

# Measure Theory

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**Text**

Real Analysis (4<sup>th</sup> edition)

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December 27, 2022



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# Chapter 1

## The Real Numbers

### 1.1 Open Sets, and $\sigma$ -Algebras

**Definition.** We call a set  $U$  of real numbers **open** provided for any  $x \in U$ , there is an  $r > 0$  such that  $(x - r, x + r) \subseteq U$ .

**Lemma 1.1.1.** *The set of real numbers  $\mathbb{R}$ , together with open sets defines a topology on  $\mathbb{R}$ .*

*Proof.* Notice that both  $\mathbb{R}$  and  $\emptyset$  are open sets. Moreover, if  $\{U_n\}$  is a collection of open sets, then so is their union. Now, consider the finite collection  $\{U_k\}_{k=1}^n$  and let  $U = \bigcap_{k=1}^n U_k$ . If  $U$  is empty, we are done. Otherwise, let  $x \in U$ . Then  $x \in U_k$  for every  $1 \leq k \leq n$ , and since each  $U_k$  is open, choose an  $r_k > 0$  for which  $(x - r_k, x + r_k) \subseteq U_k$ . Then let  $r = \min\{r_1, \dots, r_n\}$ . Then  $r > 0$ , and we have  $(x - r, x + r) \subseteq U$ , which makes  $U$  open in  $\mathbb{R}$ . ■

**Lemma 1.1.2.** *Every nonempty set is the disjoint union of a countable collection of open sets.*

*Proof.* Let  $U$  be nonempty and open in  $\mathbb{R}$ . Let  $x \in U$ . Then there is a  $y > x$  for which  $(x, y) \subseteq U$  and there is a  $z < x$  for which  $(z, x) \subseteq U$ . Now, let  $a_x = \inf\{z : (z, x) \subseteq U\}$  and  $b_x = \sup\{y : (x, y) \subseteq U\}$ , and let  $I_x = (a_x, b_x)$ . Then we have

$$x \in I_x \text{ and } a_x \notin I_x \text{ and } b_x \notin I_x$$

Let  $w \in I_x$  such that  $x < w < b_x$ . Then there is a  $y > w$  such that  $(x, y) \subseteq U$  so that  $w \in U$ . Now, if  $b_x \in U$ , then there is an  $r > 0$  for which  $(b_x - r, b_x + r) \subseteq U$ , in particular,  $(x, b_x + r) \subseteq U$ . But  $b_x$  is the least upperbound of all such numbers, and  $b_x < b_x + r$ , a contradiction. Thus  $b_x \notin U$ , and hence  $b_x \notin I_x$ . A similar argument shows that  $a_x \notin I_x$ .

Consider now the collection  $\{I_x\}_{x \in U}$ . Then  $U = \bigcup I_x$  and since  $a_x, b_x \notin I_x$  for each  $x$ , the collection  $\{I_x\}$  is a disjoint collection. Lastly, by the density of  $\mathbb{Q}$  in  $\mathbb{R}$  there is a 1-1 mapping between this collection and  $\mathbb{Q}$ , making it countable. ■

**Definition.** Let  $E \subseteq \mathbb{R}$  a set. We call a point  $x \in \mathbb{R}$  a **point of closure** of  $E$  if every open interval containing  $x$  also contains a point of  $E$ . We call the collection of all such points the **closure** of  $E$ , and denote it  $\text{cl } E$ . If  $E = \text{cl } E$ , then we say that  $E$  is **closed**.

**Lemma 1.1.3.** *For any set  $E$  of real numbers,  $\text{cl } E$  is closed; i.e.  $\text{cl } E = \text{cl}(\text{cl } E)$ . Moreover,  $\text{cl } E$  is the smallest closed set containing  $E$ .*

**Lemma 1.1.4.** *Every set  $E$  of real numbers is open if, and only if  $\mathbb{R} \setminus E$  is closed.*

**Definition.** Let  $E \subseteq \mathbb{R}$  a set. We call a collection  $\{E_\lambda\}$  a **cover** of  $E$  if  $E \subseteq \bigcup E_\lambda$ . If each  $E_\lambda$  is open, then we call this collection an **open cover** of  $E$ .

