Field Due to a Continuous Distribution of Charge:

The potential at a point *p* due to a number of charges is obtained as a simple algebraic addition or *superposition* of the potentials produced at the point by each of charges acting alone. If $q_1,q_2,q_3,....q_n$ are charges located at distances $R_1,R_2,R_3,...,R_n$, respectively, from the point *p*, the potential at *p* is given by

$$V = \frac{1}{4\pi\varepsilon}\left(\frac{q_1}{R_1} + \frac{q_2}{R_2} + ... + \frac{q_n}{R_n}\right) = \frac{1}{4\pi\varepsilon}\sum_{i=1}^{n}\frac{q_i}{R_i} \qquad \text{.....(1)}$$

If the charge is distributed continuously throughout a region, rather than being located at a discrete number of points, the region can be divided into elements of volume ΔV each containing a charge $\rho\,\Delta V$, where ρ is the charge density in the volume element. The potential at a point *p* will then be given as before by

$$V = \frac{1}{4\pi\varepsilon}\sum_{i=1}^{n}\frac{\rho_i \Delta V_i}{R_i} \qquad \text{.....(2)}$$

where R_i is the distance to *p* from the *ith* volume element. As the size of volume element chosen is allowed to become very small, the summation becomes as integration, that is

$$V = \frac{1}{4\pi\varepsilon}\int_V \frac{\rho\, dV}{R} \qquad \text{.....(3)}$$

The integration is performed over the volume where ρ has finite value. However, it must be noted that it is not valid for charge distribution which extend to infinity.
Equation (3) is often written in the form

$$V = \int_V \rho G\, dV \qquad \text{.....(4)}$$

in which $G=1/(4\pi\varepsilon R)$. The function G is the potential of a unit point charge and is often referred to as the *electrostatic Green's function for a unbounded homogeneous region*.

Electrostatic Energy:

When a capacitor is charged so that there exists a voltage V between its plates, there is a storage of energy, which can be converted into heat by discharging the capacitor through a resistance. The amount of energy stored can be found by calculating the work done in charging the capacitor. Since the potential is defined in terms of work per unit charge, the work done in moving a small charge dq against a potential difference V is Vdq. But the voltage can be expressed in terms of the capacitance C and the charge q as follows

$$V = \frac{q}{C} \qquad \text{.....(1)}$$

Therefore the work done in increasing the charge on a capacitor by an amount dq is

$$\frac{q}{C}dq \qquad \text{.....(2)}$$

The total work done in charging a capacitor to Q coulombs is

$$= \int_0^Q \frac{q}{C}dq = \frac{1}{2}\frac{Q^2}{C}$$

Therefore the energy stored by the charged capacitor

$$= \frac{1}{2}\frac{Q^2}{C} = \frac{1}{2}VQ = \frac{1}{2}CV^2 \qquad \text{.....(3)}$$

Electrostatic energy may also be looked upon as the energy necessary to establish a given charge distribution in space. Suppose that the entire space is initially charge-free and that N point charges are brought in from infinity and located at specific points. The energy expended in locating the ith charge at the point r_i (a vector indicating the point x_i, y_i, z_i where the potential is V_i) is given by

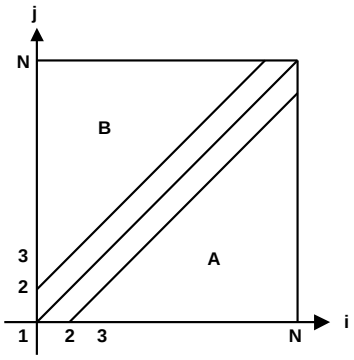
$$W_i = q_i V_i = \frac{q_i}{4\pi\varepsilon}\sum_{j=1}^{i-1}\frac{q_i}{R_{ij}} \qquad \text{.....(4)}$$

in which $R_{ij} = \left|r_i - r_j\right|$

No energy is used up in locating the first charge and thus the total energy is

$$W = \sum_{i=2}^{N} W_i = \frac{1}{4\pi\varepsilon}\sum_{i=2}^{N}\sum_{j=1}^{i-1}\frac{q_i q_j}{R_{ij}} \qquad \text{.....(5)}$$

Note that the energy necessary to form each point charge has been ignored in the above expression. The numbers over which the summation in equation (5) is carried out may be depicted as points in the *i-j* plane as shown in the figure. The summation in equation (5) is clearly over the triangular region marked A. Since the quantity being summed is symmetric in *i* and *j*, the same energy *W* would be obtained by a summation over triangle B.



