Turbulence modeling

Numerical Flow Simulation

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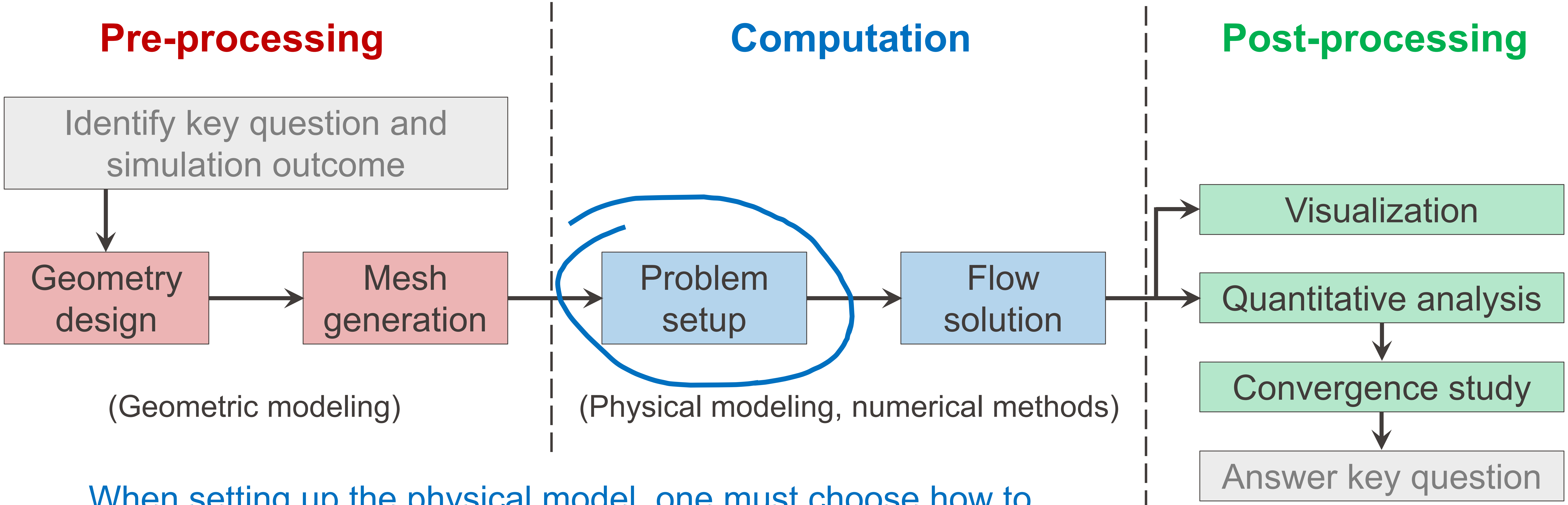
Numerical simulation workflow



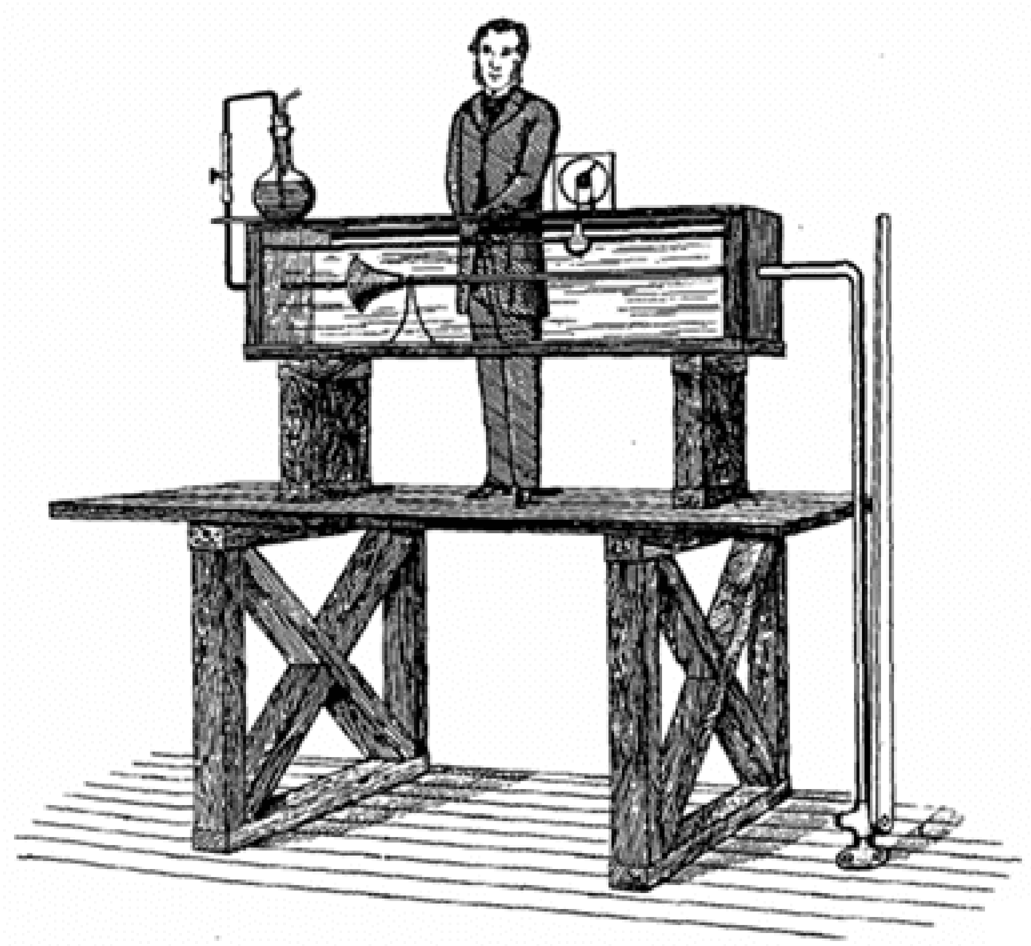
When setting up the physical model, one must choose how to compute turbulent flows.

Schematic outline of this lecture:

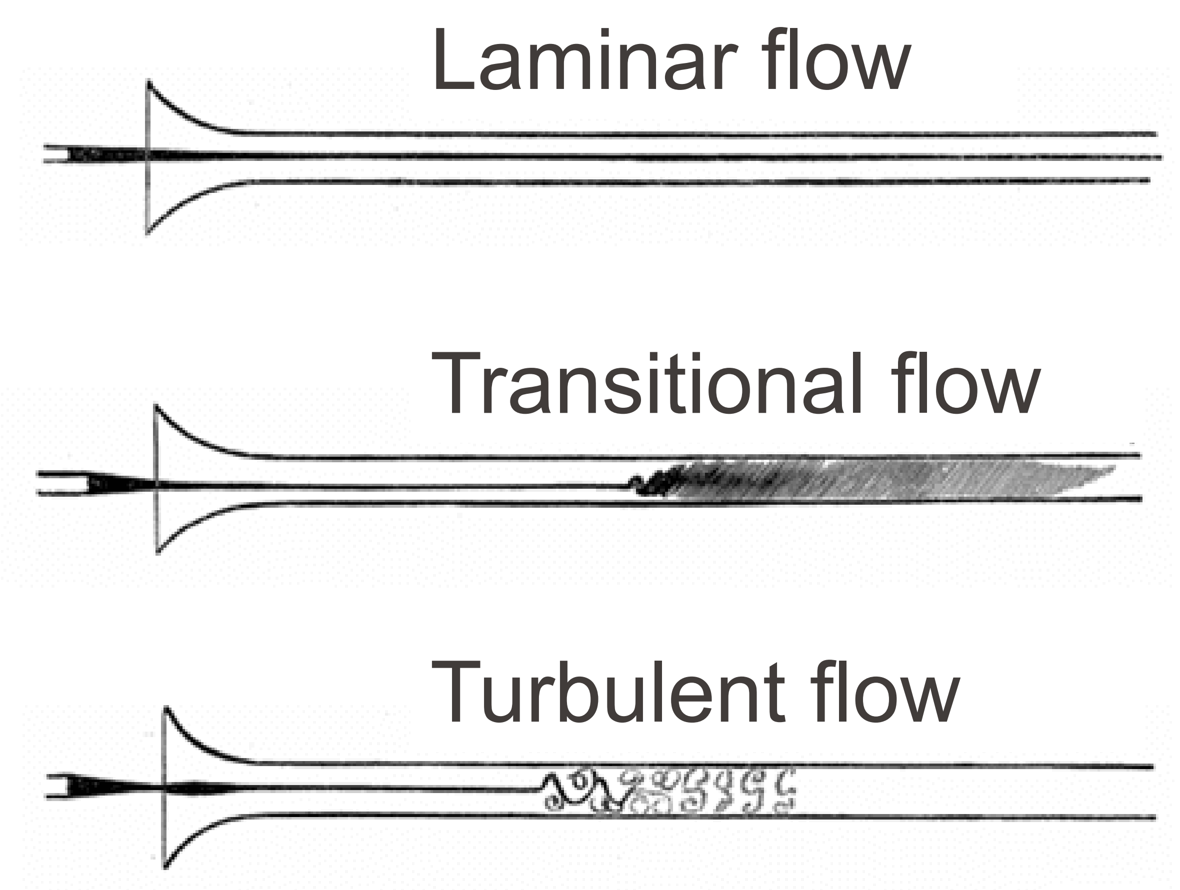
1. What is turbulence? (Brief overview)
2. Numerical simulation of turbulent flows
3. Overview of turbulence models

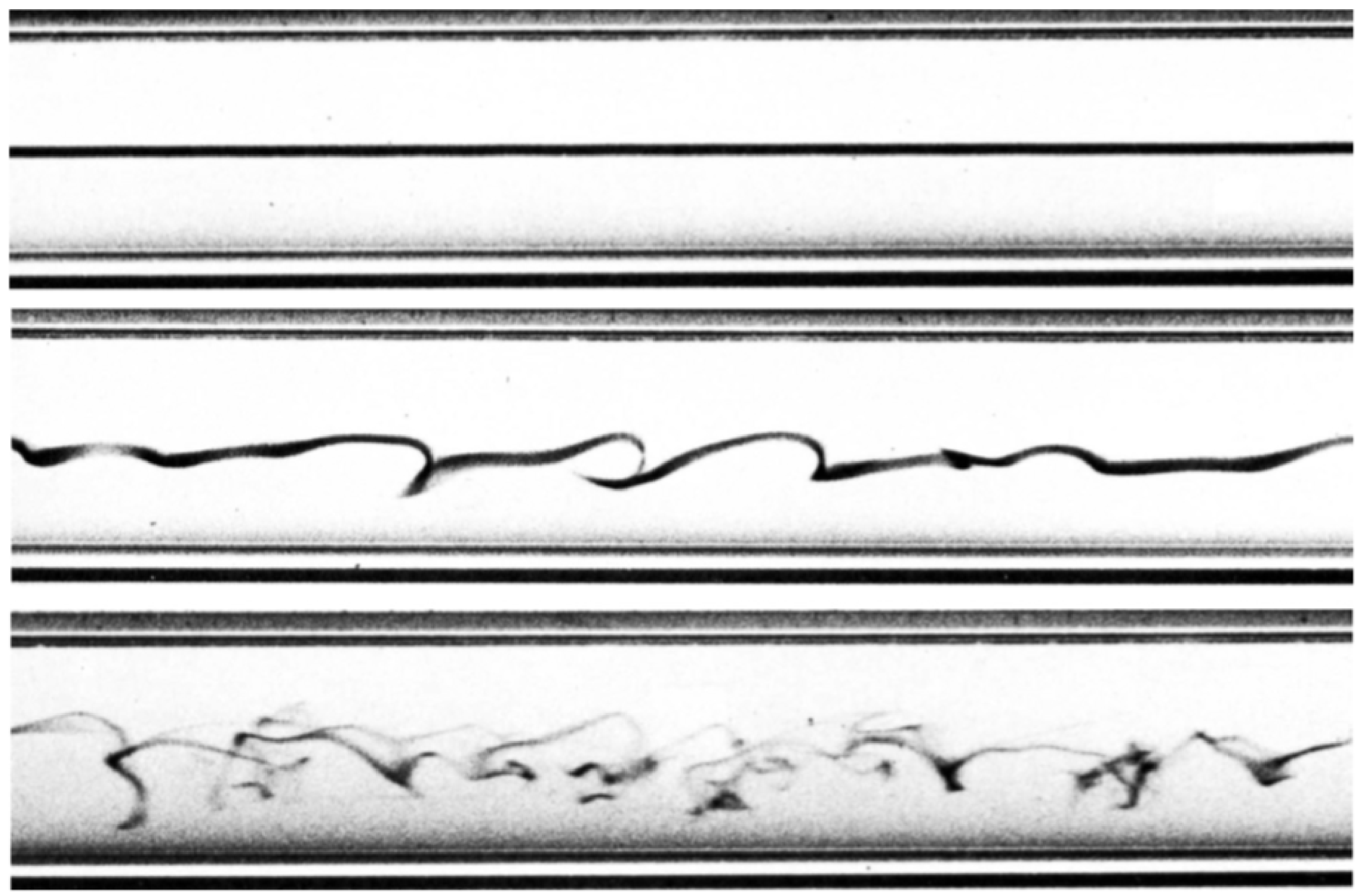
Basic characteristics of turbulence

- Experimental observation from Osborne Reynolds:



Reynolds 1883





van Dyke 1982

- Flow becomes unstable (and later turbulent) with increasing velocity and size, and with decreasing viscosity. Reynolds understood the role of “his” number:

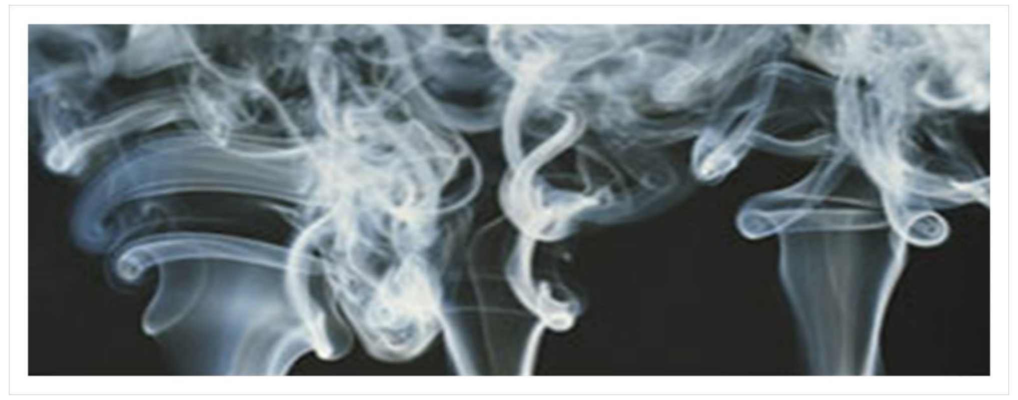
$$Re = \frac{UL}{\nu}$$

Ratio of inertial to viscous forces:

$$\frac{div(\rho \mathbf{uu}) \sim \rho U^2/L}{div(\mu\, grad(\mathbf{u})) \sim \mu U/L^2}$$

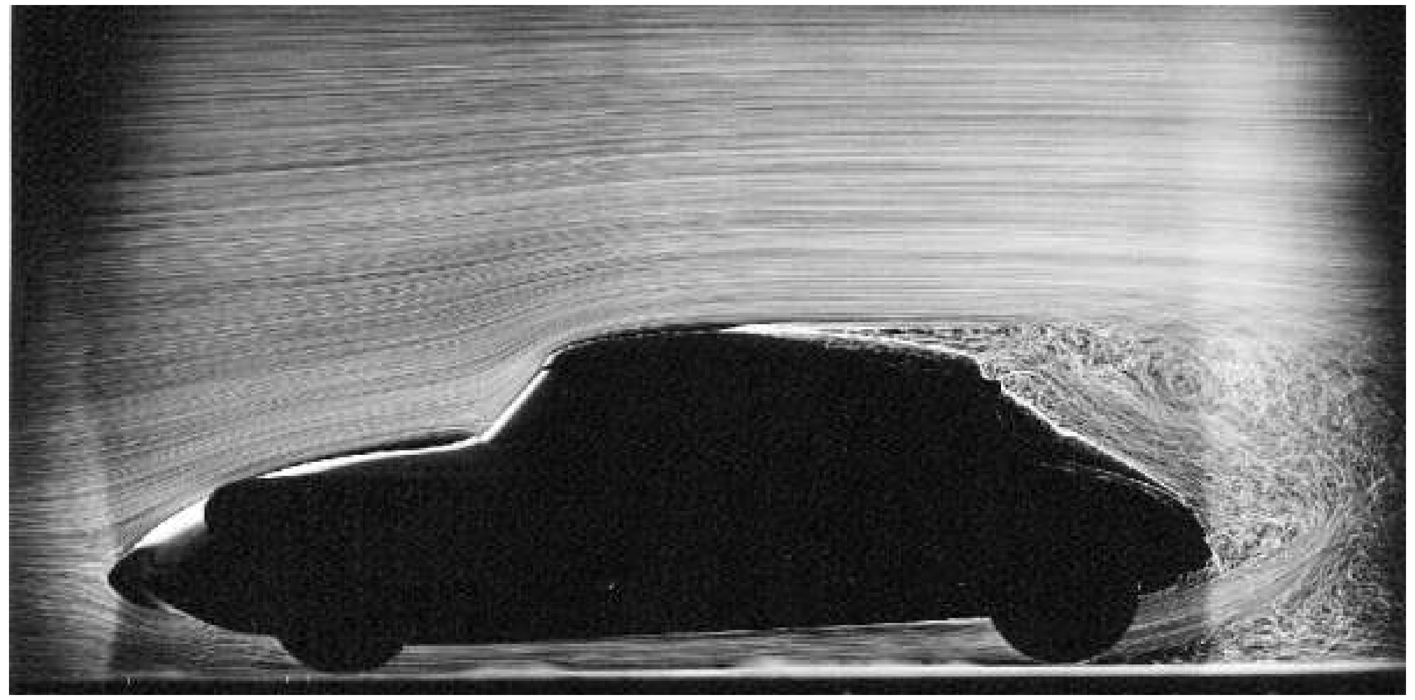
Basic characteristics of turbulence

- Flows generally turbulent when Reynolds numbers above $O(10^3\text{-}10^4)$.
- Wide variety of turbulent flows in nature and engineering:



da Vinci, 15th c.

Hiroshige, 19th c.



Numerical Flow Simulation

Quiz

Turbulence $\underset{?}{\overset{?}{\rightleftarrows}}$ Vorticity $\underset{?}{\overset{?}{\rightleftarrows}}$ “Rotation” (vortex)

Quiz

Turbulent flows contain small-scale vortices.

