Richardson Extrapolation

in central difference differentiation $f'(x_1) = \dfrac{f(x_1+h) - f(x_1-h)}{2h} = D_h$

error↓ with h↓ but rounding error↑

$$f(x_1+h) = f(x_1) + f^{(1)}(x_1)h + \frac{f^{(2)}(x_1)}{2}h^2 + \frac{f^{(3)}(x_1)}{3!}h^3 + \frac{f^{(4)}(x_1)}{4!}h^4 + O(h^5)$$

$$f(x_1-h) = f(x_1) - f^{(1)}(x_1)h + \frac{f^{(2)}(x_1)}{2}h^2 - \frac{f^{(3)}(x_1)}{3!}h^3 + \frac{f^{(4)}(x_1)}{4!}h^4 + O(h^5)$$

$$D_h = \frac{1}{2h}\left[2f^{(1)}(x_1)h + 2\frac{f^{(3)}(x_1)}{3!}h^3 + 2\frac{f^{(5)}(x_1)}{5!}h^5 + O(h^7)\right]$$

① $$D_h = f^{(1)}(x_1) + \frac{f^{(3)}(x_1)}{3!}h^2 + \frac{f^{(5)}(x_1)}{5!}h^4 + O(h^6)$$

② $$D_{h/2} = f^{(1)}(x_1) + \frac{f^{(3)}(x_1)}{3!}\left(\frac{h}{2}\right)^2 + \frac{f^{(5)}(x_1)}{5!}\left(\frac{h}{2}\right)^4 + O(h^6)$$

$\left(\frac{h}{2}\right)^6 = \frac{h^6}{2^6}$

$\propto O(h^6)$

2^2 ② − ① ⇒

$$2^2 D_{h/2} - D_h = (2^2-1)f^{(1)}(x_1) + \left(\frac{1}{2^2}-1\right)\frac{f^{(5)}(x_1)}{5!}h^4 + O(h^6)$$

$$\Rightarrow \underbrace{\frac{2^2 D_{h/2} - D_h}{(2^2-1)}}_{D_h^{(1)}} = f^{(1)}(x_1) + \frac{\left(\frac{1}{2^2}-1\right)}{(2^2-1)}\frac{f^{(5)}(x_1)}{5!}h^4 + O(h^6)$$

Error $\propto h^4$

$D_n^{(1)}$ & $D_{n/2}^{(1)}$ Taken again so that the next $D_n^{(2)}$ becomes

Error $\propto h^6$

Question 1

$f'(x) = -18e^{-3x}$ // $f'(0.5) = -4.0163$

Question # 1 : A function is given by $f(x) = 6e^{-3x}$. Now Answer the following:

1. [1 Mark] Calculate $f'(x)$ at $x = 0.5$ with $h = 0.32$ using the central difference formula.
2. [1 Mark] Calculate $f'(x)$ at $x = 0.5$ with $h = 0.16$ using the central difference formula.
3. [3 Marks] Now compute $D_{0.32}^{(1)}$ at $x = 0.5$ using Richardson extrapolation method.
4. [2 Marks] If the exact value of the derivative, $f'(0.5)$ is ~~[illegible]~~ -4.0163, find the percentage error with extrapolated value found in the previous part.

CDD,

$$f'(x) \approx \frac{f(x_0+h) - f(x_0-h)}{2h}$$

(1)

$$f'(0.5)_{h=0.32} \approx \frac{f(0.5+0.32) - f(0.5-0.32)}{2\times 0.32}$$

$$= \frac{0.51260 - 3.4964}{0.64} = -4.6621$$

(2)

$$f'(0.5)_{h=0.16} \approx \frac{f(0.5+0.16) - f(0.5-0.16)}{2\times 0.16}$$

$$= \frac{0.82841 - 2.1635}{0.32} = -4.1721$$

(3)

$$D_{0.32}^{(1)} = \frac{2^2 D_{0.16} - D_{0.32}}{2^2-1} = \frac{4(-4.1721) - (-4.6621)}{4-1}$$

$$= -4.0087$$

Truncation Error

h

$0.2 \quad 0.2$

x	1.0	1.2	1.4
$f(x)$	1.2075	1.4307	1.9506

(a) Compute $f'(1.2)$ using central difference method.

CD;

$$f'(1.2) \approx \frac{f(1.2+0.2) - f(1.2-0.2)}{2 \times 0.2}$$

$$= \frac{1.9506 - 1.2075}{2 \times 0.2}$$

$$= 1.8527$$