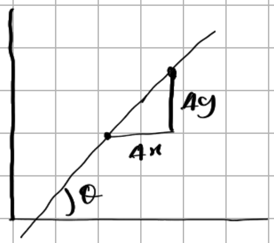
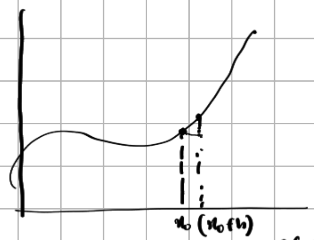
# CHAPTER 3

Numerical Differenciation




$$\tan\theta = f'(x) = \frac{\Delta y}{\Delta x} = \frac{\text{rise}}{\text{run}}$$




$$f'(x_0) = \lim_{h \to 0} \frac{f'(x_0+h) - f(x_0)}{h}$$

compare

$$f'(x_0) = \frac{f'(x_0+h) - f(x_0)}{h} \quad \cdots (1)$$

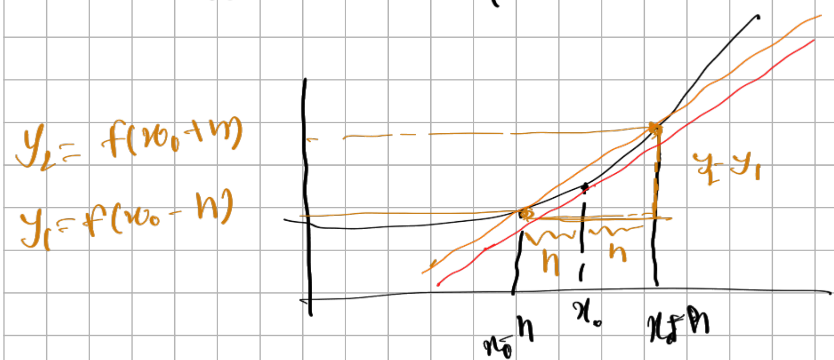
$h \downarrow$ [$f'(x)$ accuracy] $\uparrow$

In equation 1,

if $h>0$; forward difference

$h<0$; backward difference

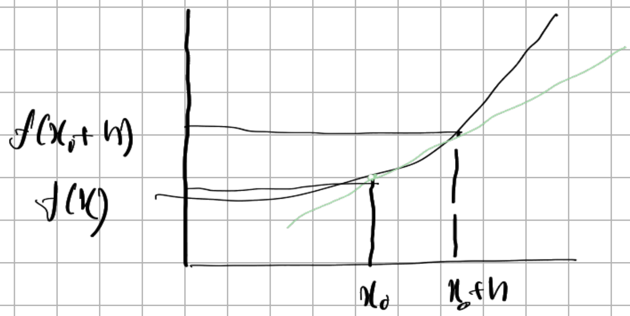
Central difference method




$$f'(x) = \frac{f(x_0+h) - f(x_0-h)}{2h}$$

Intusion, forward diff/backward diff < central diff.
in accuracy.

Forward Difference




$x_0 - f(x_0)$

$x_1 - f(x_1)$

$$P_1(x) = \frac{x - x_1}{x_0 - x_1} f(x_0) + \frac{x - x_2}{x_1 - x_2} f(x_1)$$

Cauchy's theore

$$f(x) = \frac{x - x_1}{x_0 - x_1} f(x_0) + \frac{x - x_2}{x_1 - x_2} f(x_1) + \text{Error}$$

$$= \frac{x - x_1}{x_0 - x_1} f(x_0) + \frac{x - x_2}{x_1 - x_2} f(x_1) + \frac{f''(\xi)}{2!}(x - x_0)(x - x_1)$$

$$\xi \in [x_0, x_0 + h]$$

$$f'(x) = \frac{1}{x_0 - x_1} f(x_0) + \frac{1}{x_1 - x_0} f(x_1) + \frac{f'''(\xi)}{2} \frac{d\xi}{dx}(x - x_0)(x - x_1)$$

