Topology

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Chapter 1

Topological Spaces and Continuous Functions.

1.1 Topological Spaces.

Definition. A **topology** on a set X is a collection $\mathcal{T}$ of subsets of X such that:

(1) $\emptyset \in \mathcal{T}$ and $X \in \mathcal{T}$

(2) For any collection $\{U_\alpha\}$ of subsets of X, $\bigcup_\alpha U_\alpha \in \mathcal{T}$.

(3) For any finite collection $\{U_i\}_{i=1}^n$ of subsets of X, $\bigcap_{i=1}^n U_i \in \mathcal{T}$.

We call the pair $(X, \mathcal{T})$ a **topological space**, and we call the elements of $\mathcal{T}$ **open sets**.

Example 1.1. (1) Let X be any set, the collection of all subsets of X, 2^X is a topology on X, which we call the **discrete topology**. We call the topology $\mathcal{T} = \{\emptyset, X\}$ the **indiscrete topology**.

(2) The set of three points $\{a, b, c\}$ has the 9 following topologies in figure 1.1.

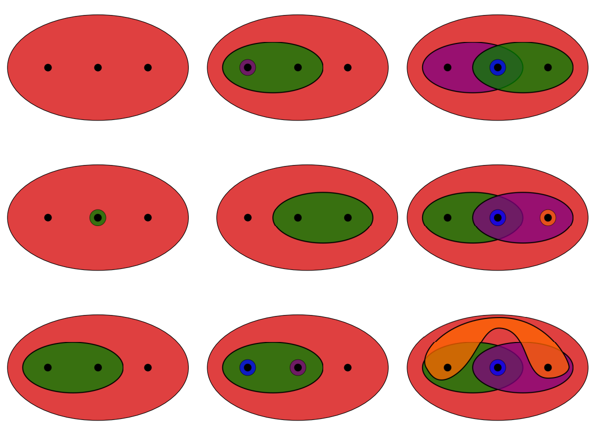

Figure 1.1: The Topologies on $\{a, b, c\}$.

(3) Let X be any set, and let $\mathcal{T}_f = \{U \subseteq X : X \backslash U \text{ is finite, or } X \backslash U = X\}$. Then $\mathcal{T}_f$ is a topology and called the **finite complement topology**.

(4) Let X be any set, and let $\mathcal{T}_c = \{U \subseteq X : X \backslash U \text{ is countable, or } X \backslash U = X\}$. Then $\mathcal{T}_c$ is a topology on X called the **countable complement topology**.

Definition. Let X be a set, and let $\mathcal{T}$ and $\mathcal{T}'$ be topologies on X. We say that $\mathcal{T}$ is **coarser** than $\mathcal{T}'$, and $\mathcal{T}'$ **finer** than $\mathcal{T}$ if $\mathcal{T} \subseteq \mathcal{T}'$. If two topologies are either coarser, or finer than each other, we call them **comparable**.

Example 1.2. The topologies $\mathcal{T}_f$ and $\mathcal{T}_c$ are comparable, and we see that $\mathcal{T}_f \subseteq \mathcal{T}_c$, so $\mathcal{T}_f$ is coarser than $\mathcal{T}_c$, and $\mathcal{T}_c$ is finer than $\mathcal{T}_f$.

1.2 The Basis and Subbasis for a Topology.

Definition. If X is a set, the **basis** for a topology on X is a collection $\mathcal{B}$ of subsets of X, called **basis elements**, such that:

(1) For every $x \in X$, there is a $B \in \mathcal{B}$ such that $x \in B$.

(2) For $B_1, B_2 \in \mathcal{B}$, if $x \in B_1 \cap B_2$, then there is a $B_3 \in \mathcal{B}$ such that $x \in B_3 \subseteq B_1 \cap B_2$

We define the topology $\mathcal{T}$ **generated** by $\mathcal{B}$ to be collection of open sets: $\mathcal{T} = \{U \subseteq X :$ for all $x \in U$, there exists a $B \in \mathcal{B}$ such that $x \in B\}$.

Theorem 1.2.1. *Let X be a set, and $\mathcal{B}$ a basis of X, then the collection of subsets of X, $\mathcal{T} = \{U \subseteq X :$ for all $x \in U$, there exists a $B \in \mathcal{B}$ such that $x \in B\}$ is a topology on X.*

Proof. Let $\mathcal{B}$ be a basis for a topology in X, and consider $\mathcal{T}$ as defined above. Cleary, $\emptyset \in X$ and so is X.

Now let $\{U_\alpha\}$ be a collection of subsets of X, and let $U = \bigcup U_\alpha$. Then if $x \in U$ for some α, there is a B_α such that $x \in B_\alpha \subseteq U_\alpha$, thus $x \in B_\alpha \subseteq U$.

Now let $x \in U_1 \cap U_2$, and choose $B_1, B_2 \in \mathcal{B}$ such that $x \in B_1 \subseteq U_1$ and $x \in B_2 \subseteq U_2$. Then by definition, there is a B_3 for which $x \in B_3 \subseteq B_1 \cap B_2$. Now suppose for arbitrary n, that $U = \bigcap_{i=1}^{n} U_i \in \mathcal{T}$, for some finite collection $\{U_i\}$ of subsets of X. Then by let $B_n, B_{n+1} \in \mathcal{B}$ such that $x \in B_n \subseteq U$ and $x \in B_{n+1} \subseteq U_{n+1}$. Then by our hypothesis, there is a B for which $x \in B \subseteq B_n \cap B_{n+1}$, thus $U \cap U_{n+1} = \bigcap_{i=1}^{n+1} U_i \in \mathcal{T}$. This make $\mathcal{T}$ a topology on X. ■

Example 1.3. (1) Let $\mathcal{B}$ be the set of all circular regions in the plane $\mathbb{R} \times \mathbb{R}$, then $\mathcal{B}$ satisfies the conditions needed for a basis.

(2) The collection $\mathcal{B}'$ in $\mathbb{R} \times \mathbb{R}$ of all rectangular region also forms a basis for a topology on $\mathbb{R} \times \mathbb{R}$.

(3) For any set X, the set of all 1-point subsets of X forms a basis for a topology on X.

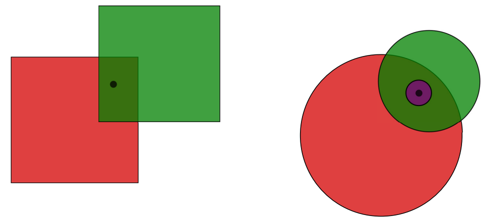

Figure 1.2: The basis for $\mathcal{B}$ and $\mathcal{B}'$ in $\mathbb{R} \times \mathbb{R}$ (see example (2)).

Lemma 1.2.2. *Let X be a set, and $\mathcal{B}$ be a basis for a topology $\mathcal{T}$ on X. Then $\mathcal{T} = \{\bigcup B : B \in \mathcal{B}\}$.*

Proof. Given a collection $\{B_i\}_{i=1}^{\infty}$ of basis elements in $\mathcal{B}$, since they are all in $\mathcal{T}$, their unions are also in $\mathcal{T}$. Conversely, given $U \in \mathcal{T}$, then for every point $x \in U$, choose a $B_x \in \mathbb{B}_x$ such that $x \in B_x \subseteq U$, then $U = \bigcup_{x \in U} B_x$. ■

Lemma 1.2.3. *Let $(X, \mathcal{T})$ be a topological space, and let $\mathcal{C} \subseteq \mathcal{T}$ be a collection of open sets of X such that for every $x \in U$, there is a $C \in \mathcal{C}$ such that $x \in C \subseteq U$. Then $\mathcal{C}$ is the basis for a $\mathcal{T}$ on X.*

Proof. Take any $x \in X$, then there is a $C \in \mathcal{C}$ such that $x \in C \subseteq U$, thus the first condition for a basis is satisfied. Now let $x \in C_1 \cap C_2$ for $C_1, C_2 \in \mathcal{C}$, since $C_1 \cap C_2$ is open in X, there is a $C_3 \in \mathcal{C}$ such that $x \in C_3 \subseteq C_1 \cap C_2$. Therefore $\mathcal{C}$ is a basis for a topology on X.

Now let $\mathcal{T}_{\mathcal{C}}$ be the the topology generated by $\mathcal{C}$, now for $U \in \mathcal{T}$, we have by the hypothesis, that $U \in \mathcal{T}_{\mathcal{C}}$; and by lemma 1.2.2, $W \in \mathcal{T}_{\mathcal{C}}$ is the union of elements of $\mathcal{C}$, which is a subcollection of $\mathcal{T}$, thus $W \in \mathcal{T}$. Therefore $\mathcal{T}_{\mathcal{C}} = \mathcal{T}$. ■

Lemma 1.2.4. *Let $\mathcal{B}$ and $\mathcal{B}'$ be bases for topologies $\mathcal{T}$ and $\mathcal{T}'$ on X. Then the $\mathcal{T} \subseteq \mathcal{T}'$ if and only if for all $x \in X$, and all $B \in \mathcal{B}$, there is a $B' \in \mathcal{B}'$ such that $x \in B' \subseteq B$.*

Proof. Suppose first that $\mathcal{T} \subseteq \mathcal{T}'$, and let $x \in X$, and choose $B \in \mathcal{B}$ such that $x \in B$, then B is open in $\mathcal{T}$, thus it is open in $\mathcal{T}'$, thus there is a $B' \in \mathcal{B}'$ such that $x \in B' \subseteq B$. Conversely, suppose there is a $B' \in \mathcal{B}'$ for which $x \in B' \subseteq B$ for all $x \in X$, $B \in \mathcal{B}$. Take $x \in U \in \mathcal{T}$, since $\mathcal{B}$ generates $\mathcal{T}$, $x \in B \subseteq U$, since $B' \subseteq B$, this implies that $U \in \mathcal{T}'$ and $\mathcal{T} \subseteq \mathcal{T}'$. ■

Definition. If $\mathcal{B}$ is the collection of open intervals (a, b) in $\mathbb{R}$, we call the topology generated by $\mathcal{B}$ the **standard topology** on $\mathbb{R}$, and we denote it simply by $\mathbb{R}$.

Definition. If $\mathcal{B}$ is the collection of half open intervals $[a, b)$ in $\mathbb{R}$, we call the topology generated by $\mathcal{B}$ the **lower limit topology** on $\mathbb{R}$, and we denote it simply by $\mathbb{R}_l$. If $\mathcal{B}'$ is the collection of all half open intervals $(a, b]$ in $\mathbb{R}$, then we call the topology generated by $\mathcal{B}'$ the **upper limt topology** on $\mathbb{R}$, and denote it $\mathbb{R}_L$.

Definition. If $\mathcal{B}$ is the collection of all open intervals of the form $(a, b) \setminus \frac{1}{\mathbb{Z}^+}$, where $\frac{1}{\mathbb{Z}^+} = \{\frac{1}{n} : n \in \mathbb{Z}^+\}$, we call the topology generated by $\mathcal{B}$ the $\frac{1}{\mathbb{Z}^+}$**-topology** on $\mathbb{R}$, and we denote it $\mathbb{R}_{\frac{1}{\mathbb{Z}^+}}$.

Lemma 1.2.5. *The topologies $\mathbb{R}_l$, $\mathbb{R}_L$, and $\mathbb{R}_{\frac{1}{\mathbb{Z}^+}}$ are all strictly finer than $\mathbb{R}$, but are not comparable with each other.*

Proof. Let (a, b) be a basis element for $\mathbb{R}$, and let $x \in (a, b)$, the basis element $[x, b) \in \mathbb{R}_l$ lies in (a, b) and contains x, however, there can be no interval (a, b) in $[x, b)$ as $x \leq a$, thus $\mathbb{R} \subset \mathbb{R}_l$; a similar argument holds for $\mathbb{R}_L$.

Similarly, for $(a, b) \in \mathbb{R}$, the basis element $(a, b) \backslash \frac{1}{\mathbb{Z}^+}$ of $\mathbb{R}_{\frac{1}{\mathbb{Z}^+}}$ lies in (a, b), however, choose the basis $B = (-1, 1) \backslash \frac{1}{\mathbb{Z}^+}$, and choose $0 \in B$, since $\mathbb{Z}^+$ is dense in $\mathbb{R}$, there is no interval (a, b) containing 0 and lying in B, thus $\mathbb{R} \subseteq \mathbb{R}_{\frac{1}{\mathbb{Z}^+}}$.

Now choose $[0, 1)$ in $\mathbb{R}_l$, and choose $\frac{1}{k} \in [0, 1)$ such that $k \in \mathbb{Z}^+$. Now $(0, 1) \subseteq [0, 1)]$, so we cannot say that $[0, 1)$ is a basis for $\mathbb{R}$, and moreover, $[0, 1) \backslash \frac{1}{\mathbb{Z}^+}$ cannot be said to be a basis in $\mathbb{R}_{\frac{1}{\mathbb{Z}^+}}$, thus $\mathbb{R}_l$ and $\mathbb{R}_{\frac{1}{\mathbb{Z}^+}}$ are incompararble, a similar argument holds for $\mathbb{R}_L$.

Lastly, let (a, b) be in $\mathbb{R}$ and choose $x \in (a, b)$. Then $(a, x]$ and $[x, b)$ are both in (a, b), however it is clear that $(a, x]$ and $[x, b)$ connot be contained in each other, thus $\mathbb{R}_l$ and $\mathbb{R}_L$ are incomparable. ■

Definition. A **subbasis**, $\mathcal{S}$, for a topology on X is a collection of subsets of X whose union equals X. We call the **topology generated by** $\mathcal{S}$ to be the collection of all unions of finite intersections of elements of $\mathcal{S}$, that is:

$$\mathcal{T} = \{\bigcup \bigcap_{i=1}^{n} S_i : S_i \in \mathcal{S} \text{ for } 1 \leq i \leq n\}$$

Theorem 1.2.6. *Let $\mathcal{S}$ be a subbasis for a topology on X. Then the collection $\mathcal{T} = \{\bigcup \bigcap_{i=1}^{n} S_i : S_i \in \mathcal{S}$ for $1 \leq i \leq n\}$ is a topology on X.*

Proof. It is sufficient to show that the collection $\mathcal{B}$ of all finite intersections of elements of $\mathcal{S}$ is a basis for a topology on X. By lemma 1.2.1, for $x \in X$, it belongs to an element S of $\mathcal{S}$, and therefore, to an element of $\mathcal{B}$. Now let $B_1 = \bigcap_{i=1}^{m} S_i$ and $B_2 = \bigcap_{j=1}^{n} S'_j$ be basis elements of $\mathcal{B}$. The intersection $\mathbb{B}_1 \cap B_2$ is a finite intersection of elements of $\mathcal{S}$, and hence also belongs in $\mathcal{B}$, and hence we can take another basis element B_3 such that $x \in B_3 \subseteq B_1 \cap B_2$. ■

1.3 The Order Topology.

Definition. Let X be a set with a simple order relation, and suppose that $|X| > 1$. Let $\mathcal{B}$ be the collection of sets of the following forms:

(1) All open intervals $(a, b) \in X$.

(2) All half open intervals $[a_0, b)$ where a_0 is the least element (if any) of X.

(3) All half open intervals of the form $(a, b_0]$ where b_0 is the greatest element (if any) of X.

Then $\mathcal{B}$ forms the basis for a topology on X called the **order topology**

Theorem 1.3.1. *The collection $\mathcal{B}$ forms a basis.*

Proof. Consider $x \in X$, if x is the least element of X, then it liess in all intervals of type (2), if it is the largest, then it lies in all intervals of type (3). If x is neither the least nor largest element, then $x \in (a_0, b_0)$ with a_0 and b_0 the least and largest elements (if any) of X. If no such elements exist, then $x \in (a, b)$, for some lowerbound a and upperbound b. Thus, in all three cases, there is a basis element containing x.

Now suppose $B_1, B_2 \in \mathcal{B}$ such that $x \in B_1 \cap B_2$. If B_1 and B_2 are both of type (1), then let $B_1 = (a, b)$, $B_2 = (c, d)$, then $B_1 \cap B_2$ is an open interval of type (1), now fix B_1 to be of type one. If B_2 is of type (2), then letting $B_2 = [a_0, c)$, then $x \in [a_0, d)$ for some $d \in X$. Likewise, if $B_2 = (c, b_0]$, is of type (3), we get a similar result. Moreover, the results are analogous if we fix B_2 and let B_1 range between intervals of the three types. Thusm in all cases, there is a $B_3 \in \mathcal{B}$ such that $x \in B_3 \subseteq B_1 \cap B_2$. ■

Example 1.4. (1) The standard topology on $\mathbb{R}$ is the order topology on $\mathbb{R}$ induced by the usual order relation. We have that $\mathbb{R}$ under this topology has no intervals of type (2), nor (3), so all bases elements in the standard topology are open intervals in $\mathbb{R}$.

(2) Consider the dictionary order on $\mathbb{R} \times \mathbb{R}$. Since $\mathbb{R} \times \mathbb{R}$ has no intervals of type (2), nor (3), the bases of $\mathbb{R} \times \mathbb{R}$ under the dictionary order are the open intervals of the form $(a \times b, c \times d)$ Where $a \leq c$, and $b < d$.

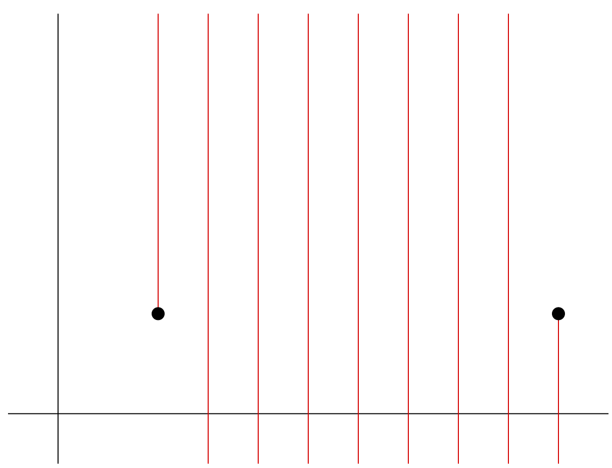

Figure 1.3: The order topology on $\mathbb{R} \times \mathbb{R}$.

(3) The positive integers $\mathbb{Z}^+$ with the least element 1 form an ordered set under the usual order. Taking $n > 1$, we see the bases of $\mathbb{Z}^+$ under the order topology are of the form $(n-1, n+1) = \{n\}$ and $[1, n) = \{1, \ldots, n-1\}$. Thus the order topology on $\mathbb{Z}^+$ is the discrete topology.

(4) The set $X = \{1, 2\} \times \mathbb{Z}^+$ over the dictionary order is also an ordered set, with the least element 1×1. Denote $1 \times n$ as a_n and $2 \times n$ as b_n. Then X consist of the elements $a_1, a_2, \ldots, b_1, b_2, \ldots$.

Now take $\{b_1\}$, then any open set containing b_1 must have a basis about b_1, and also contains points a_i with $i \in \mathbb{Z}^+$; thus the order topology on X is not the discrete topology.

Definition. Let X be an ordered set, and let $a \in X$. There are two subsets in X, $(a, \infty) = \{x \in X : x > a\}$ and $(-\infty, a) = \{x \in X : x < a\}$ called **open rays** of X. There are also two sets $[a, \infty) = \{x \in X : x \geq a\}$ and $(-\infty, a] = \{x \in X : x \leq a\}$ called **closed rays** of X.

Theorem 1.3.2. *Let X be an ordered set. Then the collection of all open rays in X form a subbasis for the order topology on X.*

Proof. Let $\mathcal{S}$ be the collection of all open rays of X, let $a < b$ and (a, ∞), $(-\infty, b) \in \mathcal{S}$, then $(a, b) = (a, \infty) \cap (-\infty, b)$. Now take:

$$S = \bigcup_{a,b \in X} (a, b)$$

then $S \subseteq X$, likewise, since S runs through all intersections of open rays of X, it contains all open intervals in X, hence $X \subseteq S$, and so $X = S$ as required. ∎

1.4 The Product Topology.

Definition. Let X and Y be topological spaces. We define the **product topology** on $X \times Y$ to be the topology having as basis the collection

$$\mathcal{B} = \{U \times V \subseteq X \times Y : U \text{ is open in } X \text{ and } V \text{ is open in } Y\}$$

Theorem 1.4.1. *The collection $\mathcal{B} = \{U \times V \subseteq X \times Y : U$ is open in X and V is open in $Y\}$ forms a basis for the product topology on $X \times Y$.*

Proof. Clearly, we have that $X \times Y$ is a basis element of $\mathcal{B}$. Now take $U_1 \times V_1$ and $U_2 \times V_2$ in $\mathcal{B}$. Since $U_1 \times V_1 \cap U_2 \times V_2 = U_1 \cap U_2 \times V_1 \cap V_2$, since $U_1 \cap U_2$ and $V_1 \cap V_2$ are open in X and Y respectively, then we have that $U_1 \times V_1 \cap U_2 \times V_2$ is a basis element as well. ∎

Theorem 1.4.2. *If $\mathcal{B}$ is the basis for a topology on X, and $\mathcal{C}$ is the basis for a topology on Y, then the collection:*

$$\mathcal{D} = \{B \times C : B \in \mathcal{B} \text{ and } C \in \mathcal{C}\}$$

Is a basis for the topology on $X \times Y$.

Proof. By lemma 1.2.3, let W be an open set of $X \times Y$, and let $x \times y \in W$. Then there is a basis $U \times V$ such that $x \times y \in U \times V \subseteq W$. Since $\mathcal{B}$ and $\mathcal{C}$ are bases of X and Y respectively, choosing $B \in \mathcal{B}$ and $C \in \mathcal{C}$, we have that $x \in B \subseteq U$, and $y \in C \subseteq Y$, thus $x \times y \in B \times C \subseteq U \times V \subseteq W$. Therefore, $\mathcal{D}$ is the basis for a topology on $X \times Y$. ∎

Example 1.5. The product of the standard topology on $\mathbb{R}$ with itself is called the **standard topology on** $\mathbb{R} \times \mathbb{R}$, and has as basis the collection of all products of open sets in $\mathbb{R}$. By theorem 1.4.2, if we take the collection of all open intervals $(a, b) \times (c, d)$ in $\mathbb{R} \times \mathbb{R}$, we form a basis. Constructing this basis geometrically gives the interior of a rectangle, whose boundaries are the intervals (a, b) and (c, d).

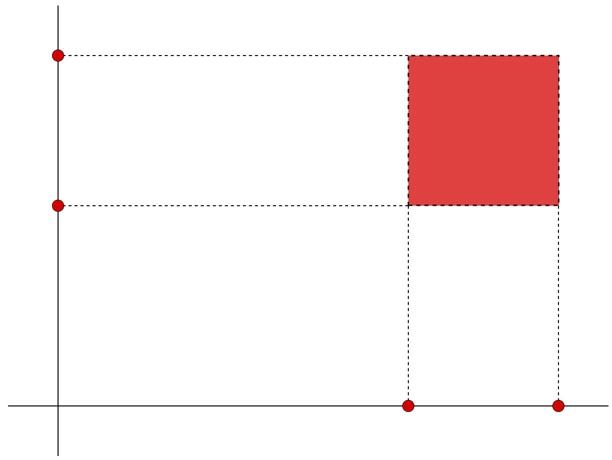

Figure 1.4: A basis element for $\mathbb{R} \times \mathbb{R}$

Definition. Let $\pi_1 : X \times Y \to X$ be defined such that $\pi_1(x, y) = x$, and define $\pi_2 : X \times Y \to Y$ such that $\pi_2(x, y) = y$. We call π_1 and π_2 **projections** of $X \times Y$ onto its first and second **factors**; that is onto X and Y, respectively.

Clearly, π_1 and π_2 are both onto. Now let U be open in X, then $\pi_1^{-1}(U) = U \times Y$ is open in $X \times Y$; similarly, $\pi_2^{-1}(V) = X \times V$ is also open in $X \times Y$, for V open in Y.

