Complex Analysis

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Contents

Chapter 1

The Complex Numbers

1.1 The Field of Complex Numbers

Definition. We define the set of **complex numbers** to be the collection of all ordered pairs of real numbers $\mathbb{C} = \{(a, b) : a, b \in \mathbb{R}\}$ together with the binary operations $+$ and $\cdot$ of **complex addition**, and **complex multiplication**, respectively defined by the rules

$$(a, b) + (c, d) = (a + c, b + d)$$
$$(a, b)(c, d) = (ac - bd, bc + ad)$$

Theorem 1.1.1. *The set of complex numbers $\mathbb{C}$ forms a field together with complex addition and complex multiplication.*

Corollary. *$\mathbb{C}$ is a field extension of the real numbers $\mathbb{R}$.*

Proof. The map $a \to (a, 0)$ from $\mathbb{R} \to \mathbb{C}$ defines an imbedding of $\mathbb{R}$ into $\mathbb{C}$. ∎

Definition. We define the element $i = (0, 1)$ of $\mathbb{C}$ so that $i^2 = -1$, and the polynomial z^2+1 has as root i. We write $(a, b) = a + ib$. If $z = a + ib$, we call a the **real part** of z, and b the **imaginary part** of z and write $\operatorname{Re} z = a$ and $\operatorname{Im} z = z$.

Definition. Let $z = a+ib \in \mathbb{C}$. We define the **norm** (or **modulus**) of z to be $\|z\| = \sqrt{a^2 + b^2}$. We define the complex **conjugate** of z to be $\overline{z} = a - ib$.

Lemma 1.1.2. *For every $z \in \mathbb{C}$, $\|z\|^2 = z\overline{z}$.*

Proof. Let $z = a+ib$. Then $\overline{z} = a-ib$, and so $z\overline{z} = (a+ib)(a-ib) = a^2+b^2 = (\sqrt{a^2 + b^2})^2 = \|z\|^2$. ∎

Corollary. *If $z \neq 0$, then $z^{-1} = \frac{1}{z} = \frac{\overline{z}}{\|z\|^2}$.*

Proof. The relation follows from above, and it remains to show that it is indeed the inverse element. Notice that if $z \in \mathbb{C}$ is nonzero, then $z\frac{\overline{z}}{\|z\|^2} = \frac{z\overline{z}}{\|z\|^2} = \frac{\|z\|^2}{\|z\|^2} = 1$. ∎

Example 1.1. (1) Let $z = a + ib$. Then we get that $\frac{1}{z} = \frac{\overline{z}}{\|z\|}$ has real part $\operatorname{Re}\frac{1}{z} = \frac{a}{a^2+b^2}$ and imaginary part $\operatorname{Im}\frac{1}{z} = -\frac{b}{a^2+b^2}$.

(2) Let $z = a + ib$, and $c \in \mathbb{R}$. Then $\frac{z-c}{z+c} = \operatorname{Re} \frac{z-c}{z+c}$, so $\operatorname{Im} \frac{z-c}{z+c} = 0$.

(3) Let $z = a + ib$, then $z^3 = a^3 - 3ab^2 + i(3a^2b - b^3)$ So that $\operatorname{Re} z^3 = a^3 - 3ab^2$ and $\operatorname{Im} z = 3a^2b - b^3$.

(4) $\frac{3+i5}{1+i7} = \frac{19}{25} - i\frac{18}{25}$.

(5) $(-\frac{1}{2} + i\frac{\sqrt{3}}{2})^3 = 1$, and hence $(-\frac{1}{2} + i\frac{\sqrt{3}}{2})^6 = 1$.

(6) Notice that $i^n = 1, i, -1, -i$ whenever $n \equiv 0 \mod 4$, $n \equiv 1 \mod 4$, $n \equiv 2 \mod 4$, and $n \equiv 3 \mod 4$. respectively.

(7) $\| -2 + i\| = \sqrt{5}$, and $\|(2 + i)(4 + i3)\| = \|5 + i10\| = 5\sqrt{5}$.

Lemma 1.1.3. *The following are true for all $z, w \in \mathbb{C}$.*

(1) $\operatorname{Re} z = \frac{1}{2}(z + \overline{z})$ *and* $\operatorname{Im} z = \frac{1}{2i}(z - \overline{z})$.

(2) $\overline{(z + w)} = \overline{z} + \overline{w}$ *and* $\overline{zw} = \overline{z}\ \overline{w}$.

(3) $\|\overline{z}\| = \|z\|$.

