Hermite Interpolation
Cauchy's Interpolation

Chapter 02: Polynomial Interpolation
CSE330 Numerical Methods

Hermite Interpolation / Derivative Conditions

degree $n \uparrow$ accuracy $\uparrow$ error $\downarrow$

$\downarrow$

node $\uparrow$

takes $f(x)$ & $f'(x)$

$f(x) \xrightarrow{(n+1)\text{ node}} P_n$

$f(x) \xrightarrow{(n+1)\text{ nodes}} P_{2n+1}$

first 3 technique $(n+1)$ nodes $\equiv (n+1)$ charac. $\equiv P_n$

Hermite tech. $(n+1)$ nodes $\equiv 2(n+1)$ " $\equiv P_{2n+1}$

$\downarrow$

$2n+2$

x	$f(x)$	$f'(x)$
x_0	$f(x_0)$	$f'(x_0)$
x_1	$f(x_1)$	$f'(x_1)$
x_2	$f(x_2)$	$f'(x_2)$
$\vdots$	$\vdots$	$\vdots$
x_n	$f(x_n)$	$f'(x_n)$

$f(x)$ column $\rightarrow (n+1)$ char. $\rightarrow$ ~~$P_n(x)$~~

$f'(x)$ column $\rightarrow (n+1)$ char

$2n+2$ char

$\downarrow$

$2n+1$ degree

P_{2n+1} $\uparrow$

Hermite Polynomial Formulas

$$P_{2n+1}(x) = \sum_{k=0}^{n} h_k(x) f(x_k) + \hat{h}_k(x) f'(x_k)$$

known

$$P_{2\times1+1}(x) = h_0(x) f(x_0) + \hat{h}_0(x) f'(x_0) + h_1(x) f(x_1) + \hat{h}_1(x) f'(x_1)$$

$\uparrow$ n

$$\boxed{\begin{aligned} h_k(x) &= 1 - 2(x-x_k)\, l_k^2(x)\, l_k'(x_k) \\ \hat{h}_k(x) &= (x-x_k)\, l_k^2(x) \end{aligned}}$$

Example

Consider a function $f(x) = \cos x$ and nodes $= \{0, \pi/2\}$. Interpolate it with a hermite polynomial.

Ans:

prev. deg, $n = 1$

hermite deg, $2n+1$

x	$f(x)$	$f'(x)$
$x_0 = 0$	1	0
$x_1 = \pi/2$	0	-1

$P_{1\times2+1} = P_3$

$$P_3(x) = h_0(x) f(x_0) + \hat{h}_0(x) f'(x_0) + h_1(x) f(x_1) + \hat{h}_1(x) f'(x_1)$$

$$= h_0(x) - \hat{h}_1(x)$$

$$l_0(x) = \frac{x - \pi/2}{0 - \pi/2}$$

$$= -\frac{2}{\pi} x + \frac{\cancel{\pi}}{\cancel{2}} \times \frac{\cancel{2}}{\cancel{\pi}}$$

$$l_0(x) = -\frac{2x}{\pi} + 1$$

$$l_0'(x) = -\frac{2}{\pi}$$

$$l_0'(0) = -\frac{2}{\pi}$$

$\underline{\underline{k=0}}$

$$h_0(x) = 1 - 2(x)\, \ell_0^2(x)\, \ell_0'(0) = 1 + 2x\left(1 - \frac{2x}{\pi}\right)^2 \times \frac{2}{\pi}$$

$\underline{\underline{k=1}}$

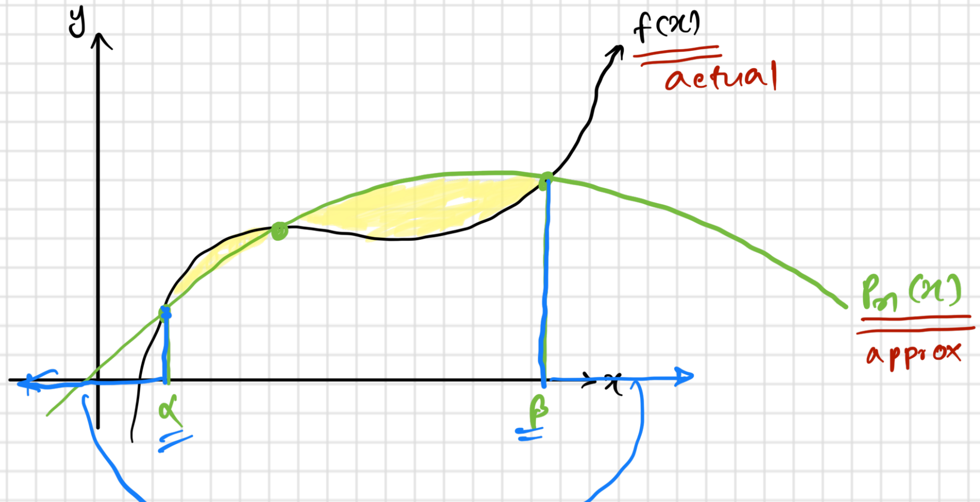
$$\hat{h}_1(x) = \left(x - \frac{\pi}{2}\right) \ell_1^2(x) = \left(x - \frac{\pi}{2}\right)\left(\frac{2x}{\pi}\right)^2$$

