CHAPTER II Polynomial Interpolation

Any polynomial can be represented in the following form,

$$P_n(x) = a_0x^0 + a_1x^1 + a_2x^2 + a_3x^3 + \ldots + a_nx^n$$

Where, variable $= x$
degree $= n$
constant coefficients $= \{a_0, a_1, a_2, \ldots, a_n\}$
basis set $= \{x^0, x^1, x^2, \ldots, x^n\}$

Some other attributes are,
dimensional space $= (n+1) =$ # of elements in the coeff set or the basis set

Thus, polynomials can be considered as vectors, where a polynomial of degree n, $P_n(x)$ belongs to a vector space of $(n+1)$ dimension.

$$P_n(x) \in V^{n+1}$$

Basis Vector
a set of vectors that spans the whole vector space

Weierstrass Approximation Theorem (has to be real & continuous)

For a continuous function $f(x)$ on a bounded interval can be uniformly approximated as closely as desired by a polynomial function, with high enough degree.

$$f(x) \Rightarrow \text{approximated} \Rightarrow P_n(x)$$

$f(x) \in V^{\infty}$ $\qquad$ $P_n(x) \in V^{n+1}$

$n\uparrow$ error$\downarrow$
$n\downarrow$ error$\uparrow$

$\max|f(x) - P_n(x)| < \epsilon$; where $\epsilon > 0$

$\max\limits_{a \le x \le b} |f(x) - P_n(x)|$

Taylor Series

As we know, any cont. func. can be converted to a polynomial of infinite dimensions.

So, $f(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + \dots$

We can also write this as,

Eq 1 $f(x) = a_0 + a_1(x - x_0) + a_2(x - x_0)^2 + a_3(x - x_0)^3 + a_4(x - x_0)^4 + \dots$
where, x_0 is a constant.

Eq 2 $f'(x) = 0 + a_1 x + 2a_2(x - x_0) + 3a_3(x - x_0)^2 + 4.a_4(x - x_0)^3 + \dots$

Eq 3 $f''(x) = 0 + 0 + 2.1.a_2 x + 3.2.a_3(x - x_0) + 4.3.a_4(x - x_0)^2 + \dots$

Eq 4 $f'''(x) = 0 + 0 + 0 + 3.2.1.a_3 x + 4.3.2\, a_4(x - x_0) + \dots$

If we consider x to be x_0 $(x = x_0)$ then,

from Eq 1 $f(x_0) = a_0 \Rightarrow a_0 = f(x_0)$

Eq 2 $f'(x_0) = a_1 x \Rightarrow a_1 = \dfrac{f'(x_0)}{1} = f'(x_0)$

Eq 3 $f''(x_0) = 2.1.a_2 x \Rightarrow a_2 = \dfrac{f''(x_0)}{2!}$

Eq 4 $f'''(x_0) = 3.2.1.a_3 x \Rightarrow a_3 = \dfrac{f'''(x_0)}{3!}$

Replacing $\{a_0, a_1, a_2, a_3, \dots\}$ in equation 1 we get,

$$f(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2 + \frac{f'''(x_0)}{3!}(x - x_0)^3 + \dots$$

General Structure of Taylor series.

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{n}(x - x_0)}{n!} (x - x_0)^n$$

Maclaurin Series is just a special case of Taylor Series where $x_0 = 0$

Thus, we get,

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots$$

Taylor's Theorem (Error Analysis)

Let f be $n+1$ differentiable on (a,b) and $f^{(n)}$ to be continuous on $[a,b]$.

If $x, x_0 \in [a,b]$ then there exists $\xi \in (a,b)$ such that,

$$f(x) = \underbrace{\sum_{k=0}^{n} \frac{f^{k}(x_0)}{k!} (x - x_0)^k}_{\text{Taylor polynomial of degree } n} + \underbrace{\frac{f^{(n+1)}(\xi)}{(n+1)!} (x - x_0)^{n+1}}_{\text{Lagrange form or remainder}}$$

Basically error

truncated a polynomial of degree ∞ to n due to the limitation of computation.

Taylor Series

$$\boxed{f(x) = \sum_{k=0}^{\infty} \frac{f^{k}(x_0)}{k!} (x - x_0)^k \quad \text{// Infinite series}}$$

$$f(x) = \sum_{k=0}^{n} \frac{f^{k}(x_0)}{k!} (x - x_0)^k + \frac{f^{n+1}(\xi)}{(n+1)!} (x - x_0)^{n+1}$$

Example

x^3 x^5 x^7 x^9

$n=6$

$$f(x)=\sin(x) = \underbrace{x - \frac{x^3}{3!} + \frac{x^5}{5!}}_{P_6(x)} \underbrace{- \frac{x^7}{7!} + \frac{x^9}{9!} + \dots}_{\frac{f^7(\xi)}{7!}(x-x_0)^7}$$

Let's consider if we truncate till $n=6$

$$\text{So, } f(x) = P_6(x) + \frac{f^7(\xi)}{7!}(x-x_0)^7$$

$f^7(\sin x) = -\sin x$

$$\Rightarrow \left| f(x) - P_6(x) \right| = \left| \frac{f^7(\xi)}{7!}(x-x_0)^7 \right|$$

$$\Rightarrow \left| \underset{0.1}{f(x)} - \underset{0.1}{P_6(x)} \right| = \frac{(0.1-0)^7}{7!} \left| -\sin(\xi) \right|$$

Considering $x_0 = 0$

$x = 0.1$

the value is bounded in $[0,1]$

taking the maximum possible error we get,

$$\underbrace{\left| f(0.1) - P_6(0.1) \right|}_{\text{Error}} \leq 1.98412 \times 10^{-11}$$

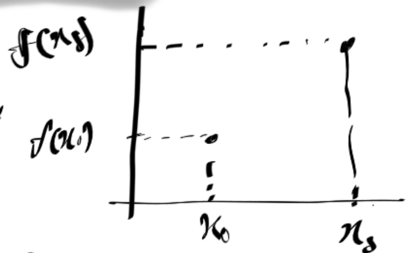
the value found from the polynomial $P_6(0.1)$ will be accurate upto atleast 11 digit after the decimal point.