The summation over both triangles must give *2W*, a result which may be stated as follows

$$W = \frac{1}{8\pi\varepsilon}\sum_{i=1}^{N}\sum_{j=1}^{N}\frac{q_i q_j}{R_{ij}},\ i \neq j \qquad \text{……(6)}$$

A continuous distribution of charge may be dealt with similarly by writing $\rho(r)dV$ in place of q_i and $\rho(r')dV'$ in place of q_j. The summation now may be replaced by integrations over volumes V and V' which must be large enough to contain all the charge present. Note that the stipulation $i \neq j$ in (6) now may be waived because point charges are not present (with their associated infinite self-energies). Thus the energy necessary to form a continuous charge distribution is

$$W = \frac{1}{8\pi\varepsilon}\int_V\int_{V'}\frac{\rho(r)\rho(r')}{R}dV'dV \qquad \text{……(7)}$$

in which $R = |r - r'|$.

Previous work on the field of a charge distribution shows that

$$V(r) = \frac{1}{4\pi\varepsilon}\int_{V'}\frac{\rho(r')}{R}dV' \qquad \text{……..(8)}$$

Substitution of (8) into (7) gives the very important formula

$$W = \frac{1}{2}\int_V \rho(r)V(r)dV \qquad \text{…….(9)}$$

The above formula for W may be represented in another form if use is made of the identity

$$\nabla.(VD) = V\nabla.D + D.\nabla V \qquad \text{……(10)}$$

Equation (9) may be transformed as follows:

$$W = \frac{1}{2}\int_V \rho V dV = \frac{1}{2}\int_V V\nabla.D dV$$

$$= \frac{1}{2}\int_V [\nabla.(VD) - D.\nabla V]dV$$

$$= \frac{1}{2}\int_S VD.dS + + \frac{1}{2}\int_V D.EdV \qquad \text{……(11)}$$

If the surface is allowed to approach infinity, the integrand in the surface integral must drop off at least as fast as r^{-3} and thus the surface integral vanishes. For this case (11) becomes

$$W = \frac{1}{2}\int_{allspace} \varepsilon E^2 dV \qquad \text{……(12)}$$

Equation (12) is often interpreted as an assertion that everywhere in space there exists an energy density w given by

$$w = \frac{1}{2}\varepsilon E^2 \qquad \text{……(13)}$$

This interpretation is frequently very useful but one must be cautious in applying it- only the energy integral (11) may be used without question. Note that energy may be computed either from the charge distribution (7) or from the electric field strength (11).

Thus the energy is said to be “associated with the electric charge” or alternatively” associated with the electric field”.

Charge on a Conducting Surface:

If charge is assembled on the surface of a conductor, that surface must be an equipotential. Under such circumstances

$$W = \frac{1}{2} V \int_s \rho_s ds$$

$$= \frac{1}{2} QV \qquad \text{……..(14)}$$

In which V is the potential of the surface and Q is the total charge on it. Note that (14) is identical to (3).

Force on a charged conductor:

The force on a charged conductor may be calculated using the expression for energy density. If an elemental area ΔS on a charge conductor is depressed in a distance Δl, the increase in stored energy is

$$\Delta W = w \Delta S \Delta l \qquad \text{…….(15)}$$

Such a depression must be carried out against a force F and thus

$$\Delta W = F \Delta l = w \Delta S \Delta l \qquad \text{……..(16)}$$

In which f is the pressure, i.e. force per unit area. Then comparison of (16) with (15) shows that

$$f = w \qquad \text{…….(17)}$$

If the surface charge density is ρ_s, then (13) and (17) give

$$f = \frac{1}{2} \varepsilon E^2 = \frac{1}{2\varepsilon} D^2 = \frac{1}{2\varepsilon} \rho_s^2 \qquad \text{…….(18)}$$

Dimensionally, the mechanical pressure on the conductor surface is equal to the product of surface charge density (ρ_s) and the electric field intensity on the conductor surface (E_s), i.e.

$$f = \rho_s * E_s \qquad \text{…… (19)}$$

So, from Eqns. (18) and (19),

$$E_s = (\rho_s / 2\varepsilon) \qquad \text{…… (20)}$$