There is vorticity is simple shear flows too.

Turbulence $\underset{\not\Leftarrow}{\Rightarrow}$ Vorticity $\underset{\Leftarrow}{\not\Rightarrow}$ "Rotation" (vortex)

Laminar flows too can have vorticity.

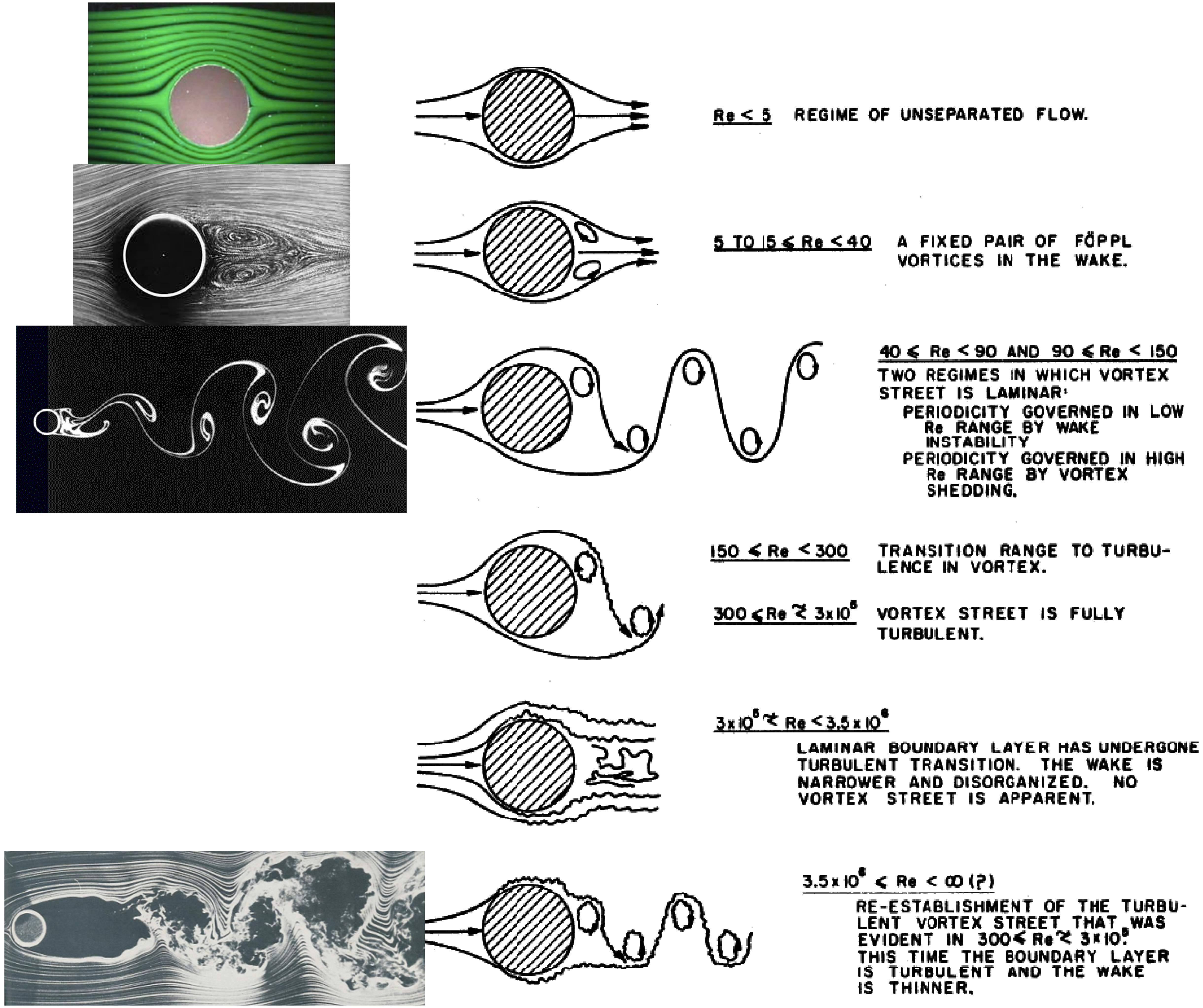
In general vortices have vorticity, but there are irrotational vortices too:

$$\begin{pmatrix} u_r \\ u_\theta \\ u_z \end{pmatrix} \propto \begin{pmatrix} 0 \\ 1/r \\ 0 \end{pmatrix} \Rightarrow \boldsymbol{\omega} = \mathbf{0}$$

Basic characteristics of turbulence

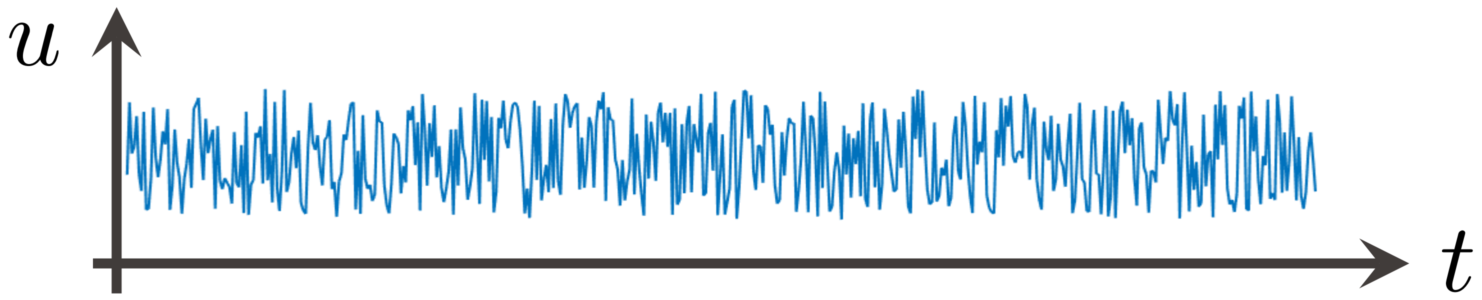
- Turbulent flows contain small-scale vortices, but **vortical flows are not necessarily turbulent.**

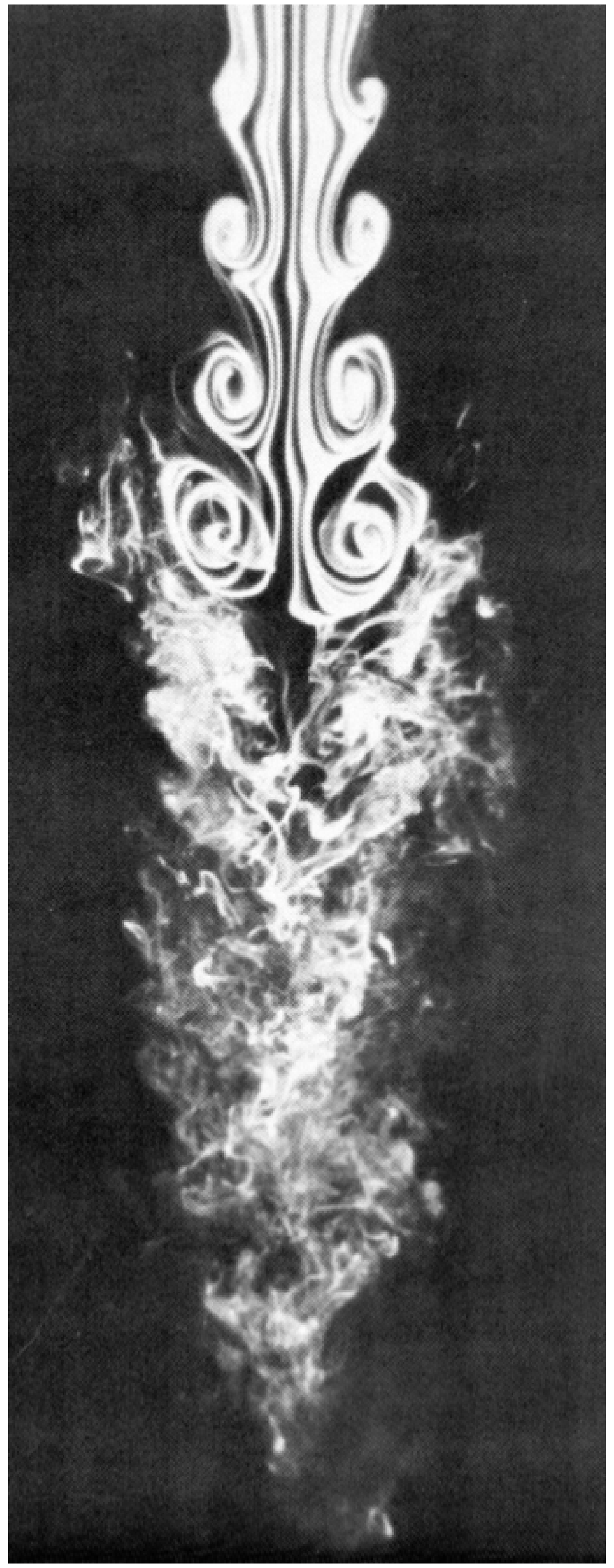
Increasing *Re*



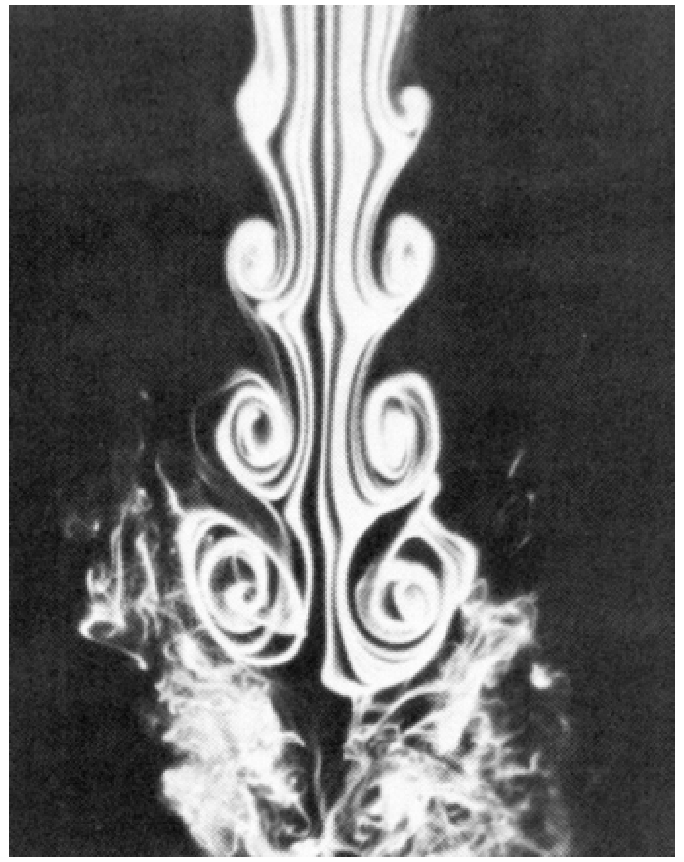
Basic characteristics of turbulence

- No universal definition, but a set of characteristics:
  1. **Always unsteady** (even with steady boundary conditions).





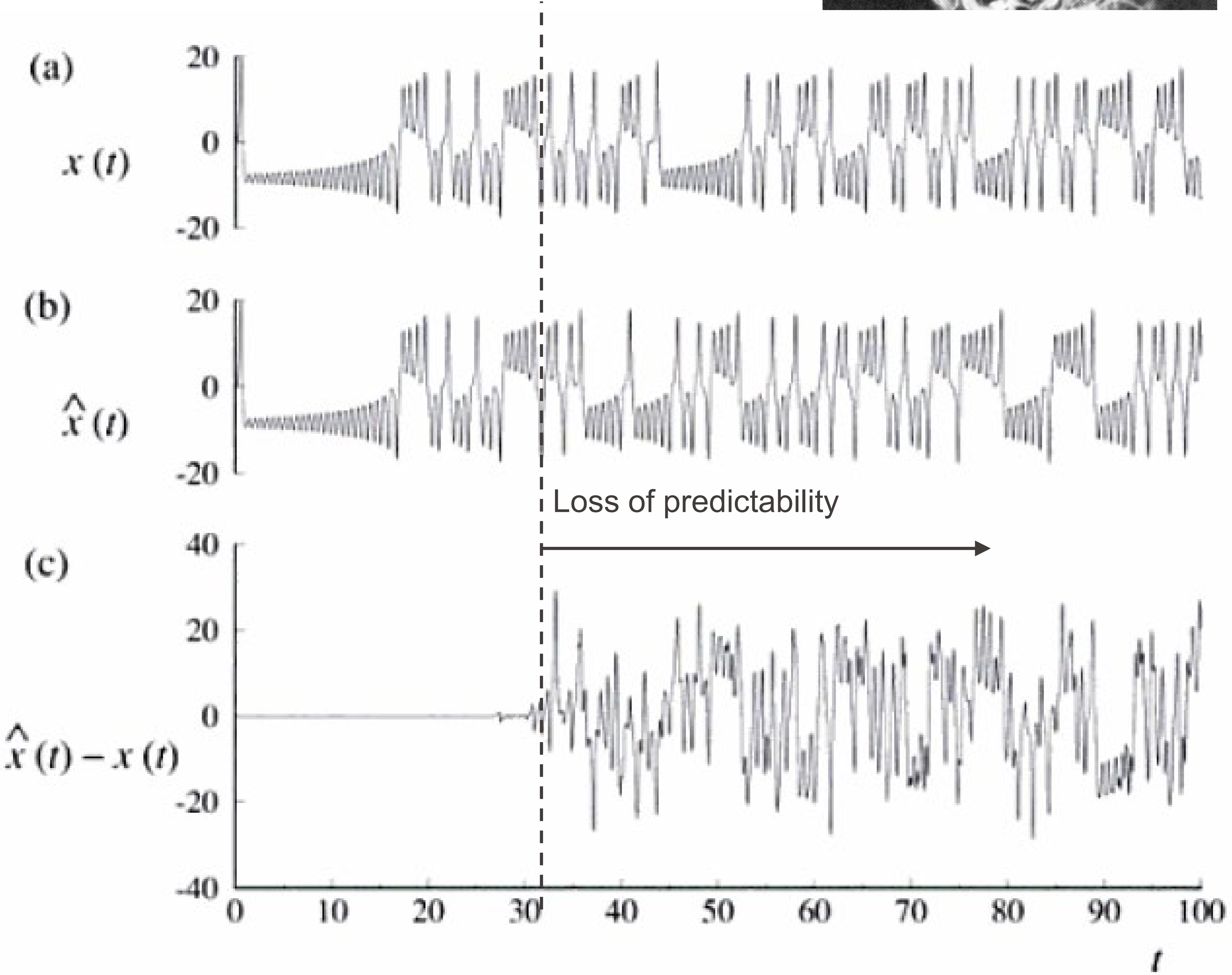
Basic characteristics of turbulence



- No universal definition, but a set of characteristics:

  2. **Random process.**
     - Signals seem unpredictable in time and space. Different measurements for different repetitions of the same experiment under same conditions. (But statistical properties *are* predictable.)
     - The NS equations are deterministic, but become extremely sensitive to perturbations as *Re* increases. Similar to chaotic systems ("butterfly effect").

Example: Lorenz' system (1963) of ODEs with only 3 degrees of freedom (*x,y,z*). Two simulations with nearby initial conditions, *x*(0)=0.1 and 0.100001.



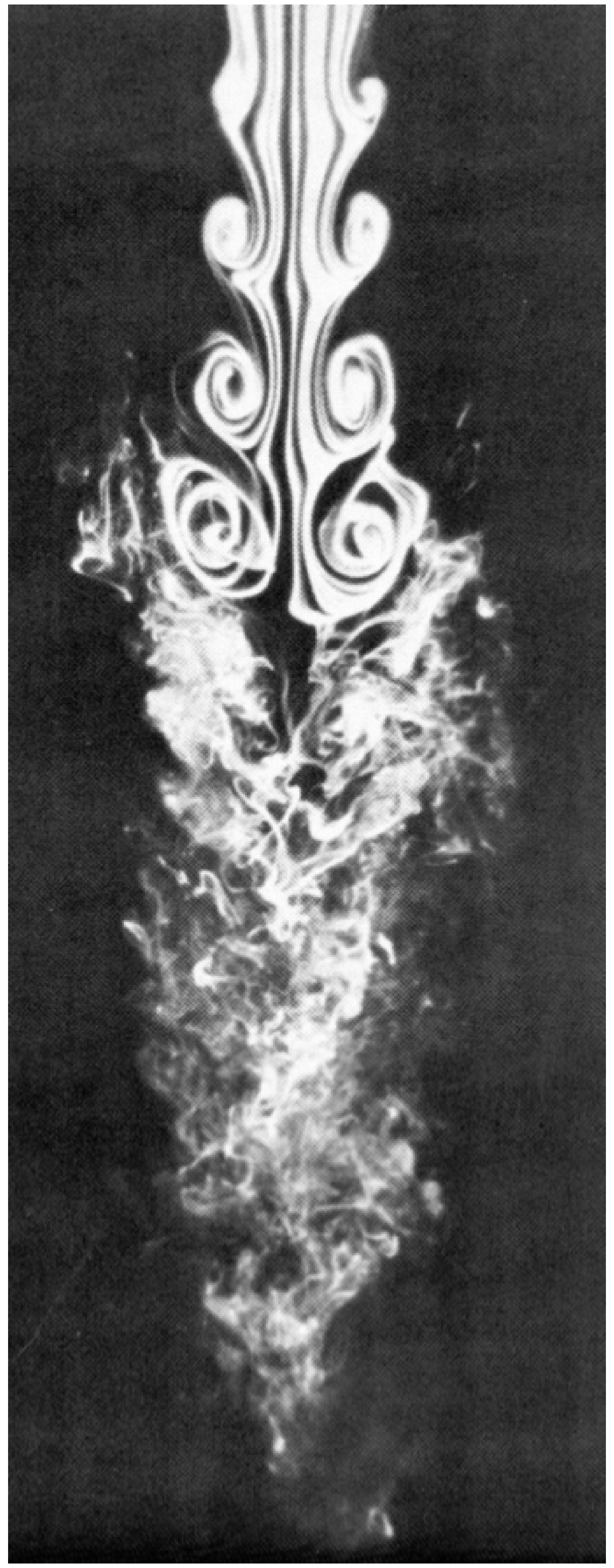
Basic characteristics of turbulence

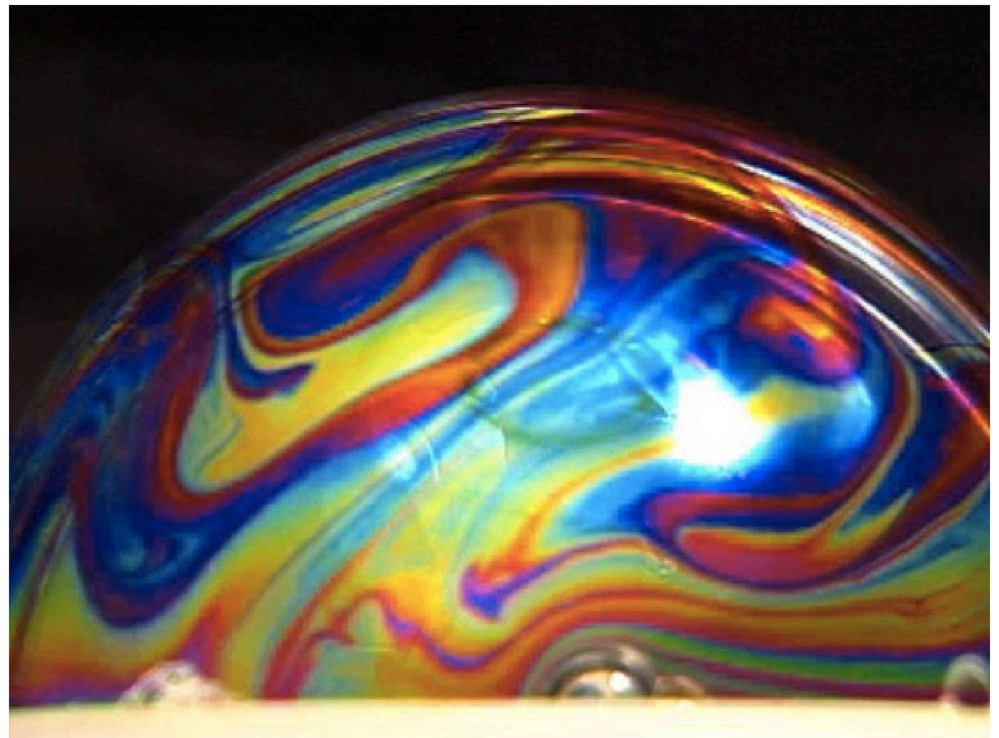
- No universal definition, but a set of characteristics:
  3. **Always 3D.**
     - Fluctuations in all directions, even in geometry invariant in 3rd direction.
     - 3D is necessary for vortex stretching (intensification) and tilting (deformation), an essential mechanism in turbulent flows.
       Recall the (incompressible) vorticity equation:

