorem 1.4.2. *Every compact* 1*-manifold admits a triangulation.*

Theorem 1.4.3 (Radó & Kerekjarto)**.** *Every compact* 2*-manifold admits a triangulation.*

Theorem 1.4.4 (Bing & Moise)**.** *Every compact* 3*-manifold admits a triangulation.*

Definition. Let K and L be simplicial complexes. We call a continuous map $\phi : \|K\| \to \|L\|$ a **simplicial map** if for every simplex f in K, there exists a simplex g in L, such that $\phi \circ f = g$.

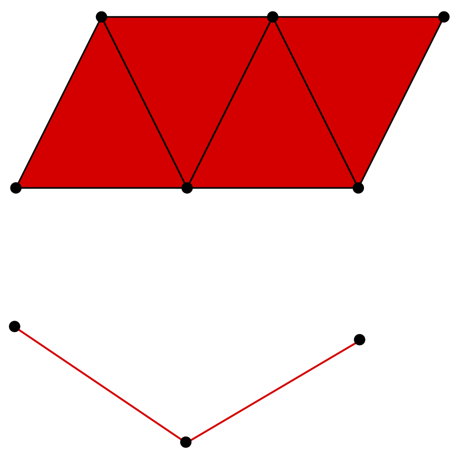

Figure 1.12: A simplicial map between a simplicial complex of 2-simplices, and a simplicial complex of 1-simplices.

Definition. Let K and L be simplicial complexes, and $\phi : \|K\| \to \|L\|$ a simplicial map. We call ϕ a simplicial isomorphism if it is a homeomorphism and we call K and L **isomorphic** if there exists a simplicial isomorphism between them,

Definition. Let (M, K) and (N, L) triangulated manifolds. We call them **isomorphic** if there exists a simplicial isomorphism $\phi : M \to N$.

Definition. We define a **subcomplex** of a simplicial complex K to be a subset L of K that is also a simplicial complex.

Definition. Let K be a finite simplicial complex and S_i the collection if simplices of dimension i. We define the **Euler characteristic** of K to be

$$\chi(K) = \sum_{i=0}^{n} (-1)|S_i|$$

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