which holds for $0 \leq x < 1$, we find that

$$\log \zeta(s) = \log \prod_p \frac{1}{1-p^{-s}} = \sum_p \log\left(\frac{1}{1-p^{-s}}\right) = \sum_{p,m} \frac{p^{-ms}}{m}.$$

Since the double sum converges absolutely, we need not specify the order of summation. See the Note at the end of this chapter. The formula then holds for all $\mathrm{Re}(s) > 1$ by analytic continuation. Note that, by Theorem 6.2 in Chapter 3, $\log \zeta(s)$ is well defined in the simply connected half-plane $\mathrm{Re}(s) > 1$, since ζ has no zeros there. Finally, it is clear that we have

$$\sum_{p,m} \frac{p^{-ms}}{m} = \sum_{n=1}^{\infty} c_n n^{-s},$$

where $c_n = 1/m$ if $n = p^m$ and $c_n = 0$ otherwise.

The proof of the theorem we shall give depends on a simple trick that is based on the following inequality.

Lemma 1.4 *If $\theta \in \mathbb{R}$, then $3 + 4\cos\theta + \cos 2\theta \geq 0$.*

This follows at once from the simple observation

$$3 + 4\cos\theta + \cos 2\theta = 2(1+\cos\theta)^2.$$

Corollary 1.5 *If $\sigma > 1$ and t is real, then*

$$\log |\zeta^3(\sigma)\zeta^4(\sigma+it)\zeta(\sigma+2it)| \geq 0.$$

Proof. Let $s = \sigma + it$ and note that

$$\mathrm{Re}(n^{-s}) = \mathrm{Re}(e^{-(\sigma+it)\log n}) = e^{-\sigma \log n}\cos(t\log n) = n^{-\sigma}\cos(t\log n).$$

Therefore,

$$\begin{aligned} &\log |\zeta^3(\sigma)\zeta^4(\sigma+it)\zeta(\sigma+2it)| \\ &= 3\log|\zeta(\sigma)| + 4\log|\zeta(\sigma+it)| + \log|\zeta(\sigma+2it)| \\ &= 3\mathrm{Re}[\log\zeta(\sigma)] + 4\mathrm{Re}[\log\zeta(\sigma+it)] + \mathrm{Re}[\log\zeta(\sigma+2it)] \\ &= \sum c_n n^{-\sigma}(3 + 4\cos\theta_n + \cos 2\theta_n), \end{aligned}$$

where $\theta_n = t\log n$. The positivity now follows from Lemma 1.4, and the fact that $c_n \geq 0$.