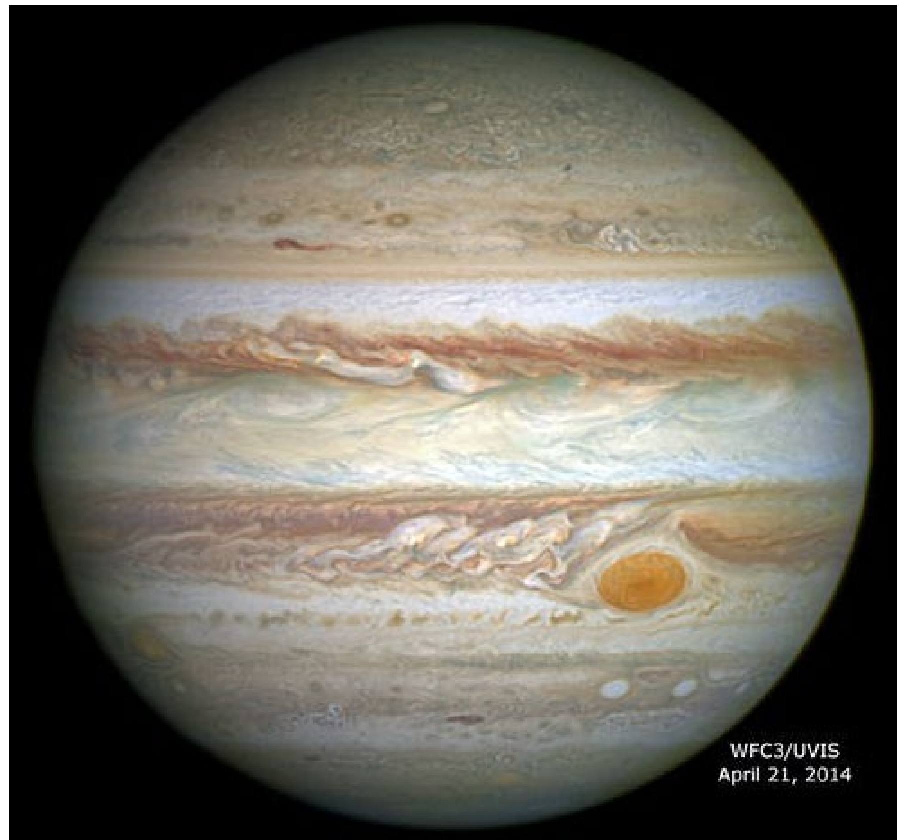
$$\frac{\partial \boldsymbol{\omega}}{\partial t} + (\mathbf{u} \cdot \nabla)\boldsymbol{\omega} = \nu \nabla^2 \boldsymbol{\omega} + (\boldsymbol{\omega} \cdot \nabla)\mathbf{u}$$

convection diffusion **vortex stretching/tilting**

     - Really 2D only in strongly confined systems (e.g. soap films, geophysical flows).





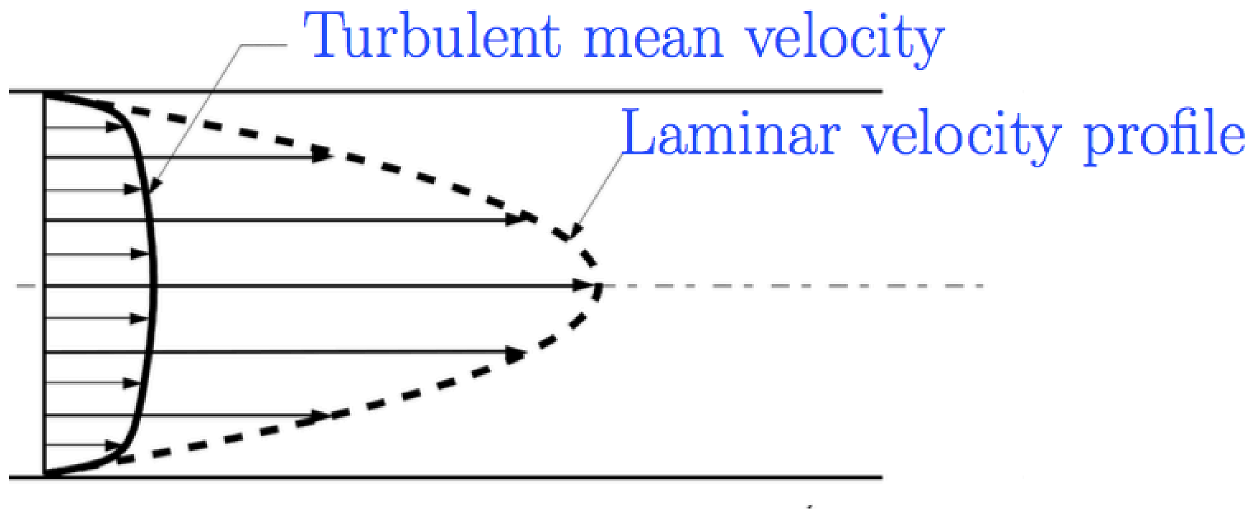


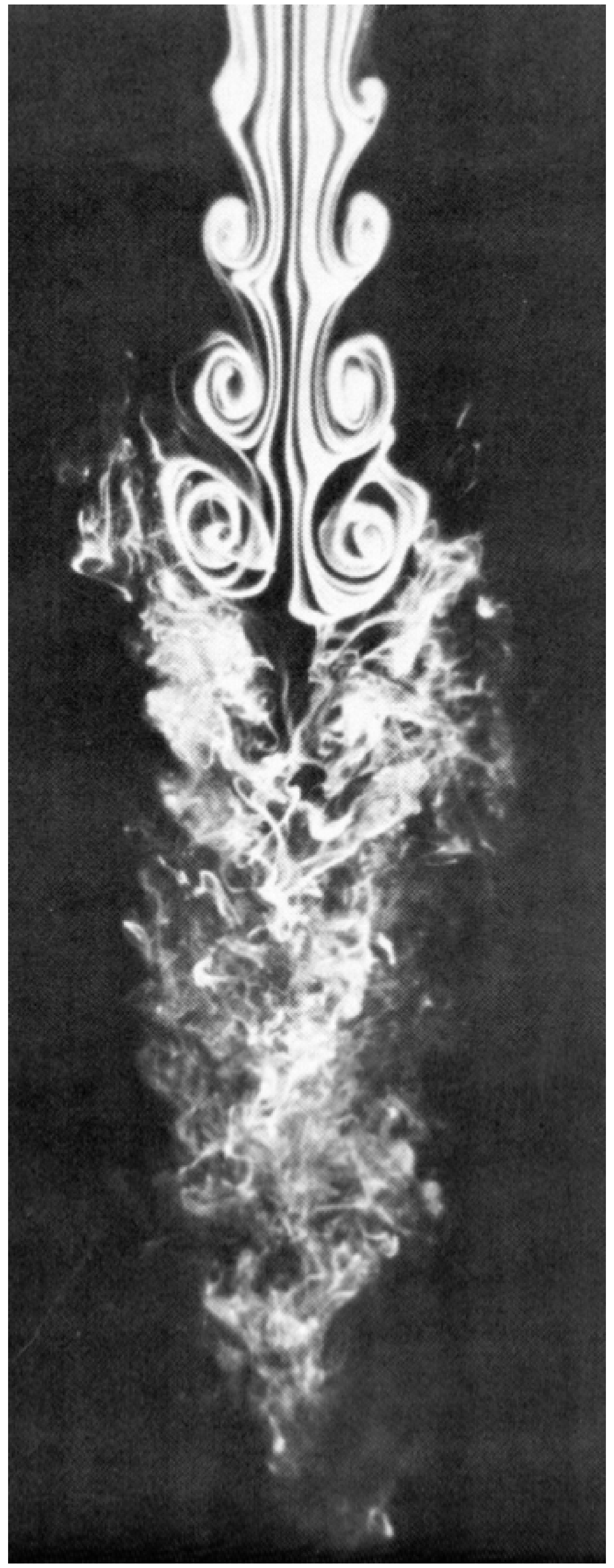
Jupiter

Basic characteristics of turbulence

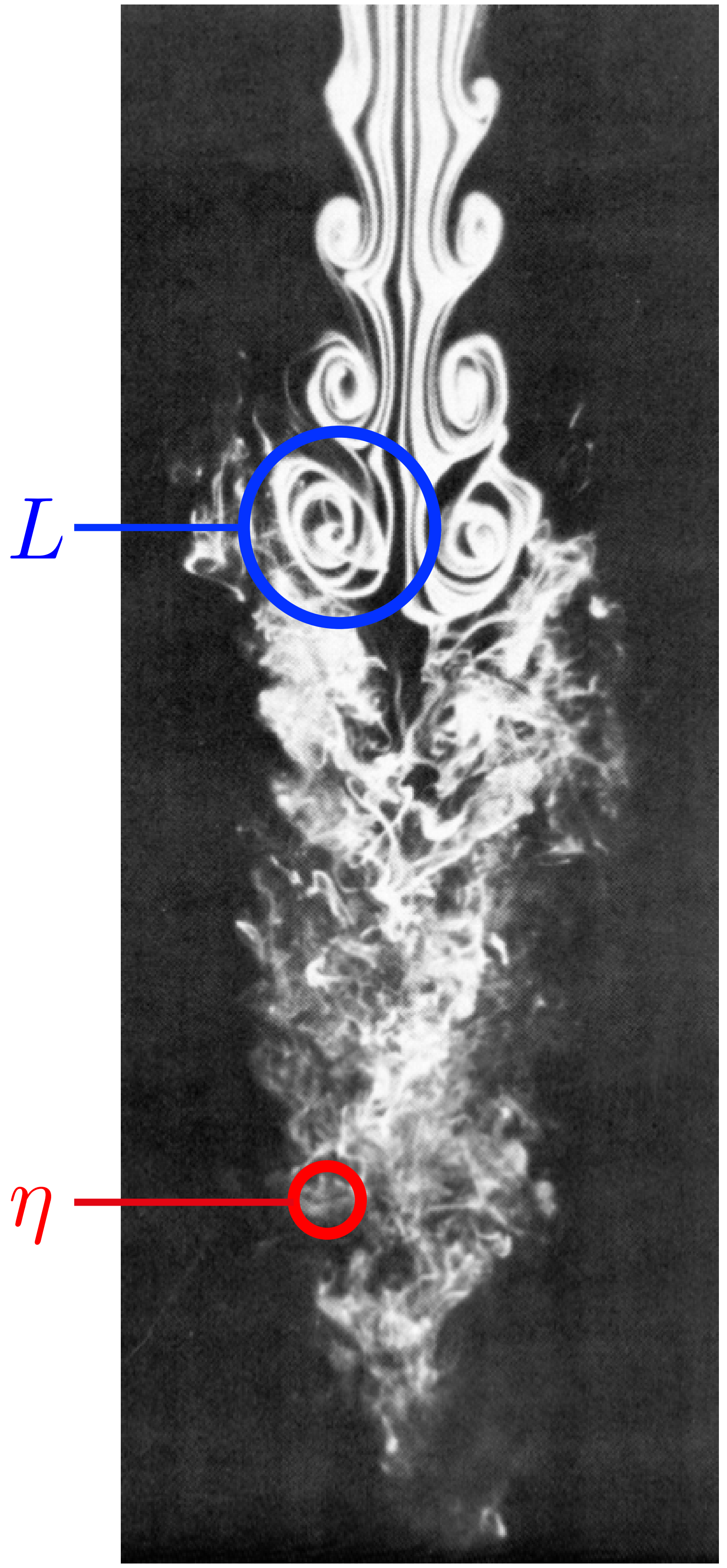
- No universal definition, but a set of characteristics:
  4. **Turbulence mixes all quantities** (mass, momentum, energy, passive scalars), because fluid particles initially far away can be brought close together by the eddying motions

  → increased mixing and heat transfer, but increased drag/losses too.





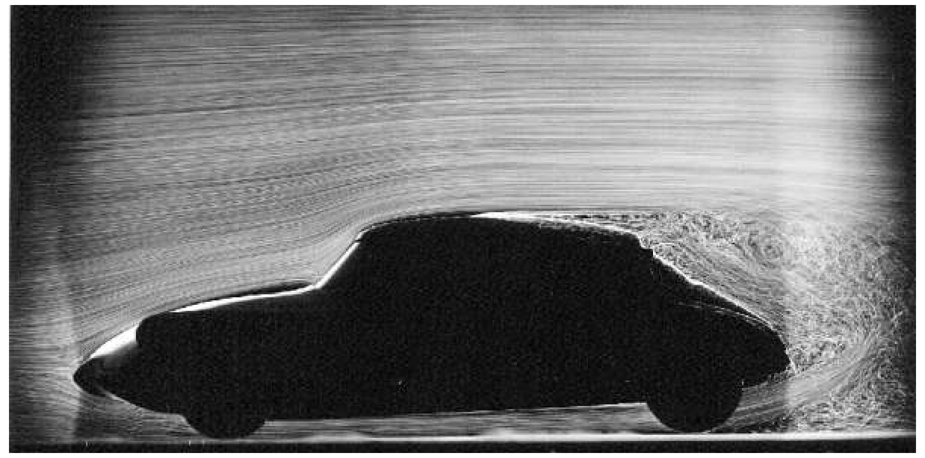
Basic characteristics of turbulence



- No universal definition, but a set of characteristics:
  5. **Wide range of length and time scales** (broadband spectra). For smallest vortices ("eddies"), Kolmogorov length scale and time scale:

$$\eta \sim \frac{L}{Re^{3/4}} \qquad \tau \sim \frac{L/U}{Re^{1/2}}$$

- Examples (air: $\nu \sim 10^{-5}\ \mathrm{m^2/s}$):



$$L \sim 1\ \mathrm{m}$$
$$U \sim 10\ \mathrm{m/s}$$
$$Re = \frac{UL}{\nu} \sim 10^6$$

$$L \sim 10^3\ \mathrm{m}$$
$$U \sim 10\ \mathrm{m/s}$$
$$Re = \frac{UL}{\nu} \sim 10^9$$

$$\frac{L}{\eta} \sim \frac{1\ \mathrm{m}}{3 \times 10^{-5}\ \mathrm{m}} \sim Re^{3/4} \sim 3 \times 10^4$$

