DERIVATIVE SECURITIES

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Contents

FIXED INCOME SECURITIES

Arithmetic of Interest Rates

If the annual nominal rate of interest is a constant *r* and interest is compounded annually, the future value V_t after *t* years of a present value V_0 is calculated as:

$$V_t = V_0(1+r)^t$$

If interest is compounded *m* times per year, then

$$V_t = V_0\left(1+\frac{r}{m}\right)^{mt}$$

In this case, the equivalent effective annual rate is given by $r_e = (1+\frac{r}{m})^m - 1$. Now, we will compute with the continuously compounded rate of interest, which is defined as $\lim_{m\to\infty}\left(1+\frac{r}{m}\right)^m = e^r$. Hence, the continuously compounded future value V_t after *t* years of a present value V_0 is calculated as:

$$V_t = V_0 e^{rt}$$

It is clear that, for any $m > 1$, $r_e > r$ and the highest equivalent effective rate is obtained with continuous compounding. When we solve these equations for V_0 (for example, in continuous discounting, $V_0 = V_t e^{-rt}$), we can compute present values.

Example: An investment of 10 now will yield cash flows of 2, 3, 4, and 3 at the end of each of the next four years. What is the annual (discrete time) rate of return?

Find the discount rate that makes the ***net present value*** (NPV) equal to zero (often called the *internal rate of return*):

$$0 = -10 + \frac{2}{(1+r)} + \frac{3}{(1+r)^2} + \frac{4}{(1+r)^3} + \frac{3}{(1+r)^4}$$

The rate of return is numerically computed as $r \approx 0.0718$. (As an alternative example, we may want to find the continually compounded rate of return by solving $0 = -10 + 2e^{-r} + 3e^{-2r} + 4e^{-3r} + 3e^{-4r}$ to get $r \approx 0.0693$. Make sure you understand why the continually compounded rate is smaller than the annually compounded rate.)

The above calculations assume that the interest rate remains constant over the time interval from 0 to t. However, interest rates change over time and it may be more realistic to use ***time-varying interest rates*** denoted by $r(t)$, where r is expressed as a function of time. To conceptualize the time dependence of rates, define $V(t + \Delta t) - V(t)$ as the change in value over the time interval from t to $t + \Delta t$ and suppose that the rate of interest remains near $r(t)$ (there are no "large" discontinuous jumps). Thus, $V(t + \Delta t) \approx V(t)(1 + r(t)\Delta t)$ and

$$\frac{V(t + \Delta t) - V(t)}{\Delta t} \approx V(t)r(t)$$

Taking the limit of both sides as $\Delta t \to 0$, we get the simple differential equation

$$\frac{dV(t)}{dt} = r(t)V(t)$$

The solution of this equation yields the future value $V(t)$ after t years of a fixed present value V_0 as

$$V(t) = V_0\, e^{\int_0^t r(x)dx}$$

The average of the time-varying interest rate over the interval $[0, t]$ is found as

$$S_{0,t} = \frac{1}{t}\int_0^t r(x)dx$$

which, under a proper definition, can be called the ***spot rate of interest*** for the time interval $[0, t]$. Hence, we can write (using continual compounding as an example)

$$V(t) = V_0\, e^{S_{0,t}\, t}$$

With annual compounding, the formula would be $V(t) = V_0\,(1 + S_{0,t})^t$.

The NPV of a sequence of discrete cash flows realized at distinct (and not necessarily equally spaced) points in time, $C_{t_1}, C_{t_2}, \ldots, C_{t_n}$, is given by

$$V_0 = \sum_{i=1}^{n} C_{t_i}(1 + S_{0,t_i})^{-t_i} \text{ , or, with continual compounding, } V_0 = \sum_{i=1}^{n} C_{t_i} e^{-S_{0,t_i} t_i}$$

In some cases, however, it may be more realistic to model the cash flows as a ***continuous stream of cash flows*** rather than as a sequence of distinct cash flows. An example may be the cash inflows from sales of a large retail company. Another example is the cash flows in and out of the clearing mechanism of a securities exchange. There is a large number of buy and sell orders at any point in time and thus a continuous flow of cash. To model this phenomenon, let the cash flow per unit time be the function $C(t)$ and also let us partition the time interval $[0, T]$ as $T = \Delta t_1 + \Delta t_2 \ldots + \Delta t_n$. Assuming a smooth function, the cash flow during a small time interval Δt will be approximately $C(t)\Delta t$. Hence, the total cash flow over the entire time interval will approximately be given by the Riemann sum:

$$\sum_{i=1}^{n} C(t_i)\Delta t_i$$

To switch to the continuous flow scenario, let $n \to \infty$ (or, $\Delta t \to 0$), the total cash flow over the interval $[0, T]$ is

$$\int_0^T C(t)\, dt$$

Using continuous compounding with time-varying interest rate, the present and future value of the total cash flow are calculated, respectively, as

$$V(0) = \int_0^T e^{-r(t)t}\, C(t)\, dt$$

$$V(T) = \int_0^T e^{r(t)\,(T-t)}\, C(t)\, dt$$

In applications (especially when rates are assumed to be time-dependent), these integrals often require numerical approximations, which often resemble discrete-time cash flows discounted by observed spot rates.

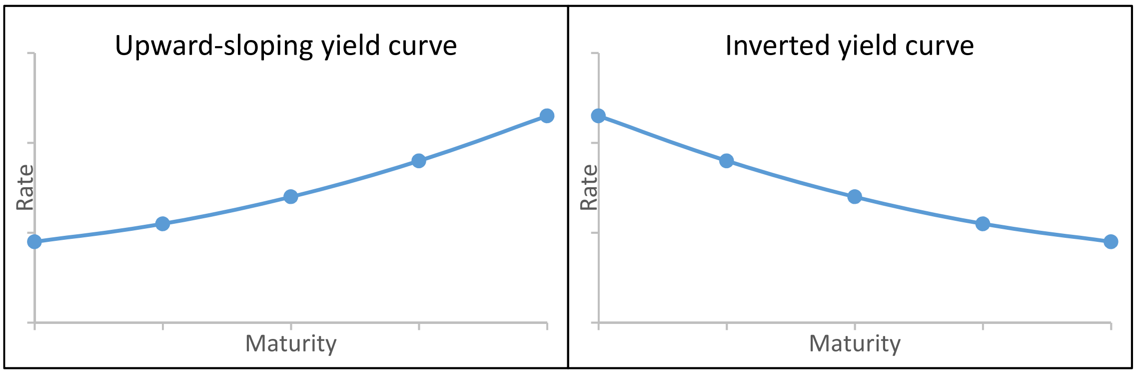
Spot and Forward Interest Rates

The **spot rate**, $S_{0,t}$, is the t-year rate of interest available at time 0 (now), for a single cash flow at time t. Formally, it is the rate of interest on a risk-free t-year zero-coupon bond (for example, a t-year Treasury bill):

$$P_{0,t} = \frac{C_t}{(1 + S_{0,t})^t}$$

where $P_{0,t}$ is the current market price and C_t is the cash flow at time t (for example, the face value of the bond). The function $d_t = 1/(1+S_{0,t})^t$ is sometimes called the **discount factor**. (Unless needed, we will not use the subscript 0 from now on.)

The series of spot rates, $\{S_1, S_2, S_3 \ldots\}$, define the **yield curve,** or the **term structure of interest rates** for a given risk class. If S_t is an increasing function of t, we say the yield curve is upward sloping, otherwise we say that the yield curve is inverted (downward sloping):



Consider a T-year fixed-income asset which will pay cash flows $C_1, C_2 \ldots C_T$, and which has a current market price of $P_{0,T}$, the spot rates satisfy the pricing equation:

$$P_{0,T} = \frac{C_1}{1+S_1} + \frac{C_2}{(1+S_2)^2} + \ldots + \frac{C_T}{(1+S_T)^T}$$

Given S_1, S_2 can be found. Given S_1 and S_2, S_3 can be found and so on.

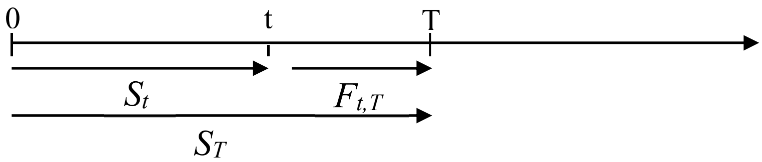
Example: The rate of interest on 1-year bills is 9%. The Treasury also has 2-year bonds with 3% annual coupon, which are currently selling for 950 TL (based on a face value of 1,000 TL). What is the 2-year spot rate?

Since $S_1 = 0.09$, $950 = \frac{30}{1+0.09} + \frac{30+1{,}000}{(1+S_2)^2}$, which gives $S_2 = 0.1166$.

Example: Bond A is a 10-year bond with a 10% coupon and selling at 98. Bond B is a 10-year bond with 8% coupon and selling at 85. The face value is 100 for both bonds. What is the 10-year spot rate?

A portfolio with –0.8 units of A and 1 unit of B would be equivalent to a bond with zero coupon, a face value of 20 and a price of 6.60. Since $6.60 \times (1+S_{10})^{10} = 20$, $S_{10} = 11.7\%$.

Implied Forward Rates: The current implied forward rate between times t and T ($T > t$) in the future, $F_{t,T}$, is the rate of interest that is consistent with a given series of spot rates.



In discrete time,

$$(1+S_{0,T})^T = (1+S_{0,t})^t (1+F_{t,T})^{T-t}, F_{t,T} = \left[\frac{(1+S_{0,T})^T}{(1+S_{0,t})^t}\right]^{1/T-t} - 1$$

In discrete time with compounding m times per year,

$$(1+\frac{S_{0,T}}{m})^{Tm} = (1+\frac{S_{0,t}}{m})^{tm}(1+\frac{F_{t,T}}{m})^{(T-t)m}, F_{t,T} = m\left[\frac{(1+\frac{S_{0,T}}{m})^T}{(1+\frac{S_{0,t}}{m})^t}\right]^{1/T-t} - m$$

In continuous time, we get a simpler formula

$$exp(\,TS_{0,T}) = exp(\,tS_{0,t})\, exp(\,(T-t)F_{t,T}), F_{t,T} = \frac{TS_T - tS_t}{T-t}$$

Example 4: If S_1 = 0.07 and S_2 = 0.08, then the forward rate for next year (with annual compounding) is found as

$$F_{1,2} = \frac{(1+0.08)^2}{(1+0.07)^1} - 1 = 0.09009$$

To understand the meaning of the forward rate, suppose that the Treasury has 1-year and 2-year zero-coupon bonds with face values of 1 TL each. The present values of these bonds are calculated as:

$$P_{0,1} = \frac{1}{(1+0.07)^1} = 0.93458\,, \quad P_{0,2} = \frac{1}{(1+0.08)^2} = 0.85734$$

The difference between $P_{0,1}$ and $P_{0,2}$ is that $P_{0,2}$ earns interest for one more year. Taking the ratio 0.93458 / 0.85734 = 1.09009, we can figure that this is the implicit rate earned on the longer maturity bond. This implicit rate is the forward rate. Another meaning of the forward rate is that it is the rate that we can contract now for a Treasury-type one-year riskless loan for next year.

Note that

- In our notation, $F_{0,t} \equiv S_{0,t}$. Forward rates spanning a single time period are sometimes called **short rates.** A series of short rates fully specifies a term structure:

$$(1+S_{0,T})^T = \left(1+S_{0,1}\right)\left(1+F_{1,2}\right)\left(1+F_{2,3}\right) \dots (1+F_{T-1,T})$$

- Given a series of spot rates S_1, S_2 , …,, we can draw the forward rate curve by calculating the implied forward rates corresponding to consecutive fixed time intervals of length Δ as:

$$F_{0,\Delta}(=S_{0,\Delta}),\, F_{\Delta,2\Delta}, F_{2\Delta,3\Delta}, \dots$$

(In some applications, the forward curve is drawn with forward rates for increasing maturities as $F_{0,\Delta}(=S_{0,\Delta}),\, F_{\Delta,2\Delta},\, F_{\Delta,3\Delta}, F_{\Delta,4\Delta}, \dots$. For now, we will adopt the former approach.)

- A yield curve should be calculated only by using data on bonds with identical risk levels and coupon rates. The common practice is to use zero-coupon risk-free government issues.

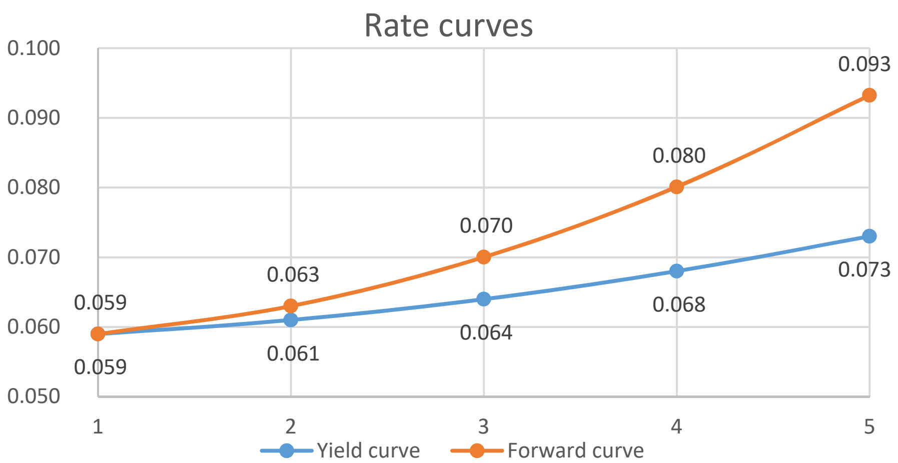
- $S_T > S_t \Rightarrow F_{t,T} > S_T > S_t$ (an upward sloping yield curve and a forward curve above it)

- $S_T < S_t \Rightarrow F_{t,T} < S_T < S_t$ (an inverted yield curve and a forward curve below it)

Example: Suppose the current spot rates on zero-coupon Treasury bonds and the implied forward rates are as follows:

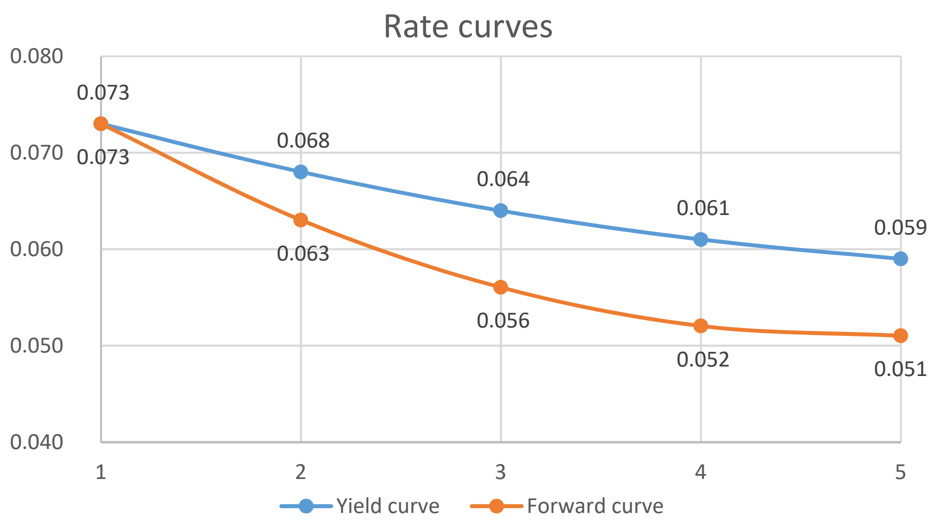
Maturity	Spot rate	Forward rate
1	0,059	0,059
2	0,061	0,063
3	0,064	0,070
4	0,068	0,080
5	0,073	0,093

where, for instance, $0.093 = (1+0.073)^5 / (1+0.068)^4 - 1$. Setting the time interval $\Delta = 1$ year, the spot rate and forward rate curves are drawn below:



If, instead, we had an inverted spot rate curve, the plots would look like:

Maturity	Spot rate	Forward rate
1	0,073	0,073
2	0,068	0,063
3	0,064	0,056
4	0,061	0,052
5	0,059	0,051



In applications, since the spot rate curve is discrete (that is, we have spot rates for certain maturities only), we do not have to worry about losing any information by choosing a "sub-optimal" time interval Δ. However, in modeling the term structure as stochastic processes, we use the concept of ***instantaneous (time-varying) forward rate*** defined as:

$$
\begin{aligned}
f(t) &= \lim_{\Delta t \to 0} F_{t,t+\Delta t} \\
&= \lim_{\Delta t \to 0} \frac{(t+\Delta t)S_{0,t+\Delta t} - tS_{0,t}}{\Delta t} \\
&= S_t + t\frac{\partial S_t}{\partial t}
\end{aligned}
$$

Remember this as a note for future use.

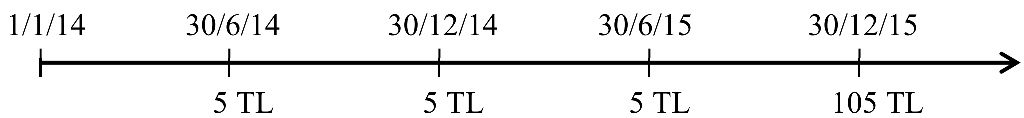
Valuation of Bonds

Notation and terminology:
P : current market price of bond
C : total coupon payment per year
F : face value of bond
T : number of years to maturity
m : number of coupon payments per year
C/F = "coupon rate"
C/P = "current yield"
"Settlement date" is the date the cash actually changes hands and "stub period" is the number of days from the settlement date to the next coupon payment

The price of a bond on a coupon date (say, "ex-coupon") is given by

$$P_{0,T} = \frac{C/m}{(1+\frac{S_{1/m}}{m})} + \frac{C/m}{(1+\frac{S_{2/m}}{m})^2} + \frac{C/m}{(1+\frac{S_{3/m}}{m})^3} + \ldots + \frac{C/m}{(1+\frac{S_T}{m})^{Tm}} + \frac{F}{(1+\frac{S_T}{m})^{Tm}}$$

Example: Suppose today is January 1st, 2014, and a 2-year Treasury bond with a face value of 100TL is issued with an annual coupon rate of 10%, paid semiannually (on June 30th and December 30th), and a maturity date of December 30th, 2015. Assume that the Treasury uses a 30/360 day count convention. If the current spot rates are $S_{1/2} = 0.11, S_1 = 0.11, S_{3/2} = 0.12$ and $S_2 = 0.13$, what should be the price of the bond?



$$P_{0,2} = \frac{5}{(1+\frac{0.11}{2})} + \frac{5}{(1+\frac{0.11}{2})^2} + \frac{5}{(1+\frac{0.12}{2})^3} + \frac{100+5}{(1+\frac{0.13}{2})^4} = 95.048TL$$

A bond's **yield-to-maturity**, r, is the internal rate of return on the bond

$$P = \frac{C/m}{(1+r/m)} + \frac{C/m}{(1+r/m)^2} + \frac{C/m}{(1+r/m)^3} + \ldots + \frac{C/m}{(1+r/m)^{Tm}} + \frac{F}{(1+r/m)^{Tm}}$$

which can be solved for by an iterative procedure. Thus, the yield-to-maturity is a complex average of the spot rates. Sometimes, it may be easier to code as:

$$P = \frac{F}{(1+r/m)^{Tm}} + \frac{C}{y}\left[1 - \frac{1}{(1+r/m)^{Tm}}\right]$$

Example: The yield of the bond in the example is found by solving for *y*

$$95.048 = \frac{5}{(1+\frac{r}{2})} + \frac{5}{(1+\frac{r}{2})^2} + \frac{5}{(1+\frac{r}{2})^3} + \frac{100+5}{(1+\frac{r}{2})^4}$$

to get $\frac{r}{2} = 0.0645$ and $r = 0.1289$ (using Excel's YIELD function).

Note that the total return on a bond is the sum of coupon payments ($C \times T$ liras in total), return on reinvested coupons (which depends on the available interest rates throughout the life of the bond), and any difference between *F* and *P*. Thus, the realized rate of return will be equal to the yield-to-maturity if *the coupons are all reinvested at the same rate as r.* This is a mathematical fact and, for this to be possible in practice, the whole yield curve and the bond's risk structure has to remain unchanged until time *T*!

Example: Suppose the above bond is bought on February 20th, 2014, 50 days after the issue date or 130 days before the next coupon payment. Calculating the stub as $130/180 = 0.722$, the value of the bond ("dirty price") on this date will be

$$\frac{5}{(1+0.0645)^{0.72}} + \frac{5}{(1+0.0645)^{1.72}} + \frac{5}{(1+0.0645)^{2.72}} + \frac{100+5}{(1+0.0645)^{3.72}} = 96.706$$

However, the seller will demand the ***accrued interest*** of $5 \times (50/180) = 1.388$, and the ***quoted price*** ("clean price") will be equal to $96.706 - 1.388 = 95.317$.

Can bond yields be used for arbitrage purposes?

Consider the following three government bonds with different maturities and coupon rates. The prices are based on the spot rates. Then, the yields are calculated:

Bond	Maturity (years)	Coupon rate	Price	Yield	Spot rates
A	1	0.05	100.86	0.0410	0.0410
B	2	0.10	110.53	0.0439	0.0440
C	3	-	86.88	0.0480	0.0480
D	3	0.15	128.08	0.0475	

Suppose the government issues a new 3 – year bond D with 15% annual coupon and at a price of 128.08. The yield of this bond is then calculated as 4.75%. Comparing this bond with bond C, which has the same maturity, can we say that this bond is "overpriced" because 4.75% < 4.80%? Can we realize an arbitrage profit by selling the "expensive" bond D and buying the "cheap" bond C?

The answer is a definite NO because, based on the following pricing equation using the spot rates, bond D is found to be correctly priced:

$$\frac{15}{1+0.041}+\frac{15}{(1+0.044)^2}+\frac{15+100}{(1+0.048)^3}=128.08$$

This example shows that yield-to-maturity is simply a numerical result but spot rates are a pricing system. Bond valuation and investment decision-making should be based on zero-coupon spot rates and not on yields. The difference in yields of C and D, which have the same maturity, is explained by the so called "coupon bias."

Example: Now suppose the price of bond D is 130 > 128.08, and hence *D* is now overpriced. This mispricing can be arbitraged away by selling (or, short selling) one unit of D and buying suitable quantities of correctly priced C*,* B and A to get the following cash flows:

Period cash flows	**0**	**1**	**2**	**3**
Sell 1 unit of D	130.00	-15.00	-15.00	-115.00
Buy 115/100 units of *C*	-99.91	0	0	115.00
Buy 15/110 units of *B*	-15.07	1.42	15.00	
Buy 13.58/105 units of A	-13.10	13.58		
Net cash flow	1.92	0.00	0.00	0.00

As a result, the net arbitrage profit is 1.92 now. Such transactions will force the price of D down until such arbitrage profits are no longer possible.

Example: "Coupon stripping" is to artificially design zero-coupon bonds out of existing coupon bonds. Suppose, using bonds A and B above, a bank wants to engineer a 2–year zero–coupon bond with a face value of 100. Solving the following equations:

$$n_A 105 + n_B 6 = 0$$
$$n_B 106 = 100$$

to get the portfolio $\{n_B = 0.9434,\ n_A = -\ 0.0539\}$ with the following cash flows:

Period cash flows	**0**	**1**	**2**
Buy 0.9434 units of *B*	-97.19	5.66	100.00
Sell 0.0539 units of *A*	5.44	-5.66	
Net cash flow	-91.75	0.00	100.00

The bank would sell this artificial 2-year zero for 91.75. It is no coincidence that this price satisfies

$$\frac{100}{(1+S_{0,2})^2} = \frac{100}{(1.044)^2} = 91.75$$

Risks in bond investment

1. **Interest rate risk:** change in bond values caused by unpredictable changes in interest rates
2. **Default risk:** probability of failure to meet scheduled interest and/or principal payments, which is evaluated by credit rating algorithms

Bond Credit Ratings

	Moody's	**Standard & Poor's**	**Fitch Ratings**
Highest credit quality	Aaa	AAA	AAA
Very high credit quality	Aa1 Aa2 Aa3	AA+ AA AA-	AA
High credit quality	A1 A2 A3	A+ A A-	A
Adequate credit quality for now	Baa1 Baa2 Baa3	BBB+ BBB BBB-	BBB
Below investment grade, but good chance of no default	Ba1 Ba2 Ba3	BB+ BB BB-	BB
Significant credit risk	B1 B2 B3	B+ B B-	B
High default risk	Caa1 Caa2 Caa3	CCC+ CCC CCC-	CCC CC C
Already in default	C	D	D

The only tool to manage default risk is diversification, preferable with models that take into account extreme events such as default.

Traditional approach to interest rate risk management

The present value ("dirty price") of a T-year riskless fixed-income asset with cash flows (coupon and principal payments) $C_1, C_2 \dots C_T$, and yield-to-maturity *r* is as before given by

$$P = \sum_{t=1}^{T} \frac{C_t}{(1+r)^t}$$

Given the contractual parameters *C* and *T,* the bond's price is a non-linear function of its yield, $P = P(r)$. If the yield changes, the bond price will change in the opposite change. In order to find the sensitivity of price to changes in yields (at a given level of *P*), we can compute the first derivative as:

$$P'(r) = \frac{dP}{dr} = -\frac{1}{1+r}\sum_{t=1}^{T} \frac{tC_t}{(1+r)^t} = -\frac{D^*P}{1+r} < 0$$

where $D^* = \frac{1}{P}\sum_{t=1}^{T} \frac{tC_t}{(1+r)^t}$ is called the bond's ***duration***. The measure of duration has two useful economic meanings. First, rearranging the above expression, we get

$$D^* = -\frac{dP/P}{dr/(1+r)} > 0$$

This shows that, for small changes in *r*, duration is minus the percentage change in price for a given percentage change in yield. We can also write $P'(r) = -DP < 0$, where

$$D = \frac{D^*}{1+r} = -\frac{dP/P}{dr}$$

is called the ***modified duration***, which is approximately the percentage change in price for a unit change in yield. In either case, duration stands as a useful measure of interest rate sensitivity of a bond. Since the sensitivity of a bond to changes in interest rates is determined not only by its maturity but also other factors, duration is a more useful measure than maturity. Indeed, two bonds with the same duration but different maturities have more in common that two bonds with the same maturity but different durations.

For a second economic interpretation of duration, define a set of weights as $w_t = (\frac{C_t}{(1+r^t})/P$ for $t = 1, \dots, T$. Note that w_t is the proportion of the bond's value due to the cash flow at time t and that $w_1 + \cdots + w_T = 1$. Then, the expression of duration takes the form $D = \sum_{t=1}^{T} w_t t$, which is easily interpreted as the weighted *average life of a bond*. Indeed, the calculation of duration may have been originally motivated as a measure of a bond's average life.

Example: What is the duration of a 3-year 10% - coupon bond with semiannual payments and selling at 95? To calculate, we first find the bond yield-to-maturity as $r = 0.12$ or semiannually as $r/2 = 0.06$, and then get duration as

$$D^* = \frac{1}{95}\left(\sum_{t=1}^{6} \frac{5t}{(1+0.06)^t} + \frac{100x6}{(1+0.06)^6}\right) = 2.65 < 3 \text{ years}$$

Some applications such as Excel use the **Macaulay** duration formula:

$$D = \left[\frac{1+y}{y} - \frac{1+y+mT(c-y)}{c((1+y)^{mT}-1)+y}\right]/m$$

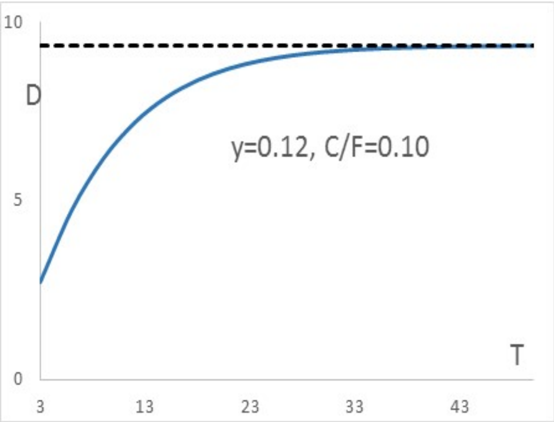
where $c = (C/m)/F$ be the coupon rate per period and $y = r/m$ be the yield per period. Finally, if we resolve the above problem with quarterly coupon payments instead of semiannual payments, we get

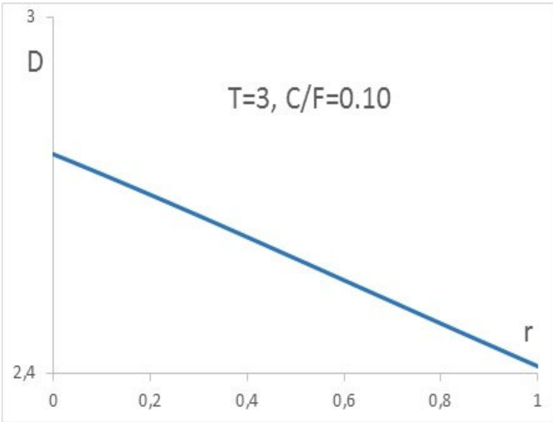
$$D = \frac{\left[\frac{1+0.03}{0.03} - \frac{1+0.03+12(0.025-0.03)}{0.025((1+0.03)^{12}-1)+0.03}\right]}{4} = 2.61 < 2.65 < 3$$

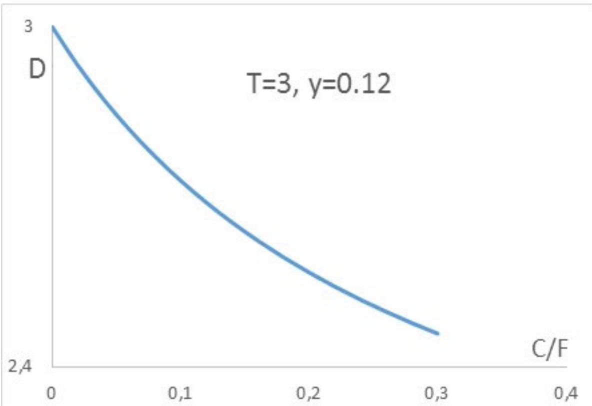
To summarize:

- Bond prices are inversely related to bond yields and long bonds are more sensitive to yield changes than short bonds
- (Except for deep discount bonds) Duration increases with maturity. In all cases, except for zero-coupon bonds when $D = T$, $D <$ T.
- The higher the coupon rate (*C/F*), the shorter is the duration. Then, bonds with higher coupon rates are less sensitive to yield changes.

- The higher the rate of interest, the shorter is the duration.
- The price increase from a decrease in yield is always higher than the price decrease from an equal increase in yield.





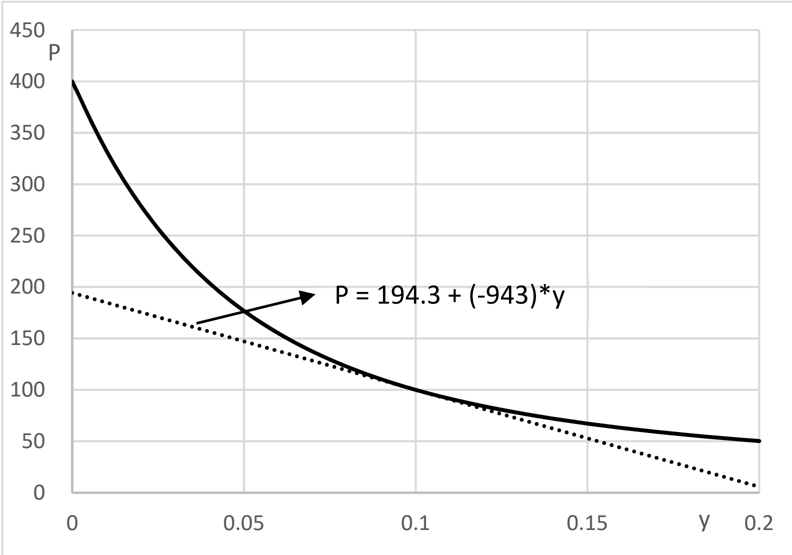


Measuring Interest Rate Sensitivity with Duration

Example: Consider a 10% 30-year bond selling at par (say, 100). Its price-yield curve is given below. The duration of this bond at the price level of 100 can be found to be 10.37. Hence, its modified duration is 10.37 / (1.10) = 9.43 years and $dP/dr = -943$.

If the yield increases by $\Delta = 0.01$ from 10% to 11%, the bond price will decline to 91.31. Using the duration measure, which is only a linear approximation to the interest rate sensitivity, the price will be approximated to decrease to 90.57 by

$(-9.43)(100)(0.01) = -9.43$ (see the graph below). This may not be acceptable as an accurate measure of interest rate risk.



In order to get a more accurate approximation to the non-linearity of the price function, we can include its second derivative:

$$P''(r) = \frac{d^2P}{dr^2} = \left[\sum_{t=1}^{T} \frac{t(t+1)C_t}{(1+r)^{t+2}}\right] > 0$$

The ***convexity*** of the bond is then defined as $K = \big(P''(r)\big)/P$ and, for the problem, it is found to be 157.28. The relationship between duration, convexity and price changes can now be seen by a Taylor series expansion of $P(r)$ around r:

$$P(r+\Delta) - P(r) = P'(r)\frac{\Delta}{1} + P''(r)\frac{\Delta^2}{2} + \mathrm{o}(\Delta^2)$$

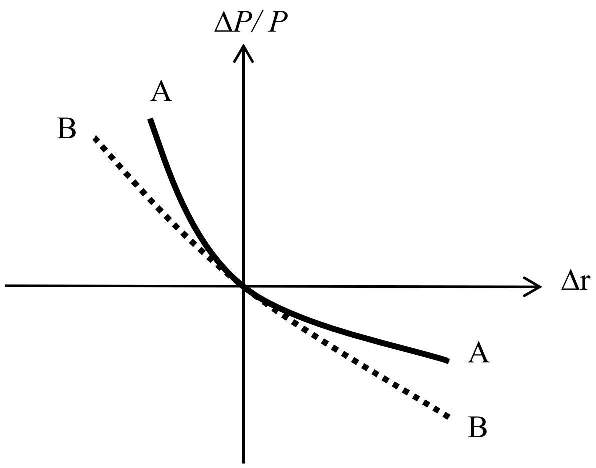
Ignoring higher order terms, we rewrite this expansion and apply to the problem on hand as:

$$P(r+\Delta) - P(r) \approx -DP\Delta + KP\frac{\Delta^2}{2}$$

$$P(r+\Delta) - P(r) = -(9.43)(100)(0.01) + (157.28)(100)\frac{(0.01)^2}{2} = -8.64$$

The new price is now approximated as $100 - 8.64 = 91.36$, which is clearly a better estimate. For small changes in yield, a first-order approximation may work fine but, for large changes, convexity correction must also be added. This is obvious because, in the Taylor series approximation, the convexity is multiplied by the square of the yield change.

In general, to see the wealth effects of convexity, consider the price-yield curves of bonds A and B in the following graph:



Bond A has more convexity than bond B: when yields decrease, its price increases by a greater percentage and, when yields increase, its price decreases by a smaller percentage amount. That is why bond traders often like convexity. Convexity is also inversely related to the degree of concentration of cash flows in time. The longer the bond, the larger is the impact of convexity.

Since duration underestimates price increases when yields decrease and overestimates price decreases when yields increase, when the yield curve changes markedly and in large magnitudes, the errors from relying on duration alone can be large. All of these have to be considered in portfolio management.

<u>Duration of a portfolio</u>

The duration a portfolio of N bonds with prices $P_1,\ldots,P_N$ and durations $D_1,\ldots,D_N$, *all calculated at the same yield y*, is found as

$$D = \sum_{i=1}^{N} \left(\frac{n_i P_i}{P} \right) D_i$$

where $P = n_1 P_1 + ... + n_N P_N$ is the total value of the portfolio. Note that for this formula to be exact, all prices and durations are to be calculated at a common discount rate. Otherwise, it will only be an approximation. The duration of a portfolio is used and interpreted in the same manner as that of a single bond.

A Different Approach to Duration

So far, we have used duration as a measure of relative price sensitivity to a change in the bond's yield. Sometimes, especially in managing large portfolios of fixed-income securities, shifts in the whole yield curve (rather than changes in a given yield) are more relevant sources of risk. Hence, it may be useful to develop a notion of duration based on the term structure of interest rates.

One such measure of duration is the so called ***Fisher-Weil duration***. Suppose we are interested in finding the duration of a cash flow stream $\{C_1, C_2 \ldots\ldots, C_T\}$ and let the spot rates defining the current yield curve be $\{S_1, S_2, \ldots\ldots, S_T\}$. The Fisher-Weil duration is defined as

$$D = \left[\sum_{t=1}^{T} t C_t e^{-S_t t} \right] \frac{1}{P}$$

where the present value of the cash flows in continuous time is computed using the spot rates as $P = \sum_{t=1}^{T} C_t e^{-S_t t}$. In discrete time, the Fisher-Weil formula may be duplicated as

$$D = \left(\sum_{t=1}^{T} \frac{t C_t}{(1 + S_t)^t} \right) \frac{1}{P}$$

Some Conceptual Examples with Duration and Convexity

Example: An option–free T–year mortgage with fixed annual payments of *M* will have a price and a duration equaling

$$P = \frac{M}{(1+r)} + \cdots + \frac{M}{(1+r)^T} \quad \Rightarrow \quad D = -\frac{dP}{dr}\frac{(1+r)}{P} = \frac{1+r}{r} - \frac{T}{(1+r)^T - 1}$$

Note that as $T \to \infty$, $D \to (1+r)/r$, the duration of a perpetuity.

Example: (Duration of equity!) Suppose that dividends are expected to grow at a constant rate of g (that is, $d_t = d_{t-1}(1+g)$) and the required rate of return on equity is a constant *R* (which is equal to the riskless rate of interest *r* plus some suitable equity risk premium). Under this scenario, the current price of equity can be expressed as

$$P = \frac{d_0(1+g)}{(1+R)} + \frac{d_0(1+g)^2}{(1+R)^2} + \cdots = \frac{d_1}{R-g}$$

with $R > g$. This is known as the "constant – growth model." Assuming that the equity risk premium is constant, then

$$D = -\frac{dP}{dr}\frac{(1+r)}{P} = -\frac{-d_1}{(R-g)^2}\frac{(1+r)}{P} = \frac{1+r}{R-g} = \frac{1+r}{d_1/P}$$

If *r* is small, then $D_{equity} = \dfrac{1}{dividend\ \ yield}$ may be used a rule of thumb.