**Theorem 1.1.5** (Heine-Borel). *For any closed and bounded set  $F$  of  $\mathbb{R}$ , every open cover of  $F$  has a finite subcover.*

*Proof.* Suppose first that  $F = [a, b]$ , for  $a \leq b$  real numbers. Then  $F$  is closed and bounded. Let  $\mathcal{F}$  be an open cover of  $[a, b]$ , and define  $E = \{x \in [a, b] : [a, x] \text{ has a finite subcover by sets in } \mathcal{F}\}$ . Notice that  $a \in E$ , so that  $E$  is nonempty. Now, since  $E$  is bounded by  $b$ , by the completeness of  $\mathbb{R}$ , let  $c = \sup \{E\}$ . Then  $c \in [a, b]$  and there is a set  $U \in \mathcal{F}$  with  $c \in U$ . Since  $U$  is open, there exists an  $\varepsilon > 0$  such that  $(c - \varepsilon, c + \varepsilon) \subseteq U$ . Now,  $c - \varepsilon$  is not an upperbound of  $E$ , so there is an  $x \in E$  with  $c - \varepsilon < x$ , and a finite collection of open sets  $\{U_i\}_{i=1}^k$  covering  $[a, x]$ . Then the collection  $\{U_i\}_{i=1}^k \cup U$  covers  $[a, x]$  so that  $c = b$ , and we have found a finite subcover of  $F$ .

Now, let  $F$  be closed and bounded. Then it is contained in a closed bounded interval  $[a, b]$ . Now, let  $U = \mathbb{R} \setminus F$  open and  $\mathcal{F}$  an open cover of  $F$ . Let  $\mathcal{F}' = \mathcal{F} \cup U$ . Since  $\mathcal{F}$  covers  $F$ ,  $\mathcal{F}'$  covers  $[a, b]$ . By above, there is a finite subcover of  $[a, b]$ , and hence of  $F$  by sets in  $\mathcal{F}'$ . Remove  $U$  from  $\mathcal{F}'$ , we get a finite subcover of  $F$  by sets in  $\mathcal{F}$ . ■

**Theorem 1.1.6** (The Nested Set Theorem). *Let  $\{F_n\}$  be a descending collection of nonempty closed sets of  $\mathbb{R}$ , for which  $F_1$  is bounded. Then*

$$\bigcap F_n \neq \emptyset$$

*Proof.* Let  $F = \bigcap F_n$ , and suppose to the contrary that  $F$  is empty. Then for all  $x \in \mathbb{R}$ , there is an  $n \in \mathbb{Z}^+$  for which  $x \notin F_n$ . So that  $x \in U_n = \mathbb{R} \setminus F_n$ . Then  $U_n = \mathbb{R}$ , and each  $U_n$  is open. So  $\{U_n\}$  is an open cover of  $\mathbb{R}$ , and hence  $F_1$ . By the theorem of Heine-Borel, there is an  $N > 0$  such that  $F \subseteq \bigcup_{n=1}^N U_n$ . Since  $\{F_n\}$  is descending, the collection  $\{U_n\}$  is ascending, and hence  $\bigcup U_n = U_N = \mathbb{R} \setminus F_N$  which makes  $F_1 \subseteq \mathbb{R} \setminus F_N$ , a contradiction. ■

**Definition.** Let  $X$  be a set. We call a collection  $\mathcal{A}$  of subsets of  $X$   **$\sigma$ -algebra** if

- (1)  $\emptyset \in \mathcal{A}$ .
- (2) For any  $A \in \mathcal{A}$ ,  $X \setminus A \in \mathcal{A}$ .
- (3) If  $\{A_n\}$  is a countable collection of elements of  $\mathcal{A}$ , then their union is an element of  $\mathcal{A}$ .

