Orthogonal Coordinate Systems and Vector Operators

A fairly good knowledge of vectors is an essential pre-requisite to the study of theory of electromagnetic fields. Therefore, before going into the details of the field analysis, it is necessary to introduce vector concepts. For the position & orientation of a space vector it is important introduce different co-ordinate systems.

A) Cartesian System of Coordinates:

The position and orientation of a space vector can be easily defined and identified by choosing an appropriate system of co-ordinate axes X, Y, Z. It is customary to make use of the right-handed system of axes, viz. a screw with right-handed thread will advance in the positive Z-direction when turned through 90 degrees from the positive X-axis towards the Y-axis, the three axes X, Y, Z being mutually perpendicular (see Fig below).

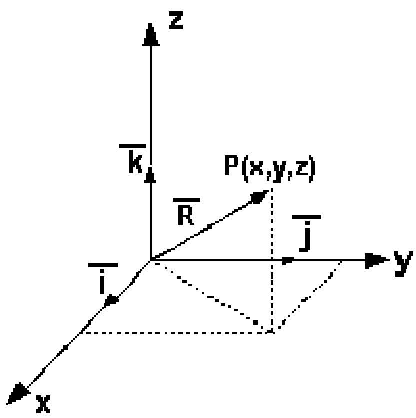


Fig. Cartesian system of coordinates

i, j, k are unit vectors in positive X, Y, Z- directions respectively.

Now |**R**| is given by $|\mathbf{R}| = (x^2+y^2+z^2)^{1/2}$

Consider any point P in space at a distance R from the origin O; **R** is a vector directed from the origin to the point P (x, y, z). If **i, j, k** are unit vectors in the positive X, Y, Z directions respectively.

$$\mathbf{A} = x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}$$

In general, if **A** is any vector whose components along X, Y, Z - axes are A_x, A_y, A_z respectively then

$$\mathbf{A} = A_x\,\mathbf{i} + A_y\,\mathbf{j} + A_z\,\mathbf{k}$$

If the unit vectors in X, Y, Z directions are written as $\boldsymbol{u_x}$, $\boldsymbol{u_y}$ and $\boldsymbol{u_z}$ respectively, with the result that A may be rewritten as follows :

$$\mathbf{A} = A_x\,\boldsymbol{u_x} + A_y\,\boldsymbol{u_y} + A_z\,\boldsymbol{u_z}$$

$$\mathbf{i} \times \mathbf{j} = \mathbf{k}, \quad \mathbf{j} \times \mathbf{k} = \mathbf{i}, \quad \mathbf{k} \times \mathbf{i} = \mathbf{j} \qquad [\,\mathbf{i} = \boldsymbol{u_x},\ \mathbf{j} = \boldsymbol{u_y},\ \mathbf{k} = \boldsymbol{u_z}\,]$$

As the cross products are non-commutative, it follows that

$$\mathbf{i} \times \mathbf{j} = -\mathbf{j} \times \mathbf{i} = \mathbf{k}$$

Similarly $\quad \mathbf{j} \times \mathbf{k} = -\mathbf{k} \times \mathbf{j} = \mathbf{i}$ etc.

Further $\mathbf{i} \times \mathbf{i} = \mathbf{j} \times \mathbf{j} = \mathbf{k} \times \mathbf{k} = 0$

The dot products between the unit vectors are

$$\mathbf{i.i} = \mathbf{j.j} = \mathbf{k.k} = 1$$
$$\mathbf{i.j} = \mathbf{j.k} = \mathbf{k.i} = 0$$

as any of two unit vectors in the orthogonal system are perpendicular to each other.