Example: (Duration of a Bank's Equity) Consider the following balance sheet of a deposit bank:

Assets (*A*)	Market Value	Duration (years)	Liabilities (*L*) and Equity (*E*)	Market Value	Duration (years)
Cash	100	0	1-year deposits	600	1
Short-term Credits	400	1.5	3-year deposits	300	3
Long-term Credits	500	5	Equity	100	
	1,000			1,000	

The duration of the assets and the liabilities are calculated as:

$$D_A = \frac{(100 \times 0) + (400 \times 1.5) + (500 \times 5)}{1{,}000} = 3.1 \; years$$

$$D_L = \frac{(600 \times 1) + (300 \times 3)}{900} = 1.7 \; years$$

Since $\frac{d\,A}{d\,r} = \frac{d\,(L+E)}{d\,r}$, then $AD_A = LD_L + ED_E$. The duration of equity is then calculated as

$$D_E = \frac{(1{,}000 \times 3.1) - (900 \times 1.7)}{100} = 15.7 \; years$$

Roughly speaking, if interest rates rise by 1%, the value of equity will decline by 15.7%. To hedge a bank's equity against interest rate changes, an approximate rule may be inferred as

$$D_A = \frac{L}{A} D_L \;\; \Rightarrow \; D_E = 0$$

In the example, the duration of the assets must be reduced from 3.1 years to $(900/1{,}000) \times 1.7 = 1.53$ years.

Example: (Convexity of a zero-coupon bond) From the definition of convexity on page 18, we can directly calculate for a zero-coupon bond

$$K = \frac{T(T+1)}{(1+r)^2}$$

Some useful comparative statics follow:

- $\frac{\partial K}{\partial T} = \frac{2T+1}{(1+r)^2} > 0$ (the longer the bond, the larger the impact of convexity)
- $\frac{\partial K}{\partial r} = \frac{-2(1+r)T(T+1)}{(1+r)^4} < 0$ (Convexity decreases with interest rate levels and also with the coupon rates.)

Example: (*Money duration*) In the duration formula, we have a division by P, which implicitly assumes that price is non-zero. However, in some applications involving arbitrage strategies, the portfolio of bonds may have a zero value by construction. In such cases, in order to measure yield sensitivity, we can use the money (TL, dollar, etc.) duration:

$$D^{(M)} = -\frac{\partial P}{\partial r} = P\ D$$

The money duration of a portfolio is similarly calculated as the value weighted average of individual money durations.

Example: (*Immunization*) Duration as a measure of interest rate risk is widely used in bond portfolio management. An example of its usage is the immunization strategy, where a target duration is set for a portfolio of bonds. The basic idea is based on the fact that a change in yields has two offsetting effects:

- **Reinvestment Effect**: Different reinvestment rates become available for subsequent cash flows,

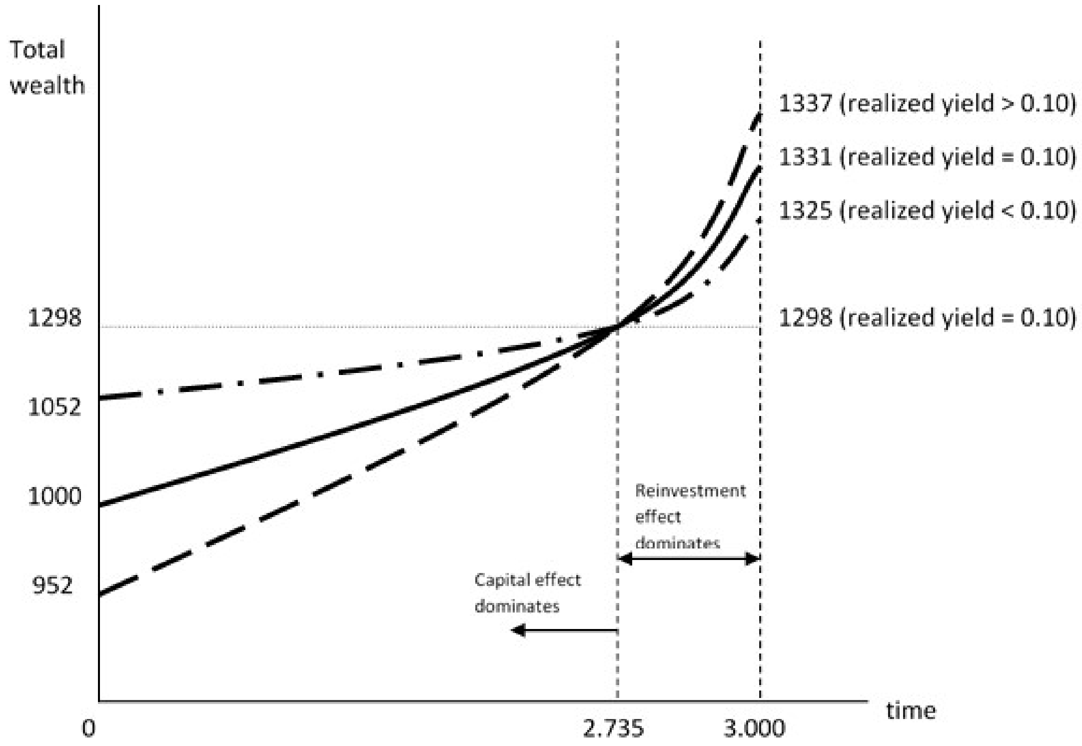
- **Capital Effect**: Bond price changes inversely with the yield.

Hence, it would seem to be possible to set up a portfolio of bonds such that these two effects cancel each other out. Indeed, if, through continual rebalancing of a portfolio, the duration is always fixed to a constant time point, then the portfolio value as of that time point will be immunized from interest rate variations. Consider a 3-year, 10% annual coupon, \$1,000 face value bond selling at par would have a duration of 2.735 years. Suppose there is a one-time change in the yield in the amount of ±0.02 from 0.10 now. The total wealth after 2.735 years under each scenario would be

$$y = 0.10: \quad 100(1.10)^{1.735} + 100(1.10)^{0.735} + \frac{1100}{(1.10)^{0.265}} = 1298.4$$

$$y = 0.12: \quad 100(1.12)^{1.735} + 100(1.12)^{0.735} + \frac{1100}{(1.12)^{0.265}} = 1298.6$$

$$y = 0.08: \quad 100(1.08)^{1.735} + 100(1.08)^{0.735} + \frac{1100}{(1.08)^{0.265}} = 1298.1$$



Example: (from Luenberger's book) Suppose there is an obligation to pay \$1 million at the end of 5 years. There are two bonds available for investment now. The first is a 12-year 6% bond with price 65.95. The second is a 5-year 10% bond with price 101.66. (These prices are calculated using the spot rates.) Using the Fisher-Weil duration formula with annual compounding, the modified durations of these bonds are found to be 7.07 years and 3.80 years, respectively. The present value and the modified duration of the obligation are in turn found as ($S_5 = 0.0975$)

$$P = \frac{1{,}000{,}000}{(1+S_5)^5} = \frac{1{,}000{,}000}{(1+0.0975)^5} \cong 628{,}000 \quad \Rightarrow \quad D^* = 5/(1.0975) = 4.56$$

Hence, to figure the allocation of funds between the two bonds so that immunization is accomplished, we solve

$$65.95 n_1 + 101.66 n_2 = 628{,}000$$

$$\left(\frac{65.95}{628{,}000}\right)(7.07) n_1 + \left(\frac{101.66}{628{,}000}\right)(3.80) n_2 = 4.56$$

to get $n_1 = 2{,}208$ and $n_2 = 4{,}744$ as the optimal quantities of bonds to buy now.

These two examples of immunization are also examples of ***duration hedging***. However, as some or all of the variables in the duration formula change, the position will have to be rebalanced accordingly to continually maintain the equality of duration to the remaining time until the investment horizon. This is called ***dynamic immunization***, which also has some shortcomings.

General problem of interest rate risk management

Consider a portfolio of two coupon bonds with a value of

$$V = n_1 P_1 + n_2 P_2$$

If the yields of these bonds change by respective amounts Δ_1 and Δ_2, the change in the value of the portfolio will be

$$\Delta V = -n_1 D_1 P_1 \Delta_1 - n_2 D_2 P_2 \Delta_2 + \frac{n_1 P_1 K_1 \Delta_1^2}{2} + \frac{n_2 P_2 K_2 \Delta_2^2}{2} + \cdots$$

As a first case, suppose that the changes in yields are small so that we can ignore the second-order and higher terms to get

$$\Delta V \approx -n_1 D_1 P_1 \Delta_1 - n_2 D_2 P_2 \Delta_2$$

We say that a bond portfolio is ***duration hedged*** when $\Delta V = 0$ so that the portfolio value is unaffected by changes in yields. To find the relative quantities of bonds to hold to get this result, we calculate

$$0 = -n_1 D_1 P_1 \Delta_1 - n_2 D_2 P_2 \Delta_2$$

$$\frac{n_1}{n_2} = -\frac{D_2 P_2 \Delta_2}{D_1 P_1 \Delta_1}$$

If Δ_1 and Δ_2 were known with certainty, then this solution would be optimal for small variations in yields. However, in reality, these yield changes cannot be predicted with certainty, making the approach of little practical usefulness. There is one exception and it is when there is a parallel shift in the yield curve. In this case, $\Delta_1 = \Delta_2$ and duration hedging is obtained by setting

$$\frac{n_1}{n_2} = -\frac{D_2 P_2}{D_1 P_1}$$

Clearly, one bond will be held long and the other will shorted. This balancing will be valid under two constraints:

- Yield changes are parallel and sufficiently small so that higher order terms can be ignored, and

- Time does not change! (If time passes as well, this riskless portfolio will earn the risk-free rate of interest if there is positive investment or it will incur a cost equaling to the risk-free rate if there is an open position.)

Example: Suppose we have n_1 units of the first bond and discover that the second bond is overpriced. To pursue a riskless arbitrage profit, we want to short the second bond without assuming any interest rate risk. Number of units of the second bond to be shorted is found by the above formula as $n_2 = -n_1 D_1 P_1 / D_2 P_2$.

The position in the example will be approximately riskless if interest rates vary in small amounts and if they all change by the same amount (parallel shift of the yield curve). The problem of large changes in interest rates may be handled by taking into account convexity in addition to duration. This strategy is called ***convexity hedging*** and it requires at least three bonds to set up a suitable position. Again, consider the value of a portfolio of three bonds:

$$V = n_1 P_1 + n_2 P_2 + n_3 P_3$$

If the yields of these bonds change by respective amounts Δ_1, Δ_2, and Δ_3, the change in the value of the portfolio will be

$$\Delta V = -n_1 D_1 P_1 \Delta_1 - n_2 D_2 P_2 \Delta_2 - n_3 D_3 P_3 \Delta_3 + \frac{n_1 P_1 K_1 \Delta_1^2}{2} + \frac{n_2 P_2 K_2 \Delta_2^2}{2} + \frac{n_3 P_3 K_3 \Delta_3^2}{2} + \cdots$$

Assuming the yields change in equal amounts so that $\Delta_1 = \Delta_2 = \Delta_3$, we say the portfolio is duration hedged and convexity hedged if $\Delta V = 0$:

$$\text{Duration hedge:} \quad -n_1 D_1 P_1 - n_2 D_2 P_2 - n_3 D_3 P_3 = 0$$

$$\text{Convexity hedge:} \quad n_1 K_1 P_1 + n_2 K_2 P_2 + n_3 K_3 P_3 = 0$$

Fixing any one of the three unknowns, we get two equations in two unknowns.

Example: Consider the following data about three coupon bonds A, B and C,

	A	B	C
Annual coupon (TL)	5	4	6
Face value (TL)	100	100	100
Maturity (years)	10	15	6
Yield-to-maturity	0.045	0.05	0.04
Price (TL)	103.96	89.62	110.48
Duration (years)	8.15	11.34	5.26
Modified duration	7.80	10.80	5.05
Convexity	76.23	148.62	32.45

Suppose Bond B is underpriced and hence we buy 1 unit of it. However, as this position will be subject to interest rate risk, we want to use other bonds to hedge against unfavorable yield changes. Consider the following three portfolios:

- Unhedged portfolio:

$$n_B = 1$$

- Duration-hedged portfolio: (short C to hedge B)

$$n_B = 1 \text{ and } n_C = -1.73$$

$$n_C = -n_B \frac{D_B P_B}{D_C P_C} = -\frac{(10.80)(89.92)}{(5.05)(110.48)} = -1.73$$

- Convexity-hedged portfolio: (using C and A to hedge B)

$$n_B = 1 \text{ and } n_A = -2.61;\ n_C = 2.06$$

$$-(10.80)(89.62) - n_A(7.80)(103.96) - n_C(5.50)(110.48) = 0$$

$$(148.62)(89.62) + n_A(76.23)(103.96) + n_C(32.45)(110.48) = 0$$

The results for various simulated changes in the yield curve are given below:

	Yields			Prices		
	A	B	C	A	B	C
0	*0,045*	*0,050*	*0,040*	*103,96*	***89,62***	*110,48*
1	0,050	0,055	0,045	100,00	**84,94**	107,74
2	0,040	0,045	0,035	108,11	**94,63**	113,32
3	0,045	0,070	0,020	103,96	**72,68**	122,41
4	0,075	0,080	0,070	82,84	**65,76**	95,23
5	0,045	0,030	0,070	103,96	**111,94**	95,23

	Unhedged	Duration-hedged		Convexity-hedged	
	Gain	Value	Gain	Value	Gain
0		-101,89		45,67	
1	**-4,68**	-101,80	0,09	45,67	0,00
2	**5,01**	-101,80	0,09	45,67	0,00
3	**-16,94**	-139,50	-37,61	53,30	7,63
4	**-23,86**	-99,31	2,58	45,56	-0,11
5	**22,32**	-53,13	48,75	36,56	-9,11

As seen, for parallel but relatively large shifts in the yield curve, convexity-hedging gives a better protection than duration-hedging only. However, for non-parallel and random changes in the yield curve, although still better, neither strategy gives the desired hedging effect.

Example: (Non-parallel change in the yield curve) Consider a Portfolio A, which is fully invested in 4-year zero, and a Portfolio B, which is 75% invested in a 2-year zero and 25% invested in a 10-year zero. Both portfolios have the same duration of 4 years. Their risk exposures are approximately

$$\frac{\Delta P_A}{P_A} \approx \frac{-4}{1+S_4}\Delta S_4, \quad \frac{\Delta P_B}{P_B} \approx 0.75\frac{-2}{1+S_2}\Delta S_2 + 0.25\frac{-10}{1+S_{10}}\Delta S_{10}$$

If the curve flattens or becomes steeper (S_2 and S_{10} change but S_4 does not), only portfolio B will be affected. If all three rates change, while one of the portfolios may lose, the other may well be hedged or even post a profit.

Can there be a parallel shift of the yield curve?

Two bond portfolios with the same value and durations may have different convexities. When the yield curve has a parallel shift, an arbitrage strategy of buying the high-convexity portfolio and selling the low-convexity portfolio will generate a net profit.

Example: Consider an upward-sloping zero-coupon yield curve with spot rates $S_1 = 0.05,\ S_5 = 0.06,\ S_{10} = 0.07$. The prices, durations and convexities are:

Maturity	Price	Modified duration	Convexity
1 year	95.24	0.95	1.81
5 years	74.73	4.72	26.70
10 years	50.83	9.35	96.08

Suppose Portfolio A has 1 unit of the 5-year bond. We then form a "barbell portfolio" B of the 1-year and the 10-year bonds such that both portfolios have the same value and modified duration. This is done by solving:

$$95.24 \times n_1 + 50.83 \times n_{10} = 74.73$$
$$95.24 \times 0.95 \times n_1 + 50.83 \times 9.35 \times n_{10} = 74.73 \times 4.72 \times 1$$

to find $n_1 = 0.43$ and $n_{10} = 0.66$. The convexities of the portfolios are calculated as $K_A = 26.70$ and $K_B = 44.09 > 26.70$. Starting with initial values of 74.73 each, consider the new values after the yield curve shifts:

Curve Shift	Spot rates	Portfolio A	Portfolio B
Upward	$S_1 = 0.06,\ S_5 = 0.07,\ S_{10} = 0.08$	71.36	71.30
Downward	$S_1 = 0.04,\ S_5 = 0.05,\ S_{10} = 0.06$	78.42	78.35

Selling the 5-year bond and buying the barbell always gives a net arbitrage profit:

Curve Shift	Profit from A	Profit from B	Net profit
Upward	71.36 – 74.73 = - 3.37	74.73 - 71.30 = 3.43	0.06
Downward	78.42 – 74.73 = 3.69	74.73 - 78.35 = - 3.62	0.07

These examples show that the conditions under which the duration model works cannot exist in equilibrium.

Term Structure Hypotheses

In practice, to set up proper hedging positions, we have to able to model the stochastic evolution of the term structure. That is, we have to discover the stochastic processes generating the interest rates. If this is not possible, we will need instruments that can provide the desired protection regardless of the stochastic nature of interest rates. All of these call for the use of derivative assets in interest rate risk management.

We will not go into the details of term structure models now but only mention some of the popular theories of the term structure. The question to address at this stage is "What do today's forward rates imply about tomorrow's spot rates?". In other words, how should we interpret today's yield curve or forward curve?

If $P_{0,T}$ is the current price of a zero with face value of 1 TL, then, for $(T > t > 0)$:

$$P_{0,T} = \frac{1}{(1+S_T)^T} = \frac{1}{(1+S_1)(1+F_{1,2})\dots(1+F_{T-1,T})}$$

$$F_{t,T} = \frac{(1+S_T)^T}{(1+S_t)^t} - 1 = \left(\frac{P_{0,t}}{P_{0,T}}\right)^{\frac{1}{T-t}} - 1$$

In a riskless economy

First, suppose that future interest rates are known with certainty. Then, the following arbitrage condition must hold:

$$F_{t,T} = S_{t,T}$$

Otherwise, for example, if $F_{t,t+1} > S_{t,t+1}$, then we short a zero maturing at t and buy $(1+F_{t,t+1})$ zeros maturing at $t+1$. The cost of the position will be $P_{0,t} - \left(1+F_{t,t+1}\right)P_{0,t+1} = P_{0,t} - (\frac{P_{0,t}}{P_{0,t+1}})(P_{0,t+1}) = 0$. The value at maturity time t will then be $-1 + \left(1+F_{t,t+1}\right)\left(1+S_{t,t+1}\right)^{-1} > 0$. This could not be the case if the no-arbitrage condition $F_{t,t+1} = S_{t,t+1}$ were true. (If $F_{t,t+1} < S_{t,t+1}$, we reverse the above position and get the same riskless arbitrage profit.)
Hence, in a world of certainty, the return on a T-year zero will be equal to the return on rolling over T one-year zeros:

$$(1+S_t)^T = (1+S_1)(1+S_{1,2}) \dots (1+S_{T-1,T})$$

However, since future interest rates cannot be known with certainty, we have to treat them as uncertain / risky and the above result will not hold as such.

Expectations Hypotheses

There are several versions of the expectations hypothesis, which differ in how risk is treated. We will summarize them under two versions: (1) The unbiased expectations version and (2) The present value version.

Both approaches assume that investors are neutral to the risk associated with term to maturity; that is, the risk of waiting until the future to see what the interest rates on borrowing or lending will be. The ***unbiased expectations hypothesis*** states that forward rates are unbiased predictors of future spot rates:

$$F_{t,T} = E_0[S_{t,T}]$$

where $E_0[S_{t,T}]$ is the expected value at time 0 of the spot rate $S_{t,T}$. In other words, based on the information available at time 0, the forward rate is the market's expectation of the spot rate for the future time interval $[t,T]$. The forward rate at which investors agree now to borrow or lend at a future date is equal to the expected spot rate in the future. They do not require a *liquidity premium* when buying longer bonds; that is, $L(t,T) \equiv F_{t,T} - E_0[S_{t,T}] = 0$. Then,

$$(1+S_t)^T = (1+S_1)\big(1+E_0[S_{1,2}]\big) \dots (1+E_0[S_{T-1,T}])$$

$$P_{0,T} = \frac{1}{(1+S_T)^T} = \frac{1}{(1+S_1)(1+E_0[S_{1,2}]) \dots (1+E_0[S_{T-1,T}])}$$

The present value version of the expectations hypothesis (also known as the ***local expectations hypothesis***) also assumes that investors are neutral to the risk associated with term to maturity and starts with conceptualizing the spot rate as the expected rate of return on a zero-coupon bond:

$$P_{0,T} = \frac{E_0[P_{t,T}]}{(1+S_t)^t}$$

Starting with $t = 1$, when $P_{0,T} = E_0[P_{1,T}]/(1 + S_1)^1$, and continuing with successive substitutions until $t = T - 1$ (remembering that, since face value is 1TL, $P_{T-1,T} = 1/\left(1 + S_{T-1,T}\right)^1$) we get

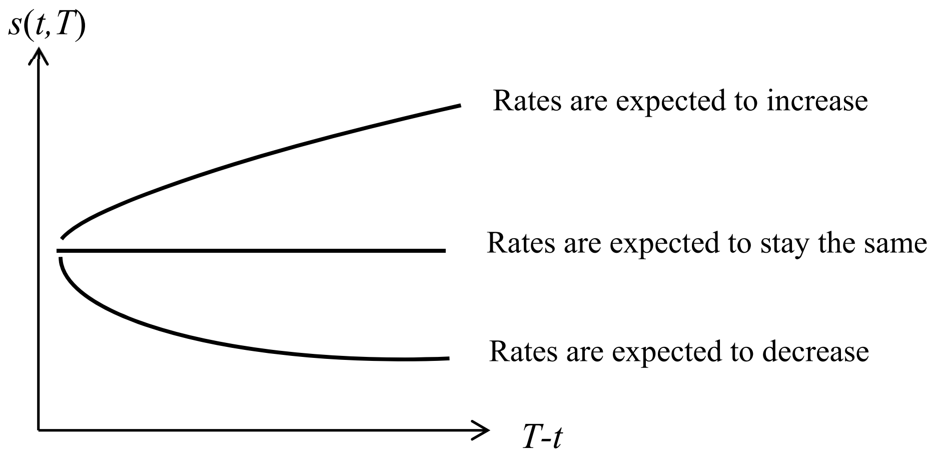
$$P_{0,T} = E_0\left[\frac{1}{(1 + S_1)(1 + S_{1,2}) \dots (1 + S_{T-1,T})}\right]$$

Finally, we compare that the two versions of the expectations hypothesis are not compatible since

$$E_0\left[\frac{1}{(1 + S_1)(1 + S_{1,2}) \dots (1 + S_{T-1,T})}\right] > \left[\frac{1}{(1 + S_1)(1 + E_0[S_{1,2}]) \dots (1 + E_0[S_{T-1,T}])}\right]$$

due to Jensen's inequality (if $f(x)$ is a convex function, then $E[f(x)] \geq f(E[x])$).

Under either version, the yield curve is interpreted as shown below:

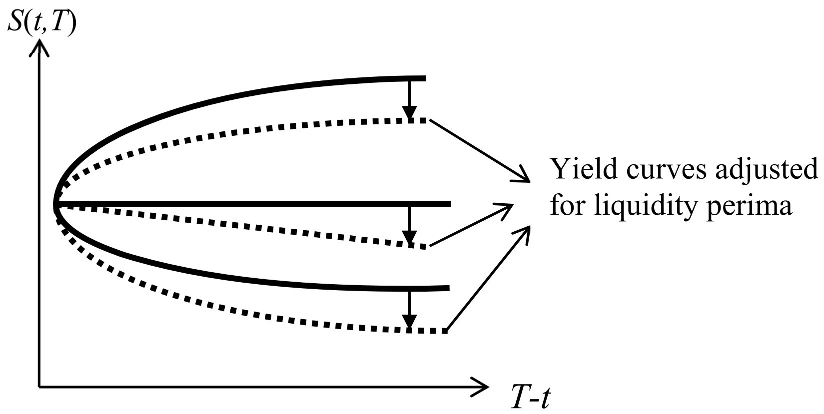


There is not much empirical evidence supporting this hypothesis. There may be two reasons for this. First, most investors are not risk neutral and hence they may require some premium for liquidity risk. Secondly, if spot rates and forward rates follow some stochastic process with time-dependent increments (for example, autocorrelation), then a given sample path of forward rates may imply a variety of different sample paths for future spot rates. Nonetheless, as the present value version attempts to base itself on a stochastic model for the whole term structure, it is a more sensible approach for more realistic models of interest rates.

As an alternative, the ***liquidity preference hypothesis*** maintains that investors require a "liquidity premium" for risk of price volatility associated with term to maturity and the liquidity premium is an increasing function to maturity. As a result, the relation between long rates and short rates takes the form

$$F_{t,T} = E_0[S_{t,T}] + L(t,T), \text{ or } (1+S_t)^t = \frac{E_0[P_{t,T}]}{P_{0,T}} + l(t,T)$$

and the yield curve is to be interpreted accordingly,



Since the yield curve is upward sloping most of the time and since investors cannot be expecting higher future rates most of the time, this hypothesis is more sensible empirically than the expectations hypotheses.

Note on Inflation

Suppose that a default-free bond's ***nominal rate of return*** over a one-year period is 8% (that is, 100TL grows to 108TL) and during the same period the realized rate of ***inflation*** has been 6%. A popular question is about the ***real rate of return*** on this investment. In other words, the question is actually about how much the ***purchasing power*** of the investment value has changed. Inflation is measured as the weighted-average change in the prices of a pre-defined basket of goods and services. If the beginning-of-period price of the basket were 100TL, then its price at the end of the period will be 106TL and the 108TL at the end of the period can purchase $108/106 = 1.019$ baskets. The investor's purchasing power has increased by 1.9%, which is then called the real rate of return.

Letting r, r_r and i denote the nominal rate of return, the real rate of return, and the rate of inflation, respectively, for the same time period, then the relation of the three numbers is given by

$$(1+r) = (1+r_r)(1+i)$$

$$r = r_r + i + ri$$

Re-arranging, we can write

$$r_r = \frac{r-i}{1+i}$$

Note that $(0.08 - 0.06)/(1 + 0.06) = 0.019$.

If τ is the tax rate on nominal interest income, then the after-tax real rate of return will be

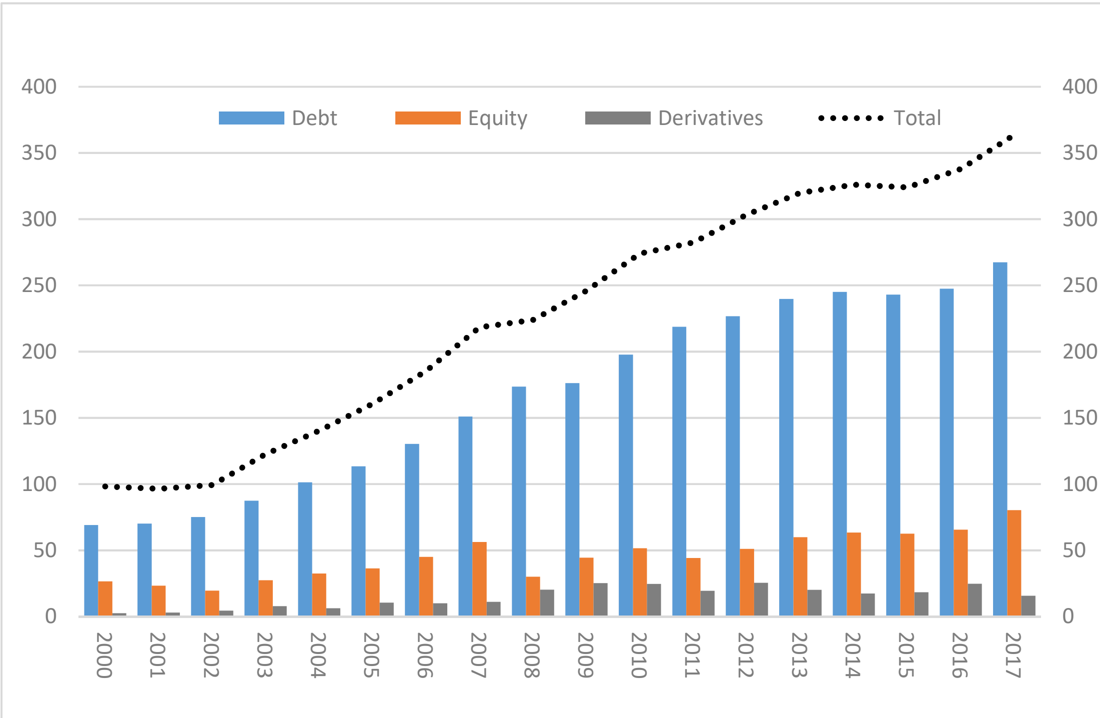
$$r_r^{(at)} = \frac{(1-\tau)r - i}{1+i} < (1-\tau)r_r$$

Thus, the real return is being taxed at a higher rate than the nominal tax rate. For example, if $\tau = 0.30$, then $r_r^{(at)} = ((1-0.3)(0.08) - 0.06)/(1+0.06) = -0.004$, which means you have lost in real terms.

Global Financial Markets

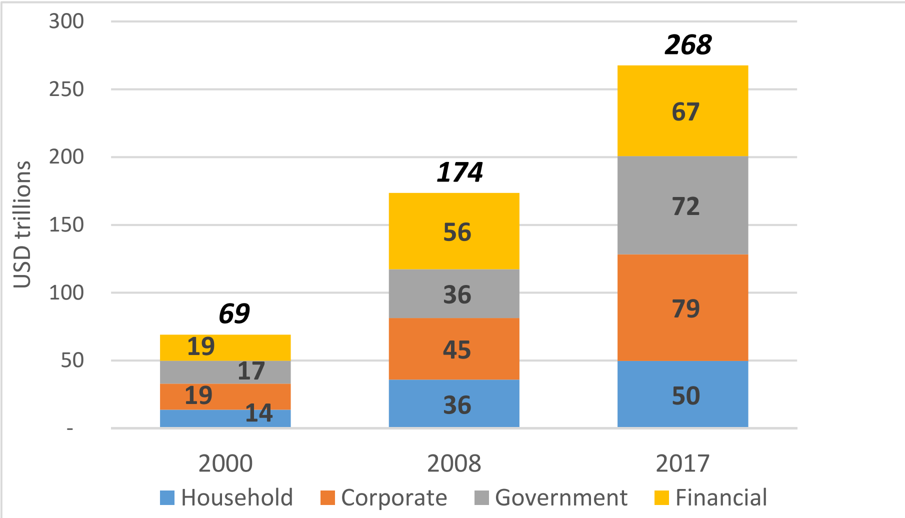
The global market values of financial assets during the period 2000 – 2017 are shown in the following chart. The total value in 2017 has reached about $370 trillion, which is about five times the world's total GDP of $81 trillion in the same year. During the period, the share of derivatives has ranged between 5% and 10%, the share of equity between 15% and 30%, and the share of debt (public debt, corporate bonds, securitized and un-securitized loans) between 70% and 80%.

Global financial assets

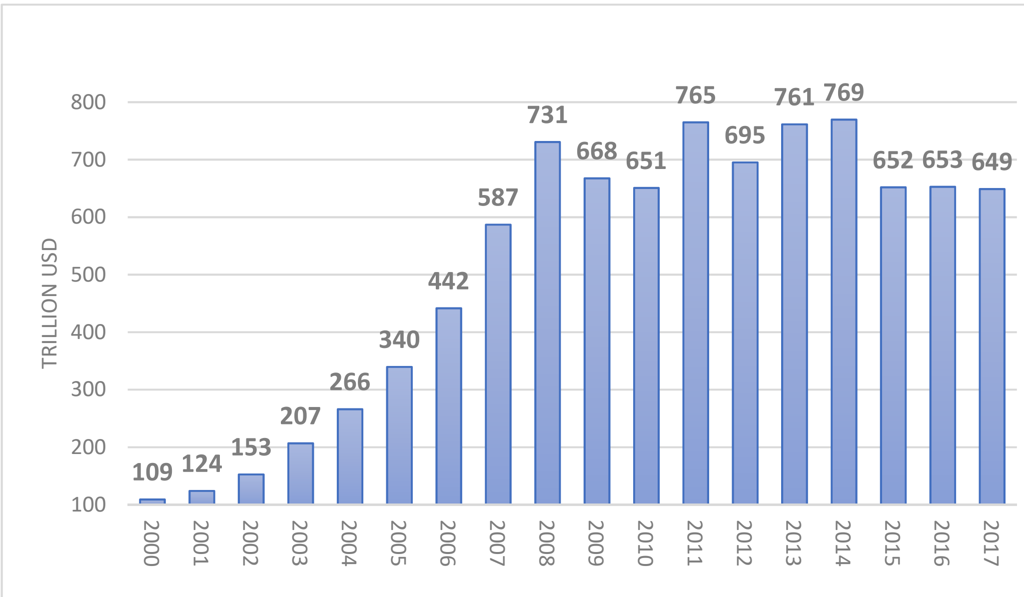


It is fair to say that the world is a world of debt, and this debt is not payable in any foreseeable length of time. It will have to be transferred from one generation to next generation.Despite all regulatory attempts, all types of debt have kept increasing even after the 2008 crisis. The following chart shows the global debt levels (which may underestimated):

Global Debt



Although the market value of derivatives is a relatively small percentage of the total financial assets, the total underlying notional value is naturally much larger. The total value of notional value traded in OTC markets and organized exchanges has reached a total of about $770 trillion in 2014, and most of it ($691 trillion) consists of OTC derivatives. By far, the biggest market is the market for interest rate derivatives, which in turn largely consists of interest rate swaps. Despite all the policy attempts and new regulations, the growth of derivatives has not stopped after the 2008 crisis, and also the desired growth in exchange trading is yet to be seen. As of 2017, global notional value was about $650 trillion.



Forward and Futures Contracts

Forward Contracts

Notation:
P_t : spot price at time t
$X_{t,T}$: forward price at time t for delivery at time T
V_t : theoretical value of a forward contract at time t
r : risk-free rate of interest

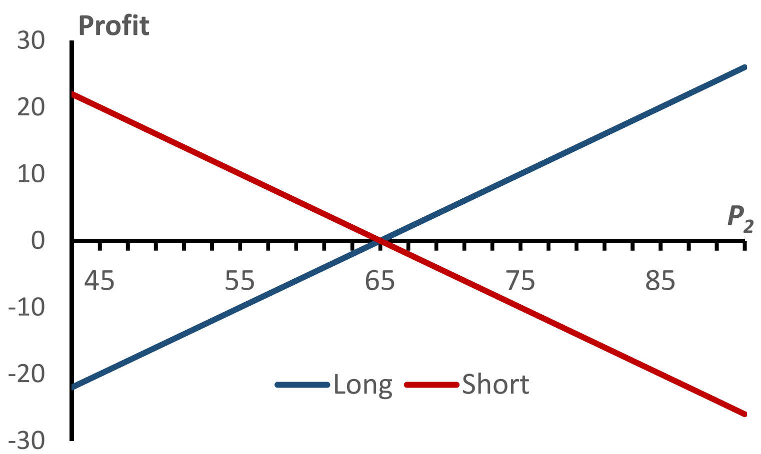
A forward contract is a contract issued at time t to buy or sell a given asset (the *underlying* asset) at a specific price $X_{t,T}$ (*delivery* or *exercise* price) at a specified date T (*delivery* or *expiration* date) in the future. In forward contracting:

- The market is over-the-counter and parties are typically a bank and its customer
- Default risk (*performance risk*) is important
- Transaction costs are usually small (commissions plus bid-ask spreads)
- Settlement is at maturity

Payoff profile:

- Long position - agrees to buy the asset and profits if $P_T > X_{0,T}$
- Short position - agrees to sell the asset and profits if $P_T < X_{0,T}$

Example: Consider a 2-year forward contract on oil with $X_{0,2} = \$65$. Suppose that the current spot price is $P_0 = \$50$. The profits of the short and long sides for various values of P_2 are shown in the following chart:



Relation between Forward and Spot Prices

The rule of no arbitrage implies that buying an asset now at time *t* and carrying it until time *T* must be equal to the forward purchase price at *T*. Thus, in general,

$$P_t e^{c(T-t)} = X_{t,T}$$

where $c = r - q + u - y$, the net annual percentage carrying cost for the time interval, is determined by

- r is the risk-free rate of interest, which is the basic cost component,
- $q \geq 0$ is the rate of cash flow to the holder of the underlying asset, with a decreasing effect on carrying cost, which may be
    - dividend yield on stocks
    - foreign interest rate on currency forwards
    - coupon yield on interest rate forwards
- $u \geq 0$ is *storage costs* (physical storage, insurance, etc.)
- $y \geq 0$ is the *convenience yield*, which is some measure of the benefit of holding the asset (usually consumable assets)

A deviation from this *no-arbitrage* or *cash-and-carry* condition implies an riskless arbitrage profit:

1. If $X_{t,T} > P_t e^{c(T-t)}$, then a net arbitrage profit of is possible through "*long arbitrage*" where we
    - At time *t*: borrow P_t liras at interest *r* to buy spot; short forward at $X_{t,T}$
    - At time *T*: pay $P_t e^{c(T-t)}$ for debt repayment and other carrying costs; deliver asset and receive $X_{t,T}$.

2. If $X_{t,T} < P_t e^{c(T-t)}$, then a net arbitrage profit of is possible through "*short arbitrage*" where we do the opposite of the above (long forward, short spot).

    (Note: If the asset is held for "consumption" (not investment), then short arbitrage may not work. In that case, as a no-arbitrage condition, $X_{t,T} \leq P_t e^{c(T-t)}$ may be all that can be said.)

Example: (Forward contract for gold) Consider a 3-month contract ($T = 0.25$) with $X_{0,0.25} = \$1{,}500/oz$ and currently $P_0 = \$1{,}400/oz, r = 0.10, u = 0.012$.