**Lemma 1.1.7.** *Let  $\mathcal{F}$  a collection of subsets of a set  $X$ . The intersection of all  $\sigma$ -algebras containing  $\mathcal{F}$  is a  $\sigma$ -algebra. Moreover, it is the smallest such  $\sigma$ -algebra.*

**Definition.** We define the **Borel sets** of  $\mathbb{R}$  to be the  $\sigma$ -algebra of  $\mathbb{R}$  containing all open sets in  $\mathbb{R}$ .

**Lemma 1.1.8.** *Every closed set of  $\mathbb{R}$  is a Borel set.*

**Definition.** We call a countable intersection of open sets of  $\mathbb{R}$  a  **$G_\delta$ -set** and we call a countable union of closed sets of  $\mathbb{R}$  an  **$F_\sigma$ -set**.

## 1.2 Sequences of Real Numbers

**Definition.** A sequence  $\{a_n\}$  of real numbers is said to **converge** to a point  $a$ , if, for every  $\varepsilon > 0$ , there is an  $N > 0$  such that

$$|a - a_n| < \varepsilon \text{ whenever } n \geq N$$

We call  $a$  the **limit** of  $\{a_n\}$  and write  $\{a_n\} \rightarrow a$ , or

$$\lim_{n \rightarrow \infty} \{a_n\} = a$$

**Lemma 1.2.1.** *Let  $\{a_n\} \rightarrow a$  a sequence of real numbers converging to  $a \in \mathbb{R}$ . Then the limit of  $\{a_n\}$  is unique,  $\{a_n\}$  is bounded, and for any  $c \in \mathbb{R}$ , if  $a_n \leq c$  for all  $n$ , then  $a \leq c$ .*

**Theorem 1.2.2** (The Monoton C Vonvergence Theorem). *A monotone sequence of real numbers converges to a point if, and only if it is bounded.*

*Proof.* Without loss of generality, suppose that the sequence  $\{a_n\}$  is increasing. If  $\{a_n\} \rightarrow a$ , by lemma 1.2.1,  $\{a_n\}$  is bounded. On the otherhand, suppose that  $\{a_n\}$  is bounded. Let  $S = \{a_n : n \in \mathbb{Z}^+\}$ , then by the completeness of  $\mathbb{R}$ , let  $a = \sup S$ . Let  $\varepsilon > 0$ . Notice that  $a_n \leq a$  for all  $n$ . Now, since  $a - \varepsilon$  is not an upperbound, there exists an  $N > 0$  for which  $a_N > a - \varepsilon$ , then since  $\{a_n\}$  is increasing,  $a_n > a - \varepsilon$  whenever  $n \geq N$ . So we get

$$|a - a_n| < \varepsilon \text{ whenever } n \geq N$$

Which makes  $\{a_n\} \rightarrow a$ . ■

**Theorem 1.2.3** (Bolzano-Weierstrass). *Every bounded sequence has a convergent subsequence.*

*Proof.* Let  $\{a_n\}$  be a bounded sequence, and let  $M > 0$  such that  $|a_n| \leq M$  for all  $n \in \mathbb{Z}^+$ . Define  $E_n = \text{cl}\{a_j : j \geq n\}$ . Then  $E_n \subseteq [-M, M]$ . Thus  $\{E_n\}$  is a decreasing sequence of closed, bounded, and nonempty sets of  $\mathbb{R}$ . By the nested set theorem, the intersection  $E = \bigcap E_n$  is nonempty. Choose an  $a \in E$ . Then for every  $k \in \mathbb{Z}^+$ ,  $a$  is a point of closure of the set  $\{a_j : j \geq k\}$ . SO that  $a_j \in (a - \frac{1}{k}, a + \frac{1}{k})$  whenever  $j \geq k$ . By induction, construct a strictly increasing sequence  $\{n_k\}$  of natural numbers for which  $|a - a_{n_k}| < \frac{1}{k}$ . Then by the principle of Archimedes,  $\{a_{n_k}\} \rightarrow a$ , and we have a convergent subsequence. ■