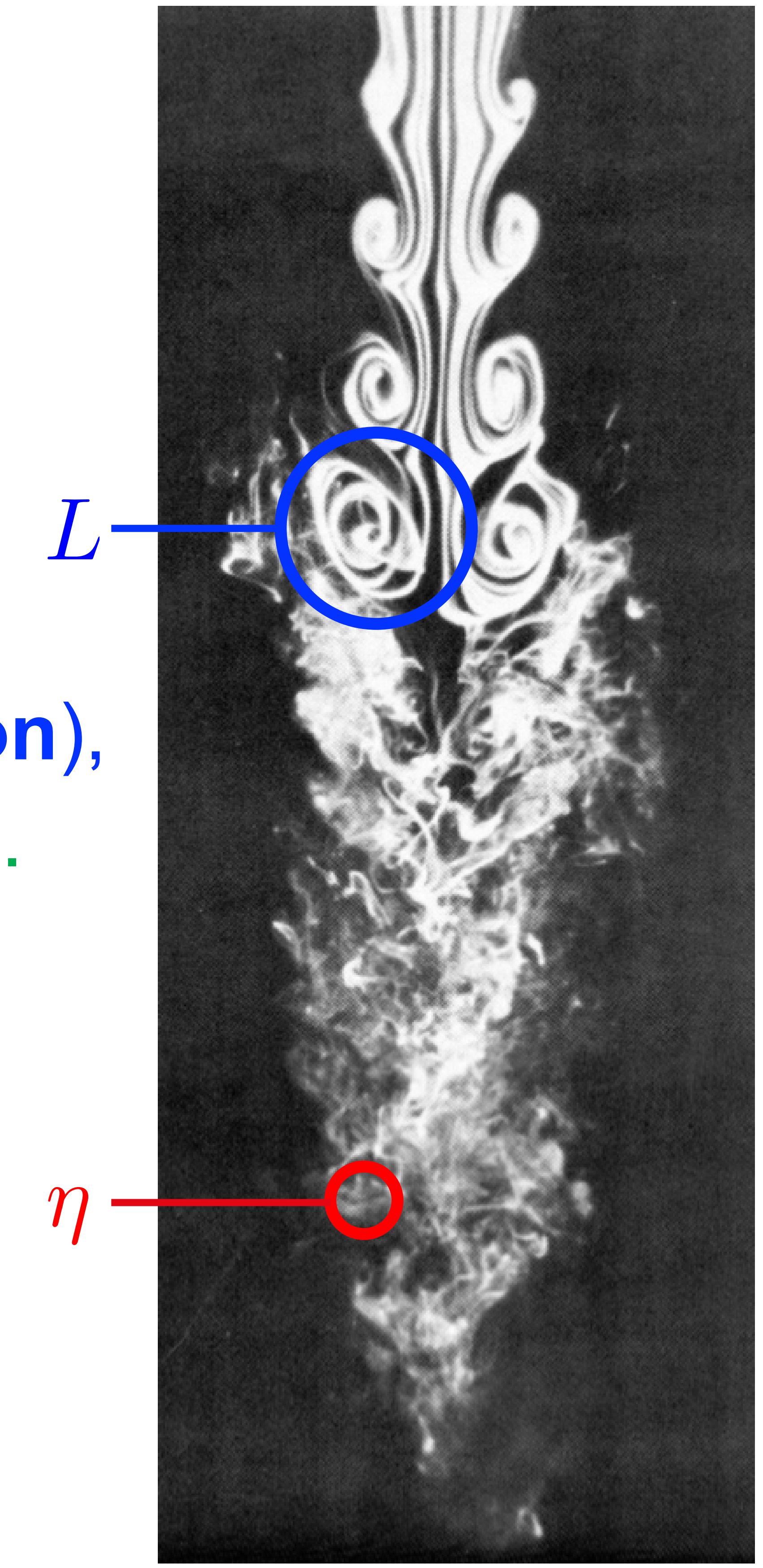
$$\frac{L/U}{\tau} \sim \frac{0.1\ \mathrm{s}}{10^{-4}\ \mathrm{s}} \sim Re^{1/2} \sim 10^3$$

$$\frac{L}{\eta} \sim \frac{10^3\ \mathrm{m}}{2 \times 10^{-4}\ \mathrm{m}} \sim Re^{3/4} \sim 6 \times 10^6$$

$$\frac{L/U}{\tau} \sim \frac{10^2\ \mathrm{s}}{3 \times 10^{-3}\ \mathrm{s}} \sim Re^{1/2} \sim 3 \times 10^4$$

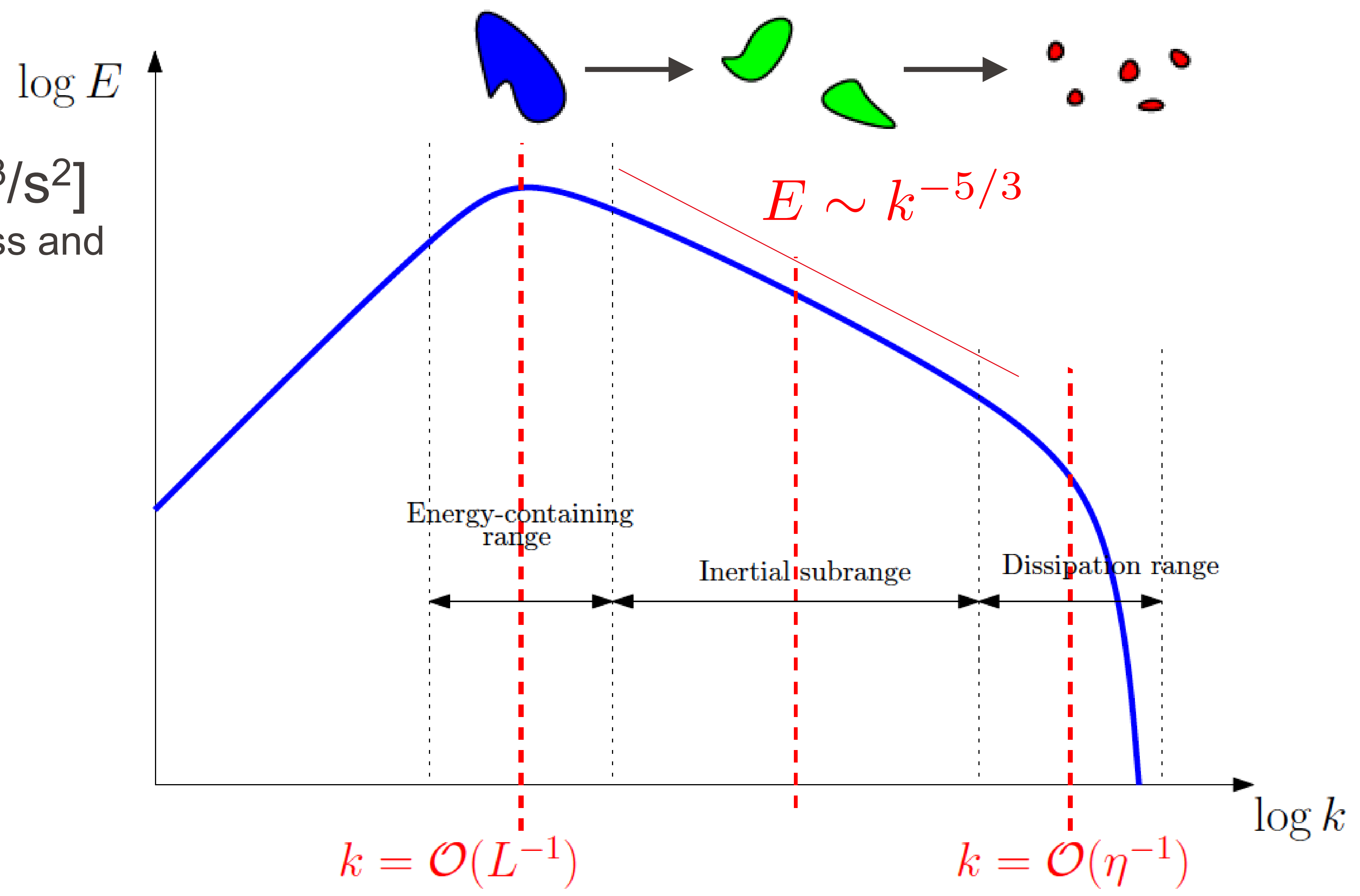
Basic characteristics of turbulence

- No universal definition, but a set of characteristics:
  6. **Energy cascade:**
     - Large structures created by hydrodynamic instabilities (**energy injection**),
     - Vortex stretching breaks up eddies into smaller ones (**energy transfer**).
     - Viscosity dissipates smallest eddies (**energy dissipation**).



Typical energy spectrum (log-log):

Spectral energy [m^3/s^2]
(kinetic energy per unit mass and unit wavenumber)



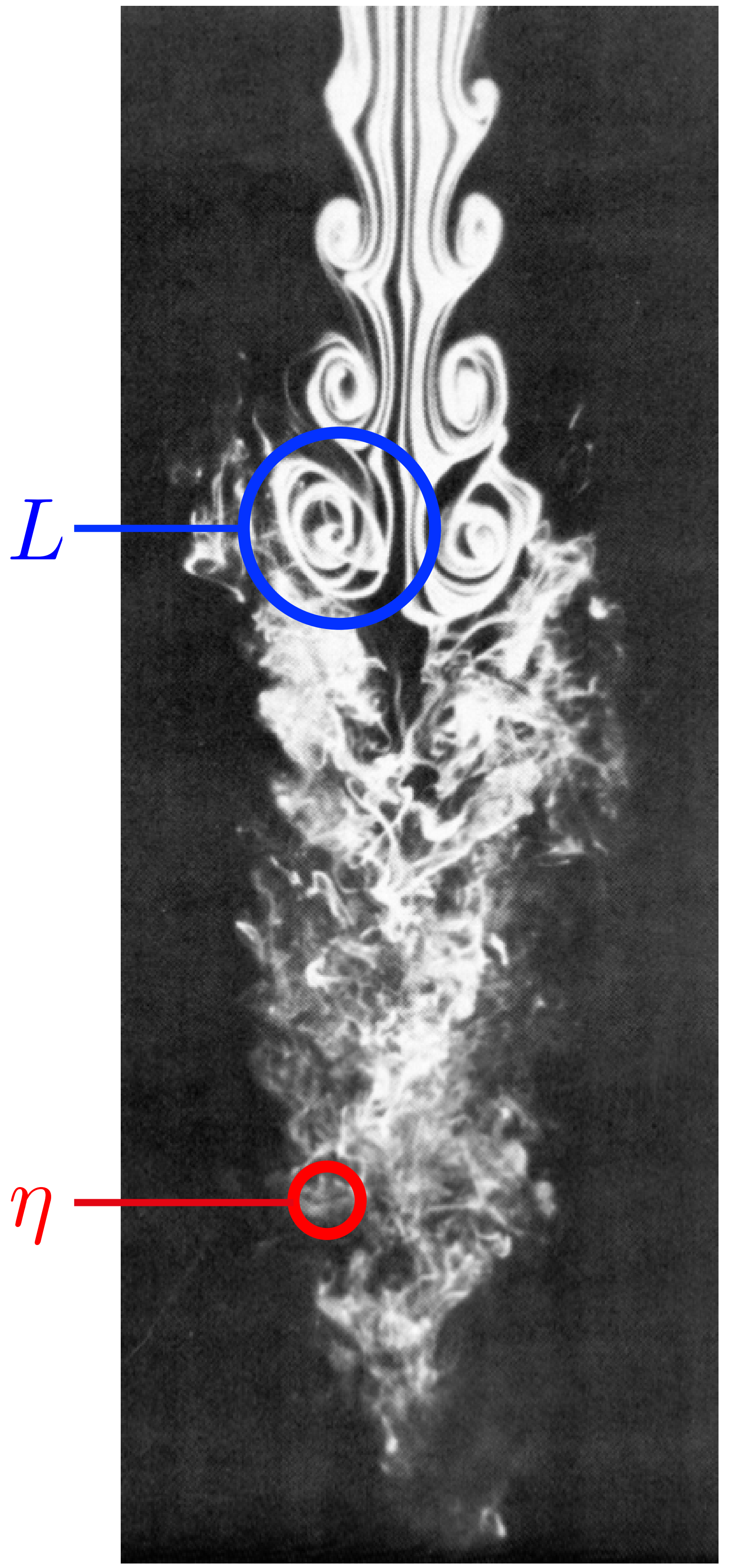
Spatial wavenumber [1/m]

Basic characteristics of turbulence

- No universal definition, but a set of characteristics:

6. **Largest eddies:**
   - Flow & geometry-dependent,
   - Most energetic,
   - Insensitive to viscosity.

Smallest eddies:
- Homogeneous and isotropic,
- Dissipate energy via viscosity,
- Become smaller as *Re* increases, but are still much larger than the molecular free path (continuum).



Typical energy spectrum (log-log):

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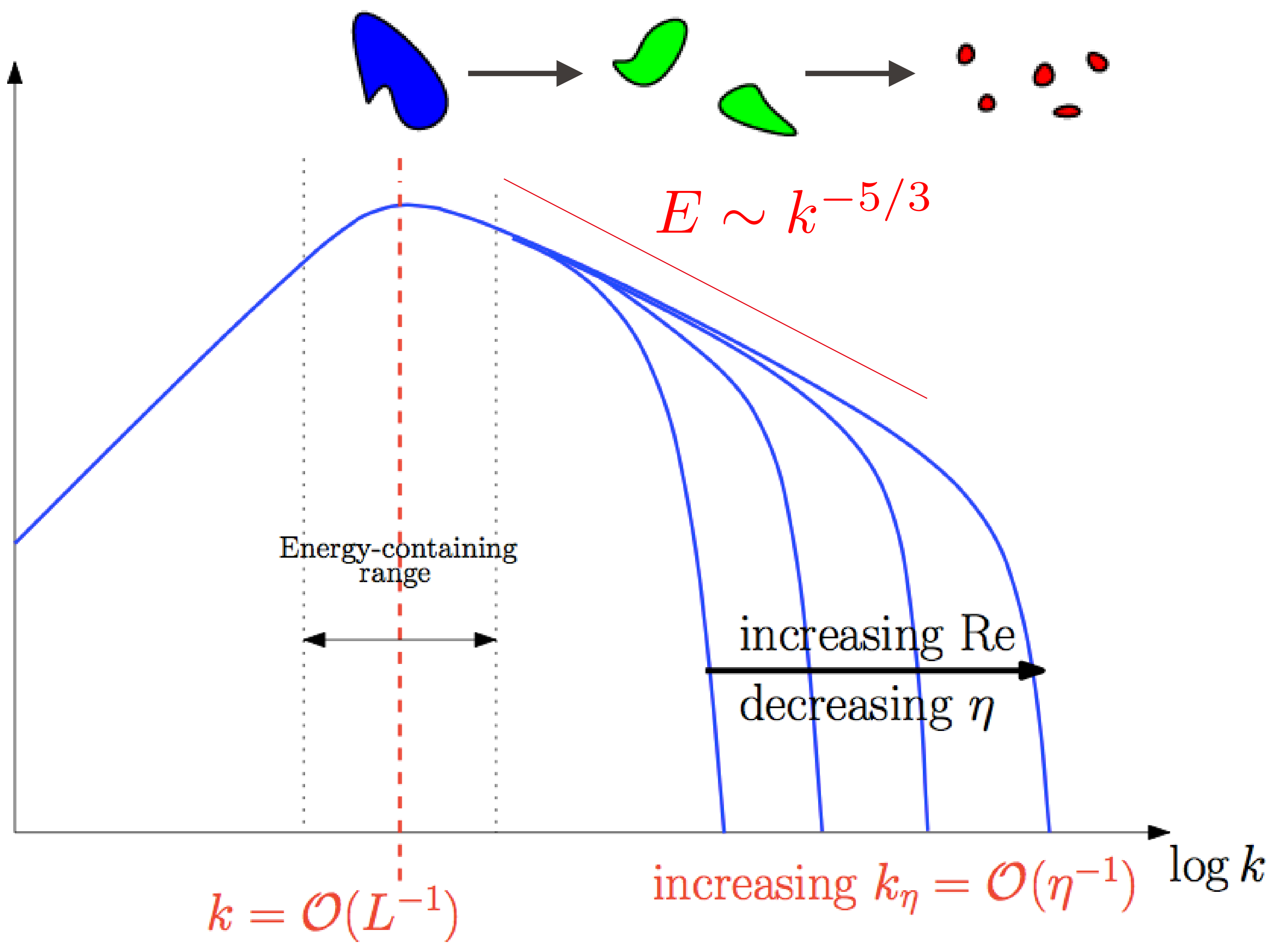


Spatial wavenumber [1/m]

Turbulence is difficult

- Some (verified and unverified) quotations:
  - *"When I meet God, I am going to ask him two questions: Why relativity? And why turbulence? I really believe he will have an answer for the first."* (Heisenberg?)
  - *"I am an old man now, and when I die and go to heaven there are two matters on which I hope for enlightenment. One is quantum electrodynamics, and the other is the turbulent motion of fluids. And about the former I am rather optimistic."* (Horace Lamb?)
  - *"The study of turbulence is not easy (…) even after various theoretical hypotheses have been absorbed, there are relatively few situations in which we can make positive predictions."* (P.A. Davidson, "Turbulence: an introduction for scientists and engineers", Oxford University Press, 2004)

Turbulence is difficult

- Why?
  - Wide range of length and time scales, that increases (dramatically) with *Re*.
  - Extreme sensitivity (like chaotic systems), random character.

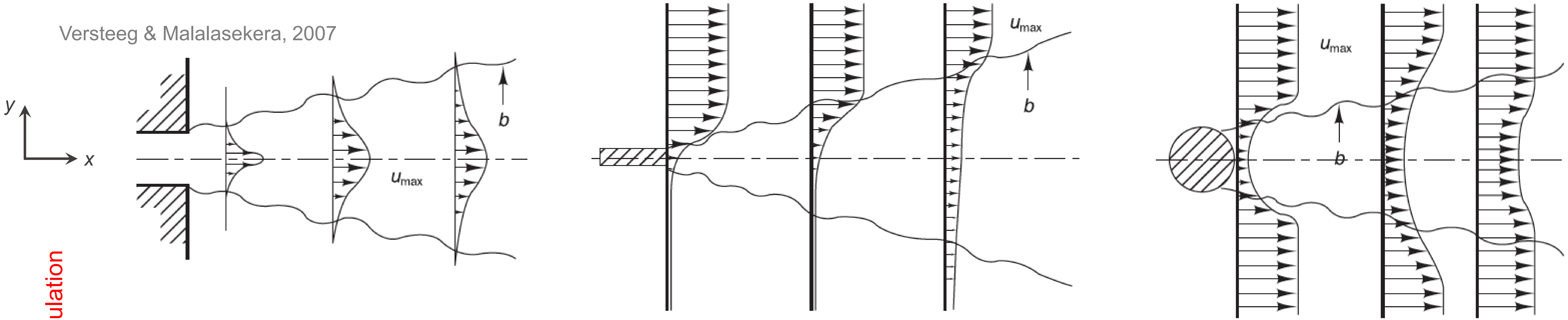
- How can we tackle it?
  - **Analytical approach**: no simple analytical closed-form theory today.
  - **Empirical approach**: describe turbulence statistically.
  - **Numerical approach**: can solve the Navier-Stokes equations directly if enough computational resources to resolve all the scales. Otherwise, need a turbulence model (approximation).