If another vector **B** is given as

$$\mathbf{B} = B_x\,\mathbf{i} + B_y\,\mathbf{j} + B_z\,\mathbf{k}$$

The dot product of the vectors **A** and **B** is given

$$\mathbf{A} \cdot \mathbf{B} = |\mathbf{A}|\,|\mathbf{B}|\,\cos(\mathbf{A}, \mathbf{B}) = A_xB_x + A_yB_y + A_zB_z$$

From the above relation, the angle between the vectors **A** & **B** can be evaluated right from the definition of the product. Accordingly

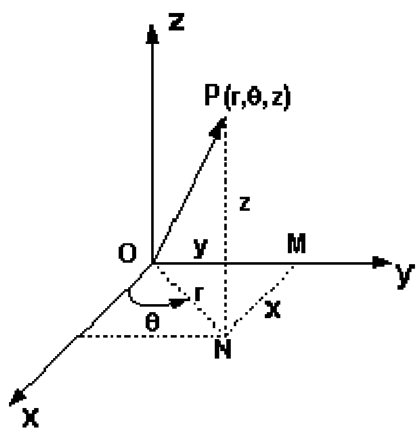
$$\cos(\mathbf{A}, \mathbf{B}) = (A_xB_x + A_yB_y + A_zB_z) \;/\; |\mathbf{A}|\,|\mathbf{B}|$$

It may be noted that $(A_x/|\mathbf{A}|)$, $(B_x/|\mathbf{B}|)$ etc are direction cosines of the line segments A and B.

The cross-product of **A** and **B** is given by

$$\mathbf{A} \times \mathbf{B} = (A_yB_z - A_zB_y)\,\mathbf{i} + (A_zB_x - A_xB_z)\,\mathbf{j} + (A_xB_y - A_yB_x)\,\mathbf{k}$$

Cylindrical Coordinate System

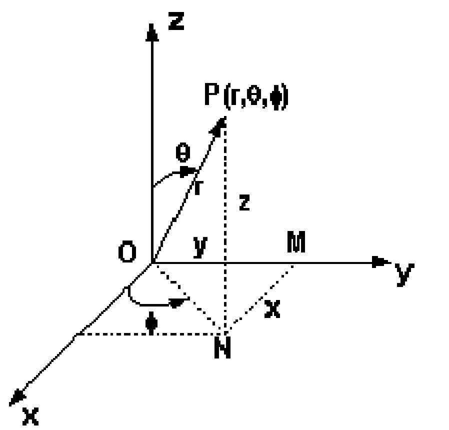


Now, ON = r So, OM = y = r sin θ
& NM = x = r cos θ
& NP = z

Special cases :

i) r = constant, this represents a circular cylinder which is generated by straight lines parallel to the z-axis.
ii) θ = constant, this represents a plane passing through the z-axis.
iii) z = constant, this represents a plane parallel to the x – y plane.

Spherical Coordinate System



Now, OP = r, $\therefore$ ON = r sin θ & NP = z = r cos θ.

Again, OM = y = ON sin φ = r sin φ sin θ

NM = x = ON cos φ = r cos φ sin θ

So, $r = (x^2+y^2+z^2)^{1/2}$

$\tan\theta = \pm\{(x^2+y^2)^{1/2}/z\}$ and $\tan\phi = (y/x)$

Special case:

i) r = constant represents a sphere.

ii) θ = constant represents a right circular cone whose vertex is at origin with z-axis as its axis.

iii) φ = constant represents a plane which makes a constant angle with x – z plane.

Note:

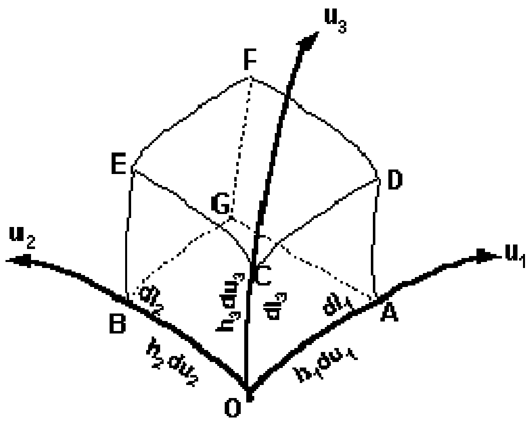
In the Cartesian coordinate system, the unit vectors **i**, **j** and **k** have fixed directions, independent of the location of P. But this is not true for cylindrical and spherical coordinate systems. Each unit vector is normal to its coordinate surface and is in the direction in which the coordinate increases. Note also that all the three systems are right handed so that

$\mathbf{i} \times \mathbf{j} = \mathbf{k}$; $\mathbf{u_r} \times \mathbf{u_\theta} = \mathbf{u_z}$; $\mathbf{u_r} \times \mathbf{u_\theta} = \mathbf{u_\phi}$

A problem, which has cylindrical or spherical symmetry, could be expressed and solved in Cartesian coordinates. However, the solution would fail to show the symmetry and in most of the cases would be very complex. Therefore, in field theory in addition to the Cartesian coordinates, the Cylindrical and Spherical polar coordinates are also used commonly.