Proof. Let $z = a + ib$ and $w = c + id$. Then notice that

$$\frac{(a + ib) + (a - ib)}{2} = \frac{2a + (ib - ib)}{2} = \frac{2a}{2} = a = \operatorname{Re} z$$

and

$$\frac{(a + ib) - (a - ib)}{2i} = \frac{(a - a) + 2ib}{2} = \frac{2ib}{2i} = b = \operatorname{Im} z$$

Moreover

$$\overline{(a + ib) + (c + id)} = \overline{(a + c) + i(b + d)} = (a + c) - i(b + d) = (a - ib) + (c - id)$$

And

$$\overline{(a + ib)(c + id)} = \overline{(ac - bd) + i(bc + ad)} = (ac - bd) - i(bc + ad) = (a - ib)(c - id)$$

so that $\overline{z + w} = \overline{z} + \overline{w}$ and $\overline{zw} = \overline{z}\ \overline{w}$.

Now, we have that $\|zw\|^2 = (zw)\overline{zw} = (zw)(\overline{z}\ \overline{w}) = (z\overline{z})(w\overline{w}) = \|z\|^2\|w\|^2$. Taking square roots, we get the result

$$\|zw\| = \|z\|\|w\|$$

Finally, notice that $\|z\|^2 = z\overline{z} = \overline{z}\ \overline{\overline{z}} = \|\overline{z}\|$. ■

Corollary. *The following are also true; provided $w \neq 0$.*

(1) $\overline{(\frac{z}{w})} = \frac{\overline{z}}{\overline{w}}$.

(2) $\|\frac{z}{w}\| = \frac{\|z\|}{\|w\|}$

Corollary. *If $z = z_1 + \cdots + z_n$, and $w = w_1 \ldots w_n$, with $z_i, w_i \in \mathbb{C}$ for all $1 \leq i \leq n$, then*

(1) $\overline{z} = \overline{z_1} + \cdots + \overline{z_n}$.

(2) $\|w\| = \|w_1\| \ldots \|w_n\|$.

Proof. We prove both results by induction on n. For $n = 2$, we have already shown that $\overline{z} = \overline{z_1} + \overline{z_2}$ and $\|w\| = \|w_1\|\|w_2\|$. Now, for all $n \geq 2$, suppose that both

$$\overline{z} = \overline{z_1} + \cdots + \overline{z_n}$$
$$\|w\| = \|w_1\| \ldots \|w_n\|$$

Then let $z' = z + z_{n+1}$ and $w' = ww_{n+1}$ for $z_{n+1}, w_{n+1} \in \mathbb{C}$. Then we have that

$$z' = z + z_{n+1} = z_1 + \cdots + z_n + z_{n+1}$$
$$w' = ww_{n+1} = w_1 \ldots w_n w_{n+1}$$

so by the induction hypothesis, we have

$$\overline{z'} = \overline{(z + z_{n+1})} = \overline{z} + \overline{z_{n+1}} = \overline{z_1} + \cdots + \overline{z_n} + \overline{z_{n+1}}$$

and that

$$\|w'\| = \|ww_{n+1}\| = \|w\|\|w_{n+1}\| = \|w_1\| \ldots \|w_n\|\|w_{n+1}\|$$

which completes the proof. ■

Lemma 1.1.4. *Let $z \in \mathbb{C}$. Then z is a real number if, and only if $z = \overline{z}$.*

Proof. If z is real, then $z = a + i0$, for some $a \in \mathbb{R}$, and hence $\overline{z} = a - i0 = z$. COnversely, suppose that $z = \overline{z}$. Then we have

$$\operatorname{Re} z = \frac{1}{2}(z + \overline{z}) = \frac{1}{2}(z + z) = z$$

so z has only a real part, and hence must be a real number. ■

Lemma 1.1.5. *The following are true for all $z, w \in \mathbb{C}$.*

(1) $\|z + w\|^2 = \|z\|^2 + 2 \operatorname{Re} z\overline{w} + \|w\|^2$.

(2) $\|z - w\|^2 = \|z\|^2 - 2 \operatorname{Re} z\overline{w} + \|w\|^2$.

(3) $\|z + w\|^2 + \|z - w\|^2 = 2(\|z\|^2 + \|w\|^2)$.

Proof. We first notice that $\|z+w\|^2 = (z+w)\overline{(z + w)} = (z+w)(\overline{z}+\overline{w}) = z\overline{z}+z\overline{w}+w\overline{z}+w\overline{w} = \|z\|^2 + z\overline{w} + w\overline{z} + \|w\|^2$. Now, let $z = a + ib$ and $w = c + id$. Then we have

$$(a + ib)(c - id) = (ac + bd) - i(ad - bc)$$
$$(c + id)(a - ib) = (ac + bd) + i(ad - bc)$$

so that $z\overline{w} + w\overline{z} = 2(ac + bd) = 2\operatorname{Re} z\overline{w}$, and we are done. To get the identity for $\|z - w\|^2$, we simply replace w by $-w$, and use the above argument.