**Definition.** We call a sequence  $\{a_n\}$  **Cauchy** if for every  $\varepsilon > 0$ , there is an  $N > 0$  for which

$$|a_m - a_n| < \varepsilon \text{ whenever } m, n \geq N$$

**Theorem 1.2.4** (The Cauchy Convergence Criterion). *A sequence of real numbers converges if, and only if it is Cauchy.*

*Proof.* Suppose that the sequence  $\{a_n\} \rightarrow a$  converges to  $a \in \mathbb{R}$ . Then for any  $m, n \in \mathbb{Z}^+$ , notice that  $|a_m - a_n| \leq |a_m - a| + |a - a_n|$ . Let  $\varepsilon > 0$  and choose  $N > 0$  such that  $|a - a_n| < \frac{\varepsilon}{2}$ , and  $|a_m - a| < \frac{\varepsilon}{2}$ . Then if  $n, m \geq N$ , we get  $|a_m - a_n| < \varepsilon$ , which makes  $\{a_n\}$  Cauchy.

Conversely, suppose that  $\{a_n\}$  is Cauchy. Let  $\varepsilon = 1$  and choose  $N > 0$  such that if  $m, n \geq N$ , then  $|a_m - a_n| < 1$ . Then we get  $|a_n| \leq 1 + |a_N|$  for all  $n \geq N$ . Define  $M = 1 + \max\{|a_1|, \dots, |a_N|\}$ . Then  $|a_n| \leq M$  for all  $n$ . This makes  $\{a_n\}$  bounded. By the theorem of Bolzano-Weierstrass,  $\{a_n\}$  has a convergent subsequence  $\{a_{n_k}\} \rightarrow a$ . Let  $\varepsilon > 0$ , since  $\{a_n\}$  is Cauchy, choose an  $N > 0$  such that  $|a_m - a_n| < \frac{\varepsilon}{2}$  whenever  $n, m \geq N$ . Likewise, we get  $|a - a_{n_k}| < \frac{\varepsilon}{2}$  and  $n_k \geq N$ . Thus we observe that  $|a_n - a| \leq |a_n - a_{n_k}| + |a - a_{n_k}| < \varepsilon$  and so  $\{a_n\} \rightarrow a$ . ■

**Theorem 1.2.5.** *Let  $\{a_n\} \rightarrow a$  and  $\{b_n\} \rightarrow b$  be convergent sequences. Then for any  $\alpha, \beta \in \mathbb{R}$ , we have that the sequence  $\{\alpha a_n + \beta b_n\}$  converges and that*

$$\lim_{n \rightarrow \infty} \{\alpha a_n + \beta b_n\} = \alpha a + \beta b$$

**Definition.** We say a sequence  $\{a_n\}$  of real numbers **converges to infinity**  $\infty \in \mathbb{R}_\infty$  if for every  $c \in \mathbb{R}$ , there is an  $N > 0$  such that  $a_n \geq c$  whenever  $n \geq N$ . We write  $\{a_n\} \rightarrow \infty$ , or

$$\lim_{n \rightarrow \infty} \{a_n\} = \infty$$

**Definition.** Let  $\{a_n\}$  be a sequence of real numbers. We define the **limit superior** of  $\{a_n\}$  to be

$$\limsup \{a_n\} = \lim_{n \rightarrow \infty} (\sup \{a_k : k \geq n\})$$

Similarly, we define the **limit inferior** of  $\{a_n\}$  to be

$$\liminf \{a_n\} = \lim_{n \rightarrow \infty} (\inf \{a_k : k \geq n\})$$

**Theorem 1.2.6.** *For any sequences  $\{a_n\}$  and  $\{b_n\}$  of real numbers, the following are true:*

- (1)  $\limsup \{a_n\} = l \in \mathbb{R}_\infty$  if, and only if for every  $\varepsilon > 0$ , there exists infinitely many  $n \in \mathbb{Z}^+$  such that  $a_n > l - \varepsilon$  and finitely many  $n \in \mathbb{Z}^+$  for which  $a_n > l + \varepsilon$ .
- (2)  $\limsup \{a_n\} = \infty$  if, and only if  $\{a_n\}$  is not bounded above.
- (3)  $\limsup \{a_n\} = -\liminf \{-a_n\}$
- (4)  $\{a_n\} \rightarrow a \in \mathbb{R}_\infty$  if, and only if  $\limsup \{a_n\} = \liminf \{a_n\}$ .
- (5) If  $a_n \leq b_n$  for all  $n$ , then  $\limsup \{a_n\} \leq \limsup \{b_n\}$ .