Orthogonal Curvilinear Coordinate System

The fundamental definitions of vector functions such as gradient or divergence do not involve any specific coordinate system. In general, the formulae are worked out in terms of orthogonal curvilinear coordinates and the expressions for any specific coordinate system may be obtained by substitution of appropriate parameters.



The figure shows the orthogonal coordinate system defined by three families of surfaces, which are orthogonal to each other. There three families of surfaces are given by

$u_1(x,y,z) = \text{constant}$; $u_2(x,y,z) = \text{constant}$; $u_3(x,y,z) = \text{constant}$

At any point (u_1,u_2,u_3) three unit vectors $\mathbf{a_1}$, $\mathbf{a_2}$ and $\mathbf{a_3}$ are defined, which are tangential to the corresponding coordinate line at that point. For the right handed orthogonal system, these three unit vectors are mutually perpendicular such that $\mathbf{a_1} \times \mathbf{a_2} = \mathbf{a_3}$. Then any vector field **A** can be expressed in terms of components A1, A2 and A3 along these three unit vectors such that

$$\mathbf{A} = \mathbf{a_1} A1 + \mathbf{a_2} A2 + \mathbf{a_3} A3$$

Consider the infinitesimal parallel piped as shown in the above figure. Then the differential elements of length are expressed as:

$$dl_1 = h_1\, du_1 \;;\quad dl_2 = h_2\, du_2 \;;\quad dl_3 = h_3\, du_3$$

where, h_1, h_2, h_3 are suitable scale factors and are also known as **metric functions** of u_1, u_2 and u_3.

The volume of this parallel piped is then $h_1 h_2 h_3\, du_1\, du_2\, du_3$.

Gradient:

Let, $\phi\,(u_1, u_2, u_3)$ be a scalar function. Then the u_1 component of $\nabla\phi$ is given by,

$$(\nabla\phi)_1 = \frac{\partial\phi}{\partial l_1} = \frac{\partial\phi}{(h_1 du_1)} = \frac{1}{h_1}\frac{\partial\phi}{\partial u_1}$$

$$\text{similarly,}\quad (\nabla\phi)_2 = \frac{1}{h_2}\frac{\partial\phi}{\partial u_2} \;\text{ and }\; (\nabla\phi)_3 = \frac{1}{h_3}\frac{\partial\phi}{\partial u_3}$$

Therefore the resultant expression for $\nabla\phi$ is

$$\nabla\phi = \frac{1}{h_1}\frac{\partial\phi}{\partial u_1}\,\vec{a}_1 + \frac{1}{h_2}\frac{\partial\phi}{\partial u_2}\,\vec{a}_2 + \frac{1}{h_3}\frac{\partial\phi}{\partial u_3}\,\vec{a}_3$$

Divergence:

It has already been discussed that the divergence of a vector field **A** may be evaluated by finding the net flux coming out per unit volume.

Let, a point (u'_1, u'_2, u'_3) be located at the center of the volume element $dl_1\, dl_2\, dl_3$ and A_1 be the u_1 component of **A**. Then the Taylor series expansion of A_1 at any point ($u_1, u_2\, u_3$) in the vicinity of the point (u'_1, u'_2, u'_3) yields

$$A_1(u_1,u_2,u_3) = A_1(u'_1,u'_2,u'_3) + \frac{\partial A_1}{h_1\partial u_1}(u_1 - u'_1)h_1 + \frac{\partial A_1}{h_2\partial u_2}(u_2 - u'_2)h_2$$

$$+\frac{\partial A_1}{h_3\partial u_3}(u_3 - u'_3)h_3 + \text{higher order terms}$$

$$\left[\frac{\partial A_1}{\partial l_1}(l_1 - l'_1) = \frac{\partial A_1}{h_1\partial u_1}h_1(u_1 - u'_1) = \frac{\partial A_1}{\partial u_1}(u_1 - u'_1)\right]$$

where, the partial derivation $\frac{\partial A_1}{\partial u_1}, \frac{\partial A_1}{\partial u_2}$ and $\frac{\partial A_1}{\partial u_3}$ are evaluated at the point (u'_1, u'_2, u'_3).