Since $P_0 e^{c(T-t)} = 1{,}400e^{(0.10+0.012)(0.25)} = 1{,}440 < 1{,}500$, long arbitrage is called for:

- At time 0: {borrow \$1,400 at 10% to buy spot gold; store for 3 months; short 3-month forward at \$1,500}, which requires zero net investment
- 3 months later: {deliver gold and receive \$1,500; pay back debt plus interest plus storage costs totaling \$1,440}, which gives a net profit of \$60

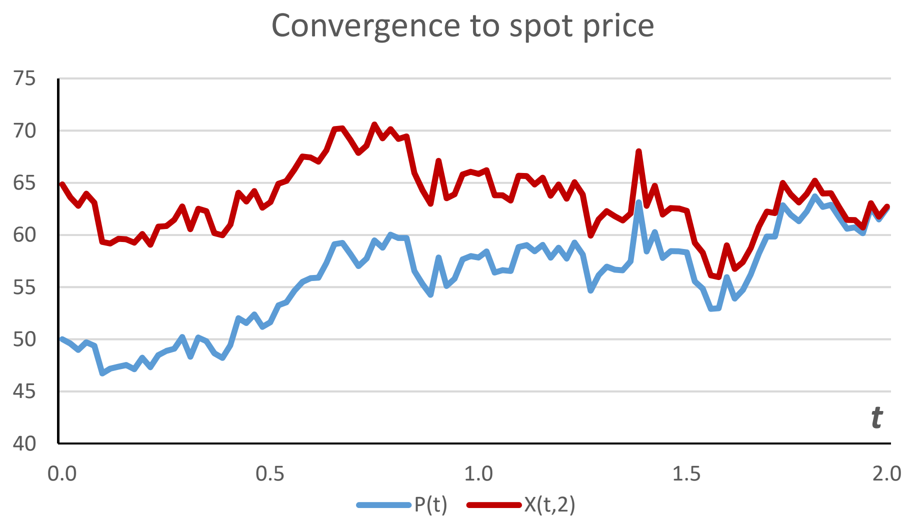
Note that this position is established at zero cost and there is no risk because all are contracted. If a sufficient number of such arbitrageurs trade so, then (keeping *r* and *u* constant), spot price will go up and/or forward price will come down until such arbitrage profits are no longer possible.

Example: Continuing with the example on oil contracts, suppose $r = 0.10$ and storage cost $u = 0.03$. Then, the 2-year forward price must be equal $50e^{(0.10+0.03)(2)} = 65$ and it is. In other words, the forward price is set such that $V_0 = 0$ and, at delivery date, $V_T = |P_T - X_{0,T}|$. At any other time, $V_t = P_t - X_{0,T}e^{-c(T-t)}$. For example, keeping u constant, consider then following change scenarios from the perspective of the long party:

$t = 0.5, P_{0.5} = 55, r = 0.10$: $V_{0.5} = 55 - 65e^{-(0.10+0.03)(2-0.5)} = 1.52$
$t = 0.5, P_{0.5} = 50, r = 0.12$: $V_{0.5} = 50 - 65e^{-(0.12+0.03)(2-0.5)} = -1.90$
$t = 1.5, P_{1.5} = 50, r = 0.10$: $V_{1.5} = 50 - 65e^{-(0.10+0.03)(2-1.5)} = -10.91$

However, since settlement will be at maturity *T*, these are only non-tradable theoretical values.

The difference between the forward price and the spot price ($P_t e^{c(T-t)} - P_t$) is known as the ***basis***. It is clear that as $t \to T$, the forward price on a newly issued contracts will converge towards the spot price and finally $X_{T,T} = P_T$ at delivery date. The following graph shows a simulation based on the previous example, where both spot prices and also interest rates change randomly.



As the delivery date approaches, the basis convergences towards zero but, depending on changes in interest rates and other relevant economic factors, it may fluctuate until then. It may even change sign. This unpredictable component of the basis is termed ***basis risk***, and it has to be taken into account in risk management with forward and especially futures contracts.

Example: A **F**orward **R**ate **A**greement (FRA) is a contract between two parties to exchange payments on a *notional principal*, on the basis of a short-term interest rate (the *reference rate*) over a predetermined time period at a future date. For example, {US$ 3X6 10%} denotes a contract to borrow (lend) 3 months from now a "notional principal" of $100m at a rate of $F_{0.25,0.50} = 10\%$ for 3 months. The borrower (contract buyer) is hedging against a rise in interest rates, and vice versa. Consider the following scenarios 3 months later ($t = 0.25$):

- $S_{0.25,0.50} = 11\%$: The long side (borrower) is paid by the short side (lender)

$$(100m)(0.11 - 0.10)(\tfrac{92}{360})/(1 + 0.11 \times \tfrac{92}{360}) = \$248{,}568$$

- $S_{0.25,0.50} = 9\%$: The long side (borrower) pays the short side (lender)

$$(100m)(0.10 - 0.09)(\tfrac{92}{360})/(1 + 0.09 \times \tfrac{92}{360}) = \$249{,}810$$

Example: (hedging interest rate risk) In the above example, if $100m will actually be needed 3 months from now and for a period of 2 months, one can buy a {US$ 3X5 10%} contract. This is equivalent to selling a zero maturing 5 months from now at a 3-month forward price of $\$98.3m = (\$100m)/(1 + 0.10 \times \frac{61}{360})$ and thus locking in a 10% borrowing cost.

Example: (Cash-Futures Arbitrage in the Treasury bill market) Suppose that the spot price of a 1-year T-bill is 970TL and the forward price for delivery in 23 days is 975TL. The *implied "repo" rate* is $(975/970)^{365/23} - 1 = 0.085$. Consider the following two disequilibrium scenarios:

- $S_{23/365} = 0.08 < 0.085$ ➔ "long arbitrage" (reverse repo, short forward):

Now	Borrow 970 at 8% to buy a 1-year T-bill	0
	Short a 23-day forward contract	0
	Net cash flow	0
23 days later	Pay back loan (970*1.08^23/365)	-974.72
	Deliver T-bill	975.00
	Net cash flow	0.28

Note that, since a short forward position is fully hedged by buying a T-bill with borrowed money, it is equivalent to selling a T-bill and lending the proceeds. In other words, a short forward position is the same as a reverse repo.

- $S_{23/365} = 0.09 > 0.085$ ➔ "short arbitrage" (repo, long forward):

Now	Sell short a T-bill and lend proceeds at 9%	0
	Long a 23-day forward contract	0
	Net cash flow	0
23 days later	Receive payment on loan (970*1.09^23/365)	975,28
	Take delivery of T-bill	-975.00
	Net cash flow	0,28

Note that, since a long forward position is fully hedged by selling a T-bill and lending the proceeds, it is equivalent to buying a T-bill with borrowed money. In other words, a long forward position is the same as a repo.

It is clear that the forward price of the zero must satisfy the no-arbitrage parity:

$$X_{23/365} = 970(1 + S_{23/365})$$

Example: (Interest rate parity) The no-arbitrage condition for forward contracts $P_t e^{c(T-t)} = X_{t,T}$ can be written for currency contracts as follows:

$$P_t e^{(r-r_f)(T-t)} = X_{t,T}$$

Let r is the local currency rate of interest, r_f is the foreign currency rate of interest, P_t is the spot exchange rate (local currency per foreign currency such as TL/\$) at time t, and $X_{t,T}$ is the forward exchange rate at time t for delivery at time T. This equilibrium condition is also known as ***interest rate parity.*** (Note that if P and X were defined in terms of \$/TL, then the equation would have to be written as $P_t e^{(r_f-r)(T-t)} = X_{t,T}$)

Setting $t = 0$ and $T = 1$, we can express the 1-year forward TL price of 1 USD in discrete time as

$$\frac{X_{0,1}}{P_0} = \frac{1 + r_{TL}}{1 + r_{\$}}$$

Suppose that the one-year spot rates on TL and USD zeros are $r_{TL} = 0.08$ and $r_{\$} = 0.02$. If the current spot exchange rate TL/\$ is $P_0 = 2.40$, then the 1-year forward exchange rate must be equal to

$$X_{0,1} = 2.40 \times \frac{1.08}{1.02} = 2.54$$

As an arbitrage example, suppose it is quoted that $X_{0,1} = 2.60 > 2.54$. This calls for a long arbitrage:

- Now: borrow 2.40TL at 8% to buy \$1 to invest at 2%, short 1.02 forward contracts at 2.60TL/\$
- One year later: Deliver \$1.02 to receive $\$1.02 \times 2.60 = 2.652$TL and pay back $1.08 \times 2.40 = 2.592$TL to make a net arbitrage profit of 0.06TL.

Given the interest rates, either the spot exchange rate is relatively low, or the quoted forward exchange rate is relatively too high, or both. These point to the nature of the expected market correction(s).

Example: (no-arbitrage parity with transaction costs) In practice, because of bid-ask spreads and other transactions costs, there will not be a unique forward price but rather an interval of no-arbitrage forward prices. For clarity of notation, let $t = 0$ and define:

- $P^{(a)}$ and $P^{(b)}$ are the ask price (price when selling) and the bid price (price when buying). In general, $P^{(a)} > P^{(b)}$ and the difference is called the spread.
- $r^{(b)}$ and $r^{(l)}$ are the borrowing rate and the lending rate of interest. In general, for most investors, $r^{(b)} > r^{(l)}$.
- z is the execution cost per transaction (spot or forward), if relevant.

Then, the no-arbitrage condition $Pe^{rT} = X_T$ (where, for simplicity, we let $c = r$ only) can be written as

$$(P^{(b)} - 2z)e^{r^{(l)}T} \le X_T \le (P^{(a)} + 2z)e^{r^{(b)}T}$$

Transaction costs give rise to no-arbitrage intervals, rather than no-arbitrage prices. The proof is left as an exercise.

Example: (Calculating forward exchange rates with bid-ask spreads) Consider the following data and assume $z = 0.01$ TL/trade:

	Bid	**Ask**
Spot TL/$	$P^{(b)} = 2.40$	$P^{(a)} = 2.41$
LIBOR($)	$r_{\$}^{(b)} = 0.025$	$r_{\$}^{(l)} = 0.020$
LIBOR(TL)	$r_{TL}^{(b)} = 0.086$	$r_{TL}^{(l)} = 0.080$

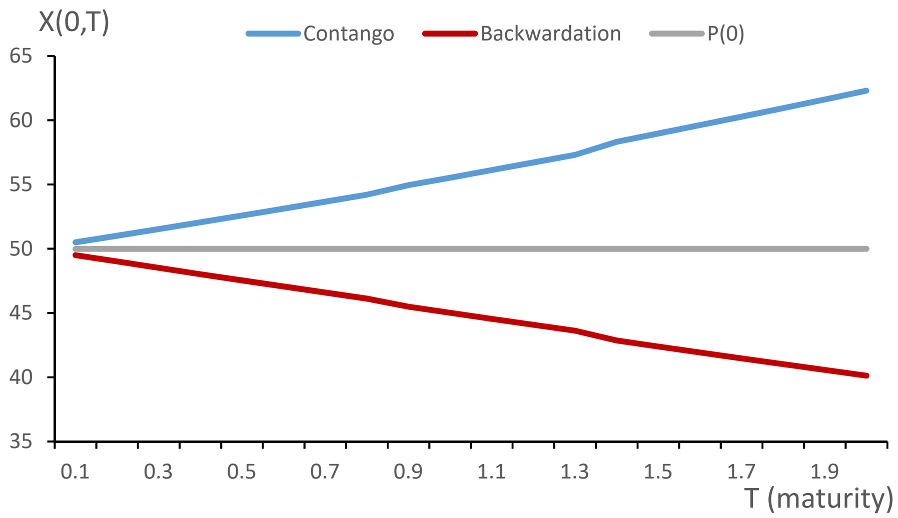
$$(2.40 - 2 \times 0.01)e^{(0.086-0.020)} \le X_1 \le (2.41 + 2 \times 0.01)e^{(0.08-0.025)}$$

$$X_1^{(b)} \equiv 2.542 \le X_1 \le 2.567 \equiv X_1^{(a)}$$

This example shows that transaction costs should be taken into account when evaluating arbitrage opportunities.

As a final note, the no-arbitrage condition $P_t e^{c(T-t)} = X_{t,T}$ often implies that $P_t < X_{t,T}$ because $c > 0$. The graph of forward prices as a function of maturity T would therefore be upward sloping. Such a market is said to be in ***contango*** (or, forwardation).

In practice, however, it may often be impractical to carry out arbitrage trading (especially in physical commodity contracts) because it involves spot buying, storage, transportation etc. Therefore, we may sometimes observe $P_t > X_{t,T}$ (even when $c > 0$), which results in a downward sloping forward price curve as a function of maturity. Such a market is said to be in ***backwardation***. These are shown in the following graph.



Futures Contracts

The definition of a futures contracts is the same as that of a forward contract. However, forwards and futures are markedly different contracts by design and by the way they are traded. The following table summarizes the major differences.

Feature	Forward Contract	Futures Contract
Market	Over-the-counter	Organized exchanges
Parties	Buyer vs seller	Buyer/seller vs the clearing house
Specifications	Freely designed by the two parties	Standardized contracts (price, quantity, quality, delivery procedures etc.)
Settlement	At maturity	Daily marking-to-market
Collateral	Decided by the parties	Margin requirements set by the exchange
Price limits	No	Daily limits
Default risk	Risk that parties involved may default	Risk that the clearing house may default
Trading	Parties have to wait until maturity to settle	Positions can be closed anytime by opposite trades

It is clear that a futures contract can be seen a series of daily forward contracts (with the additional flexibility of trading at any time during the day as well).

In futures, the risk that the buyer or the seller will default (performance risk in forwards) does not exist because the clearing house is the counter party in every contract (buyer to the seller and seller to the buyer). The risk that the clearing house itself will default is largely handled by daily settlement, required margins and daily price limits. Nevertheless, the crisis of 2008 has raised some concern about the default risk of clearing houses. Triggered by a systemic event or a market-wide panic, a sufficient number of investors may fail to respond to margin calls, making a clearing house unable to honor its commitments.

Example: To illustrate trading in a futures contract, consider buying 1,000 units of a 10-day futures contract with a futures price $X_{0,10/365} = 100\text{TL}$ and suppose the required margin is 5% of notional value at all times. If the annual rate of interest is 10% (assumed to stay the same for 10 days), the margin flows for a simulated series of futures prices are calculated below:

day	no of contracts	futures price	price change	notional value	required margin	margin balance	margin call
0	1,000	100.00		100,000	5,000	5,000	
1	1,000	100.39	0.39	100,390	5,020	5,392	0
2	1,000	100.02	- 0.37	100,020	5,001	5,023	0
3	1,000	99.67	- 0.35	99,673	4,984	4,677	306
4	1,000	99.16	- 0.51	99,159	4,958	4,470	488
5	1,000	98.00	- 1.15	98,005	4,900	3,805	1095
6	1,000	98.49	0.48	98,488	4,924	5,384	0
7	1,000	98.79	0.30	98,790	4,940	5,688	0
8	1,000	98.30	- 0.49	98,305	4,915	5,204	0
9	1,000	97.24	- 1.07	97,240	4,862	4,141	721
10	1,000	96.93	- 0.31	96,934	4,847	4,557	290

When the margin balance drops below the required level, there will be a margin call. In the example, this first happens when the forward price is 99.67 and an extra margin of $306 = 4{,}984 - 4{,}677$ (4,984 is 5% of 99,673) has to be deposited. It is assumed that the lending and borrowing rates are the same rate of 10%. The final payoff at maturity is calculated as:

$$4{,}557 - 5{,}000 + 290 - FV\left(r^{(b)} = 10\%; \{306, \dots, 290\}\right) = -3{,}056$$

Note that, if this were a forward contract, the net payoff at maturity would be equal to $96{,}934 - 100{,}000 = -3{,}066$. This is different from the future's payoff because of daily marking-to-market and margin requirements. (In practice, though, there may also be various transaction costs in forward contracting.)

Futures vs Forward Prices

Futures prices and forward prices may not be equal for two main reasons:

- The net payoff from marking-to-market in futures contracts carries the risk of the term structure of interest rates, which does not exist in forward contracts. This extra uncertainty in futures contracts may be good or bad, depending on the correlation between interest rates and futures prices.
- Forward contracts have an inherent risk of default by counter parties, which does not exist in futures contracts.

To compare the two prices, first consider the simple case where the cash flows to 2-period forward and futures contracts (lower-case x is the futures price) are:

	$t = 0$	$t = 1$	$t = 2$
Forward price	$X_{0,2}$		P_2
Cash flow	0	0	$P_2 - X_{0,2}$
Value of contract	0		$P_2 - X_{0,2}$
Futures price	$x_{0,2}$	$x_{1,2}$	P_2
Cash flow	0	$x_{1,2} - x_{0,2}$	$P_2 - x_{1,2}$
Value of contract	0		$(P_2 - x_{1,2}) + (x_{1,2} - x_{0,2})(1 + S_{1,2})$ $= (P_2 - x_{0,2}) + (x_{1,2} - x_{0,2})S_{1,2}$

When we compare the values of the contracts at $t = 2$, we see that the difference is due to the second term $(x_{1,2} - x_{0,2})S_{1,2}$ in the expression for the futures contract. This term has an uncertain value because the future spot rate $S_{1,2}$ and the futures price $x_{1,2}$ at $t = 1$ are both unknown. By definition, the present values (V for the forward and v for the futures) of the two contracts should both zero:

$$V_0 = \mathrm{PV}\{P_2\} - \frac{X_{0,2}}{\left(1 + S_{0,2}\right)^2} = 0$$

$$v_0 = \mathrm{PV}\{P_2\} - \frac{x_{0,2}}{\left(1 + S_{0,2}\right)^2} + \mathrm{PV}\{(x_{1,2} - x_{0,2})S_{1,2}\} = 0$$

Substituting the first expression into the second and rearranging, we get

$$X_{0,2} = x_{0,2} - (1 + S_{0,2})^2\ \mathrm{PV}\{(x_{1,2} - x_{0,2})S_{1,2}\}$$

Stated differently, $\mathrm{PV}\{(x_{1,2} - x_{0,2})S_{1,2}\}$ is the present value of the interest earned or spent from marking-to-market in the second period and this may be negative or positive. This present value is the properly discounted value of

$$\mathrm{E}[(x_{1,2} - x_{0,2})S_{1,2}] = \mathrm{E}[S_{1,2}]\,\mathrm{E}[(x_{1,2} - x_{0,2})] + Cov[x_{1,2}, S_{1,2}]$$

where, on the average, the first term will be positive. Hence, we can roughly conclude that

- If $Cov(x, S) > 0$, then $x_{0,T} > X_{0,T}$, the futures price will be greater than the forward price (contango)
- If $Cov(x, S) < 0$, then $x_{0,T} < X_{0,T}$, the futures price will be smaller than the forward price (backwardation)

For example, since bond prices are inversely related to interest rates, normal backwardation would seem to be the case for interest rate futures contracts.

Exercise: When future interest rates are known with certainty, show that forward and futures prices must be equal.

Despite the above theoretical result, empirical studies have found little or no significant differences between forward and futures prices. However, we should not overlook the fact that forwards and futures are different contracts, largely due to the cash flows generated by mark-to-market in futures. In applications of hedging, this difference may be very important to consider.

Eurodollar Futures

A Eurodollar contract is a cash-settlement contract on the interest rate on a notional Eurodollar deposit. The most popular type on the CME is a \$1 million 90-day contract based on the exchange-set LIBOR and minimum price movement is \$25 for 1 basis point (1 basis point is equivalent to 0.01 units of price or 0.01% in units of interest rate), or ½ or ¼ of \$25 depending on the expiry. The long contract (the buyer) agrees to lend and the short contract (the seller) agrees to borrow. Hence, a decrease in price (increase in rate) is a loss for the long and a gain for the short party. It is a convenient instrument to hedge short-term fluctuations in interest rates.

Example: Suppose today is March 17 and consider a 90-day contract expiring in March 30. The convention is to quote the "contract price" as 100 minus 100 times the 90-day LIBOR. Suppose the current contract price is 90.00, meaning that the 90-day futures rate is 10%. (The contract price here is not the price of a 90-day zero on the 10% rate, which would be $100/(1 + 0.10 \times \frac{90}{360}) = 97.56$) So, the Eurodollar contract price is meaningful only as a means of quoting the rate.

This contract commits the seller to borrow at 10% on March 30 for a term of 90 days. Assuming a required margin of 5%, or $50,000 per contract, a lending rate of 10% and a borrowing rate of 12%, the margin flows for a simulated series of contract prices are calculated below:

Date	Contract Price	Futures Rate	Price change (bps)	Mark to Market	Margin Balance	Margin call
17.3	90.00	0.100			50,000	
18.3	90.07	0.099	6.8	- 170.5	49,843	157
19.3	89.99	0.100	- 8.0	200.4	50,214	0
20.3	90.40	0.096	41.6	- 1,039.3	49,189	811
21.3	90.79	0.092	38.6	- 965.2	49,048	952
22.3	91.37	0.086	57.9	- 1,446.7	48,567	1,433
23.3	91.00	0.090	- 37.0	924.1	50,938	0
24.3	90.88	0.091	- 11.7	291.5	51,243	0
25.3	90.38	0.096	- 49.8	1,244.8	52,502	0
26.3	90.44	0.096	5.8	- 143.9	52,373	0
27.3	90.40	0.096	- 4.2	105.5	52,493	0
28.3	91.25	0.088	84.6	- 2,115.0	50,393	0
29.3	91.63	0.084	38.5	- 962.9	49,444	556
30.3	92.08	0.079	45.3	- 1,131.3	48,882	1,118

When the last mark-to-market payment is made on the expiration date, the long and short have no remaining obligations (cash settlement). The net profit of the short contract as of March 30 will be

$$48{,}882 - 50{,}000 + 1{,}118 - FV\left(r^{(b)} = 12\%; \{157, \dots, 1{,}118\}\right) = -\$5{,}036$$

Exercise: Prepare a similar table for the long contract and show that the net profit of the long side will be $5,395. Explain why this is different from $5,036.

Example: Suppose that, on March 30, a firm will actually need $100 million for a period of 90 days. The loan rate will be the 3-month LIBOR plus 200 bps at that time. To hedge the risk of a high LIBOR then, the firm will sell 100 March 30 contracts. Based on the above scenario, where the 90-day spot LIBOR on March 30 turns out to be 7.9%, the result will be:

Interest to be paid to bank after 90 days: $(0.079 + 0.02) \times \frac{90}{360} \times 100m = 2{,}479{,}148$

Futures profit on March 30: $-5{,}036 \times 100 = -503{,}643$

FV after 90 days: $-503{,}643 \times (1 + (0.02 + 0.079) * \frac{90}{360})) = -516{,}129$

Net interest cost: $2{,}479{,}148 + 516{,}129 = \ 2{,}995{,}277$

Effective annual rate: $\left(\frac{2{,}995{,}277}{100{,}000{,}000}\right) \times 4 = 11.98\% \neq 12\%$

Although the CFO of the firm may be fired because of the loss in futures trading in this scenario, the firm could lock in an approximate borrowing cost of 10% plus 200 bps. The CFO could defend the trade by showing that a similar borrowing cost would still prevail if the rates has increased instead and that this is indeed what hedging means.

Note that if, instead of Eurodollar futures, an FRA had been used for hedging purposes, the result would be:

Interest to be paid to bank after 90 days: $(0.079 + 0.02) \times \frac{90}{360} \times 100m = 2{,}479{,}148$

Forward profit: $-(92.08 - 90.00) \times 100 \times 25 \times 100 = -520{,}852$

Net interest cost: $2{,}479{,}148 + 520{,}852 = \ 3{,}000{,}000$

Effective annual rate: $\left(\frac{3{,}000{,}000}{100{,}000{,}000}\right) \times 4 = 12\%$

There is a difference between the futures hedge and the forward hedge ($11.98\% \neq 12\%$) and it is because of the fact that mark-to-market flows in futures trading are immediate but forward contracts are settled only once at maturity. It is possible to adjust the number of futures contracts so as to have a closer hedging effectiveness to forward contracts and this technique is called "*tailing the hedge.*"

As long as traders have fair access to all markets (exchanges and also OTC) and markets are competitive, the net hedging cost in futures contracts in equilibrium cannot be significantly different from that in forward contracts.

Bond Futures

Example: 30-Year U.S. Treasury Bonds (US, ZB) futures contract is a contract to deliver a nominal 6%, $100,000 face value T-bond with at least 15 years to maturity or to the first call date. As such, there is no one underlying asset but a basket of bonds, from which the seller can choose. This is why the exchange announces conversion factors. The invoice price (cash received by the short) equals the quoted futures price times the conversion factor for the bond delivered plus accrued interest. Details can be found at www.cme.com .

The conversion factor is the price of the delivered bond ($1 par value) to yield 6 percent. Suppose the bond to be delivered is an 10% semi-annual coupon T-bond with 20 years and 9 months to maturity. The value of this bond 3 months later is calculated as

$$0.05 + \sum_{t=1}^{41} \frac{0.05}{(1+0.03)^t} + \frac{1}{(1+0.03)^{41}} = 1.2499$$

The bond's value today is then $1.2499/(1.03)^{0.5} = 1.2316$. Subtracting the 3-month's accrued interest of $0.025, we get the bond price as $1.2066, which is the conversion factor.

Suppose the futures price is $1.0502. Then, the short will invoice the long for the following amount:

$$1.0502 \times 1.2066 + 0.025 = 1.2922 \;\text{ or }\; \$129{,}217$$

The cost to the short of obtaining the bond to deliver is the bond's price plus accrued interest, which was found as 1.2316. Therefore, the net return to the seller is $1.2922 - 1.2316 = 0.06057$, or $6,506.

As the short will want to maximize this return, he will choose the *cheapest-to-deliver* (CTD) bond among the ones available at settlement. The flexibility may be viewed as an embedded option held by the short party, which may be expected to have some positive value. The outcome would seem to depend on shape of the term structure of interest rates. In practice, there are two more embedded options in T-bond futures:

- *Wild card option*: The futures market closes at 2:00 pm, the cash market closes later, and the deadline for notifying about delivery is 8:00 pm. During this 6-hour period, cash prices may drop.
- *Timing option*: The futures contract stops trading on the seventh day preceding the last day of the month. During this period, all deliveries have to be made and the seller may benefit from any decline in cash prices during this period.

All of these embedded options seem to imply a positive value for the short side. Therefore, in an efficient market, the larger the value of these options, the lower should be the futures prices. The exact relationship is varying and it complicates the determination of futures prices for bonds. This example shows how financial instruments can be made unnecessarily complicated in time.

Stock Index Futures

A futures contract on a "notional" stock portfolio whose value is calculated as

$$\text{Index Value} \times \text{Multiplier} = \text{Notional value}$$

Example: The definition of the SP500 index from www.cme.com is "The S&P 500 Stock Index has long been the benchmark by which professionals measure portfolio performance. The S & P Corporation designed and maintains the S&P 500 to be an accurate proxy for a diversified equity portfolio. The Index is based on the stock prices of 500 large-capitalization companies. The S&P 500 is capitalization-weighted, representing the market value of all outstanding common shares of the firms listed (share price x shares outstanding). This means that a change in the price of any one stock influences the index in proportion to the relative market value of that firm's outstanding shares." The futures contract is specified as

Ticker Symbol	SP
Contract Size	$250 times S&P 500 futures price
Price Limits	5%, 10%, 15% and 20%
Minimum Price (Tick)	0.10 index points = $25 per contract
Contract Months	Mar, Jun, Sep, Dec
Regular Trading Hours	8:30 - 3:15 locally, and 3:45 - 8:15 on the GLOBEX
Last Trading Day	The Thursday prior to the third Friday (settlement) of the contract month
Position Limits	20,000 net long or short in all contract months combined

Example: The definition of the BIST30 (XU30) index can be found at www.borsaistanbul.com. The futures contract on this index is specified as:

Ticker Symbol	XU
Contract Size	TL100 times (BIST30 Index / 1,000)
Price Limits	15%
Minimum Price (Tick)	25 index points = TL 2.5 per contract
Contract Months	Feb, Apr, Jun, Aug, Oct, Dec
Regular Trading Hours	9:10 - 17:45 with a no-trade period from 12:30 to 13:55
Last Trading Day	Last business day of the contract month, also the settlement date

Index arbitrage

In what follows, daily mark-to-market flows are ignored for ease of discussion. Then, like in forwards, the no-arbitrage cost-of-carry condition for stock index futures can be written as:

$$P_0 e^{(r-d)T)} = X_{0,T}$$

where d is the annual dividend yield of the index stocks over the T-day period, and r is the T-day spot rate of interest as before. This relation is often used in *index arbitrage*, and its violation signals *program trading*:

- If $X_{0,T} > P_0 e^{(r-d)T)}$, a *short arbitrage* is signaled: Buy the index stocks (borrowing at r) and short futures.

- If $X_{0,T} < P_0 e^{(r-d)T)}$, a *long arbitrage* is signaled: sell or short the stocks in the index, invest the proceeds at r, and long futures.

The short-sale rule poses difficulties for long arbitrage in practice. Consider an arbitrageur attempting to sell short the S&P 500 stocks. He must wait for an up-tick in each of the 500 stocks! However, institutional investors, that already own the stocks in the index, sell those stocks, put the proceeds in debt instruments, and replace the stocks with futures contracts.

Some facts to use in various trading strategies involving index futures:

- If $r > d > 0$, then $X_{0,T} > P_0$, and vice versa.

- Sometimes, it may be needed to calculate the *implied dividend yield* as

$$d = r - \ln(X_{0,T}/P_0)/T$$

- Note that $\frac{\partial X_{0,T}}{\partial d} = -TX_{0,T}$ shows that higher dividend yield implies a lower futures price. Therefore, if, for example, the expected dividend yield until the expiry date is greater than the implied rate, then the futures price is relatively higher than what it should be (given the spot price and the rate of interest). This calls for a short position in the futures contract. This position, in turn, may be partially hedged by a long position in another contract, which is either correctly priced or underpriced.

- The sensitivity of the futures price to interest rate fluctuations may be estimated using

$$\frac{\partial X_{0,T}}{\partial r} = TX_{0,T}$$

  showing that contracts with longer maturities are more interest sensitive. This fact may be handy for purposes of *spread trading*. For example, suppose that we want to capitalize on an expectation of declining interest rates. For this purpose, we go long a short-term index futures contract and short another contract with a longer term. If the rates decline as expected, the futures price on the long-term contract will go down by more than that of the short-term one, resulting in a net profit.

- Using futures contracts on indexes with different compositions, we can trade on the relative performance of one market segment compared to another. Suppose that there are futures contracts on an energy-sector index and also on a financial-sector index. Suppose we currently anticipate the energy stocks to perform better than the bank stocks in general. To capitalize on this expectation, the proper position would be to go long on energy-index contracts and short financial-index contracts.

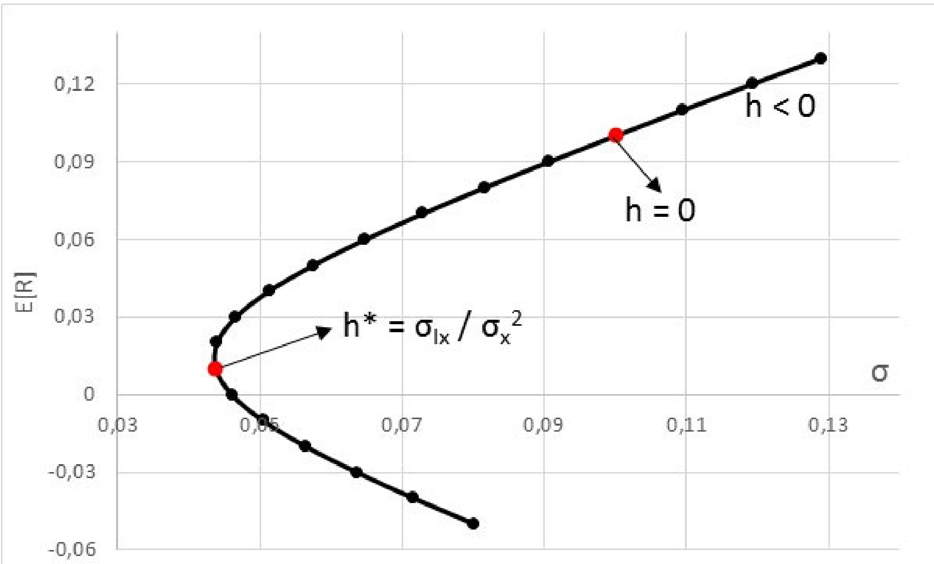
Hedging with Index Futures

Consider investing in an index mutual fund, which is identical in composition to the XU30 Index, and hedging it by selling h XU30 futures contracts ("short hedge"). Since the payoffs from the short futures and from the long stocks are inversely related, the hedged portfolio will have lower risk and lower expected return than an equivalent unhedged portfolio. Define $R_I = (I_t - I_{t-1})/I_{t-1}$ as the rate of return on the mutual fund portfolio (or, the index) and $R_x = (X_{t,T} - X_{t-1,T})/I_{t-1}$ as the rate of return on the futures position. Then, the expected return and variance of the hedged portfolio are

$$\mathrm{E}[R_h] = \mathrm{E}[R_I] - h\mathrm{E}[R_x]$$

$$\sigma_h^2 = \sigma_I^2 + h^2\sigma_x^2 - 2h\sigma_{Ix}$$

By changing h, different combinations of $E[R_h]$ and σ_h^2 can be obtained as shown in the following graph:



Solving $\partial\sigma_h^2/\partial h = 0$, we get $h^* = \sigma_{Ix}/\sigma_x^2 = \rho_{Ix}\sigma_I/\sigma_x$ as the minimum-risk hedge. Note that $\sigma_{h^*} = \sigma_I\sqrt{1-\rho_{Ix}^2} > 0$ when spot and futures prices are not perfectly positively correlated (that is, $\rho_{Ix} < 1$). This is part of the *basis risk*. In practice, though, the correlation between the index and index futures is often seen to be very close to one, $\rho_{Ix} \approx 1$. This is especially true when returns are calculated over short intervals (say, by using liquid nearest-to-maturity futures such as those with one month or less to expiry) and when the duration of the hedge is long. This is because the size of the changes in the basis become small relative to changes in the index over long periods of time.

The above result is important in that, without changing the composition of the stock portfolio, different risk-return combinations can be generated. For instance, if the stock market volatility is expected to increase, a high h value (if possible, h^*) would give protection on the downside. As another case in point, consider "insuring" the realized profits on the index fund against a probable market decline. However, it should be understood that, whatever the purpose may be, the cost of risk reduction by hedging is always a decrease in the expected rate of return on the position. This is the usual risk-return tradeoff.

To generalize, define the "hedge ratio" as

$$h = \frac{\text{(notional value/contract)} \times \text{(\# short contracts)}}{\text{(value of spot portfolio)}}$$

where, under this definition, $h > 0$ in a short hedge and $h < 0$ in a long hedge.

Example: Suppose the XU30 index is at 80,000, and the value of the portfolio being hedged is TL5m. If 500 contracts are sold, then

$$h = 500 \times 8{,}000/5{,}000{,}000 = 0.80$$

Now, suppose we want a hedge ratio of 0.50. For this, we have to go short 312 or 313 contracts as

$$0.50 \times 5{,}000{,}000/8{,}000 = 312.5$$

Note that, to maintain a fixed hedge ratio over time, the number of short contracts has to be continually adjusted as the index level and the portfolio value change.

Example: (switching from stocks to cash) Suppose that we currently have 1.6m TL invested in an XU30 index fund but want to exit from the stock market temporarily for a year and invest in Treasury bills. The index is at 80,000 and hence the notional value of one futures is 8,000TL. Assume the stocks in the index will pay no dividends this year and the 1-year T-bill rate is 10%. The 1-year futures price (with discrete compounding) must therefore be $X_{0,1} = 8{,}000(1 + 0.1) = 8{,}800$. If we short 200 contracts ($h = 1$), the cash flows will be:

	Cash flows	
	t = 0	t = 1
Index fund value	P_0 = 1,600,000	P_1
Short 200 futures	0	1,760,000 – P_1
Net position value	1,600,000	1,760,000

As a result, instead of going through the trouble of selling the stocks, we keep them and short futures. After one year, the rate of return is the risk-free 10%, regardless of the value of the stocks then. A higher hedge ratio will not change the result, but a lower hedge ratio would give a mixed result.

Now, suppose a one-factor market model such as the CAPM describes the behavior of stock prices. In empirical form of the CAPM, the excess rate of return on any portfolio p and its total risk (that is, the variance of the excess return) are given by

$$R_p - r = \alpha_p + \beta_p(R_I - r) + \varepsilon_p$$

$$\sigma_p^2 = \beta_p^2\sigma_I^2 + \sigma_\varepsilon^2$$

where r is the short-term risk-free rate of interest (such as the spot rate on 3-month Treasury bills), and $\beta_p = \sigma_{pI}/\sigma_I^2 = \rho_{pI}\sigma_p/\sigma_I$ is the "beta" of the portfolio relative to the index *I.* Since the theoretical market portfolio of the CAPM is not observable, the stock index is used here as a proxy for the market portfolio. Beta measures the sensitivity of the portfolio's returns to the returns on the market (index).

Setting $r = 0$ for simplicity, we can breakdown total return and risk into market-related and portfolio-specific components:

$$R_p = \{\beta_p R_I\} + \{\alpha_p + \varepsilon_p\} = \{\text{market-related return}\} + \{\text{specific return}\}$$

$$\sigma_p^2 = \{\beta_p^2\sigma_I^2\} + \{\sigma_\varepsilon^2\} = \{\text{market risk}\} + \{\text{specific risk}\}$$

For a well-diversified portfolio, $\alpha_p + \varepsilon_p \to 0$, $\beta_p \to 1$, and $\sigma_\varepsilon^2 \to 0$, and hence the portfolio will be very similar to the index in performance. Therefore, in order to get an above-average performance (that is, “to beat the market”), one or both of the following may be needed:

- **Market timing** is increasing the portfolio’s market exposure by raising its beta when the market is expected to go up, and vice versa. If a bull market is expected, a long hedge ($h < 0$) is called for. If a bear market is expected, a short hedge ($h > 0$) is called for.
- **Stock Selection** is buying underpriced stocks ($\alpha > 0$) and selling, or at least avoiding, overpriced stocks ($\alpha < 0$). If the beta (or variance of return) of a portfolio is changed as a result of selectively increasing the weights of undervalued stocks and decreasing the weights of overvalued stocks, a proper hedge ratio will bring back the beta to its original value.