Theorem 1.4.3. *The collection* $\mathcal{S} = \{\pi_1^{-1}(U) : U$ *is open in* $X\} \cup \{\pi_2^{-1}(V) : V$ *is open in* $Y\}$ *is a subbasis for the product topology on* X*.*

Proof. Let $\mathcal{T}$ be the product topology on $X \times Y$, and let $\mathcal{T}'$ be the topology generated by $\mathcal{S}$. Since every element of $\mathcal{S}$ is open in $\mathcal{T}$, $\mathcal{T} \subseteq \mathcal{T}'$. Conversely, consider the basis element $U \times V$ of $\mathcal{T}$, then $\pi_1^{-1}(U) \cap \pi_2^{-1}(V) = U \times Y \cap X \times V = U \times V$, thus $\mathcal{T} \subseteq \mathcal{T}'$. Therefore, $\mathcal{S}$ is a subbasis for the product topology. ■

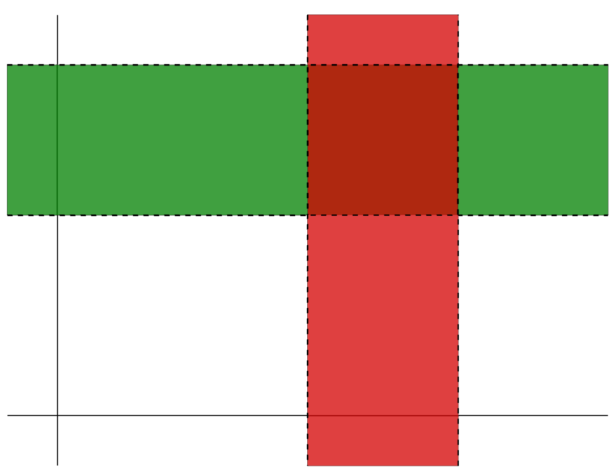

Figure 1.5: The inverse images, $\pi_1^{-1}(U)$ and $\pi_2^{-1}(V)$, of the projections π_1 and π_2 onto the $X \times Y$ plane.

1.5 The Subspace Topology.

Theorem 1.5.1. *Let X be a topological space with topology $\mathcal{T}$, and let $Y \subseteq X$. Then the collection:*

$$\mathcal{T}_y = \{Y \cap U : U \in \mathcal{T}\}$$

forms a topology on Y.

Proof. Cleary, $Y \cap \emptyset = \emptyset \in \mathcal{T}_Y$ and $Y \cap X = Y \in \mathcal{T}_Y$. Now consider the collection $\{U_\alpha\}$. Then $\bigcup(Y \cap U_\alpha) = Y \cap \bigcup U_\alpha$, similarly, for $\{U_i\}_{i=1}^n$, $\bigcap(Y \cap U_i) = Y \cap \bigcap U_i$, hence $\mathcal{T}$ is a topology on Y. ■

Definition. Let X be a topological space, and let $Y \subseteq X$. We call the $\mathcal{T}$ defined in theorem 1.5.1 the **subspace topology** on Y. We say that $U \subseteq Y$ is **open in** Y if $U \in \mathcal{T}_Y$.

Lemma 1.5.2. *Let $\mathcal{B}$ be the basis for a topology on X. Then the collection $\mathcal{B}_Y = \{B \cap Y : B \in \mathcal{B}\}$, where $Y \subseteq X$, is a basis for the subspace topology on Y.*

Proof. Let U be open in X, and let $y \in Y \cap U$, and choose $B \in \mathcal{B}$ such that $y \in B \subseteq U$, then $y \in B \cap Y \subseteq U \cap Y$, then by lemma 1.2.2, $\mathcal{B}_y$ is the basis fpr the subspace topology on Y. ■

Lemma 1.5.3. *Let Y be a subspace of X, If $U \subseteq Y$ is open in Y and Y is open in X, then U is open in X.*

Proof. Let $U \in \mathcal{T}_Y$, then for some $V \subseteq X$, $U = Y \cap V$. Now since Y is open in X, and so is V, then it follows that U is also open in X. ■

Remark. What this lemma says is that given a topological space X, and a subspace Y of X, then the subspace topology of Y is courser than the topology on X, i.e. $\mathcal{T}_Y \subseteq \mathcal{T}$.

Theorem 1.5.4. *If A is a subspace of X, and B is a subspace of Y, then the product topology on $A \times B$ is the topology that $A \times B$ inherits as a subspace of $X \times Y$.*

Proof. We have that $U \times V$ is the basis element for $X \times Y$, with U open in X, and V open in Y. Thus $(U \times V) \cap (A \times B) = (U \cap A) \times (V \cap B)$ is a basis element for the subspace topology on $X \times Y$. Since $U \cap A$ and $V \cap B$ are open in the subspace topologies of A and B respectively, then $(U \cap A) \times (V \cap B)$ is a basis for the product topology on $A \times B$. ■

Example 1.6. (1) Consider $[0,1] \subseteq \mathbb{R}$. In the subspace topology of $[0,1]$, we have as basis elements of the form $(a,b) \cap [0,1]$, with $(a,b) \subseteq \mathbb{R}$. If we have that $(a,b) \subseteq [0,1]$, then $(a,b) \cap [0,1] = (a,b)$. On the other hand, if $a \in [0,1]$ or $b \in [0,1]$, then we get $(a,b) \cap [0,1] = (a,1]$ or $(a,b) \cap [0,1] = [0,b)$, lastly if neither a nor b are in $[0,1]$, then we have $(a,b) \cap [0,1] = [0,1]$ only if $[0,1] \subseteq (a,b)$, and $(a,b) \cap [0,1] = \emptyset$ otherwise.

The only one of these sets open in $\mathbb{R}$ under the standard topology is $(0,1)$.

(2) For $[0,1) \cup \{2\} \subseteq \mathbb{R}$, the singletoun $\{2\}$ is open in the subspace topology on $[0,1) \cup \{2\}$; for observe, that $(\frac{3}{5}, \frac{5}{2}) \cap ([0,1) \cup \{2\}) = \{2\}$, however, in the order topology, on that same set, $\{2\}$ is not open. Any basis element on $[0,1) \cup \{2\}$ containing 2 is of the form $(a,2]$, where $a \in [0,1) \cup \{2\}$.

(3) The dictionary order on $[0,1] \times [0,1]$ is a restriction of the dictionary order on $\mathbb{R} \times \mathbb{R}$. Now the set $\{\frac{1}{2}\} \times (\frac{1}{2}, 1]$ is open in the subspace topology on $[0,1] \times [0,1]$, but it is not open in the dictionary order on the same set.

Definition. We call the set $[0,1] \times [0,1]$ on the dictionary odere the **ordered square**, and we denote it by I_0^2.

Definition. Let X be an ordered set. We say that a nonempty subset $Y \subset X$ is **convex** in X if for each pair of points $a, b \in Y$, with $a < b$, then the open interval $(a, b) \subseteq X$ is also contained in Y.

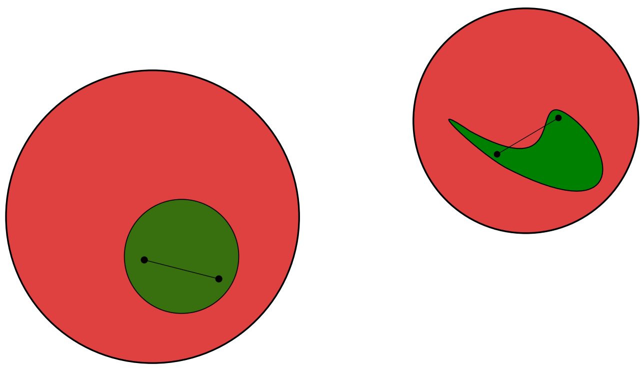

Figure 1.6: A convex set, and a nonconvex set.

Example 1.7. Let X be any ordered set. Then by definition, all open intervals and rays in X are convex in X.

Theorem 1.5.5. *Let X be an ordered set on the order toplogy, and let $Y \subseteq X$ be convex in X. Then the order topology on Y is the same as the subspace topology on Y.*

Proof. Consider $(a, \infty) \subseteq X$. If $a \in Y$, then $(a, \infty) \cap Y = \{x \in Y : x > a\}$, which is by definition an open ray on Y. Now if $a \notin Y$, then a is either a lowerbound, or an upperbound. Then $(a, \infty) \cap Y = \emptyset$ and $(-\infty, a) \cap Y = Y$ if a is an upperbound, similarly, if a is a lowerbound we get $(a, \infty) \cap Y = Y$ and $(-\infty, a) \cap Y = \emptyset$.

Since $(a, \infty) \cap Y$ and $(-\infty, a) \cap Y$ form a subbasis on the subspace topology on Y, and since they are also open in the order topology, then the order topology contains the subspace topology.

Now if (a, ∞) is an open ray in Y, then $(a, \infty) = (b, \infty) \cap Y$, with (b, ∞) some open ray in X, hence (a, ∞) is open in the subspace topology of Y, and since it also forms the subbasis for the order topology, we have that the order topology is contained within the subspace topology. Thus both topologies are equal. ∎

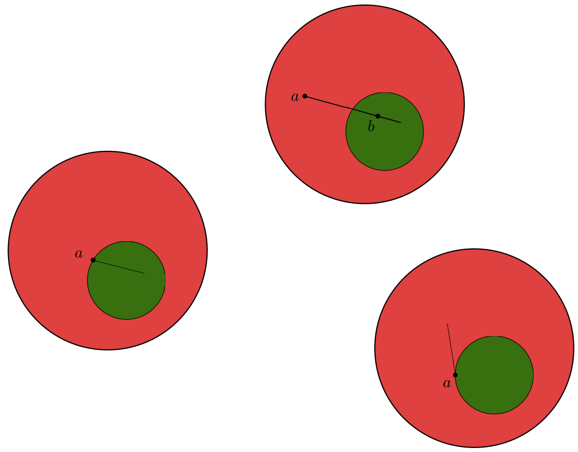

Figure 1.7: An illustration of theorem 1.5.5.

1.6 Closed Sets and Limit Points.

Definition. A subset A of a topological space X is said to be **closed** if $X\backslash A$ is open.

Example 1.8. (1) Consider $[a,b] \subseteq \mathbb{R}$, we have that $\mathbb{R}\backslash[a,b] = (-\infty, a) \cup (b, \infty)$ which is open in $\mathbb{R}$. So $[a,b]$ is closed.

(2) In $\mathbb{R} \times \mathbb{R}$, the set $A = \{x \times y : x, y \geq 0\}$ (i.e the first quadtant of the plane) is closed, for $\mathbb{R} \times \mathbb{R}\backslash A = ((-\infty, 0) \times \mathbb{R}) \cup (\mathbb{R} \times (-\infty, 0))$, which is open in $\mathbb{R} \times \mathbb{R}$.

(3) Consider the finite complement topology $\mathcal{T}_f$ on a set X. We have that $X\backslash X = \emptyset \in \mathcal{T}_f$, so X is closed, similarly, $\emptyset$ is also closed. Likewise, if $A \subseteq X$ is a finite set, then $X\backslash A$ is also finite, and hence A is also closed. Thus, we have that all the closed sets of $\mathcal{T}_f$ are those finite subsets of X. As a consequence, this example also illustrates that sets can be both closed and open.

(4) In the discrete topology 2^X, every open set is closed. This is another example where open sets are also closed sets.

(4) Consider $[0,1] \cup (2,3)$ in the subspace topology on $\mathbb{R}$. We have that $[0,1]$ is open in the subspace topology on $\mathbb{R}$; $[0,1] = [0,1] \cup (2,3) \cap (-\frac{2}{3}, \frac{3}{2})$, similarly, $(2,3)$ is also open. Now taking $[0,1] \cup (2,3)\backslash(2,3) = [0,1]$, which is open, so $[0,1]$ is closed in the subspace topology on $\mathbb{R}$, by the same reasoning, so is $(2,3)$.

Theorem 1.6.1. *Let X be a topological space. Then:*

(1) $\emptyset$ and X are closed.

(2) Arbitrary intersections of closed sets are closed.

(3) Finite unions of closed sets are closed.

Proof. We have that $X\backslash\emptyset = X$ and $X\backslash X = \emptyset$, both of which are open in X, so they are also closed in X. Now let $\{U_\alpha\}$ be a collection of closed sets of X. We have that:

$$X\backslash\bigcap_\alpha U_\alpha = \bigcup_\alpha X\backslash U_\alpha.$$

Similarly, for $\{U_i\}_{i=1}^n$, we have

$$X\backslash\bigcup_{i=1}^n U_i = \bigcap_{i=1}^n X\backslash U_i.$$

Both of which are open in X. This completes the proof. ■

Definition. If Y is a subspace of X, we say that A is **closed in** Y if $A \subseteq Y$ and A is closed in the subspace topology on Y.

Theorem 1.6.2. *Let Y be a subspace of X. Then A is closed in Y if and only if A equals the intersection of a closed set of X with Y.*

Proof. Suppose that A is closed in Y, then $Y\backslash A$ is open in Y, hence we have that $Y\backslash A = U \cap Y$ for some open set U of X. Now $X\backslash U$ is closed in X, and with $A \subseteq Y$, we have that $A = Y \cap (X\backslash U)$.

Conversely, suppose that $A = C \cap Y$, with C closed in X. Then $X\backslash C$ is open in X, hence $(X\backslash C) \cap Y$ is open in Y, now since $(X\backslash C) \cap Y = Y\backslash A$, which is open, we have that A is closed in Y. ■

Theorem 1.6.3. *Let Y be a subspace of X. If A is closed in Y, and Y is closed in X, then A is closed in X; that is, closure is transitive.*

Proof. By theorem 1.6.2, if A is closed in Y, then $A = C \cap Y$ with C closed in X, now since Y is closed in X, then $Y = D \cap X$ with D closed in X. Thus $A = (C \cap D) \cap X$, therefore, A is closed in X. ■

We now go over the concepts of the closure, and the interior of a set.

Definition. Let $A \subseteq X$, with X a topological space. The **interior** of A is defined to be the union of all open sets in A. The **closure** of A is defined to be the intersection of all closed sets containing A. We denote the interior and the closure of A as $\operatorname{Int} A$ and $\operatorname{cl} A$ respectively

We have by the very definitions that $\operatorname{Int} A \subseteq A \subseteq \operatorname{cl} A$

Lemma 1.6.4. $\operatorname{Int} A = A$ *only when A is open, and* $\operatorname{cl} A = A$ *only when A is closed.*

Proof. Now, if A is open, then it is in the union of all open sets of A, hence $A \subseteq \operatorname{Int} A$, likewise, if A is closed, then since $\operatorname{cl} A$ is the intersection of all closed sets containing A, we get $\operatorname{cl} A \subseteq A$. ■

Corollary. *A is closed and open if and only if* $\operatorname{Int} A = \operatorname{cl} A$.

Theorem 1.6.5. *Let Y be a subspace of X, and let $A \subseteq Y$. Then $\operatorname{cl} A \cap Y$ is the closure of A in Y.*

Proof. Let $\operatorname{cl} A$ be the closure of A in X. Since $\operatorname{cl} A$ is closed in X, by theorem 1.6.2, $\operatorname{cl} A \cap Y$ is closed in Y, now we have that $A \subseteq \operatorname{cl} A \cap Y$, and since $\operatorname{cl} A = \bigcap U$, then $\operatorname{cl} A \subseteq \operatorname{cl} A \cap Y$.

Conversely, suppose that $\operatorname{cl} A$ is closed in Y, again by theorem 1.6.2, we have that $\operatorname{cl} A = C \cap Y$, where C is closed in X, since $A \subseteq \operatorname{cl} A$, then $A \subseteq C$, and since C is closed, then $\operatorname{cl} A \subseteq C$, thus $\operatorname{cl} A \cap Y \subseteq \operatorname{cl} A$. ■

Definition. Let X be a topological space, and let $x \in X$. We call an open set U of X a **neighborhood** of x if $x \in U$.

Theorem 1.6.6. *If $A \subseteq X$, with X a topological space, then $\operatorname{cl} A$ is a neighborhood of $x \in X$ if and only if for every neighborhood U of x, $A \cap U \neq \emptyset$.*

Proof. We prove the contrapositve. If $x \notin \operatorname{cl} A$, then $U = X \setminus \operatorname{cl} A$ is an open set containing A, disjoint from A. Conversely, suppose there is a neighborhood U of x, with U disjoint from A, then $X \setminus U$ is closed, and therefore contains the closure of A, thus $x \notin \operatorname{cl} A$ ■

Corollary. $\operatorname{cl} A$ *is a neighborhood of x if and only if for every basis element B of X, containing x, intersects A.*

Proof. This is a direct application of theorem 1.6.6, since basis elements are open sets. ■

Example 1.9. (1) We have the closure of $(0,1]$ in $\mathbb{R}$ is the closed interval $[0,1]$, since every neighborhood of 0 intersects $(0,1]$. Now every point outside of $[0,1]$ has a neighborhood disjoint from $[0,1]$ (take the neighborhood $(2,3)$ of 2).

(2) $\operatorname{cl} \frac{1}{\mathbb{Z}^+} = \{0\} \cup \frac{1}{\mathbb{Z}^+}$ and $\operatorname{cl} \{0\} \cup (1,2)) = \{0\} \cup [1,2]$.

(3) $\operatorname{cl} \mathbb{Q} = \mathbb{R}$, $\operatorname{cl} \mathbb{Z}^+ = \mathbb{Z}^+$, $\operatorname{cl} \mathbb{R}^+ = \mathbb{R}^+ \cup \{0\}$. This first follows from the density of $\mathbb{Q}$ in $\mathbb{R}$. Every neighborhood $n \in \mathbb{Z}^+$ intersects $\mathbb{Z}^+$, so $\operatorname{cl} \mathbb{Z}^+ \subseteq \mathbb{Z}^+$, and we have that the neighborhood $(0,1)$ of 0 intersects $\mathbb{R}^+$, so $\operatorname{cl} \mathbb{R}^+ \subseteq \mathbb{R}^+ \cup \{0\}$.

Definition. If $A \subseteq X$, with X a topological space, and if $x \in X$, we say that x is a **limit point** of A if every neighborhood of x intersects A at some distinct point. That is: $x \in \operatorname{cl} X \setminus \{x\}$.

Example 1.10. (1) Consider $(0,1]$, we have that $0 \in [0,1] = \operatorname{cl}(0,1] = \operatorname{cl}[0,1] \setminus \{0\}$, so 0 is a limit point of $(0,1]$, the same can be said for any $x \in (0,1]$.

(2) For $\frac{1}{\mathbb{Z}^+}$, 0 is once again a limit point. Let $x \in \mathbb{R}$ be nonzero, and let $[x,b)$ be the neighborhood of x in the lower limit topology. Then $[x,b) \cap \frac{1}{\mathbb{Z}^+} = \emptyset$ or $\{x\}$, hence, 0 is the only limit point of $\frac{1}{\mathbb{Z}^+}$.

(3) $\operatorname{cl} \{0\} \cup (1,2) = \{0\} \cup [1,2]$ has all of its limit points in $[1,2]$. Likewise, every point in $\mathbb{R}$ is a limit point of $\mathbb{Q}$. $\mathbb{Z}^+$ has no limit points in $\mathbb{R}$, and the limit points of $\mathbb{R}^+$ are all the points of $\operatorname{cl} \mathbb{R}^+$.

Theorem 1.6.7. *Let $A \subseteq X$, X a topological space, and let A' be the set of all limit points in A. Then $\operatorname{cl} A = A \cup A'$.*

Proof. Let $x \in A'$, then every neighborhood of x intersects A at some distinct point x', by definition, so by theorem 1.6.6, $x \in \text{cl}\, A$, hence $A' \subseteq \text{cl}\, A$, so $A \cup A' \subseteq \text{cl}\, A$. Now, let $x \in \text{cl}\, A$. If $x \in A$, we are done. Otherwise, since every neighborhood of x intersects A, we have that they intersect at distinct points, thus $x \in A'$, therefore $\text{cl}\, A \subseteq A \cup A'$. ■

Corollary. *$A \subseteq X$ is closed if and only if $A' \subseteq A$.*

Proof. If A is closed, then $\text{cl}\, A = A = A \cup A'$, thus $A' \subseteq A$. The converse is obvious. ■

Definition. Let X be a topological space. A sequence of points of X $\{x_n\}$ is said to **converge** to a point $x \in X$ if for every neighborhood U of x, there is an $N \in \mathbb{Z}^+$ such that $x_n \in U$ for all $n \geq N$.

Example 1.11. Consider the following topological space on $\{a, b, c\}$ in figure **??**, and define

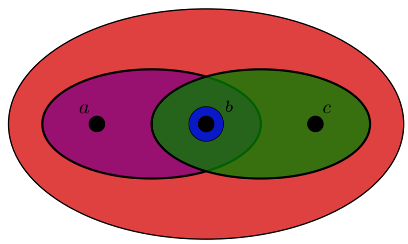

Figure 1.8: A topology on $\{a, b, c\}$, is not a Hausdorff space.

the sequence $\{x_n\}$ by $x_n = b$ for all $n \in \mathbb{Z}^+$. The neighborhoods of a, b, and c are $U_a = \{a, b\}$, $U_b = \{b\}$, and $U_c = \{b, c\}$. Now let $N > 0$, then we see that for all $n \geq N$, that $b \in U_a$, $b \in U_b$, and $b \in U_c$ thus b converges to a and to c, and itself,

Definition. A topological space X is called a **Hausdorff space** if for each pair of distinct points x_1 , and x_2, there are neighborhoods U_1 and U_2 of x_1 and x_2 respectively such that U_1 and U_2 are disjoint.

Example 1.12. The topology of the previous example in figure 1.8 is not a Hausdorff space.

Theorem 1.6.8. *Every finite point set in a Hausdorff space is closed.*

Proof. Let X be a Hausdorff space, and let $x_0 \in X$. We have that $\text{cl}\,\{x_0\} = \bigcap_{\{x_0\} \in U} U$. Now let $x \neq x_0 \in X$. Since $x\{x_0\}$, and X is Hausdorff, the intersections of the neighborhoods of x and x_0 is empty, thus $x \notin \text{cl}\,\{x_0\}$, therefore $\text{cl}\,\{x_0\} = \{x_0\}$.

We can then extend this proof to finite point sets of size n by induction. ■

Now the condition that finite point sets be closed need not depend on whether or not X is a Hausdorff space. In fact, we can assume the following for some topoltopological spaces.

Axiom 1.6.1 (The T_1 Axiom)**.** *In any topological space, every finite point set of X is closed.*

Theorem 1.6.9. *Let X be a topological space satisfying the T_1 axiom, and let $A \subseteq X$. Then a point x is a limit point of A if and only if every neighborhood of x contains infinitely many points of A.*

Proof. Let U_x be a neighborhood of x. If U_x intersects A at infinitely many points of A, then it intersects A at a point distinct from x, thus x is a limit point of A.

Conversely suppose that x is a limit point of A, and let $U_x \cap A$ be finite, then $U_x \cap (A \backslash \{x\})$ is also finite. Now let $U_x \cap (A \backslash \{x\}) = \{x_1, \ldots x_m\}$. By the T_1 axiom, $\{x_1, \ldots, x_m\}$ is closed, so $X \backslash \{x_1, \ldots, x_m\}$ is open, thus $U_x \cap (X \backslash \{x_1, \ldots, x_m\})$ is a neighborhood of x that does not intersect $A \backslash \{x\}$, which contradicts that x is a limit point. ■

Theorem 1.6.10. *If X is a Hausdorff space, then a sequence of points of X converges to at most one point in X.*

Proof. Let $\{x_n\}$ be a sequence of points converging to x, and let $y \neq x$ and let U_x and U_y be neighborhoods of x and y respectively. Then $U_x \cap U_y = \emptyset$. Now since $\{x_n\}$ converges to x, we have that for $N > 0$, $x_n \in U_x$ whenever $n \geq N$. Then $x_n \notin U_y$, and so $\{x_n\}$ cannot converge to y. ■

Definition. Let $\{x_n\}$ be a sequence in a Hausdorff space X. If $\{x_n\}$ converges to a point $x \in X$, we call x the **limit** of $\{x_n\}$ as n **approaches** ∞ and we write $\lim_{n \to \infty} x_n = x$ or $\{x_n\} \to x$ as $n \to \infty$.

