Tugas 5
Matematika 2 Kelas 67

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Latihan 2.1

1h) Selesaikan $\int t^2 \sin t \, dt$

Penyelesaian:

Misalkan

$$u = t^2 \qquad \text{dan} \qquad dv = \sin t \, dt$$

maka

$$\frac{du}{dt} = 2t \qquad \text{atau} \qquad du = 2t \, dt$$

dan

$$v = \int \sin t \, dt = -\cos t$$

sehingga

$$\begin{aligned} \int t^2 \sin t \, dt &= \int u \, dv \\ &= uv - \int v \, du \\ &= -t^2 \cos t + 2 \int t \cos t \, dt \end{aligned} \tag{1.1}$$

Kemudian kita hitung $\int t \cos t \, dt$

Misalkan

$$u = t \qquad \text{dan} \qquad dv = \cos t \, dt$$

maka

$$\frac{du}{dt} = 1 \qquad \text{atau} \qquad du = dt$$

dan

$$v = \int \cos t \, dt = \sin t$$

sehingga

$$\begin{aligned} \int t \cos t \, dt &= \int u \, dv \\ &= uv - \int v \, du \\ &= t \sin t - \int \sin t \, dt \\ &= t \sin t + \cos t + C_1 \end{aligned}$$

Substitusikan ke 1.1, diperoleh

$$\begin{aligned} \int t^2 \sin t \, dt &= -t^2 \cos t + 2(t \sin t + \cos t + C_1) \\ &= -t^2 \cos t + 2t \sin t + 2 \cos t + C \end{aligned}$$

13m) Selesaikan $\int \cos^4 2x \sin^3 2x\, dx$

Penyelesaian:

$$\int \cos^4 2x \sin^3 2x\, dx = \int \cos^4 2x(1 - \cos^2 2x) \sin 2x\, dx$$

Misalkan $u = \cos 2x$, sehingga $\dfrac{du}{dx} = -2\sin 2x$ atau $-\dfrac{du}{2} = \sin 2x\, dx$, diperoleh

$$\begin{aligned}
\int \cos^4 2x \sin^3 2x\, dx &= -\frac{1}{2}\int u^4(1 - u^2)\, du \\
&= -\frac{1}{2}\int u^4 - u^6\, du \\
&= -\frac{1}{2}\left[\frac{u^5}{5} - \frac{u^7}{7}\right] + C \\
&= \frac{u^7}{14} - \frac{u^5}{10} + C \\
&= \frac{\cos^7 2x}{14} - \frac{\cos^5 2x}{10} + C
\end{aligned}$$

14p) Selesaikan $\int \tan\theta \sec^5\theta\, d\theta$

Penyelesaian:

Misalkan $\sec\theta = u$, sehingga $\dfrac{du}{d\theta} = \tan\theta \sec\theta\, d\theta$, diperoleh

$$\begin{aligned}
\int \tan\theta \sec^5\theta\, d\theta &= \int u^4\, du \\
&= \frac{u^5}{5} + C \\
&= \frac{1}{5}\sec^5\theta + C
\end{aligned}$$

Latihan 2.2

2k) Selesaikan $\int \frac{x^5 + 2xr + 1}{x^3 + x}\, dx$
Penyelesaian:
Perhatikan bahwa

$$\begin{aligned}\frac{x^5 + 2x^2 + 1}{x^3 + x} &= \frac{x^5 + x^3 - x^3 - x + 2x^2 + x + 1}{x^3 + x} \\ &= \frac{x^2(x^3 + x) - (x^3 + x) + 2x^2 + x + 1}{x^3 + x} \\ &= x^2 - 1 + \frac{2x^2 + x + 1}{x(x^2 + 1)}\end{aligned}$$

Dengan aturan faktor linear dan faktor kuadratik, diperoleh dekomposisi pecahannya adalah

$$\frac{2x^2 + x + 1}{x(x^2 + 1)} = \frac{A}{x} + \frac{Bx + C}{x^2 + 1}$$

Kemudian

$$\begin{aligned}2x^2 + x + 1 &\equiv A(x^2 + 1) + x(Bx + C) \\ &\equiv Ax^2 + A + Bx^2 + Cx \\ &\equiv (A + B)x^2 + Cx + A\end{aligned}$$