**Definition.** Let  $\{a_n\}$  a sequence of real numbers. We call the series  $\sum_{k=1}^{\infty} a_k$  **summable** if the sequence of partial sums  $\{s_n = \sum_{k=1}^n a_k\} \rightarrow s$  converges to a point  $s \in \mathbb{R}$ .

**Lemma 1.2.7.** *Let  $\{a_n\}$  a sequence of real numbers. Then the following are true.*

- (1) The series  $\sum a_k$  is summable if, and only if for every  $\varepsilon > 0$ , there is an  $N > 0$  such that

$$\left| \sum_{k=n}^{n+m} a_k \right| < \varepsilon \text{ for all } m \in \mathbb{Z}^+ \text{ whenever } n \geq N$$

- (2) If  $\sum |a_k|$  is summable, then so is  $\sum a_k$ .
- (3) If  $a_k \geq 0$ , then  $\sum a_k$  is summable if, and only if the sequence of partial sums  $\{s_n\}$  is bounded.



## 1.3 Continuous Functions of a Real Variable.

**Definition.** A realvalued function  $f$  on a domain  $E$  is said to be **continuous** at a point  $x \in E$  provided for any  $\varepsilon > 0$  there is a  $\delta > 0$  for which

$$|f(x) - f(y)| < \varepsilon \text{ whenever } |x - y| < \delta \text{ for any } y \in E$$

We call  $f$  **continuous** on  $E$  if it is continuous at every point in  $E$ . We call  $f$  **Lipschitz continuous** if there is a  $c \geq 0$  for which

$$|f(x) - f(y)| \leq c|x - y| \text{ for all } x, y \in E$$

**Lemma 1.3.1.** *A Lipschitz continuous function on a domain is continuous on that domain.*

**Lemma 1.3.2** (The Sequential Criterion). *A realvalued function  $f$  defined on a domain  $E$  is continuous at a point  $x \in E$  if, and only if for any ssequence  $\{x_n\} \rightarrow x$  of points in  $E$ , converging to  $x$ , that the sequence  $\{f(x_n)\} \rightarrow f(x)$  converges to  $f(x)$ .*

**Theorem 1.3.3** (The Extreme Value Theorem). *A continuous realvalued function defined on a nonempty, closed and bounded domain takes on a maximum value, and a minimum value on that domain.*

*Proof.* Let  $f$  be a continuous realvalued function defined on the domain  $E$ , where  $E$  is nonempty, closed, and bounded. Let  $x \in E$  and  $\delta > 0$  and  $\varepsilon = 1$ . Define the open interval  $I_x = (x - \delta, x + \delta)$ . Then if  $y \in E \cap I_x$ , then  $|f(x) - f(y)| < 1$ . So that  $|f(y)| \leq |f(x)| + 1$ . Notice also that the collection  $\{I_x\}$  is an open cover of  $E$ . By the theorem of Heine-Borel, there is a finite subcover of  $E$ ,  $\{I_{x_k}\}_{k=1}^n$ . Define, then,  $M = 1 + \max \{|f(x_1)|, \dots, |f(x_n)|\}$ . Then we get that  $|f(x)| \leq M$  and  $f$  is bounded.