Now, we have that $\|z+w\|^2 = \|z^2\|+2\operatorname{Re} z\overline{w}+\|w\|^2$, and $\|z-w\|^2 = \|z^2\|-2\operatorname{Re} z\overline{w}+\|w\|^2$, so that adding them together, the terms $2\operatorname{Re} z\overline{w}$ cancel out and we are left with

$$\|z + w\|^2 + \|z - w\|^2 = 2(\|z\|^2 + \|w\|^2)$$

■

Lemma 1.1.6. *Let $R(z) \in \mathbb{C}(z)$ a rational function in z. Then if R has coefficients in $\mathbb{R}$, then $\overline{R(z)} = R(\overline{z})$.*

Proof. We first observe the polynomial $f \in \mathbb{C}[z]$, of finite degree $\deg f = n$, and of the form

$$f(z) = a_0 + a_1 z + \cdots + a_n z^n$$

Then if f has all coefficients in $\mathbb{R}$; i.e. $f \in \mathbb{R}[z]$, where $z \in \mathbb{C}$ is treated as indeterminant, then we have that since each $a_i \in \mathbb{R}$, then $\overline{a_i z^i} = \overline{a_i}\overline{z^i} = a_i \overline{z}^i$. So that

$$\overline{f(z)} = \overline{(a_0 + a_1 z + \cdots + a_n z^n)} = a_0 + a_1\overline{z} + \cdots + a_n \overline{z}^n$$

which makes $\overline{f(z)} = f(\overline{z})$. Now, one can also extend f to a polynomial of infinite degree by taking $n \to \infty$, and the same holds.

Now, let $R(z) \in \mathbb{C}(z)$ a rational function. Recall that $R(z)$ is of the form

$$R(z) = \frac{f(z)}{g(z)} \text{ with } g \neq 0$$

for some polynomials $f, g \in \mathbb{C}[z]$. Then if R has all real coefficients, so do f and g, and hence we get

$$\overline{R(z)} = \frac{\overline{f(z)}}{\overline{g(z)}} = \frac{f(\overline{z})}{g(\overline{z})} = R(\overline{z})$$

which completes the proof. ■

1.2 The Complex Plane

Definition. We define the **complex plane** to be the space of points (x, y) of $\mathbb{R}^2$ for which $z = x + iy$.

Lemma 1.2.1. *For every $z, w \in \mathbb{C}$ $\|z + w\| \leq \|z\| + \|w\|$.*

Proof. Observe that $-\|z\| \leq \operatorname{Re} z \leq \|z\|$ for all $z \in \mathbb{C}$, so that $\operatorname{Re} z\overline{w} \leq \|z\overline{w}\| = \|z\|\|w\|$. So we get

$$\|z + w\|^2 = \|z\|^2 + \operatorname{Re} z\overline{w} + \|\overline{w}\| \leq \|z\|^2 + \|z\|\|w\| + \|\overline{w}\| = (\|z\| + \|w\|)^2$$

Taking square roots gives us the result. ■

Corollary. $\|z+w\| = \|z\| + \|w\|$ *if* $z = tw$ *for some* $t \geq 0$.

Corollary. *If* $z_1, \ldots, z_n \in \mathbb{C}$, *then* $\|z_1 + \cdots + z_n\| \leq \|z_1\| + \cdots + \|z_n\|$.

Proof. By induction on n. ■

Corollary. *For all* $z, w \in \mathbb{C}$, $\big|\|z\| - \|w\|\big| \leq \|z - w\|$.

Proof. We have that $\|z\| \leq \|z-w\| + \|w\|$, and $\|w\| \leq \|z-w\| + \|z\|$. So we get $\|z\| - \|w\| \leq \|z - w\|$ and $-\|z-w\| \leq \|w\| - \|z\|$, so that $\big|\|z\| - \|w\|\big| \leq \|z - w\|$. ■

Definition. We define the **polar form** of a complex number $z \in \mathbb{C}$ to be the polar coordinates (r, θ) where $r = \|z\|$ and θ is the angle between the line segment from 0 to z and the positive real axis. We call r the **modulus** of z, and θ the **argument** of z. We write $\theta = \arg z$.