Kita peroleh $A = 1, B = 1$ dan $C = 1$, sehingga

$$\begin{aligned}\int \frac{x^5 + 2x^2 + 1}{x^3 + x}\, dx &= \int \left[x^2 - 1 + \frac{1}{x} + \frac{x}{x^2 + 1} + \frac{1}{x^2 + 1}\right] dx \\ &= \frac{x^3}{3} - x + \ln x + \frac{1}{2}\ln(x^2 + 1) + \tan^{-1} x + C\end{aligned}$$

6) Selesaikan $\int \frac{1}{16x^3 - 4x^2 + 4x - 1}\, dx$
Penyelesaian:
Perhatikan bahwa

$$\begin{aligned}\frac{1}{16x^3 - 4x^2 + 4x - 1} &= \frac{1}{4x^2(4x - 1) + (4x - 1)} \\ &= \frac{1}{(4x^2 + 1)(4x - 1)}\end{aligned}$$

Misalkan $u = 2x$, sehingga $\frac{du}{dx} = 2$ atau $\frac{du}{2} = dx$, maka

$$\int \frac{1}{16x^3 - 4x^2 + 4x - 1}\, dx = \frac{1}{2}\int \frac{1}{(u^2 + 1)(2u - 1)}\, du$$

Dengan aturan faktor linear dan faktor kuadratik, diperoleh dekomposisi pecahannya adalah

$$\frac{1}{(u^2 + 1)(2u - 1)} = \frac{A}{2u - 1} + \frac{Bu + C}{u^2 + 1}$$

Kemudian

$$\begin{aligned}
1 &\equiv A(u^2+1)+(2u-1)(Bu+C) \\
&\equiv Au^2+A+2Bu^2+2Cu-Bu-C \\
&\equiv (A+2B)u^2+(2C-B)u+A-C
\end{aligned}$$

Kita peroleh $A+2B=0$, $2C-B=0$, dan $A-C=1$. Kemudian diperoleh bahwa $A=\frac{4}{5}$, $B=-\frac{2}{5}$, $C=-\frac{1}{5}$, sehingga

$$\begin{aligned}
\int \frac{1}{16x^3-4x^2+4x-1}\,dx &= \frac{1}{2}\int \frac{1}{(u^2+1)(2u-1)}\,du \\
&= \frac{1}{2}\int \left[\frac{4}{5(2u-1)}+\frac{-2u-1}{5(u^2+1)}\right] du \\
&= \frac{2}{5}\int \frac{1}{2u-1}\,du-\frac{1}{5}\int \frac{u}{u^2+1}\,du-\frac{1}{10}\int \frac{1}{u^2+1}\,du \\
&= \frac{1}{5}\ln|2u-1|-\frac{1}{10}\ln|u^2+1|-\frac{1}{10}\tan^{-1}u+C \\
&= \frac{1}{5}\ln|4x-1|-\frac{1}{10}\ln|4x^2+1|-\frac{1}{10}\tan^{-1}2x+C
\end{aligned}$$

Latihan 2.3

1h) Selesaikan $\int \frac{1}{t^2\sqrt{t^2-9}}\,dt$
Penyelesaian:
Misalkan $t = 3\sec u$, sehingga $\frac{dt}{du} = 3\sec u \tan u$ atau $dt = 3\sec u \tan u\, du$, maka

$$\begin{aligned}
\int \frac{1}{t^2\sqrt{t^2-9}}\,dt &= \int \frac{3\sec u \tan u}{9\sec^2 u\sqrt{9(\sec^2 u - 1)}}\,du \\
&= \frac{1}{3}\int \frac{\tan u}{\sec u\sqrt{9\tan^2 u}}\,du \\
&= \frac{1}{9}\int cosu\, du \\
&= \frac{1}{9}\sin u + C
\end{aligned}$$