Now, let  $m = \sup f(E)$ . If  $f$  does not take the value  $m$  for any points in  $E$ , then the function  $x \rightarrow \frac{1}{f(x)-m}$  is a continuous unbounded function on  $E$ ; which is impossible. So there is an  $x \in E$  with  $f(x) = m$  and  $m$  is a maximum value. Now, since  $f$  is continuous, then so is  $-f$ , and hence  $-m$  defines a minimum value on  $f$ . ■

**Theorem 1.3.4** (The Intermediate Value Theorem). *If  $f$  is a continuous realvalued function on a closed bounded interval  $[a, b]$ , for which  $f(a) < c < f(b)$ , then there exists an  $x_0 \in (a, b)$  for which  $f(x_0) = c$ .*

*Proof.* Define  $a_1 = a$  and  $b_1 = b$  and let  $m_1$  be the midpoint of the interval  $[a_1, b_1]$ . If  $c < f(m_1)$ , define  $a_2 = a_1$  and  $b_2 = m_1$ , otherwise define  $a_2 = m_1$  and  $b_2 = b_1$ , so that in either case we get  $f(a_2) \leq c \leq f(b_2)$  and  $b_2 - a_2 = \frac{b-a}{2}$ . By induction, construct the collection of closed bounded intervals  $\{[a_n, b_n]\}$  such that  $f(a_n) \leq c \leq f(b_n)$  and  $b_n - a_n = \frac{b-a}{2^{n-1}}$ . This collection is a descending collection, so by the nested set theorem, the intersection  $I = \bigcap [a_n, b_n]$  is nonempty. Choose an  $x_0 \in I$ , and observe that

$$|a_n - x_0| \leq b_n - a_n = \frac{b-a}{2^{n-1}}$$

So the sequence  $\{a_n\} \rightarrow x_0$ . By the sequential criterion, since  $f$  is continuous at  $x_0$ , we get the sequence  $\{f(a_n)\} \rightarrow f(x_0)$ . Since  $f(a_n) \leq c$ , and  $(-\infty, c]$  is closed, we also get  $f(x_0) \leq c$ .

By similar reasoning to the argument provided above, we also get that  $f(x_0) \geq c$  so that equality is established. ■

**Definition.** A realvalued function  $f$  on a domain  $E$  is said to be **uniformly continuous** if for every  $\varepsilon > 0$  there is a  $\delta > 0$  such that

$$|f(x) - f(y)| < \varepsilon \text{ whenever } |x - y| < \delta \text{ for all } x, y \in E$$

**Lemma 1.3.5.** *If  $f$  is a uniformly continuous function on a domain  $E$ , then it is continuous on  $E$ .*

**Theorem 1.3.6.** *A continuous realvalued function on a closed and bounded domain is uniformly continuous.*

*Proof.* Let  $f$  be continuous on  $E$ , and  $E$  a closed and bounded domain. Let  $\varepsilon > 0$ . For every  $x \in E$ , there is a  $\delta_x > 0$  for which  $|f(x) - f(y)| < \varepsilon$  whenever  $|x - y| < \delta_x$  for some  $y \in E$ . Define  $I_x = (x - \frac{\delta_x}{2}, x + \frac{\delta_x}{2})$ . Then  $\{I_x\}$  is an open cover for  $E$ , so that by the theorem of Heine-Borel, there is a finite subcover  $\{I_{x_k}\}_{k=1}^n$  of  $E$ . Define  $\delta = \frac{1}{2} \min \{\frac{\delta_{x_1}}{2}, \dots, \frac{\delta_{x_n}}{2}\}$ . Then  $\delta > 0$  moreover, if  $x, y \in E$ , with  $|x - y| < \delta$ , then asince  $\{I_{x_k}\}$  covers  $E$ , there is a  $k > 0$  such that

$$|x - x_k| < \frac{\delta_{x_k}}{2} \text{ and } |x_{x_k} - y| < \frac{\delta_{x_k}}{2}$$

Then we have  $|f(x) - f(x_k)| < \frac{\varepsilon}{2}$  and  $|f(x_k) - f(y)| < \frac{\varepsilon}{2}$  so that  $|f(x) - f(y)| < \varepsilon$ , which makes  $f$  unifomrly continuous. ■

# Chapter 2

## Lebesgue Measure

### 2.1 Lebesgue Outermeasure



# Bibliography

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