Diversification, market timing and stock selection must be conducted in such a way that they will not counteract against each other and bring in unnecessary risks. This is easily accomplished using futures contracts and at minimum transaction costs.

For any portfolio *p,* the minimum-risk hedge ratio is calculated as before:

$$h^* = \frac{\sigma_{px}}{\sigma_x^2} = \rho_{px}\frac{\sigma_p}{\sigma_x}$$

which is basically the “beta of the portfolio relative to the futures price”. This can be easily estimated using historical data. In principal, this hedge ratio will not be equal to the traditional CAPM beta $\beta_p = \rho_{pI}\sigma_p/\sigma_I$, because $\rho_{pI} \neq \rho_{px}$ and also $\sigma_I \neq \sigma_x$. Typically, because of basis volatility, futures prices are more volatile than spot prices and $\sigma_I < \sigma_x$. Again, when the hedge is for long horizons, the magnitude of the changes in the basis will be small relative to price levels, and it can then be assumed that $h^* \approx \beta_p$.

If we short h futures to hedge this portfolio, the expected rate of return on the hedged portfolio will be given by

$$\mathrm{E}[R_h] = \mathrm{E}[R_I] - h\mathrm{E}[R_x]$$

Applying the CAPM on the both sides of the equation,

$$r + \beta_h(E[R_I] - r) = r + \beta_p(E[R_I] - r) - h(r + \beta_x(E[R_I] - r))$$

Assuming that hedging is for long horizons so that $\beta_x \approx \beta_I = 1$, the adjusted beta of a portfolio hedged at a ratio h is given by $\beta_h = \beta_p - h(E[R_I]/(E[R_I] - r))$. If r is sufficiently small relative to the market return, then

$$\beta_h \approx \beta_p - h$$

To increase the portfolio's beta, set $h < 0$. To decrease the beta, set $h > 0$. This rule-of-thumb can be very handy in managing market risk due to market timing and stock selection strategies.

An Application in BIST

We will use monthly data during the time period from Jan 2009 to Jan 2013, and seven relatively liquid common stocks (AKB, GRB, TCELL, KOCH, BIM, THY, ARC). The underlying index is XU30 and the nearest-to-maturity futures contract will be the hedging instrument. The price data is partially given below:

Date	Expiry	X	P	AKB	GRB	TCELL	BIM	KOCH	THY	ARC
1.30.09	2.27.09	33,650	33,503	4,78	2,25	8,85	35,50	2,31	6,10	1,76
2.27.09	4.30.09	30,400	30,691	3,92	2,07	8,45	33,50	2,12	5,60	1,74
3.31.09	4.30.09	32,100	32,815	4,88	2,36	8,15	35,25	2,34	6,55	1,83
4.30.09	6.30.09	39,300	40,390	6,20	3,36	8,20	44,00	2,92	7,85	2,68
5.29.09	6.30.09	44,175	44,409	6,45	3,84	8,15	48,75	3,52	8,85	2,19
6.30.09	8.31.09	46,600	46,699	6,90	4,18	8,55	54,00	2,66	2,32	2,33
7.31.09	8.31.09	54,425	54,472	8,30	5,20	9,35	58,00	3,46	2,32	3,14
8.31.09	10.30.09	59,575	59,053	8,60	5,55	9,75	56,00	3,98	2,88	4,30
9.30.09	10.30.09	60,375	60,604	8,60	5,60	10,60	60,50	3,90	3,86	4,30
10.30.09	12.31.09	60,225	59,593	8,20	5,50	10,00	55,00	3,84	4,22	4,95
11.26.09	12.31.09	57,200	57,647	8,20	5,20	9,30	60,00	3,74	4,86	5,05
12.31.09	2.27.10	67,700	66,992	9,45	6,35	10,60	69,50	4,42	5,70	5,85
					....					
12.31.10	2.27.11	82,300	81,338	8,58	7,82	10,55	52,50	7,52	5,40	7,80
					....					
12.30.11	2.29.12	62,350	61,698	6,02	5,90	8,88	52,50	5,68	2,12	6,12
					....					
12.31.12	2.28.13	98,475	97,728	8,80	9,26	11,55	87,25	9,26	6,26	11,70

The period 2009-2010 is the estimation period and 2011- 2012 is the application period. The first finding is that the correlation between the returns on the market index and the futures contract is almost perfect ($\rho_{Ix} = 0{,}992$). This implies almost zero basis risk and also makes the XU30 contract a convenient tool of hedging. The average value of the relative basis $(I - X)/X$ from 2009 to 2012 is around 0.1%.

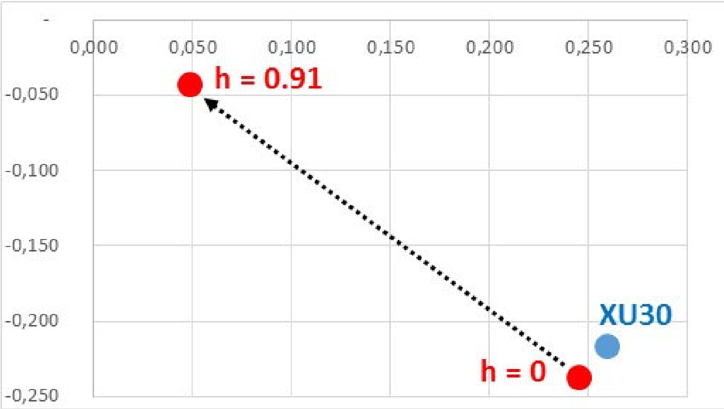
The calculated parameters during the estimation period (2009-2010) and the subsequent hedging period (2011) are given in the following tables:

	Estimation period Jan 2009 – Dec 2010			
	R	***σ***	***h******	***β***
AKB	0.298	0.435	1.00	1.10
GRB	0.636	0.409	1.22	1.36
TCELL	0.091	0.254	0.49	0.53
BIM	0.200	0.603	0.94	1.07
KOCH	0.605	0.417	0.90	1.00
THY	- 0.057	1.065	0.59	0.70
ARCLK	0.765	0.458	0.93	1.05
Portfolio	**0.425**	**0.532**	0.91	1.02
Index	**0.503**	**0.280**		

The portfolio in the example is an equally-weighted portfolio of the seven stocks. The estimates of the average return R and the standard deviation σ of the return are annualized sample estimates. The beta is the estimate of the slope coefficient b_1 in the empirical one-factor model $R_{p,t} = b_0 + b_1 R_{I,t} + \varepsilon_{p,t}$ (with a non-zero intercept) and h^* is calculated as the sample estimate of $h^* = \sigma_{IX}/\sigma_X^2$.

	Hedging period Jan 2011 – Dec 2011					
	$\boldsymbol{\sigma_h}$	$\boldsymbol{\sigma_p}$	$\boldsymbol{R_h}$	$\boldsymbol{R_p}$	$\boldsymbol{\beta_p}$	$\boldsymbol{\beta_h}$
AKB	0.124	0.282	- 0.065	- 0.271	0.98	-0.03
GRB	0.109	0.323	0.073	- 0.207	1.18	-0.05
TCELL	0.168	0.225	- 0.026	- 0.138	0.59	0.09
BIM	0.286	0.165	0.273	0.012	0.04	-0.91
KOCH	0.161	0.343	- 0.002	- 0.202	1.21	0.30
THY	0.288	0.384	- 0.504	- 0.575	1.06	0.46
ARCLK	0.274	0.441	0.079	- 0.142	1.36	0.42
Portfolio	**0.049**	**0.246**	**- 0.043**	**- 0.238**	**0.92**	**0.00**
Index		**0.259**		**- 0.217**		

The market index in 2011 was down by a -21.7% annual rate of return and, as a result, this straight hedge turned out to be beneficial. All of the hedged returns on individual stocks are higher than their unhedged returns, and all of the standard deviations of hedged returns are lower than those of unhedged returns (except for BIM, where volatility actually goes up). This shows a significant improvement in the risk-return tradeoff, which is shown in the following chart for the equally-weighted portfolio (red) versus the XU30 Index (blue).



As expected, hedging worked fine in a down market. The beta of the portfolio during the estimation period 1/2009 – 12/2010 was 1.02, and it is estimated as 0.92 during the hedging period of 1/2011 to 12/2011. The beta of the hedged portfolio was very close to 0 during the same period. Now suppose that, for 2012, we expect an up-turn in the market and therefore we want to increase the beta of the portfolio to $1.10 = 0.92 \times 1.20$ (20 percent higher exposure than current level) . This requires a hedge ratio of $-0.18 = 0.92 - 1.10$ (using the approximation $\beta_h \approx \beta_p - h$), which is a long position in the XU30 futures contracts. The result is a success and it is shown below:

	Market timing period Jan 2012 – Dec 2012					
	σ_h	σ_p	R_h	R_p	β_p	β_h
Portfolio	**0,278**	**0,237**	**0,965**	**0,804**	**0,97**	**1,15**
Index		**0,239**		**0,623**		

Naturally, we were lucky in that the year 2012 indeed turned up to be a bull market. But then, since we are managing the position with liquid futures contracts and no spot trading, we can easily rebalance and change exposure at any point in time during the year as new information arrives.

Swaps

A financial swap is a contract between two *counterparties* to exchange, or swap, a series of well-defined future cash flows. The currency swap between IBM and the World Bank in 1981 is often cited as the start of the swap market, and it is a big OTC market (the notional volume in 2014 was more than 500 trillion dollars):

- Interest rate swaps
- Currency swaps
- Equity swaps
- Credit default swaps
- Commodity swaps

For a swap contract to exist, there seem to be to requirements:

- A net advantage to both counterparties
- An intermediary to arrange the deal

Interest Rate Swaps

The two parties exchange the interest payments on a notional principal. Typically, one side pays a fixed interest rate and the other party pays a floating interest rate (usually, the 6-month LIBOR).

Example: (to show the comparative advantage argument) Suppose that two firms, A and B, can borrow in the market at the following rates:

	Firm A	Firm B	Comparative Advantage
Fixed rate of interest	0.10	0.12	0.02 (200 bps)
Floating rate of interest	LIBOR + 0.01	LIBOR + 0.015	0.005 (50 bps)

The two firms enter into a swap contract as:

	A's cash flows	B's cash flows
A borrows at the fixed rate	-0.10	
B borrows at the floating rate		- (LIBOR + 0.015)
A receives fixed rate and pays floating rate to B	+0.10 – LIBOR	
B receives floating rate and pays fixed rate to A		+LIBOR – 0.10
Net interest cost	- LIBOR	-0.115
Net advantage	+0.01	+0.005

Of course, the intermediary will also take part of the action.

Example: (*plain vanilla swap*) Party A pays a fixed rate of 6% per year on a semi-annual basis, and receives from Party B LIBOR. The current 6-month LIBOR is 5.5% per year. The swap contract is a 2-year swap on a notional principal of $1m and it is initiated on 01/01/2015. The cash flows to both parties are calculated as follows:

First fixed payment: $(\$1\text{m}) \times \left(0.06 \times \frac{182}{365}\right) = \$29{,}918$

First floating payment: $(\$1\text{m}) \times \left(0.055 \times \frac{182}{360}\right) = \$27{,}806$

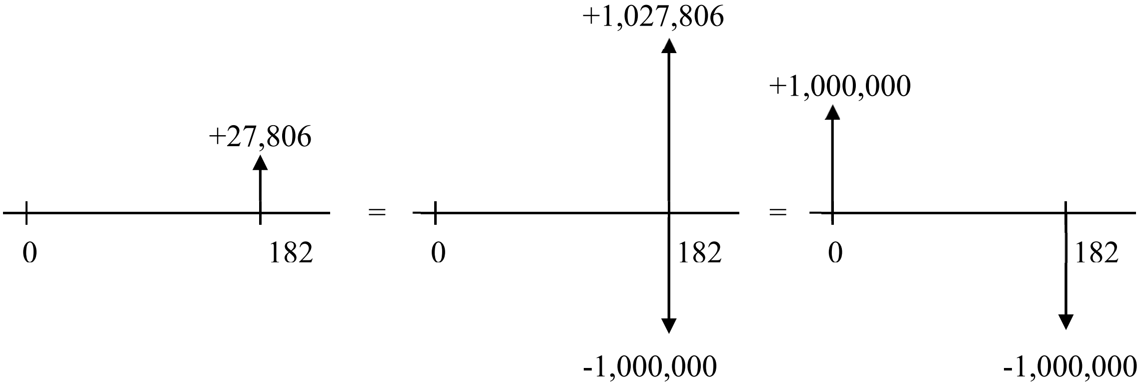
Then, payments are netted on 01/06/2015 and A pays to B $2,112. Note also: (1) The first floating payment is based on the LIBOR rate on the initiation date and hence it is a known amount (there are also *LIBOR-in-arrears swaps*, where the first floating payment would be based on the LIBOR rate on 01/06), and the fixed rate calculation is based on a 365-day year and the floating rate calculation is based on a 360-day year. These two points are important in the pricing of swaps. The cash flows for both parties are calculated below, where $l_{t,k}$ is the k-day LIBOR at time *t* ($l_{0,182} = 0.055$):

t (days)	Date	A to B	B to A	Net A - B
182	01/06/15	29,918	27,806	2,112
365	01/01/16	30,082	$(\$1\text{m}) \times l_{182,183} \times 183/360$	?
547	01/06/17	29,918	$(\$1\text{m}) \times l_{365,182} \times 182/360$	?
730	01/01/17	30,082	$(\$1\text{m}) \times l_{547,183} \times 183/360$	?

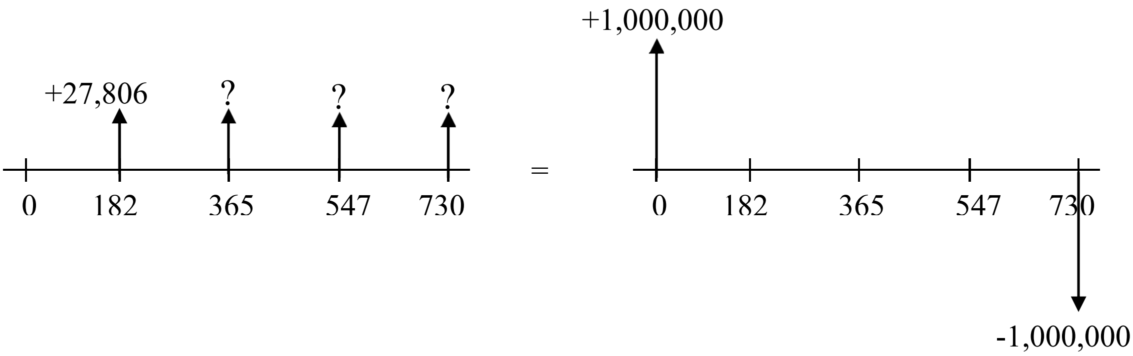
Let us now try to find the value of the swap from B's perspective (B is receiving fixed and paying floating). The value of the swap should then be equal to the PV of the fixed side minus the PV of the floating side. The PV of the fixed side is simply found by using the spot rates s_t as in any bond valuation:

$$V_{fix} = \frac{29{,}918}{(1 + S_{182})^{182/365}} + \cdots + \frac{30{,}082}{(1 + S_{730})^{730/365}}$$

The PV of the floating side is similar to that of a floating-rate bond, which sells at par immediately after the rate is set at any point in time, except that there is no principal payment. To see this, consider the first floating payment of 27,806 and how we can transform it into two zero-coupon bonds:



Thus, given the current LIBOR of 0.055, receiving 27,806 in 182 days is equivalent to receiving 1,000,000 now and paying 1,000,000 in 182 days. Similarly, receiving $1{,}000{,}000 \times l_{182,183} \times 183/360$ in 365 days is equivalent to receiving 1,000,000 in 182 days and paying 1,000,000 in 365 days, and so on. Thus, the floating side can be pictured as



The PV of the floating side is then found as

$$V_{float} = 1{,}000{,}000 - \frac{1{,}000{,}000}{(1 + l_{0,730})^{730/360}}$$

Then, the value of the swap is $V = V_{fix} - V_{float}$, which can be positive or negative depending on the interest rates.

Exercise: In the above example, we derived the value of a swap as the difference between two bonds. As an alternative, derive a swap valuation framework, where the swap contract is viewed as a portfolio of forward contracts.

To generalize, suppose the number of days between payments is constant and define F as the notional principal, r as the swap rate (that is, the fixed rate of interest per payment period), N as the number of payments, and t_i as the number of days to the i^{th} payment. Also define the PV of a risk-free zero-coupon bond which pays \$1 in T days as $B(0,T) = (1 + S_{0,T})^{-T/365}$ and the PV of a Eurodollar bond which pays \$1 in T days as $L(0,T) = (1 + l_{0,T})^{-T/360}$. Then, the PV of the swap to the party that receives fixed and pays float may be expressed as

$$V = \{Fr \sum_{i=1}^{N} B(0,t_i)\} - \{F(1 - L(0,t_N))\}$$

A ***par swap*** is a swap with a net present value of zero (that is, $V = 0$) and the ***quoted swap rate*** is the rate which gives the result:

$$r = \frac{1 - L(0,t_N)}{\sum_{i=1}^{N} B(0,t_i)}$$

Example: Continuing with the previous example, suppose we have the current prices of various T-bills and 2-year Eurobonds:

$\boldsymbol{t_i}$ (days)	$\boldsymbol{P(0,t_i)}$	$\boldsymbol{L(0,t_i)}$
182	0.983	
365	0.962	
547	0.932	
730	0.907	0.898

where $0.898 = 1 - (1 + 0{,}055)^{-730/360}$. (Note that, as in the example, we usually observe $S_{0,t} < l_{0,t}$, which shows that US Treasury bonds are perceived to be less risky than highest-rated Eurodollar bonds. The difference is known as the Treasury – Eurodollar (TED) spread, which is often small in calm markets and which also gets larger in volatile times.) Solving for the par swap rate, we find

$$r = \frac{1 - 0.898}{0.983 + \cdots + 0.907} \times 2 = 0.054$$

In the absence of any transaction costs, party A would pay the fixed rate of 5.4% to party B on a semiannual basis.

Published ***swap curves***, swap rates for various maturities, are based on these rates. The swap market is very liquid and swaps are available for almost the whole spectrum of maturities. For this reason, many people use the swap rates to calculate the prices of zeros and draw the yield curve, and not vice versa. In doing so, it has to be assumed that $B(0,t) = L(0,t)$, which will be true only if both the T-bill rates and also the LIBORs refer to the same underlying stochastic process of spot rates and both rates are quoted in the same currency. Empirically, since the TED spread is positive, this assumption is technically incorrect. Still, as a first-step approximation, suppose that $r_{t_N} \approx (1 - B(0,t_N))/\sum_{i=1}^{N} B(0,t_i)$. Starting with $r_{t_1} = (1 - B(0,t_1))/B(0,t_1)$, we get $B(0,t_1) = (1 + r_{t_1})^{-1}$. Then, by successive substitutions using the observed swap rates $r_{t_1}, r_{t_2}, \dots, r_{t_N}$, we can solve for zero prices as

$$B(0,t_j) = \frac{1 - r_{t_j} \sum_{i=1}^{j-1} B(0,t_i)}{1 + r_{t_j}}$$

to finally get the spot rates as

$$S_{0,t_j} = \left(\frac{1}{B(0,t_j)}\right)^{1/j} - 1$$

for all $j = 1, \dots, N$. This technique is called "*bootstrapping*" the yield curve.

There are several versions of interest rate swaps, some of which are:

- Amortizing / accreting swap: notional principal is reduced / increased over the life of the swap
- Constant yield swaps: both sides are floating
- Rate-capped swaps: the floating rate is capped
- Puttable and callable swaps: parties have the option to cancel the swap at certain times
- Forward (or, deferred) swaps: regular swap to be initiated at a future date
- Extendible swaps: the maturity may be extended by the holder at the original swap rate

The valuation of these "exotic" swaps, which contain embedded option-like features, is not straightforward because they are model dependent in that the cash flows are not defined precisely. We will address these after we discuss option pricing.

Currency Swaps

A currency swap is a contract to swap a series of cash flows in one currency with a series of cash flows in another currency. Typically, it involves the exchange of both interest payments and also principal.

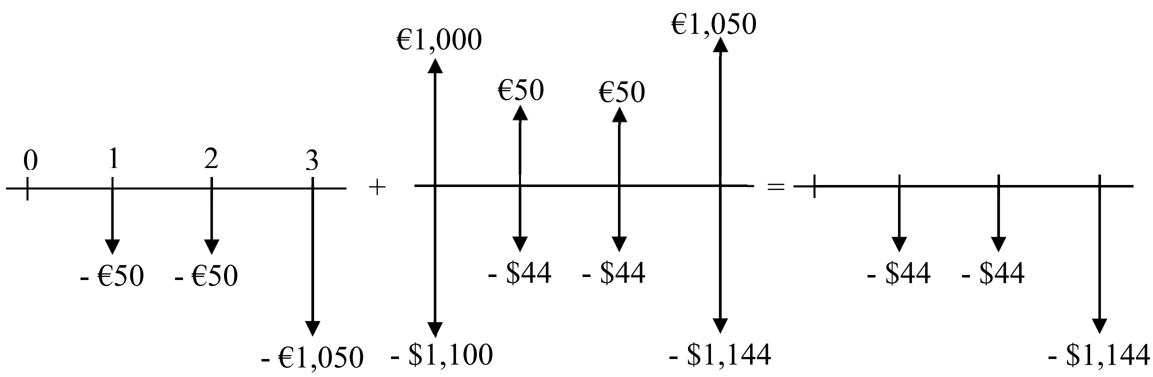
Example: A dollar-based firm has issued a 3-year 5% euro-denominated bond with a face value of €1,000 and selling at par. The firm wishes to guarantee the dollar value of the interest and principal payments. The spot exchange rate is $1.10/€ and the dollar-denominated rate of interest is 4%.To hedge against the currency exposure, the firm can buy euros forward. The following table gives the forward exchange rates and the position cash flows.

Year	Forward exchange rate	Unhedged cash flow (€)	Hedged cash flow ($)
1	1.09	50	54.5
2	1.08	50	54.0
3	1.07	1,050	1,123.5

Note that the PV of the dollar cash flows

$$54.5(1.04)^{-1} + 54.0(1.04)^{-2} + 1.123.5(1.04)^{-3} = \$1{,}100 = €1{,}000 \times \$1.10/€$$

as it should be. Hedging does not change the value of the obligation. As an alternative to hedging, the firm can enter into a currency swap, where it pays dollars (4% on a $1,100 bond) and receives euros (5% on a €1,000 bond). This eliminates the euro exposure as shown below:



We can use the same method in valuing currency swaps as in interest rate swaps, except we have translate the PVs into the reference currency:

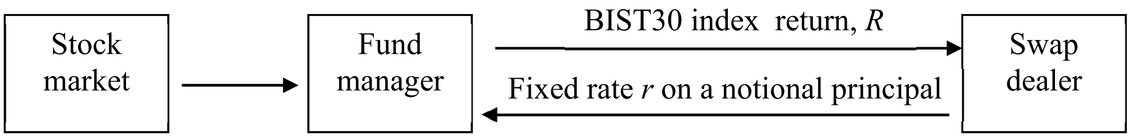
$$V = PV_{\$} - 1.10 \times PV_{€}$$

which is set to be equal to zero in the example.

In the example, both interest rates were fixed. There may also be currency swaps where one or both of the interest rates are floating, or where the rates are defined as a function of certain pre-specified market rates such as a LIBOR.

Equity Swaps

Here, counterparty A pays a fixed rate of interest on a given notional principal and counterparty B pays a variable rate of return pegged to the return on some stock market index like the S&P500 or the BIST30. Equity swaps may be used to change the risk exposure of a fund, which is closely correlated with the reference index. For example, the following diagram shows how a fund manager decreases the equity risk exposure of a fund:



We can evaluate the value of the swap to the fund manager. For this, define the rate of return on the index at time t as $R_t = (I_t/I_{t-1}) - 1$ and r be the fixed rate of interest on a notional principal F. If it is a T-year swap with annual payments, then its value at time zero will be

$$V = V_{fixed} - V_{variable}$$

$$V = \{rF \times \sum_{t=1}^{T} B(0,t)\} - \{\mathrm{PV}_0(F \times \sum_{t=1}^{T} R_t)\}$$

Note that the value of the index at time $T-1$ is the discounted value of the index at time T, $\mathrm{PV}_{T-1}(I_T) = I_{T-1}$, which implies that

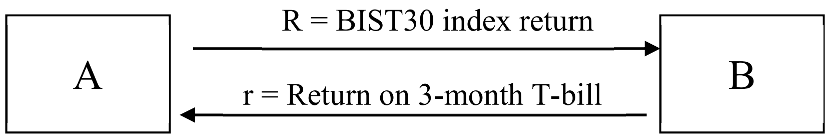
$$\mathrm{PV}_{T-1}\left(\frac{I_T}{I_{T-1}}\right) = 1 \Rightarrow \ \mathrm{PV}_{T-1}(R_T) = 1 - \mathrm{PV}_{T-1}(1) = 1 - B(T-1,T)$$

Repeating this result recursively, we compute the value of the swap as

$$V = \{rF \sum_{t=1}^{T} B(0,t)\} - \{F(1 - B(0,T))\}$$

which is usually set equal to zero by a suitable choice of the swap rate *r*.

Example: Suppose two parties A and B engage in the following monthly swap:



Since it is expected that $R > r$, how can the value of this swap be equal to zero? Why would not party A, who is giving up the higher return, require a up-front price to enter the swap? To solve this "paradox", consider the following transactions:

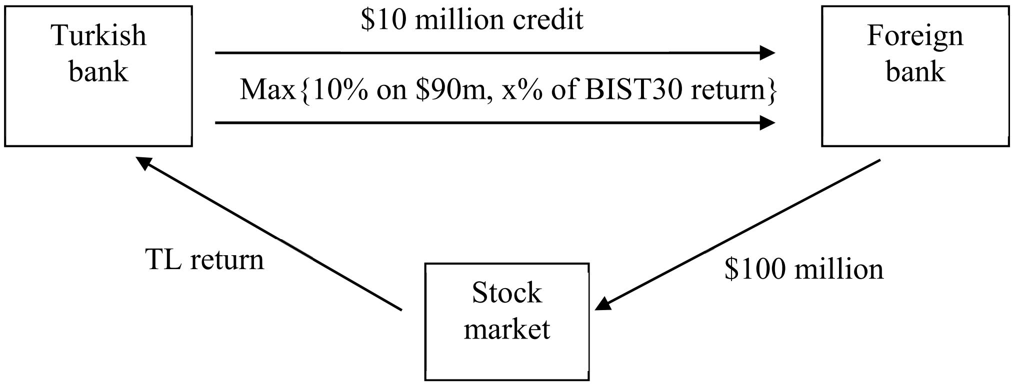
- Party A buys 10 million TL worth of the stocks in the BIST index and gives them to B. Party B now will receive the rate of return *R* on the index.
- Party B buys 10 million TL worth of 3-month T-bills and gives them to A. Party A now will receive the monthly rate of return *r* on the T-bill.

Note that these two simultaneous transactions exactly replicate the swap. The parties are basically exchanging 10 million TL. Hence, the swap should have a fair value of zero for both parties. This is basically an example of the well-known no-arbitrage condition: *The risk-adjusted rates of return on all financial assets in equilibrium must be equal to a unique risk-free rate of return.*

There are several variants of equity swaps:

- Instead of a fixed rate, a floating rate based on the LIBOR may be used.
- A variable notional principal may be used.
- A cap and/or a floor may be placed on the index return.
- The return on one index (S&P500) may be swapped with the return on another index (BIST30).
- The return on a single stock may be swapped with the return on another stock (sometimes, misleadingly called "credit swap")

Example: A deposit bank using an "equity swap" to invest in BIST at a level beyond the legally allowed limit:

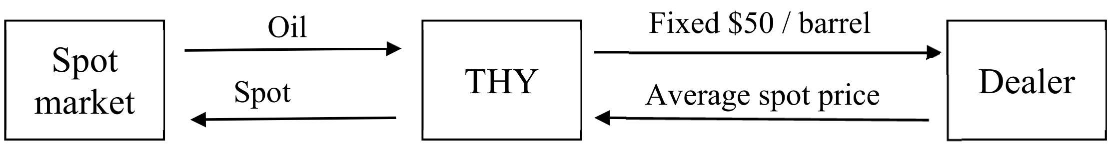


What are the risks faced by the Turkish bank? Could they be managed?

Commodity Swaps

A commodity swap is a contract where a variable price based on a notional quantity of an underlying commodity is swapped for a fixed price on the same commodity over a specified time period. There is no physical delivery at expiry. The variable price is typically an average spot price over a predefined period. The most popular commodity swaps are on gas, oil and electricity.

Example: THY, a consumer of 100,000 barrels of oil per month, is exposed to volatility in spot oil prices and enters into a 2-year commodity swap with a swap dealer in Geneva. The notional quantity is set at 100,000 barrels per month. The current spot price is $50 / barrel. THY agrees to pay a monthly amount to the dealer at a rate of $50 / barrel, and the dealer pays THY the average price during the preceding month. This average price is typically based on a widely used price index. These monthly payments continue for two years.



Through this swap deal, THY has not totally eliminated the uncertainty in oil prices but has clearly reduced it.

To derive the value of a commodity swap, let F be the notional quantity, P_{fix} be the fixed price the buyer pays the dealer and P_{t_i} be the "average" spot price the dealer pays the buyer at time t_i for $i = 1, \ldots, N$. If there is only one payment at one expiration time (say, t), the swap contract will be identical to a forward contract with $X_t = P_{fix}$ and a present value $V_0 = (\mathrm{PV}_0(P_t) - e^{-rt}X_t)F$. The contract is designed so that $V_0 = 0$, which implies $\mathrm{PV}_0(P_t) = e^{-rt}X_t$. Since there are N payment periods, the present value of a commodity swap will be equal to the sum of the present values

$$V_0 = \sum_{i=1}^{N} (\mathrm{PV}_0(P_{t_i}) - e^{-rt_i}P_{fix})F = \sum_{i=1}^{N} e^{-rt_i}(X_{t_i} - P_{fix})F$$

Setting $V_0 = 0$, we get the swap price as determined in the market:

$$P_{fix} = \sum_{i=1}^{N} e^{-rt_i}X_{t_i} \Big/ \sum_{i=1}^{N} e^{-rt_i} \equiv \sum_{i=1}^{N} B(0,t_i)X_{t_i} \Big/ \sum_{i=1}^{N} B(0,t_i)$$

OPTIONS

A call < put > option is a contractual right giving its owner the right to buy < sell > the underlying asset at a given price (strike or exercise price) at or before a given date (expiration date). The buyer pays for the option (price, or premium), and the writer (seller) deposits margin.

- Exercise styles
  - European (exercise only at expiry)
  - American (exercise at any time until expiry)
  - Bermudan (American with restricted early exercise)
- Markets
  - Exchanges (CBOE, CME, Intercontinental, NASDAQ OMX, BIST etc) for standard options
  - Over-the-counter markets for non-standard options such as Asian options, Bermudan options and other exotic options
- Types by underlying asset
  - Stock options
  - Index options
    - Broad-based market index options
    - Sector index options
    - Options on Exchange Traded Funds (ETFs)
    - Volatility indexes (such as VIX at the CBOE)
  - Long Term Equity Anticipation Securities (LEAPS)
  - Flexible exchange traded options (FLEX options)
  - Currency options
  - Interest rate options
  - Commodity and energy options
  - Options on futures
  - Options on options
- Option-like instruments
  - Warrants (options issued by a firm on its own equity)
  - Convertible bond (straight bond with option to convert into equity)

In the following tables are some examples of option trading data from the CBOE in early April. Consider the relation of option prices with the price of the underlying asset, the strike price and time to expiration.

Microsoft							**Underlying stock price: 25.72**
Expiration	Strike	Call			Put		
		Last	Volume	Open Interest	Last	Volume	Open Interest
Apr	20.00	5.70	554	5269	0.05	48	22039
May	20.00	5.70	301	874	0.15	36	365
Jul	20.00	6.10	11	2239	0.50	34	20872
Oct	20.00	6.60	60	759	0.90	64	6259
Apr	25.00	1.25	8938	101671	0.55	5527	84727
May	25.00	1.80	2309	13648	1.10	3587	5306
Jul	25.00	2.55	1039	40156	1.85	2273	49236
Oct	25.00	3.30	468	4523	2.55	43	3059
Apr	30.00	0.05	248	90003	4.50	60	11024
May	30.00	0.15	233	1494	...	...	106
Jul	30.00	0.60	3483	89392	5.00	6	9755
Oct	30.00	1.35	1072	22061	5.40	10	4603

Nasdaq 100 (NDX)						
Underlying Index	High	Low	Close	Net Change	From Dec.31	%Change
Nasdaq 100	**1066.90**	**1047.39**	**1063.46**	40.83	79.10	8.04

	Strike	Volume	Last	Net Change	Open Interest
Apr	1025.00 put	76	16.00	-16.00	853
Apr	1025.00 call	9	55.80	+25.80	966
May	900.00 put	256	8.50	-5.50	643
May	900.00 call	1	171.00	-8.90	10
May	1100.00 put	5	73.00	-12.00	69
May	1100.00 call	213	37.00	+11.00	530
Jun	1050.00 call	26	76.00	+15.00	1,687
Jun	1050.00 put	2	69.00	-14.00	1,191
Jun	1200.00 call	21	22.00	+7.00	1,339
Jun	1225.00 call	93	17.40	+5.70	270
Jun	1300.00 call	3	7.50	+2.50	908
Jun	1325.00 call	70	5.30	-3.70	190

Call Vol.	3,008	Open Int.	40,695
Put Vol.	2,639	Open Int.	49,528

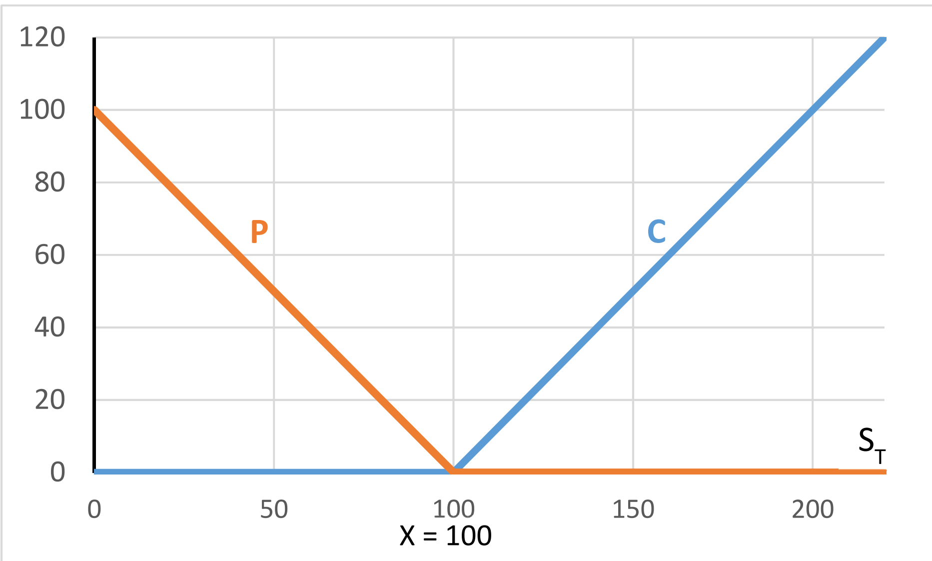
Value of an option at expiration

Notation:
S_t = current spot price of the underlying asset (at time t)
X = exercise price
T = time to expiration date
r = risk-free rate of interest
C = value of a European call (short for $C(S, t; T, X)$)
C_A = value of an American call
P = value of a European put
P_A = value of an American put

First, note that, at expiration, American and European options are identical. The value of an option at the expiration date will be:

$$C = \max(0, S_T - X)$$

$$P = \max(0, X - S_T)$$

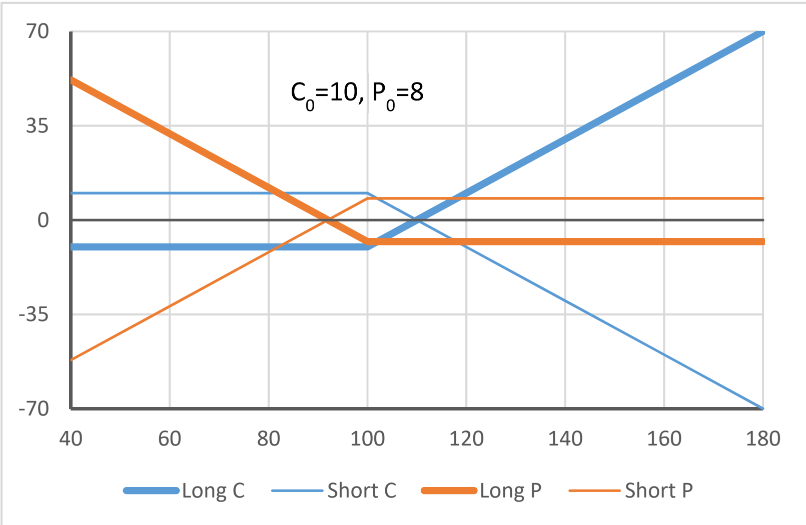


Note that that, since $0 \leq S_T < \infty$, $C < \infty$ but $P \leq X$.