Theorem 1.6.11. *The following are true:*

(1) Every simply oredered set under the order topology is Hausdorff.

(2) The product of two Hausdorff spaces is Hausdorff.

(3) The subspace of a Hausdorff space is Hausdorff.

Proof. (1) Let X be an ordered set under the order topology. Take $x, y \in X$ distinct, and suppose without loss of generality that $x < y$. Then consider the neighborhoods $(-\infty, x]$ and $[y, \infty)$ of x and y respectively. Then $(-\infty, x] \cap [y, \infty) = \emptyset$.

(2) Let X and Y be Hausdorff, and consider $X \times Y$ in the product topology. Let $x_1 \times y_1$ and $x_2 \times y_2$ be distinct points, and let U_{x_1}, U_{x_2}, V_{y_1} and V_{y_2} be basis elements of x_1, x_2, y_1, and y_2 respectively. Then they are neighborhoods of those elements respectively.

Now we have that $U_{x_1} \times V_{y_1}$ and $U_{x_2} \times V_{y_2}$ are basis elements of $x_1 \times y_1$ and $x_2 \times y_2$, respectively, and hence neighborhoods of those elements respectively.Then we have $(U_{x_1} \times V_{y_1}) \cap (U_{x_2} \times V_{y_2}) = (U_{x_1} \cap U_{x_2}) \times (V_{y_1} \cap V_{y_2}) = \emptyset \times \emptyset = \emptyset$.

(3) Let X be Hausdorff, and let Y be a subspace of X. Let x_1 and x_2 be distinct points, and let U_{x_1} and U_{x_2} be their neighborhoods. Since Y is open in X, then so are $Y \cap U_{x_1}$ and $Y \cap U_{x_2}$, so they are also neighborhoods of x_1 and x_2 respectively. Then $(Y \cap U_{x_1}) \cap (Y \cap U_{x_2}) = Y \cap (U_{x_1} \cap U_{x_2}) = \emptyset$.

■

1.7 Continuous Functions.

Definition. Let X and Y be topological spaces. We say that a mapping $f : X \to Y$ is **continuous** if for each open set V in Y, $f^{-1}(V)$ is open in X.

Now if $f : X \to Y$ is continuous, the for every open set V of Y, $f^{-1}(V)$ is open in X. Now suppose that $\mathcal{B}$ is a basis of Y, then $V = B_\alpha$, hence $f^{-1}(B_\alpha)$, is open in X

Similarly, if $\mathcal{S}$ is a subbasis of Y, then for any basis element B of Y, $B = \bigcap_{i=1}^{n} S_i$, which then implies that $f^{-1}(B) = \bigcap_{i=1}^{n} f^{-1}(S_i)$, thus $f^{-1}(S_i)$ is also open in X for $1 \leq i \leq n$.

Example 1.13. (1) Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous realvalued function. Then for each open interval $I \subseteq \mathbb{R}$, $f^{-1}(I)$ is an open interval in $\mathbb{R}$, so take $x_0 \in \mathbb{R}$ and $\varepsilon > 0$, and let $I = (f(x_0) - \varepsilon, f(x_0) + \varepsilon)$, then since $x_0 \in f^{-1}(I)$, there is a basis $(a, b) \subseteq f^{-1}(I)$ about x_0. Then take $\delta = \min\{x_0 - a, x_0 - b\}$, then $x \in (a, b)$ whenever $0 < |x - x_0| < \delta$, and we get that $f(x) \in I$, that is, $|f(x) - f(x_0)| < \varepsilon$. This is the definition of continuity defined in the real analysis. We can prove that the converse holds also.

If $f : \mathbb{R} \to \mathbb{R}$ is continuous at a point x_0, then for every $\varepsilon > 0$, there is a $\delta > 0$ such that $|f(x) - f(x_0)| < \varepsilon$ whenever $0 < |x - x_0| < \delta$. Then we notice that x and x_0 are distinct, furthermore, $x_0 - \delta < x < x_0 + \delta$, hence $x \in (x_0 - \delta, x_0 + \delta)$ implies that $f(x) \in (f(x_0) - \varepsilon, f(x_0) + \varepsilon)$. Letting $V_\delta(x_0) = (x_0 - \delta, x_0 + \delta)$ and $V_\varepsilon(f(x_0)) = (f(x_0) - \varepsilon, f(x_0) + \varepsilon)$, we have that whenever $x \in V_\delta(x_0)$, then $f(x) \in V_\varepsilon(f(x_0)) \subseteq f^{-1}(V_\delta(x_0))$. And so the topological definition of continuity is equivialent to the real analytic definition of continuity.

(2) Let $f : \mathbb{R} \to \mathbb{R}_l$ be defined such that $f(x) = x$ for all $x \in \mathbb{R}$. Take $[a, b) \subseteq \mathbb{R}_l$, we have that $f^{-1}([a, b)) = [a, b)$, which is not open in $\mathbb{R}$ (under the standard topology), hence f is not continuous. However, the map $g : \mathbb{R}_l \to \mathbb{R}$ defined the same way is continuous since $g^{-1}((a, b))$ is open in $\mathbb{R}_l$.

Theorem 1.7.1. *Let X and Y be topological spaces, and let $f : X \to Y$ be a mapping of X into Y. Then the following are equivalent:*

(1) f is continuous.

(2) For every $A \subseteq X$, $f(\operatorname{cl} A) \subseteq \operatorname{cl} f(A)$.

(3) For every closed set $B \subseteq Y$, $f^{-1}(B)$ is closed in X.

(4) For each $x \in X$ and each neighborhood V of $f(x)$, there is a neighborhood U of x such that $f(U) \subseteq V$.

Proof. Let f be continuous and let $A \subseteq X$. Consider the neighborhood V of $f(x)$, then $f^{-1}(V)$ is open in X, and intersects A at a point y. Then $V \cap f(A) = f(y)$, thus $f(x) \in \operatorname{cl} f(A)$.

Now let B be closed in Y, and let $A = f^{-1}(B)$. Then we have that $f(A) = f(f^{-1}(B)) \subseteq B$, thus $x \in \operatorname{cl} A$.

Now let V be open in Y, so that $B = Y \backslash V$ is closed in Y, and $f^{-1}(B) = f^{-1}(Y) \backslash f^{-1}(V) = X \backslash f^{-1}(V)$ which is closed in X, hence $f^{-1}(V)$ is open in X.

Now let $x \in X$, and let V be a neighborhood of $f(x)$. Then $U = f^{-1}(V)$ is a neighborhood of x for which $f(U) \subseteq V$. Finally let V be open in Y, and let $x \in f^{-1}(V)$, then $f(x) \in V$, so there is a neighborhood U_x of x for which $f(U_x) \subseteq V$, then $U_x \subseteq f^{-1}(V)$, then $f^{-1}(V)$ is a union of open sets, and hence open in X. ■

Definition. Let X and Y be topological spaces, and $f : X \to Y$ be a $1-1$ mapping of X onto Y. We call f a **homeomorphism** if both f and f^{-1} are continuous.

Lemma 1.7.2. *Let X and Y be topological spaces and let $f : X \to Y$ be a homeomorphism. Then $f(U)$ is open if and only if U is open.*

Proof. We have that both $f : X \to Y$ and $f^{-1} : Y \to X$ are continuous $1-1$ of X and Y onto each other (respectively). Now let U be open in X, then $U = f^{-1}(V)$, for some set V open in Y. Notice then, that $f(U) = f(f^{-1}(V)) = V$, thus $f(U)$ is open in Y. Conversely, let $V = f(U)$ be open in Y for some open set U in X, then $U = f^{-1}(V)$, so by definition of continuity, U is open in X. ■

Definition. Let X and Y be topological spaces and let $f : X \to Y$ be a contniuous $1-1$ maping of X into Y, and consider $f(X)$ as a subspace of Y. We call $f : X \to f(X)$ a **topological imbedding** if f is a homeomorphism of X onto $f(X)$.

Example 1.14. (1) The map $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = 3x + 1$ is a homeomorphism whose inverse is $f^{-1}(y) - \frac{1}{3}(y - 3)$, both f and f^{-1} are contiuous.

(2) The map $f : (-1, 1) \to \mathbb{R}$ defined by $f(x) = \frac{x^2}{1-x^2}$ has as its inverse the map $f : \mathbb{R} \to (-1, 1)$ defined by $f^{-1}(y) = \frac{2y}{1+\sqrt{1+4y^2}}$. Both f and f^{-1} are contiuous, so f is a homeomorphism.

(3) The map $g : \mathbb{R}_l \to \mathbb{R}$ defined by $g(x) = x$ is not a homeomorphism, despite being contiuous, as $g^{-1}(1)$ is undefined.

(4) Let S^1 be the unit circle in $\mathbb{R}$, which is a subspace of $\mathbb{R}$, and define $f : [0, 1) \to S^1$ by $f(t) = (\cos(t), \sin(t))$. Clearly f is $1-1$ onto S^1, and contiuous, however f^{-1} is not contiuous as $f([0, \frac{1}{4}))$ is not open in S^1 as $f(0)$ is in no open set of $\mathbb{R}^2$ such that $U \cap S^1 = f([0, 1))$.

(5) Consider the mappings $g : [0, 1) \to \mathbb{R}^2$ by $f(t) = (\cos(2t\pi), \sin(2t\pi)))$. Now g is $1-1$ and continuous, and we have that $g([0, 1)) \subseteq S^1$, however since g is not a homeomorphism, g fails to be a topological embedding.

Theorem 1.7.3 (Constructions for continuous functions.)**.** *Let X and Y be topological spaces, then:*

(1) (Constant construction) If $f : X \to Y$ maps $x \to y_0$ for all $x \in X$, then f is continuous.

(2) (Inclusion) If $A \subseteq X$ is a subspace, then the inclusion mapping $\iota : A \to Y$ is continuous.

(3) *(Construction by composition) If $f : X \to Y$ and $g : Y \to Z$ are continuous, then $f \circ g : X \to Z$ is also continuous.*

(4) *(Domain restriction) If $f : X \to Y$ is continuous and $A \subseteq X$, then $f : A \to Y$ is continuous.*

(5) *(Range restriction) if $f : X \to Y$, and $Z \subseteq Y$ such that $f(X) \subseteq Y$, then $f : X \to Z$ is continuous.*

(6) *(Range exapnsion) If $f : X \to Y$ is continuous, and $Y \subseteq Z$ is a subspace of Z, then $f : X \to Z$ is continuous.*

(7) *(Local Formulation) The map $f : X \to Y$ is continuous if X can be written as the union of open sets U_α such that $f : U_\alpha \to Y$ is continuous for all α.*

Proof. (1) Let $f(x) = y_0$ for all $x \in X$, and let V be open in Y, then $f^{-1}(V) = X$ or $\emptyset$ depending on if $y_0 \in V$ or noy. In either case, $f^{-1}(V)$ is open.

(2) If U us open in X, then $f^{-1}(U) = U \cap A$ which is open in the subspace topology of X.

(3) If U is open in Z, $g^{-1}(U)$ is open in Y, hence $f^{-1}g^{-1}(U)$ is open in X.

(4) Notice that $f_A = \iota \circ f = f : A \to Y$ which is continuous by (2) and (3).

(5) Let $f : A \to Y$ be continuous and let $f(X) \subseteq Z \subseteq Y$. Let B be open in Z, so $B = Z \cap U$ for some U open in Y. Now by hypothesis, we have that $f^{-1}(U) \subseteq f^{-1}(B)$, hence $f^{-1}(B)$ is open in X, thus $f : X \to Z$ is continuous.

(6) Let f be as in (5), and let $Y \subseteq Z$ be a subspace of Z. Then the mapping $h : X \to Z$ defined by $h = \iota \circ f$ is continuous.

(7) Let $X = U_\alpha$ where U_α is open in X, and $f : U_\alpha \to Y$ is continuous for all α. Let V be open in Y, then $f^{-1}(V) \cap U_\alpha = f_U^{-1}\alpha(V)$, and since f is continuous on U_α, then $f^{-1}(V) = \bigcup f_{U_\alpha}^{-1}(V)$ is open in X.

■

Theorem 1.7.4 (The pasting lemma)**.** *Let $X = A \cup B$ with A and B closed in X, and let $f : A \to Y$ and $g : B \to Y$ be continuous. If $f(x) = g(x)$ for all $x \in A \cap B$, then we can construct a mapping $h : X \to Y$ defined by $h(x) = \begin{cases} f(x), & x \in A \\ g(x), & x \in B \end{cases}$. Then h is continuous.*

Proof. Let C be closed in Y, then $h^{-1}(C) = f^{-1}(C) \cup g^{-1}(C)$. Since f and g are continuous, then $f^{-1}(C)$ and $g^{-1}(C)$ are closed in A and B, respectively. Thus $h^{-1}(C)$ is closed in X. ■

Example 1.15. Define $h : \mathbb{R} \to \mathbb{R}$ by $h(x) = \begin{cases} x, & x \leq 0 \\ \frac{x}{2}, & x \geq 0 \end{cases}$. We have that x and $\frac{x}{2}$ are continuous on their respective domains, intersecting at 0, i.e. $x : (-\infty, 0] \to \mathbb{R}$, $\frac{x}{2} : [0, \infty \to \mathbb{R}$, and $\{0\} = (-\infty, 0] \cap [0, \infty)$. Thus h is continuous on $\mathbb{R}$.

However, $k, l : \mathbb{R} \to \mathbb{R}$ defined by $k(x) = \begin{cases} x-2, & x \leq 0 \\ x+2, & x \geq 0 \end{cases}$ and $l(x) = \begin{cases} x-2, & x < 0 \\ x+2, & x \geq 0 \end{cases}$ are not continuous. We have that their domains intersect at 0, but that $k(0) = \pm_2$, (so k isn't even a function). Likewise, $(-\infty, 0) \cap [0, \infty) = \emptyset$, which is open in $\mathbb{R}$ see 1.9.

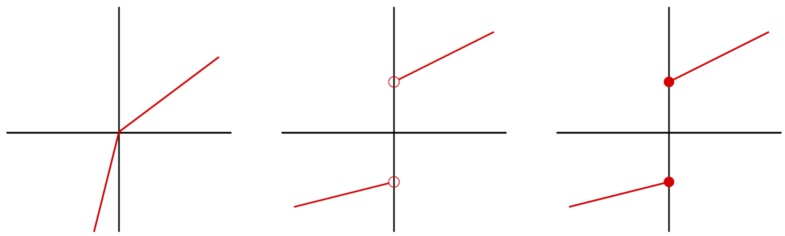

Figure 1.9: The mappings h, k, and l.

Theorem 1.7.5. *Let $f : A \to X \times Y$ be defined by $f(a) = (f_1(a), f_2(a))$, where $f_1 : A \to X$ and $f_2 : A \to Y$. Then f is continuous if and only if f_1 and f_2 are continuous.*

Proof. Let $\pi_1 : X \times Y \to X$ and $\pi_2 : X \times Y \to Y$ be projections onto X and Y respectively. Since $\pi_1^{-1}(U) = U \times Y$ and $\pi_2^{-1}(V) = X \times V$ are both open in $X \times Y$, π_1 and π_2 are continuous. Then notice that $f_1(a) = \pi_1 \circ f(a)$ and $f_2(a) = \pi_2 \circ f(a)$, both of which are continuous.

Now suppose that f_1 and f_2 are continuous..We have that $a \in f^{-1}(U \times V)$ if and only if $f_1(a) \in U$ and $f_2(a) \in V$, then $f_1^{-1}(U)$ and $f_2^{-1}(V)$ are open in A, hence so is $f^{-1}(U \times V) = f_1^{-1}(U) \cap f_2^{-1}(V)$. ■

Definition. We define the **parametrized curve** of the plane $\mathbb{R}^2$ to be the continuous function $f : [a, b] \to \mathbb{R}^2$ defined by $f(t) = (x(t), y(t))$. If f is in a vector field, then wwe define $f(t) = x(t)i + y(t)j$ where $i = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $j = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

Example 1.16. The function $f(t) = ((\cos(t)), \sin(t))$ is a parametrization of the curve $x^2 + y^2 = 1$, i.e. the unit circle S^1.

Chapter 2

More on Topological Spaces

2.1 The Product Space and Box Topologies.

We now explore more about the product topology.

Definition. Let J be an indexed set, and let X be a set. We define a **J-tuple** of elements of X to be a map $x : J \to X$, where if $\alpha \in J$, then $x(\alpha) = x_\alpha$, and we call it the **α-th coordinate** of x. We write $(x_\alpha)_\alpha \in J$, or just simply (x_α)

Definition. Let $\{A_\alpha\}$ be an indexed family, and let $X = \bigcup_{\alpha \in J} A_\alpha$. We define the **cartesian product** of $\{A_\alpha\}$, $\prod_{\alpha \in J} A_\alpha$ to be the set of all J-tuples (x_α) of elements of X, where $x_\alpha \in A_\alpha$

Theorem 2.1.1. *Let $\{X_\alpha\}$ be a family of topological spaces, and consider the cartesian product $\prod X_\alpha$. Then the collection of all cartesian products $\prod U_\alpha$, where U_α is open in X_α, for all α, forms a basis for the topology on $\prod X_\alpha$.*

Proof. Clearly $\prod X_\alpha$ itself is a basis element by the first condition. Now consider $\prod U_\alpha$ and $\prod V_\alpha$, then $\prod U_\alpha \cap \prod V_\alpha = \prod (U_\alpha \cap V_\alpha)$, which is also a basis element. ■

Definition. Let $\{X_\alpha\}$ be a family of topological spaces, and take as basis the collection of all products $\prod U_\alpha$ where U_α, where U_α is open in X_α. We call the topology generated by this basis the **box topology** on $\prod X_\alpha$.

Definition. Let $\pi_\beta : \prod X_\alpha \to X_\beta$ be defined by $\pi_\beta((x_\alpha)) = x_\beta$. We call this map the **projection mapping** of $\prod X_\alpha$ onto X_β

Theorem 2.1.2. *Let $\mathcal{S}_\beta = \{\pi_\beta^{-1}(U_\beta) : U_\beta$ is open in $X_\beta\}$, and let $\mathcal{S} = \bigcup \mathcal{S}_\beta$. Then $\mathcal{S}$ forms the basis for a topology on $\prod X_\alpha$.*

Proof. Since U_β is open in X_β, $\pi_\beta^{-1}(U_\beta)) \subseteq \prod X_\alpha$. Taking $\bigcup \mathcal{S}$, we get that $\bigcup \pi_\beta^{-1}(U_\beta)) = \prod X_\beta$ for all β. Thus $\mathcal{S}$ is a subbasis. ■

Definition. Let π_β be a projection mapping of $\prod X_\alpha$ onto X_β, and take as subbasis the collection of all $\pi_\beta^{-1}(U_\beta)$ where U_β is open in X_β. We call the topology generated by this subbasis the **product space topology**, or more generally the **product topology** on $\prod X_\alpha$.

Theorem 2.1.3. *The box topology on $\prod X_\alpha$ has as basis all sets of the form $\prod U_\alpha$, where U_α is open in X_α, and the product space topology has as basis all sets of the form $\prod U_\alpha$, where U_α is open in X_α, and $U_\alpha = X_\alpha$ except only for finitely many α.*

Proof. That the box topology has as a basis all sets of the form $\prod U_\alpha$ is clear. Now consider the basis $\mathcal{B}$ that $\mathcal{S}$ generates, and let $\beta_1, \ldots \beta_n$ be a finite set of distinct indices and let U_{β_i} be open in X_{β_i}, and $U_\alpha = X_\alpha$ for all other α. Since $B \in \mathcal{B}$ is a finite intersection of elements of $\mathcal{S}$, we have that $B = \bigcap_{i=1}^n \text{\_}\beta_i{}^{-1}(U_{\beta_i})$.

Now a point $x = (x_\alpha) \in B$ if and only if the β_i-th coordinate is in U_{β_i}, for $1 \le i \le n$, hence membership depends only on a finite number of α, thus $B = \prod U_\alpha$ where $U_\alpha = X_\alpha$ for $\alpha \ne \beta_i$ for $1 \le i \le n$. ■

Corollary. *The box topology on $\prod X_\alpha$ is finer than the product topology on $\prod X_\alpha$; moreover, if $\{X_i\}_{i=1}^n$ is a finite family of topologies, then the box toplogy, and the product topology on $\prod_{i=1}^n X_i$ are equal.*

For convention, from now on when we consider the product $\prod X_\alpha$, we assume that it is under the product space topology.

Theorem 2.1.4. *Suppose the topology on X_α is given by a basis $\mathcal{B}_\alpha$. The collection of all sets $\prod B_\alpha$ where $B_\alpha \in \mathcal{B}_\alpha$ for each α is a basis for the box topology on $\prod X_\alpha$.*

The same collection for a finite number of α, and where $B_\alpha = X_\alpha$ for all other α forms a basis for the product space topology on $\prod X_\alpha$.

Proof. Let $\mathcal{B}$ be the collection of all $\prod B_\alpha$, where $B_\alpha \in \mathcal{B}_\alpha$. Now each X_α is already its own basis, hence so is $\prod X_\alpha$. Now let $\prod U_\alpha$ and $\prod V_\alpha$ be basis elements. Since $\prod U_\alpha \cap \prod V_\alpha = \prod U_\alpha \cap V_\alpha$, for finite alpha, and since $\prod U_\alpha \cap \prod V_\alpha = \prod X_\alpha$ for all other α (in the case of the product space topology), we get another basis element. Hence $\mathcal{B}$ is a basis for the box topology, and, provided the necessary condidition, is also a basis for the product topology. ■

Theorem 2.1.5. *Let A_α be a subspace of X_α. Then $\prod A_\alpha$ is a subspace of $\prod X_\alpha$ under both the box and product space topologies.*

Proof. Since $\prod A_\alpha \cap \prod U_\alpha = \prod (A_\alpha \cap U_\alpha)$, and $A_\alpha \cap U_\alpha$ is a basis element for X_α under the subspace topology, then it follows that $\prod (A_\alpha \cap U_\alpha)$ is a basis element for the same topology on $\prod X_\alpha$, thus $\prod A_\alpha$ is a subspace. ■

Theorem 2.1.6. *If X_α is a Hausdorff space, then so is $\prod X_\alpha$ under both the box and product space topologies.*

Proof. Since X_α is a Hausdorff space, a sequence of points of X_α, $\{x_{\alpha_n}\}$ converges to at most one point. Now construct a sequence $\{x_n\}$ where $x_i = x_{\alpha_i}$ and x_{α_i} is the i-th term of (x_α), we see that $\{x_{\alpha_n}\}$ is a subsequence of $\{x_n\}$, by deifinition, and hence $\{x_n\}$ must also converge at at most one point. ■

Example 2.1. For Euclidean space $\mathbb{R}^n$, a basis consists of all products of the form $(a_1, b_1) \times \cdots \times (a_n, b_n)$ where (a_i, b_i) is an open interval for all $1 \le i \le n$. Since $\mathbb{R}^n$ is a finite product space, both the box and product topologies on $\mathbb{R}^n$ are the same.

Theorem 2.1.7. *If $A_\alpha \subseteq X_\alpha$, then $\prod \operatorname{cl} A_\alpha = \operatorname{cl} \prod A_\alpha$*

Proof. Let $x = (x_\alpha) \in \prod \operatorname{cl} A_\alpha$ and let $U = \prod U_\alpha$ be a basis element (for either topology) Choosing $y_\alpha \in U_\alpha \cap A_\alpha$, for each α, let $y = (y_\alpha)$. Then $y \in U$, and $y \in \prod A_\alpha$, hence $x \in \operatorname{cl} \prod A_\alpha$.

