# Exercise 1:

a) We have

$$\|x_j - x^*\|_2^2 = \left\|A_u^\dagger(b_u - A_u x_{j-1}) + x_{j-1} - x^*\right\|_2^2 = \left\|A_u^\dagger b_u - A_u^\dagger A_u x_{j-1} + x_{j-1} - x^*\right\|_2^2$$

And

$$\left\|(I - A_u^\dagger A_u)(x_{j-1} - x^*)\right\|_2^2 = \left\|A_u^\dagger A_u x^* - A_u^\dagger A_u x_{j-1} + x_{j-1} - x^*\right\|_2^2$$

Since $x^*$ is the optimal solution and since we assume the least square problem is consistent we have

$$Ax^* = b \Rightarrow A_u x^* = b_u$$

Which proves

$$\|x_j - x^*\|_2^2 = \left\|(I - A_u^\dagger A_u)(x_{j-1} - x^*)\right\|_2^2$$

□

b) We have that $A_u^\dagger A_u$ is an orthogonal projector (since it's idempotent and symmetric). Then any vector $y \in \mathbb{R}^n$ can be rewritten as $y = A_u^\dagger A_u y + (I - A_u^\dagger A_u)y$, so $(I - A_u^\dagger A_u)$ is the orthogonal complement projection matrix to $A_u^\dagger A_u$.
This implies

$$\|y\|_2^2 = \left\|A_u^\dagger A_u y\right\|_2^2 + \left\|(I - A_u^\dagger A_u)y\right\|_2^2$$

And hence

$$\left\|(I - A_u^\dagger A_u)y\right\|_2^2 = \|y\|_2^2 - \left\|A_u^\dagger A_u y\right\|_2^2$$

Then we have:

$$\frac{\sigma_{min}^2(A)}{\beta m}\|y\|_2^2 \leqslant \frac{\sum_{j=1}^m \sigma_{min}^2(A_{u_j})}{\beta m}\|y\|_2^2$$

Using the boundedness of singular values:

$$\|A_u\|_2^2 = \sigma_{max}(A_u)^2 = \lambda_{max}(A_u A_u^\top) \leqslant \beta$$

We get

$$\frac{\sum_{j=1}^m \sigma_{min}^2(A_{u_j})}{\beta m}\|y\|_2^2 \leqslant \sum_{j=1}^m \frac{\sigma_{min}^2(A_{u_j})}{\sigma_{max}^2(A_{u_j})m}\|y\|_2^2 = \sum_{j=1}^m \frac{1}{\kappa^2(A_{u_j})m}\|y\|_2^2$$

Where $\kappa(A)$ denotes the consistency of $A$.
Using the following consistency bound:

$$\left\|A^\dagger A y\right\|_2 \geqslant \frac{\|y\|_2}{\kappa(A)} = \frac{\|y\|_2}{\|A\|_2\|A^\dagger\|_2},$$

we finally get:

$$\sum_{j=1}^{m} \frac{1}{\kappa^2(A_{u_j})m}\|y\|_2^2 \leqslant \sum_{j=1}^{m} \frac{1}{m}\left\|A_{u_j}^\dagger A_{u_j} y\right\|_2^2 = \mathbb{E}\left\|A_u^\dagger A_u y\right\|_2^2$$

Since each $u_j$ is chosen uniformly at random. Putting everything together we have

$$\mathbb{E}\left\|(I - A_u^\dagger A_u)y\right\|_2^2 = \|y\|_2^2 - \mathbb{E}\left\|A_u^\dagger A_u y\right\|_2^2 \leqslant (1 - \frac{\sigma_{min}^2(A)}{\beta m})\|y\|_2^2$$

Which is the desired bound! □

Using part a) and this result (and supposing that the choice of u is independent from one iteration to another) we finally have

$$\mathbb{E}\|x_j - x^*\|_2^2 = \mathbb{E}\left\|(I - A_u^\dagger A_u)(x_{j-1} - x^*)\right\|_2^2 \leqslant (1 - \frac{\sigma_{min}^2(A)}{\beta m})\mathbb{E}\|x_{j-1} - x^*\|_2^2$$

$$\leqslant (1 - \frac{\sigma_{min}^2(A)}{\beta m})^j \|x_0 - x^*\|_2^2$$

Which completes the proof. □

# Exercise 2:

a) Let us prove that $\psi(L)$ is an orthogonal projection.
We first prove that $\psi(L)$ is idempotent:

$$\psi(L)^2 = \left((L^\dagger)^{\frac{1}{2}}L(L^\dagger)^{\frac{1}{2}}\right)\left((L^\dagger)^{\frac{1}{2}}L(L^\dagger)^{\frac{1}{2}}\right) = (L^\dagger)^{\frac{1}{2}}LL^\dagger L(L^\dagger)^{\frac{1}{2}} = (L^\dagger)^{\frac{1}{2}}L(L^\dagger)^{\frac{1}{2}} = \psi(L)$$