Ignoring any transaction costs and time value of money, the net profits at expiry would be calculated as:

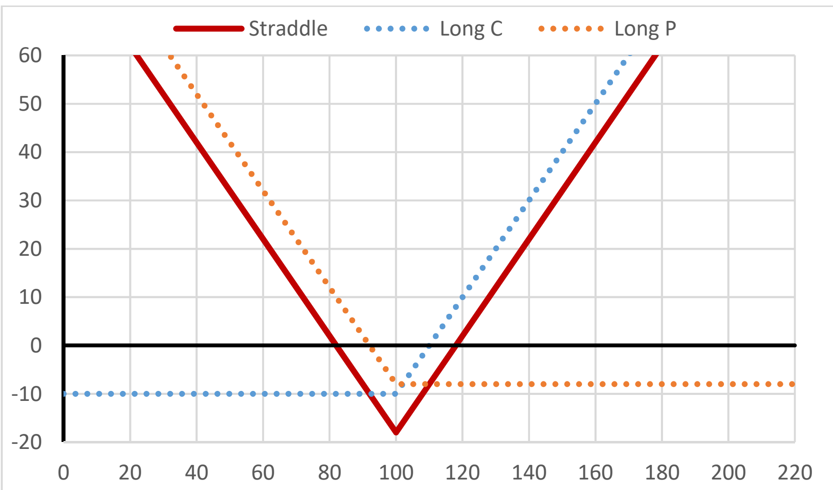
- Buy a call (long call) : $\pi = -C_0 + \max(0, S_T - X)$
- Write a call (short call): $\pi = C_0 - \max(0, S_T - X)$

- Buy a put (long put) : $\pi = -P_0 + \max(0, X - S_T)$
- Write a put (short put): $\pi = P_0 - \max(0, X - S_T)$



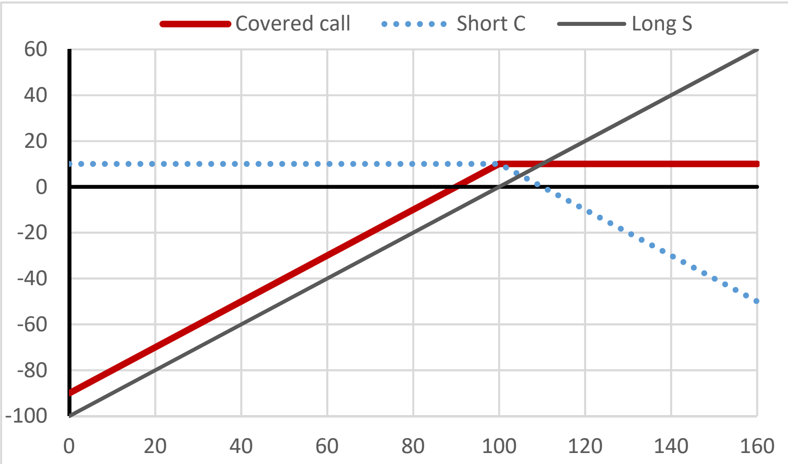
Example (straddle): Buy a call and a put on the same asset, with same X and T

$$\pi = \max(0, |S_T - X|) - (C_0 + P_0)$$



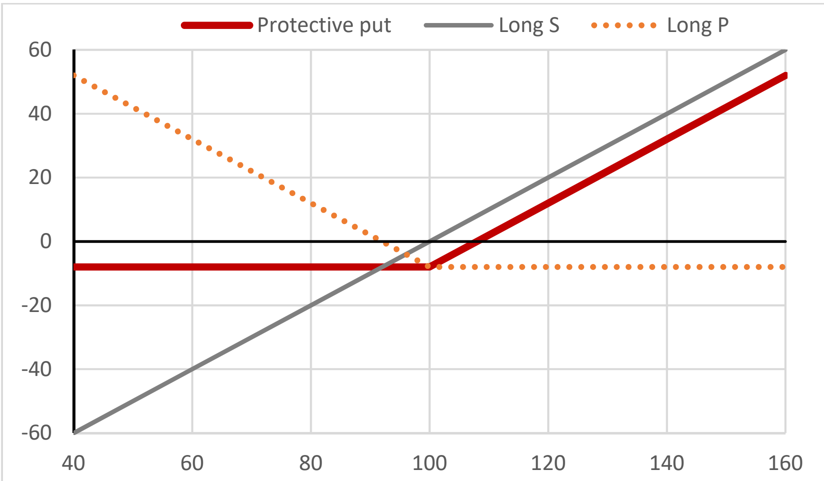
Example (covered call): Buy 1 unit of asset and short 1 call

$$\pi = C_0 + (S_T - S_0) - \max(0, S_T - X)$$



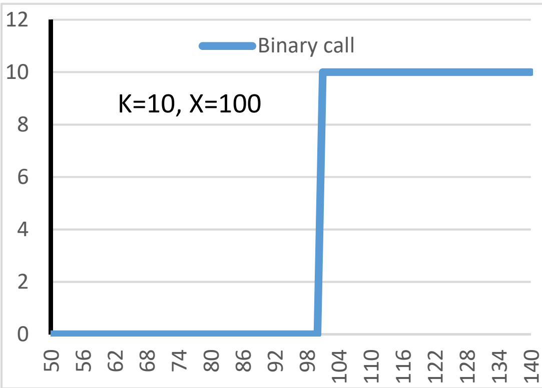
Example (protective put): Buy 1 unit of asset and long 1 put

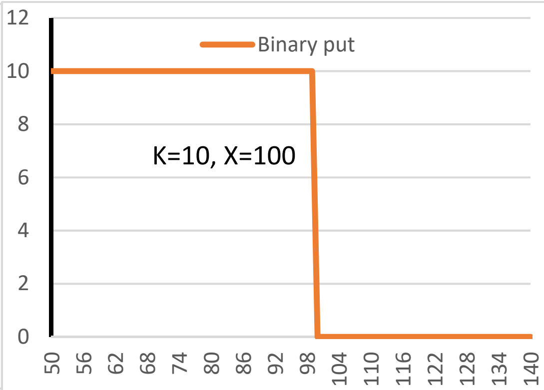
$$\pi = -P_0 + (S_T - S_0) + \max(0, X - S_T)$$



Example (binary options): Binary or digital options have binomial payoffs at expiration (K is a constant amount):

$$C = \begin{cases} K & \text{if } S_T > X \\ 0 & \text{if } S_T \leq X \end{cases} \qquad P = \begin{cases} K & \text{if } S_T < X \\ 0 & \text{if } S_T \geq X \end{cases}$$





You should be convinced that

- Any feasible payoff can be generated with the appropriate option strategy
- Any "option" can be generated through an appropriate trading strategy.

Exercise: What should be the present value of a portfolio of one binary call and one binary put? Is that portfolio fully hedged?

Exercise (butterfly spread): Buy 1 call with $X_{\max} > S_0$ (out-of-the-money), write 2 calls with $X_{mid} = S_0$ (at-the-money), buy 1 call with $X_{\min} < S_0$ (in-the-money), all with the same T. Assuming $X_{\max} - X_{mid} = X_{mid} - X_{\min}$. Calculate the payoff at T. How would the solution change if $X_{\max} - X_{mid} \neq X_{mid} - X_{\min}$?

Exercise (risk reversal): Long call with $X_{call} > S_0$ and short put with $X_{put} < S_0$, with the same T. Calculate the payoff at T. If $X_{call} = X_{put}$, how would the payoff profile look like?

Example: Compare three investment alternatives for a sum of 5,000:

- Portfolio A: 100 shares of stock at 50 per share
- Portfolio B: 5,000 in 10% treasury bills
- Portfolio C: 1,000 at-the-money calls at a price of 5 per call

	A		B		C	
Stock price	**Profit**	**Return**	**Profit**	**Return**	**Profit**	**Return**
80	3,000	60%	500	10%	25,000	500%
70	2,000	40%	500	10%	15,000	300%
62	1,200	24%	500	10%	7,000	140%
54	400	8%	500	10%	-1,000	-20%
50	0	0%	500	10%	-5,000	-100%
46	-400	-8%	500	10%	-5,000	-100%
38	-1,200	-24%	500	10%	-5,000	-100%
Mean return		14%		10%		89%
Standard deviation		29%		0%		236%

Options are risky business!

Boundary and Final Conditions on Option Prices

1. Options have "limited liability" in that $C \geq 0$ and $P \geq 0$.

2. At expiration, an option will be exercised only if it has positive "intrinsic value": $C = \max(0, S_T - X)$ and $P = \max(0, X - S_T)$. Otherwise, it will be discarded as useless.

3. American options must sell for at least their intrinsic value, or an immediate arbitrage profit could be made by buying the option and exercising.

$$C_t \geq S_t - X, \qquad P_t \geq X - S_t$$

This does not have to hold for European options since they cannot be exercised prior to expiration, making this arbitrage impossible.

4. Options with longer maturities are more valuable. For $T_2 > T_1$,

$$C(T_2) \geq C(T_1), \qquad P(T_2) \geq P(T_1)$$

The additional right to exercise during the interval $[T_2, T_1]$ cannot have a negative value. For the same reason, the additional right to exercise strictly before T cannot have negative value and therefore American options will be more valuable than their European counterparts.

5. Calls < puts > are non-increasing < non-decreasing > functions of the exercise price. For $X_2 > X_1$,

$$C(X_2) < C(X_1), \qquad P(X_2) > P(X_1)$$

To prove, if $C(X_2)$ is exercised for a profit of $S - X_2$, then $C(X_1)$ can be exercised for a greater profit $S - X_1$. Therefore, the second option cannot be less valuable.

6. The three propositions above imply that the value of a perpetual call with zero exercise price must be equal to the value of the underlying asset. Continuing, a call with a finite maturity T and positive exercise price X cannot have a greater value than S.

$$S_t = C_t(T = \infty, X = 0) \geq C_t(T < \infty, X > 0)$$

An implication is that, since $X > 0$ and the option has limited liability, a call on an asset with zero value will also have zero value: $C_t(S_t = 0, T, X) = 0$.

For put options, the most that can said is

$$0 = P_t(T = \infty, X = 0) \leq P_t(T = \infty, X > 0) \leq P_t(T < \infty, X > 0)$$

7. If the asset has no distributions such as dividends between *t* and *T*, then

$$C_t \geq S_t - Xe^{-r(T-t)}$$

To prove, compare the payoffs of two portfolios:

		Value at T	
Portfolio	Value at t	$S_T < X$	$S_T \geq X$
A: one unit of the underlying asset	S_t	S_T	S_T
B: one call and a bond promising X at T	$C_t + Xe^{-r(T-t)}$	X	$(S_T - X) + X = S_T$
		$V_A < V_B$	$V_A = V_B$

Since, at T, $V_A \leq V_B$, then $C_t + Xe^{-r(T-t)} \geq S_t$.

Since $C_t \geq S_t - Xe^{-r(T-t)} > S_t - X$, an option will always sell above its *intrinsic value*. The difference $C_t - \max(0, S_t - X)$ may be called the *time value* of the option.

An important implication of this result is that an American call on an asset with no distributions will never be exercised before maturity and hence it will be equivalent to a European option. Since $C_t > S_t - X$, the option is worth more "alive" than "dead".

8. For $X_2 > X_1$, the spread $C(X_1) - C(X_2) < (X_2 - X_1)e^{-r(T-t)}$ for calls and $P(X_2) - P(X_1) < (X_2 - X_1)e^{-r(T-t)}$ for puts.

Exercise: To prove, compare portfolio A $\{\text{long}\, C(X_1), \text{short}\, C(X_2)\}$ with portfolio B which has a bond promising $(X_2 - X_1)$ at time *T*.

9. A call option is a convex function of the exercise price. If $X_1 < X_2$, then, for any $0 < a < 1$, $C(aX_1 + (1-a)X_2) \leq aC(X_1) + (1-a)C(X_2)$. This also holds for puts.

Exercise: To prove, compare portfolio A {long $C(X)$ where $X_1 < X < X_2$} with portfolio B {$aC(X_1) + (1-a)C(X_2)$}.

10. ***The Put-Call Parity*** for European Options:

$$P_t = C_t - (S_t - Xe^{-r(T-t)})$$

To prove, consider Portfolio A: {long S_t, long P_t} and Portfolio B: {long C_t a bond paying X at T}:

		Value at T	
Portfolio	Value at t	$S_T > X$	$S_T \leq X$
A	$S_t + P_t$	S_T	X
B	$C_t + Xe^{-r(T-t)}$	S_T	X
		$V_A = V_B$	$V_A = V_B$

Since $V_A = V_B$ at time T, then $P_t + S_t = C_t + Xe^{-r(T-t)})$ at time t.

11. If a stock pays dividends and if the dividend is large enough, then early exercise of a call may be optimal. If the call is exercised, the payoff is $(S_{t^-} - X) + D$. If not exercised, the call will be worth $C_{t^+}(S_{t^+}, X, T)$. So, the call will be exercised immediately before the stock goes ex-dividend, if it is known beforehand that $D > C_{t^+}(S_{t^+}, X, T) - (S_{t^-} - X)$.

12.Regardless of distributions on the underlying asset, an American put may be exercised any time before maturity if the stock price drops low enough. To illustrate, the price drops to a point where $S_t < X - Xe^{-r(T-t)})$. In this case, the value of immediate exercise is $X - S_t$, which is greater than the present value of maximum possible gain at expiration $Xe^{-r(T-t)}$. Therefore, it is optimal not to wait until expiration.

13. Implied by the above, a put- call relation for American options may be stated as:

$$P_t \geq C_t - (S_t - Xe^{-r(T-t)})$$

To prove in the case of no distributions, it is sufficient to note that an American put will be at least as valuable as a comparable European put and that calls of either type will have equal values.

Exercise: If there are distributions, compare the following portfolios to prove the statement:

- Portfolio A: long one call, short one share, and buy bond paying X at T
- Portfolio B: long one put

14. A portfolio of long options with the same T is worth at least as much as an option on the portfolio of the same number of shares of the underlying assets. That is, for $n_i > 0$ for all i,

$$\sum_i n_i C^{(i)}\left(S^{(i)}, T, X^{(i)}\right) \geq C\left(\sum_i n_i S^{(i)}, T, \sum_i n_i X^{(i)}\right)$$

To illustrate, consider the following example involving options on stocks A and B, and another option on an equally-weighted portfolio of them:

	$S_T^{(A)} = S_T^{(B)} = 110$	$S_T^{(A)} = S_T^{(B)} = 90$	$S_T^{(A)} = 90, S_T^{(B)} = 110$
$C^{(A)}(100)$	10	0	10
$C^{(B)}(100)$	10	0	0
$C^{(A)} + C^{(B)}$	20	0	10
$C^{(A+B)}(200)$	20	0	0

Clearly, $C^{(A)} + C^{(B)} \geq C^{(A+B)}$. The reduction of risk (through diversification in this case) has lowered the value of the option. This also holds for puts.

Binomial Option Pricing

The binomial random walk

A discrete random variable X follows a Bernoulli distribution with parameter p when $\Pr(X = 1) = 1 - \Pr(X = 0) = p$. Consider a stochastic process where each realization is the result of a Bernoulli trial. Then, the probability of exactly k 1's (or, $n - k$ 0's) is given by the binomial distribution with a probability mass function

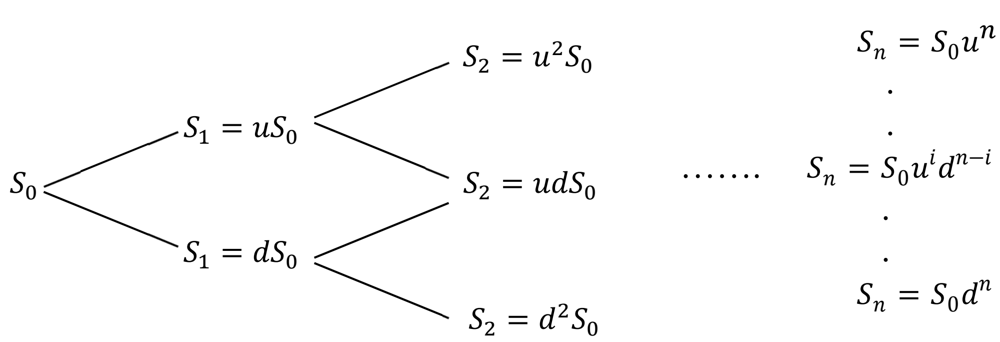
$$\mathbb{B}(n, k; p) = \binom{n}{k} p^k (1-p)^{n-k}$$

Imagine that we measure stock prices at discrete time intervals of equal length $\Delta t \equiv 1$, as $S_0, S_1, \dots, S_n$, where S_n is the price at the n^{th} time point ($n\Delta t$) and we assume for now that the stock pays no dividends. To come up with a computable scenario, suppose that each price change over the interval Δt follows a Bernoulli distribution with $\Pr(S_i/S_{i-1} = u) = 1 - \Pr(S_i/S_{i-1} = d) = p$, where u is one plus the up-tick rate of return and d is one plus the down-tick rate of return. Hence, the probability of an up-tick is p and that of a down-tick is $1 - p$. If $r \geq 0$ is the riskless rate of interest, we set $u > 1 + r \geq d > 0$.

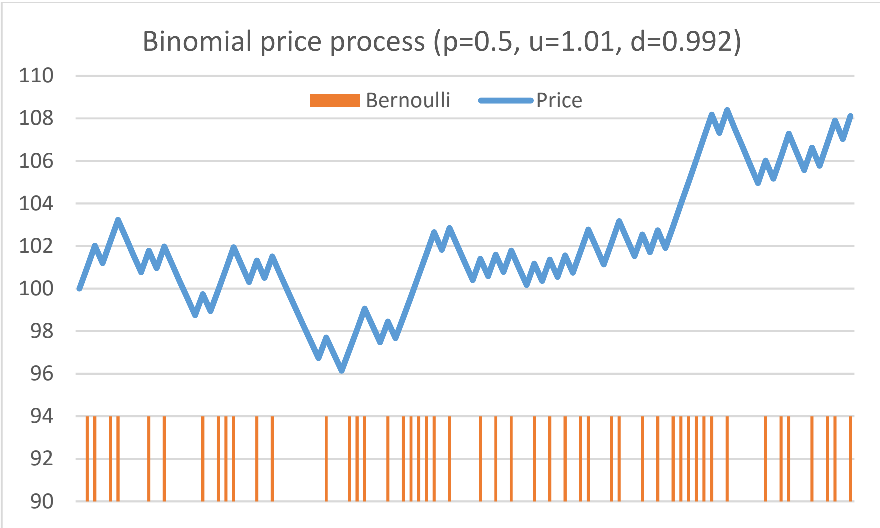
Fixing the initial price S_0, it is easily seen that the random price at the n^{th} time point is given

$$S_n = S_0 u^i d^{n-i} \quad \text{with probability} \quad \binom{n}{i} p^i (1-p)^{n-i}$$

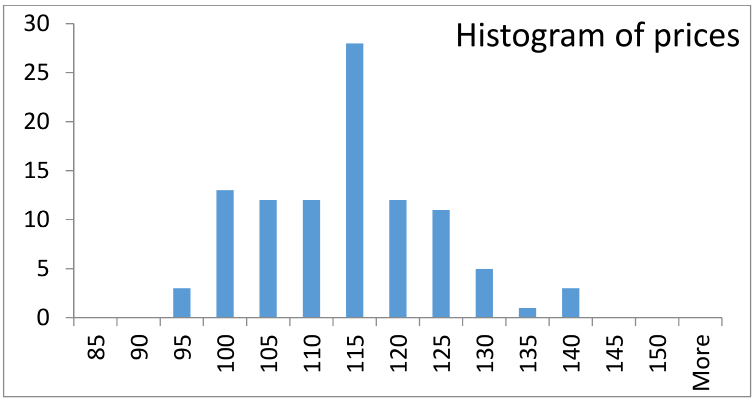
where $i \leq n$ is the number of up-ticks and S_i follows a binomial distribution with $E(S_n) = S_0(pu + (1-p)d)^n$. Hence, the name "binomial tree":



A sample realization of the price process with 100 independent Bernoulli trials is shown in the following chart:



The corresponding histogram for the final price S_{100} is also shown:



Exercise: How will the histogram look like in the limit as $n \to \infty$?

The expected return on the stock over any single time step is

$$E\left(\frac{S_{t+1}}{S_t}\right) = 1 + E(R_t) = pu + (1-p)d$$

Equivalently, the present value of the price is the usual discounted value of the expected price next period:

$$S_t = \frac{E(S_{t+1})}{1 + E(R_t)} = \frac{E(S_{t+1})}{pu + (1-p)d}$$

Instead of the actual probability p, we can clearly imagine of a "pseudo-probability" $0 < q < 1$ such that $qu + (1-q)d = 1 + r$, calculate expected values using this q, and hence define a new price process as $E_q(S_{t+1}) = S_t(qu + (1-q)d) = S_t(1+r)$. In other words, we seek a q probability that will allow us to discount a risky future value with the known and unique risk-free rate such that

$$S_t = \frac{E(S_{t+1})}{1 + E(R_t)} = \frac{E_q(S_{t+1})}{1 + r}$$

If $u > 1 + r \geq d > 0$ as assumed, then such a value will uniquely exist and solving $qu + (1-q)d = 1 + r$, we get

$$q = \frac{(1+r) - d}{u - d} > 0 \text{ and } 1 - q = \frac{u - (1+r)}{u - d} > 0$$

Under this probability measure q, stock prices will follow a sub-martingale in that

$$E_q(S_{t+1} | S_t) = S_t(1 + r) \geq S_t$$

Equivalently, if r is constant, the discounted price process will be a martingale:

$$E_q\left(\frac{S_{t+1}}{1+r}\middle| S_t\right) = S_t$$

The pseudo – probability q is sometimes called "risk – neutral probability" or "martingale probability."

Exercise: How would the above result change if $P = 1$ and hence $1 - p = 0$?

Simple examples

Suppose that we want to find the present value C_0 of a call option on a non-dividend stock, whose price follows a simple one-step binomial process:

$$
S_0 \begin{array}{l} \nearrow S_1^u = uS_0 \\ \\ \searrow S_1^d = dS_0 \end{array} \quad \rightarrow \quad C_0 = ? \begin{array}{l} \nearrow C_1^u = \max(0, uS_0 - X) \\ \\ \searrow C_1^d = \max(0, dS_0 - X) \end{array}
$$

Pricing by replication

We buy δ shares of stock, partially financed by borrowing M at an interest rate of r, to replicate the call option. The current value of this portfolio is $\delta S_0 + M$. One period later, we want those values of δ and M, which yield

$$\delta u S_0 - (1 + r)M = C_1^u \quad \text{and} \quad \delta d S_0 + (1 + r)M = C_1^d$$

These are two equations in two unknowns. Solving, we get

$$\delta = \frac{C_1^u - C_1^d}{S_0(u - d)} = \frac{C_1^u - C_1^d}{S_1^u - S_1^d} \quad \text{and} \quad M = -\frac{dC_1^u - uC_1^d}{(1 + r)(u - d)}$$

Substituting these into $C_0 = \delta S_0 + M$, we find the current call price

$$C_0 = \frac{(1 + r - d)C_1^u - (u - 1 - r)C_1^d}{(1 + r)(u - d)}$$

Using the pseudo – probability $q = (1 + r - d)/(u - d)$, we rewrite

$$C_0 = \frac{qC_1^u + (1 - q)C_1^d}{1 + r} = \frac{E_q[C_1]}{1 + r}$$

In conclusion, note that we did not need the true probabilities of an up-tick or a down-tick. All needed was the existence of the pseudo – probabilities that allowed risk-free discounting under the new measure. Hence, discounted options prices will follow a martingale process whenever pseudo-probabilities exist and are unique:

$$E_q\left(\frac{C_{t+1}}{1+r}\middle|C_t\right) = C_t$$

Pricing by hedging

Here, we construct a portfolio of short one call and long δ shares of stock with a current value of $V_0 = \delta S_0 - C_0$. To form a riskless portfolio, we choose δ to set the portfolio value next period to a known constant regardless of the direction of price change:

$$V_1 = \delta(uS_0) - C_1^u = \delta(dS_0) - C_1^d \quad \text{to get} \quad \delta = \frac{C_1^u - C_1^d}{S_1^u - S_1^d}$$

Substituting this into either one of the two ending values, we find that, after one period, the value of this portfolio will be with certainty,

$$V_1 = \frac{dC_1^u - uC_1^d}{u-d}$$

To avoid arbitrage, we must have $V_1/(1+r) = V_0$, or

$$\frac{dC_1^u - uC_1^d}{(1+r)(u-d)} = \delta S_0 - C_0$$

Solving for C_0, we find the same result as in the previous approach:

$$C_0 = \frac{(1+r-d)C_1^u - (u-1-r)C_1^d}{(1+r)(u-d)} = \frac{qC_1^u + (1-q)C_1^d}{1+r} = \frac{E_q[C_1]}{1+r}$$

All of the above arguments also apply to put options and indeed to any type of derivative security with a known payoff function at expiry.

To generalize to an *n*-step binomial tree, for fixed *u*, *d*, and *r*, we can use

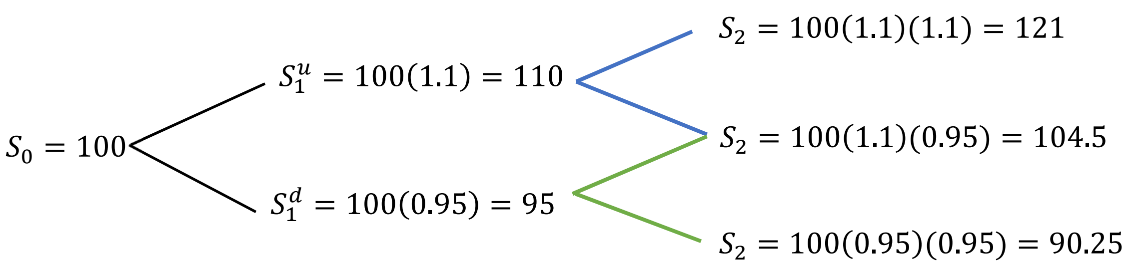
$$C_i = \frac{qC_{i+1}^u + (1-q)C_{i+1}^d}{1+r} \quad \text{and} \quad \delta_i^{u,d} = \frac{C_{i+1}^u - C_{i+1}^d}{S_i(u-d)} \quad \text{for} \quad i = 0,1,\dots,n-1$$

where the superscript over C_{i+1} denotes the direction of price change over the interval $[i, i+1]$, and the superscript over δ_i is the the direction of price change over the interval $[i-1, i]$. Start with $i = n-1$ and work for each binomial node recursively down to $i = 0$. The risk – neutral probability q is calculated as usual.

Notes:

1. To find option prices, we did not need the actual probability of an up-tick or a down-tick. This means that option prices do **not** depend on the expected rate of return on the underlying asset. As long as it is possible to hedge stock price movements, the probability of price change in any direction is not relevant. Bulls and bears must agree on the same option price.
2. Given the initial price S_0 of the underlying asset, option prices depend on the volatility of the asset's price $(u-d)$, the risk-free rate of interest (r), time to expiration (n), and the exercise price (X).
3. Any two of the triplet {underlying asset, risk-free bond, option} can be used to replicate the third. Such a market is called a "complete" market.

Example: (2-period option tree) Suppose $S_0 = 100, u = 1.1, d = 0.95, r = 0.05$ and we want to find the equilibrium current price of a call option with $X = 100$ and 2 periods to expiry.



First, calculate $q = (1.05 - 0.95)/(1.1 - 0.95) = 2/3$. Then, do the following calculations:

$$S_1 = 100(1.1) = 110$$

$$\delta_1 = \frac{21 - 4.5}{121 - 104.5} = 1$$

$$C_1^u = \frac{(2/3)(21) + (1/3)(4.5)}{1.05} = 14.76$$

$$S_2^u = 100(1.1)(1.1) = 121$$

$$C_2^u = 121 - 100 = 21$$

$$S_2^d = 100(1.1)(0.95) = 104.5$$

$$C_2^d = 104.5 - 100 = 4.5$$

$$S_1 = 100(0.95) = 95$$

$$\delta_1 = \frac{4.5 - 0}{104.5 - 90.25} = 0.315$$

$$C_1^d = \frac{(2/3)(4.5) + (1/3)(0)}{1.05} = 2.86$$

$$S_2^u = 100(1.1)(0.95) = 104.5$$

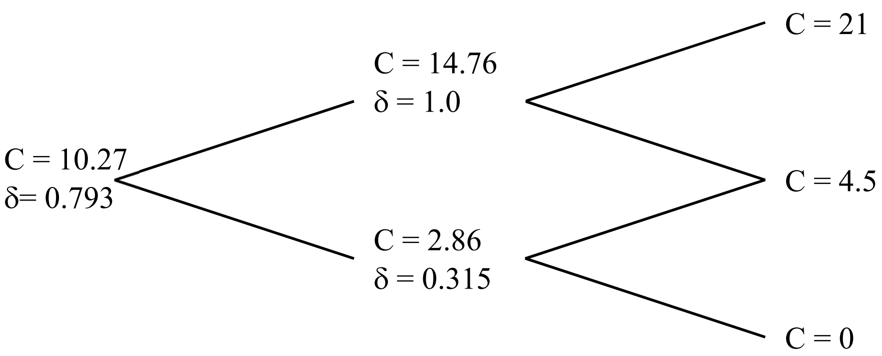
$$C_2^u = 104.5 - 100 = 4.5$$

$$S_2^d = 100(0.95)(0.95) = 90.25$$

$$C_2^d = 0$$

Finally, $\delta_0 = \frac{14.76-2.86}{110-95} = 0.793$ and $C_0 = \frac{(2/3)(14.76)+(1/3)(2.86)}{1.05} = 10.27.$

As a result, in equilibrium, the option price should follow the following path and any deviation from these equilibrium values will imply an arbitrage opportunity.



Self – financing replicating portfolio

Suppose we can set up a portfolio of δ units of the stock and M units (liras) of the riskless bond to exactly *replicate* one unit of the call option at every step of the tree and also that this portfolio is *self – financing*. This means that new purchases of assets (stock and bond) are fully financed by gains in asset prices, and vice versa, resulting in zero net cash injection or withdrawal. With multiple time steps, this strategy defines a *trading process*, as illustrated in the lattice below:

$$\delta_1^u S_2^u + M_1^u R = C_2\{u,u\}$$

(1)

$$\delta_0 S_1^u + M_0 R = \delta_1^u S_1^u + M_1^u$$

$$\delta_1^u S_2^d + M_1^u R = C_2\{\text{u,d}\}$$

$$\delta_0 S_0 + M_0 \qquad (3)$$

$$\delta_1^d S_2^d + M_1^d R = C_2\{\text{d,u}\}$$

(2)

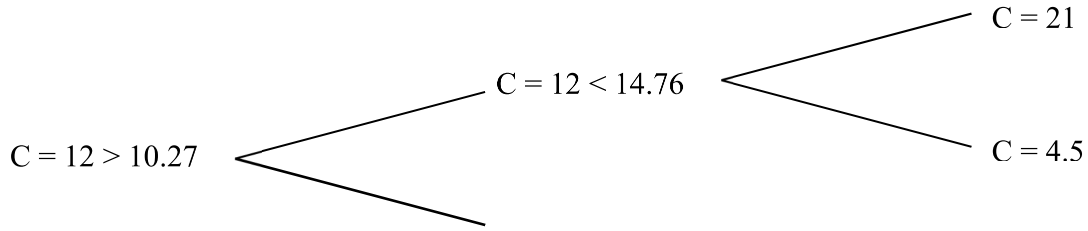
$$\delta_0 S_1^d + M_0 R = \delta_1^d S_1^d + M_1^d$$

$$\delta_1^d S_2^d + M_1^d R = C_2\{d,d\}$$

First solve the equations (1) to get $\delta_u^1 = 1, M_1^u = -95.24$, then solve the equations (2) to get $\delta_d^1 = 0.316, M_1^d = -27.14$, and finally substitute these into equations (3) to get $\delta_0 = 0.793, M_0 = -69.08$. These make up the trading strategy over the full binomial tree.

	Portfolio value	Rebalancing
$t = 0$	$0.793 \times 100 - 69.08 = 10.279$	
$t = 1$ (u)	$0.793 \times 110 - 69.08 \times 1.05 = 14.76$	$1 \times 110 - 95.24 = 14.76$
$t = 1$ (d)	$0.793 \times 95 - 69.08 \times 1.05 = 2.86$	$0.316 \times 95 - 27.14 = 2.86$
$t = 2$ (uu)	$1 \times 121 - 95.24 \times 1.05 = 21$	
$t = 2$ (ud)	$1 \times 104.5 - 95.24 \times 1.05 = 4.5$	
$t = 2$ (dd)	$0.316 \times 90.25 - 27.14 \times 1.05 = 0$	

To illustrate a mispricing scenario and resulting arbitrage trades, suppose that the observed prices in a given path deviate from the theoretical prices as follows:



Then, the following trade will yield arbitrage profits:

At $t = 0$, 12 > 10.27 (overvalued), δ = 0.793	
Sell 1000 calls at 12	12,000
Buy 793 shares at 100	-79,300
Borrow 79,300-12,000	67,300
Net investment	0

At $t = 1$, 12 < 14.76 (undervalued), δ = 1	
Buy 2,000 calls at 12	-24,000
Sell 1,793 shares at 110	197,230
Invest 197,230 – 24,000	173,230
Net investment	0

Note that this is a self-financing replicating strategy, with no net investment in the first two steps. However, it yields a riskless arbitrage profit at expiry.

At $t = 2$			
If S = 121		If S = 104.5	
Sell the remaining 1,000 calls	21,000	Sell the remaining 1,000 calls	4,500
Buy 1,000 shares at 121	-121,000	Buy 1,000 shares at 104.5	-104,500
Pay back loan $67{,}300(1.05)^2$	-74,198	Pay back loan $67{,}300(1.05)^2$	-74,198
Receive 173,230(1.05)	181,891	Receive 173,230(1.05)	181,891
Net arbitrage profit	**7,693**	**Net arbitrage profit**	**7,693**

These transactions will force prices to converge to their equilibrium values, the no-arbitrage prices.

Theoretical terminology

State Prices
In state pricing models, equilibrium (the absence of arbitrage profits) is defined as the asset price equalling the sum of the products of its price with the state price across all possible states. In the one-step binomial model above, we define the "*state prices*" in the up and down states as

$$a_u = \frac{q}{(1+r)} = \frac{S_0 - S_1^d(1+r)^{-1}}{S_1^u - S_1^d} \quad \text{and} \quad a_d = \frac{1-q}{(1+r)} = \frac{S_1^u(1+r)^{-1} - S_0}{S_1^u - S_1^d}$$

in order to write the system of equations below:

$$S_0 = a_u S_1^u + a_d S_1^d$$
$$B(0,1) = a_u + a_d$$

These are two equations in two unknowns, a_u and a_d. The first equation is for the stock and the second is for the bond (remember that $B(0,t) = (1+r)^{-t}$ is the price of a risk-free zero-coupon bond with a face value of 1TL at time *t*). As in this case, equilibrium (that is, positive state prices, $a_u > 0$ and $a_d > 0$) will exist when the number of basic securities is equal to the number of states of the world. Such a market is said to be "*complete*."

The same equation holds for any derivative security. For example, for one-step calls and puts, we can write

$$C_0 = a_u C_1^u + a_d C_1^d$$
$$P_0 = a_u P_1^u + a_d P_1^d.$$

Probability Measure and Change of Numeraire
Since the risk-neutral probabilities are $q = a_u(1+r)$ and $1-q = a_d(1+r)$, we can also write as before:

$$C_0 = (qC_1^u + (1-q)C_1^d)/(1+r)$$
$$S_0 = (qS_1^u + (1-q)S_1^d)/(1+r)$$
$$1 = q + (1-q)$$

Dividing both sides by the bond price ($B(0,1)$ now and 1TL one step later), we can rewrite the above as

$$\frac{C_0}{B(0,1)} = q\frac{C_1^u}{1} + (1-q)\frac{C_1^d}{1}$$
$$\frac{S_0}{B(0,1)} = q\frac{S_1^u}{1} + (1-q)\frac{S_1^d}{1}$$
$$1 = q + (1-q)$$

In this formulation,

- The sample space is $\Omega(u,d)$ with $0 \le q \le 1$ (other values are meaningles), and the probability set function is $\mathcal{P}(\Omega) = \Pr(u) + \Pr(d) = 1$ (the sample space includes all that can happen). Hence, we have a well defined "*probability measure*", with a proper assigment of probabilities to events.
- The "*numeraire*" is the bond price $B(0,t)$, which is denominator in the ratio of the asset price to the bond price. In other words, it is, for example, the price of a call option in units of the bond's price.

Instead of the bond's price, we could use also the stock price S_t as the numeraire. This would give

$$\frac{C_0}{S_0} = q^*\frac{C_1^u}{S_1^u} + (1-q^*)\frac{C_1^d}{S_1^d}$$
$$1 = q^* + (1-q^*)$$
$$\frac{B(0,1)}{S_0} = q^*\frac{1}{S_1^u} + (1-q^*)\frac{1}{S_1^d}$$

Where now $q^* = a_u(S_1^u/S_0)$ and $1-q^* = a_d(S_1^d/S_0)$. By changing the numeraire, we have switched to another probability measure q^*, which is also a proper probability measure.

These examples show that, given the state prices, we can price derivatives under different probability measures. A change of measure means a change in the probability distribution (such as from q to $q*$) but not a change in the random variables themselves ($\{S_t\}$ do not change). In practice, for ease of solution, we prefer the measure which allows for discounting at the risk-free rate.

Martingales
A stochastic process $\{X_t\}$, adapted to an information sequence $\{\mathrm{I}_t\}$ ("filtration"), is a *martingale with respect to* $\{\mathrm{I}_t\}$, if

$$\mathrm{E}(X_t \mid \mathrm{I}_{t-1}) = X_{t-1}$$

In continuous time processes, we will write $\mathrm{E}(X_t \mid \mathrm{I}_\tau) = X_\tau$ for all $t > \tau$. ("Martingale" is the last name of the mathematician who first defined this property.)

In financial work, a remarkable property of martingales is very useful: The expectation function of a martingale is constant:

$$\mathrm{E}(X_{t-1}) = \mathrm{E}(\mathrm{E}(X_t \mid \mathrm{I}_{t-1})) = \mathrm{E}(X_t)$$

In particular, it is interesting that $\mathrm{E}(X_t) = \mathrm{E}(X_0) = X_0$ for any $t \geq 0$. The expected future value is always equal to the current value!