Now suppose that $x \in \prod A_\alpha$ (in either topology). Let V_β be an open set of X_β containing x_β. Since $\pi_\beta^{-1}(V_\beta)$ is open in $\prod X_\alpha$ (in either topology), it containts a point $y = (y_\alpha)$ of $\prod A_\alpha$. Then $y_\beta \in V_\beta \cap A_\beta$, hence $x \in \operatorname{cl} A_\beta$. ■

Theorem 2.1.8. *Let $f : A \to \prod X_\alpha$ be defined by $f(a) = (f_\alpha(a))$, where $f_\alpha : A \to X_\alpha$. Letting $\prod X_\alpha$ have the product space topoplogy, f is continous if and only if f_α is continous for each α.*

Proof. We know that the projection mapping π_β is continous. Now suppose that f is continuous, and notice that $f_\beta = \pi_\beta \circ f$, which makes f_β continuous for each β.

On the other hand, suppose that f_β is continuous for each β. Notice that $f_\beta^{-1} = f^{-1} \circ \pi_\beta^{-1}$, since $\pi_\beta^{-1}(U_\beta)$ is open in $\prod X_\alpha$, then so is $f^{-1} \circ \pi_\beta^{-1}(U_\beta) = f_\beta^{-1}(U_\beta)$. This makes f continuous. ■

Example 2.2. Theorem 2.1.8 holds only for the product space topology and fails in general for the box topology. Consider $\mathbb{R}^\omega$ and define the map $f : \mathbb{R} \to \mathbb{R}^\omega$ by $f(t) = (t, t, t, \dots)$. We have that $f_n(t) = t$ is continuous, which makes f continuous under the product topology. Now consider the box topology: let $B = (-1, 1) \times (-\frac{1}{2}, \frac{1}{2}) \times (-\frac{1}{3}, \frac{1}{3}) \times \dots$, and supppose that $f^{-1}(B)$ were open. Then it contains some interval $(-\delta, \delta)$, about 0, thus $\pi_\beta \circ f((-\delta, \delta)) = f_\beta((-\delta, \delta)) = (-\delta, \delta) \subseteq (-\frac{1}{n}, \frac{1}{n})$, which is absurd. Thus the only implication of the theorem that holds for the box topology is that f_α is continuous only when f is continuou is continuous.

2.2 The Metric Topology

Definition. A **metric** (or **distance function**) on a set X is a map $d : X \times X \to \mathbb{R}$ satisfying the following for all $x, y, z \in X$:

(1) $d(x, y) \geq 0$ and $d(x, y) = 0$ if and only if $x = y$.

(2) $d(x, y) = d(y, x)$.

(3) $d(x, y) \leq d(x, z) + d(z, y)$ (The Triangle Inequality).

We call $d(x, y)$ the **distance** between x and y, and given $\varepsilon > 0$, we define the **ε-ball centered about** x to be the set $B_d(x, \varepsilon) = \{y \in X : d(x, y) < \varepsilon\}$.

Lemma 2.2.1. *Let d be a metric on X. For $x, y \in X$, and $B_d(x, \varepsilon)$ an ε-ball centered about x, there is a δ-ball centered about y, $B_d(y, \delta)$ such that $B_d(y, \delta) \subseteq B_d(x, \varepsilon)$.*

Proof. Let $y \in B_d(x, \varepsilon)$ and let $\delta = \varepsilon - d(x, y)$, and take $z \in B_d(y, \delta)$, then we have that $d(y, z) < \varepsilon - d(x, y)$, thus $d(x, x) \leq d(x, y) + d(y, z) < \varepsilon$ which completes the proof. ■

Theorem 2.2.2. *Let d be a metric on X. Then the colletion of all ε-balls about x, for some $x \in X$ forms the basis for a topology on X.*

Proof. Clearly $x \in B_d(x, \varepsilon)$, by definition, so it remains to show that the intersection of two ε-balls contains an ε-ball. Let B_1 and B_2 be ε-balls about x, and let $y \in B_1 \cap B_2$. By lemma 2.2.1, there are $\delta_1, \delta_2 > 0$ such that $B_d(y, \delta_1) \subseteq B_1$ and $B_d(y, \delta_2) \subseteq B_2$. Now take $\delta = \min\{\delta_1, \delta_2\}$, then we see that $B_d(y, \delta) \subseteq B_1 \cap B_2$. ■

Definition. If d is a metric on X, we call the topology having as basis the collection of all ε-balls centered about x, for some $x \in X$ and $\varepsilon > 0$, the **metric topology** induced by d.

Corollary. *A set U is open in the metric topology induced by d if and only if for each $y \in U$, and $\delta > 0$, there is a δ-ball centered about y contained in U.*

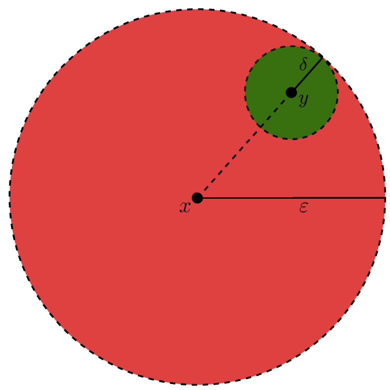

Figure 2.1: All ε-balls centered about x are open in the metric topology by lemma 2.2.1.

Example 2.3. (1) Define $d(x, y) = 1$ if $x \neq y$ and $d(x, y) = 0$ if $x = y$. Clearly d is a metric on X, and induces the discrete topology on X. The basis $B_d(x, 1) = \{x\}$

(2) The standard metric on $\mathbb{R}$ is defined to be $d(x, y) = |x - y|$ and is a metric on $\mathbb{R}$ (that is, the absolute $|\cdot|$ is a metric on $\mathbb{R}$). This metric induces the standard topology on $\mathbb{R}$ as we see that it has basis $B_d(x, \varepsilon) = \{y \in \mathbb{R} : |x - y| < \varepsilon\} = \{y \in \mathbb{R} : y - \varepsilon < x < y + \varepsilon\} = (y - \varepsilon, y + \varepsilon)$.

Definition. If X is a topological space, we call X **metrizable** if there is a metric d which induces the topology on X. A **metric space** is a metrizable space X together with the metric inducing the topology of X.

Definition. Let X be a metric space with metric d. A subset $A \subseteq X$ is said to be **bounded** if there is an $M > 0$ such that $d(a_1, a_2) \leq M$ for all $a_1, a_2 \in X$. We define the **diameter** of a bounded set A to be $\operatorname{diam} A = \sup\{d(a_1, a_2) : a_1, a_2 \in A\}$.

It is easy to see that boundedness of a set does not depend on the topology of X, but on the metric.

Theorem 2.2.3. *Let X be a metric space with metric d and define $\overline{d} : X \times X \to \mathbb{R}$ by $\overline{d} = \min\{d(x, y), 1\}$ for all $x, y \in X$. Then $\overline{d}$ is a metric on X that induces the same topology as d.*

Proof. Clearly we have that $0 \leq \overline{d}(x, y) \leq 1$, and that $\overline{d}(x, y) = \min\{d(x, y), 1\} = \min\{d(y, x), 1\} = \overline{d}(y, x)$. It remains to show the triangly inequality.

Now if $d(x, z) \leq 1$ and $d(z, y) \leq 1$, then by the triangle inequality $d(x, y) \leq 1$ and $\overline{d}(x, y) \leq \overline{d}(x, z) + \overline{d}(z, y)$. Now if $d(x, z) < 1$ and $d(z, y) < 1$, we get the same conclusion. Thus we see that $\overline{d}$ is a metric on X.

Now take as basis the collection of all ε-balls with $0 < \varepsilon < 1$, and any basis element of x contains such an ε-ball, thus $\overline{d}$ induces the same topology as d. ■

Definition. Let X be a metric space with metric d. We define the **standard bounded metric** to be the metric $\overline{d} : X \times X \to \mathbb{R}$ defined by the rule $\overline{d}(x, y) = \min\{d(x, y), 1\}$ for any $x, y \in X$.

Definition. Let $x \in \mathbb{R}^n$. We define the **norm** of x, $||x||$, to be $||x|| = \sqrt{x_1^2 + \cdots + x_n^2}$. We define the **square metric** ρ on $\mathbb{R}^n$ to be $\rho(x, y) = \{|x_1 - y_1|, \ldots |x_n - y_n|\}$.

Before we show that $|| \cdot ||$ and ρ are metrics, we introduce the following:

Definition. Let $x, y \in \mathbb{R}^n$. We define the **inner product** of x and y to be:

$$\langle x, y\rangle = x_1 y_1 + \cdots + x_n y_n \tag{2.1}$$

Lemma 2.2.4. *For $x, y \in \mathbb{R}^n$, $\langle x, y + z\rangle = \langle x, y\rangle + \langle x, z\rangle$ and $|\langle x, y\rangle| \leq ||x||||y||$.*

Proof. We have $\langle x, y + z\rangle = x_1(y_1 + z_1) + \cdots + x_n(y_n + z_n) = (x_1 y_1 + \cdots + x_n y_n) + (x_1 z_1 + \cdots + x_n z_n) = \langle x, y\rangle + \langle x, z\rangle$.

Now if $x = 0$ and $y = 0$, then $|\langle x, y\rangle| = ||x||||y|| = 0$, so suppose that both $x, y \neq 0$, and let $a = \frac{1}{||x||}$ and $b = \frac{1}{||y||}$. Notice that $||ax + by|| \geq 0$ and $||ax - by|| \geq 0$, then $||ax + by||^2 ||ax - by||^2 = ||x||^2 ||y^2|| - |\langle x, y\rangle|^2 \geq 0$, hence $|\langle x, y\rangle| \leq ||x||||y||$. ■

Remark. We call the last relation in the lemma the **Cauchy-Schwarz inequality**.

Theorem 2.2.5. *Both the norm and square metrics make $\mathbb{R}^n$ into a metric space.*

Proof. We start with the norm. Now clearly, since $\sqrt{x} \geq 0$ (for real numbers), $||x - y|| \geq 0$, and if $||x - y|| = 0$ then $(x_1 - y_1)^2 + \cdots + (x_n - y_n)^2 = 0$ hence $x_i = y_i$ for all $1 \leq i \leq n$, and if $x_i = y_i$, then clearly $||x - y|| = 0$. We also see that $||x - y|| = ||y - x||$.

Now consider $z \in \mathbb{R}^n$, notice that $||x + y||^2 = \langle x + y, x + y\rangle = \langle x + y, x\rangle + \langle x + y, y\rangle = \langle x, x\rangle + \langle y, y\rangle \leq ||x||^2 + ||y||^2$. Using this we have $||x - z|| + ||z - y|| \geq ||x - y||$ (square the left hand side and evaluate), so $|| \cdot ||$ is a metric on $\mathbb{R}^n$.

Now consider the square metric. Clearly we have that $\rho(x, y) \geq 0$ and that $\rho(x, y) = 0$ if and only if $x = y$ (since $| \cdot |$ is also a metric), we also see that $\rho(x, y) = \rho(y, x)$.

Now let $x \in \mathbb{R}^n$, and we have for all $1 \leq i \leq n$ that $|x_i - y_i| \leq |x_i - z_i| + |z_i - y_i|$, hence by definition $|x_i - y_i| \leq \rho(x, z) + \rho(z, y)$, thus $\rho(x, y) \leq \rho(x, z) + \rho(z, y)$ and ρ is a metric. ■

Lemma 2.2.6. *Let d and d' be metrics on X and let $\mathcal{T}$ and $\mathcal{T}'$ be the topologies induced by d and d' respectively. $\mathcal{T} \subseteq \mathcal{T}'$ if and only if for each $x \in X$, and $\varepsilon > 0$, there is a $\delta > 0$ such that $B_{d'}(x, \delta) \subseteq B_d(x, \varepsilon)$.*

Proof. Suppose that $\mathcal{T} \subseteq \mathcal{T}'$, and take $B_d(x, \varepsilon)$ in $\mathcal{T}$, then by lemma 2.2.1, there is a B' in $\mathcal{T}'$ or which $B' \subseteq B_d(x, \varepsilon)$, hence there is a δ-ball about x for which $B_{d'}(x, \delta) \subseteq B'$.

Conversly, suppose for $x \in X$ and $\varepsilon > 0$, that ther is a $\delta > 0$ for which $B_{d'}(x, \delta) \subseteq B_d(x, \varepsilon)$. Given a basis B of $\mathcal{T}$, ther is an ε-ball about x contained in B, hence $B_d(x, \varepsilon)$ is also in B, thus we have that $\mathcal{T} \subseteq \mathcal{T}'$. ■

Theorem 2.2.7. *The norm and the square metric both induce the product topology on $\mathbb{R}^n$.*

Proof. Notics that $\rho(x, y) \leq ||x-y|| \leq \rho(x, y)\sqrt{n}$. This first inequality shows that $B_{||\cdot||}(x, \varepsilon) \subseteq B_\rho(x, \varepsilon)$, and the second shows that $B_\rho(x, \frac{\varepsilon}{\sqrt{n}}) \subseteq B_{||\cdot||}(x, \varepsilon)$, thus both $|| \cdot ||$ and ρ induce the same topology.

Now let $B = (a_1, b_1) \times \cdots \times (a_n, b_n)$ be a basis for the product topology on $\mathbb{R}^n$. Since, for each i, there is an $\varepsilon_i > 0$ such that $(x_i - \varepsilon_i, x_i + \varepsilon_i) \subseteq (a_i, b_i)$, choosing $\varepsilon = \min\{\varepsilon_1, \ldots, \varepsilon_n\}$, we see that $B_\rho(x, \varepsilon) \subseteq B$. Conversely, given $y \in B_\rho(x, \varepsilon)$, notice that $B_\rho(x, \varepsilon) = (x_1 - \varepsilon, x_1 + \varepsilon) \times \cdots \times (x_n - \varepsilon, x_n + \varepsilon)$, thus $B \subseteq B_\rho(x, \varepsilon)$. Thus the topologies are the same. ■

Definition. Given an index set J and given points $x = (x_\alpha)$ and $y = (y_\alpha)$ of $\mathbb{R}^J$, we define the **uniform metric** $\overline{\rho}$ on $\mathbb{R}^J$ by $\overline{\rho}(x, y) = \sup\{\overline{d}(x_\alpha, y_\alpha) : \alpha \in J\}$, where $\overline{d}$ is the standard bounded metric on $\mathbb{R}^J$. We call the topology induced by $\overline{\rho}$ the **uniform topology**.

Theorem 2.2.8. *The uniform topology on $\mathbb{R}^J$ is finer than the product topology on $\mathbb{R}^J$ and coarser than the box topology on $\mathbb{R}^J$, and all three topologies are different if J is infinite.*

Proof. Let $x = (x_\alpha)$ and $\prod U_\alpha$ be a basis element for the product topology, and let $\alpha_1, \ldots, \alpha_n$ be the indices for which $U_\alpha \neq \mathbb{R}$, and for each i, choose $\varepsilon_i > 0$ such that the ε_i-ball about x_{α_i}, $B_{\overline{d}}(x_{\alpha_i}, \varepsilon_i) \subseteq U_\alpha$. Let $\varepsilon = \min\{\varepsilon_1, \ldots, \varepsilon_n\}$, and let $z = (z_\alpha) \in \mathbb{R}^J$ be such that $\overline{\rho}(x, z) < \varepsilon$. Then $\overline{d}(x_\alpha, z_\alpha) < \varepsilon$ for all α. Hence $B_{\overline{\rho}} \subseteq \prod U_\alpha$ for all α; so the uniform topology is finer.

Likewise, consider B the ε-ball about x in the $\overline{\rho}$ metric. Then the box $U = \prod (x_\alpha - \frac{\varepsilon}{2}, x_\alpha + \frac{\varepsilon}{2})$ and $y \in U$ if $\overline{d}(x_\alpha, y_\alpha < \frac{\varepsilon}{2})$, then $\overline{\rho}(x, y) \leq \frac{\varepsilon}{2}$, so $U \subseteq B$, and the uniform topology is coarser.

Now in the case where J is infinite, if J is uncountable, we are done, since there is no way to map the indices of J onto $\mathbb{Z}^+$. So consider the case where J is countable, and map $J \to \mathbb{Z}^+$ by $\alpha_i \to i$. Let $U = \prod (x_i - \varepsilon, x_i + \varepsilon)$ and consider a base $B_{\overline{\rho}}(x, \varepsilon)$. We have that for $y \in B_{\overline{\rho}}(x, \varepsilon)$ that $\overline{d}(x_\alpha, y_\alpha) = \min\{\rho(x_i, y_i), 1\}$, we have that $\overline{d}(x_i, y_i) = \overline{\rho}(x_i, y_i)$ or 1, and if we choose $0 < \varepsilon < 1$, then $\overline{\rho}$ fails to put $B_{\overline{\rho}}(x, \varepsilon)$ inside of U. Likewise, the basis $\prod U_\alpha$ (in the prodct topology) failes to be contained in $B_{\overline{\rho}}(x, \varepsilon)$ by the same argument. Thus the uniform topology is not necessarily finer than the box topology, nor coarser than the product topology in $\mathbb{R}^J$, when J is infinite. ■

Remark. Clearly the box, product and uniform topologies on $\mathbb{R}^J$ are the same when J is finite, as the box and product topologies are the same for finite product spaces.

Theorem 2.2.9. *Let $\overline{d}(a, b) = \min\{|a-b|, 1\}$ be the standard bounded metric on $\mathbb{R}$, and for $x, y \in \mathbb{R}^\omega$, define:*

$$D(x, y) = \sup\{\frac{\overline{d}(x_i, y_i)}{i}\} \tag{2.2}$$

Then D is a metric that induces the product space topology on $\mathbb{R}^\omega$

Proof. Since $\overline{d}$ is a metric, D satisfies the conditions for a metric, it is worth looking into the case for the triangle inequality. Notice that for all i, $D(x,y) \leq \frac{\overline{d}(x_i,y_i)}{i} \leq \frac{\overline{d}(x_i,z_i)}{i} + \frac{\overline{d}(z_i,y_i)}{i} \leq D(x,z) + D(z,y)$. Therefore $D(x,y) \leq D(x,z) + D(z,y)$.

Now let U be open in the metric topology induced by D, and let $x \in U$. Choose an ε-ball $B_D(x,\varepsilon) \subseteq U$ and choose $N > 0$ arbitrarily large such that $\frac{1}{N} < \varepsilon$ and let $V = (x_1 - \varepsilon, x_1 + \varepsilon) \times \ldots (x_N - \varepsilon, x_N + \varepsilon) \times \mathbb{R} \times \cdots \times \mathbb{R} \ldots$. Given $y \in \mathbb{R}^\omega$, we have $\frac{\overline{d}(x_i,y_i)}{i} \leq \frac{1}{N}$ for all $i \geq N$, thus $D(x,y) \leq \max\{\frac{\overline{d}(x_1,y_1)}{1}, \ldots, \frac{\overline{d}(x_N,y_N)}{N}, \frac{1}{N}\}$ Now if $y \in V$, then the expression is less than ε, so $V \subseteq B_D(x,\varepsilon)$.

Conversely let $U = \prod U_i$ where U_i is open in $\mathbb{R}$ for $i = \alpha_1, \ldots, \alpha_n$, and $U_i = \mathbb{R}$ for all other indices. Given $x \in U$, for $i = \alpha_1, \ldots, \alpha_n$, and leting $\varepsilon = \min\{\frac{\varepsilon_i}{i} : i = \alpha_1, \ldots, \alpha_n\}$ where $\frac{1}{N} < \varepsilon_i$ for each i. Letting $B_d(x,\varepsilon)$ be the corresponding ε-ball under the metric d, we see that $\frac{\overline{d}(x_i,y_i)}{i} \leq D(x,y) < \varepsilon$. Now if $i = \alpha_1, \ldots, \alpha_n$, then $\overline{d}(x_i,y_i) < \varepsilon_i \leq 1$, hence $|x_i - y_i| < \varepsilon_i$, putting $y \in U$. ■

2.3 More on Metric Spaces

We go more in depth on metric spaces here.

Theorem 2.3.1. *If A is a subspace of a metric space X, with metric d, then d restricted to $A \times A$ makes A into a metric space.*

Proof. Clearly $d : A \times A \to \mathbb{R}$ is a metric. So consider the ε-ball about x, $B_d(x,\varepsilon)$ in X; restricting d to $A \times A$, consider $A \cup B_d(x,\varepsilon)$. For $y \in A$, there is a δ-ball about y such that $B_d(y,\delta) \subseteq B_d(x,\varepsilon)$; then $B_d(y,\delta) \subseteq B_d(x,\varepsilon)$. This makes A as a subspace, into a metric space. ■

Theorem 2.3.2. *The Hausdorff axiom is satisfied in every metric space.*

Proof. If $x, y \in X$ are distinct, let $\varepsilon = \frac{1}{2}d(x,y)$, by the triangle inequality, we have that $B_d(x,\varepsilon)$ and $B_d(y,\varepsilon)$ are disjoint. ■

Theorem 2.3.3. *Countable products of metrizable spaces are metrizable.*

Proof. Let X be a metric space with metric d. Define $\overline{d}(x,y) = \min\{d(x,y), 1\}$ on X and define $D(x,y) = \sup\{\frac{\overline{d}(x_i,y_i)}{i}\}$ on X^ω. It is clear that both $\overline{d}$ and D are metrics on X and X^ω respectively. We would like to show that D induces the product topology on X^ω.

Let U be open and let $x \in U$. Choose $B_D(x,\varepsilon) \subseteq U$ and choose N large enough such that $\frac{1}{N} < \varepsilon$. Now let $V = (x_1 - \varepsilon, x_1 + \varepsilon) \times \cdots \times (x_N - \varepsilon, x_N + \varepsilon) \times X \times \ldots$ be a basis in the product topology on X^ω. Given $y \in X^\omega$, such that $\frac{\overline{d}(x_i,y_i)}{i} \leq \frac{1}{N}$, we have by definition that $D(x,y) \leq \max\{\overline{d}(x_1,y_1), \frac{\overline{d}(x_2,y_2)}{2}, \ldots, \frac{\overline{d}(x_N,y_N)}{N}, \frac{1}{N}\}$. If $y \in V$, we get that $V \subseteq B_D(x,\varepsilon)$ and we are done.

Conversely let $U = U_i$ be a basis of the product topology where U_i is open in X for $i = 1, \ldots, n$ and $U_i = X$ for all other indices. Now let $x \in U$ and choose an interval about x_i, $(x_i - \varepsilon_i, x_i + \varepsilon_i)$ lying in U_i with $0 < \varepsilon_i \leq 1$ for all i. Choose $\varepsilon = \min\{\varepsilon_1, \frac{\varepsilon_2}{2}, \ldots, \frac{\varepsilon_n}{n}\}$. Then $x \in B_D(x,\varepsilon) \subseteq U$, for if $y \in B_D(x,\varepsilon)$, we have that $\frac{\overline{d}(x_i,y_i)}{i} \leq D(x,y) < \varepsilon$, hence $\varepsilon \leq \frac{\varepsilon_i}{i}$ and $d(x_i,y_i) < \varepsilon_i$, and so $y \in U_i$. Therefore D induces the product space topology ■

Remark. This theorem generalizes theorem 2.2.9 for any countable product space of a metric space X. Hence we can take theorem 2.2.9 as a corollary to this theorem.

We would now like to study continuous functions in metric spaces, which brings us into the realm of analysis. We show that the "ε-δ" definition, and the sequence definiton of continuity carry over.

Theorem 2.3.4. *Let $f : X \to Y$ with X and Y metric spaces with metric d_X and d_Y respectively. Then f is continuous if and only if for $x \in X$, a nd $\varepsilon > 0$, there is a $\delta > 0$ such that $d_Y(f(x), f(y)) < \varepsilon$ whenever $d_X(x, y) < \delta$.*

Proof. Suppose that f is continous and consider $f^{-1}(B(f(x), \varepsilon))$ open in X. Then it contains a δ-ball $B(x, \delta)$ about x. If $y \in B(x, \delta)$, then $f(y) \in B(f(x), \varepsilon)$, as is required.