Kemudian, kita kembalikan u ke t. Perhatikan bahwa $\cos u = \frac{3}{t}$, maka

$$\begin{aligned}
\sin u &= \sqrt{1-\cos^2 u} \\
&= \sqrt{1-\frac{9}{t^2}} \\
&= \frac{1}{t}\sqrt{t^2-9}
\end{aligned}$$

Jadi

$$\int \frac{1}{t^2\sqrt{t^2-9}}\,dt = \frac{\sqrt{t^2-9}}{9t} + C$$

5aa) Selesaikan $\int \frac{t^{2/3}}{t+1}\,dt$
Penyelesaian:
Misalkan $u = t^{1/3}$, sehingga $u^2 = t^{2/3}$ dan $u^3 = t$.
Kemudian

$$\frac{du}{dt} = \frac{1}{3t^{2/3}} \qquad \text{atau} \qquad dt = 3t^{2/3}\,du \qquad \text{atau} \qquad dt = 3u^2\,du$$

Kita peroleh

$$\begin{aligned}
\int \frac{t^{2/3}}{t+1}\,dt &= \int \frac{u^4}{u^3+1}\,du \\
&= \int \left[\frac{u(u^3+1)}{u^3+1} - \frac{u}{u^3+1}\right] du
\end{aligned}$$

Kita selesaikan dulu $\int \frac{u}{u^3+1}\,du$. Perhatikan bahwa $u^3+1 = (u+1)(u^2-u+1)$ sehingga dekomposisinya adalah

$$\frac{u}{(u+1)(u^2-u+1)} = \frac{A}{u+1} + \frac{Bu+C}{u^2-u+1}$$

Kemudian

$$\begin{aligned}u &\equiv A(u^2-u+1)+(u+1)(Bu+C)\\ &\equiv Au^2-Au+A+Bu^2+Cu+Bu+C\\ &\equiv (A+B)u^2+(B+C-A)u+A+C\end{aligned}$$

Kita peroleh $A+B=0$, $B+C-A=1$, dan $A+C=0$, sehingga $(A,B,C)=\Big(-\dfrac{1}{3},\dfrac{1}{3},\dfrac{1}{3}\Big)$. Jadi

$$\begin{aligned}\int \frac{u}{u^3+1}\,du &= \int \left[-\frac{1}{3(u+1)}+\frac{u+1}{3(u^2-u+1)}\right] du\\ &= \frac{1}{3}\int \frac{u+1}{u^2-u+1}\,du-\frac{1}{3}\int \frac{1}{u+1}\,du\\ &= \frac{1}{6}\int \frac{2u-1}{u^2-u+1}\,du+\frac{1}{2}\int \frac{1}{u^2-u+1}\,du-\frac{1}{3}\int \frac{1}{u+1}\,du\end{aligned}$$

Kemudian kita selesaikan $\displaystyle\int \frac{1}{u^2-u+1}\,du$. Perhatikan bahwa $u^2-u+1=\Big(u-\dfrac{1}{2}\Big)^2+\dfrac{3}{4}$. Misalkan $u-\dfrac{1}{2}=p$ sehingga $du=dp$, maka

$$\begin{aligned}\int \frac{1}{u^2-u+1}\,du &= \int \frac{1}{p^2+\frac{3}{4}}\,dp\\ &= 4\int \frac{1}{4p^2+3}\,dp\end{aligned}$$

Misalkan $p=\dfrac{q\sqrt{3}}{2}$, sehingga $dp=\dfrac{\sqrt{3}}{2}\,dq$, maka

$$\begin{aligned}4\int \frac{1}{4p^2+3}\,dp &= 2\sqrt{3}\int \frac{1}{3q^2+3}\,dq\\ &= \frac{2\sqrt{3}}{3}\int \frac{1}{q^2+1}\,dq\\ &= \frac{2\sqrt{3}}{3}\tan^{-1} q\\ &= \frac{2\sqrt{3}}{3}\tan^{-1}\Big(\frac{2p}{\sqrt{3}}\Big)\end{aligned}$$