In our examples of one-step binomial tree, note that, under the *q* measure,

$$q\frac{S_1^u}{1} + (1-q)\frac{S_1^d}{1} = E_q(\frac{S_1}{1}) \quad \Rightarrow \quad E_q(\frac{S_1}{1}) = E_q(\frac{S_0}{B(0,1)}) = \frac{S_0}{B(0,1)}$$

which basically says that, based on the information at time 0 ($\mathrm{I}_0 = \{S_0, u, d, r\}$), the best forecast of the discounted value of the stock price at time 1 is the price at time 0: $E_q(S_1 B(0,1) \mid \mathrm{I}_0) = S_0$. The same argument applies to the derivative asset:

$$q\frac{C_1^u}{1} + (1-q)\frac{C_1^d}{1} = E_q(\frac{C_1}{1}) \quad \Rightarrow \quad E_q(\frac{C_1}{1}) = E_q(\frac{C_0}{B(0,1)}) = \frac{C_0}{B(0,1)}$$

Similarly, under the q^* measure,

$$q^*\frac{C_1^u}{S_1^u} + (1-q^*)\frac{C_1^d}{S_1^d} = E_{q^*}\left(\frac{C_1}{S_1}\right) \quad \Rightarrow \quad E_{q^*}\left(\frac{C_1}{S_1}\right) = E_{q^*}\left(\frac{C_0}{S_0}\right) = \frac{C_0}{S_0}$$

Hence, both processes, $\{C_t/B(0,t)\}$ and $\{C_t/S_t\}$, are martingales. Putting it all together, we can state two equivalent versions of the *fundamental theorem of arbitrage pricing*:

(1) In the absence of arbitrage opportunities, there exists at least one probability measure under which the ratio of the price of a primary asset to the price of another primary asset (numeraire) is a martingale.
(2) If there is a probability measure (on the same support as the actual probability measure) under which ratios of the prices of two primary assets are martingales, then there is no arbitrage.

If the mentioned probability measure is unique, the market is said to be complete.

In practice, we would like to fine a suitable risk-neutral probability measure so that future expected values may be discounted at the risk-free rate of interest. Since this is a market-determined probability measure, there can only be one risk-free rate in equilibrium. In this case, the value of an attainable derivative asset is given by

$$Der_0 = \frac{E_q(V_t)}{(1+r)^t} \equiv E_q(V_t) \times B(0,t)$$

where V_t is the value of a self-financing replicating portfolio and its expectation is calculated under the risk-neutral measure.

A final note here is that the risk-free rate of interest above was either a constant or at least deterministic. To allow for stochastic interest rates, you will need the Radon-Nikodym derivatives and the Girsanov theorem for suitable change-of-measure tricks, which fall right into the scope of the fascinating course of financial calculus.

Example of an incomplete market

Suppose that the stock price follows a "trinomial" process of the form:

$$S_0 \begin{cases} S_1^u = uS_0 \\ S_1^o = oS_0 \\ S_1^d = dS_0 \end{cases} \rightarrow \quad C_0 = ? \begin{cases} C_1^u = \max(0, uS_0 - X) \\ C_1^o = \max(0, oS_0 - X) \\ C_1^d = \max(0, dS_0 - X) \end{cases}$$

Further suppose that we can find state prices to satisfy:

$$S_0 = a_u S_1^u + a_o S_1^o + a_d S_1^d$$
$$B(0,1) = a_u + a_o + a_d$$

This system has two equations and three unknowns and hence there is no unique solution. There will be more than one replicating portfolio and as many different derivative prices. In other words, there will be many no-arbitrage prices and no market equilibrium. To solve the problem, we need a third unknown variable to plug in the system, and typically we use "*investor preference functions*" to derive equilibrium models in finance.

Another example of an incomplete market is where we have *stochastic volatility*. In the lattice model, this happens when u and d are jointly or independently random.

A general binomial model

If the rate of interest r and the price movements u and d are constant over time, than the probability mass function for the stock price S_n after n periods is

$$\Pr\left(S_n = S_0 u^i d^{n-i}\right) = \mathbb{B}(n, i; q) = \binom{n}{i} q^i (1-q)^{n-i}$$
$$= \Pr\{i \text{ upticks in } n \text{ periods}\}$$

Using the results of the previous section on risk-neutral pricing,

$$\begin{aligned} C_0 &= \frac{E_q[\max(0, S_n - X)]}{(1+r)^n} \\ &= \frac{\sum_{i=0}^{n} \mathbb{B}(n, i; q) \max\left(0, S_0 u^i d^{n-i} - X\right)}{(1+r)^n} \\ &= \frac{\sum_{i=k}^{n} \mathbb{B}(n, i; q)\left(S_0 u^i d^{n-i} - X\right)}{(1+r)^n} \end{aligned}$$

where k is the minimum number of upticks in the stock price for the call to finish in the money. It is the smallest integer satisfying $S_0 u^i d^{n-i} > X$. For the option to be able to expire in the money, we need to set $k \leq n$. Continuing,

$$C_0 = S_0 \left(\sum_{i=k}^{n} \mathbb{B}(n, i; q) \left(\frac{u^i d^{n-i}}{(1+r)^n} \right) \right) - \frac{X}{(1+r)^n} \left(\sum_{i=k}^{n} \mathbb{B}(n, i; q) \right)$$

Noting that $\mathbb{B}(n,i;q)\left(\frac{u^i d^{n-i}}{(1+r)^n}\right) = \binom{n}{i} q^i (1-q)^{n-i} \left(\frac{u^i d^{n-i}}{(1+r)^n}\right) = \binom{n}{i}\left(\frac{uq}{(1+r)}\right)^i (1-\frac{uq}{(1+r)})^{n-i}$, and defining $c = uq/(1+r)$ we get

$$C_0 = S_0\big(1 - \widehat{\mathbf{B}}(n, k-1; c)\big) - \frac{X}{(1+r)^n}\Big(1 - \widehat{\mathbf{B}}(n, k-1; q)\Big)$$

where $\widehat{\mathbf{B}}(n, k-1; .) = \Pr(\text{number of upticks in } n \text{ steps } \leq k-1)$ is the binomial cumulative probability distribution function. Then, $1 - \widehat{\mathbf{B}}(n, k-1; .)$ is the probability of k or more upticks in n price steps, sometimes called the binomial *survival probability*. The value of a put option can be derived as

$$P_0 = -S_0\widehat{\mathbf{B}}(n, k-1; c) + \frac{X}{(1+r)^n}\Big(1 - \widehat{\mathbf{B}}(n, k-1; q)\Big)$$

These are historically known as the Cox-Ross-Rubinstein (CRR) formulas. Although the price equation looks complex, it has an intuitive interpretation. Denoting the stock price after *i* upticks in *n* steps as $S_n^{(i)} \equiv S_0 u^i d^{n-i}$, and noting that

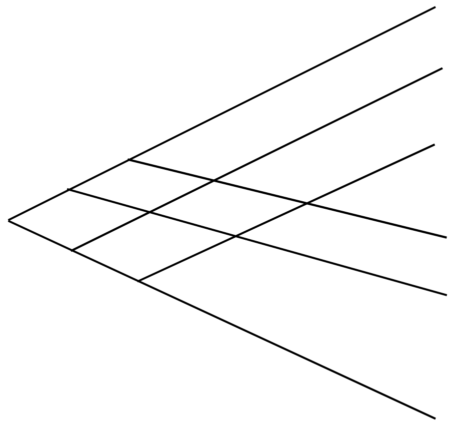
$$\sum_{i=k}^{n} \mathbb{B}(n,i;q)\left(S_n^{(i)} - X\right) = \Pr\{S_n = S_n^{(i)}\text{ , given } i \geq k\}$$

we can rewrite the above equation as

$$C_0 = \frac{E_q(S_n - X | S_n > X)}{(1+r)^n}$$

The current price of the option is found by discounting at the constant risk-free rate the expected positive payoff at expiry, where the expectation function is under the risk-neutral *q* probability measure.

The ending values of the binomial tree are shown graphically below:



Probability	Payoff
$\binom{n}{n}q^n(1-q)^0$	$S_0u^n - X$
$\binom{n}{n-1}q^{n-1}(1-q)^1$	$S_0u^{n-1}d^1 - X$
$\binom{n}{n-2}q^{n-2}(1-q)^2$	$S_0u^{n-2}d^2 - X$
.	
.	
$\binom{n}{k}q^k(1-q)^{n-k}$	$S_0u^kd^{n-k} - X$
$\binom{n}{k-1}q^{k-1}(1-q)^{n-k-1}$	0
.	.
.	.
$\binom{n}{0}q^0(1-q)^n$	0

How realistic is the model? What are reasonable values for u, d, r and Δt? Should they be assumed to be constant or time-dependent or stochastic?

The limit of the binomial random walk

In this chapter, we will first show that the simple binomial pricing model converges to a continuous-time pricing model, where the continuous-time rates of return on the underlying asset are normally distributed. Although the central limit theorem guarantees that the limiting distribution of the sum of binomial random variables is normal, a more intuitive limit scenario is presented here.

In the binomial random walk, the stock price after n discrete time intervals, each with length Δt, is given by $S_n = S_0 \prod_{i=1}^{n} R_i$, where the discrete-time return $R_i = S_{i+1}/S_i$ and it is either u or d with some probabilities. Defining the continuous-time rates of return as $u \equiv \ln(u)$, $d \equiv \ln(d)$ and $R_t \equiv \ln(S_{t+\Delta t}/S_t)$, we get a new representation of the random walk

$$S_T = S_0 \exp(\textstyle\sum_{t=0}^{n} R_t)$$

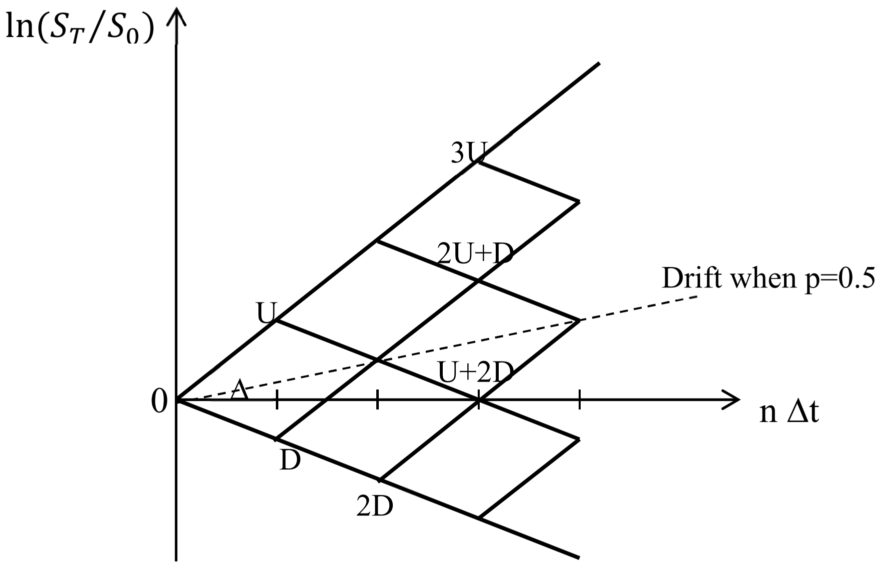
and

$$R_T \equiv \ln(S_T/S_0) = \textstyle\sum_{t=0}^{n} R_t$$

Hence, the distance travelled by the random walk in *N* time-steps is the change in log price during the time interval from 0 to $T = n\,\Delta t$. The moments of this binomial random walk with $\Pr(R = u) = 1 - \Pr(R = d) = p$ are as before

$$E(R) = pu + (1-p)d, \quad Var(R) = p(1-p)(u-d)^2$$

The binomial-tree representation of the random walk is shown below:



Under this notation, if Δt is one day (one year), then *T* is *n* days (years) and so on. The expected value and variance of R_T, which is the sum of n iid random variables, are then calculated as

$$E(R_T) = n\,E(R) = T\,(E(R)/\Delta t) \equiv m\,T$$

$$Var(R_T) = n\,Var(R) = T\,(Var(R)/\Delta t) \equiv v^2\,T$$

Over a single time-step ($n = 1$ and hence $T = \Delta t$), these reduce down to

$$E(R) = m\,\Delta t, \;\; \mathrm{Var}(R) = v^2 \Delta t, \text{and } \; \mathrm{StdDev}(R) = v\sqrt{\Delta t}$$

The parameters m and v^2 are called the *instantaneous mean (or, drift) and variance* of the return over Δt. The standard deviation v is termed *volatility*. We are interested in the limit as the number of time steps $n \to \infty$, or equivalently, as the length of the time step $\Delta t \to 0$ (naturally, u and d will also become very small). We will find the limiting distribution in two alternative ways:

(1) Noting that Δt will be very small in the limit, we can ignore numbers of order $(\Delta t)^2$ and smaller. Then, the variance of the return over one time-step will be

$$Var(R) = E(R^2) - \big(E(R)\big)^2 = \{pu^2 + (1-p)d^2\} - \{\mu^2(\Delta t)^2\}$$
$$\sigma^2\Delta t \approx pu^2 + (1-p)d^2$$

The moment generating function (mgf) of the binomial random walk is given $\mathbf{M}(j) = pe^{ju} + (1-p)e^{jd}$. Since the mgf of the sum of *n* iid rv's is the product of the individual mgf's, the logarithm of the mgf of the position of the random walk after *n* steps is (skipping some algebra)

$$\ln\big(\boldsymbol{M_n}(j)\big) = \ln\big(\big(pe^{ju} + (1-p)e^{jd}\big)^n\big)$$
$$\ln\big(\boldsymbol{M_n}(j)\big) = n\ln\big(pe^{ju} + (1-p)e^{jd}\big)$$
$$\ln\big(\boldsymbol{M_n}(j)\big) = \frac{T}{\Delta t}\ln\left(1 + \left(mj + \frac{1}{2}v^2j^2\right)\Delta t + O(\Delta t)^2 + \cdots\right)$$
$$\ln\big(\boldsymbol{M_n}(j)\big) \approx T\,(mj + \frac{1}{2}v^2j^2)$$

Hence, $\boldsymbol{M_n}(j) = \exp(T\,(mj + \frac{1}{2}v^2j^2)$, which is the mgf of a normal distribution with mean mT and variance v^2T .

(2) Since $\{R_i\}$ are iid Bernoulli rv's with finite mean $m\,\Delta t$ and finite variance $v^2\Delta t$, the *central limit theorem* can be applied on the sum $R_T = R_1 + \cdots + R_n$. As $n \to \infty$,

$$\frac{R_T - n(m\Delta t)}{\sqrt{n}\,v\sqrt{\Delta t}} = \frac{R_T - mT}{v\sqrt{T}} \xrightarrow{dist} \mathrm{N}(0,1)$$

Hence, the continuous-time limit of the return on the asset over the time interval $[0, T]$ is a normal random variable: $R_T = \ln(S_T/S_0) \sim \mathrm{N}(mT, v^2T)$. Hence, $E(\ln(S_T/S_0)) = mT$ and $Var(\ln(S_T/S_0)) = v^2T$.

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In passing, remember that, if $R_T \sim \mathrm{N}(mT, v^2T)$, then $\exp(R_T) = S_T/S_0$ is distributed *lognormally* with expected value $E(S_T/S_0) = \exp(\mu T)$ and variance $Var(S_T/S_0) = \exp(2\mu T)(\exp(v^2T) - 1)$, where $\mu = m + \frac{1}{2}v^2$.

*****

Since $R_T \sim \mathrm{N}(mT, v^2T)$, then the standardized value $Z = (R_T - mT)/v\sqrt{T}$ is a normal random variable with mean 0 and variance 1. Since $R_T = mT + \left(v\sqrt{T}\right)z$, we can write the stochastic process for the stock price at time *T*, given a fixed initial price, as

$$S_T = S_0\, e^{mT + (v\sqrt{T})Z}$$

This can be compared with the following discrete-time version $S_n = S_0 e^{mn + (v)W_n}$ where W_n is a symmetric random walk with $W_n = b_1 + \cdots + b_n$, where $b_i = 1$ or $b_i = -1$ each with equal probability $p = 0.5$ and $\Delta t = 1$.

In discrete time, the behaviour of the random walk was described im terms of three parameters u, d and p. In continuous time, it is described by only two parameters *m* and v^2. Clearly, there is an infinite number of values of (*u, d, p*) to yield a given (m, v^2) pair to describe the same random behaviour. For example, a high m can be achieved by choosing either $\{p = 0.5, u > d\}$ or equivalently by $\{\, p > 0.5, u = d\ \}$. So, different people have chosen different combinations in numerical applications, two of which seem to be more popular:

- Convergence to the normal limit is fastest when $p = 0.5$ because $p(1 - p)$ is maximed at this value and the normal density is symmetric. Then, the parameters are related as $u = m\,\Delta t + v\sqrt{\Delta t}$ and $d = m\,\Delta t - v\sqrt{\Delta t}$

- Setting $u = -d$, giving $u = v\sqrt{\Delta t} = -d$ and $p = \frac{1}{2} + \frac{m\sqrt{\Delta t}}{2v}$

Example: Consider a call option with $X = 80$ and 3 months (say, $n = 75$ days) to expiry on the XU100 index with a current value $S_0 = 82$. Using daily data over the last 3 years (20.04.12 – 20.04.15) with 746 obervations, we first calculate the sample mean and variance of the log returns $R_t = \ln(S_t/S_{t-1})$ for $t = 1, \ldots, 745 = n$.

DATE	S(t)	R(t)
20.04.12	60.123	
24.04.12	59.706	- 0,0070
25.04.12	59.941	0,0039
26.04.12	60.678	0,0122
27.04.12	60.597	- 0,0013
	....	
13.04.15	81.938	- 0,0086
14.04.15	82.320	0,0047
15.04.15	81.358	- 0,0118
16.04.15	82.437	0,0132
17.04.15	82.418	- 0,0002
20.04.15	82.303	- 0,0014

$$\Delta t = 1 \text{ day}$$

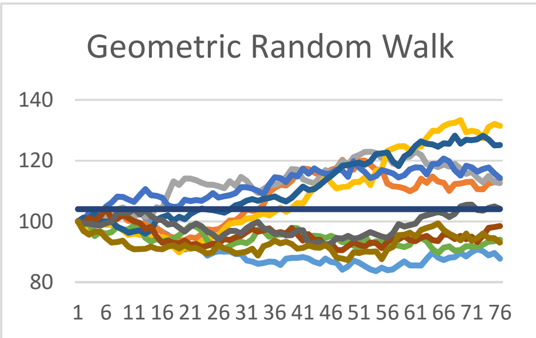
$$m = \frac{1}{745}\sum_{t=1}^{745} R_t = 0.000421$$

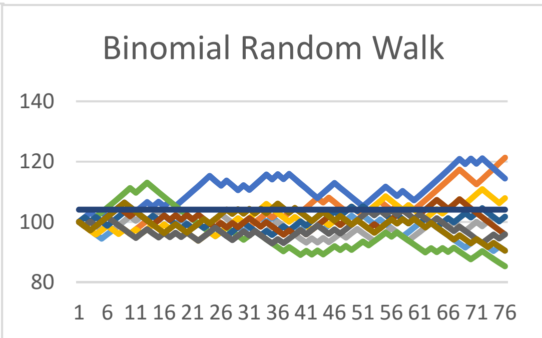
$$v^2 = \frac{1}{745}\sum_{t=1}^{745}(R_t - \mu)^2 = 0.000216$$

$$v = \sqrt{0.000216} = 0.014682$$

(Note also that $m_{week} = 5m$, $v_{week}^2 = 5v^2$ and $v_{week} = v\sqrt{5}$)

Let us calculate $u = m\,\Delta t + v\sqrt{\Delta t} = 0.000421 + 0.014682$ =0.015103 and $d = m\,\Delta t - v\sqrt{\Delta t} = 0.000421 - 0.014682 = -0.01426$. To compare, 10 simulated random walks of each type are shown in the following charts. Note that the geometric random ("Brownian motion") tends to wander away more than the discrete random walk.





The binomial tree for computation with daily steps is given below:

$$S_0 = 82 \quad \begin{cases} S_1^u = 82e^u \\ S_1^d = 82e^d \end{cases} \quad \cdots\cdots\cdots\cdots \quad \begin{cases} S_{75} = 82e^{75u} \\ S_{75} = 82e^{74u+1d} \\ \cdots\cdots\cdots\cdots \\ S_{75} = 82e^{1u+74d} \\ S_{75} = 82e^{75d} \end{cases}$$

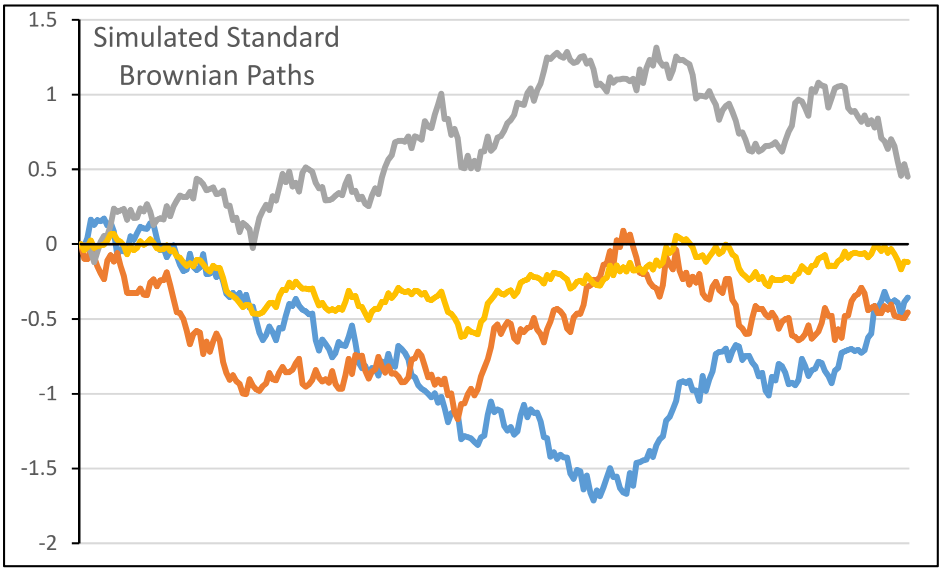
Option Pricing in Continuous Time

Brownian Motion

A standard Brownian motion, or a Wiener process, is a stochastic process $\{W(t), t \geq 0\}$ with the following properties:

1. It starts at the origin, $W(0) = 0$, and the sample paths are continuous,
2. Every increments $W(t + \Delta t) - W(t)$ is distributed as $\mathrm{N}(0, \Delta t)$,
3. It has stationary independent increments.

Simply put, it is a random process that evolves continuously in time and its change over any time interval is a normal variate with mean zero and variance equalling the length of the time interval. Four random simulations of the paths of a standard Brownian motion (with $\Delta t = 1/250$ and $W(n) = W(n-1) + z\sqrt{\Delta t}$), where z is a pseudo-random standard normal variate, $W(0) = 0$, and $n = 1, \dots, 250$) are shown below:



Some important properties of the Brownian motion are:

- Brownian motion is a Markov process because the third property implies that $W(t) - W(\tau)$ is independent of $W(s)$ for any $s < \tau < t$. In other words, the conditional distribution of $W(t) - W(s)$, given information up until time point s, depends only on $W(s)$.

- Since the mean is zero, the increments of a Brownian motion constitute a martingale in that

$$E(W(t)|W(s)) = W(s)$$

- Normality property means that (below $n(0,t;x) = e^{-x^2/2t}/\sqrt{2\pi t}$ is the normal density function for mean zero and variance *t*)

$$\Pr(W(t) \leq a|W(0) = 0) = \Pr(W(t) - W(0) \leq a) = \int_{-\infty}^{a} n(0,t;x)\,dx$$

- Brownian motion is "*self-similar*," meaning that patterns of sample paths are similar but never identical. Although it may be continuous, the sample paths of a self-similar process are nowhere differentiable. In the neighborhood of any point, the function changes its shape in a completely random way.

- Consider a discrete partitioning of the process $0 = t_0 < t_1 < \cdots < t_n = T$ of any time interval $[0, T]$. Brownian sample paths move "too fast" and, hence, they have unbounded "*total variation*" over this time interval: $\sup \sum_{i=1}^{n} |W(t_i) - W(t_{i-1})| = \infty$. Physically, total variation measures the total distance travelled by the random variable. This distance is infinite even during a very small interval of time and hence a Brownian motion may be physically impossible for the human brain. But it is still a good approximation to many real-life phenomena largely because it emerges as a limit process to several sum processes with finite variances.

  In practical applications, it is more convenient to scale the increments with the square root of the length of a time interval. The "*quadratic variation*" of a Brownian motion is measured by the limit, as $t_i - t_{i-1} \to 0$, $\lim \sum_{i=1}^{n} \left(W(t_i) - W(t_{i-1})\right)^2 = t$ almost surely. The quadratic variation is equal to the length of the time interval over which it is measured. (This result is to be compared with the quadratic variations of differentiable functions of time, which are always zero.) So, why is the Brownian motion, which is a strange process with continuous jiggling, widely used in financial models? The answer is given by *Levy's theorem*, which says that continuous-time martingales must have infinite total variation and that such martingale will be Brownian motions if their quadratic variation over any time interval $[0, T]$ is equal to to *T*. And we need martingales in asset pricing!

- Despite its strange behaviour and aside from reasons of computational convenience, Brownian motion is used as the basic model in asset pricing because all financial prices are driven by the same common random information and it is realistic to imagine this as a normal random variable. Then, different processes may be built on this core event.

Stock Prices as a Brownian Motion

Let the stock price follow the process

$$S(t) = S(0)e^{Y(t)}$$

where $Y(t) = mt + vW(t)$ is a "*Brownian motion with drift*" with drift parameter m and diffusion parameter $v > 0$. Now, every increment $Y(t + \Delta t) - Y(t)$ is distributed as $\mathrm{N}(m\Delta t, v^2\Delta t)$. As it may be that $m \neq 0$, this process is not a martingale. All other properties remain unchanged.

The process $S(t)$ is called a "*geometric Brownian motion*", which is the solution of the stochastic differential equation

$$\frac{dS}{S} = (m + \tfrac{1}{2}v^2)\, dt + v\, dW$$

*****

Note: It can be shown that $\lim_{t\to\infty} \Pr\{Y(t) < 0 | Y(0) > 0\} = 1/2$. This is why a standard arithmetic Brownian motion such as $S(t) = S(0)\, Y(t)$ cannot be an admissible process for stock prices and also why we use the geometric Brownian motion $S(t) = S(0)\ \exp(Y(t))$. Since $e^a > 0$ for any *a*, this agrees with non-negative prices.

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Taking the logarithms of both sides of the stock price process, we get $\ln(S(t)/S(0)) = mt + vW(t)$, which, for given $S(0)$, is the solution of

$$d\, lnS = m\, dt + v\, dW$$

Hence, $\frac{dS}{S} = \mu\, dt + v\, dW$ (where $\mu = m + \frac{1}{2}v^2$) and $d\, lnS = m\, dt + v\, dW$ are equivalent equations and hence they both have as a solution the same geometric Brownian motion.

Estimation of the Parameters of a Brownian Motion

Suppose we observe stock prices at time points $0 = t_0 < t_1 < \cdots < t_n = T$, where $t_i - t_{i-1} = \Delta t$ is fixed and hence $T = n\, \Delta t$, as $\{S_0, S_1, \dots, S_n\}$. We can then calculate the series of continuous rates of return as $\{r_1, r_2 \dots, r_n\}$, where $r_i = \ln(S_i/S_{i-1})$. The sample mean and variance, estimators of $m\Delta t$ and $v^2\Delta t$, are calculated as:

$$\widehat{m} = \frac{1}{T}\sum_{i=1}^{n} r_i = \frac{\ln(S_n/S_0)}{T} \quad \text{and} \quad \hat{v}^2 = \frac{1}{T-1}\sum_{i=1}^{n}(r_i - \widehat{m})^2$$

These are the estimates of the expected value and the variance of the rate of return over the time interval Δt.

If $\Delta t = 1$ day and the sample contains n daily rates of return, the daily rate of return will have an estimated mean and variance given by

$$\widehat{m} = \frac{1}{n}\sum_{i=1}^{n} r_i = \frac{\ln(S_n/S_0)}{n} \quad \text{and} \quad \hat{v}^2 = \frac{1}{n-1}\sum_{i=1}^{n}(r_i - \widehat{m})^2$$

Assuming that one year contains 250 trading days, then the annual parameters would be calculated as

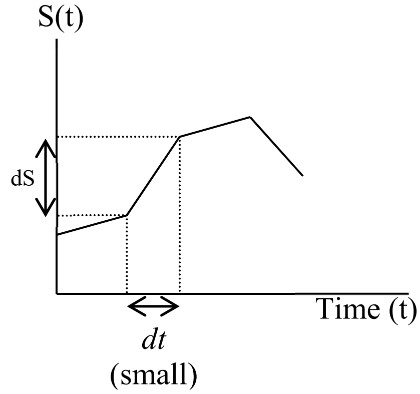
$$\widehat{m}_{year} = \frac{250}{n}\sum_{i=1}^{n} r_i = \frac{250 \times \ln(S_n/S_0)}{n} \quad \text{and} \quad \hat{v}_{year}^2 = \frac{250}{n-1}\sum_{i=1}^{n}(r_i - \widehat{m})^2$$

As a warning, the standard deviation of the estimate $\widehat{m}$ is $v/\sqrt{T}$, which can make the estimate quite unreliable. For example, using XU100 daily data over the period 4/2013 – 4/2015 with 530 obervations, we find $\widehat{m} = 0.00013$ and $\hat{v}/\sqrt{T} = 0.0007$. A routine 95% confidence interval around the estimate will be $[-0.00126, 0.00152]$, which makes the estimate of 0.00013 practically useless. Estimation of expected rates of return from financial data is a problematic issue. Fortunately, in derivatives applications, we are interested in only the estimation of volatility. If volatility is constant, this can be done quite precisely with the above approach. However, if volatility is not constant over time, the historical estimate will be of limited validity. We will come back to this problem later.

Solution of Stochastic Differential Equations

The rate of return $\frac{dS}{S}$ over a small time interval dt is represented as:

$$\frac{dS}{S} = \underbrace{\mu\, dt}_{\text{Predictable portion}} + \underbrace{v\, dW}_{\text{Random surprise portion}}$$



where dW is an increment of a Brownian motion process in a small time interval dt, and it may also be written as $z\sqrt{dt}$. Hence, given the initial price S, the expected value and variance of the rate of return are $E(dS/S) = \mu dt$ and $Var(dS/S) = v^2 dt$.

If we take out the random term, the rate of return will be μdt and we are left with an ordinary (non-stochastic) differential equation

$$\frac{dS}{S} = \mu\, dt$$

If μ is constant, the solution in discrete time is easily found to be an exponential growth model

$$S(t + \Delta t) = S(t)e^{\mu \Delta t}$$

However, the solution of stochastic differential equations cannot proceed in the same way as ordinary differential equations because of the presence of a term like dW, a small increment of a Brownian motion. Brownian paths are continuous everywhere (that is, no "jumps") and they are nowhere differentiable. We need a way to transform stochastic equations into a format managable by procedures of ordinary differential equations.

Ito Processes

Consider a smooth (twice-differentiable) function $f(S,t)$, where $S(t)$ is the solution (due to Ito) of the stochastic differential equation,

$$dS = a(S,t)\, dt + b(S,t)\, dW$$

where the drift *a* and diffusion *b* are now allowed to functions of *S* and *t*. Again, dW is a normal Brownian increment with mean zero and variance equalling dt (in discrete time, $\Delta W = z\sqrt{\Delta t}$). A special case of this is when $a(S,t) = \mu S$ and $b(S,t) = vS$, which is our original expression for a geometric Brownian motion.

Given the smoothness assumption, a small change in $S(t)$ will lead to a small change in $f(S,t)$. The total differential of $f(S,t)$ can be evaluated by applying a Taylor series expansion:

$$df = \frac{\partial f}{\partial S} dS + \frac{\partial f}{\partial t} dt + \frac{1}{2}\frac{\partial^2 f}{\partial S^2}(dS)^2 + \frac{\partial^2 f}{\partial S\,\partial t} dSdt + \frac{1}{2}\frac{\partial^2 f}{\partial t^2}(dt)^2 + \cdots$$

The key to Ito's solution is to see that, in the limit, as $dt \to 0$, $dW \to \sqrt{dt}$. Any other order of magnitude leads to trivial and uninteresting properties for the price process. If $dW \gg \sqrt{dt}$, the process goes immediately to zero or infinity, and, if $dW \ll \sqrt{dt}$, the random portion vanishes as $dt \to 0$. Therefore, we set $dW = \sqrt{dt}$ in the limit and retain only those terms that are larger than $o(dt)$. The rules of thumb to remember in Ito-type work are: $(dt)^2 = 0, (dt)(dW) = 0, (dW)^2 = dt$.

Substituting for $dS = a\, dt + b\, dW$, and invoking these order rules, we obtain:

$$df = \frac{\partial f}{\partial S} dS + \frac{\partial f}{\partial t} dt + \frac{1}{2}\frac{\partial^2 f}{\partial S^2}(dS)^2$$

Noting that $(dS)^2 = (a\, dt + b\, dW)^2 = b^2 dt$ and substituting, we obtain *Ito's lemma*:

$$df = \left(b\frac{\partial f}{\partial S}\right) dW + \left(\frac{\partial f}{\partial t} + a\frac{\partial f}{\partial S} + \frac{b^2}{2}\frac{\partial^2 f}{\partial S^2}\right) dt$$

Note that the first term is a random component proportional to dW and the second term is deterministic proportional to dt. Thus, if $S(t)$ is a random walk, $f(S,t)$ is also a random walk, driven by the same Brownian "source of randomness" $W(t)$. Ito's lemma simply measures a small change in $f(S,t)$ caused a small change in the random variable $S(t)$ itself, which in turn is triggered by $W(t)$.

Example: (geometric Brownian motion) Suppose $f(S(t)) = \ln(S)$, $\frac{dS}{S} = adt + bdW$ and the parameters a and b are constant. Applying Ito's lemma, we get

$$dln(s) = \left(bS\frac{1}{S}\right)dW + \left(0 + aS\frac{1}{S} - \frac{(bS)^2}{2}\frac{1}{S^2}\right)dt$$

$$dln(s) = v\,dW + mdt$$

which is our familiar geometric Brownian motion, where we have used the usual notation as $v = b$ and $m = a - \frac{1}{2}b^2$.

Example: (hedging) Since $f(S(t))$ and $S(t)$ share the same source of randomness $W(t)$, we can possibly construct a third variable, $V = f + \delta S$, as a function of these two random variables but which is totally deterministic. Here, δ is an arbitrary hedging parameter. Holding δ constant during the time interval dt (so that we do not have to worry about $d\delta/dt$), we find $dV = df + \delta\, dS$. Using $dS/S = \mu dt + vdW$ and applying Ito's lemma on f, we get

$$dV = \left(vS\frac{\partial f}{\partial S}\right)dW + \left(\frac{\partial f}{\partial t} + \mu S\frac{\partial f}{\partial S} + \frac{(vS)^2}{2}\frac{\partial^2 f}{\partial S^2}\right)dt + \delta(\mu S\,dt + vS\,dW)$$

$$dV = vS\left(\frac{\partial f}{\partial S} + \delta\right)dW + \left(\frac{\partial f}{\partial t} + \mu S\left(\frac{\partial f}{\partial S} + \delta\right) + \frac{(vS)^2}{2}\frac{\partial^2 f}{\partial S^2}\right)dt$$

If we set the hedging parameter $\delta = -\partial f/\partial S$, we are left with

$$dV = \left(\frac{\partial f}{\partial t} + \frac{(vS)^2}{2}\frac{\partial^2 f}{\partial S^2}\right)dt$$

where the term dW, the source of uncertainty, has vanished. This means that the porfolio V is riskless. Furthermore, the drift μ has also vanished, meaning that, whenever a riskless trading strategy can be designed, knowledge of the expected rate of return is not needed. These results must be familiar.

Several Ito Processes

Let $f(g_1, \dots, g_n, t)$ be a twice-differentiable function where each random variable g_i follows an Ito process satisfying

$$dg_i = a_i(g_i, t)\, dt + b_i(g_i, t)\, dW$$

then $f(g_1, \dots, g_n, t)$ will also be an Ito process and

$$df = \frac{\partial f}{\partial t} dt + \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} dx_i + \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{\partial^2 f}{\partial x_i \partial x_j} dx_i dx_j$$

Note that all of the above Ito processes are driven by the same Brownian motion $W(t)$. The volatility of the random variable f will depend on the volatilities of the random variables $\{g_i\}$ and also the correlations between them.

Example: (two variables) Suppose random variables X and Y are both Ito processes with constant drift and volatility and let $Z = f(X, Y, t)$:

$$dX = \mu_x\, dt + v_x\, dW_x$$
$$dY = \mu_y\, dt + v_y\, dW_y$$

Then, $Z(t)$ will be an Ito process and

$$df = \frac{\partial f}{\partial t} dt + \frac{\partial f}{\partial x} dX + \frac{\partial f}{\partial y} dY + \frac{1}{2} \frac{\partial^2 f}{\partial x^2} (dX)^2 + \frac{1}{2} \frac{\partial^2 f}{\partial y^2} (dY)^2 + \frac{\partial^2 f}{\partial x \partial y} dX dY$$

Note that the order rules $(dt)^2 = 0, (dt)(dW) = 0,\ (dW)^2 = dt$ imply

$$(dX)^2 = v_x^2 dt, (dY)^2 = v_y^2 dt \ \text{ and } \ dXdY = v_x v_y\, \rho\, dt$$

where ρ is the correlation between W_x and W_y. We will use these in the following examples.

Example: If $Z = XY$, then $dZ = XdY + YdX + dXdY$ and, using the above result, we can rewrite this as

$$\frac{dZ}{Z} = \frac{dX}{X} + \frac{dY}{Y} + \frac{dX}{X}\frac{dY}{Y}$$

The variance of dZ/Z is then calculated as $(dZ/Z)^2 = \left(v_x^2 + v_y^2 + 2v_x v_y\,\rho\right)dt$. Hence, the volatility of the product XY is $\sqrt{v_x^2 + v_y^2 + 2v_x v_y\,\rho}$.

Example: If $Z = \exp(X)$, then

$$\frac{dZ}{Z} = dX + \frac{(dX)^2}{2}$$

The volatility of e^X can be easily shown to be equal to v, a familiar result.

Exercise: As in the above examples, find expressions for dZ/Z when $Z = X/Y$ and when $Z = \ln(X)$. Also find their volatilities.

