Geometry.

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Contents

Chapter 1

Hilbert's Axioms for Geometry.

1.1 The Incidence Axioms.

Definition. We define a set G of **points** together with a set of subsets of G which we call **lines** to be an **incidence geometry** if the following properites are satisfied:

(I1) For any two distinct points A, B, there is a unique line l containing A and B.

(I2) Every line contains at least two points.

(I3) There exist three noncolinear points. That is not all three points are contained in a line.

Proposition 1.1.1. *Two distinct lines can have at most one point in common.*

Proof. Let l, m be lines and suppose they have at least 2 points in common, A, B with $A \neq B$. Then by axiom (I1), there is a unique line containing both points, making $l = m$. ■

Example 1.1. 1 Consider the set of points in $\mathbb{R}^2$. Define a line to be a subset of points of $(x, y) \in \mathbb{R}^2$ such that $ax + by + c = 0$ for some $a, b, c \in \mathbb{R}$. Then $\mathbb{R}^2$ together with this collection of lines forms an incidence geometry.

2 Consider the finite set $G = \{A, B, C\}$. Then take the collection of lines to be the subsets:

$$\{A, B\}$$
$$\{A, C\}$$
$$\{B, C\}$$

Then G together with this collection forms an incidence geometry. It is the smallest possible incidence geomtry.

Definition. In any incidence geometry, we call two lines **parellel** if they contain no points in commonm, or they are equal.

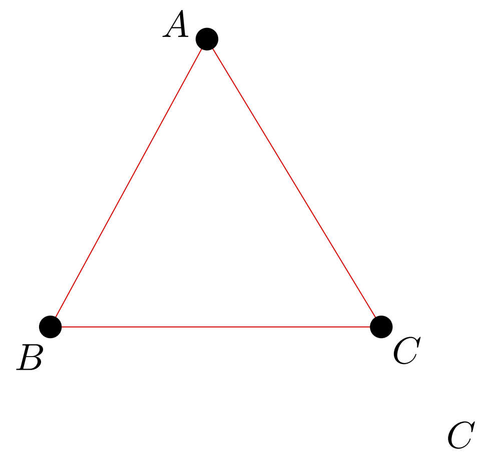

Figure 1.1: The incidence geometry on 3 points.

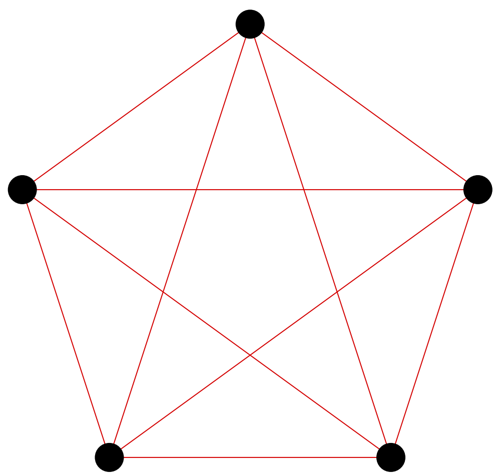

Figure 1.2: An incidence geometry on 5 points.

Example 1.2. The following figure 1.2 shows an incidence geometry on 5 points, and in which there exists parallel lines. Note that the following proposition is not satisfied in this geometry.

We have the following proposition which we accept without proof.

Proposition 1.1.2 (Pla$_yfair'sAxiom$). *For each point A in an incidence geometry, and a line l, there is at most one line passing through A and parallel to l.*

Definition. Let G and H be incidence geometries. We say that G and H are **isomorphic** if there exists a $1-1$ map $\phi : G \to H$ of G onto H such that if l is a line in G, then $\phi(l)$ is a line in H. We call ϕ an **isomorphism** and we write $G \simeq H$. If $H = G$, then we say that ϕ defines an **automorphism**.

Proposition 1.1.3. *The axioms (I1), (I2), (I3), and Playfair's axiom are all independent of one another.*

Proof. Example 1.2 describes a geometry in where the incidence axioms are satisfied, but Playfair's axiom is not.

Alternatively, taking a two element set $G = \{A, B\}$ with its set of lines being G satifies (I1), (I2), and Playfair's axiom, but not (I3).

For a geomtry satisfiying (I1), (I3) and Playfair's axiom take the geometry of example 1.1(2) with a line passing through the point A.

Finally for a geometry satisfying (I2), (I3), and Playfair's axiom, take $G = \{A, B, C\}$ with collection of lines the emptyset $\emptyset$. ■

Bibliography

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