Lemma 1.2.2. *Let* $z = r_1 \cos\theta_1 + r_1 i \sin\theta_1$ *and* $w = r_2 \cos\theta_2 + r_2 i \sin\theta_2$. *Then*

$$zw = r_1 r_2 \cos(\theta_1 + \theta_2) + r_1 r_2 i \sin(\theta_1 + \theta_2)$$

so that $\arg zw = \arg z + \arg w$.

Proof. We multiply the expanded forms of z and w together and use the trigonometric identities to get the result. ■

Corollary. *If* $z_k = r_k \cos\theta_k + r_k i \sin\theta_k$, *then*

$$z_1 \ldots z_n = (r_1 \ldots r_n)\cos(\theta_1 + \cdots + \theta_n) + (r_1 \ldots r_n) i \sin(\theta_1 + \cdots + \theta_n)$$

Proof. By induction on n. ■

Theorem 1.2.3 (DeMoivre's Theorem)**.** *For all integers* $n \geq 0$, *if* $z = \cos\theta + i\sin\theta$, *then*

$$z^n = \cos n\theta + i \sin n\theta$$

Proof. We use the corollory to lemma 1.2.2 recursively on z^n. ■

Lemma 1.2.4. *FOr each nonzero* $a \in \mathbb{C}$, *and integer* $n \geq 2$, *the polynomial* $z^n - a$ *has has roots all* z *of the form*

$$z = \|a\|^{\frac{1}{n}}(\cos\frac{\alpha + 2k\pi}{n} + i\sin\frac{\alpha + 2k\pi}{n}) \text{ for all } 0 \leq k \leq n-1$$

where $a = \|a\|\cos\alpha + \|a\| i \sin\alpha$

Proof. Let $a = \|a\|\cos\alpha + \|a\| i \sin\alpha$. Then we have $z^n - a = 0$ has as solution

$$z' = \|a\|^{\frac{1}{n}}(\cos\frac{\alpha + 2\pi}{n} + i\sin\frac{\alpha + 2\pi}{n})$$

The rest of the solutions are obtained by noting that $(z')^n - a = 0$. ■

Definition. Let $a \in \mathbb{C}$ a nonzero complex number. We call the roots of the the polynomial $z^n - a \in \mathbb{C}[z]$ the **n-th roots** of a. We call the roots of $z^n - 1 \in \mathbb{C}[z]$ the **n-th roots of unity**.

Example 1.2. The n-th roots of unity are all complex numbers of the form

$$z = \cos \frac{2k\pi}{n} + i \sin \frac{2k\pi}{n} \text{ for all } 0 \leq k \leq n-1$$

Lemma 1.2.5. *Let $L \subseteq \mathbb{C}$ a straight line in $\mathbb{C}$. Then $L = \{z \in \mathbb{C} : \operatorname{Im} \frac{z-a}{b} = 0\}$, where $z = a + tb$ for some $t \in \mathbb{R}$.*

Proof. Let a be any point in L, and b the direction vector of L. Then if $z \in L$ $z = a + tb$ for some $t \in \mathbb{R}$. Since $b \neq 0$, $\operatorname{Im} \frac{z-a}{b} = 0$, since $t = \frac{z-a}{b}$, and $t \in \mathbb{R}$. ■

Corollary. *Let $H_a = \{z \in \mathbb{C} : \operatorname{Im} \frac{z-a}{b} > 0\}$ and $K_a = \{z \in \mathbb{C} : \operatorname{Im} \frac{z-a}{b} < 0\}$. Then $H_a = a + H_0$ and $K_a = a - K_0$.*

Proof. Suppose that $\|b\| = 1$, and let $a = 0$, then $H_0 = \{z \in \mathbb{C} : \operatorname{Im} \frac{z}{b} > 0\}$. Now, $b = \cos\beta + i \sin\beta$. If $z = r\cos\theta + ri\sin\theta$, then $\frac{z}{b} = r\cos(\theta - \beta) + ri\sin(\theta - \beta)$. So $z \in H_0$ if, and only if $\sin(\theta - \beta) > 0$; that is $\beta < \theta < \pi + \beta$, which makes H_0 the upper half plane about L.