Hence neglecting the higher order terms, the flux inflow through the surface OBEC

$$\int_{OBEC} A_1\, ds = dl_2\, dl_3\left[A_1 - \frac{\partial A_1}{\partial u_1}\frac{du_1}{2}\right]$$

as $\frac{\partial A_1}{\partial u_2}$ and $\frac{\partial A_1}{\partial u_3}$ are zero

Further, the net flux outflow through the surface AGFD,

$$\int_{AGFD} A_1\, ds = dl_2\, dl_3\left[A_1 + \frac{\partial A_1}{\partial u_1}\frac{du_1}{2}\right]$$

So, the net flux coming out through the surfaces normal to u_1 direction,

$$\int_{AGFD} A_1\, ds - \int_{OBEC} A_1\, ds = \frac{\partial A_1}{\partial u_1} dl_2\, dl_3 du_1$$

$$= \frac{\partial}{\partial u_1}(h_2 h_3 A_1)\, du_1\, du_2\, du_3$$

Similarly, the outflow through the other two pairs of surfaces can be determined. Thus the net flux outflow through the volume is

$$\oint_s \vec{A}\, ds = \left[\frac{\partial}{\partial u_1}(h_2 h_3 A_1) + \frac{\partial}{\partial u_2}(h_3 h_1 A_2) + \frac{\partial}{\partial u_3}(h_1 h_2 A_3)\right] du_1\, du_2\, du_3$$

Thus, the divergence is given by

$$\vec{\nabla}.\vec{A} = \frac{\oint_s \vec{A}\, ds}{\mathrm{h}_1 h_2 h_3 du_1 du_2 du_3} = \frac{1}{h_1 h_2 h_3}\left[\frac{\partial}{\partial u_1}(h_2 h_3 A_1) + \frac{\partial}{\partial u_2}(h_3 h_1 A_2) + \frac{\partial}{\partial u_3}(h_1 h_2 A_3)\right]$$

<u>The Laplacian:</u>

In the equation for divergence, if we put **A** = $\nabla\phi$

$$\vec{\nabla} \cdot \vec{\nabla}\phi = \vec{\nabla}^2\phi$$

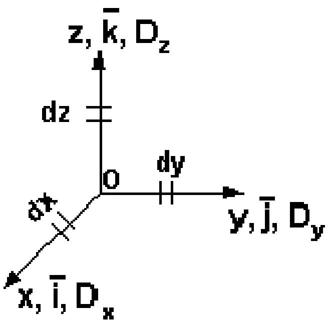
$$= \frac{1}{h_1 h_2 h_3}\left[\frac{\partial}{\partial u_1}\left(\frac{h_2 h_3}{h_1}\frac{\partial\phi}{\partial u_1}\right) + \frac{\partial}{\partial u_2}\left(\frac{h_3 h_1}{h_2}\frac{\partial\phi}{\partial u_2}\right) + \frac{\partial}{\partial u_3}\left(\frac{h_1 h_2}{h_3}\frac{\partial\phi}{\partial u_3}\right)\right]$$

Vector Functions in Cartesian Coordinates:

Here, $u_1 = x$, $dl_1 = dx$, i.e. $h_1 = 1$, $\mathbf{a_1} = \mathbf{I}$

$u_2 = y$, $dl_2 = dy$, i.e. $h_2 = 1$, $\mathbf{a_2} = \mathbf{j}$

$u_3 = z$, $dl_3 = dz$, i.e. $h_3 = 1$, $\mathbf{a_3} = \mathbf{k}$



$$\therefore\ \vec{\nabla}\phi = \frac{\partial\phi}{\partial x}\vec{i} + \frac{\partial\phi}{\partial y}\vec{j} + \frac{\partial\phi}{\partial z}\vec{k}$$

$$\vec{\nabla}\cdot\vec{D} = \frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z}$$