Now suppose that for $x \in X$ and $\varepsilon > 0$, that there is a $\delta > 0$ such that $d_Y(f(x), f(y)) < \varepsilon$ whenever $d_X(x, y) < \delta$, for $x \in X$. Let V be open in Y and let $x \in f^{-1}(V)$. Then $f(x) \in V$, hence there is an ε-ball $B(f(x), \varepsilon) \subseteq V$. By hypothesis, there is a $\delta > 0$ such that $f(B(x, \delta)) \subseteq B(f(x), \varepsilon)$, hence $B(x, \delta) \subseteq f^{-1}(V)$ which makes $f^{-1}(V)$ open. ■

Lemma 2.3.5 (The Sequence Lemma)**.** *Let X be a topological space and let $A \subseteq X$. If there is a sequence of points of A converging to $x \in X$, then $x \in \operatorname{cl} A$. The converse holds if X is metrizable.*

Proof. Suppose for some sequence $\{x_n\} \subseteq A$ that $x_n \to x$. By theorem 1.6.6, we have every neighborhood of x contains points of A, hence $x \in \operatorname{cl} A$. Conversely, suppose that X is metrizable with metric d, and let $x \in \operatorname{cl} A$. For $n \in \mathbb{Z}^+$, take $B_d(x, \frac{1}{n})$ and take $\{x_n\} = B_d(x, \frac{1}{n}) \cap A$. Then $x_n \to n$, for: any open set U of x contains an ε-ball about x, $B_d(x, \varepsilon)$, so choose N large enough so that $\frac{1}{N} < \varepsilon$, hence U contains x_i for all $i \geq N$. ■

Theorem 2.3.6 (The Sequential Criterion)**.** *Let $f : X \to Y$ be continuous, then for every convergent sequence $\{x_n\}$ converging to $x \in X$, the sequence $\{f(x_n)\}$ converges to $f(x)$. the converse holds if X is metrizable.*

Proof. Let f be continuous and suppose that $x_n \to x$. Let V be a neighborhood of $f(x)$, then $f^{-1}(V)$ is a neighborhood of x; hence there is an $N > 0$ such that $x_n \in f^{-1}(V)$ whenever $n \geq N$, thus $f(x_n) \in V$ whenever $n \geq N$.

Conversely suppose that for every $\{x_n\}$ converging to x, that $\{f(x_n)\}$ converges to $f(x)$, and let $A \subseteq X$. if $x \in \operatorname{cl} A$, by the sequence lemma, there is a sequence $\{x_n\} \subseteq A$ converging to X. Now since $f(x_n) \to f(x)$. and $f(x_n) \in f(A)$, by the sequence lemma again, $f(x) \in \operatorname{cl} f(A)$. Thus $f(\operatorname{cl} A) \subseteq \operatorname{cl} f(A)$ and we are done. ■

We now consder methods for constructing continous functions on metric spaces.

Lemma 2.3.7. *The additions, subtraction, and multiplication operations are continous from $\mathbb{R} \times \mathbb{R}$ to $\mathbb{R}$. The quotient operation is continous from $\mathbb{R} \times \mathbb{R} \backslash \{0\}$ to $\mathbb{R}$.*

Theorem 2.3.8. *If X is a topological space and if $f, g : X \to \mathbb{R}$ are continous, then $f + g$, $f - g$, a nd fg are continuous; moreover if $g(x) \neq 0$ for all $x \in X$, then $\frac{f}{g}$ is also continuous.*

Proof. The map $h : X \to \mathbb{R} \times \mathbb{R}$ defined by $h(x) = f(x) \times g(x)$ is continuous. Then notice that $f + g(x) = +(f(x), g(x)) = + \circ h(x)$, so by the above lemma, we get that $f + g$ is contincontinuous. We also have that $f - g$ is continuous for $f - g(x) = +(f(x), -g(x))$. The same argument holds for fg and $\frac{f}{g}$. ■

Definition. Let $f_n : X \to Y$ be a sequence of functions from X to the metric space Y, with metric d. We say that the sequence $\{f_n\}$ **converges uniformly** to the function $f : X \to Y$ if for $\varepsilon > 0$, there is an integer $N > 0$ such that $d(f_n(x), f(x)) < \varepsilon$ whenever $n \geq N$, for all $x \in X$.

Theorem 2.3.9. *Let $f_n : X \to Y$ be a sequence of continuous functions from the topological space X to the metric space Y. If $\{f_n\}$ converges uniformly to f, then f is continuous.*

Proof. Let V be open in Y and let $x_0 \in f^{-1}(V)$. Let $y_0 = f(x_0)$ and choose $\varepsilon > 0$ such that $B(y_0, \varepsilon) \subseteq V$. By uniform convergence, choosing $N > 0$ so that whenever $n \geq N$, $d(f_n(x), f(x)) < \frac{\varepsilon}{3}$ for all $x \in X$. By the continuity of f_N, choose a neighborhood U of x_0 such that $f_N(U) \subseteq B(f_N(x_0), \frac{\varepsilon}{3})$. Then if $x \in U$, we have $d(f(x), f_N(x)) < \frac{\varepsilon}{3}$, $d(f_N(x), f_N(x_0)) < \frac{\varepsilon}{3}$, and $d(f_N(x_0), f(x_0)) < \frac{\varepsilon}{3}$ by the triangle inequality we get $d(f(x), f(x_0)) < \varepsilon$ which completes the proof. ■

Example 2.4. (1) $\mathbb{R}^\omega$ is not metrizable in the box topology. Let $A = \{(x_1, x_2, \dots) \in \mathbb{R}^\omega : x_i > 0\}$ and consider $0 = (0, 0, \dots) \in \mathbb{R}^\omega$. $0 \in \operatorname{cl} A$ if for any basis element $B = (a_1, b_1) \times (a_2, b_2) \times \dots$, $0 \in B$; then $B \cap A \neq \emptyset$ (take the point $\frac{1}{2}b \in \mathbb{R}^\omega$). Now let $\{a_n\}$ be a sequence of points of A with $a_n = (x_{1n}, x_{2n}, \dots, x_{in}, \dots)$, since $x_{in} > 0$, construct a basis element $B' = (-x_{11}, x_{11}) \times (-x_{22}, x_{22}) \times \dots$. Then $0 \in B'$, but $\{a_n\} \nsubseteq B'$ for the point $x_{nn} \notin (-x_{nn}, x_{nn})$. Thus $a_n \not\to 0$.

(2) An uncountable product of $\mathbb{R}$ with itself is not metrizable. Let J be uncountable, and let $A = \{(x_\alpha) \in \mathbb{R}^J : x_\alpha = 1 \text{ for all but finitely many } \alpha\}$. Consider $0 \in \mathbb{R}^J$ and let U be a basis for containing 0. Now $U_\alpha \neq \mathbb{R}$ for $\alpha_1, \dots, \alpha_n$, so let $(x_\alpha) \in A$ be defined by letting $x_\alpha = 0$ for $\alpha_1, \dots, \alpha_n$ and $x_{\alpha=1}$ for all other indices. Then $(x_\alpha) \in A \cap U_\alpha$.

Nowe let $\{a_n\} \subseteq A$ and for $n \in \mathbb{Z}^+$ elt $J_n = \{\alpha \in J : \alpha_{\alpha n} \neq 1\}$. Then we see that $\bigcup J_n$ is a countable union of finite sets, and hence countable itself. Now since J is uncountable, there is a $\beta \in J$ for which $\beta \notin \bigcup J_n$, so $a_{\beta n} \neq 1$. Lettng $U_\beta = (-1, 1)$ in $\mathbb{R}$ let $U = \pi_\beta^{-1}(U_\beta)$ in $\mathbb{R}^J$. Then U is a neighborhood of 0 not containing any points of $\{a_n\}$, so $a_n \not\to 0$.

2.4 The Quotient Topology

Definition. Let X and Y be topological spaces, and let $p : X \to Y$ be onto. We say that p is a **quotient map** if a subset $U \subseteq Y$ is open if and only if $p^{-1}(U) \subseteq X$ is open. We say that a subset C is **saturated** with respect to p if for every $p^{-1}(\{y\})$ that intersects C, $p^{-1}(\{y\}) \subseteq C$; that is $C \cap p^{-1}(\{y\}) = p^{-1}(\{y\})$.

Definition. Let $f : X \to Y$ be a map, with X and Y topological spaces. We say that f is an **open map** if for each open subset $U \subseteq X$, $f(U) \subseteq Y$ is also open. We say f is a **closed map** if for each closed subset $U \subseteq X$, $f(U) \subseteq Y$ is closed.

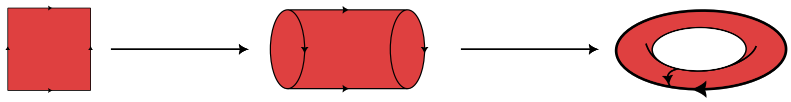

Figure 2.2: Gluing the edges of the unit square to form a cylinder, and then a torus.

Lemma 2.4.1. *If $p : X \to Y$ is a continuous map of X onto Y, for topological spaces X and Y, that is either open or closed, then p is a quotient map.*

Remark. A quotient map need not be open nor closed.

Example 2.5. (1) Let X be the subspace $[0,1] \cup [2,3]$ and Y be the subspace $[0,2]$ in $\mathbb{R}$, and defined $p : X \to Y$ by $p(x) = \begin{cases} x, & x \in [0,1] \\ x-1, & x \in [2,3] \end{cases}$. We have that p is continuous onto, by the pasting lemma, furthermore, p is also closed; hence p is a quotient map. p is not open however, as $p([0,1]) = [0,1]$ is closed in Y.

Now if A is the subspace $[0,1) \cup [2,3]$ of X, then $A \to Y$ is continuous onto, but fails to be closed. So $p|_A$ is not a quotient map, depsite the fact that $[2,3]$ is open in A and saturated with respect to $p|_A$.

(2) Let $\pi_1 : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be the projection map that takes $x \times y \to x$. Clearly π_1 is continuous onto, and π_1 is also open as $\pi_1(U \times V) = U$ which is open in Y; hence π_1 is a quotient map. Now consider the closed set $C = \{x \times y : xy = 1\}$ in $\mathbb{R} \times \mathbb{R}$. $\pi_1(C) = \mathbb{R} \backslash \{0\}$ which is not closed.

Theorem 2.4.2. *Let X be a topological space, and A a set, and let $p : X \to Y$ be onto. Define $\mathcal{T}$ to be the collection of subsets U of A for which $p^{-1}(U)$ is open in X. Then $\mathcal{T}$ is a unique topology for which p is a quotient map.*

Proof. Clearly $\emptyset, A \in \mathcal{T}$, for $p^{-1}(\emptyset) = \emptyset$ and $p^{-1}(A) = X$. Now let $\{U_\alpha\}$ and $\{U_i\}_{i=1}^n$ be collections of subsets of A. Then

$$p^{-1}(\bigcup U_\alpha) = \bigcup p^{-1}(U_\alpha)$$

and

$$p^{-1}(\bigcap_{i=1}^n U_i) = \bigcap_{i=1}^n p^{-1}(U_i).$$

Thus $\mathcal{T}$ is a topology on A. Now notice that this also makes p into a quotient map.

Now suppose that there is another topology $\mathcal{T}'$ for which p is a quotient map. Clearly $\mathcal{T}\mathcal{T}'$, now if p is open, then for U open in X, $p(U)$ is open in A, hence so are their preimages, and since p is continuous and onto this makes $\mathcal{T}' \subseteq \mathcal{T}$, thus $\mathcal{T}$ is unique. Likewise, by similar reasoning with closed sets, if p is closed, $\mathcal{T}$ is still unique. ∎

Definition. If X is a topologicas space, and A a set, and $p : X \to A$ is onto, then there is exactly one topology $\mathcal{T}$ on A for which p is a quotient map. We call this topology the **quotient topology** on A induced by p.

Example 2.6. Let $p : \mathbb{R} \to A$, with $A = \{a, b, c\}$ be defined by

$$p(x) = \begin{cases} a, & x > 0 \\ b, & x < 0 \\ c, & x = 0 \end{cases}$$

Then the quotient topology on A is the topology $\mathcal{T} = \{\emptyset, \{a\}, \{b\}, \{a, b\}, A\}$.

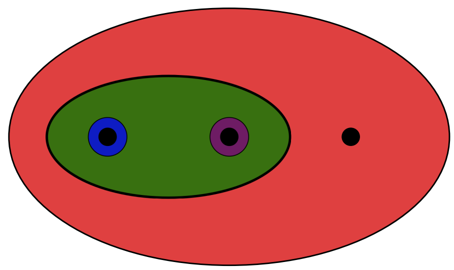

Figure 2.3: The quotient topology on $A = \{a, b, c\}$ induced by p.

Definition. Let X be a topological space and X/p be a partition of X into disjoint subsets whos disjoint union is X. Let $p : X \to X/p$ be onto such that $p : x \to U$ if $x \in U$. We call X/p in the quotient topology induced by p the **quotient space** of X, or the **decomposition space** of x.

We can take an equivalence relation $\sim$ on X by taking $x \sim y$ if $x, y \in U$ for $U \in X/p$. Then the quotient space is the collection of equivalence classes of X; that is we can think of obtaining X^* by "identifying" those pairs of equivalent points. Similarly, we can also describe the quotient space X/p by noting a subset U of equivalence classes where $p^{-1}(U) = \bigcup_{V \in U} V$ is the union of all equivalence classes in U. We will denote the quotient space by X/p or $X/\sim$.

Example 2.7. (1) Let $X = \{x \times y : x^2 + y^2 \leq 1\}$ be the closed unit ball in $\mathbb{R}^2$ and let $X/\sim$ be the quotient space of X by taking its equivalence classes to be all one point sets $\{x \times y\}$ for which $x^2 + y^2 < 1$, and the unit circle S^1. Take $p : X \to X/\sim$ by the rule

$$p : x \times y \to \begin{cases} \{x \times y\}, & x^2 + y^2 < 1 \\ S^1 & x^2 + y^2 = 1 \end{cases}$$

Clearly p is onto by definition, and we have that for any one point sets and S_1 in $X/\sim$ that $p^{-1}(\{x \times y\}) = B_{||\cdot||}(x, 1)$, and $p^{-1}(S^1) = S_1$, thus open sets in X are open in $X/\sim$ (likewise closed), so p is open which makes it into a quotient map. Now let $S^2 = \{x \times y \times z : x^2 + y^2 + z^2 = 1\}$ be the **unit sphere** in $\mathbb{R}^3$, then we can map $X/\sim \to S^2$ by taking $\{x \times y\} \to \{x \times y \times z : x^2 + y^2 + z^2 < 1\}$ and $S^1 \to S^1$. Then we have that $X/\sim$ is homeomorphic to the unit sphere.

(2) Let X be the unit square $[0, 1] \times [0, 1]$ and define the quotient space $X/\sim$ of X to be the collection of all point sets $\{x \times y\}$ for which $x \times y \in (0, 1) \times (0, 1)$, together with the sets $\{x \times 0, x \times 1\}, \{0 \times y, 1 \times y\}$ with $x \times y \in (0, 1) \times (0, 1)$, and the set $\{0 \times 0, 0 \times 1, 1 \times 0, 1 \times 1\}$, then we can show that $X/\sim$ is homeomorphic to a torus.

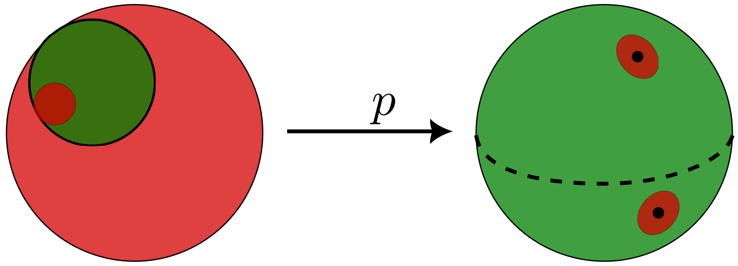

Figure 2.4: Transforming the unit disk into the unit sphere

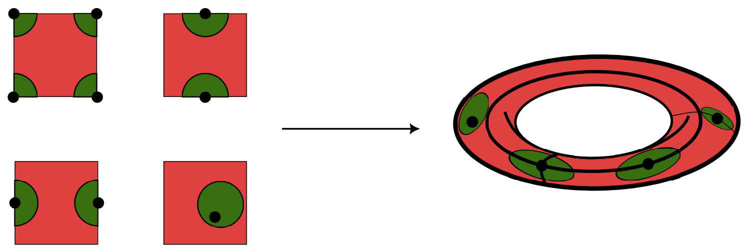

Figure 2.5: Transforming the unit square into torus

Theorem 2.4.3. *Let $p : X \to Y$ be a quotient map and let A be a subspace of X saturated with respect to p. Let $q : A \to p(A)$, then:*

(1) If A is either open or closed, then q is a quotient map.

(2) If p is either open or closed, then q is a quotient map.

Proof. We first have that $q^{-1}(V) = p^{-1}(V)$ since $V \subseteq A$ and A is saturated, well $p^{-1}(V) \subseteq A$. We also have that $p(U \cap A) = U) \cap p(A)$. Now suppose that $y = p(u) = p(a)$ for $u \in U$ and $a \in A$. Since A is saturated, it contains $p^{-1}(p(a)) = p^{-1}(p(u))$, hence $y = p(u)$ with $u \in U \cap A$, so $P(U) \cap p(A) \subseteq P(U \cap A)$.

Now suppose that A, or that p is open. Given $V \subseteq p(A)$, suppose $q^{-1}(V) \subseteq A$ is open. If A is open in X, then $q^{-1}(V)$ is also open in X, and since $q^{-1}(V) = p^{-1}(V)$, then V is open in Y, and since p is a quotient map, then V is open in $p(A)$. Now if p is open, since $q^{-1}(V) = p^{-1}(V)$ and $q^{-1}(V)$ is open in A, then $p^{-1}(V) = U \cap A$, thus $V = p(p^{-1}(V)) = p(U \cap A) = p(U) \cap p(A)$. Now $p(U)$ is open in Y, so V is open in $p(A)$. In either case e have that q is a quotient map. ■

Lemma 2.4.4. *The composition of two quotient maps is a quotient map,*

Proof. Let $p : X \to Y$ and $q : Y \to Z$ be quotient maps and consider $q \circ p : X \to Z$. Let $V \subseteq Z$ be open, then $q^{-1}(V) = U$ is open in Y which implies that $p^{-1}(U)$ is open in X. Hence $p^{-1}(q^{-1}(V))$ is open in X. ■

Theorem 2.4.5. *Let $p : X \to Y$ be a quotient map and let Z be a topological space with $g : X \to Z$ a map constant on all $p^{-1}(\{y\})$ for each $y \in Y$. Then g induces a map $f : Y \to Z$ such that $f \circ p = g$, where f is continuous if and only if g is continuous, and where f is a quotient map if and only if g is a quotient map.*

$$\begin{tikzcd} X \arrow[d, "p"'] \arrow[dr, "g"] & \\ Y \arrow[r, dotted, "f"'] & Z \end{tikzcd}$$

Proof. For each $y \in Y$, since g is constant on all $p^{-1}(\{y\})$, $g(p^{-1}(\{y\}))$ is a one point set in Z; let it be $f(y)$. Then we have a map $f : Y \to Z$ such that $f(p(x)) = g(x)$ for each x. Now if f is continuous, then so is g; likewise if f is a quotient map.

Conversely, suppose that g is continuous, given $V \subseteq Z$ open, $g^{-1}(V)$ is open in X. Now $g^{-1}(V) = p^{-1}(f^{-1}(V))$, and since p is a quotient map, $f^{-1}(V)$ is open in Y, hence f is continuous. Now suppose that g is a quotient map, for $V \subseteq Z$, if $f^{-1}(V)$ is open in Y, then $p^{-1}(f^{-1}(V))$ is open in X by continuity of p, thus $g^{-1}(V)$ is open in X, then V is open in Z; this makes f a quotient map. ■

Corollary. *Let $g : X \to Z$ be continuous and let $X/\sim = \{g^{-1}(\{z\}) : z \in Z\}$ ' be the following quotient space for X. Then*

(1) The map g induces a continuous $1-1$ map $f : X/\sim \to Z$ of $X/\sim$ onto Z which is a homeomorphism if and only if g is a quotient map.

$$\begin{tikzcd} X \arrow[d, "p"'] \arrow[dr, "g"] & \\ X/\sim \arrow[r, "f"'] & Z \end{tikzcd}$$

(2) If Z is Hausdorff, then so is $X/\sim$.

Proof. By the theorem 2.4.5, g induces a continuous map $f : X/\sim \to Z$ which is $1-1$ and (clearly) onto. Suppose that f is a homeomorphism. Then both f and $p : X \to X/\sim$ are quotient maps, which makes g a quotient map. On the other hand, if g is a quotient map, then so is f, which makes f $1-1$ continuous, and hence a homeomorphism.

Now supose that Z is Hausdorff, given points of $X/\sim$, their images under f are distinct, and hence they posses disjoint neighborhoods U and V. Then $f^{-1}(U) \cap f^{-1}(V) = \emptyset$, which makes $X/\sim$ Hausdorff. ■

We conclude with some examples.

Example 2.8. (1) Let $X = \{[0,1] \times n : n \in Z^+\}$ be the subspace of $\mathbb{R}^2$ and let $Z = \{x \times \frac{x}{n} : x \in [0,1]\ n \in \mathbb{Z}^+\}$ be another subspace of $\mathbb{R}^2$. We have that X is the union of countably many disjoint line segments in $\mathbb{R}^2$ and Z is the union of countably many line segments in $\mathbb{R}^2$ having a common endpoint.

Now define $g : X \to Z$ by $g : x \times n \to x \times \frac{x}{n}$. We have that g is continuous onto. Now consider the quotient space X/g whose elements are $g^{-1}(\{z\})$ to be the space obtained from X by identifying the equivalence classes of $\{0\} \times \mathbb{Z}^+$ to be points. Now we have that g induces a $1-1$ continuous map $f : X/g \to Z$ of X/g onto Z; but f is not a homeomorphism. It is sufficient to show that g is not a quotient map consider the sequence $\{\frac{1}{n} \times n\}$ of points of X. This sequence is closed in X since it has no limit points, it is also saturated with respect to g. However $g(A)$ is not closed in Z for it consists of the sequence of points $\{\frac{1}{n} \times \frac{1}{n^2}\}$ whose limit point is the origin (see 2.6).

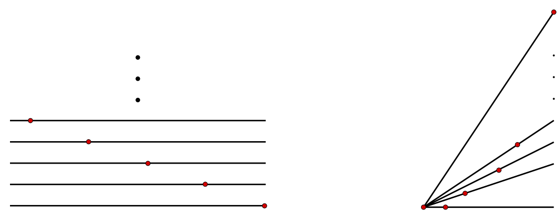

Figure 2.6:

(2) Not it is not in general true that the product of two quotient maps is itself a quotient map. Consider $\mathbb{R}$ and let $\mathbb{R}/\sim$ be the quotient space obtained from $\mathbb{R}$ by taking $\mathbb{Z}^+ \sim b$ for some point b. Now let $p : \mathbb{R} \to \mathbb{R}/\sim$ be the quotient map and consider $\mathbb{Q}$ as a subspace of $\mathbb{R}$. Let $i : \mathbb{Q} \to \mathbb{Q}$ be the identity map, then $p \times i : \mathbb{R} \times \mathbb{Q} \to \mathbb{R}/\sim \times \mathbb{Q}$ is not a quotient map. For each n, take the sequence $\{\frac{\sqrt{2}}{n}\}$ and consider the straight lines of slope ± 1 in $\mathbb{R}^2$ through the point $n \times \frac{\sqrt{2}}{n}$. Let U_n be the set of all $\mathbb{R} \times \mathbb{Q}$ points lying above or beneth both these line, and between the lines $x = n \pm \frac{1}{4}$. We have that U_n is open in $\mathbb{R} \times Q$ as it contains the set $\{n\} \times \mathbb{Q}$.