Example: (change of numeraire to obtain a martingale) If a stock price process is geometric Brownian motion with non-zero drift such as in $dS/S = \mu\,dt + v\,dW$, it will not be a martingale. We want to find a numeraire asset to convert the price process into a martingale so that risk-neutral probabilities can be used. Let the continuous risk-free rate of interest on a *t*-maturity zero be $r = \frac{1}{t}\int_o^t r(u)du$ and hence the price of the zero be $B(0,t) = \exp(-rt)$. Using $B(0,t)$ as a numeraire, define a new variable $Z(t) = S(t)/B(0,t)$. Then, by the product rule of calculus,

$$\frac{dZ}{Z} = -r\,dt + \frac{dS}{S} = -r\,dt + (\mu\,dt + v\,dW) = (\mu - r)\,dt + v\,dW$$

To get a martingale, the drift $(\mu - r)$ of dZ/Z must be zero, which requires the drift of dS/S to be equal to $\mu = r$. Hence, we must have a random process such as

$$dS/S = r\,dt + v\,dW^*$$

where $dW^* = ((\mu - r)/v)\,dt + dW$. The drift $(\mu - r)/v$ of this new process is the familiar market price of risk. After some algebra, its quadratic variation is found as $(dW^*)^2 = dt$. Hence, by Levy's theorem, W^* is also a Brownian motion. So, the change of numeraire worked fine.

To formalize these findings, we state the popular

Girsanov's Theorem: Let $W(t)$ be a Brownian motion with a probability measure P on a sample space Ω and u be a "nice" function. There exists an equivalent probability measure Q on the same sample space Ω such that $W^*(t) = W(t) + ut$ is also a Brownian motion. Equivalently, $dW^* = u\,dt + dW$.

Applying this theorem on,

$$\frac{dZ}{Z} = (\mu - r)\,dt + v\,dW$$

we get

$$\frac{dZ}{Z} = (\mu - r)\,dt + v\,(dW^* - u\,dt) = (\mu - r - vu)dt + v\,dW^*$$

To get a zero drift, we set $u = (\mu - r)/v$ and obtain the martingale process

$$dZ = d\left(\frac{S}{B}\right) = \ v\left(\frac{S}{B}\right)dW^*$$

Since $dB = rB\,dt$, we have $dS/S = r\,dt + v\,dW^*$. Girsanov's theorem guarantees the existence of an equivalent measure Q, which allows valuation at the riskless rate *r*. In fact, an amazing property of the Brownian motion is that, by a proper change of the supporting probability measure, we can make the process to have any drift we want. And, for risk-neutral pricing, we choose the riskless rate of interest as the drift (discount rate).

The Black – Scholes – Merton Model

The simplest version of the model has the following asumptions:

- The stock price follows a geometric Brownian motion with constant drift and volatility: $dS/S = \mu\, dt + v\, dW$
- A constant riskless rate of interest
- Perfect markets
- European option and no cash distributions to holders of the underlying asset

Given the exercise price X, the expiration date T and the riskless rate of interest r, the price of an option will be a function of the price of the underlying asset. As an example, let us use a call option on a no-dividend common stock. Then, $C = C(S,t)$ and we can apply Ito's lemma to get

$$dC = \left(vS\frac{\partial C}{\partial S}\right)dW + \left(\frac{\partial C}{\partial t} + \mu S\frac{\partial C}{\partial S} + \frac{(vS)^2}{2}\frac{\partial^2 C}{\partial S^2}\right)dt$$

The first term accounts for the change in C associated with a small change in the stock price and it is stochastic due to the Brownian increment dW. The second term is related to the change in the call price caused by a small change in the time to expiration, dt . For given S, this term is deterministic.

Since $C(S,t)$ and $S(t)$ share the same source of randomness $W(t)$ and functionally related, we can construct a third variable, $V = C + \delta S$, as a function of these two random variables but which is totally deterministic. Here, δ is the number shares of stock for each unit of the call. If we fix the value of δ during one time-step dt (we do this in order not to have the terms $d\delta$), the change in the value of this portfolio will be

$$dV = \left(vS\frac{\partial C}{\partial S}\right)dW + \left(\frac{\partial C}{\partial t} + \mu S\frac{\partial C}{\partial S} + \frac{(vS)^2}{2}\frac{\partial^2 C}{\partial S^2}\right)dt + \delta(\mu S dt + vS dW\ )$$

$$dV = vS\left(\frac{\partial C}{\partial S} + \delta\right)dW + \left(\frac{\partial C}{\partial t} + \mu S\left(\frac{\partial C}{\partial S} + \delta\right) + \frac{(vS)^2}{2}\frac{\partial^2 C}{\partial S^2}\right)dt$$

If we set $\delta = -\,\partial C/\partial S$ (evaluated at the beginning of dt), the stochastic term with dW and the term with the drift μ vanish. We are then left with a non-stochastic differential equation:

$$dV = \left(\frac{\partial C}{\partial t} + \frac{(vS)^2}{2}\frac{\partial^2 C}{\partial S^2}\right)dt$$

Since the portfolio is totally hedged during dt, in the absence of arbitrage, its rate of return must be the riskless rate, $dV/V = r\,dt$. Hence,

$$dV = \left(C + \frac{\partial C}{\partial S}S\right)r\,dt = \left(\frac{\partial C}{\partial t} + \frac{(vS)^2}{2}\frac{\partial^2 C}{\partial S^2}\right)dt$$

Dividing both sides by dt, we arrive at the Black-Scholes partial differential equation:

$$\frac{\partial C}{\partial t} + \frac{(vS)^2}{2}\frac{\partial^2 C}{\partial S^2} + rS\frac{\partial C}{\partial S}S + rC = 0$$

which is a parabolic differential equation with many possible solutions. Some boundary conditions must be imposed to get a unique solution. For a European call, the needed initial and boundary conditions are

- $C(S,T) = \max\{0, S(T) - X\}$
- $C(0,t) = 0$
- $\lim_{S\to\infty} C(S,t) = S$

An explicit solution for the call price is then found as (via Fourier transforms or other methods)

$$C(S,t) = S(t)N(d_1) - Xe^{-r(T-t)}N(d_2)$$

where

$$d_1 = \frac{\ln\left(\frac{S(t)}{X}\right) + (r + \frac{v^2}{2})(T-t)}{v\sqrt{T-t}}$$

$d_2 = d_1 - v\sqrt{T-t}$

$N(Z) = \Pr(z(0,1) \leq Z)$ (the normal probability distribution function).

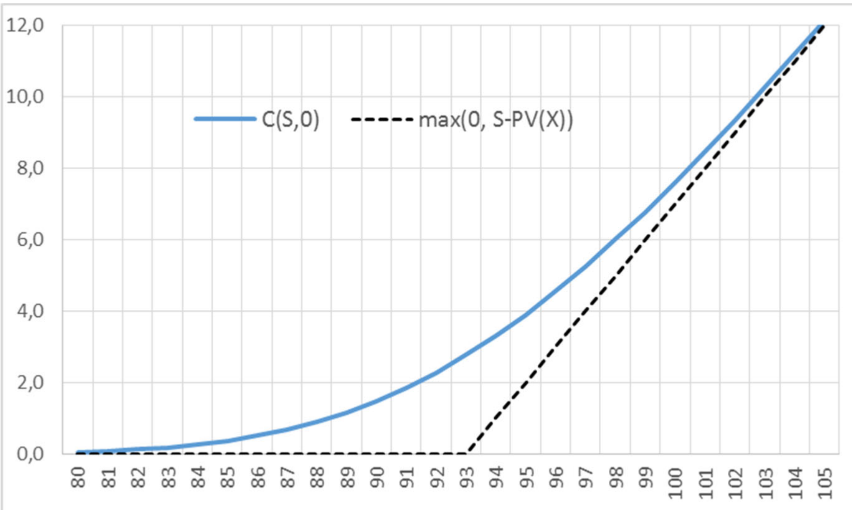
The BSM formula will give call values between 0 (when $S \ll X$) as the lower limit and $S - Xe^{-r(T-t)}$ (when $S \gg X$) as the upper limit. To see this, consider two extreme cases:

- $S \ll X$ (deep out-of-the-money): In this case, $\ln(S/X) \ll 0$, which implies $N(d_1) \approx 0$ and $N(d_2) \approx 0$. As a result, $C \approx 0$.
- $S \gg X$ (deep out-of-the-money): In this case, $\ln(S/X) \gg 1$, which implies $N(d_1) \approx 1$ and $N(d_2) \approx 1$. As a result, $C \approx S - Xe^{-r(T-t)}$.

Example: Suppose the interest rate $r = 0.07$ and a common stock is currently trading at $S(0) = 95$ per share. The standard deviation of the stock's rate of return has been estimated at $v = 0.25$ per year. To find the value for a call option on this stock with $X = 100$ and one month to expiration (that is, $T - 0 = 1/12 = 0.083$ years), first calculate $d_1 = -0.594, d_2 = -0.666$ to get the normal cumulative probabilities $N(-0.594) = 0.276, N(-0.666) = 0.253$, and then plug in the BSM formula to get the call price:

$$C(95,0) = 95 \times 0.276 - (100e^{-0.07\times 0.083} \times 0.253) = 1.48$$

Call prices $C(S, 0)$ for several stock prices are shown in the following graph:

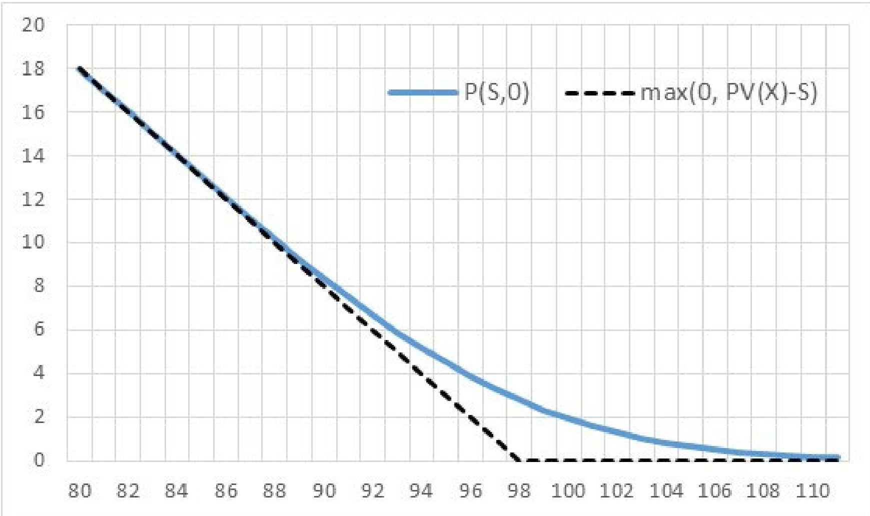


In the graph, note how $C(S, 0)$ behaves as it goes deep out-of-the-money ($S \to 0$) and also as it becomes far-in-the-money ($S - X \to \infty$). The function $C(S, 0)$, which is a convex function, becomes almost linear when S is far from X.

For a European put option on a non-dividend stock, we can find the price of a put option using the put-call parity $P(S,t) = C(S,t) - S + Xe^{-r(T-t)}$. Alternatively, after noting that $N(d) = 1 - N(-d)$ due to the symmetry of the normal density, we can write:

$$P(S,t) = -S(t)N(-d_1) + Xe^{-r(T-t)}N(-d_2)$$

Using the same example, various put values as a function of S are shown below:



In the graph, note how $P(S,0)$ behaves as it goes deep out-of-the-money ($S \to \infty$) and also as it becomes far-in-the-money ($X - S \to X$). The function $P(S,0)$, which is a convex function, becomes almost linear when S is far from X.

Some Comparative Statics

The BSM formula identifies the price of a European option on a non-dividend paying asset as a function of five variables: $S, T-t, X, r$ and v. In this section, we will calculate the sensitivity of option prices to each of these variables.

Delta

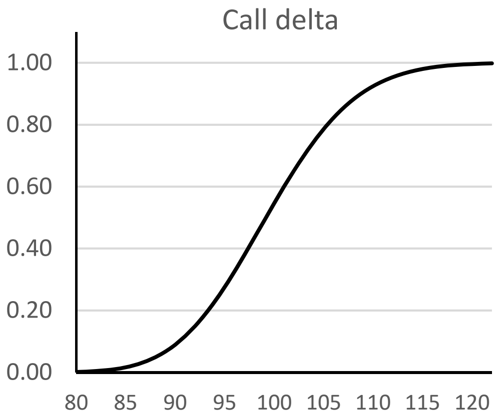
The "delta" of an option is its sensitivity to the price of the underlying asset. Calculating the partial derivative of the BSM call function, we find

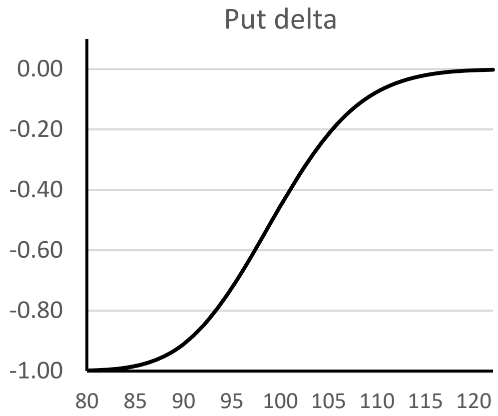
$$1 > \delta = \frac{\partial C}{\partial S} = N(d_1) > 0$$

Similarly, for a put option, we find

$$-1 < \delta = \frac{\partial P}{\partial S} = N(d_1) - 1 < 0$$

For the example, these are shown for various stock prices in the following graphs:





The call's delta at $S = 95$ is found to be 0.276, meaning that if the stock price changes by a small unit ΔS, the call price will change in the same direction by approximately $0.276 \times \Delta S$ units. At the same price level, the put's delta is calculated as -0.724, showing the inverse relationship between the put price and the stock price.

It is seen that, for a call, as $S/X \to \infty, \delta \to 1$ and, as $S/X \to 0, \delta \to 0$. For a put option, as $S/X \to \infty, \delta \to 0$ and, as $S/X \to 0, \delta \to -1$. As an option becomes more in-the-money ($S/X \approx 1$), its absolute price movements become more similar in magnitude to movements in the price of the underlying asset.

A measure directly related to delta is the option's *lambda*, Λ, which measures the option's price elasticity,

$$\Lambda = \frac{\partial C/C}{\partial S/S} = \delta \frac{S}{C} > 1$$

For a put option, lambda is similarly given by $\Lambda = \frac{\partial P/P}{\partial S/S} = \delta \frac{S}{P} < -1$.

The percentage change in the option price will always be greater than that in the underlying asset's price. For a call, as $S/X \to \infty, \Lambda \to 1$ and, as $S/X \to 0, \Lambda \to \infty$. For a put option, as $S/X \to \infty, \Lambda \to \infty$ and, as $S/X \to 0, \Lambda \to -1$. This means that the leverage effect is more pronounced for options that are more out-of-the-money.

Example: The value of a portfolio of one short call and δ long shares, defined as $V = -C + \delta S$, would have a "position delta" equalling

$$\delta_V = \frac{dV}{dS} = -\frac{\partial C}{\partial S} + \delta = -\delta + \delta = 0$$

This position is called "delta neutral", meaning that, during an infinitesimal interval dt, the position is a complete hedge. This is why delta is often called the option's "hedge ratio." (Remember that the Black-Scholes derivation was based on maintaining continuously such a delta-neutral position.)

Example: In general, the delta of a portfolio measures the exposure of its value to price risk and is calculated as the weighted sum of the deltas of its components, $\delta_V = \sum_i n_i \delta_i$. Hence, a similar delta-neutral position can also be established using two options (on the same asset but with different strikes and/or expiries) when one option is used to hedge the other option. Consider a portfolio of n_1 units of C_1 and n_2 units of C_2:

$$V = n_1 C_1 + n_2 C_2 \implies \delta_V = n_1 \delta_1 + n_2 \delta_2$$

To get a delta-neutral position with $\delta_V = 0$, we simply choose n_1 and n_2 such that $n_1/n_2 = -\delta_2/\delta_1$.

Delta is a function of five parameters $S, T - t, X, r$ and v. Therefore, any change in one or more of these variables will cause a change in delta. The following hold for both calls and puts:

- Even if nothing else changes, the delta of an option will change with the passage of time:

$$\frac{\partial \delta}{\partial t} < 0 \text{ if } S \leq X, \text{ and } \frac{\partial \delta}{\partial t} > 0 \text{ if } S > X$$

- When interest rates change, an option's delta will change in the same direction. Call values are more sensitive and put values are less sensitive to changes in price of the underlying asset at higher interest rates, and vice versa.

$$\frac{\partial \delta}{\partial r} > 0$$

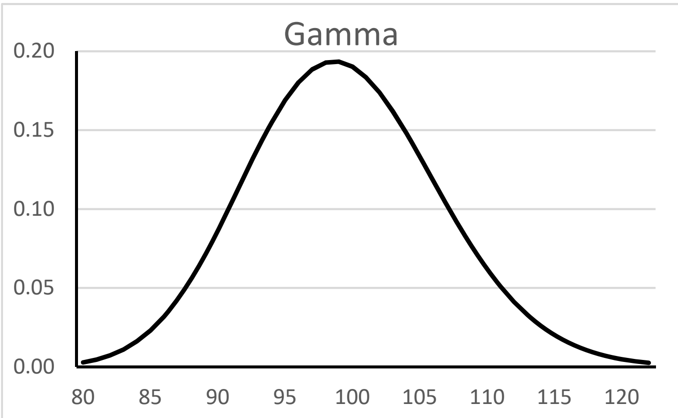
- Changes in the volatility of the underlying asset will also affect the delta of an option.

$$\frac{\partial \delta}{\partial v} > 0 \quad \text{if } S < X, \text{ and } \quad \frac{\partial \delta}{\partial v} < 0 \quad \text{if } S > X$$

- ***Gamma***, Γ, is the more interesting parameter and it measures the sensitivity of an option's delta to changes in the price of the underlying asset:

$$\Gamma = \frac{\partial \delta}{\partial S} = \frac{\partial^2 \delta}{\partial S^2} = \frac{n(d_1)}{Sv\sqrt{T-t}} > 0$$

  Where $n(d_1) = \partial N(d_1)/\partial d_1$ is the standard normal probability density function. Gamma values for several stock prices are shown below:



  If the option is around being at-the-money, its gamma is greatest. This means that the position's price exposure δ is likely to change a lot when the stock price moves around the level of the strike price. For deep in- or out-of-the-money options, gamma is close to zero and delta does not change much with the stock price.

Example: To establish a delta-neutral and a gamma-neutral position (that is, a position where delta neutrality is maintained for changing stock prices), at least two options are needed. Consider a portfolio of two calls $V = n_1 C_1 + n_2 C_2$. To establish the desired position, we set

$$\delta_V = n_1\delta_1 + n_2\delta_2 = 0$$
$$\Gamma_V = n_1\Gamma_1 + n_2\Gamma_2 = 0$$

These are two equations in two unknowns, n_1 and n_2. Fixing the other variables, these values will give a delta-gamma-neutral position.

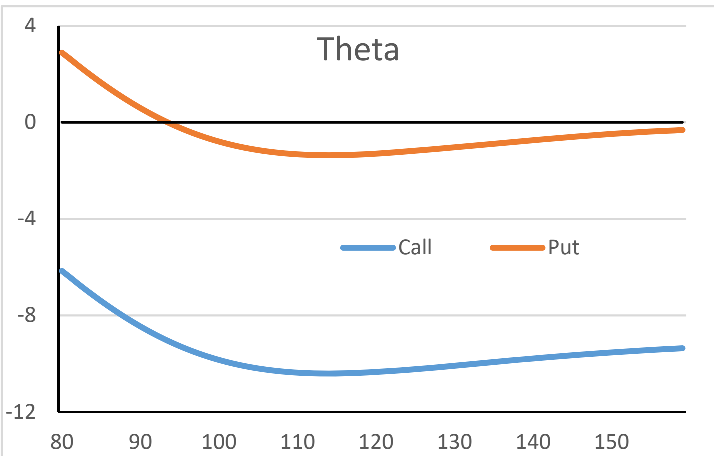
Theta

Theta, θ, measures the change in option price with respect to a change in time to expiration:

$$\theta = \frac{\partial C}{\partial t} = -S\,n(d_1)\frac{v}{2\sqrt{T-t}} - rXe^{-r(T-t)}N(d_2) < 0 \implies \frac{\partial C}{\partial(T-t)} > 0$$

$$\theta = \frac{\partial P}{\partial t} = -S\,n(d_1)\frac{v}{2\sqrt{T-t}} + rXe^{-r(T-t)}N(-d_2) \lesseqgtr 0 \implies \frac{\partial P}{\partial(T-t)} \lesseqgtr 0$$

The theta values for various stock prices are shown in the following graph (with $r = 0.10, T - t = 1$ to make the chart for illustrative):



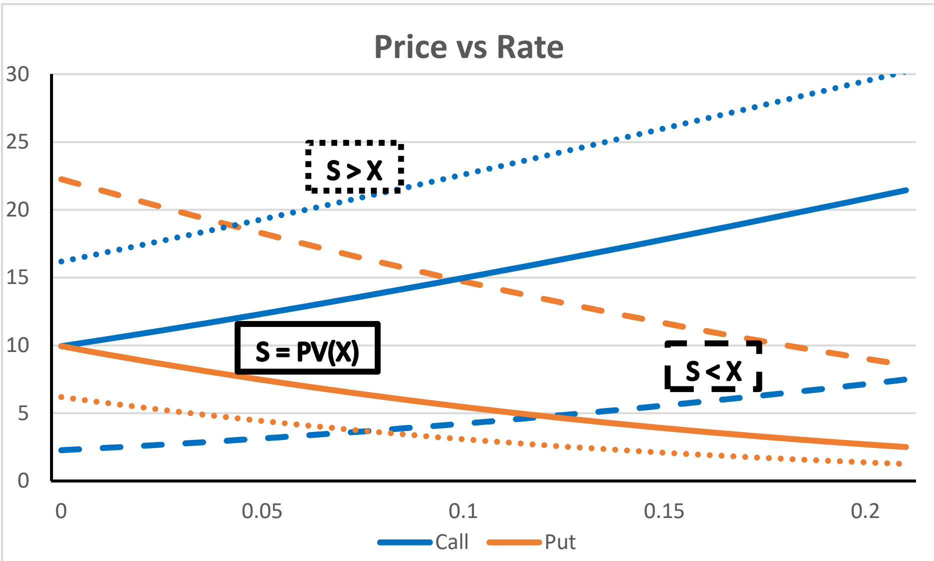
Theta is not related to any "risk" that we are interested in hedging. For all call options and for puts not deep in-the-money, it shows the decay in the time value of the option over time. For put options with $X \gg S$, theta is positive meaning that the shorter the time to expiration, the more valuable the option will be.

Rho
This derivative measures the sensitivity of option value to changes in interest rates:

$$\Upsilon = \frac{\partial C}{\partial r} = (T-t)Xe^{-r(T-t)}N(d_2) > 0$$

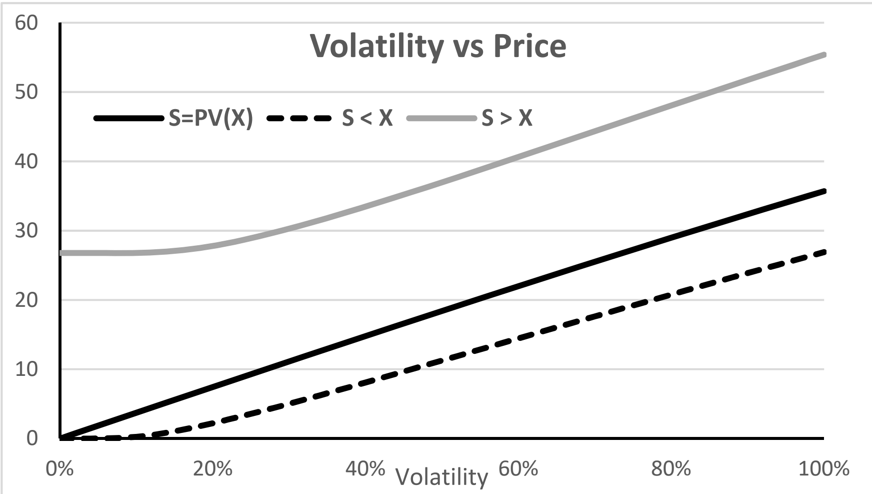
$$\Upsilon = \frac{\partial P}{\partial r} = -(T-t)Xe^{-r(T-t)}N(-d_2) < 0$$

Holding the other parameters constant, as interest rates increase, call prices will increase and put prices will decrease. Intuitively, a higher rate of interest means a lower present value of the exercise price, implying a higher intrinsic value for the call and a lower value for the put. For the example at hand, option prices at various rates are shown below:



Vega
This is sometimes referred to as the *kappa* of an option, and it measures the effect on option price of a change in the volatility of the underlying asset. (Incidentally, unlike all the other partial derivatives conventionally denoted by Greek letters, vega is not a Greek letter. It is said to be the Arabic name of a bright northern star.)

Volatility is at the core of most financial models and certainly more so in derivatives pricing. In the BSM model, it is the only parameter that needs to be estimated. It is worthwhile to look more closely at the relation between volatility and option prices. The graph below shows call prices as a function of volatility v.

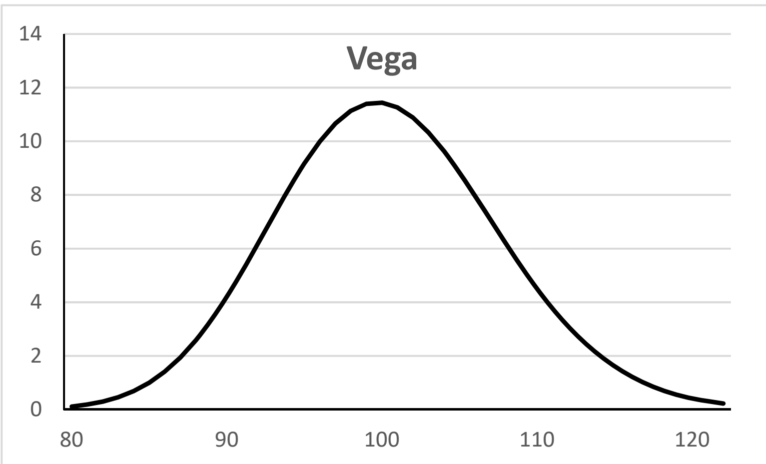


A few observations are informative. First, for at-the-money calls ($S = Xe^{-r(T-t)}$), the call price is almost a linear function of volatility. (Indeed, we will later derive the approximation $C \approx Sv\sqrt{T-t}/\sqrt{2\pi}$.) This must be why option trading is often called "volatility trading" in the market's jargon. Second, no matter how low volatility may be, option price should always be greater than intrinsic value.

For both puts and calls, the vega of an option is calculated as

$$\boldsymbol{\mathcal{v}} = \frac{\partial C}{\partial v} = \frac{\partial P}{\partial v} = S\,n(d_1)\sqrt{T-t} = Xe^{-r(T-t)}n(d_2) > 0$$

For the option in the example, vega values for various stock prices are below:



Vega is highest when option is trading around at the money and it becomes smaller when the stock price moves away from the strike price in either direction.

Although the Black-Scholes model is based on the assumption of constant volatility, volatility is estimated rather than measured. Hence, $\boldsymbol{\nu}$ may be used as a measure of exposure to estimation risk. Similarly, in markets with non-constant volatility of prices, vega may be a good first-step estimate of option's sensitivity to changes in underlying volatility. In such markets, options with lower vega values may be thought to be "less risky." Finally, if two volatilities give the same price for two (otherwise identical) options, then they must be equal. This must be why it is common market practice to quote volatilities rather than prices.

A (Slightly) Generalized BSM Model

For European options, if the underlying asset has distributions, the original BSM formula may be modified as follows:

$$C(S,t) = e^{(g-r)(T-t)}S(t)N(d_1) - Xe^{-r(T-t)}N(d_2)$$

where now $d_1 = \frac{\ln\left(\frac{S(t)}{X}\right)+(g+\frac{v^2}{2})(T-t)}{v\sqrt{T-t}}$ and $d_2 = d_1 - v\sqrt{T-t}$. The parameter g is interpreted as the constant cost-of-carry rate of holding the underlying asset until expiration. Note that setting $g = r$ (that is, the only carrying cost is the interest rate) gives the original formula. The formula for a put option is similarly modified.

Examples follow:

Example: (European call with a constant continuous dividend yield) First, to describe this dividend process, define the dividend yield q as the constant proportion of the asset price paid out to stockholders per unit of time and thus $qS(t)dt$ as the TL dividend during the instant dt. If all dividends are reinvested in the same stock, then one share of stock at time 0 will grow to e^{qt} shares by time t and hence the value of this trading portfolio will be equal to $S(t)e^{qt}$. In other words, an investment of $S(t)$ in the underlying stock at time t will grow in value to $S(T)e^{q(T-t)}$ by time T with dividends being reinvested. This is equivalent to an investment of $S(t)e^{-q(T-t)}$ at time t growing to $S(T)$ by time T.

Since the option value at expiry will depend on $S(T)$, we can interpret $S(t)e^{-q(T-t)}$ as the price of the stock that pays no dividends but has the same expected final value $S(T)$ as a stock that pays a constant continuous dividend yield q. Thus, we can modify the BSM formula as

$$C(S,t) = (S(t)e^{-q(T-t)})N(d_1) - Xe^{-r(T-t)}N(d_2)$$

with $d_1 = \frac{\ln\left(\frac{S(t)}{X}\right) + (r - q + \frac{v^2}{2})(T-t)}{v\sqrt{T-t}}$ and $d_2 = d_1 - v\sqrt{T-t}$.

Alternatively, we could arrive at the same result by noting that the cost-of-carry in this case of a constant dividend yield will be $g = r - q$ and substituting this in the general formula above. The formula for a put option is similarly modified.

Example: (Stock index options) As usual, these are options on an index value times a multiplier. If it is a stock market index and the option is European, the BSM formula is valid with

$S(t)$ = the index value at time *t* times the multiplier
v^2 = the variance of the log return on the index
q = estimated average dividend yield on the stocks in the index

Example: (Currency options) These are options to buy or sell a foreign currency. Define the parameters as

$S(t)$ = price of one unit of foreign currency in domestic currency, or the exchange rate (e.g. $S = 2.50$ TL/\$)
v^2 = variance of the rate of change in the exchange rate S
r_f = the riskless interest rate on foreign currency instruments (which is analogous to dividends on stock)
X = the strike price in domestic currency

Here, the carrying cost is $g = r - r_f$ and the BSM formula for a currency call becomes

$$C(S,t) = (S(t)e^{-r_f(T-t)})N(d_1) - Xe^{-r(T-t)}N(d_2)$$

with $d_1 = \frac{\ln\left(\frac{S(t)}{X}\right) + (r - r_f + \frac{v^2}{2})(T-t)}{v\sqrt{T-t}}$ and $d_2 = d_1 - v\sqrt{T-t}$.

Alternatively, since interest rate parity implies the forward exchange rate $F_{t,T} = S(t)e^{(r-r_f)(T-t)}$, the same formula is sometimes written as

$$C(S,t) = e^{-r(T-t)}\big(F_{t,T}N(d_1) - XN(d_2)\big)$$

where now $d_1 = \frac{\ln\left(\frac{F_{t,T}}{X}\right)+\frac{v^2}{2}(T-t)}{v\sqrt{T-t}}$ and $d_2 = d_1 - v\sqrt{T-t}$.

Since a put on the TL/$ rate is equivalent to a call on the $/TL rate, valuing a currency put is identical to valuing a currency call with proper definitions.

Example: (Options on forward contracts) These are options (expiry T and strike X) to enter into a forward contract with an expiry $\tau \geq T$ and a forward price equalling the strike price of the option. Remembering the parity $F_{t,\tau} = S(t)e^{(r+u-q-y)(\tau-t)}$ and assuming that the carrying cost parameters (r,u,q,y) are constant, the forward price F will follow the same stochastic process as S, with a different drift rate but the same variance. The Black-Scholes partial differential equation can be derived for forward options in exactly the same way as it was derived for ordinary options. Through the usual hedging arguments, the drift term will cancel out making $g = 0$. Hence, the BSM formula becomes:

$$C(S,t) = (F_{t,T}e^{-r(\tau-t)})N(d_1) - Xe^{-r(\tau-t)}N(d_2)$$

where now $d_1 = \frac{\ln\left(\frac{F_{t,\tau}}{X}\right)+\frac{v^2}{2}(\tau-t)}{v\sqrt{\tau-t}}$ and $d_2 = d_1 - v\sqrt{\tau-t}$. If the underlying asset is a futures contract instead of a forward contract, there will be marking to market on date T, the expiry of the option. That is, cash flows from the strike of the option will be realized on this date. Therefore, for options on futures, we have to replace τ with T in the above formula.

Example: (Valuation of employee stock options, from Merton et al (1995)) A publicly held company has the following securities outstanding:

- bonds maturing in one year with a face value of $100 million and current market price of $85 million
- 1 million shares of common stock with a total market value of $20 million
- 100,000 employee stock options expiring in one year with an exercise price of $20 per share (thus, optionholders have the right to buy 1/11 of the firm's

value a year from now for a total payment of \$2 million). They have no quoted market value because they are not traded.

If the rate of interest is 6%, what are the implied values of the stock options and the entire firm (including the employee stock options)?

Letting V_1 denote the yearend value of the firm, employees will exercise their options if $2 \leq (V_1 - 100 + 2)/11 \Rightarrow V_1 \geq 120$. Hence, the stock option's payoff at maturity is $\max(0, V_1 - 120)/11$ and their current value is given by the BSM formula

$$O_0 = (V_0 N(d_3) - 120e^{-0.06} N(d_4))/11$$

where $d_3 = (\ln(V_0/120) + 0.06 + v^2/2)/v$, $d_4 = d_3 - v$ and v is the volatility of the rate of return on the entire firm value.

The first step to solve the problem is to see that the equity of the firm may be viewed as a call option written on the value of the firm, with an exercise price equalling to the face value of debt and expiration equalling the maturity of debt. In other words, equityholders will pay back the debt if the value of the company is greater than the amount of debt or else they will choose not to pay. Hence, at expiration, the payoff will be $\max(0, V_1 - 100)$ and its BSM value now can be calculated as

$$E_0 = V_0 N(d_1) - 100e^{-0.06} N(d_2)$$

where now $d_1 = (\ln(V_0/100) + 0.06 + v^2/2)/v$ and $d_2 = d_1 - v$. Hence, the market value of the debt satisfies

$$85 = V_0 - E_0 = V_0 - [V_0 N(d_1) - 100e^{-0.06} N(d_2)]$$

The market value of the equity satisfies $20 = E_0 + O_0$. Solving these two equations for V_0 and v, we find $V_0 = 106.14$ and $v = 0.3621$. Hence, the market value of the employee stock options is $106.14 - 85 - 20 = \$1.14m$, or \$11.40 per option.

Deviations from the Black-Scholes-Merton price

Remember that the Black-Scholes analysis identifies the price of an option as a function of five parameters (S, T, X, r, σ) under the following assumptions:

- The stock price follows a geometric Brownian motion with constant drift and volatility $dS/S = mdt + \sigma dW$
- There is a constant riskless rate of interest
- Perfect markets (no transaction costs, etc.)
- European option and no cash distributions to holders of the underlying asset (in the previous class, the no-distribution assumption was relaxed for certain options under the condition that the implications of the other assumptions are unchanged.)

These assumptions make possible continuous hedging and a unique option price emerges. In other words, it is a solution for complete markets. Clearly, this is an idealization and it is no surprise that real market prices often deviate from the Black-Scholes prices.

Of the parameters in the formula, *T* and *X* are specified in the option contract. The rate of interest *r* can be taken as given. Then, the only parameter to be estimated is the volatility σ, which is directly related to the underlying stochastic process of prices. Hence, an inappropriate assumption about the stochastic process and a related mis-estimation of volatility can make the BS prices deviate from actual prices.

Remember that an application of Ito's lemma on $dS/S = mdt + \sigma dW$ gives the expression for log-prices as $d\ln(S_t) = \mu dt + \sigma\, dW_t$ where $\mu = m - 0.5\sigma^2$. Defining the continuous-time rate of return as $R_t = \ln(S_t) - \ln(S_{t-1})$, we get by direct integration

$$R_t = \mu + \sigma(W_t - W_{t-1})$$

Since the Brownian increments $W_t - W_{t-1}$ are standard normal variates, this is a normal random walk with drift.

In other words, the discrete-time version of the continuous-time geometric Brownian motion is a normal random walk with drift for the log-price series. In practice, since prices are only observed at discrete points in time, we have to use this version for estimation purposes.

Given a series of prices on a stock $S_0, S_1, ..., S_N$, separated by a constant time interval Δt, the maximum likelihood estimates of the parameters are:

$$\hat{\mu} = \frac{1}{N\,\Delta t}\sum_{i=1}^{N}\ln(S_i/S_{i-1}) \quad and \quad \hat{\sigma}^2 = \frac{1}{N\,\Delta t}\sum_{i=1}^{N}\left(\ln(S_i/S_{i-1}) - \hat{\mu}\right)^2$$

We know that the MLEs are unbiased and asymptotically normal with variances

$$Var(\hat{\mu}) = \frac{\sigma^2}{N\,\Delta t} + \frac{\sigma^4}{2N} \quad and \quad Var(\hat{\sigma}^2) = \frac{2\sigma^4}{N}$$

(In passing, note the presence of Δt in the asymptotic variance of $\hat{\mu}$. This shows that the drift cannot be estimated consistently even from very large samples.)

When we use the MLE of volatility in the BS formula, we get “some estimate” of the option price

$$\hat{C}_t = C(S_t, X, T, r, \hat{\sigma})$$

Since $C(S_t, X, T, r, \hat{\sigma})$ is a non-linear function of $\hat{\sigma}$, $\hat{C}_t$ is not an unbiased estimate of $C = C(S_t, X, T, r, \sigma)$, the BS call value calculated with the “true” σ. We can construct asymptotic confidence intervals around $\hat{C}_t$ as

$$\hat{C}_t \pm 2\mathrm{K}\sqrt{2\hat{\sigma}^4 / N}$$

where $\mathrm{K} = \left.\frac{\partial C}{\partial \sigma}\right|_{\sigma=\hat{\sigma}}$ is the kappa (vega) of the call. The length of the confidence interval will depend on the kappa value, the sensitivity of the option with respect to the volatility.

Suppose for a moment that all the assumptions of the BS approach are empirically supported. When we use the *historical volatility* estimate in the BS formula, we can only get an estimate of the true option value, whose statistical properties are not very clear. Hence, this approach may not be suitable for trading purposes. We first have to check how actual option prices are determined in the marketplace and then see how the BS approach can be used in practice.