See how the curve changes with the increased value of k

https://www.desmos.com/calculator/7zxy20wqbc

Polynomial Interpolation & Vandermonde Matrix

So far we predicted the value of any random point x point on a $f(x)$ curve

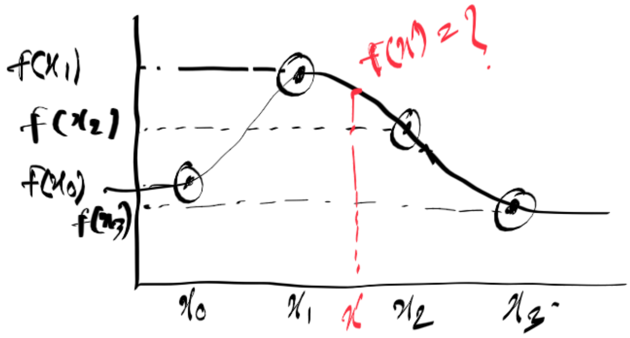


Known values: $f(x_0)$, $f'(x_0)$, $f''(x_0)$, $f'''(x_0)$ ....

But now we have multiple points $(x_0, f(x_0))$, $(x_1, f(x_1))$, $(x_2, f(x_2))$, ... then we would like to find any polynomial that will go through all of them.

Polynomial Interpolation – Populating some polynomial of n degree where $(n+1)$ nodes are known, and the polynomial will go through all the points

Node Set, $N = \underbrace{\{(x_0, f(x_0)), (x_1, f(x_1)), \dots, (x_n, f(x_n))\}}_{\text{item, } n+1}$



$$
\begin{aligned}
x_0 &\longrightarrow f(x_0) \\
x_1 &\longrightarrow f(x_1) \\
x_2 &\longrightarrow f(x_2) \\
&\vdots \\
x_n &\longrightarrow f(x_n)
\end{aligned}
$$

known values

the polynomial $P_n(x)$ will have to go through all these points

$N \longrightarrow$ will generate $\longrightarrow P_n(x)$

So, $\boxed{P_n(x') = f(x')}$

Let's consider, if there were 2 nodes, so the polynomial would be of degree $n=1$

Nodes

x_0	$f(x_0)$
x_1	$f(x_1)$

So, the polynomial will be in the form,

$P_1(x) = a_0 + a_1 x$

these are unknown.

$$
\begin{cases}
P_1(x_0) = a_0 + a_1 x_0 = f(x_0) & \dots eq(1) \\
P_1(x_1) = a_0 + a_1 x_1 = f(x_1) & \dots eq(2)
\end{cases}
$$

$$
\begin{bmatrix} 1 & x_0 \\ 1 & x_1 \end{bmatrix}
\begin{bmatrix} a_0 \\ a_1 \end{bmatrix}
=
\begin{bmatrix} f(x_0) \\ f(x_1) \end{bmatrix}
$$

$$\Rightarrow \begin{bmatrix} a_0 \\ a_1 \end{bmatrix} = \begin{bmatrix} 1 & x_0 \\ 1 & x_1 \end{bmatrix}^{-1} \begin{bmatrix} f(x_0) \\ f(x_1) \end{bmatrix}$$

Now, if we consider $(n+1)$ number of nodes, we can come up with,

$$\begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \\ \vdots \\ a_n \end{bmatrix} = \begin{bmatrix} 1 & x_0 & x_0^2 & x_0^3 & \dots & x_0^n \\ 1 & x_1 & x_1^2 & x_1^3 & \dots & x_1^n \\ 1 & x_2 & x_2^2 & x_2^3 & \dots & x_2^n \\ 1 & x_3 & x_3^2 & x_3^3 & \dots & x_3^n \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & x_n & x_n^2 & x_n^3 & \dots & x_n^n \end{bmatrix}^{-1} \begin{bmatrix} f(x_0) \\ f(x_1) \\ f(x_2) \\ f(x_3) \\ \vdots \\ f(x_n) \end{bmatrix}$$

(A) Vandermonde Matrix (V) (F)

So, $A = V^{-1} F$ Issue: inverse matrix requires $O(n^3)$

Uniqueness aka Existance Theorem

given $(n+1)$ nodes there exists a polynomial $P_n \in V^{n+1}$ that interpolates the function $f(x)$

Let's consider any function could be interpolated by both $P_n(x)$ and $Q_n(x)$

$$r_n(x) := P_n(x) - q_n(x) \;\|\; \text{where, } r_n \text{ has degree} \leq n$$

So, for, $r_n(x_i) = P_n(x_i) - q_n(x_i) = 0$ cmt for all points $i = 0, \dots, n$

$r_n(x) = 0 \Rightarrow$ so, $r_n(x)$ should have $(n+1)$ roots

but, a n degree non zero polynomial should have n roots, but the $r_n(x)$ should have n roots.

So, $\underline{r_n(x) = 0}$