Now let $U = \bigcup U_n$, then U is open in $\mathbb{R} \times \mathbb{Q}$, and it is saturated with respect to $p \times i$, as $\mathbb{Z}^+ \times q \in \mathbb{R} \times \mathbb{Q}$ for all $q \in \mathbb{Q}$, so assume that $U' = p \times i(U)$ is open in $\mathbb{R}/\sim \times \mathbb{Q}$. Now since $\mathbb{Z}^+ \times 0 \subseteq U$, the point $b \times 0 \in U'$, so U' contains an open set of the form $W \times I_\delta$ where W is a neighborhood of b in $\mathbb{R}/\sim$, and $I_\delta = \{y \in \mathbb{Q} : |y| < \delta\}$. Then $p^{-1}(W) \times I_\delta \subseteq U$. Choosing n large enough so that $\frac{\sqrt{2}}{n} < \delta$, since $p^{-1}(W)$ is open in $\mathbb{R}$ and contains $\mathbb{Z}^+$, choose $0 < \varepsilon < \frac{1}{4}$ so that $(n - \varepsilon, n + \varepsilon) \subseteq p^{-1}(W)$. Then U contains the subset $(n - \varepsilon, n + \varepsilon) \times I_\delta \subseteq \mathbb{R} \times \mathbb{Q}$, however there are points that need not lie in U, which contradicts that U' is open in $\mathbb{R}/\sim$ So $p \times i$ cannot be a quotient map.

2.5 Topological Groups.

Chapter 3

Connectedness and Compactness

3.1 Connected Spaces.

Definition. Let X be a topological space. We define a **seperation** of X to be a pair U, V of disjoint open sets in X who's union equals X. We say that X is **connected** if no such seperation exists.

That is to say that X is connected if one cannot partition X into two open sets.

Lemma 3.1.1. *A topological space is connected if and only if the only open and closed sets of X are X itself and $\emptyset$.*

Proof. Let $A \subseteq X$ be both open and closed in X. Then A and $X \backslash A$ are both nonempty disjoint open subsets of X with $A \cup X \backslash A = X$, and hence form a seperation. Conversely if U and V form a seperation of X, with $U \neq X$, we see that $U = X \backslash V$ is also closed. ∎

Lemma 3.1.2. *If Y is a subspace of a topological space X, a seperation of Y is a pair of nonempty disjoint sets A, B who's union is Y; such that A shares no limit point with B and B shares no limit point with A i.e.* $\operatorname{cl} A \cap B = \emptyset$ *and* $A \cap \operatorname{cl} B = \emptyset$

Proof. Let $A, B \subseteq Y$ be nonempty such that $A \cup B = Y$. Suppose that neither share any limit points with eachother; that is if a is a limit point of A and b a limit point of B then $a \notin B$ and $b \notin A$. Then $\operatorname{cl} A \cap B = \emptyset$ and $A \cap \operatorname{cl} B = \emptyset$. Hence $\operatorname{cl} A \cap Y = A$ and $clB \cap Y = B$, so A and B are both open and closed. On the otherhand, $Y \backslash A = B$ and $Y \backslash B = A$, so they are both open as well. Hence they form a seperation. ∎

Example 3.1. (1) If X is a 2-point set in the indescrete topology, then clearly by lemma 3.1.1, X is connected.

(2) Consider the subspace $[-1, 0) \cup (0, 1]$ in $\mathbb{R}$. We have that the intervals $[-1, 0)$ and $(0, 1]$ form a seperation of $\mathbb{R}$ (also notice that $[-1, 0)$ and $(0, 1]$ share no limitpoints).

(3) The sets $[-1, 0]$ and $(0, 1]$ are disjoint in the subspace $[-1, 1]$ of $\mathbb{R}$, however, they are not a seperation of $[-1, 1]$ as $[-1, 0]$ is closed (and also shares a limit point with $(0, 1]$).

(4) The field of rationals $\mathbb{Q}$ is not connected. Let Y be a subspace of $\mathbb{Q}$ and consider $Y \cap (-\infty, a)$ and $Y \cap (a, \infty)$, for $a \in \mathbb{R} \backslash \mathbb{Q}$. These two sets form a seperation of $\mathbb{Q}$.

(5) Consider $X = \{x \times x \in \mathbb{R}^2 : y = 0\} \cup \{x \times y \in \mathbb{R} : x \geq 0 \text{ and } y = \frac{1}{x}\}$. X is not connected, the subsets in the definition form a seperation.

Lemma 3.1.3. *If X is a topological space and C and D form a seperation of X and if Y is a connected subspace of X, then either $Y \subseteq C$ or $Y \subseteq D$, but not both.*

Proof. That $Y \nsubseteq C \cap D$ is obvious. Now since C and D are open in X, $C \cap Y$ and $D \cap Y$ are open in Y. Now since $(C \cap Y) \cap (D \cap Y) = (C \cap D) \cap Y = \emptyset$, and $(C \cap Y) \cup (D \cap Y) = Y$. Now if both $C \cap Y$ and $D \cap Y$ are nonempty, then they form a seperation of Y, which is imposible, hence $Y \subseteq C$ or $Y \subseteq D$. ■

Theorem 3.1.4. *The union of a collection of connected subspaces that have a common point is connected.*

Proof. Let $\{A_\alpha\}$ be a collection of connected subspaces and let $p \in \bigcap A_\alpha$ and let $Y = \bigcup A_\alpha$. Suppose that $Y = C \cup D$ is seperation of Y. Then $p \in C$ or $p \in D$ (but not both). Suppose that $p \in C$, then since A_α is connected for all α, $A_\alpha \subseteq C$ or $A_\alpha \subseteq D$, but not both. Now since $p \in C$, $A_\alpha \subseteq C$ implying $D = \emptyset$, a contradiction. Hence Y is also connected. ■

Theorem 3.1.5. *Let A be a connected subspace of X. If $A \subseteq B \subseteq \operatorname{cl} A$. Then B is connected.*

Proof. Let A be connected and let $A \subseteq B \subseteq \operatorname{cl} A$. If $B = C \cup D$ is a seperation of B, then $A \subseteq C$ or $A \subseteq D$, but not both. If $A \subseteq C$, then $\operatorname{cl} A \subseteq \operatorname{cl} C$ and since $\operatorname{cl} C \cap D = \emptyset$, so $B \cap D = \emptyset$, hence $D = \emptyset$, a contradiction. ■

Remark. What this theorem says is that we can construct a connected space from a given connected subspace A by adjoining limit points of A to itself.

Theorem 3.1.6. *The image of a connected subspace under a continuous map is connected.*

Proof. Let $f : X \to Y$ be continuous and let X be connected consider $f : X \to f(X)$ and suppose that $f(X) = C \cup D$ is a seperation of $f(X)$. Then $f^{-1}(C)$ and $f^{-1}(D)$ are also nonempty, disjoint open sets of X who's union is X, and so they form a seperation of X which cannot happen. Hence $f(X)$ is connected. ■

Theorem 3.1.7. *Finite products of connected spaces are connected.*

Proof. Suppose first that X and Y are connected, and let $a \times b \in X \times Y$. Notice that $X \times b$ is connected and homeomorphic to X via π_1, and $a \times Y$ is connected and homeomorphic to Y via π_2. Then the space $T_x = (X \times b) \cup (x \times Y)$ for each $x \in X$ is connected, then taking $\bigcup T_x = X \times Y$, since T_x is connected for al x and share a common point, then $X \times Y$ is also connected.

Now suppose that $\prod_{i=1}^n X_i$ is connected for all $n \geq 1$. We have that $\prod_{i=1}^{n+1} X_i$ is homeomorphic to $\prod_{i=1}^n X_i \times X_{n+1}$, which is connected by hypothesis. Thus $\prod_{i=1}^{n+1} X_i$ is connected. ■

Example 3.2. (1) Consider $\mathbb{R}^\omega$ in the box topology. Let A be the set of all bounded sequences, and B the set of all unbounded sequences. Then $\mathbb{R}^\omega = A \cup B$ is a seperation of $\mathbb{R}^\omega$, for if $U = \prod (a_i - 1, a_i + 1)$ for the sequence $\{a_n\}$, then U is bounded if $\{a_n\}$ is bounded, and unbounded if $\{a_n\}$ is unbounded.

(2) $\mathbb{R}^\omega$ under the product topology is connected. Let $\hat{\mathbb{R}}^n$ be the subspace of $\mathbb{R}^\omega$ of all sequences that are eventually 0, i.e. $x_i = 0$ whenever $i > n$. $\hat{\mathbb{R}}^n$ is homeomorphic to $\mathbb{R}^n$, so $\hat{\mathbb{R}}^n$ is connected. Then $\mathbb{R}^\infty = \bigcup \hat{\mathbb{R}}^n$ is also connected since 0 is a common point.

Now let $a \in \mathbb{R}^\omega$ and $U = \prod U_i$ be a basis element about a. There is an $N \in \mathbb{Z}^+$ such that $U_i = \mathbb{R}$ for all $i > N$,=. Then the point $x = (a_1, \dots, a_N, 0, 0, \dots) \in \mathbb{R}^\infty \cap U$, since $U_i \in U$ for all i and $0 \in U_i$ whenever $i > N$. So $\operatorname{cl} \mathbb{R}^\infty = \mathbb{R}^\omega$, and so $\mathbb{R}^\omega$ is indeed connected.

3.2 Connected Spaces of The Real Line.

Definition. We call a simply ordered set L with $|L| > 1$ a **ordered contunuum** if:

(1) L has the least upperbound property.

(2) If $x < y$, then there exists a z such that $x < z < y$.

Theorem 3.2.1. *If L is a linear continuum in the order topology, then L is connected, and so are the open sets of L (the intervals and rays in L).*

Proof. We show that convex sets are connected. Let $Y = A \cup B$ be a seperation, and choose $a \in A$, $b \in B$ with $a < b$. We have that the interval of points in L, $[a, b] \subseteq Y$; and we also have that $[a, b] \subseteq A_0 \cup B_0$ with $A_0 = A \cap [a, b]$ and $B_0 = B \cap [a, b]$. Now $A_0, B_0 \neq \emptyset$, so $[a, b] = A_0 \cup B_0$ is a seperation of $[a, b]$. Now let $c = \sup A_0$. Suppose first that $c \in B_0$, then $c \neq a$, so either $c = b$ or $a < c < b$. Since B_0 is open in $[a, b]$ as a subspace of Y, there is some interval $(d, c] \subseteq B_0$.

If $c = b$, then $d < c$ is an upperbound of A_0, which contradicts that c is the least upperbound. Now suppose that $c < b$. We have that since $c, b \in B_0$, $(c, b] \cap A_0 = \emptyset$, then $(b, d] \cap A_0 = (d, c] \cap (c, b] \cap A_0 = \emptyset$, and again we have $d < c$ which gives us the contradiction. So $c \notin B$. By similar reasoning $c \notin A_0$. ■

Corollary. $\mathbb{R}$ *is connected and so are the intervals and rays of* $\mathbb{R}$.

Proof. $\mathbb{R}$ is a linear continuum. ■

Theorem 3.2.2 (The Intermediate Value Theorem)**.** *Let $f : X \to Y$ be continuous with X connected, and Y an ordered set under the order topology. If $a, b \in X$, and if $r \in Y$ such that $f(a) < r < f(b)$ or $f(b) < r < f(a)$, then there exists a $c \in X$ for which $f(c) = r$.*

Proof. Let $r \in Y$ such that $f(a) < r < f(b)$, without loss of generality. We have that $A = f(X) \cap (-\infty, r)$ and $B = f(X) \cap (r, \infty)$ are disjoint, nonempty sets open if $f(X)$ as a subspace of Y. Now suppose there is no $c \in X$ for which $f(c) = r$, then $f(X) = A \cup B$ is a seperation of $f(X)$, which contradicts theorem 3.1.6. ■

Example 3.3. (1) The ordered square I_0^2 is a linear continuum. Let $A \subseteq I_0^2$ and consider the projection $\pi_1 : I_0^2 \to I_0^2$. Let $b = \sup \pi_1(A)$, now if $b \in \pi_1(A)$ then $A \cap (b \times I_0) \neq \emptyset$, and since $I_0 \subseteq \mathbb{R}$, $A \cap (b \times I_0)$ has a least upperbound, $b \times c$, where $c = \sup I_0$, which is also the least upperbound of A. Now if we have $a \times c < b \times d$, then $a < b$ and $c < d$; and since $\mathbb{R}$ is a linear continuum, there are $y, z \in \mathbb{R}$ for which $a < y < b$ and $c < z < d$. Hence $a \times c < y \times z < b \times d$; which makes I_0^2 into a linear continuum.

(2) If X is a well ordered set, then $X\times[0,1)$ is a linear contiuum in the dictionary order. Let $A \subseteq X \times [0,1)$ and consider the projection $\pi_2 : X \times [0,1) \to [0,1)$. If $b = \sup \pi_2(A)$, then $A \cap (b \times [0,1)) \neq 0$, and so $A \cap (b \times [0,1)$ has a least upperbound $b \times c$ with $c = \sup [0,1)$, which is also a least upperbound of A.

Now since $\mathbb{R}$ is a linear continuum, if $x \times a < y \times b$, under the dictionary order, then $x \leq y$ and $a < b$. Then there are $c, z \in \mathbb{R}$ such that $x \leq z \leq y$ and $a < c < b$, so that $x \times a < z \times c < y \times b$.

Definition. Let X be a topological space with $x, y \in X$. An xy-**path** in X from x to y is a continuous map $f : [a,b] \to X$, with $[a,b] \subseteq \mathbb{R}$ such that $f(a) = x$, and $f(b) = y$. We call X **path connected** if there exists an xy-path in X for every $x, y \in X$.

Theorem 3.2.3. *Path connected spaces are connected*

Proof. Let X be path connected, and suppose that $X = A \cup B$ is a seperation of X. Let $f : [a,b] \to X$ be some path in X. Since f is continuous, and $[a,b] \subseteq \mathbb{R}$, by theorem 3.1.6, $f([a,b]) \subseteq X$ is a connected subspace, so either $f([a,b]) \subseteq A$ or $f([a,b]) \subseteq B$, so there is no path from a point in A to a point in B. But X is path connected; a contradiction. Therefore X must be connected. ∎

Example 3.4. (1) Define the **unit ball** in $\mathbb{R}^n$ under $||\cdot||$ to be $B^n = \{x \in \mathbb{R}^n : ||x|| \leq 1\}$. B^n is path connected. Consider $f[0,1] \to B^n$ by $f(t) = (1-t)x + ty$, then $||f(t)|| = (1-t)||x|| + t||y|| \leq 1$, hence $f(t) \in B^n$. Extending this to arbitrariy balls, for $\varepsilon > 0$, $B(x,\varepsilon)$ and $clB(x,\varepsilon)$ are also path connected. The function f also shows that the unit ball, and open balls (as well as their closure) are convex.

(2) Define **punctured Euclidean space** to be $\mathbb{R}^n\backslash\{0\}$. If $n > 1$, $\mathbb{R}\backslash\{0\}$ is path connected. Connect the points $x, y \in \mathbb{R}^n\backslash\{0\}$ by a straight line not passing through 0, or choose a point $z \in \mathbb{R}^n\backslash\{0\}$ on that line and form a path by adjoining the lines from x to z and from z to y

(3) Consider the unit sphere $S^{n-1} = \{x \in \mathbb{R}^n : ||x|| = 1\}$ in $\mathbb{R}^n$. S^{n-1} is path connected for $n > 1$. Take the map $g : \mathbb{R}^n\backslash\{0\} \to S^{n-1}$ by $g : x \to \frac{x}{||x||}$.

1. The ordered square I_0^2 is connected, but it is not path connected/ Let $p = 0 \times 0$, $1 = 1 \times 1$ and let $f : [a,b] \to I_0^2$ be a path joining p and q. We have that $f([a,b])$ must contain all $x \times y \in I_0^2$ by the intermediate value theorem. Hence for each $x \in I_0$, $U_x = f^{-1}(x \times I_0) \neq \emptyset$ and it is also open in $[a,b]$. Now choose for each $x \in I_0$, $q_x \in \mathbb{Q}$ such that $q_x \in U_x$. Since $\bigcap_{x \in I_0} U_x = \emptyset$, the map $x \to q_x$ is $1-1$ of I_0 onto $\mathbb{Q}$. which makes I_0 countable; a contradiction.

(4) Let $S = \{x \times \sin\frac{1}{x} : 0 < x \leq 1\}$. This is the image of the continuous map $x \to x \times \sin\frac{1}{x}$ from $(0,1] \to \mathbb{R}^2$, Since $(0,1]$ is connected, then so is S. by theorem 3.1.6. We call $\operatorname{cl} S = S \cup (0 \times [-1,1])$ the **topologist's sine curve** (see figure 3.2).

Suppose that $f : [a,c] \to \operatorname{cl} S$ is a path begining at 0 and ending at a point in S. The set $T = \{t \in \mathbb{R} : f(t) \in 0 \times [-1,1]\}$ is closed, so it has a largest element b. Then f is a path mapping $b \to 0 \times [-1,1]$ and taking all other points to S. Suppose that

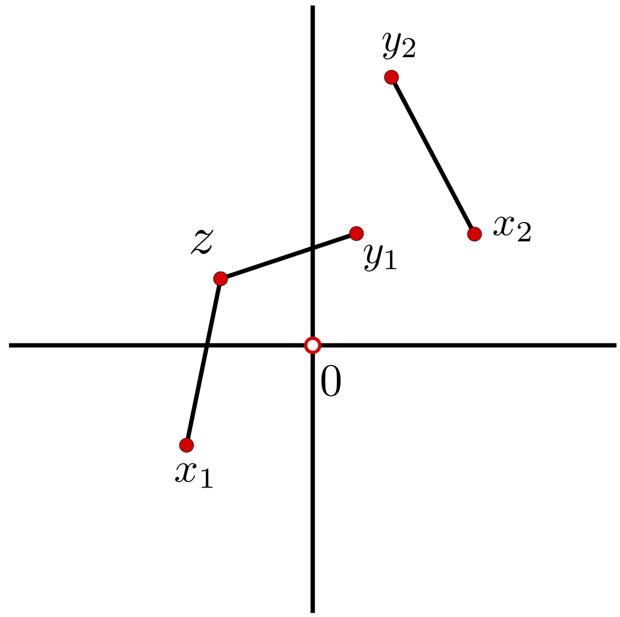

Figure 3.1: Punctures 2-space $\mathbb{R}^2 \backslash \{0\}$.

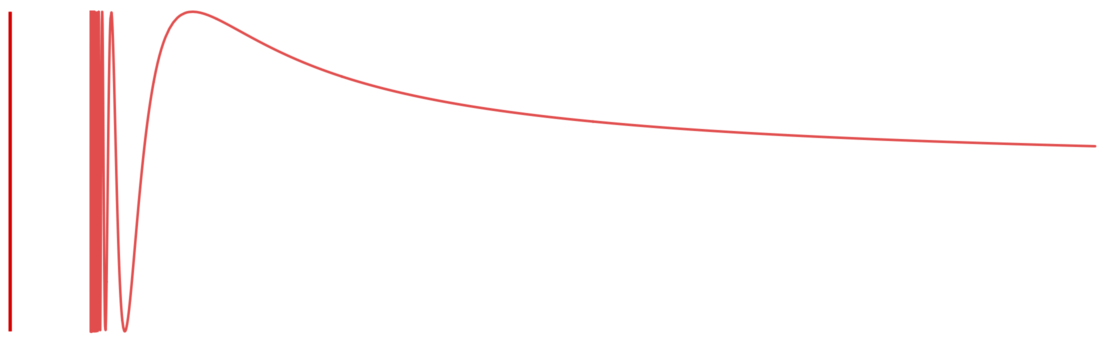

Figure 3.2: The topologists Sine Curve, defined by $x \times \sin \frac{1}{x}$.

$[b, c] = [0, 1]$ and let $f(t) = (x(t), y(t))$ where $x(0) = 0$, $x(t) > 0$ and $y(t) = \sin \frac{1}{x(t)}$ for all $t > 0$. For $n \in \mathbb{Z}^+$, choose $0 < u < x(\frac{1}{n})$ such that $\sin \frac{1}{u} = (-1)^n$. By the intermediate value theorem, there are $0 < t_n < \frac{1}{n}$ with $x(t_n) = u$. Then the sequence $\{t_n\} \to 0$, and $x(t_n) = (-1)^n$, which diverges; a contradiciton since $x(0) = 0$ and $\{t_n\} \to 0$. Hence $\operatorname{cl} S$ is not path connected.

3.3 Components and Local Connectedness.

Proposition 3.3.1. *Let $\sim$ be a relation defined on the topological space X by: $x \sim y$ if there exists a connected subspace containing both x and y. Then $\sim$ is an equivalence relation on X.*

Proof. Clearly, $x \sim x$. Now suppose that $x \sim y$, then there is a connected subspace containing both x and y, by definition, $y \sim x$. Now suppose that $x \sim y$ and $y \sim z$. Then there are connected subspaces U and V with $x, y \in U$, $y, z \in V$. Since $y \in U \cap V$, by theorem 3.1.4, $U \cup V$ is a connected subspace with $x, z \in U \cup V$. That is $x \sim z$. ■

Definition. Let X be a topological space. Define an equivalence relation $\sim$ on X by taking $x \sim y$ if there is a onnected subspace of X containing x and y. We call the equivalence classes of $X/\sim$ **connected components** (or **components**) of X.

The following theorem gives us the equivalence classes of X.

Theorem 3.3.2. *The components of X are disjoint nonempty subspaces of X, whose union is X such that each nonempty connected subspace of X intersects only one of them.*

Proof. Since components are equivalence classes of X under $\sim$ (as defined in proposition 3.3.1), they partition X. Then they are disoint, and their union is X.

Now let A be a connected subspace intersecting at least two components C_1 and C_2 at elements x_1 and x_2 respectively. Then $x_1, x_2 \in A$; by definition of $\sim$, $x_1 \sim x_2$. This makes $C_1 = C_2$ since they are equivalence classes.

Now let C be a component with $x_0 \in C$ then for all $x \in C$, $x_0 \sim x$, then there is a connected subspace A_x of X with $x_0, x \in A_x$. Then $A \subseteq C$, moreover

$$C = \bigcup_{x \in C} A_x$$

Since A_x is connected for each x, and $x_0 \in \bigcap A_x$, by theorem 3.1.4, C is connected. ■

Corollary. *If X is a connected topological space, then it has only one component.*

Proof. Notice that X is a subspace of itself. So if X intersects components C_1 and C_2 at points x_1 and x_2 respectively, then $C_1 = C_2$. ■

Lemma 3.3.3. *Components of a topological space are closed.*

Proof. Let C be a component of a topological space X. Since C is connected, $\operatorname{cl} C$ is connected, and is a subspace of X containing (any) $x, y \in C$ hence by definition $clC \subseteq C$. ■

Example 3.5. Consider $\mathbb{Q}$ as a subspace of $\mathbb{R}$ under $|\cdot|$. $\mathbb{Q}$ is Hausdorff, so consider a subspace A of $\mathbb{Q}$ with atleast two elements q_1, q_2 with $q_1 < q_2$. Since $\mathbb{R}\backslash\mathbb{Q}$ is dense in $\mathbb{R}$, then there exists an $i \in \mathbb{R}\backslash\mathbb{Q}$ such that $q_1 < i < q_2$. Now define $U = (-\infty, i)$ and $V = (i, \infty)$. We have $q_1 \in U$, $q_2 \in V$, $U \cap V = \emptyset$ and $A = U \cup V$, that is U and V form a seperation of A. Thus if A is to be connected, A must be a singleton. That is all the components of $\mathbb{Q}$ are singletons.

Now if A was an open component in $\mathbb{Q}$, then the subspace topology on $\mathbb{Q}$ would be $2^{\mathbb{Q}}$. Now notice that the sequence $\{\frac{1}{n}\}_{\mathbb{Z}^+}$ has a limit point at 0 in the topology induced by $|\cdot|$, but it does not have a limit point at 0 under $2^{\mathbb{Q}}$. So none of the components of $\mathbb{Q}$ are open.

We also define equivalence classes of a topological space using path connectedness.

