CSE330
Numerical Methods
Polynomial Interpolation

Newton's Method

$$P_n(x) = a_0 + a_1(x-x_0) + a_2(x-x_0)(x-x_1) + \dots + a_n(x-x_0)(x-x_1)\dots(x-x_{n-1})$$

$a_0 = f[x_0]$

$a_1 = f[x_0, x_1]$

$a_2 = f[x_0, x_1, x_2]$

$\vdots$

$a_n = f[x_0, x_1, x_2 \dots x_n]$

$$\left.\begin{array}{ll} x_0 & f(x_0) \\ x_1 & f(x_1) \\ x_2 & f(x_2) \end{array}\right\} P_2(x)$$

So, Any expression of degree 3 could be written as,

$$P_3(x) = f[x_0] + f[x_0, x_1](x-x_0) + f[x_0, x_1, x_2](x-x_0)(x-x_1)$$
$$+ f[x_0, x_1, x_2, x_3](x-x_0)(x-x_1)(x-x_2)$$

So, let's say we have

x	$f(x)$
x_0	a
x_1	b
x_2	c
x_3	d

x_0, x_1, x_2: $P_2(x)$ would represent the func

x_3: New node

But for the added node we need $P_3(x)$, where,

$$P_3(x) = P_2(x) + f[x_0, x_1, x_2, x_3](x-x_0)(x-x_1)(x-x_2)$$

$x_0 \quad f[x_0] = a$

$x_1 \quad f[x_1] = b$

$x_2 \quad f[x_2] = c$

$x_3 \quad f[x_3] = d$

$$f[x_0, x_1] = \frac{b-a}{x_1 - x_0}$$

$$f[x_1, x_2] = \frac{c-a}{x_2 - x_1}$$

$$f[x_2, x_3] = \frac{d-c}{x_3 - x_2}$$

$$f[x_0, x_1, x_2] = \frac{f[x_1, x_2] - f[x_0, x_1]}{x_2 - x_1}$$

$$f[x_1, x_2, x_3] = \frac{f[x_2, x_3] - f(x_1, x_2)}{x_3 - x_2}$$

$$f[x_0, x_1, x_2] \rightarrow f[x_0, x_1, x_2, x_3] = \frac{f[x_0, x_1, x_2] - f[x_1, x_2, x_3]}{x_3 - x_0}$$

$$f[x_1, x_2, x_3] \rightarrow f[x_0, x_1, x_2, x_3]$$

Example

	x	$f(x)$
0	-1	5
1	0	1
2	1	3
3	2	11

$P_2(x) = f[x_0] + f[x_0, x_1](x - x_0) + f[x_0, x_1, x_2](x - x_0)(x - x_1)$

But for the added node we need $P_3(x)$

$x_0 = -1$ $\quad f[x_0] = 5$

$$f[x_0, x_1] = \frac{1-5}{0-(-1)} = -4$$

$x_1 = 0$ $\quad f[x_1] = 1$

$$f(x_0, x_1, x_2) = \frac{2+4}{1+1} = 3$$

$$\underset{x_0}{\uparrow} - \underset{x_1}{\uparrow}$$

$$f[x_1, x_2] = \frac{3-1}{1-0} = 2$$

$$f[x_1, x_2, x_3] = \frac{8-2}{2-0} = 3$$

$x_2 = 1$ $\quad f[x_2] = 3$

$$f[x_2, x_3] = \frac{11-3}{2-1} = 8$$

$x_3 = 2$ $\quad f[x_3] = 11$

$f[x_0, x_1, x_2] = 3$

$$f[x_0, x_1, x_2, x_3] = \frac{3-3}{2-(-1)}$$

$= 0$

$f[x_1, x_2, x_3] = 3$

Cauchy's Theorem

If we interpolate a function $f(x) \in V^{\infty}$ to a Polynomial, of degree n, $P_n(x) \in V^{n+1}$, then the error would be

$$|f(x) - P_n(x)| = \left| \frac{f^{n+1}(\xi)}{(n+1)!} (x-x_0)(x-x_1)\cdots(x-x_n) \right|$$

Example

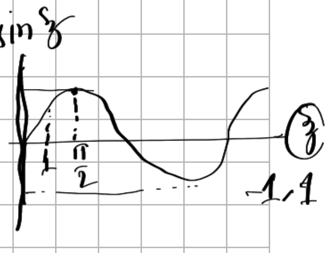
$f(x) = \cos x$ | nodes $= \underbrace{\left\{ \overset{x_0}{-\frac{\pi}{4}}, \overset{x_1}{0}, \overset{x_2}{\pi/4} \right\}}_{P_2(x)}$ | $x, \xi \in [-1, 1]$

Calculate the upper bound of error using cauchy's theorem,

$$|f(x) - P_2(x)| = \left| \frac{f^3(\xi)}{3!} (x-x_0)(x-x_1)(x-x_2) \right|$$

$\cos(x) \Rightarrow -\sin x$ 1
$-\sin x \Rightarrow -\cos x$ 2
$-\cos x \Rightarrow \sin x$ 3

$$= \left| \frac{\sin \xi}{3!} \right| \left| (x+\frac{\pi}{4})(x-0)(x-\frac{\pi}{4}) \right|$$



$$\boxed{\therefore |f(x) - P_2(x)| = \left| \frac{\sin 1}{3!} \right| \; |0.383|}$$ max possible error