Let us now prove symmetry:

$$\psi(L)^\top = ((L^\dagger)^{\frac{1}{2}})^\top L^\top ((L^\dagger)^{\frac{1}{2}})^\top$$

Since G is undirected, $L$ is symmetric.
We then have from the properties of the pseudo-inverse:

$$\begin{cases} (LL^\dagger)^\top = LL^\dagger \Rightarrow (L^\dagger)^\top L = LL^\dagger \\ (L^\dagger L)^\top = L^\dagger L \Rightarrow L(L^\dagger)^\top = L^\dagger L \end{cases}$$

Now recall that the pseudo-inverse is uniquely defined, let us check that $(L^\dagger)^\top$ satisfies all for defining properties of the pseudo-inverse of $L$:

$$\begin{cases} L(L^\dagger)^\top L = (LL^\dagger L)^\top = (L)^\top = L \\ (L^\dagger)^\top L(L^\dagger)^\top = (L^\dagger LL^\dagger)^\top = (L^\dagger)^\top \\ ((L^\dagger)^\top L)^\top = LL^\dagger = (L^\dagger)^\top L \\ (L(L^\dagger)^\top)^\top = L^\dagger L = L(L^\dagger)^\top \end{cases}$$

By uniqueness of the pseudo-inverse, this implies $(L^\dagger)^\top = (L^\dagger)$.
Then since $L$ is a graph Laplacian, it is positive semi-definite (PSD), which implies that $L^\dagger$ is also PSD, hence $(L^\dagger)^{\frac{1}{2}}$ is also symmetric (and PSD).
Thus

$$\psi(L)^\top = \psi(L)$$

Which proves that $\psi(L)$ is an orthogonal projector. $\square$
For the second part of the question, we start with the definition of S being an $\epsilon$-spectral approximation of L:

$$(1-\epsilon)L \leqslant S \leqslant (1+\epsilon)L \quad \Leftrightarrow \quad -\epsilon L \leqslant S - L \leqslant \epsilon L$$

Now let us prove that $\psi$ is a monotone operator (i.e. it preserves inequalities). Let $B \geqslant A$, ($B - A$ is PSD), we have $\psi(B) - \psi(A) = \psi(B - A)$. Since $(L^\dagger)^{\frac{1}{2}}$ is PSD, $\psi(B - A)$ must be PSD as well which proves $\psi$ is monotone.
We then have,

$$-\epsilon L \leqslant S - L \leqslant \epsilon L \quad \Leftrightarrow \quad -\epsilon\psi(L) \leqslant \psi(S-L) \leqslant \epsilon\psi(L)$$

$$\Leftrightarrow \|\psi(S-L)\|_2 \leqslant \epsilon\|\psi(L)\|_2 = \epsilon,$$

since $\psi(L)$ is an orthogonal projector, $\|\psi(L)\| = 1$.

b) In this section we shorthand $E := E_{ii} + E_{jj} - E_{ij} - E_{ji}$.
We first notice that $E$ is a rank-one matrix.
Indeed, it has only two non-zero columns, one equal to $e_i - e_j$ the other $e_j - e_i$, where $e_i$ denotes a zero-vector with a one in the $i$th coordinate. Multiplying one column by $-1$ gives the other column, hence they are linear multiples of each other, and the matrix is rank one.
Since $(L^\dagger)^{\frac{1}{2}}$ is non-zero, it has rank at least one, and thus $\psi(E)$ has rank one.
This means all of its eigenvalues but one are equal to zero.
The non-zero eigenvalue is equal to two, indeed:

$$E(e_i - e_j) = 2(e_i - e_j)$$

which shows $E$ is PSD.
Using similar arguments as in part a, this implies that $\psi(E)$ is also PSD.
Since $\psi(E)$ has only one non-zero eigenvalue,

$$\operatorname{tr}(\psi(E)) = \lambda_{max}(\psi(E)) = \|\psi(E)\|_2.$$

We then recall that the matrix representing the graph Laplacian $L$ for a connected graph has rank $n-1$.
Then recall from part a that $\psi(L)$ is an orthogonal projector. This implies that it has

eigenvalues either $0$ or $1$.
Combining these two facts, we have that

$$\mathrm{tr}(\psi(L)) = \sum_{i=1}^{n-1} 1 = n-1.$$

We then have

$$\sum_{(i,j)\in E} p_{ij} = \sum_{(i,j)\in E} \frac{w_{ij}}{n-1}\|\psi(E_{ii}+E_{jj}-E_{ij}-E_{ji})\| = \sum_{(i,j)\in E} \frac{w_{ij}\,\mathrm{tr}\left(\psi(E_{ii}+E_{jj}-E_{ij}-E_{ji})\right)}{n-1}$$

$$= \frac{\mathrm{tr}(\psi(\sum_{(i,j)\in E} w_{ij}(E_{ii}+E_{jj}-E_{ij}-E_{ji})))}{n-1} = \frac{\mathrm{tr}(\psi(L))}{n-1} = 1$$

□

c)

## Exercise 3:

a)

b)

c)

d)

e)