Proposition 3.3.4. *Let X be a topological space and define $\sim$ on X by $x \sim y$ if there is an xy-path in X. Then $\sim$ is an equivalence relation on X.*

Proof. The function $f : [a, b] \to X$ defined by $f : t \to t$ defines a path from x to x, so $x \sim x$. Now suppose $x \sim y$, then there is a path $f : [0, 1] \to X$ from x to y. Define $g : [0, 1] \to X$ by $g(t) = f(1-t)$. Since f is a path, g is continuous, and $g(0) = f(1) = y$ and $g(1) = f(0) = x$, so g is a path from y to x and $y \sim x$. Lastly, suppose $x \sim y$ and $y \sim z$. Then there are paths $f : [0, 1] \to X$ and $g : [1, 2] \to X$ from x to y and y to z respectively. Since $f(1) = g(1) = y$ for $\{1\} = [0, 1] \cap [1, 2]$, and f and g are continuous; construct, by the pasting lemma, a continuous function $h : [0, 2] \to X$. Then $h(0) = f(0) = x$ and $h(2) = g(2) = z$, hence h is a path from x to z. That is $x \sim z$. ■

Definition. Let X be a topological space and $\sim$ the equivalence relation defined by $x \sim y$ if there is an xy-path in X. Then the equivalence classes of X are called **path components**.

Theorem 3.3.5. *The path components of a topological space X are disjoint path connected subspaces of X whose union is X, such that each nonempty path connected subspace of X intersects only one of them.*

Proof. That path components are disjoint and form X in union is guaranteed by them being equivalence classes. Now let P_1 and P_2 be path components and let A be a path connected subspace intersecting P_1 and P_2 at x_1 and x_2 respectivley. Then $x_1, x_2 \in A$, and since A is path connected, $x_1 \sim x_2$, therefore $P_1 = P_2$.

Now let P be a path component and choose $x_0 \in P$. Then for all $x \in P$, $x_0 \sim x$, that is there is a path connected subspace A_x with $x_0, x \in X$. This make $A_x \subseteq P$. Since there exists a path from x_0 to x, both of which are in P, this makes P path connected. ■

Corollary. *If X is path connected, then it has only one path component.*

Example 3.6. The topologist's sine curve, $\operatorname{cl} S$, with $S = \{x \times \sin \frac{1}{x} : 0 < x \leq 1\}$ is connected, and hence only has one component. However, $\operatorname{cl} S$ is not path connected, notice that S is a path component, and $0 \times [-1, 1]$ is another path component. So $\operatorname{cl} S$ has two path components. Here S is open, in $\operatorname{cl} S$ (for otherwise, $\operatorname{cl} S = S$ which cannot happen), and $0 \times [-1, 1]$ is closed in $\operatorname{cl} S$.

Now construct the space S' from $\operatorname{cl} S$ by deleting all points with rational second coordinates in $[-1, 1]$. Then S' has one component, but uncountably many path components; since countably many points were deleted and leaving only those points with second coordinat in $[-1, 1] \backslash \mathbb{Q}$ (which is homeomorphic to $\mathbb{R} \backslash \mathbb{Q}$, which is uncountable).

We now define connectedness in the local.

Definition. A topological space is **locally connected** at x if for every neighborhood U of x, there is a connected neighborhood V with $x \in V \subseteq U$. We call X **locally connected** if it is locally connected at every point.

Definition. A topological space is **locally path connected** at x if for every neighborhood U of x, there is a path connected neighborhood V with $x \in V \subseteq U$. We call X **locally path connected** if it is locally path connected at every point.

Theorem 3.3.6. *A topological space X is locally connected if and only if for every open set U of X, each component of U is open in X.*

Proof. Suppose that X is locally connected, and let U be open in X; and let C be a component of U. If $x \in C$, choose a neighborhood V in X with $x \in V \subseteq U$, since V is connected, $V \subseteq C$, thus C is open in X.

Now suppose that for every component C of U, that C is open in X. Let $x \in X$, and a neighborhood U of x, and let C be a component of U with $x \in C$. Then we see that C is a connected neighborhood with $x \in C \subseteq U$. Therefore X is locally connected. ■

Theorem 3.3.7. *A topological space X is locally path connected if and only if for every open set U of X, each path component of U is open in X.*

Proof. Suppose that X is locally path connected, and let U be open in X; and let P be a path component of U. If $x \in P$, choose a path connected neighborhood V in X with $x \in V \subseteq U$, since V is path connected, $V \subseteq P$, thus P is open in X.

Now suppose that for every path component P of U, that P is open in X. Let $x \in X$, and a neighborhood U of x, and let P be a path component of U with $x \in P$. Then we see that P is a path connected neighborhood with $x \in P \subseteq U$. Therefore X is locally path connected. ■

Theorem 3.3.8. *If X is a topological space, each path component of X lies in a component of X. If X is locally path connected, then path components are components.*

Proof. Since path components are path connected, they are connected, and hence must lie within componenpath components are path connected, they are connected, and hence must lie within components.

Now let X be locally path connected and P a path component, and C a component of X sucht that $P \subseteq C$, but $P \neq C$. Define $Q = \bigcup P'$, where each $P' \neq P$ is a path component intersecting C, then $P' \subseteq C$. So $C = P \cup Q$. Since X is locally path connected, each path component is open in X, thus $P, Q \neq \emptyset$ are open, and disjoint, hence they form a seperation of C, which contradicts its connectedness. Therefore $P = C$. ■

3.4 Compact Spaces.

Definition. A collection $\mathcal{A} = \{A_\alpha\}$ of subsets of a topological space X is said to be a **cover**, or a **covering** of X if $X \subseteq \bigcup A_\alpha$. We call $\{A_\alpha\}$ an **open cover** if each A_α is open in X. If $\{A'_\alpha\}$ is a subcollection of $\mathcal{A}$ that also covers X, we call $\{A'_\alpha\}$ a **subcover** of X.

Definition. We call a topological space X **compact** if for every open cover of X, there is a finite subcover of X.

Example 3.7. (1) $\mathbb{R}$ is not compact. Consider the following cover of $\mathbb{R}$:

$$\mathcal{A} = \{(n, n+2) : n \in \mathbb{Z}\}$$

however, there is no finite subcollection of $\mathcal{A}$ that is a subcover of $\mathbb{R}$.

(2) The subspace $X = \{0\} \cup \{\frac{1}{n} : n \in \mathbb{Z}\}$ of $\mathbb{R}$ is compact. Let $\mathcal{A}$ be a cover of X. There is a $U \in \mathcal{A}$ with $0 \in U$ and U contains all but finitely many of the points $\frac{1}{n}$. Now choose for each $\frac{1}{n} \in X \backslash U$ an element of $\mathcal{A}$ A_n containing it. Then for all n, $\{A_n\}$ is finite and covers X.

(3) If X is a finite topological space, then it is compact, since every open cover of X is finite.

(4) The interval $(0,1]$ is not compact. The open cover $\mathcal{A} = \{(\frac{1}{n}, 1] : n \in \mathbb{Z}^+\}$ contains no finite subcollection of $\mathcal{A}$ that covers $(0,1]$. Likewise, $(0,1)$ is not compact by the same argument. However, $[0,1]$ is compact.

Lemma 3.4.1. *Let Y be a subspace of a topological space X. Y is compact if and only if every open cover of Y, by open sets of X has a finite subcover of Y.*

Proof. Suppose Y is compact and let $\{A_\alpha\}$ be a cover of Y with A_α open in X for all α. Since $\{A_\alpha\}$ covers Y, so does the collection $\{A_\alpha \cap Y\}$, where $A_\alpha \cap Y$ is open in Y for all α. Since Y is compact, choose the finite subcollection $\{A_i \cap Y\}_{i=1}^n$ to be a finite subcover of Y; i.e. $\{A_i \cap Y\}_{i=1}^n \subseteq \{A_\alpha\}$.

Conversely, suppose that for every cover $\{A_\alpha\}$, open in X, of Y that $\{A_\alpha\}$ contains a finite subcover of Y. Choose $A'_\alpha = A_\alpha \cap Y$ for all α; then $\{A'_\alpha\}$ is an open cover of Y. By hypothesis, there is a finite subcover $\{A_i\}_{i=1}^n$, and by our assignment, we get that $\{A'_i\}_{i=1}^n \subseteq \{A'_\alpha\}$ is also a finite subcover of Y. Therefore Y is compact as a subspace of X. ■

Theorem 3.4.2. *Every closed subspace of a compact space is compact.*

Proof. Let Y be a closed subspace of a compact space X. Let $\{A_\alpha\}$ be an open cover of Y with A_α open in X for all α. Consider $\{B_\alpha\} = \{A_\alpha\} \cup X \backslash Y$. Since $\{A_\alpha\}$ covers Y, $\{B_\alpha\}$ covers X. Now take some finite subcollection $\{B_i\}_{i=1}^n$. If it contains $X\backslash Y$, then just consider $\{B_i\}\backslash(X\backslash Y)$. Then this collection is a finite subcover of Y. ■

Theorem 3.4.3. *Every compact subspace of a Hausdorff space is closed.*

Proof. Let Y be a compact subsapce of a Hausdorff space X. Choose $x_0 \in X\backslash Y$ and for each $y \in Y$ choose disjoint neighborhoods U_y and V_y of x_0 and y respectively. The collection $\{V_y\}$ covers Y by open sets in X, hence there is a finite subcover, by theorem 3.4.2, $\{V_{y_n}\}$ of Y. Take $V = \bigcup_{i=1}^n V_{y_i}$ to be open in X, and take $U = \bigcap_{i=1}^n U_{y_i}$ also open in X. Then $Y \subseteq V$ and $V \cap U = \emptyset$; for take $z \in V$, $z \in V_{y_i}$ for some $1 \leq i \leq n$. Hence $z \notin U_{y_i}$, so $z \notin U$. Then U is a neighborhood of X, disjoint from Y, making $X\backslash Y$ open in X. Therefore Y is closed in X. ■

Corollary. *If Y is a compact subspace of a Hausdorff space X, and $x_0 \in X\backslash Y$, then there exists opensets U and V in X with $x_0 \in U$ and $Y \subseteq V$, respectively.*

Example 3.8. (1) Since $[a,b]$ is compact in $\mathbb{R}$, so is any closed subspace of $[a,b]$. The subspaces $(a,b]$, $[a,b)$, and (a,b), however, are not compact; as they are not closed in $\mathbb{R}$ as a Hausdorff space.

(2) Consider the finite complement topology on $\mathbb{R}$. Every proper finite subset of $\mathbb{R}$ is closed under finite complements, however every subset of $\mathbb{R}$ is compact in the finite complement topology.

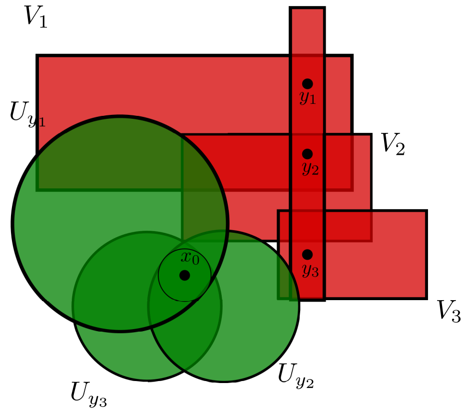

Figure 3.3: Closed sets of a compact Hausdorff space (See theorem 3.4.3).

Theorem 3.4.4. *If X is a compact space, and Y is a topological space, and $f : X \to Y$ is a continuous map, then $f(X)$ is compact.*

Proof. Let $f : X \to Y$ be continuous, and let X be compact. Let $\mathcal{C}$ be a cover of $f(X)$ by sets ib Y, and consider the collection $\{f^{-1}(C)\}_{C \in \mathcal{C}}$. We have that $f^{-1}(C)$ is open in X by continuity, hence there is a finite subcollection $\{f(C_i)\}_{i=1}^n$ covering X (by compactness) That is $X \subseteq \bigcup_{i=1}^n f(C_i)$, hence $f(X) \subseteq \bigcup C_i$, which makes $\{C_i\}_{i=1}^n$ a finite subcover of $f(X)$. ■

Theorem 3.4.5. *Let X be compact, and Y be Hausdorff, and let $f : X \to Y$ be a $1-1$ continuous map of X onto Y. Then f is a homomorphism.*

Proof. If $A \subseteq X$ is closed, then A is compact by theorem 3.4.2, making $f(X)$ compact in Y. Since Y is Hausdorff, $f(A)$ closed is in Y. This makes f^{-1} continuous. ■

Definition. Let X be a topological space and let Y be compact. Let $x_0 \in X$ and $N \subseteq X \times Y$. If W is a neighbourhood of $x_0 \times Y$ such that $W \times Y \subseteq N$, then we call $W \times Y$ a **tube** about $x_0 \times Y$.

The following theorem lemma states the existence of such tubes, and helps establish the next theorem

Lemma 3.4.6 (The Tube Lemma)**.** *Let X be a topological space, and let Y be compact. Choose $x_0 \in X$ and $N \subseteq X \times Y$ open. Then there exists a tube $W \times Y$ about $x_0 \times Y$, for some nieghbourhood W of x_0.*

Proof. Take $x_0 \in X$ and let $N \times Y$ be open. We show there exists a tube about $x_0 \times Y$. Cover $x_0 \times Y$ by basis elements $U \times V \subseteq N$. We have $x_0 \times Y$ is compact since $x_0 \times Y$ is homeomorphic to Y; hence there is a finite subcollection $\{U_i \times V_i\}_{i=1}^n$ of the $U \times V \subseteq N$ covering $x_0 \times Y$, supposing that $U_i \times V_i$ intersects $x_0 \times Y$ for all i. Now define $W = \bigcap_{i=1}^n U_i$. W is open in X, and contains x_0.

Now let x $\times y \in$W $\times Y$ and $x_0 \times y \in x_0 \times Y$. We have that $x_0 \times y \in U_i \times v_i$, hence $y \in V_i$ for $1 \leq i \leq n$. We also get that $x \in U_j$ for all $1 \leq j \leq n$, since $x \in W$ by definition. Hence $x \times y \in U_i \times V_i$. Since $W \subseteq U_i \times V_i \subseteq N$, for all $1 \leq i \leq n$, we get the required result. ■

Theorem 3.4.7. *Finite products of compact spaces are compact.*

Proof. Let $\mathcal{C}$ be an open cover of $X \times Y$, with X and Y compact. Take $x_0 \in X$ and consider $x_0 \times Y$ compact in $X \times Y$. Then $x_0 \times Y$ can be covered by a finite subcollection $\{C_i\}_{i=1}^n \subseteq \mathcal{C}$. Take $N = \bigcup_{i=1}^n C_i$. Then $x_0 \times Y \subseteq N$. By the tube lemma, there is a tube $W \times Y$ about $x_0 \times Y$ with W a neighbourhood of x_0, so $W \times Y \subseteq N$.

Now for each $x \in X$, choose W_x such that $W_x \times Y$ can be covered by a finite subcollection of $\mathcal{C}$. $\{W_x\}$ forms an open cover of X, hence by compactness there is a finite subcover of X $\{W_i\}_{i=1}^n\{W_x\}$. Take $\{W_i \times Y\}_{i=1}^n$ and their union contains $X \times Y$. ■

Example 3.9. Let $Y = 0 \times \mathbb{R}$ and let $N = \{x \times y : |x| < \frac{1}{y^2+1}\}$. N is open in $\mathbb{R}^2$ and $Y \subseteq N$; however there is no tube about Y, so the tube lemma fails when Y is not compact.

Definition. A collection $\mathcal{C}$ of subsets of a set X is said to satisfy the **finite intersection property** (or **FIP**) if every for finite subcollection $\{C_i\}_{i=1}^n$ of $\mathcal{C}$, $\bigcap_{i=1}^n C_i \neq \emptyset$.

Theorem 3.4.8. *Let X be a topological space. X is compact if and only if for every collection $\mathcal{C}$ of closed sets satisfying the FIP, the intersection $\bigcap_{C \in \mathcal{C}} C \neq \emptyset$.*

Proof. Let $\mathcal{A}$ be a collection of subsets of X, and define $\mathcal{C} = \{X \backslash A : A \in \mathcal{A}\}$. The following hold:

(1) $\mathcal{A}$ is a collection of open sets if and only if $\mathcal{C}$ is a collection of closed sets.

(2) $\mathcal{A}$ covers X if and only if $\bigcap_{C \in \mathcal{C}} C = \emptyset$.

(3) A finite subcollection $\{A_i\}_{i=1}^n$ of $\mathcal{A}$ covers X if and only if $\bigcap X \backslash A_i = \emptyset$.

Take the contrapositive and then the complements of the sets. ■

3.5 Tychonoff's Theorem

This section is entirely devoted to proving that arbitrary (or infinite) products of compact spaces are also compact. Unlike its finite counterpart, the proof for this is not so well behaved. It can be achieved using the Well ordering principle, or using Zorn's lemma. We present the proof using Zorn's lemma, which we state without proof.

Lemma 3.5.1 (Zorn's Lemma)**.** *Every strictly partially ordered set, whose simply oredered sets have upperbounds, has a maximal element.*

Lemma 3.5.2. *Let X be a set, and $\mathcal{A}$ be a collection of subsets satisfying the finite intersection property. There exists a collection $\mathcal{D}$ of subsets of X such that $\mathcal{A} \subseteq \mathcal{D}$ and $\mathcal{D}$ has the finite intersection property, and is maximal with respect to it.*

Proof. We construct $\mathcal{D}$ using Zorn's lemma on a set whose elements are collections of subsets of X; let us denote is as a **superset**. Let $\mathcal{A} \subseteq 2^X$ satisfy the FIP. Define a partial order $\subset$ on supersets by $\mathcal{A} \subset \mathcal{B}$ if $\mathcal{A} \subseteq \mathcal{B}$ and $\mathcal{A} \neq \mathcal{B}$. Let $\mathbb{A}$ be the superset of all collections $\mathcal{B} \subseteq 2^X$ such that $\mathcal{A} \subset \mathcal{B}$ and $\mathcal{B}$ satisfies the FIP.

Let $\mathbb{B} \subseteq \mathbb{A}$ be ordered by inclusion and let $\mathcal{C} = \bigcup_{\mathcal{B}\in\mathbb{B}} \mathcal{B}$. Since $\mathbb{B} \subseteq \mathbb{A}$, we get $\mathcal{A} \subset \mathcal{C}$. Now let $C_1, \dots, C_n \in \mathcal{C}$. By definition, there is a $\mathcal{B}_i \in \mathbb{B}$ such that $C_i \in \mathcal{B}_i$ for $1 \leq i \leq n$. Then we have the superset $\{\mathcal{B}_i\}_{i=1}^n \subseteq \mathbb{B}$ is finite, hence there is a largest element, i.e. there is a $\mathcal{B}_k$ such that $\mathcal{B}_i \subset \mathcal{B}_k$ for all $1 \leq i \leq n$, $i \neq k$. Then all $C_i \in \mathcal{B}_k$, and since $\mathcal{B}_k$ has the FIP, we get $\bigcap_{i=1}^n C_i \neq \emptyset$. So $\mathcal{C}$ also has the finite intersection property, putting $\mathcal{C} \in \mathbb{A}$. Also notice that $\mathcal{C}$ is an upper bound of $\mathbb{B}$. ∎

Lemma 3.5.3. *Let X be a set and let $\mathcal{D} \subseteq 2^X$ be maximal with respect to the finite intersection property. Then:*

(1) Any finite interesection of elements of $\mathcal{D}$ is an element of $\mathcal{D}$.

(2) If $A \subseteq X$ intersects every element of $\mathcal{D}$, then $A \in \mathcal{D}$.

Proof. Let $B = \bigcap_{i=1}^n D_i$ where $D_i \in \mathcal{D}$ for all $1 \leq i \leq n$. Define a collection $\mathcal{E} = \mathcal{D} \cup \{B\}$. Now take $\{E_j\}_{j=1}^m \subseteq \mathcal{E}$. If $E_j \neq B$ for all j, then $\bigcap E_j \neq \emptyset$, by the FIP of $\mathcal{D}$. If $E_j = B$ for some $1 \leq j \leq m$, then $\bigcap E_j = E_1 \cap \dots \cap E_m \cap B = \bigcap E_j \cap \bigcap D_i \neq \emptyset$. Either way $\mathcal{E}$ satisfies the FIP. Now since $\mathcal{D} \subseteq \mathcal{E}$ and $\mathcal{D}$ is maximal with respect to the FIP, this makes $\mathcal{D} = \mathcal{E}$.

Now let $A \subseteq X$ and define $\mathcal{E} = \mathcal{D} \cup \{A\}$. Take $\{E_i\}_{i=1}^n \subseteq \mathcal{E}$. If $E_i \neq A$ for all i, then $\bigcap E_i$ is a finite intersection consisting of elements of $\mathcal{D}$ and (nonempty) intersections of elements of $\mathcal{D}$ and A, so $\bigcap E_i \neq \emptyset$. On the other hand, if $E_i = A$ for some $1 \leq i \leq n$, then $\bigcap E_i = E_1 \cap \dots \cap E_n \cap A$. Since A intersects every element of $\mathcal{D}$, we get $\bigcap E_i \neq \emptyset$ again. HEnce $\mathcal{E}$ has the FIP, which makes $A \in \mathcal{D}$. ∎

Theorem 3.5.4 (Tychonoff's Theorem)**.** *Arbitrary products of compact spaces are compact in the product topology.*

Proof. Let $X = \prod X_\alpha$ with X_α compact for all α. Take $\mathcal{A}$ the collection of all subsets of X satisfying the finite intersection property, and consider $\bigcap_{A\in\mathcal{A}} \operatorname{cl} A$.

By lemma 3.5.2, there is a collection $\mathcal{D} \subseteq 2^X$ such that $\mathcal{A} \subset \mathcal{D}$ (refer to the partial order defined in lemma 3.5.2) and $\mathcal{D}$ is maximal with respect to the FIP.

Let $\pi_\alpha : X \to X_\alpha$ be the projection of $x \to x_\alpha$ and consder the collection $\{\pi_\alpha(D)\}_{D\in\mathcal{D}}$ of subsets of X_α. Since $\mathcal{D}$ has the FIP, so does $\{\pi_\alpha(D)\}$. Now by the compactness of X_α, take $x_\alpha \in X_\alpha$ such that $x_\alpha \in \bigcap \operatorname{cl} \pi_\alpha(D)$ for each α. Let $x \in X$. We wish to show $x \in \operatorname{cl} D$.

Let U_β be a neighbourhood of $x_\beta \in X_\beta$, and considet the subbasis element $\pi_\beta^{-1}(U_\beta)$. Since $x_\beta \in \operatorname{cl} \pi_\beta(D)$, we have $U_\beta \cap D = \pi_\beta(y)$, with $y \in D$. Then by lemma 3.5.3, we have that every basis element containing x is in $\mathcal{D}$. Then every basis element containing x intersects every $D \in \mathcal{D}$, thus $x \in \operatorname{cl} D$. This shows that X is compact. ∎

3.6 Compact Subspaces of $\mathbb{R}$.

We now construct new compact spaces from old ones and prove some results from real analysis.

Theorem 3.6.1. *Let X be a simply ordered set satisfying the least upper bound property. Then every closed interval in X is compact under the order topology.*

Proof. Let $a < b$ for $a, b \in X$ and let $\mathcal{A}$ be an open covering of $[a, b]$ of sets open in $[a, b]$ under the subspace topology. Let $x \in [a, b]$ with $x \neq b$, and suppose that x has an immediate successor $y > x$, that is there is no $z \in [a, b]$ with $y > z > x$. Then $[x, y] = \{x, y\}$ so it can be covered by at most two elements of $\mathcal{A}$. Now, if no such y exists in X, then choose an $A \in \mathcal{A}$ with $x \in A$, since $x \neq b$, we get that A is open, and that A contains an interval of the form $[x, c)$ for some $c \in [a, b]$, with $x < c$. Then choose a point $y \in (x, y)$ then the interval $[x, y]$ is covered by A, since $x < y < c$.