Given the other parameters, the BS formula gives an exact deterministic relationship between volatility and the option value. Consequently, the following system of non-linear equations must have a unique solution in volatility:

$$C^{(i,j)} = C(S, X^{(i)}, T^{(j)}, r, \sigma)$$

for all different observed option prices, $C^{(i,j)}$, defined by different exercise prices $X^{(i)}$ and maturities $T^{(j)}$. Yet, empirical evidence on actual option prices shows that this system has no unique solution. In fact, there seem to be as many option prices as the number of $(X^{(i)}, T^{(j)})$ pairs. As a result, the BS formula is rejected by the data. It seems to be only an aproximate formula such as:

$$C^{(i,j)} = C(S, X^{(i)}, T^{(j)}, r, \sigma) + e_{ij}$$

where the error term e_{ij} accounts for the deviation of the actual price from the BS price. The average magnitude of the error seems to get larger as the difference $|S - X|$ becomes larger and it wanders as T changes. The formula works best when the option is around being at-the-money and also close to expiration.

Before investigating why the BS formula does not hold exactly in reality, it is worth looking into volatility more closely. For a given market price C, there is a unique value of σ, say v, that sets the BS formula equal to the market price:

$$C = C(S, X, T, r, v)$$

This value v is called ***implied volatility*** and it is calculated numerically by inverting the BS formula

$$v = v(S, X, T, r, C)$$

Given the other parameters, there is a one-to-one correspondence between C and v and hence they are often quoted together. (Despite the name, implied volatility may also be viewed as a normalized option price.)

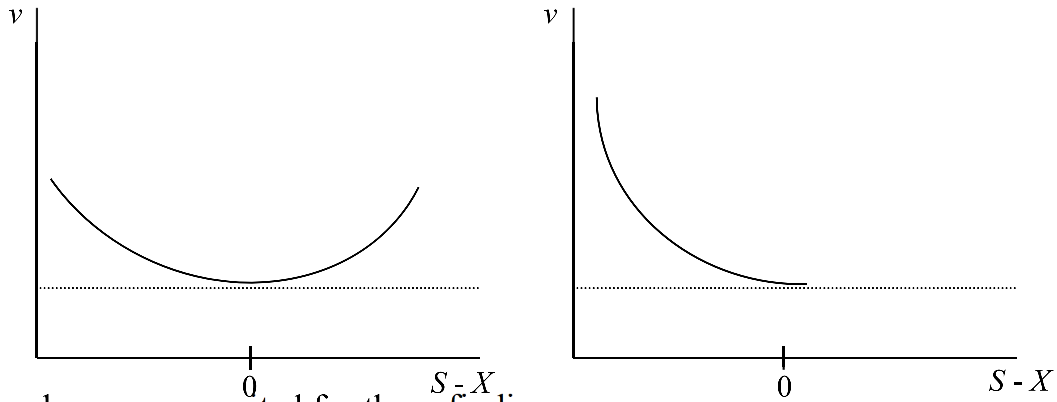
Implied volatility is computed by trial-and-error and some of the easier methods are the classic Newton-Raphson method (when kappa is analytically computable) and the bisection method (when kappa cannot be computed analytically). Excel's GOALSEEK function may also be used. Corrado and Miller (1996, Journal of

Banking and Finance) give a simple formula to compute a good approximation to implied volatility.

If the BS theory were rigorously correct, the implied volatility would be equal to the historical volatility and both would be constant. We mentioned before that this is not the case. Empirical evidence shows the following:

- The implied volatility depends both on X and T. The function $v(X,T)$ (given S, C and r) defines the so called *volatility surface*.
- For at-the-money options, the implied volatility seems to be constant. If we then take the implied volatility for at-the-money options as the "true" volatility, then it seems that the BS price $< C$ for $S > X$ and BS price $> C$ for $S < X$.

When implied volatilities are calculated from market data, a number of patterns such as "*volatility smile*" and "*volatility skew*" are observed.



Several reasons are cited for these findings:

- The BS model does not take into account transaction costs such as bid-ask spreads and commissions. However, this argument is weak for two reasons: (1) Options are typically very liquid assets with very low relative transaction costs and (2) There is no apparent reason why transaction costs should have different impacts on options with different characteristics.

- Volatility is not constant in time and in space. Hence, market participants may value options using different volatility values based on strike price and maturity. Models of stochastic volatility have been suggested to model this behaviour.

- Empirical properties of data is not in full agreement with the assumed Brownian motion. Empirical densities (especially those measured over short time intervals) exhibit leptokurtosis and often skewness. The central limit theorem may guarantee that the normal limit will be eventually reached but a very big sum may be required to get there. Options, however, are liquid assets and they have short maturities. Furthermore, the risk-neutral hedging argument in the BS model requires that the normality assumption should hold for all time intervals and horizons.

Several probability models have been suggested as alternatives to the geometric Brownian motion model. Examples are stable Paretian models and mixed diffusion-jump models. Both have the capability of explaining the empirical tail behaviour of data.

As an example, suppose that the return-generating process is a mixed diffusion-jump process with Poisson jumps of random size:

$$\ln(S(t)) - \ln(S(0)) = \underbrace{\mu_B t + \sigma_B (W(t) - W(0))}_{\text{diffusion process}} + \underbrace{\sum_{n=1}^{N(t)} J(n)}_{\text{jump process}}$$

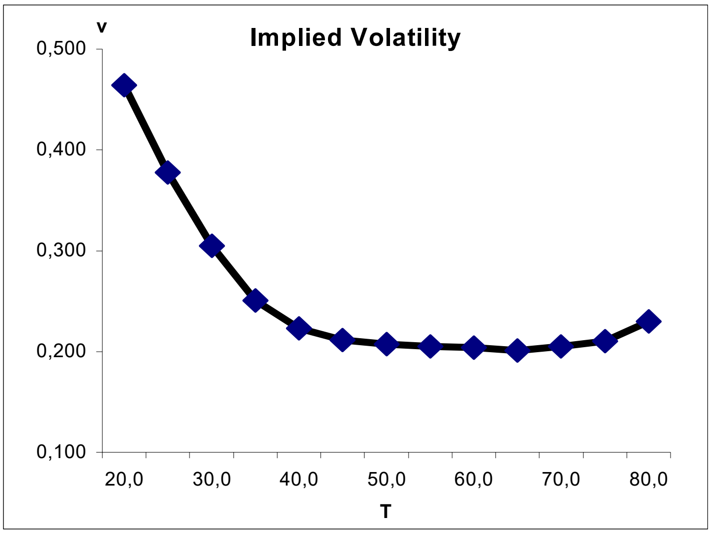
where $N(t)$ is a Poisson random variable with parameter λt and the logarithm of the size of the n^{th} jump $\ln(J(n))$ is normally distributed with mean μ_J and variance σ_J^2. Assuming that the diffusion process, the Poisson process and the jump sizes are independent, the mean and variance of the mixed random variable can be shown to be:

$$E(\ln(S(t)/S(0))) = (\mu_B + \lambda\mu_J)t \equiv \mu t$$
$$Var(\ln(S(t)/S(0))) = (\sigma_B^2 + \lambda\sigma_J^2 + \lambda\mu_J^2)t \equiv \sigma^2 t$$

These distributions are leptokurtic for $\lambda > 0$ and skewed for $\mu_J \neq 0$. Assuming that jump risk is diversifiable, Merton (1976, JFE) has shown that, keeping the other assumptions of the BS model and with $\mu_J = 0$, the price of a call option will be equal to

$$C^*(S,T,X,r,\sigma_B;\lambda,\sigma_J) = \sum_{n=0}^{\infty} \underbrace{\frac{e^{\lambda T}(\lambda T)^n}{n!}}_{\text{Poisson probability}} \cdot \underbrace{C(S,T,X,r,\sigma_B^2 + \frac{n\sigma_J^2}{T})}_{\text{BS call formula}}$$

Suppose that, using the above formula, we simulate call prices for various T and X, and then calculate the corresponding implied probabilities pretending that the BS model is correct. When we do this for $\lambda = 0.01$ per year, $\sigma = 0.20$, $r = 0.05$ and $S = 50$, and calculate the implied probabilities by inverting the BS formula, the following "volatility smile" is obtained:



These findings show that a possible explanation for the incomplete fit of the BS model to actual data is the presence of fat tails in empirical distributions. In general, any type of deviation of empirical distributions from normality may be a reason for the poor fit of the BS model.

So what do we do?

Despite its statistically poor fit to data, the BS model still remains a valuable tool for pricing options and for comparing the prices of different options. Derivative markets are active and liquid markets with well-informed traders. As such, derivative prices should have a rich informational content. As a case in point, the implied volatility from the BS model also carries much useful information.

Volatility is a necessary parameter in almost all of the practical models in finance. One can either use the historical volatility estimate or an alternative such as the implied volatility. When the pros and cons of historical and implied volatility measures are compared, the following observations are noted:

- Volatility is not stationary over time. There are more volatile periods and less volatile periods. The empirical success of econometric models such as GARCH models is no surprise.
- Using historical volatility as an estimate of future volatility does not take into account current facts and market's expectations about the future. For example, the historical volatility based on a sample of observations before the Iraq War cannot be a good estimate of volatility during the war period. Even a conditional volatility model such GARCH will not work in this case.
- Several studies have shown that implied volatility calculated from current option prices is a better indicator of current volatility than historical volatility and even conditional volatility models.

But then, given the whole volatility surface $v(X,T)$, which implied volatility from the BS model shall we use? We should pick the one that is least biased. For this, we first note that, when the exercise price equals the forward stock price $X = Se^{rT}$ (at-the-money forward), call and put values are identical by parity and both are approximately linear functions:

$$C = P \approx \frac{S\sigma\sqrt{T}}{2\pi} \approx 0.4S\sigma\sqrt{T}$$

Then, the implied volatility will be a linear function of the option price

$$\sigma = \frac{C\sqrt{2\pi}}{S\sqrt{T}} \approx \frac{2.5C}{S\sqrt{T}}$$

This estimate will be almost unbiased statistically. The CBOE uses this measure to publish a volatility index (VIX) and also as a basis for newly offered option and futures contracts for relatively short maturities.

There are several uses of implied volatility:

- portfolio selection (derivation of the efficient frontier)
- valuation via *contingent claims approach* (an example is on next page)
- risk management (swaps, caps, collars, floors, etc.)
- management of financial guarantees (like deposit insurance and credit guarantees) by governments
- monetary and fiscal policy measures requiring as input market volatility

American options

The valuation formula for European options was obtained by solving the Black-Scholes equation by applying certain boundary conditions. An analytical solution could be obtained because the boundary conditions were known and, most importantly, it was known where these conditions were relevant. However, since American options can be exercised at any time, we cannot know a priori where the boundary conditions are to be imposed. At each time point, we have to derive not only the value of the option but also, for each asset price, we have to determine whether or not the option should be exercised. In math, these types of equations are known as "free boundary" problems and they usually require the solution of complex variational calculus equations.

Since $C_a \geq C \; and \; P_a \geq P$, the Black-Scholes formula will generally undervalue American options. This is due the added possibility of early exercise. At each point in time, it must be decided whether the option is worth more alive or dead.

The domain of the Black-Scholes equation divides into two parts separated by a boundary $S_b(t)$; a continuation region, where the option is kept, and a stopping region, where the option is exercised.

Put Options

It is optimal to exercise if the asset price falls in the stopping region $0 \leq S \leq S_b(t)$ and

$$P = X - S, \qquad \frac{\partial P}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 P}{\partial S^2} + rS\frac{\partial P}{\partial S} - rP < 0$$

The continuation region is therefore $S_b(t) < S < \infty$ and

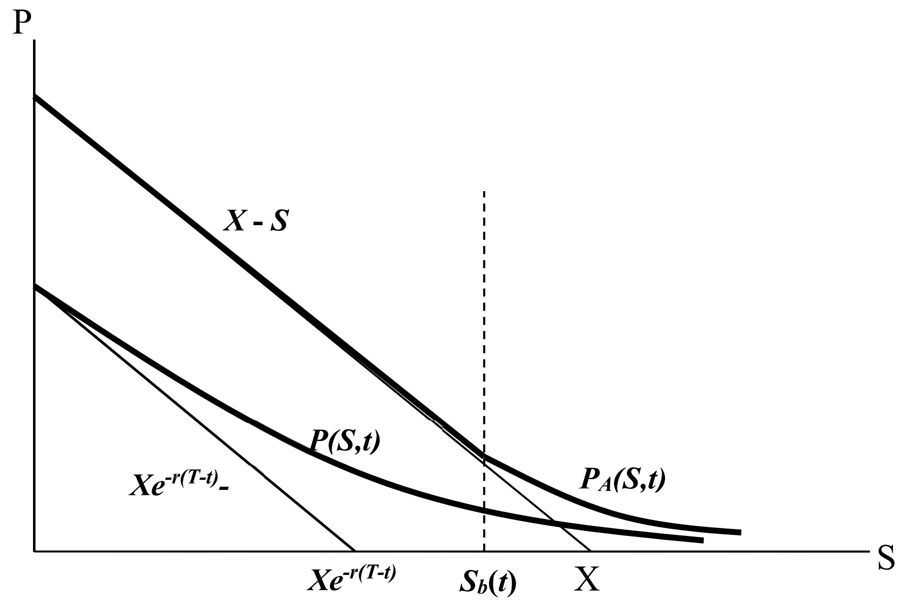
$$P > X - S, \qquad \frac{\partial P}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 P}{\partial S^2} + rS\frac{\partial P}{\partial S} - rP = 0$$

So, in this region, the Black-Scholes formula is valid. The free (exercise) boundary conditions are given by

$$P(S_b, t) = \max(0, X - S_b), \qquad \left.\frac{\partial P}{\partial S}\right|_{S_b(t)} = -1$$

In summary, the value of an American put is given by

$$P_A = \begin{cases} X - S(t) & \text{if} \quad S(t) < S_b(t) \\ P_A(S,t) & \text{if} \quad S(t) > S_b(t) \end{cases}$$



Call Options

If the asset price is high enough just prior to a distribution, the call may be exercised. The Black-Scholes pde is modified with new free boundaries as follows:

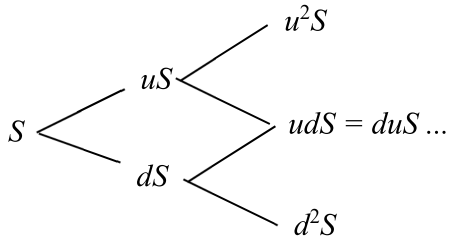
$$\frac{\partial C}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 C}{\partial S^2} + rS\frac{\partial C}{\partial S} - rC \le 0$$

$$C(S_b, t) = \max(0, S_b - X), \quad \left.\frac{\partial C}{\partial S}\right|_{S_b(t)} = 1$$

In the case of American options (except calls on non-dividend assets), there are no closed-form analytic solutions for option values. Consequently, numerical solutions are called for: (1) Finite difference methods, or (2) Binomial solutions

Binomial Methods

As described before, the asset price changes at discrete times $\Delta t, 2\Delta t, 3\Delta t, \ldots$ up to $N\Delta t = T$, the expiration date. If the asset price at time $n\Delta t$ is $S(n)$, then at time $(n+1)\Delta t$, it may go up to $S(n+1) = uS(n)$ or $S(n+1) = dS(n)$ with probabilities p and $1-p$, respectively. Then, a "tree" of all asset prices can be drawn:



Note that, after N price changes, there are $(N+1)$ possible prices at time T.

We would like to choose the parameters u, d and p of this discrete-time binomial process such that it will reflect the important statistical properties of the continuous-time Brownian motion process as closely as possible. We had mentioned before some sensible values of u, d and p in terms μ and σ. Also recall that the drift parameter μ drops out in the derivation of the Black-Scholes formula. Hence, we can arbitrarily set $\mu = r$, the risk-free rate of interest, and obtain the same result. In fact, this was the basic idea behind risk-neutral valuation.

Let $C(n,i)$ the i^{th} possible value of a call at time-step $n\Delta t$. At expiration time $N\Delta t = T$,

$$C(N,i) = \max\left(0, S(N,i) - X\right)$$

For a European option (which does not permit early exercise), given $C(N,i)$ for $i = 1,2,\ldots,N+1$, we had shown before that

$$C(N-1,i-1) = e^{-r\Delta t}\,\hat{E}(C(N,i)) = e^{-r\Delta t}\,(\hat{p}C(N,i) + (1-\hat{p})C(N,i-1))$$

where $\hat{p} = (e^{r\Delta t} - d)/(u-d)$ is the risk-neutral probability. Continuing similarly for $N-2, N-3, \ldots, 0$, we can find $C(0)$, the current value of the call. A similar recursion can be applied for put options.

For an American option, however, we have to determine whether or not early exercise is optimal at each time-step. That is, we have to determine whether the option is more valuable "dead" or "alive". There are two possibilites at each time point

- If the call is not exercised, its value at time-step $n \le N$ is given by

  $$C(n, i-1) = e^{-r\Delta t}\,(\hat{p}C(n+1, i) + (1-\hat{p})C(n+1, i-1))$$

- If the call is exercised, its payoff will be $S(n, i-1) - X$. Therefore, the call value in general will be (for $i = 1, \ldots, n+1$)

  $$C(n, i) = \max\left\{S(n, i) - X,\ e^{-r\Delta t}\,(\hat{p}C(n+1, i+1) + (1-\hat{p})C(n+1, i))\right\}$$

As a result, some branches will be stopped before T and, working recursively with continuing branches, $C(0)$ can be obtained. A similar procedure applies for put options as well.

Example: Suppose we have at-the-money options with $S(0)=100$, $X=100$, $r=0.05$ per year, $\sigma=0.20$ per year, $T=0.5$ (6 months).

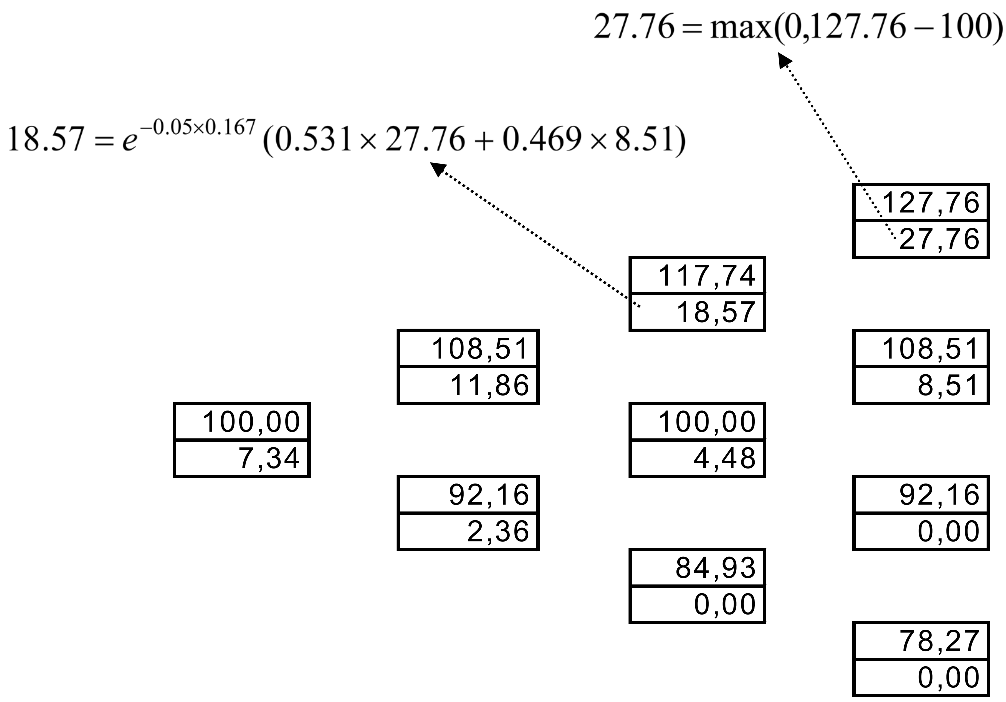
Using the BS formula, we would find

$$C(0)=100\cdot \mathrm{N}(0.2475)-100\cdot e^{-0.05\times 0.5}\mathrm{N}(0.1061)=6.89 \text{ and } P(0)=4.42$$

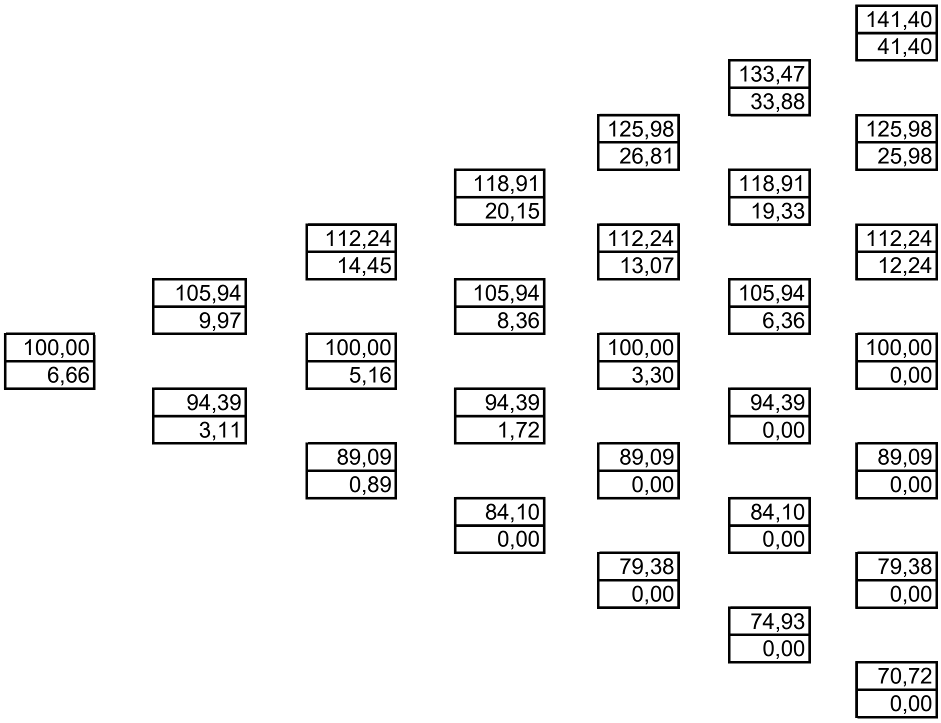
In the binomial approximation, let us use the approximations $u=d^{-1}$ with $u=e^{\sigma\sqrt{\Delta t}}$, $d=e^{-\sigma\sqrt{\Delta t}}$ and $\hat{p}=\dfrac{e^{0.05\times 0.167}-d}{u-d}$.

Let us first use a 3-step binomial tree to get:

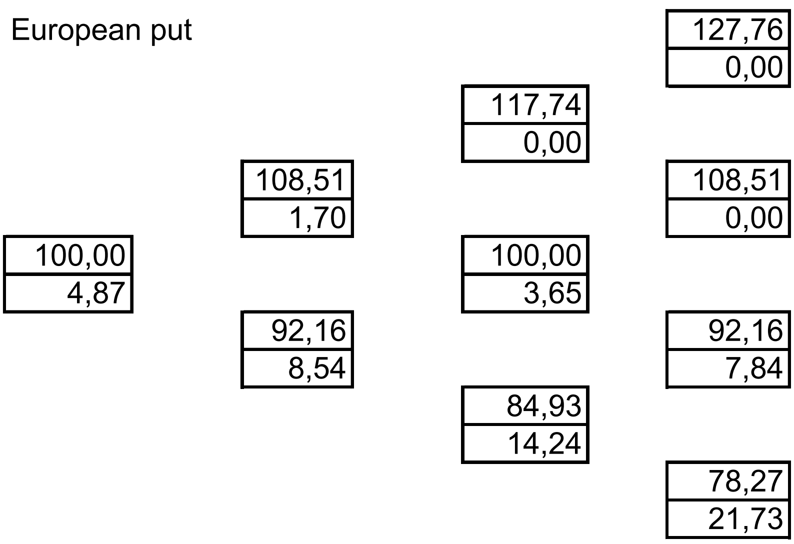
$\Delta t=T/N=0.5/3=0.167$, $u=e^{0.20\sqrt{0.167}}=1.085$, $d=e^{-0.20\sqrt{0.167}}=0.922$ and the risk-neutral probabilities $\hat{p}=\dfrac{e^{0.05\times 0.167}-0.922}{1.085-0.922}=0.531$ and $1-\hat{p}=0.469$.



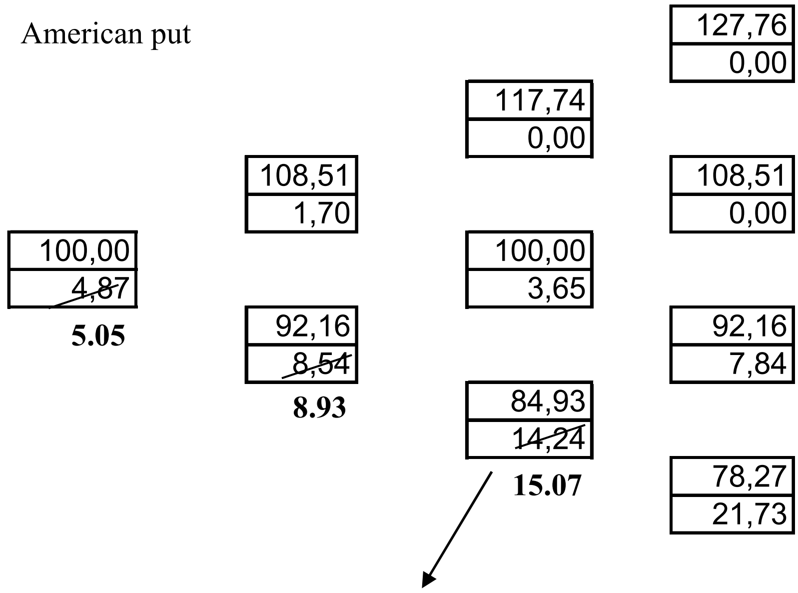
Instead, if we use a 6-step binomial tree, we will get a better approximation to the BS value. In this case, the parameters turn out to be $\Delta t = 0.0833,\ u = 1.059,\ d = 0.944,\ \hat{p} = 0.522$ and $1 - \hat{p} = 0.478$.



For a European-type put option, the 3-step binomial tree is given below:



If it were an American option, at each node, we would calculate the option value after a stop-or-go decision:



Here, since 100 – 84.93 = 15.07 > 14.24, the put is exercised.

Exotic options

Simple types

Forward start option: This is an option starting at some future time t, with expiry at T and exercise price $X = aS_t$. The value of this option at time t will be

$$g(S_t, t; aS_t, T-t) = S_t g(1, t; a, T-t)$$

The rhs is the value of $g(.)$ shares of stock purchased now at a price S_0. Therefore, the current value of the forward-start option must be equal to the current value of this position:

$$g(S_0, 0; aS_0, T-t) = S_0 g(1, 0; a, T-t)$$

The function $g(1,0;a,T-t)$ is non-stochastic and it can be calculated easily with the BS formula.

Chooser option: At some future time t, the optionholder will decide whether it will be a call or a put. Then, its value will be (assuming European type)

$$V_t = \max\{C(S_t, t; X, T-t), P(S_t, t; X, T-t)\}$$
$$V_t = \max\left\{C(S_t, t; X, T-t), C(S_t, t; X, T-t) + Xe^{-r(T-t)} - S_t\right\}$$
$$V_t = \underbrace{C(S_t, t; X, T-t)}_{\text{call at t with maturity T-t}} + \underbrace{\max\left\{0, Xe^{-r(T-t)} - S_t\right\}}_{\text{put at t with maturity at t and exercise } Xe^{-r(T-t)}}$$
$$V_t = C(S_t, t : X, T) + P(S_t, t; Xe^{-r(T-t)}, t)$$

$$\Rightarrow \quad V_0 = C(S_0, 0 : X, T) + P(S_0, 0; Xe^{-r(T-t)}, t)$$

Exercise: Derive the chooser value when the call and put have different exercise prices and maturities.

Binary (digital) options: First, remember that, in the BS formula for European calls $C(S_0, 0; X, T) = S_0 \mathrm{N}(d_1) - Xe^{-rT}\mathrm{N}(d_2)$, the term $\mathrm{N}(d_2)$ is the risk-neutral probability that $S_T > X$ and hence $e^{-rT}\mathrm{N}(d_2)$ is the PV of this probability. This can be interpreted as the PV of a bet, which pays 1 if $S_T > X$ and 0 otherwise. This is a **cash-or-nothing** type of binary call and its current value is

$$C_{\text{cash}} = e^{-rT}\mathrm{N}(d_2)$$

By the same reasoning, the cash-or-nothing binary put will have a value

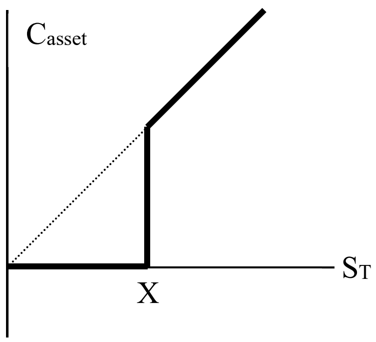
$$P_{\text{cash}} = e^{-rT}(1 - \mathrm{N}(d_2))$$

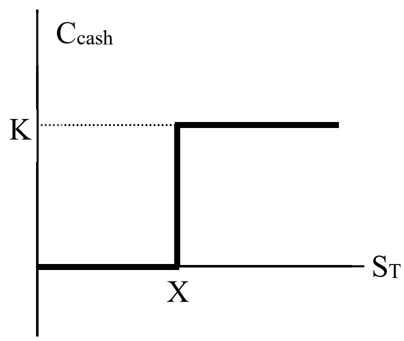
The first term $S_0\mathrm{N}(d_1)$ in the BS formula, evaluated at S_T, will have a value of 0 if $S_T < X$ and a value of S_T if $S_T > X$. Hence, viewed as a call option, this is an **asset-or-nothing** type of binary call with current value

$$C_{\text{asset}} = S_0\mathrm{N}(d_1)$$

Similarly, $P_{\text{asset}} = S_0(1 - \mathrm{N}(d_1))$ for an asset-or-nothing put.

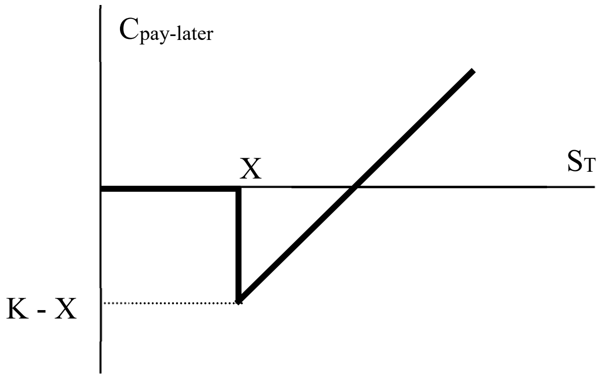
The payoffs at *T* of an asset-or-nothing call and a cash-or-nothing call (with payoff of *K* if $S_T > X$ and 0 otherwise) are shown below:





A **gap option** is defined as $C_{gap} = C_{asset} - C_{cash} = S_0\mathrm{N}(d_1) - Ke^{-rT}\mathrm{N}(d_2)$.

- If $K = X$, we obtain a standard call option.
- If the values are such that $C_{gap} = 0$ (which will be possible if $K > X$), it is called a **pay-later** option with a payoff profile at maturity as follows:



Power options: These are designed by FE junkies who would like to become rich at the pace of a power function. The payoffs at maturity of some popular examples are:

$$C_T = (\max(0, S_T - X))^k \quad \text{and} \quad C_T = \max(0, (S_T)^k - X^k)$$

Exercise: When $k = 2$, show that the two options are related through

$$(\max(0, S_T - X))^2 = \max(0, (S_T)^2 - X^2) - 2X \max(0, S_T - X)$$

Example: An even faster way to get rich would be through a call with payoff at maturity equalling $C_T = \max(0, (S_T)^k - X)$. If the underlying stochastic process for $\ln S$ is a Brownian motion W_T with $W_0 = 0$, then, in a risk-neutral framework,

$$\begin{aligned}(S_T)^k &= (S_0 e^{(r-0.5\sigma^2)T + \sigma W_T})^k \\ &= (S_0)^k e^{((r-q)-0.5v^2)T + vW_T}\end{aligned}$$

where $q = (1-k)(r + 0.5k\sigma^2)$ and $v = k\sigma$. Hence, the BS formula can be used substituting $r - q$ for *r*, *v* for σ, and S^k for *S*.

Complex types

- Currency translated options (compos and quantos)
- Multi-asset options
- Compound options (options on options and extendible options)
- Barrier options (knock-in, knock-out and rebate options)

- Asian options (arithmetic and geometric average options)
- Lookback options
- Passport options

The most tricky (conceptually, not mathematically) one is the currency translated option and we will finish this course with a short description of compos and quantos. For the other ones, you can read your textbook or wait for a future opportunity to discuss them.

We first define an **exchange option** as an option whose payoff depends on the prices of two distinct assets as $\max(0, q_s S_T - q_p P_T)$. The name comes from the interpretation of this option as: (1) An option to buy q_s units of S in exchange for q_p units of P, or (2) An option to sell q_p units of P in exchange for q_s units of S. Setting $q_s = q_p = 1$ for simplicity, the option value may be written as

$$g(S_0, P_0) = P_0 g(\frac{S_0}{P_0}, 1) \equiv P_0 g(f_0, 1)$$

where f may be viewed as the price of asset S in terms of the price of asset P. Then, $g(f,1)$ is a simple call option with an exercise price of one and the BS model may be applied. Two observations on this problem are critical:

1. The underlying random variable is f, which is the quotient of two lognormal random variables. Hence, f is also lognormal with variance

$$\sigma_f^2 = \sigma_s^2 - 2\rho_{sp}\sigma_s\sigma_p + \sigma_p^2$$

2. Since prices are not denominated in terms of regular money (cash) but rather in terms of the price *P*, we have to set the interest rate in the BS formula equal to zero ($r = 0$).

Based on these observations, the value of an exchange call may be written as

$$C_{exchange} = P_0 g(f_0, 1) = P_0(f_0 \mathrm{N}(d_1) - \mathrm{N}(d_2))$$
$$C_{exchange} = S_0 \mathrm{N}(d_1) - P_0 \mathrm{N}(d_2)$$

where $d_1 = \dfrac{\ln(f_0) + 0.5\sigma_f^2 T}{\sigma_f \sqrt{T}}$ and $d_2 = d_1 - \sigma_f \sqrt{T}$. To explain why the rate of interest does not appear in the solution, remember that a replicating portfolio would include borrowing $B_t(S_t, P_t)$ and investing in suitable quantities of the two assets to satisfy

$$g(S_t, P_t) = B_t(S_t, P_t) - \left[\frac{\partial g_t}{\partial S_t} S_t + \frac{\partial g_t}{\partial P_t} P_t \right]$$

Since the function $g(S_t, P_t)$ is homogeneous in S and P, the term in the brackets is identically equal to $g(S_t, P_t)$. Hence, $B_t = 0$ continuously for all t, meaning that we do not have to borrow money for hedging purposes and the rate of interest is irrelevant. (To replicate the option, we just long one asset and short a suitable quantity of the other.)

To describe currency-translated options, consider the scenario of an American investor buying a call option on a Turkish common stock. If all of the relevant prices are quoted in TL and if the investor is willing to assume the exchange rate risk at the beginning and at the end of the option contract, then such an option is simply called a **flexo** and it is no different from an ordinary call traded among the Turkish investors. However, complications arise when some prices are quoted in TL and others in \$.

Compo (domestic currency strike): Here, the American investor wants the exercise price fixed in \$. Then, the payoff at maturity takes the form

$$\max(0, S_T - X^{\$}) = \max(0, S_T - f_T X^{\$})$$

where f_T is the exchange rate of one \$ for TL (e.g., 1,500,000 TL/\$). Since it is a random variable, the product $P_T \equiv f_T X^{\$}$ is also a random variable price from the viewpoint of the Turkish writer of the option. The framework for exchange options explained above can then be applied on this problem giving an explicit call value in TL as

$$C_{compo}^{TL} = S_0 \mathrm{N}(d_1) - (X^{\$} f_0 e^{r_{\$} T}) \mathrm{N}(d_2)$$

with $d_1 = \dfrac{\ln(S_0 / X^{\$} f_0 e^{r_{\$}T}) + 0.5v^2 T}{v\sqrt{T}}$, $d_2 = d_1 - v\sqrt{T}$, $v^2 = \sigma_s^2 - 2\rho_{sf}\sigma_s\sigma_f + \sigma_f^2$

To find the dollar value of this compo, consider the payoff at maturity from the viewpoint of the American investor

$$\max(0, S_T - X^{\$}) = \max(0, S_T / f_T - X^{\$})$$

By the same line of reasoning, the value turns out to be

$$C_{compo}^{\$} = (S_0 / f_0)\,\mathrm{N}(d_1) - X^{\$} e^{-r_{\$}T}\,\mathrm{N}(d_2)$$

where $d_1 = \dfrac{\ln(S_0 / X^{\$} f_0 e^{-r_{\$}T}) + 0.5v^2 T}{v\sqrt{T}}$ and $d_2 = d_1 - v\sqrt{T}$. The volatility term here is $v^2 = \sigma_s^2 + 2\rho_{s(1/f)}\sigma_s\sigma_{(1/f)} + \sigma_{(1/f)}^2$.

Quanto (foreign currency strike with fixed exchange rate): Here, all of the parameters are quoted in TL but the final payoff is guaranteed by a fixed exchange rate between TL and \$, say k. The payoff in TL at expiry becomes

$$f_T \max(0, S_T - X)k = \max(0, f_T S_T - f_T X)k$$

Again, applying the exchange-option methodology, we can derive

$$C_{quanto}^{TL} = f_0 \left[S_0 e^{-(r_{\$} - r_{TL} - \rho_{sf}\sigma_s\sigma_f)T}\,\mathrm{N}(d_1) - X e^{-r_{\$}T}\,\mathrm{N}(d_2) \right] k$$

where $d_1 = \dfrac{\ln(S_0 e^{-(r_{\$} - r_{TL} - \rho_{sf}\sigma_s\sigma_f)T} / X e^{-r_{\$}T}) + 0.5\sigma_s^2 T}{\sigma_s\sqrt{T}}$ and $d_2 = d_1 - \sigma_s\sqrt{T}$.

The \$ price can be calculated by simply dividing the TL price by f_0.