Category Theory

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Contents

Chapter 1

Categories and Functors.

1.1 Metacategories

Definition. A **metagraph** is a collection of objects together with a collection of **arrows** between objects, and operations dom and cod such that if f is an arrow, then $a = \operatorname{dom} f$ and $b = \operatorname{cod} f$ are objects. We write $f : a \to b$, and call a the **domain**, and b the **codomain**.

Definition. A **metacategory** is a metagraph together with an operation $\circ$, called **composition**, and arrows Id_a corresponding to each object a, such that

(1) $\circ$ corresponds the pair of arrows (g, f) to the arrow $g \circ f$, called the **composition**, where $\operatorname{cod} f = \operatorname{dom} g$; i.e. if $f : a \to b$, and $g : b \to c$, then $g \circ f : a \to c$.

(2) $\operatorname{dom} \mathrm{Id}_a = \operatorname{cod} \mathrm{Id}_a = a$. Moreover, if $f : a \to b$, and $g : b \to c$, then $\mathrm{Id}_b \circ f = f$ and $g \circ \mathrm{Id}_b = g$.

(3) If $f : a \to b$, $g : b \to c$, and $h : c \to d$ are arrows, then the compositions $k \circ (g \circ f)$ and $(k \circ g) \circ f$ are equal whenever one of them is defined.

Definition. A **diagram** is a directed graph whos vertices consists of objects of a metacategory, and whos edjes are arrows between those objects. We say a diagram **commutes** if for any pair c, c' of vertices, any two paths from c to c' gives, by composition of the labels, equal arrows.

Example 1.1. (1) We represent arrows of metagraphs pictoraly by the diagram

$$a \xrightarrow{f} b$$

(2) The following is a diagram of the composition law of a metacategory

$$\begin{tikzcd} a \arrow[d, "f"'] \arrow[dr, "g \circ f"] & \\ b \arrow[r, "g"'] & c \end{tikzcd}$$

Moreover, the associative law, and identity laws of metagategories are represented by diagrams as

$$\begin{tikzcd} a \arrow[r, "h\circ(g\circ f)=(h\circ g)\circ f"] \arrow[d, "f"'] \arrow[dr, "g\circ f"] & d \\ b \arrow[r, "g"'] \arrow[ur, "h\circ g"] & c \arrow[u, "h"'] \end{tikzcd} \qquad \begin{tikzcd} a \arrow[r, "f"] \arrow[dr, "f"'] & b \arrow[d, dashed, "\mathrm{Id}_b"] \arrow[dr, "g"] & \\ & b \arrow[r, "g"'] & c \end{tikzcd}$$

(3) The metacategory of sets has as objects all sets, and as arrows, all functions. The identity arrow is the identity function, and the composition law is the usual function composition.

Definition. An **arrows-only metacategory** is a collection of arrows together with pairs of arrows called **composable pairs** and an operation assigning the composable pair (g, f) to the arrow gf such that if $(hg)f$ and $h(gf)$ are defined, then they are equal, and moreover hgf is defined whenever hg and gf are defined. We define the **identity arrow** to be the arrow u such that $fu = f$ and $ug = g$, whenever defined. Moreover, for every arrow g, there are arrows u, u' such that $u'g$ and gu are defined.

1.2 Categories

Bibliography

[1] S. Mac Lane, *Categories for the Working Mathematician.* New York: Springer, 1998.