Now let C be the set of all $y > a$ of $[a, b]$ such that the interval $[a, y]$ is covered by finitely many elements of $\mathcal{A}$. By the above case there exists at least one such y, so that $C \neq \emptyset$. Now, let c be the least upperbound of C, then $a < c \leq b$.

Now, choose an element $A \in \mathcal{A}$ with $c \in A$. Since A is open, it contains an interval of the form $(d, c]$ for some $d \in [a, b]$. Now, if $c \notin C$ then there exists a $z \in C$ with $z \in (d, c)$; for otherwise, d would be an upperbound of C which cannot happen. Then $[z, c]$ can be covered by n elements of $\mathcal{A}$. Now, $[z.c] \subseteq A$, so $[a, c] = [a, z] \cup [z, c]$ can be covered by $n + 1$ elements of $\mathcal{A}$, which puts $c \in C$; which contradicts that $c \notin C$.

Now suppose that $c < b$ then by the first argument, with $x = c$ there is a $y > c$ of $[a, b]$ such that $[c, y]$ can be covered by finitely many elements of $\mathcal{A}$. Since $c \in C$, we have that $[a, c]$ can also be covered by finitely many elements of $\mathcal{A}$; so then can the interval

$$[a, y] = [a, c] \cup [c, y]$$

This makes $y \in C$, which cotradicts that c is the least upperbound of C. Therefore, X must be compact. ∎

Corollary. *Every closed interval in $\mathbb{R}$ is compact.*

Theorem 3.6.2. *A subspace A of $\mathbb{R}^n$ is compact if, and only if it is closed and is bounded in the Euclidean metric or the Square metric.*

Proof. Let ρ the Square metric. Then recall that:

$$\rho(x, y) \leq \|x - y\| \leq \rho(x, y)\sqrt{n} \tag{3.1}$$

This inequality implies that the subspace A is bounded under $\|\cdot\|$ if and only if it is bounded under ρ.

Now suppose that A is compact. Since $\mathbb{R}^n$ is Hausdorff, by theorem 3.4.3, A is closed. Consider the the collection of all open balls $B_\rho(0, m)$ where $m \in \mathbb{Z}^+$ whose union of elements is all of $\mathbb{R}^n$, that is:

$$\mathbb{R}^n = \bigcup_{m \in \mathbb{Z}^+} B_\rho(0, m)$$

By hypothesis, some finite subcollection covers A. Then we have that for some M, $A \subseteq B_\rho(0, M)$; then for any two points $x, y \in A$, $\rho(x, y) \leq 2M$, which bounds A under ρ. Now,since $\rho(x, y) \leq \|x - y\|$, this makes A bounded under $\|\cdot\|$.

Conversely, supposed that A is closed, and bounded under ρ (and consequently $\|\cdot\|$), and that $\rho(x, y) \leq N$ for all $x, y \in A$. Now, choose an $x_0 \in A$ such that $\rho(x_0, 0) = b$, then by the triangle inequality, $\rho(x, 0) \leq \rho(x, x_0) + \rho(x_0, 0) = N + b$ for all $x \in A$. Now, we get that A is a subset of the cube $[-b - N, N + b]$ which is compact in $\mathbb{R}$. Since A is closed, we get that A is compact. ∎

Example 3.10. (1) The unit ball S^{n-1} and closed unit ball B^n in $\mathbb{R}^n$ are compact by the above theorem, since they are both closed and bouned.

(2) The curve $A = \{x \times \frac{1}{x} : 0 < x \leq 1\}$ in $\mathbb{R}^2$ is closed, but not bounded, so it is not compact by the above theorem.

(3) The curve $A = \{x \times \sin\frac{1}{x} : 0 < x \leq 1\}$ in $\mathbb{R}^2$ is bounded, but not closed, so it is not compact.

Theorem 3.6.3 (The Extreme Value Theorem)**.** *X be a topological space, and Let Y be an ordered set under the order topology, and let $f : X \to Y$ be a continuous finction. If X is compact, then there exist points $c, d \in X$ such that $f(c) \leq f(x) \leq f(d)$ for all $x \in X$.*

Proof. Suppose that X is compact, then since f is continuous, then $A = f(X)$ is compact by theorem 3.4.4. Now if A has no largest element, then the collection of all rays $\mathcal{A} = \{(-\infty, a) : a \in A\}$ forms an open covering of A, then by compactness, there is a finite subcollection $\{(-\infty, a_1), \ldots, (-\infty, a_n)\}$ that covers A. If a_i is the largest element, then a_i is in none of these sets, which contradicts that they cover A. Then A has a largest element $M = f(d)$ for some $d \in X$.

By similar reasoning, taking $\mathcal{A} = \{(a, \infty) : a \in A\}$, we see that A has a smallest element $m = f(c)$ for $c \in X$. ■

Definition. Let X be a metric space with metric d, and let $A \subseteq X$ be nonempty. For each $x \in X$, we define the **distance** from x to A to be:

$$d(x, A) = \inf\{d(x, a) : a \in A\} \tag{3.2}$$

That is, it is the greatest lowerbound of all the distances between x and elements of A.

Lemma 3.6.4. *Let X be a metric space with metric d, and $A \subseteq X$ nonempty, then for all $x \in X$, $d(x, A)$ is a continuous function.*

Proof. Fix $x \in X$ and take $y \in X$. Then by definition and the triangle inequality:

$$d(x, A) \leq d(x, a) \leq d(x, y) + d(y, a)$$

for each $a \in A$, so we get:

$$d(x, A) - d(x, y) \leq \inf d(y, a) = d(y, A)$$

and so $d(x, A)$ is continuous. ■

Remark. Do not understand, please work thriugh the proof and rewrite it.

Lemma 3.6.5 (The Lebesgue Number Lemma)**.** *Let $\mathcal{A}$ be an open covering of a metric space X with metric d. If X is compact, then there exists a $\delta > 0$ such that for each subset of X with diameter less than δ, there exists an element of $\mathcal{A}$ containing it.*

Proof. Let $\mathcal{A}$ be an open covering of X. If $X \in \mathcal{A}$, then any positive real number is a Lebesgue number for $\mathcal{A}$. Now, suppose that $X \notin \mathcal{A}$. Then choose a finite subcollection $\{A_1, \ldots, A_n\}$ of $\mathcal{A}$ covering X. Then for each i, let $C_i = X \backslash A_i$ and define $f : X \to \mathbb{R}$ by the rule:

$$f(x) = \frac{1}{n} \sum_{i=1}^{n} d(x, C_i) \tag{3.3}$$

Then $f(x) > 0$ for all $x \in X$; for, fix x and choose an i so that $x \in A_i$, and choose ε so that $B_d(x, \varepsilon) \subseteq A_i$, then $d(x, C_i) \geq \varepsilon$ so that $f(x) > \frac{\varepsilon}{n}$.

Now, we have f is continuous since $d(x, C_i)$ is continuous. Then f has minimum value $\delta > 0$. Let $B \subseteq A$ have diameter $\operatorname{diam} B < \delta$ and choose a point $x_0 \in B$. Then $B \subseteq B_d(x_0, \delta)$. Now, we have that

$$\delta \leq f(x_0) \leq d(x_0, C_m)$$

where $d(x_0, C_m)$ is the largest of the $d(x_0, C_i)$. Then we see that $B_d(x_0, \delta) \subseteq A_m = X \backslash C_m$. ■

Definition. Let $\mathcal{A}$ be an open covering of a compact metric space X with metric d. We call a number $\delta > 0$ such that for each subseet of X with diam $< \delta$, there exists an element of $\mathcal{A}$ containing it a **Lebesgue number** of $\mathcal{A}$.

Definition. Let X and Y be metric spaces with metrics d_X, and d_Y repsectivelty. A function $f : X \to Y$ is said to be **uniformly continuous** if given $\varepsilon > 0$, there is a $\delta > 0$ such that for every pair of points $x, y \in X$, $d_X(x, y) < \delta$ implies that $d_Y(f(x), f(y)) < \varepsilon$.

Theorem 3.6.6 (The Uniform Continuity Theorem)**.** *Let X and Y be metric spaces with metrics d_X, and d_Y repsectivelty, and let X be compact. Then the function $f : X \to Y$ is uniformly continuous only if it is continous.*

Proof. Let $\varepsilon > 0$ and let $\mathcal{A} = \{B(y, \frac{\varepsilon}{2})\}_{y \in Y}$ be an open covering of Y. Then $f^{-1}(\mathcal{A}) = \{f^{-1}(B(y, \frac{\varepsilon}{2}))\}$ is an open covering of X. Now, choose $\delta > 0$ a Lebesgue number for $f^{-1}(\mathcal{A})$, then for $x, z \in X$ with $d_X(x, x) < \delta$, the set $\{x, z\}$ has $\operatorname{diam}\{x, z\} < \delta$, so $\{(x), f(z)\} \subseteq B(y, \frac{\varepsilon}{2})$ for $y \in Y$. Thus $d_Y(f(x), f(z)) < \varepsilon$. ■

Definition. If X is a toplogical space, then a point $x \in X$ is called an **isolated point** if the one-point set $\{x\}$ is open in X.

Theorem 3.6.7. *Let X be a nonepmty compact Hausdorff space. Then if X has no isolated points, X is uncountable.*

Proof. Let $U \neq \emptyset$ be an open set of X, and let $x \in X$. Choose then a $y \in X$ with $y \neq x$. Choose also disjoint neighborhoods W_1, and W_2 of x and y, respectively, (figure 3.4) and let $V = W_2 \cap U$. Then $V \subseteq U$ is nonempty and $x \notin \operatorname{cl} V$.

Now let $f : \mathbb{Z}^+ \to X$ be a function, and let $x_n = f(n)$. By the preceding argument, let $U = X$ and choose $V_1 \subseteq U$ nonempty such that $x_1 \notin \operatorname{cl} V_1$ for $x_1 \in X$. Then proceeding inductively, we can get find $V_{n+1} \subseteq V_n$ nonempty with $x_n \notin \operatorname{cl} V_{n+1}$. Now, consider the nested sequence of nonempty sdubsets:

$$\operatorname{cl} V_1 \supseteq \operatorname{cl} V_2 \supseteq \cdots \supseteq \operatorname{cl} V_{n+1} \supseteq \ldots$$

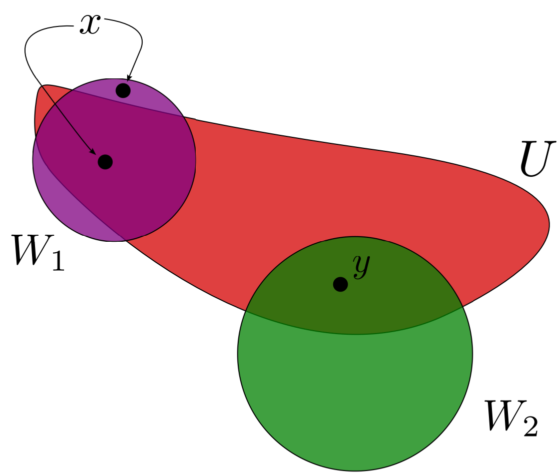

Figure 3.4: We choose disjoint neighborhoods W_1 and W_2 of x and y respectively. Here we want $y \in W_2 \cap U$, but x may be in $W_1 \cap U$ or simply $W_1 \backslash U$.

Since X is compact, there is an

$$x \in \bigcap_{i \in \mathbb{Z}^+} \operatorname{cl} V_i$$

Then $x \neq x_n$ since $x \in \operatorname{cl} V_n$ and $x_n \notin \operatorname{cl} V_n$. This means that f is not onto, and hence X is uncountable. ∎

Corollary. *Every closed set in $\mathbb{R}$ is uncountable.*

3.7 Limit Point Compactnes

Definition. We say a topological space X is **limit point compact** if every infinite subset of X has a limit point.

Theorem 3.7.1. *A compact topological space is limit point compact.*

Proof. Let X be a compact topological space, and let $A \subseteq X$ be an infinite subset of X. Supposing that A has no limit point then $A' \subseteq A$, making A closed. Now let $a \in A$ and U_a a neighborhood of a such that $U_a \cap A = \{a\}$. Now both $X \backslash A$ and U_a cover X for each a, and since X is compact, X can be covered by finitely many sets of the collection $(X \backslash A) \cup \{U_a\}_{a \in A}$. Now, $(X \backslash A) \cap A = \emptyset$ and only one a is in each U_a. By finiteness of the covering, A is finite, which contradicts the assumption that it is infinite. Therefore A must have a limit point, therefore X must be limit point compact. ∎

Example 3.11. (1) Let Y be a topological space under the indescrete topology $\mathcal{T} = \{\emptyset, Y\}$, with $|Y| = 2$. Let $X = Y \times \mathbb{Z}^+$. Then X is limit point compact since the infinite subsets $y_1 \times \mathbb{Z}^+$ and $y_2 \times \mathbb{Z}^+$ have limit points. However, X is not compact, since the open cover $\{Y \times \{n\}\}_{n \in \mathbb{Z}^+}$ has no finite open subcover.

(2) Let S_ω be the minimal uncountable well ordered set under the order topology. S_ω is not compact as it has no greatest element. Now, let $A \subseteq S_\omega$ be an infinite subset, and let

$B \subseteq A$ be countably finite. Then B has an upperbound bS_ω, so that $B \subseteq [a_0, b]$, where a_0 is the least element of S_ω. Since S_ω is well ordered, it has the least upperbound property, so $[a_0, b]$ is compact by theorem 3.6.1 Thus B has a limit point, making A have a limit point. Therefore by theorem 3.7.1, S_ω is limit point compact.

Definition. Let X be a topological space, and $\{x_n\}$ a sequence of points of X. If $n_1 < n_2 < \dots$ is an increasing sequence of nonnegative integers, then we call the sequence $\{y_m\}$ defined by $y_i = x_{n_i}$ a **subsequence** of $\{x_n\}$.

Definition. We call a topological space **sequentially compact** if every sequence of points in the space has a convergent subsequence.

Theorem 3.7.2. *For any metrizable space X, the following are equivalent:*

(1) X is compact.

(2) X is limit point compact.

(3) X is sequentially compact.

Proof. Let X be a metrizable space. By theorem 3.7.1 we have that (1) implies (2).

Now, suppose that X is limit point compact; that is, every infinite subset of X has a limit point. Consider the sequence $\{x_n\}_{n \in \mathbb{Z}^+}$. If $\{x_n\}$ is finite, then there is a point $x \in X$ for which $x = x_n$ for infinitely many n. Then the sequence $\{x\}$ is a convergent subsequence of $\{x_n\}$ by this fact, making X sequentially compact.

Now, if $\{x_n\}$ is an infinite sequence, then $\{x_n\}$ has a limit point $x \in X$ by hypothesis. Define, then, the sequence $\{y_m\}$ such that

$$y_i = x_{n_i} = B(x, \frac{1}{i})$$

which works since every open ball interesect $\{x_n\}$ at infinitely many points. Then as $i \to \infty$, $\frac{1}{i} \to 0$, so that $\{y_m\} \to x$, making X sequentially compact.

Now, suppose that X is sequentially compact. Then every sequence $\{x_n\}$ of points of X has a convergent subsequence. Let $\mathcal{A}$ be an open cover for X and suppose that $\mathcal{A}$ has no Lebesgue number; i.e. there is no $\delta > 0$ for which any set with diam $< \delta$ has an element of $\mathcal{A}$ containing it. Then for each $n \in \mathbb{Z}^+$, there are sets with diam $< \frac{1}{n}$, not contained in any member of $\mathcal{A}$. Denote these sets C_n and choose $x_n \in C_n$. Then for the sequence $\{x_n\}_{x_n \in C_n}$, there is a subsequence $\{y_m\}$ with $y_i = x_{n_i}$ converging to a point $a \in A$ for some $A \in \mathcal{A}$. Now, choose $\varepsilon > 0$ such that $B(a, \varepsilon) \subseteq A$, if i is sufficiently large such that $\frac{1}{n_i} < \frac{\varepsilon}{2}$. Then take $C_{n_i} \subseteq B(x_{n_i}, \frac{\varepsilon}{2})$, if i is also chosen such that $d(x_{n_i}, a) < \frac{\varepsilon}{2}$, then $C_{n_i} \subseteq B(a, \varepsilon)$ which makes $C_{n_i} \subseteq A$, a contradiction on the definition of the C_n. This forces $\mathcal{A}$ to satisfy the Lebesgue number lemma.

Now, suppose there exists an $\varepsilon > 0$ such that X cannot be covered by finitely many ε-balls. Define the sequence of points of X, $\{x_n\}$ by the following steps:

Step 1: Choose any $x_1 \in X$.

Step 2: Choose $x_2 \in X$ with $x_2 \notin B(x_1, \varepsilon)$.

Step 3: Proceed inductively and choose $x_n \in X$ with $x_n \notin \bigcup B(x_i, \varepsilon)$

Since the collection $\{B(x_i, \varepsilon)\}_{i=1}^{n}$ does not cover X, and $d(x_{n+1}, x_i) \geq \varepsilon$, then the sequence $\{x_n\}$ has no convergent subsequence. This contradicts that X is sequentially compact. So there must be a finite collection of ε-balls covering X.

Finally, by the above, we have that $\mathcal{A}$ has a Lebesgue number δ. Let $\varepsilon = \frac{\delta}{2}$, then by above, we can find a finite collection of ε-balls, $\{B(x_i, \varepsilon)\}$, covering X where $\operatorname{diam} B(x_i, \varepsilon) \leq \frac{2}{3}\delta$. So $B(x_i, \varepsilon) \subseteq A$ for some $A \in \mathcal{A}$, for each x_i, and so $\{B(x_i, \varepsilon)\}$ forms a finite subcover of X. Therefore X is compact. ■

3.8 Local Compactnes

Definition. A topological space X is said to be **locally compact** at a point x, if there exists a compact subspace C of X containing a neighborhood of x. If X is locally compact at each of its points, we call X **locally compact**.

Example 3.12. (1) $\mathbb{R}$ is locally compact, for every $x \in \mathbb{R}$ lies in an interval $(a, b) \subseteq [a, b]$, where $a, b \in \mathbb{R}$ and $a < b$. Since $[a, b]$ is closed, it is a compact subspace of $\mathbb{R}$ containing (a, b), a neighborhood of x.

(2) $\mathbb{R}^n$ is locally compact only for finite n, so $\mathbb{R}^\omega$ is not locally compact.

(3) Every simply ordered set with the least uppebound property is locally compact. Every basis of X is contained in a closed interval of X, which is compact.

Theorem 3.8.1. *Let X be a topological space. X is a locally compact Hausdorff space if, and only if there is a topological space Y such that:*

(1) X is a subspace of Y.

(2) $Y \backslash X$ is a singleton.

(3) Y is a compact Hausdorff space.

Proof. Suppose that X is a locally compact Hausdorff space, and let $Y = X \cup \{\infty\}$ where $\infty \notin X$. COnsider the collection of subsets of Y consisting of all U open in X together with all sets of the form $Y \backslash C$ where C is a compact subspace of X. This collection forms a topology on Y.

Now given V open in Y, either $V = U$, U open in X or $V = Y \backslash C$, C compact in X. Then whe have that $U \cap X = U$ which is open in X or $(Y \backslash C) \cap X = X \backslash C$ also open in X (since C is closed). Thus in either case, X is a subspace of Y.

Now, let $\mathcal{A}$ be an open cover of Y. then $\mathcal{A}$ contains sets of the form $V = U$, U open in X, and $V = Y \backslash C$, C compact in X. Now, none of the sets of the form $V = U$ contain the point ∞. So, take all members of $\mathcal{A}$ different from $Y \backslash C$ and intersect them with X. Then they form an open cover of C. By compactness of C, we have that finitely many of these members also cover C, then the corresponding finite cover of C together with all $Y \backslash C$ forms an finite open subcover of Y. Therefore, Y is compact.

Now, let $x, y \in Y$. If both $x, y \in X$, then since X is Hausdorff, there are disjoint neighborhoods U and V of x and y, respectively. Since X is a subspace, Y is Hausdorff consequently. Alternatively, if $x \in X$, and $y = \infty$, then choose a C compact in X containing a neighborhood U of x, and $V = Y \backslash C$. Then $x \in U$, $\infty \in V$ and $U \cap V = \emptyset$, therefore Y is Hausdorff. The result is the same for $y \in X$ and $x = \infty$.

Finally, conversely, suppose that Y is a topological space satisfying properties (1)-(3). Then since Y is Hausdorff, and X is a subspace of Y, then X is Hausdorff. Now, let $x \in X$ and choose U and V disjoint open sets of Y containing x and the point $\{p\} = Y \backslash X$, respectively. Take $C = Y \backslash V$, closed in Y. Then C is compact in Y, and since $C \subseteq X$, it is also compact in X. Then C contains the neighborhood U of x, making X locally compact. ∎

Definition. If Y is a compact Hausdorff space, and $X \subseteq Y$ is a proper subspace of Y, with $\operatorname{cl} X = Y$, then Y is called a **compactification** of X. If $Y \backslash X$ is a singleton, we call Y a **one-point compactification** of X.

Lemma 3.8.2. *Let X be a topological space, if Y and Y' are two one-point compactifications of X, then there is a homeomorphism $h : Y \to Y'$ where h is the identity map on X.*

Proof. Let X be a topological space, and let Y and Y' be one-point compactifications of X. define the map $h : Y \to Y'$ by taking $h(p) = q$ where $Y \backslash X = \{p\}$ and $Y' \backslash X = \{q\}$, and where h is the identity everywhere on X. Now, let U be open in Y. If $p \notin U$ then $U \subseteq X$ and $h(U) = U$. So $h(U)$ is open in X which is open in Y' so $h(U)$ is open in Y'.

Now, is $p \in U$, take $C = Y \backslash U$ closed in Y. Then C is compact in Y, moreover since $C \subseteq X$ (because $p \in U$), we have that C is compact in X, and hence compact in Y'. Since Y' is Hausdorff, then C is closed and $h(U) = Y' \backslash C$ which is open in Y'. In both cases, h is a homeomorphism. ∎

Remark. What this lemma states is that one-point compactifications of topological spaces are unique up to homeomorphism.

Example 3.13. The one-point compactification of $\mathbb{R}$ is homeomorphic to the unit circle S^1. The one-point compactification of $\mathbb{R}^2$ is the unit sphere S^2. The one-point compactification of $\mathbb{C}$ is denoted $\mathbb{C} \cup \infty$ and is called the **Riemann sphere**.

Theorem 3.8.3. *Let X be a Hausdorff space. Then X is locally compact if, and only if given $x \in X$, and a neighborhood U of x, there is a neighborhood V of x such that $\operatorname{cl} V \subseteq U$ and $\operatorname{cl} V$ is compact.*

Proof. First, we have that $\operatorname{cl} V$ is a compact subspace of X containing a neighborhood of x, this makes X locally compact.

Now suppose that X is locally compact. Then letting $x \in X$, and U a neighborhood of x, take the one-point compactification Y of X, and let $C = Y \backslash U$. Then C is closed in Y and hence compact in Y. Then by the corollary to theorem 3.4.3, choose V, W disjoint open sets of Y with $x \in V$ and $C \subseteq W$. Then $\operatorname{cl} V \subseteq Y$ is compact and $C \cap \operatorname{cl} V = \emptyset$, so that $\operatorname{cl} V \subseteq U$. ∎

Corollary. *If X is a locally compact Hausdorff space, and A is a subspace of X, then if A is open, or closed in X, A is locally compact in X.*

Proof. Suppose that A is open in X. Let $x \in A$, then choose a neighborhood V of x in X such that $\operatorname{cl} V \subseteq A$ and $\operatorname{cl} V$ is compact. Then $\operatorname{cl} V$ is a compact subspace of A containing V.

Now, if A is closed, let $x \in A$ and C a compact subspace of X containing a neighborhood U of x. Then $C \cap A$ is closed in C, and hence compact. It also contains the neighborhood $U \cap A$ of x. ■

Corollary. *A topological space X is homeomorphic to an open subspace of a compact Hausdorff space if, and only if X is a locally compact Hausdorff space.*

3.9 Nets

Bibliography

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