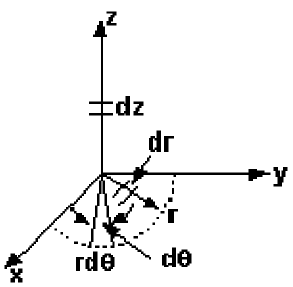
$$\vec{\nabla}^2\phi = \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2}$$

Vector Functions in Cylindrical Coordinates:

Here, $u_1 = r$, $dl_1 = dr$, i.e. $h_1 = 1$, $\mathbf{a_1} = \mathbf{a_r}$ and $A_1 = D_r$

$u_2 = \theta$, $dl_2 = rd\theta$, i.e. $h_2 = r$, $\mathbf{a_2} = \mathbf{a_\theta}$ and $A_2 = D_\theta$

$u_3 = z$, $dl_3 = dz$, i.e. $h_3 = 1$, $\mathbf{a_3} = \mathbf{a_z}$ and $A_3 = D_z$



$$\vec{\nabla}\phi = \frac{1}{h_1}\frac{\partial\phi}{\partial u_1}\vec{a}_1 + \frac{1}{h_2}\frac{\partial\phi}{\partial u_2}\vec{a}_2 + \frac{1}{h_3}\frac{\partial\phi}{\partial u_3}\vec{a}_3$$

$$or,\ \vec{\nabla}\phi = \frac{\partial\phi}{\partial r}\vec{a}_r + \frac{1}{r}\frac{\partial\phi}{\partial\theta}\vec{a}_\theta + \frac{\partial\phi}{\partial z}\vec{a}_Z$$

$$\vec{\nabla}\cdot\vec{D} = \frac{1}{r}\frac{\partial}{\partial r}(rD_r) + \frac{1}{r}\frac{\partial D_\theta}{\partial\theta} + \frac{\partial D_z}{\partial z}$$

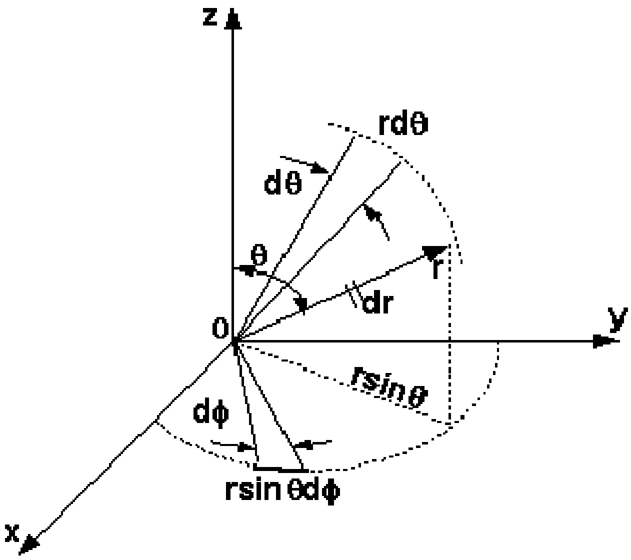
$$\&\ \ \vec{\nabla}^2\phi = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\phi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2} + \frac{\partial^2\phi}{\partial z^2}$$

Vector Functions in Spherical Coordinates:

Here, $u_1 = r$, $dl_1 = dr$, i.e. $h_1 = 1$, $a_1 = a_r$ and $A_1 = D_r$

$u_2 = \theta$, $dl_2 = rd\theta$, i.e. $h_2 = r$, $a_2 = a_\theta$ and $A_2 = D_\theta$

$u_3 = \phi$, $dl_3 = R \sin\theta \, d\phi$, i.e., $h_3 = r \sin\theta$, $a_3 = a_\phi$ and $A_3 = D_\phi$



$$\vec{\nabla}\phi = \frac{\partial \phi}{\partial r}\vec{a}_r + \frac{1}{r}\frac{\partial \phi}{\partial \theta}\vec{a}_\theta + \frac{1}{r\sin\theta}\frac{\partial \phi}{\partial \phi}\vec{a}_\phi$$

$$\vec{\nabla}.\vec{D} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 D_r\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta D_\theta\right) + \frac{1}{r\sin\theta}\frac{\partial D_\phi}{\partial \phi}$$

$$\vec{\nabla}^2\phi = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial \phi}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial \phi}{\partial \theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 \phi}{\partial \phi^2}$$