Simple turbulent flows

- Free turbulent flows:

Jet | Mixing layer | Wake

Versteeg & Malalasekera, 2007



- Mean flow and turbulent structures are **self-similar** (far enough downstream). For instance, in jets:

Mean streamwise velocity: $$\frac{\overline{u}}{\overline{u}_{max}} = f\left(\frac{y}{b}\right)$$

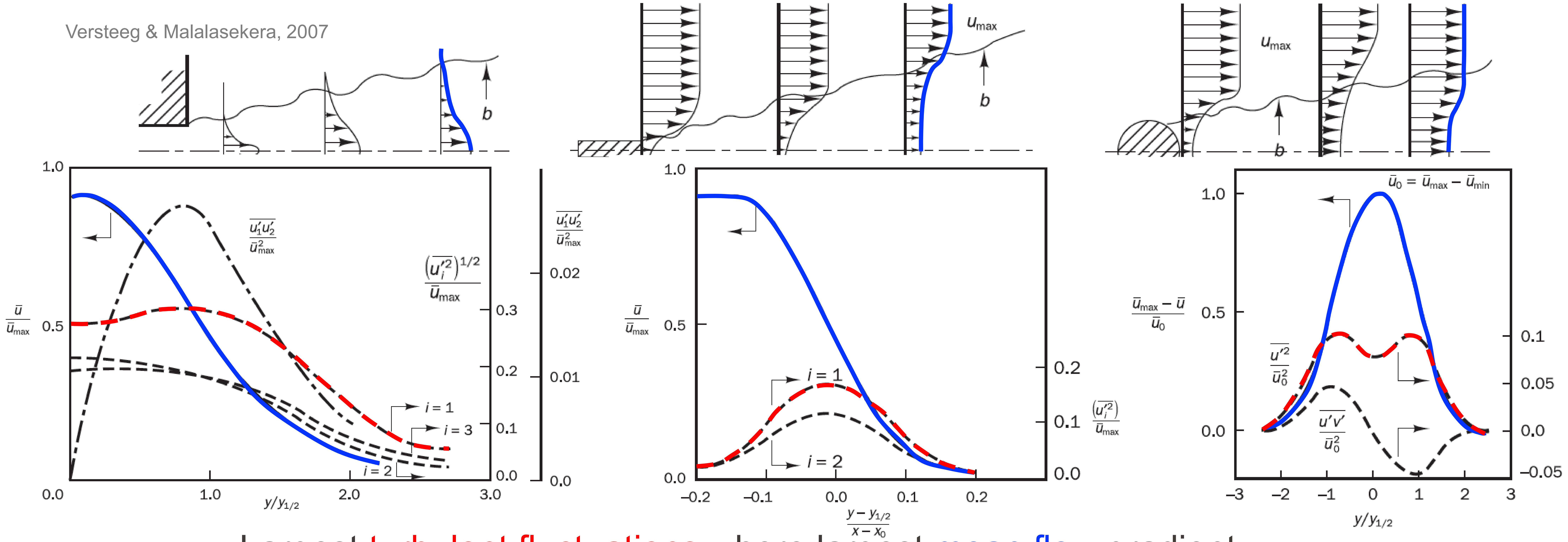
Turbulent correlations: $$\frac{\overline{u'^2}}{\overline{u}_{max}} = g\left(\frac{y}{b}\right) \qquad \frac{\overline{u'v'}}{\overline{u}_{max}} = h\left(\frac{y}{b}\right)$$ etc.

Simple turbulent flows

- Free turbulent flows:

Jet Mixing layer Wake

Versteeg & Malalasekera, 2007



- Largest turbulent fluctuations where largest mean flow gradient.
- Turbulence is anisotropic (e.g. $\overline{u'^2} > \overline{v'^2}$) in these flows.

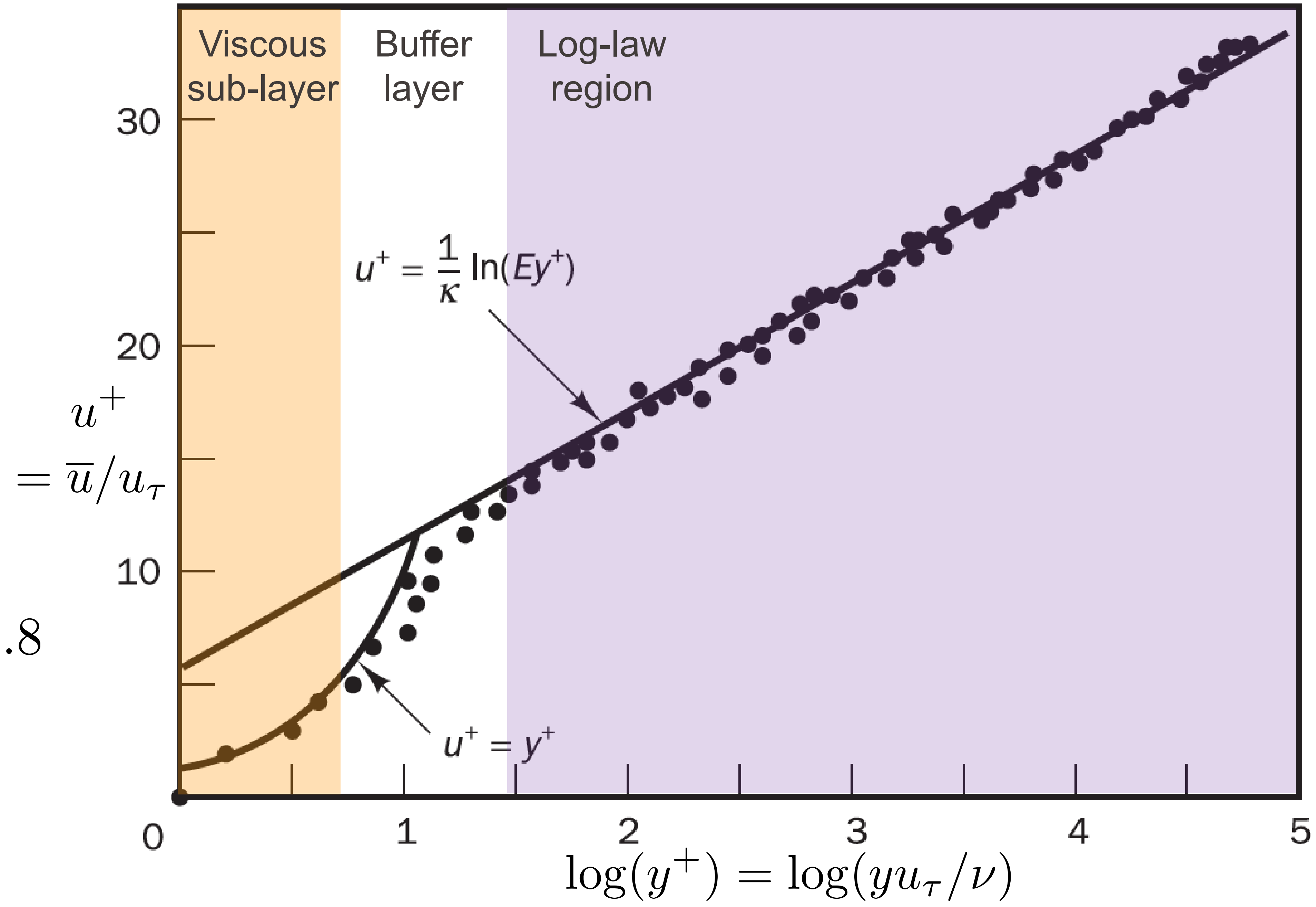
Simple turbulent flows

- Wall-bounded flows: flat plate boundary layer, pipe flow, channel flow…
  - Close to the wall, flow influenced by viscous effects and independent of free-stream parameters → mean flow velocity depends on distance from the wall y and wall shear stress τ_w. From dimensional analysis:

Law of the wall: $$\frac{\overline{u}}{u_\tau} = f\left(\frac{y u_\tau}{\nu}\right)$$ with velocity scale $u_\tau = \sqrt{\frac{\tau_w}{\rho}} = \sqrt{\nu \frac{\partial u_x}{\partial y}}$ ("friction velocity")

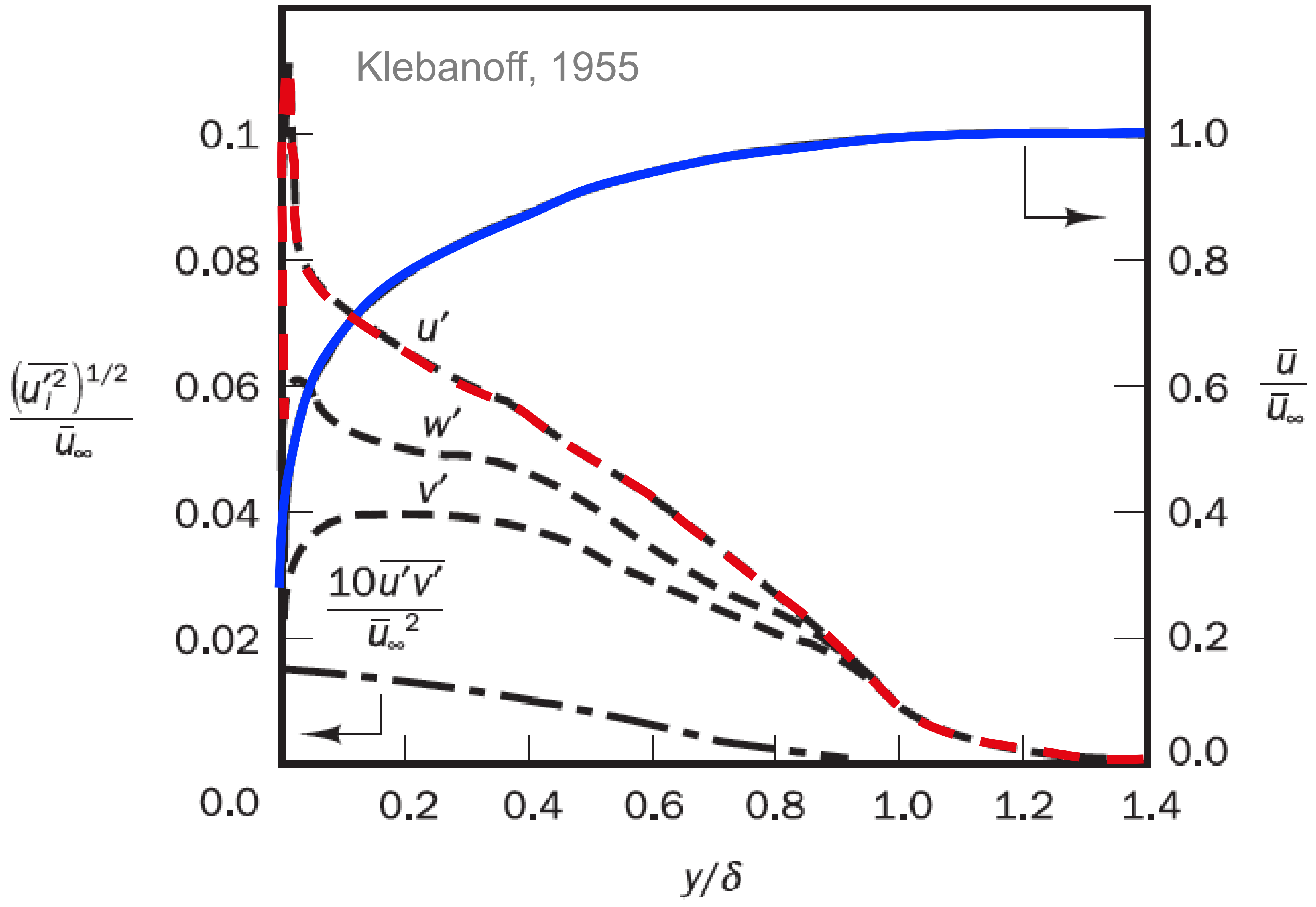
$$u^+ = f(y^+)$$

  - Viscous sub-layer (y^+<5): dominated by viscous effects, very thin, linear relationship
$$u^+ = y^+$$
  - Log-law layer (30<y^+, y/δ<0.3): logarithmic relationship
$$u^+ = \frac{1}{\kappa}\ln(Ey^+) \qquad \kappa \approx 0.4,\ E \approx 9.8$$
  - Outer layer (50<y^+): viscous effects negligible, fully turbulent.



Simple turbulent flows

- Wall-bounded flows
  - Example: flat-plate boundary layer (without pressure gradient)



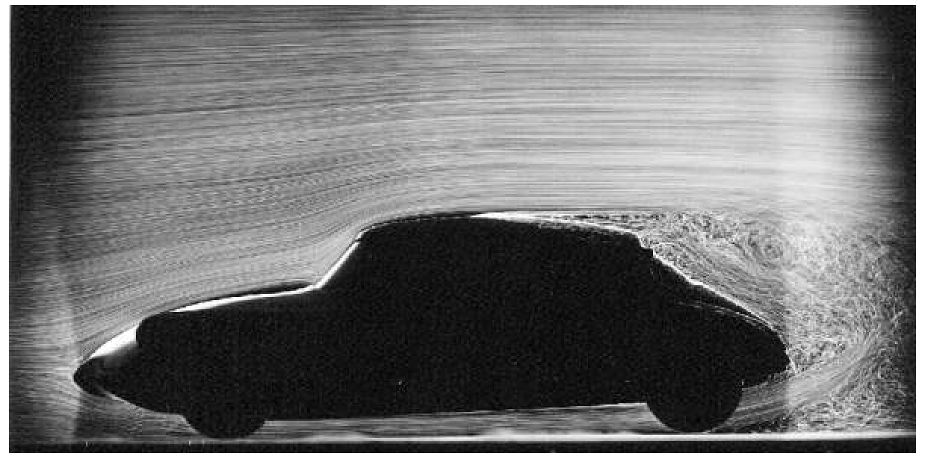
  - High turbulent production where large mean flow gradient.
  - Turbulence strongly anisotropic near the wall, becomes isotropic farther away.

Numerical simulation of turbulent flows

- Challenging because of the wide range of length and time scales
  - Recall the ratio of large scales to Kolmogorov scales: $\frac{L}{\eta} \sim Re^{3/4}$ $\quad \frac{L/U}{\tau} \sim Re^{1/2}$
  - Number of operations needed to resolve all scales:

$$N = N_{\mathbf{x}} N_t \sim \left(\frac{L}{\eta}\right)^3 \left(\frac{L/U}{\tau}\right) \sim Re^{11/4}$$

  - Examples:



$L \sim 1\ \mathrm{m}$

$U \sim 10\ \mathrm{m/s}$

$Re = \frac{UL}{\nu} \sim 10^6$

$N \sim 10^{33/2} \sim 10^{16.5}$

$L \sim 10^3\ \mathrm{m}$

$U \sim 10\ \mathrm{m/s}$

$Re = \frac{UL}{\nu} \sim 10^9$

$N \sim 10^{99/4} \sim 10^{25}$

  - Starts to become accessible to fastest supercomputers, but not quite to regular computers used in the industry.

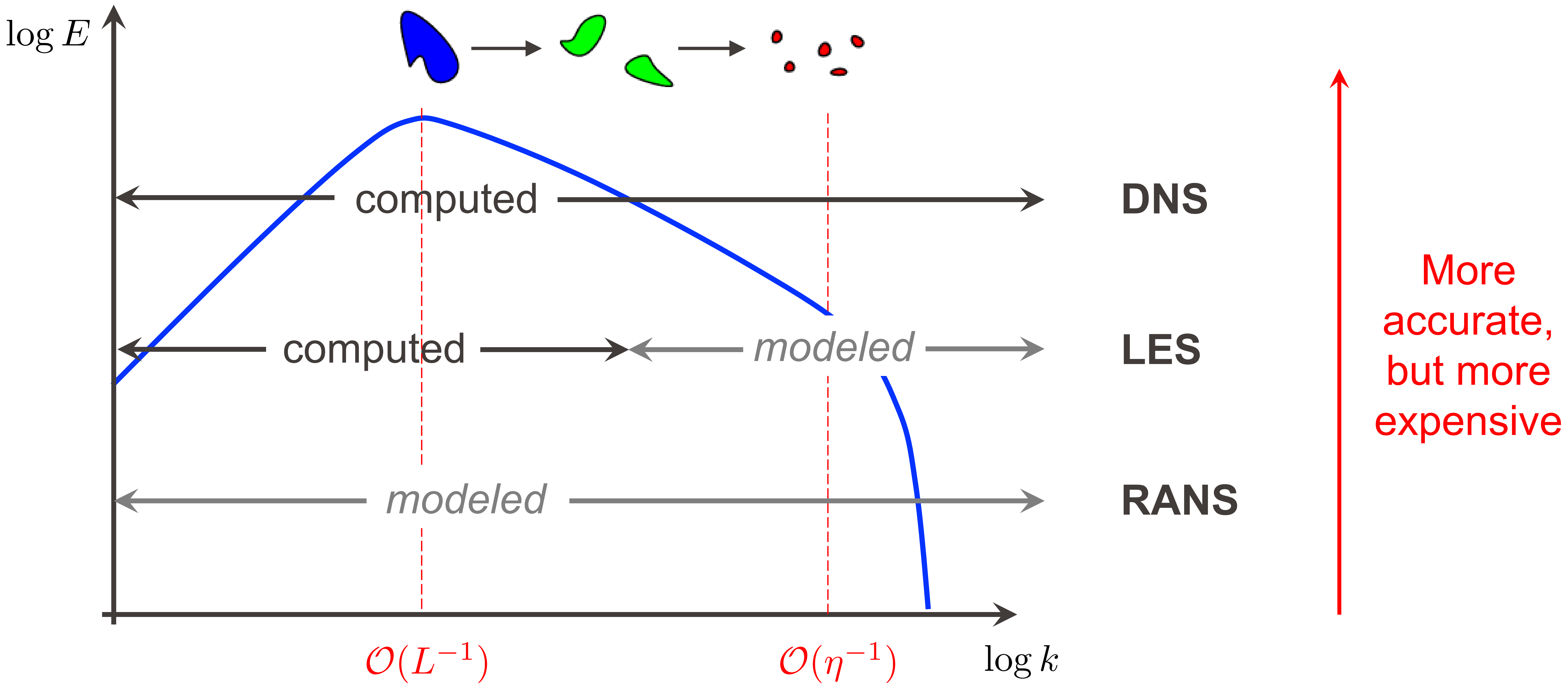
Numerical simulation of turbulent flows

- Three main methods:
  1. **Direct numerical simulation (DNS):**
     - Resolve all scales, all the way down to Kolmogorov scale
  2. **Large-eddy simulation (LES):**
     - Resolve larger eddies (flow & geometry-dependent) down to a size in the inertial subrange
     - Model the effect of smaller eddies (isotropic, universal behavior)
  3. **Reynolds-averaged Navier-Stokes (RANS) eqs.** + turbulence models
     - Solve for mean flow (ensemble average)
     - Model fluctuations at all scales

More accurate, but more expensive

Numerical simulation of turbulent flows

- Three main methods:

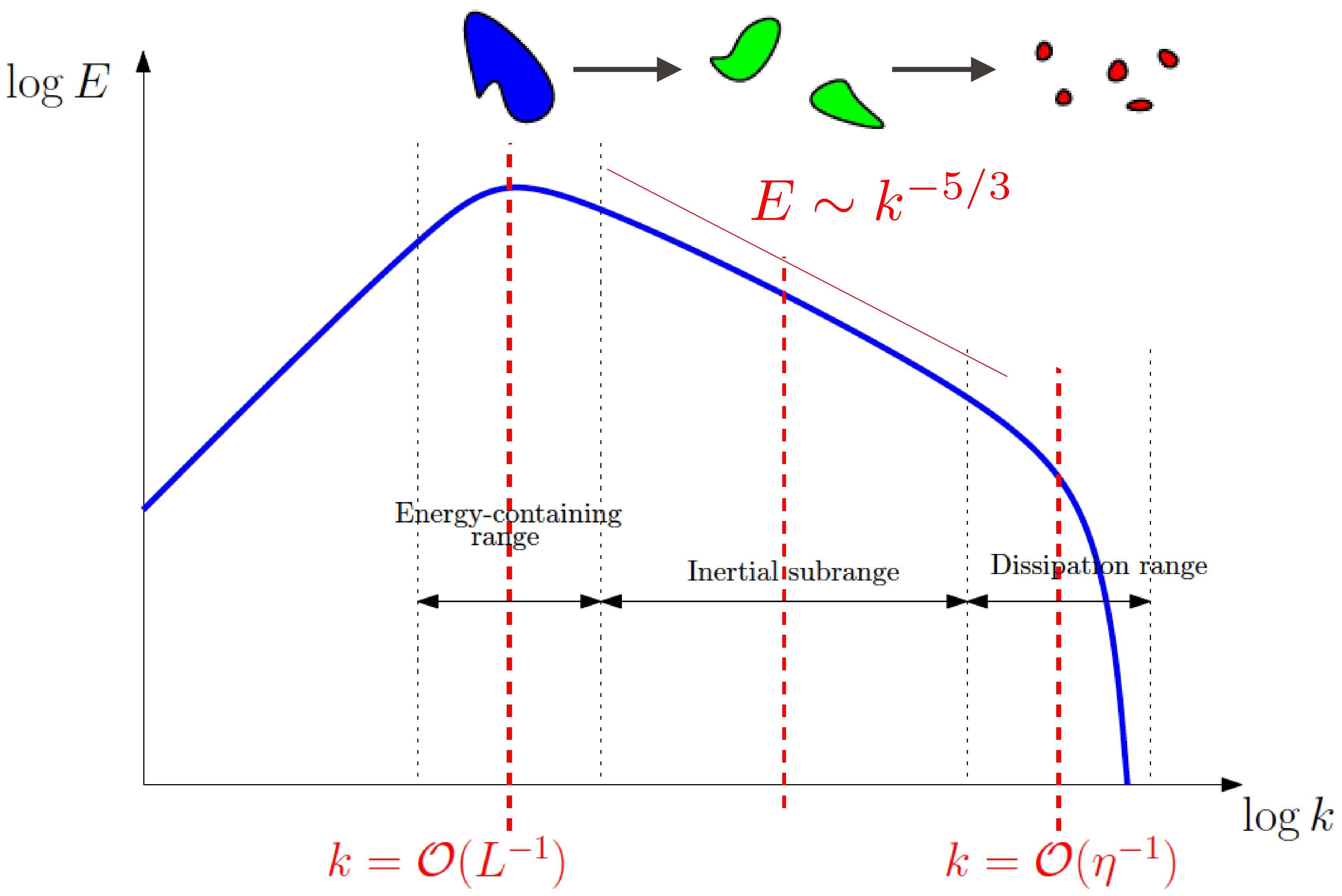


Direct numerical simulation (DNS)

- Resolve all scales, all the way down to Kolmogorov scale. No modeling.
- Assessment:
  - Highest possible level of description (all scales are resolved, all orders of the statistical moments can be computed)
  - Complete by nature (no flow-specific parameter)
  - Prohibitive cost at high *Re*
  - Range of applicability: theoretically unlimited
  - Accuracy: theoretically unlimited (practically limited by the precision of the numerical method)

Large-eddy simulations (LES)

- Compute explicitly large (energy-containing) eddies. Their dynamics is flow and geometry-dependent.
- Model the effect of small (dissipative) eddies. They have a more isotropic and flow-independent dynamics → easier to find a universal model.
- Large savings compared to DNS (which spends most computational effort on smallest dissipative scales).



Numerical Flow Simulation

Large-eddy simulations (LES)

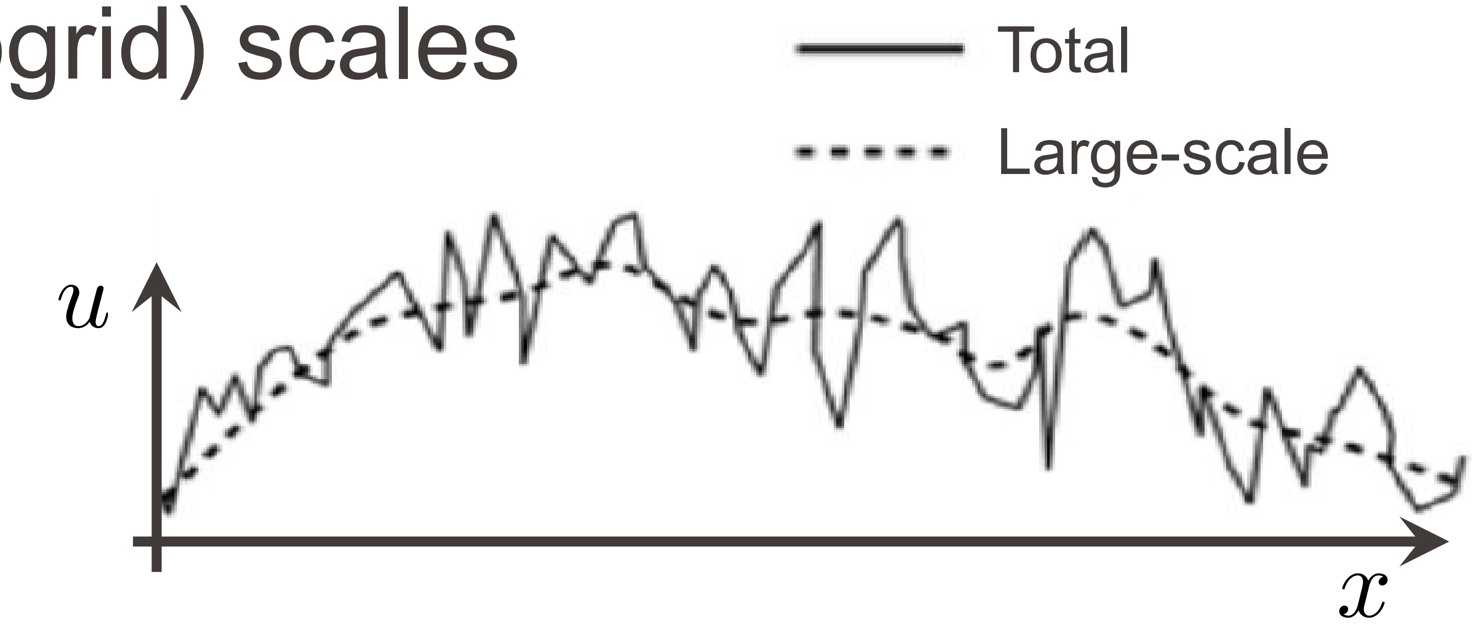
- Two main steps:
  1. Choice of a filtering operation to separate resolved scales and modeled scales
     → spatial filtering of the Navier-Stokes equations.
  2. Choice of a closure model: subgrid-scale (SGS) modeling.

1. Spatial filtering:

- Total flow = filtered (resolved) scales + residual (subgrid) scales

$$\mathbf{u}(\mathbf{x},t) = \overline{\mathbf{u}}(\mathbf{x},t) + \mathbf{u}^{\dagger}(\mathbf{x},t)$$

$$\overline{\mathbf{u}}(\mathbf{x},t) = \int \mathbf{u}(\mathbf{x}',t)G(\mathbf{x}-\mathbf{x}')d\mathbf{x}'$$



- Filtering = convolution in physical space
  = multiplication in spectral space

  G: filter, or convolution kernel. Many types: sharp, Gaussian... Determines the size of the resolved eddies.

Large-eddy simulations (LES)

1. **Filtered** NS eqs. (incompressible): eqs. for the resolved, large-scale eddies

$$\nabla \cdot \overline{\mathbf{u}} = 0, \qquad \frac{\partial \overline{\mathbf{u}}}{\partial t} + (\overline{\mathbf{u}} \cdot \nabla)\overline{\mathbf{u}} = -\frac{1}{\rho}\nabla \overline{p} + \nu \nabla^2 \overline{\mathbf{u}} - \boxed{\nabla \cdot \boldsymbol{\tau}}$$

- Additional term: **subgrid-scale (SGS) stress tensor** $\boldsymbol{\tau}_{ij} = \overline{u_i u_j} - \overline{u}_i \overline{u}_j$
- Effect of the small eddies on the dynamics of the resolved, large-scale eddies.
- **Closure problem**: SGS stress unknown (non-resolved terms)! Must be modeled.

2. **SGS modeling**

- Most common choice: eddy viscosity model. SGS stress proportional to rate of deformation of the resolved flow, and same coefficient of proportionality for all components.

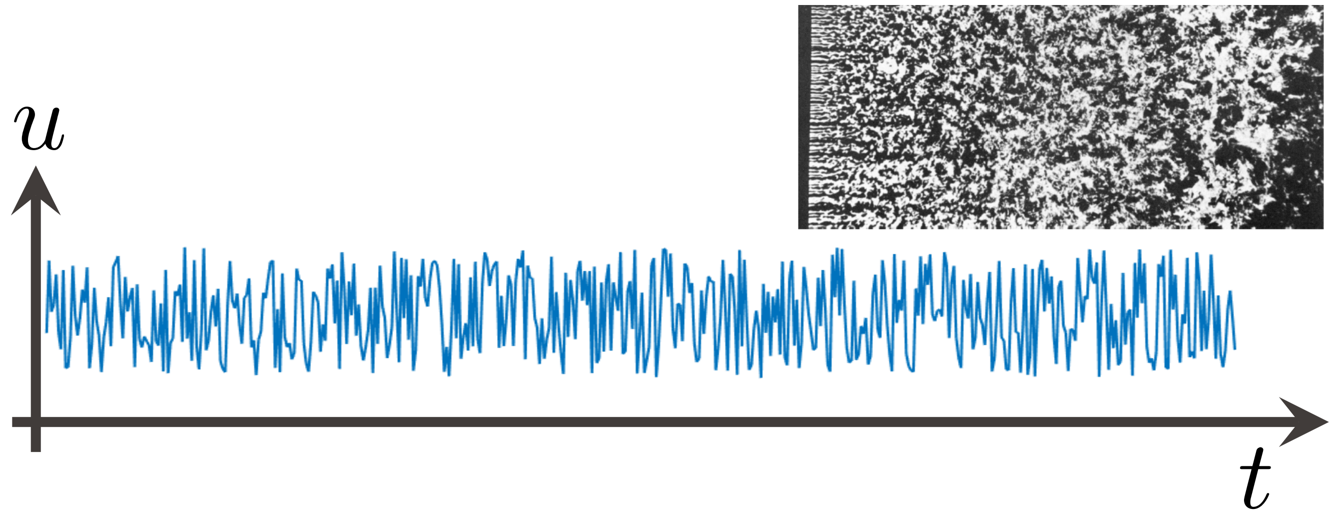
$$\boldsymbol{\tau}_{ij} - \frac{1}{3}\boldsymbol{\tau}_{kk}\delta_{ij} = -\nu_{SGS}\left(\nabla \overline{\mathbf{u}} + \nabla \overline{\mathbf{u}}^T\right)_{ij}$$

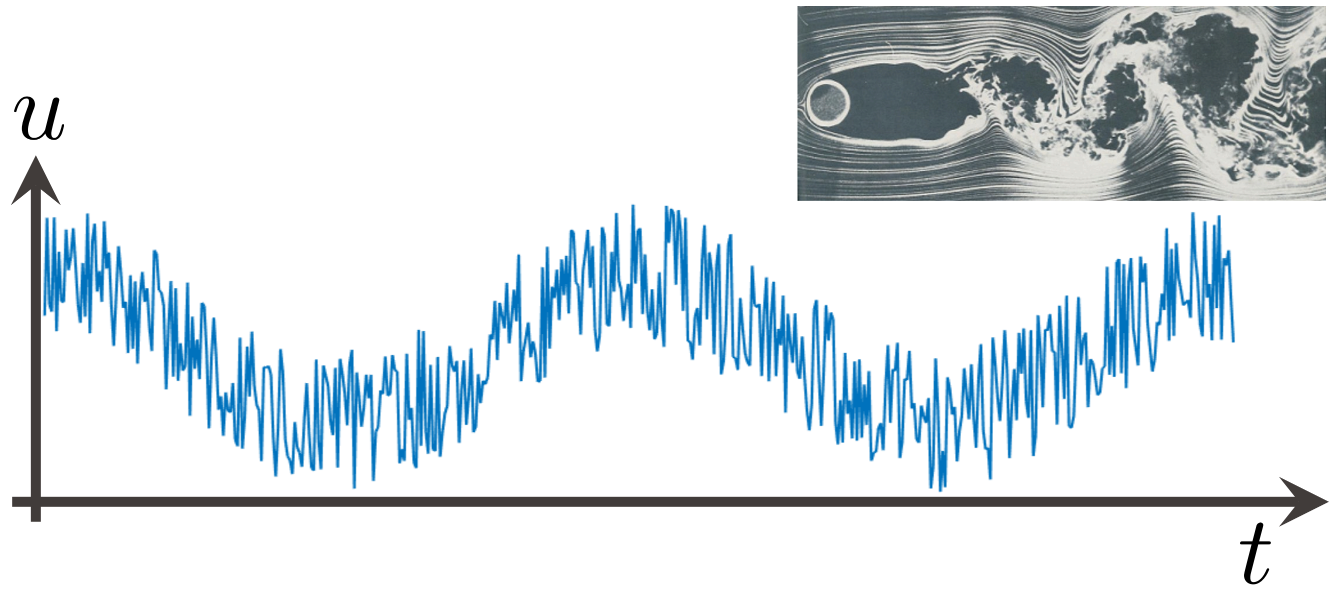
ν_{SGS} : eddy viscosity [m^2/s]

- Many SGS models proposed for eddy viscosity.
For example: Smagorinsky-Lilly, dynamic Smagorinsky-Lilly, wall-adapted local eddy-viscosity (WALE), dynamic kinetic energy subgrid-scale (all available in Fluent).

Reynolds-averaged Navier-Stokes (RANS) equations

- Reynolds decomposition
  - Total flow = mean flow (ensemble average) + fluctuations





  - Averaging properties:

$$\overline{f'} = 0 \qquad \overline{\overline{f}g} = \overline{f}\,\overline{g}$$

$$\overline{\overline{f}g'} = 0 \qquad \overline{\frac{\partial f}{\partial x}} = \frac{\partial \overline{f}}{\partial x} \qquad \overline{\frac{\partial f}{\partial t}} = \frac{\partial \overline{f}}{\partial t}$$

- Incompressible NS equations
  - Continuity eq.: $\overline{\nabla \cdot \mathbf{u} = 0} \longrightarrow \nabla \cdot \overline{\mathbf{u}} = 0$ and $\nabla \cdot \mathbf{u}' = 0$

    The mean flow and fluctuations are incompressible.

Reynolds-averaged Navier-Stokes (RANS) equations

- Incompressible NS equations
  - Momentum eq.:

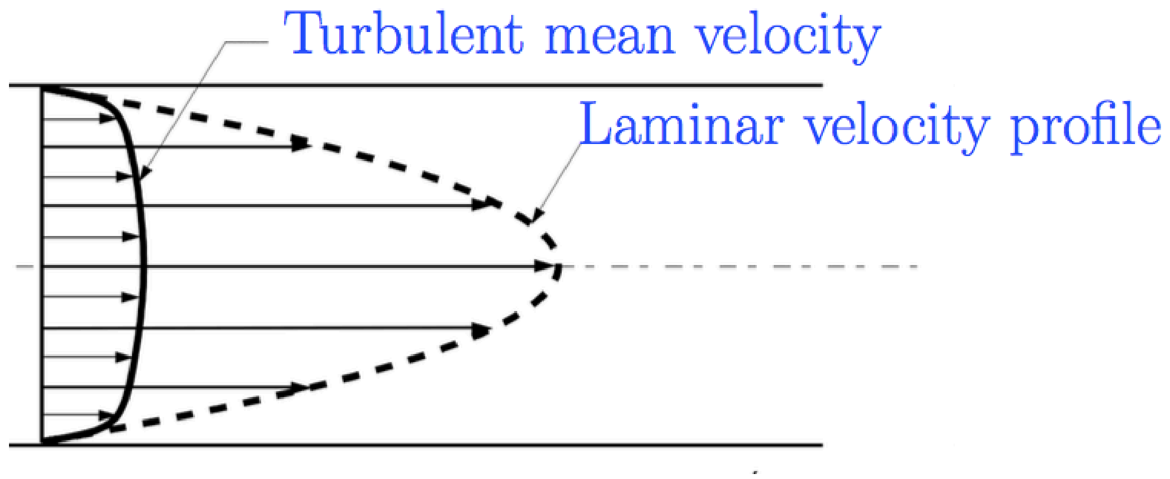
$$\overline{\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u}\cdot\nabla)\mathbf{u} = -\frac{1}{\rho}\nabla p + \nu\nabla^2\mathbf{u}}$$

$$\overline{\frac{\partial(\overline{\mathbf{u}}+\mathbf{u}')}{\partial t} + ((\overline{\mathbf{u}}+\mathbf{u}')\cdot\nabla)(\overline{\mathbf{u}}+\mathbf{u}') = -\frac{1}{\rho}\nabla(\overline{p}+p') + \nu\nabla^2(\overline{\mathbf{u}}+\mathbf{u}')}$$

$$\frac{\partial\overline{\mathbf{u}}}{\partial t} + \overline{(\overline{\mathbf{u}}\cdot\nabla)\overline{\mathbf{u}}} + \cancel{\overline{(\overline{\mathbf{u}}\cdot\nabla)\mathbf{u}'}} + \cancel{\overline{(\mathbf{u}'\cdot\nabla)\overline{\mathbf{u}}}} + \overline{(\mathbf{u}'\cdot\nabla)\mathbf{u}'} = -\frac{1}{\rho}\nabla\overline{p} + \nu\nabla^2\overline{\mathbf{u}}$$

$$\frac{\partial\overline{\mathbf{u}}}{\partial t} + (\overline{\mathbf{u}}\cdot\nabla)\overline{\mathbf{u}} = -\frac{1}{\rho}\nabla\overline{p} + \nu\nabla^2\overline{\mathbf{u}} \overbrace{\boxed{-\,\overline{(\mathbf{u}'\cdot\nabla)\mathbf{u}'}}}^{=\,\nabla\cdot\boldsymbol{\tau}}$$

  - RANS eqs. for the mean flow: almost like NS eqs., but divergence of an additional stress, the **Reynolds stress** tensor $\boldsymbol{\tau} = -\overline{\mathbf{u}'\mathbf{u}'}$ (variance of fluctuations, 6 independent components). Makes all the difference!