Jadi

$$\begin{aligned}\int \frac{1}{u^2-u+1}\,du &= 4\int \frac{1}{4p^2+3}\,dp\\ &= \frac{2\sqrt{3}}{3}\tan^{-1}\Big(\frac{2p}{\sqrt{3}}\Big)\\ &= \frac{2\sqrt{3}}{3}\tan^{-1}\Big(\frac{2u-1}{\sqrt{3}}\Big)\end{aligned}$$

Kemudian

$$\begin{aligned}\int \frac{u}{u^3+1}\,du &= \frac{1}{6}\int \frac{2u-1}{u^2-u+1}\,du + \frac{1}{2}\int \frac{1}{u^2-u+1}\,du - \frac{1}{3}\int \frac{1}{u+1}\,du \\ &= \frac{1}{6}\ln|u^2-u+1| + \frac{2\sqrt{3}}{3}\tan^{-1}\left(\frac{2u-1}{\sqrt{3}}\right) - \frac{1}{3}\ln|u+1|\end{aligned}$$

Sehingga

$$\begin{aligned}\int \frac{t^{2/3}}{t+1}\,dt &= \int \frac{u^4}{u^3+1}\,du \\ &= \int u\,du - \int \frac{u}{u^3+1}\,du \\ &= \frac{u^2}{2} - \left[\frac{1}{6}\ln|u^2-u+1| + \frac{2\sqrt{3}}{3}\tan^{-1}\left(\frac{2u-1}{\sqrt{3}}\right) - \frac{1}{3}\ln|u+1|\right] + C \\ &= \frac{u^2}{2} + \frac{1}{3}\ln|u+1| - \frac{1}{6}\ln|u^2-u+1| - \frac{2\sqrt{3}}{3}\tan^{-1}\left(\frac{2u-1}{\sqrt{3}}\right) + C \\ &= \frac{\sqrt[3]{t^2}}{2} + \frac{1}{3}\ln|\sqrt[3]{t}+1| - \frac{1}{6}\ln|\sqrt[3]{t^2}-\sqrt[3]{t}+1| - \frac{2\sqrt{3}}{3}\tan^{-1}\left(\frac{2\sqrt[3]{t}-1}{\sqrt{3}}\right) + C\end{aligned}$$

5jj) Selesaikan $\displaystyle\int \frac{1}{1+\sin x+\cos x}\,dx$

Penyelesaian:

Misalkan $u = \tan\left(\frac{x}{2}\right)$, sehingga $x = 2\tan^{-1} u$, dan $dx = \dfrac{2}{1+u^2}\,du$.

Ingat bahwa $\tan\left(\frac{x}{2}\right) = \sqrt{\dfrac{1-\cos x}{1+\cos x}}$, sehingga

$$\begin{aligned}u^2 &= \frac{1-\cos x}{1+\cos x} \\ u^2 + u^2\cos x &= 1-\cos x \\ (u^2+1)\cos x &= 1-u^2 \\ \cos x &= \frac{1-u^2}{1+u^2}\end{aligned}$$

Kemudian

$$\begin{aligned}\sin x &= \sqrt{1-\left(\frac{1-u^2}{1+u^2}\right)^2} \\ &= \sqrt{\frac{(1+u^2)^2-(1-u^2)^2}{(1+u^2)^2}} \\ &= \sqrt{\frac{(1+u^2+1-u^2)(1+u^2-1+u^2)}{(1+u^2)^2}} \\ &= \sqrt{\frac{4u^2}{(1+u^2)^2}} \\ &= \frac{2u}{1+u^2}\end{aligned}$$

Kita peroleh

$$\begin{aligned}
\int \frac{1}{1+\sin x+\cos x}\,dx &= \int \frac{1}{1+\left(\dfrac{2u}{1+u^2}\right)+\left(\dfrac{1-u^2}{1+u^2}\right)}\left(\frac{2}{1+u^2}\right)du \\
&= \int \frac{2}{(1+u^2)+2u+(1-u^2)}\,du \\
&= \int \frac{1}{1+u}\,du \\
&= \ln|1+u|+C \\
&= \ln\left|1+\tan\left(\frac{x}{2}\right)\right|+C
\end{aligned}$$