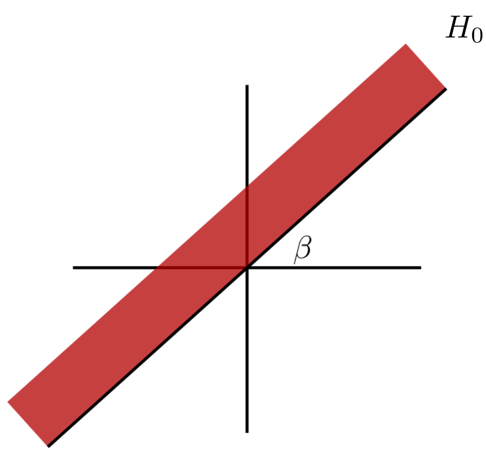

Figure 1.1:

Putting $H_a = \{z \in \mathbb{C} : \operatorname{Im} \frac{z-a}{b} > 0\}$, we get $H_a = a + H_0$. By similar reasoning, we get $K_a = a - K_0$, where $K_a = \{z \in \mathbb{C} : \operatorname{Im} \frac{z-a}{b} < 0\}$. ■

1.3 The Extended Complex Numbers

Definition. We define the **extended complex numbers** to be the set $\mathbb{C}_\infty = \mathbb{C} \cup \{\infty\}$.

Lemma 1.3.1. *$\mathbb{C}_\infty$ is homeomorphic to the unit sphere S^2 of $\mathbb{R}^3$.*

Proof. Identify $\mathbb{C}$ with the plane $\mathbb{R}^2$ as a subset of $\mathbb{R}^3$. Then $\mathbb{C}$ cuts the sphere S^2 along the equator. Now, let $N = (0, 0, 1)$ be the noth pole of S^2. For $z \in \mathbb{C}$, let L_z the line passing through z and N, and hence cuts S^3 at exactly one point $Z \neq N$. If $\|z\| > 1$, Z is in the northern hemisphere of S^2, and if $\|z\| < 1$, then Z is in the southern hemisphere. If $\|z\| = 1$, then $Z = z$. Then notice that as $\|z\| \to \infty$, then $Z \to N$; and so identify N with ∞ in $\mathbb{C}_\infty$.

Now, let $z = x+iy$ and $Z = (x_1, x_2, x_3)$ a point on S^2. Then $L_z = \{tN+(1-t)z : t \in \mathbb{R}\}$. Observe then that

$$L_z = \{((1-t)x, (1-t)y, t) : t \in \mathbb{R}\}$$

Then we get

$$1 = (1-t)^2\|z\|^2 + t^2$$

Taking $t \neq 1$ so that $z \neq \infty$

$$Z = \Big(\frac{2x}{\|z\|^2+1}, \frac{2y}{\|z\|^2+1}, \frac{\|z\|^2-1}{\|z\|^2+1}\Big)$$

additionally

$$Z = \Big(\frac{z+\overline{z}}{\|z\|^2+1}, -i\frac{z-\overline{z}}{\|z\|^2+1}, \frac{\|z\|^2-1}{\|z\|^2+1}\Big)$$

Taking $Z \neq N$ and $t = x_1$, we also get by definition of L_z, that

$$z = \frac{x_1+ix_2}{1-x_3}$$

Define now, the metric d on $\mathbb{C}_\infty$ by $d(z, w)$ is the distance between the points $Z = (x_1, x_2, x_3)$ and $W = (y_1, y_2, y_3)$ on S^2. Then we get

$$d(z, w) = \sqrt{(x_1-y_1)^2 + (x_2-y_2)^2 + (x_3-y_3)^2}$$

Squaring both sides we ovserve tha

$$d(z, w) = 2 - 2(x_1y_1 + x_2y_2 + x_3y_3)$$

Using the previous derivations of the components of Z, we finally obtain

$$d(z, w) = \frac{z\|z-w\|}{\sqrt{(\|z\|^2+1)(\|w\|^2+1)}} \text{ for all } z, w \in \mathbb{C}$$

When $w = \infty$, we have

$$d(z, \infty) = \frac{z}{\sqrt{\|z\|^2+1}}$$

Then d is the required homeomorphism. ■

Definition. We call the correspondence between S^2 and $\mathbb{C}_\infty$ the **stereographic projection** of S^2 onto $\mathbb{C}_\infty$.

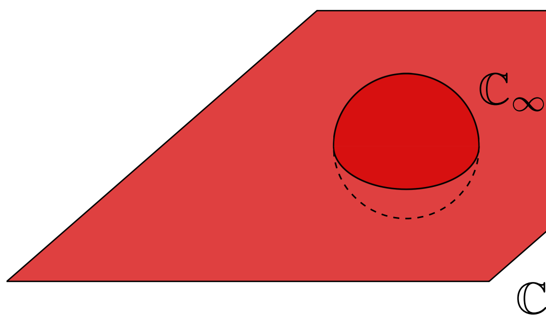

Figure 1.2: The Extended Complex Numbers.

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