  - **Closure problem**: Reynolds stress unknown! More unknown than equations. To solve the RANS eqs., must **model** the Reynolds stress.

Reynolds-averaged Navier-Stokes (RANS) equations

- Must model the Reynolds stress: $\boldsymbol{\tau} = -\overline{\mathbf{u}'\mathbf{u}'} = ?$
- Several types of turbulence models of increasing complexity
  - Turbulent viscosity models:
    - 0-equation models (algebraic models): mixing length models
    - 1-equation models: Spalart-Allmaras, turbulent kinetic energy model
    - 2-equation models: k-ε, k-ω models
  - Reynolds-stress eq. models (RSM): 7 additional transport eqs.

More accurate or general, but more expensive

- *"Essentially, all models are wrong but some are useful."* (G. Box)
- *"Models of turbulence are inevitably incomplete."* (S. Pope)

RANS + turbulent viscosity models

- To close the RANS equations, assume that the Reynolds stress is related to the mean flow, i.e. $\boldsymbol{\tau} = -\overline{\mathbf{u}'\mathbf{u}'}$ can be expressed as a function of $\overline{\mathbf{u}}$.
- **Boussinesq hypothesis** (1877): Reynolds stress proportional to mean rate of deformation, and same coefficient of proportionality for all components

$$\boldsymbol{\tau} + \frac{2k}{3}\mathbf{I} = \nu_t \left(\nabla\overline{\mathbf{u}} + \nabla\overline{\mathbf{u}}^T \right)$$

  where $k = \frac{1}{2}\left(\overline{u'^2} + \overline{v'^2} + \overline{w'^2}\right)$ is the turbulent kinetic energy (TKE) [m^2/s^2],

  and ν_t is the turbulent viscosity (eddy viscosity) [m^2/s].

- Analogy with viscous stress (proportional to rate of deformation) and molecular viscosity:

$$\boldsymbol{\sigma} + p\mathbf{I} = \mu \left(\nabla\mathbf{u} + \nabla\mathbf{u}^T \right)$$

- Central issue: how to determine ν_t? Specific to each model (mixing length, Spalart-Allmaras, k-ε, k-ω…)

RANS + turbulent viscosity models

- To close the RANS equations, assume that the Reynolds stress is related to the mean flow, i.e. $\boldsymbol{\tau} = -\overline{\mathbf{u}'\mathbf{u}'}$ can be expressed as a function of $\overline{\mathbf{u}}$.
- **Boussinesq hypothesis** (1877): Reynolds stress proportional to mean rate of deformation, and same coefficient of proportionality for all components

$$\boldsymbol{\tau} + \frac{2k}{3}\mathbf{I} = \nu_t\left(\nabla\overline{\mathbf{u}} + \nabla\overline{\mathbf{u}}^T\right)$$

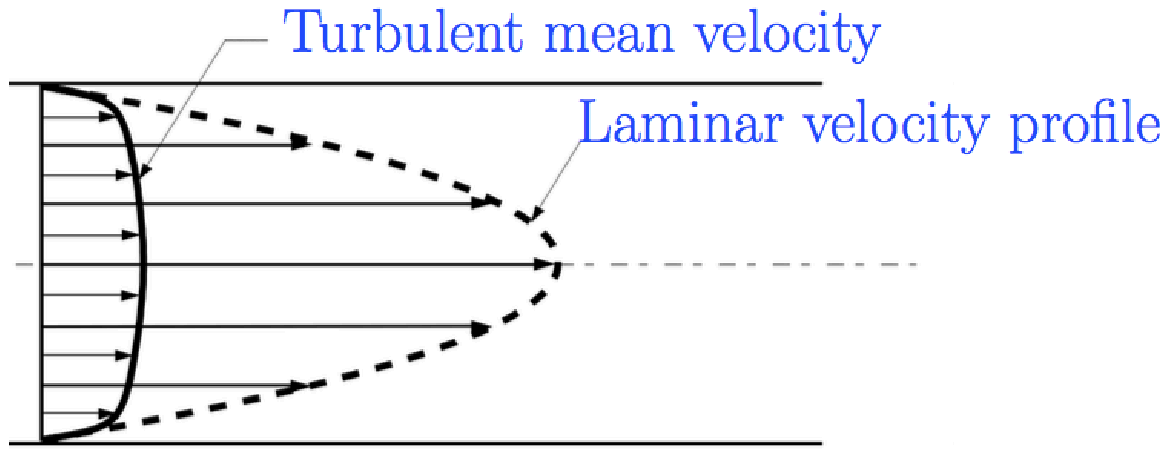
- With this hypothesis, the RANS eqs. for the mean flow

$$\frac{\partial\overline{\mathbf{u}}}{\partial t} + (\overline{\mathbf{u}}\cdot\nabla)\overline{\mathbf{u}} = -\frac{1}{\rho}\nabla\overline{p} + \nu\nabla^2\overline{\mathbf{u}} \overset{= \nabla\cdot\boldsymbol{\tau}}{\boxed{-\overline{(\mathbf{u}'\cdot\nabla)\mathbf{u}'}}}$$

are closed:

$$\frac{\partial\overline{\mathbf{u}}}{\partial t} + (\overline{\mathbf{u}}\cdot\nabla)\overline{\mathbf{u}} = -\frac{1}{\rho}\nabla\left(\overline{p} + \frac{2k}{3}\right) + (\nu+\nu_t)\nabla^2\overline{\mathbf{u}}$$

- Note: turbulent viscosity must depend on space, $\nu_t(\mathbf{x})$ (otherwise, RANS eqs. would simply be equal to NS eqs. with a different viscosity/different *Re*, which is not true).



RANS + turbulent viscosity models

Boussinesq hypothesis:

$$\boldsymbol{\tau} + \frac{2k}{3}\mathbf{I} = \nu_t \left(\nabla \overline{\mathbf{u}} + \nabla \overline{\mathbf{u}}^T\right)$$

- Zero-equation models (algebraic models): mixing length models
  - From dimensional analysis, turbulent viscosity [m^2/s] = length * velocity: $\nu_t \sim l_t u_t$
  - Assume strong connection between mean flow and behavior of largest (energy-containing) eddies → build velocity scale on mean gradients:

$$l_t = l_m$$

Length scale = "mixing length", to be specified (flow-dependent)

$$u_t = l_m ||\nabla \overline{\mathbf{u}}|| = l_m \left(\sum_{i,j} \left(\frac{\partial \overline{u_i}}{\partial x_j} \right)^2 \right)^{1/2} \longrightarrow \nu_t = l_m^2 ||\nabla \overline{\mathbf{u}}||$$

Velocity scale

  - Simple 2D flows: only one significant mean gradient: $u_t = l_m \left| \frac{\partial \overline{u}}{\partial y} \right| \longrightarrow \nu_t = l_m^2 \left| \frac{\partial \overline{u}}{\partial y} \right|$
    + only one significant Reynolds stress:

$$\tau_{xy} = \tau_{yx} = -\overline{u'v'} = \nu_t \frac{\partial \overline{u}}{\partial y} = l_m^2 \left| \frac{\partial \overline{u}}{\partial y} \right| \frac{\partial \overline{u}}{\partial y}$$

RANS + turbulent viscosity models

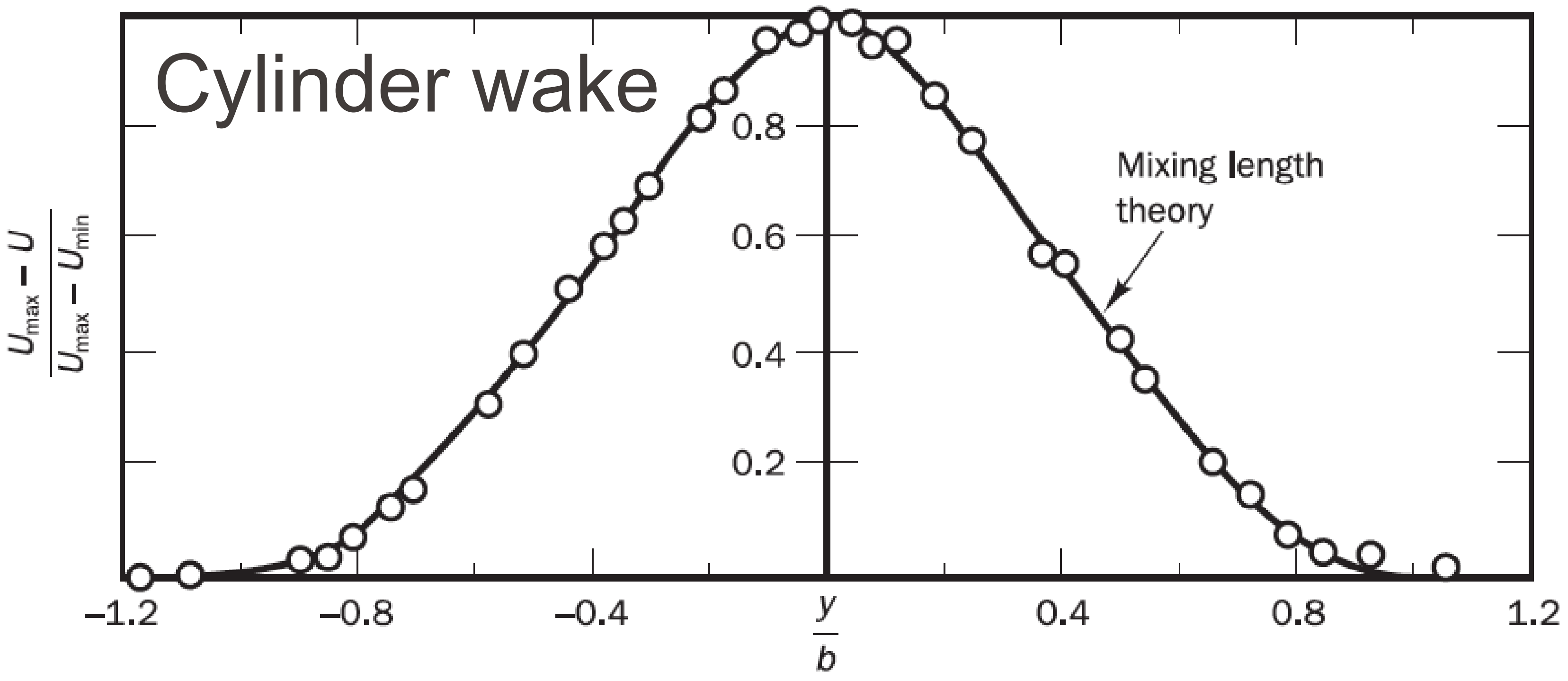
Boussinesq hypothesis:

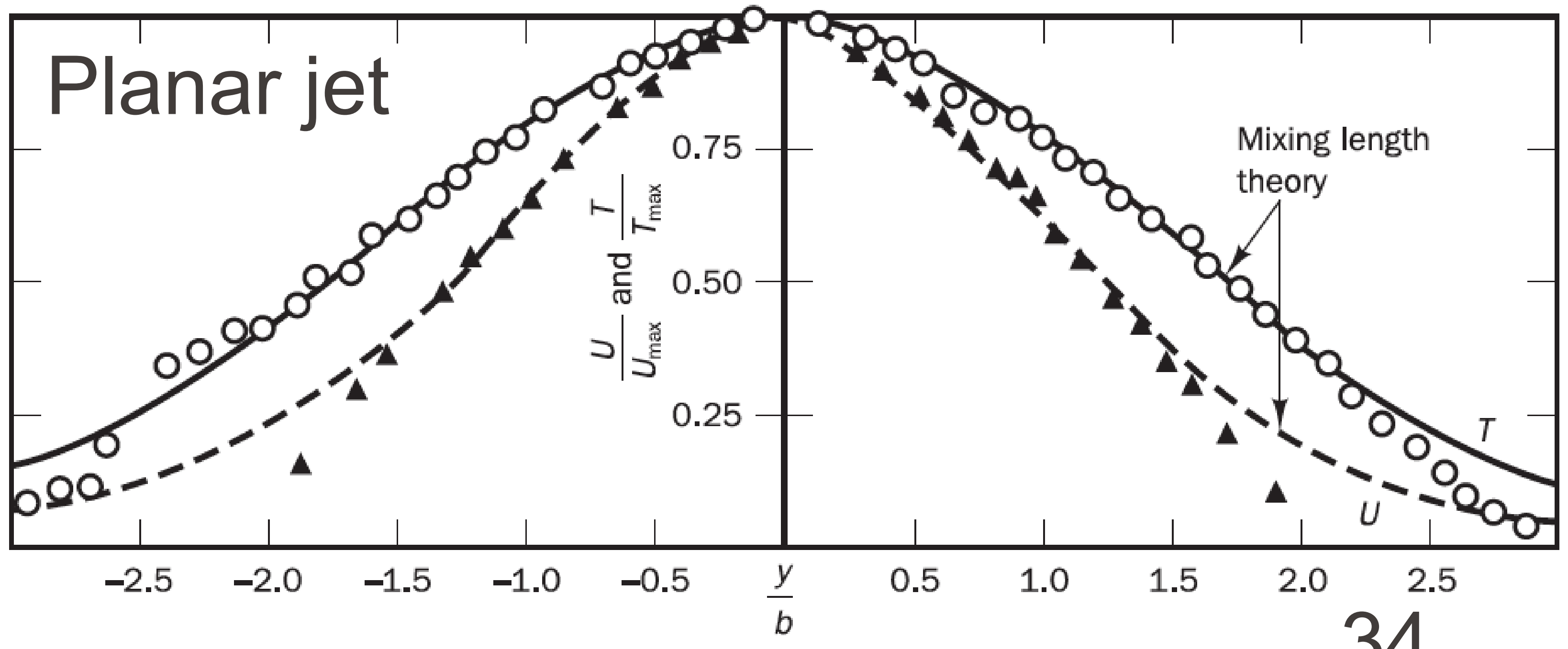
$$\boldsymbol{\tau} + \frac{2k}{3}\mathbf{I} = \nu_t \left(\nabla\bar{\mathbf{u}} + \nabla\bar{\mathbf{u}}^T \right)$$

- Zero-equation models (algebraic models): mixing length models
  - Easy to implement, and computationally cheap.
  - Works well only in simple 2D flows: mixing layers, wakes, jets, boundary layers… Cannot describe flows with separation/recirculation.

Flow	Mixing length ℓ_m	L
Mixing layer	$0.07L$	Layer width
Jet	$0.09L$	Jet half width
Wake	$0.16L$	Wake half width
Axisymmetric jet	$0.075L$	Jet half width
Boundary layer ($\partial p/\partial x = 0$)		
viscous sub-layer and	$\kappa y[1 - \exp(-y^+/26)]$	
log-law layer ($y/L \le 0.22$)		Boundary layer
outer layer ($y/L \ge 0.22$)	$0.09L$	thickness
Pipes and channels		Pipe radius or
(fully developed flow)	$L[0.14 - 0.08(1 - y/L)^2 - 0.06(1 - y/L)^4]$	channel half width

Rodi, 1980





Schlichting, 1979

Numerical Flow Simulation

RANS + turbulent viscosity models

Boussinesq hypothesis:

$$\boldsymbol{\tau} + \frac{2k}{3}\mathbf{I} = \nu_t \left(\nabla\overline{\mathbf{u}} + \nabla\overline{\mathbf{u}}^T\right)$$

- One-equation models: Spalart-Allmaras model
  - Specifically designed for external aerodynamics (boundary layers with adverse pressure gradients)
  - Solve one transport equation for an "eddy viscosity parameter" $\tilde{\nu}$ [m²/s]

$$\underbrace{\frac{\partial(\rho\tilde{\nu})}{\partial t}}_{\text{rate of change}} + \underbrace{\nabla\cdot(\rho\tilde{\nu}\overline{\mathbf{u}})}_{\text{convection}} = \underbrace{\frac{1}{\sigma_\nu}\left\{\nabla\cdot\left[\rho(\nu+\tilde{\nu})\nabla\tilde{\nu}\right] + C_{b2}\rho(\nabla\tilde{\nu})^2\right\}}_{\text{molecular and turbulent diffusion}} + \underbrace{C_{b1}\rho\tilde{\nu}\tilde{\Omega}}_{\text{production}} - \underbrace{C_{w1}\rho\left(\frac{\tilde{\nu}}{\kappa y}\right)^2 f_w}_{\text{dissipation}}$$

y : distance to closest wall

$\sigma_\nu, C_{b2}, C_{b1}, C_{w1}, \kappa$: empirical constants (flow-independent)

$\tilde{\Omega}, f_w$: functions containing other empirical constants

  - Then, turbulent viscosity obtained as: $\nu_t = \tilde{\nu} f_{\nu 1}$

$f_{\nu 1}$: function that tends to 1 for large *Re*

RANS + turbulent viscosity models

Boussinesq hypothesis:

$$\boldsymbol{\tau} + \frac{2k}{3}\mathbf{I} = \nu_t \left(\nabla\overline{\mathbf{u}} + \nabla\overline{\mathbf{u}}^T\right)$$

- One-equation models: turbulent kinetic energy model
  - Improvement on mixing length model
  - Turbulent viscosity: still $\nu_t \sim l_t u_t$, and $l_t = l_m$, but build velocity scale as $u_t = c\sqrt{k}$ with *c* a constant and *k* the turbulent kinetic energy [m²/s²]

$$\longrightarrow \quad \nu_t = c l_m \sqrt{k}$$

  - An equation for *k* can be derived rigorously from NS (same method as for RANS eqs.):

$$\underbrace{\frac{\partial(\rho k)}{\partial t}}_{\text{rate of change}} + \underbrace{\nabla\cdot(\rho k\overline{\mathbf{u}})}_{\text{convection}} = \underbrace{\nabla\cdot\left[\boxed{-\overline{p'\mathbf{u}'} - \rho\overline{\mathbf{u}'^2\mathbf{u}'}} + \rho\nu\nabla k\right]}_{\text{diffusion}} + \underbrace{\rho\boldsymbol{\tau}:\nabla\overline{\mathbf{u}}}_{\text{production}} - \underbrace{\boxed{\rho\nu\overline{||\nabla\mathbf{u}'||^2}}}_{\text{dissipation}}$$

Note (see later): $\epsilon = \nu\overline{||\nabla\mathbf{u}'||^2}$ is the rate of dissipation of TKE per unit mass [m²/s³]

  - Again, **closure problem**: unknown diffusion and dissipation terms.