$w(x) = (x + \pi/4)(x)(x - \pi/4)$

$= (x^2 - \frac{\pi^2}{16})(x)$

$\Rightarrow w(x) = x^3 - \frac{\pi^2}{16} x$

@ maxpoint $w'(x) = 3x^2 - \frac{\pi^2}{16} = 0$

$\Rightarrow x = \pm \frac{\pi}{4\sqrt{3}}$

x	$\lvert w(x) \rvert$
$-\pi/4\sqrt{3}$	0.186
$\pi/4\sqrt{3}$	0.186
-1	0.383
1	0.383

if range is provided

Hermite Interpolation

Hermite suggests that for given $(n+1)$ nodes there exists an unique polynomial of degree $(2n+1)$ P_{2n+1}, that interpolates both $f(x)$ and $f'(x)$.

Example $\quad f(x) = \sin(x) \qquad f'(x) = \cos(x)$

	x	$f(x)$	$f'(x)$
0	0	0	1
1	$\pi/2$	1	0

Previously we could only get $P_1(x)$ that would use $(x_i, f(x_i))$ But now it will use $f'(x)$ values to increase the degree of the polynomial ($P_3(x)$) and interpolate with more accuracy.

The polynomial in hermite basis

$$\boxed{\begin{aligned} P_3(x) = {} & h_0(x)\,\cancel{f(x_0)}^{0} + h_1(x)\,\cancel{f(x_1)}^{1} + \\ & \hat{h}_0(x)\,\cancel{f'(x_0)}^{1} + \hat{h}_1(x)\,\cancel{f'(x_1)}^{0} \end{aligned}}$$

$$P_3(x) = h_1(x) + \hat{h}_0(x)$$

$$\begin{cases} h_k(x) = \left(1 - 2(x - x_k)\,l_k'(x_k)\right)\left(l_k(x)\right)^2 \\ \hat{h}_k(x) = (x - x_k)\left(l_k(x)\right)^2 \end{cases}$$

$$l_1(x) = \frac{x - x_0}{x_1 - x_0} = \frac{2}{\pi}x$$

$$l_1'(x) = 2/\pi$$

$$l_1'(\pi/2) = 2/\pi$$

$$\begin{aligned} h_1(x) &= \left(1 - 2\left(x - \tfrac{\pi}{2}\right)\tfrac{2}{\pi}\right)\left(\tfrac{2}{\pi}x\right)^2 \\ &= \left(1 - \tfrac{4x}{\pi} + 2\right)\left(\tfrac{4}{\pi^2}x^2\right) \\ &= \left(3 - \tfrac{4}{\pi}x\right)\left(\tfrac{4}{\pi^2}x^2\right) \end{aligned}$$

$$l_0(x) = \frac{x - x_1}{x_0 - x_1} = \frac{x - \pi/2}{0 - \pi/2} = 1 - \frac{2}{\pi}x$$

$$\hat{h}_0(x) = (x - 0)\left(1 - \frac{2}{\pi}x\right)^2$$

Runge Function

$$f(x) = \frac{3}{4+9x^2} \quad ; \text{ interval} \in [-4, 4] \; ; \; n = 3$$

ⓐ Calculate the values of the equally angled points/ θ

ⓑ Calculate the value of chebyshev nodes.

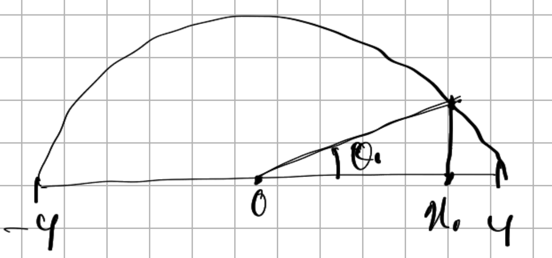
ⓒ Find the lagrange basis, $\ell_2(x)$

ⓐ $$\theta_i = \frac{(2i+1)\pi}{2(n+1)} \; ; \; i = \underbrace{0, 1, 2, 3}_{n=3}$$

$$\theta_0 = \frac{(0+1)\pi}{2(3+1)} = \frac{\pi}{8} \qquad \theta_2 = \frac{5}{8}\pi$$

$$\theta_1 = \frac{(2\times1+1)\pi}{2(3+1)} = \frac{3}{8}\pi \qquad \theta_3 = \frac{7}{8}\pi$$

ⓑ



$$x_i = \underbrace{\max(\text{interval})}_{4} \; \underbrace{\cos(\theta_i)}_{\theta_{0,1,2,3}}$$

$x_0 = 4\cos(\theta_0)$ // $x_1 = 4\cos(\theta_1)$ // $x_2 = 4\cos(\theta_2)$ // $x_3 = 4\cos(\theta_3)$

$\hookrightarrow f(x_0)$ $\quad \hookrightarrow f(x_1)$ $\quad \hookrightarrow f(x_2)$ $\quad \hookrightarrow f(x_3)$

ⓒ $$\ell_2(x) = \frac{x - x_0}{x_2 - x_0} \times \frac{x - x_1}{x_2 - x_1} \times \frac{x - x_3}{x_2 - x_3} \quad \Bigg| \begin{matrix} 0 \\ 1 \\ \cancel{2} \\ 3 \end{matrix}$$