$$+ \frac{f''(\xi)}{2}(2x - x_0 - x_1)$$

$$f'(x_0) = \frac{f(x_1) - f(x_0)}{x_1 - x_0} + \frac{f''(\xi)}{2}(x_0 - x_1)$$

$$f'(x_0) = \frac{f(x_0+h) - f(x_0)}{h} + \frac{f''(\xi)}{2}(-h)$$

Truncation Error

* error ∝ h
* error ∝ $f''(\xi)$

Example

$\ln(x)$ @ $x = 2$

$\frac{d}{dx}(\ln(x))$

$= \frac{1}{x}$

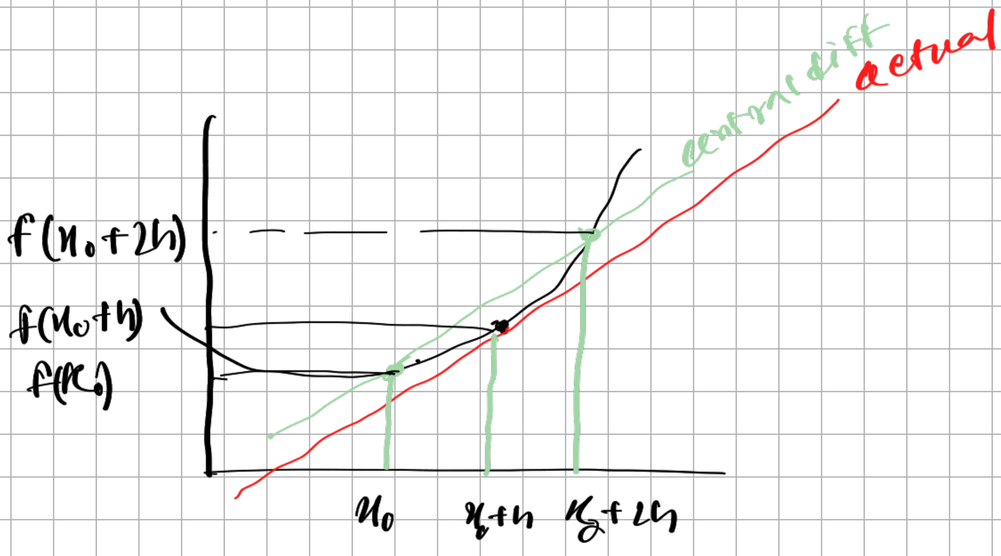
@ $x = 2$

$\frac{1}{2}$

$$\frac{\ln(2+h) - \ln(2)}{h}$$

| h | Forward diff | Truncation Err. |
|---|---|---|
| 1 | 0.405469 | 0.5 − 0.405469 = 0.0945 |
| 0.1 | 0.487902 | 0.0120984 |
| 0.01 | 0.498754 | 0.00124585 |
| 0.001 | 0.499825 | 0.000124958 |

# Central Difference




$x_0$

$x_1 = x_0+h$

$x_2 = x_0+2h$

$$f(x) = \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)} f(x_0) + \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)} f(x_1)$$

$$+ \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)} f(x_2) + \frac{f'''(\xi)}{3!}(x-x_0)(x-x_1)(x-x_2)$$

$$f'(x) = \frac{2x - x_1 - x_2}{(x_0 - x_1)(x_0 - x_2)} f(x_0) + \frac{2x - x_0 - x_2}{(x_1 - x_0)(x_1 - x_2)} f(x_1)$$

$$+ \frac{2x - x_0 - x_1}{(x_2 - x_0)(x_2 - x_1)} f(x_2) + \frac{f^{(3)}(\xi)}{3!} \Big[ (x - x_1)(x - x_2)$$

$$+ (x - x_0)(x - x_2) + (x - x_1)(x - x_2) \Big]$$

$$+ \frac{f^{4}(\xi)}{3!} \frac{d\xi}{dx} (x - x_0)(x - x_1)(x - x_2)$$

$$f'(x_1) = \frac{x_1 - x_2}{(x_0 - x_1)(x_0 - x_2)} f(x_0) + \frac{2x_1 - x_0 - x_2}{(x_1 - x_0)(x_1 - x_2)} f(x_1) + \frac{x_1 - x_0}{(x_2 - x_0)(x_2 - x_1)} f(x_2)$$

$$+ \frac{f^{(3)}(\xi)}{3!} (x_1 - x_0)(x_1 - x_2)$$

$$f'(x) = -\frac{1}{2h} f(x_0) + \frac{1}{2h} f(x_0 + 2h) - \frac{f^{(3)}(\xi)}{3!} h^2$$

$$= \frac{f(x_0 + 2h) - f(x_0)}{2h} - \underbrace{\frac{f^{(3)}(\xi)}{3!} h^2}_{\text{Truncation error}}$$

# Central diff. differenciation error $\propto h$

Example

| $x$ | 2.4 | 2.3 | 2.5 | 2.7 |
|---|---|---|---|---|
| $f(x)$ | 14.25 | 18.64 | 20.90 | 24.00 |

(a) Compute the $f'(2.3)$ using central diff. method

(b) Evaluate truncation error for $f(x) = 12 \ln x$ @ 2.4 using $h = 0.1$ in forward diff. method.

## Rounding Error

$$f'(x) \approx \frac{f(x+h) - f(x)}{h} \quad \text{forward/backward differentiation}$$

$\rightarrow$ error $\propto h$

$$f'(x) \approx \frac{f(x+h) - f(x-h)}{2h} \quad \text{Central differenciation}$$

$\rightarrow$ error $\propto h^2$

Apart from truncation error we need to consider Rounding Error as well as two close numbers $f(x+h)$ & $f(x-h)$ has to be subtracted

$$fl[f(x_1+h)] = \underbrace{f(x_1+h)}_{\text{actual}} + \delta_1 f(x_1+h) = (\delta_1+1) f(x_1+h)$$

invarient erro

sea

$$|\delta_1|, |\delta_2| \le \epsilon_m, 2h$$

$$fl[f(x_1-h)] = (\delta_2+1) f(x_1-h)$$

$$\text{Total Error} = \left| f'(x_1) - \frac{fl[f(x_1+h)] - fl[f(x_1-h)]}{2h} \right.$$

$$= \left| \frac{f(x_1+h) - f(x_1-h)}{2h} - \frac{f'''(\xi)}{3!} h^2 - \frac{(1+\delta_1) f(x_1+h) - (1-\delta_2) f(x_1-h)}{2h} \right|$$

$$= \left| - \frac{f'''(\xi)}{3!} h^2 - \frac{\delta_1 f(x_1+h) - \delta_2 f(x_1-h)}{2h} \right|$$

$$\le \frac{|f'''(\xi)|}{3!} h^2 + \frac{|\delta_1 f(x_1+h) - \delta_2 f(x_1-h)|}{2h}$$

now, $|\delta_1|, |\delta_2| \leq \epsilon_m$

$$\leq \underbrace{\frac{|f'''(\xi)|}{6} h^2}_{\text{Truncation Error} \propto h^2} + \underbrace{\epsilon_m \frac{|f(x_1+h)| + f(x_1-h)|}{2h}}_{\text{Rounding Error} \propto \frac{1}{h}}$$