$$\ell_1(x) = \frac{x - x_0}{x_1 - x_0} = \frac{x - 0}{\frac{\pi}{2} - 0} = \frac{2x}{\pi}$$

$$\boxed{\begin{aligned} h_k(x) &= 1 - 2(x - x_k)\, \ell_k^2(x)\, \ell_k'(x_k) \\ \hat{h}_k(x) &= (x - x_k)\, \ell_k^2(x) \end{aligned}}$$

$$P_3(x) = \left[1 + 2x\left(1 - \frac{2x}{\pi}\right)^2 \frac{2}{\pi}\right] - \left[\left(x - \frac{\pi}{2}\right)\left(\frac{2x}{\pi}\right)^2\right]$$

Cauchy's Theorem // Interpolation Error



node $\{x_0, x_1, x_2, \ldots, x_n\}$

error ≅ ∞

$\xi_x \in [\alpha, \beta]$

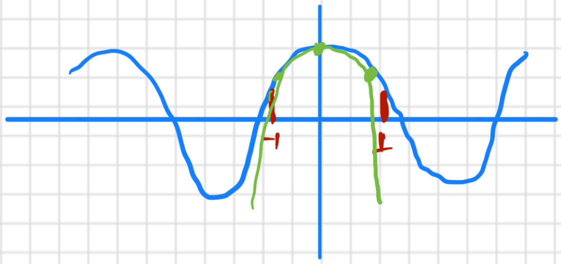
$x \in [\alpha, \beta]$

$$|f(x) - P_n(x)| \leq \left| \frac{f^{(n+1)}(\xi_x)}{(n+1)!} (x - x_0)(x - x_1)(x - x_2)\ldots(x - x_n) \right|_{max}$$

Example

Consider a function $f(x) = \cos x$ (rad) and nodes $\in \{-\frac{\pi}{4}, 0, \frac{\pi}{4}\}$.

Find the maximum possible error/upper bound of error in the interval $x \in [-1, 1]$



$$|f(x) - P_2(x)| \le \left| \frac{f'''(\xi)}{3!} (x+\frac{\pi}{4})(x)(x-\frac{\pi}{4}) \right|_{max}$$

$$\le \frac{1}{3!} \, |f'''(\xi)|_{max} \, |(x^3 - \frac{\pi^2}{16}x)|_{max}$$

$$\le \frac{1}{3!} (0.841) \times 0.383$$

$$\le 0.05368$$

$f(\xi) = \cos \xi$

$f'(\xi) = -\sin \xi$

$f''(\xi) = -\cos \xi$

$f'''(\xi) = \sin \xi$

$$\underbrace{\xi \in [-1, 1]}$$

$|\sin(-1)| = 0.841$

$|\sin(1)| = 0.841$

$|f'''(\xi)|_{max} = 0.841$

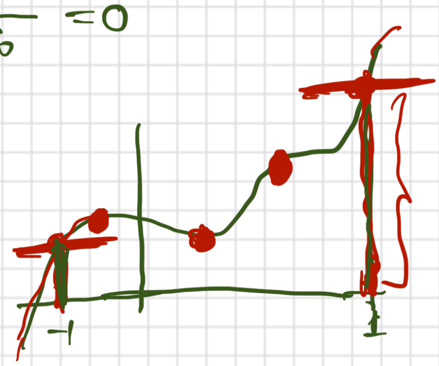
$$w(x) = x^3 - \frac{\pi^2}{16}x$$

$$\Rightarrow w'(x) = 3x^2 - \frac{\pi^2}{16} = 0$$

$$\Rightarrow 3x^2 = \frac{\pi^2}{16}$$

$$\Rightarrow x^2 = \frac{\pi^2}{3 \times 16}$$

$$\Rightarrow x = \pm \frac{\pi}{4\sqrt{3}} \quad \text{[critical point]}$$



x	$\lvert w(x)\rvert_{max}$	
$\pi/4\sqrt{3}$	0.186	critical
$-\pi/4\sqrt{3}$	0.186	

$$\left.\begin{array}{ll} 1 & 0'383 \\ -1 & 0'383 \end{array}\right\} \text{interval}$$

$$|\omega(x)|_{max} = 0'383$$