RANS + turbulent viscosity models

Boussinesq hypothesis:

$$\boldsymbol{\tau} + \frac{2k}{3}\mathbf{I} = \nu_t \left(\nabla\bar{\mathbf{u}} + \nabla\bar{\mathbf{u}}^T\right)$$

- One-equation models: turbulent kinetic energy model

$$\frac{\partial(\rho k)}{\partial t} + \nabla\cdot(\rho k\bar{\mathbf{u}}) = \nabla\cdot\left[-\overline{p'\mathbf{u}'} - \rho\overline{\mathbf{u}'^2\mathbf{u}'} + \rho\nu\nabla k\right] + \rho\boldsymbol{\tau}:\nabla\bar{\mathbf{u}} - \rho\nu\overline{||\nabla\mathbf{u}'||^2}$$

  - Solve a model transport equation for *k*:
    - On dimensional ground, write $\epsilon \sim U^3/L$ and choose $\epsilon = C_D\dfrac{k^{3/2}}{l_m}$ with C_D a constant.
    - Model diffusive flux as $\rho\dfrac{\nu_t}{\sigma_k}\nabla k$, with σ_k a constant.

$$\frac{\partial(\rho k)}{\partial t} + \nabla\cdot(\rho k\bar{\mathbf{u}}) = \nabla\cdot\left[\rho\frac{\nu_t}{\sigma_k}\nabla k\right] + \rho\boldsymbol{\tau}:\nabla\bar{\mathbf{u}} - \rho C_D\frac{k^{3/2}}{l_m}$$

  - Major drawback: still need to specify mixing length.

RANS + turbulent viscosity models

Boussinesq hypothesis:

$$\boldsymbol{\tau} + \frac{2k}{3}\mathbf{I} = \nu_t \left(\nabla\overline{\mathbf{u}} + \nabla\overline{\mathbf{u}}^T\right)$$

- Two-equation models: *k*-ε model (standard)
  - Improvement on one-equation TKE model (no need to specify a mixing length)
  - Turbulent viscosity: on dimensional ground $\nu_t \sim k^2/\epsilon \longrightarrow$ let $\nu_t = C_\mu \frac{k^2}{\epsilon}$
  - Solve 2 model transport equations for *k* (same as TKE: theoretical eq. + closure approx.) and for ε (empirical, but based on the analysis of homogeneous turbulence, turbulent free-shear flows, and wall-bounded turbulence):

$$\frac{\partial(\rho k)}{\partial t} + \nabla\cdot(\rho k\overline{\mathbf{u}}) = \nabla\cdot\left[\rho\frac{\nu_t}{\sigma_k}\nabla k\right] + \rho\boldsymbol{\tau} : \nabla\overline{\mathbf{u}} - \rho\epsilon$$

$$\frac{\partial(\rho \epsilon)}{\partial t} + \nabla\cdot(\rho \epsilon\overline{\mathbf{u}}) = \nabla\cdot\left[\rho\frac{\nu_t}{\sigma_\epsilon}\nabla \epsilon\right] + \rho C_{1\epsilon}\boldsymbol{\tau} : \nabla\overline{\mathbf{u}}\frac{\epsilon}{k} - \rho C_{2\epsilon}\frac{\epsilon^2}{k}$$

rate of change — convection — diffusion — production — dissipation

$\sigma_k, \sigma_\epsilon, C_{1\epsilon}, C_{2\epsilon}$ empirical constants (flow-independent)

  - Need special treatment at the wall (see later).
  - Well established, widely validated, well calibrated for internal flows. However, poor performance in unconfined flows, swirling/rotating flows, curved boundary layers.
  - Three versions in Fluent: standard (Launder & Spalding, 1974), RNG, realizable.

RANS + turbulent viscosity models

Boussinesq hypothesis:

$$\boldsymbol{\tau} + \frac{2k}{3}\mathbf{I} = \nu_t \left(\nabla\overline{\mathbf{u}} + \nabla\overline{\mathbf{u}}^T \right)$$

- Two-equation models: *k*-ω model (standard)
  - Instead of ε, use turbulence frequency $\omega = \epsilon/k$ [1/s] (not vorticity) as 2nd variable.
  - Turbulent viscosity: on dimensional ground, $\nu_t \sim k/\omega$
  - Solve 2 model transport equations for *k* (similar to that of *k*-ε) and for ω.
  - Unlike *k*-ε, no need for special treatment at the wall.
  - More widely applicable than *k*-ε. But problematic in some complex external flows.
  - Several versions in Fluent: standard (Wilcox, 1998), SST (Menter, 1992; combines *k*-ε in the freestream and standard *k*-ω near the wall), GEKO, BSL.

RANS + Reynolds-stress model (RSM)

Boussinesq hypothesis:

$$\boldsymbol{\tau} + \frac{2k}{3}\mathbf{I} = \nu_t \left(\nabla\overline{\mathbf{u}} + \nabla\overline{\mathbf{u}}^T\right)$$

- Reynolds-stress model
  - Don't use Boussinesq hypothesis (isotropic turbulent viscosity). Instead, solve explicitly transport eqs. for the 6 Reynolds stresses (+1 eq. for ε).
  - Can derive a rigorous eq. for $\boldsymbol{\tau} = -\overline{\mathbf{u}'\mathbf{u}'}$:

$$\underbrace{\frac{\partial \overline{u_i' u_j'}}{\partial t}}_{\text{rate of change}} + \underbrace{\overline{u_k}\partial_k \overline{u_i' u_j'}}_{\text{convection}} = \underbrace{-\left(\overline{u_i' u_k'}\partial_k \overline{u_j} + \overline{u_j' u_k'}\partial_k \overline{u_i}\right)}_{\text{production}} \underbrace{-\partial_k \boxed{\overline{u_i' u_j' u_k'}} + \nu \partial_{kk} \overline{u_i' u_j'}}_{\text{turbulent and molecular diffusion}}$$

$$\underbrace{-2\nu \boxed{\overline{\partial_k u_i' \partial_k u_j'}}}_{\text{dissipation}} \underbrace{-\frac{1}{\rho}\left(\boxed{\overline{u_i'\partial_j p'} + \overline{u_j'\partial_i p'}}\right)}_{\text{pressure-strain interaction}}$$

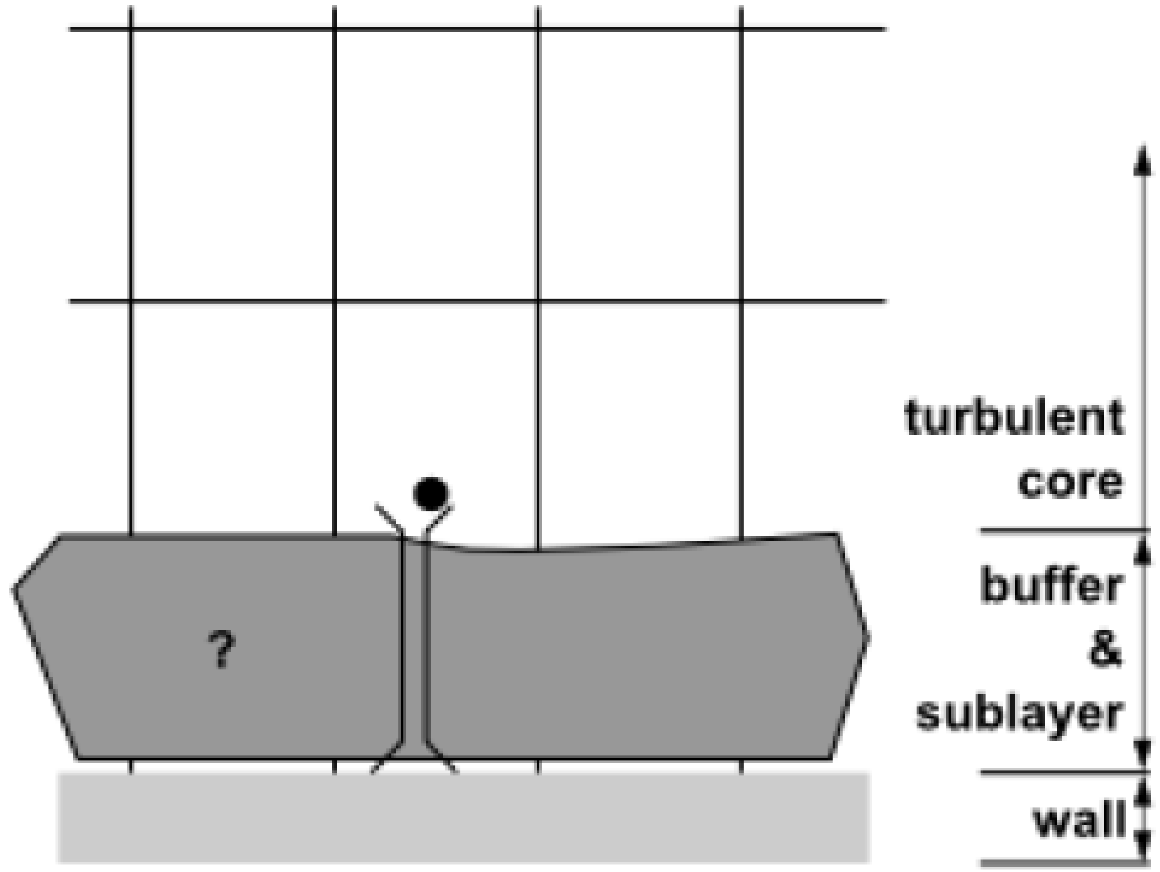
  - Again, **closure problem**: must model the terms with unknown correlations.
  - Allow for anisotropic turbulence. Accurate in simple and complex flows.
  - However, less validated than k-ε.

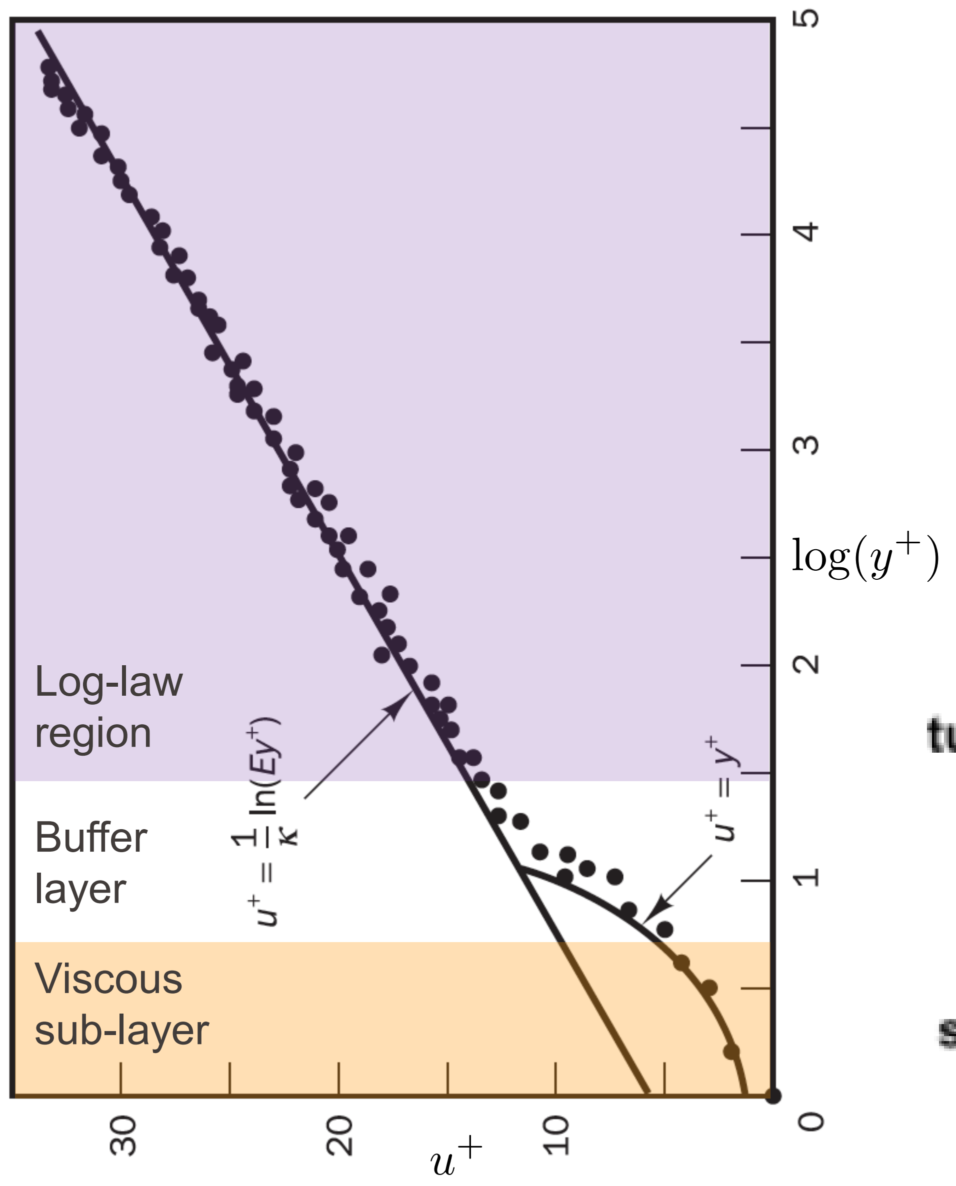
Near-wall treatment

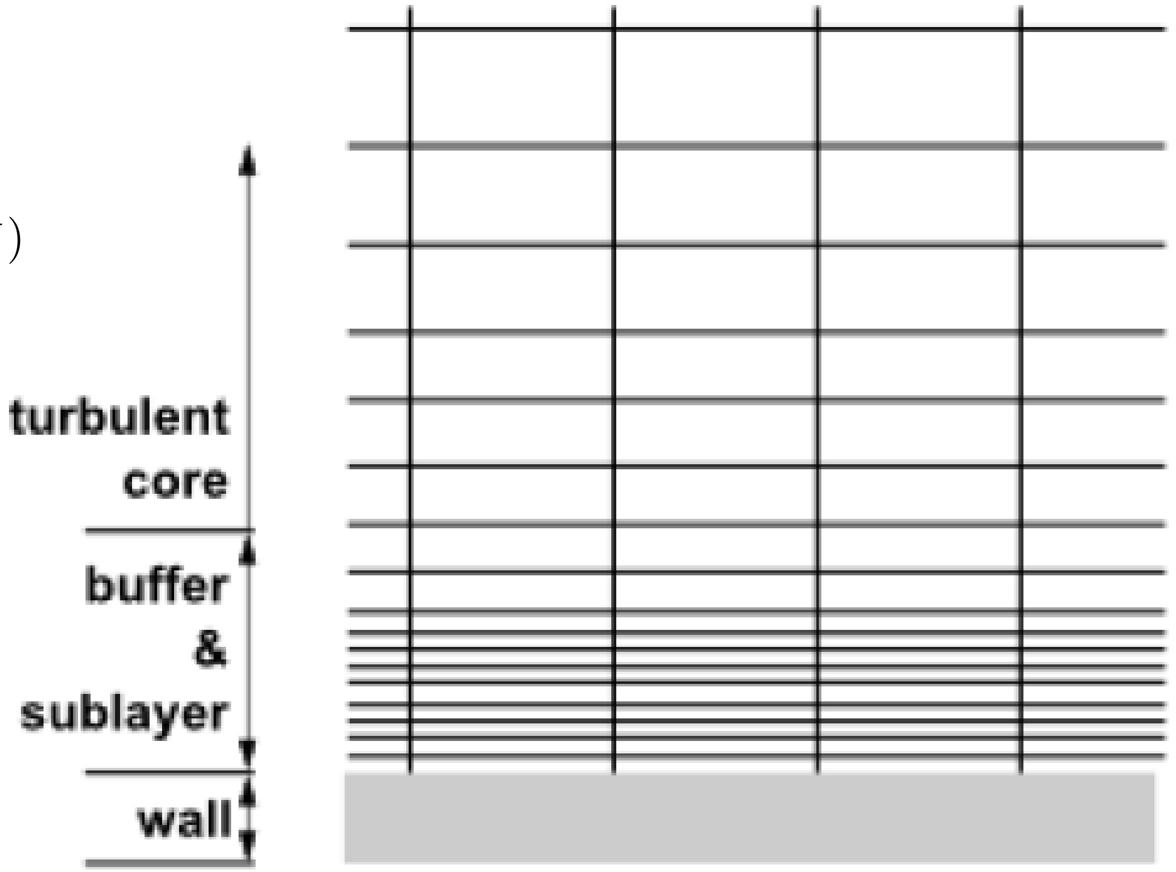
- Walls are the main source of mean vorticity and turbulence → strong impact on turbulence.
- Some hypotheses used to develop turbulence models are not valid near the walls → near-wall modeling significantly affects the accuracy of turbulent flow simulations.

Near-wall treatment

- Numerically, two approaches:







Wall functions:

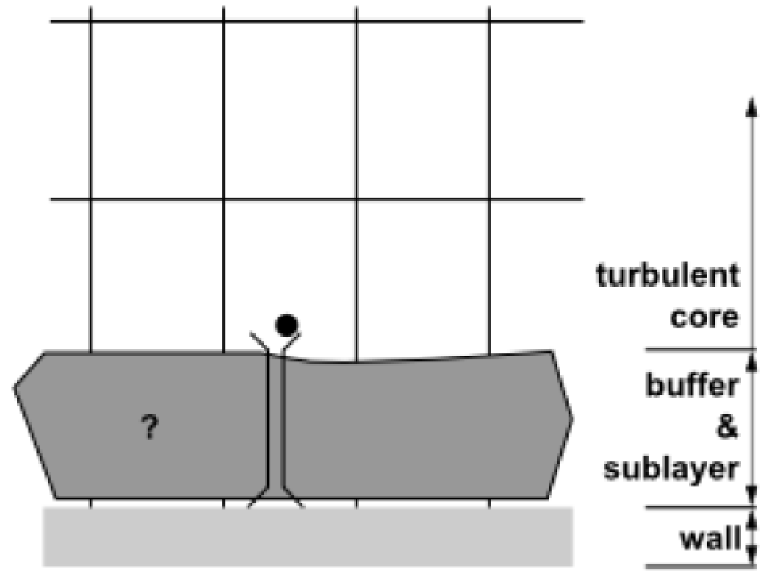
- Viscosity-affected region (viscous sub-layer + buffer layer) is not resolved, but bridged with semi-empirical “wall functions”.
- Outside this region, turbulent models can be used “as is”.
- Cheaper, but less accurate.

Near-wall models:

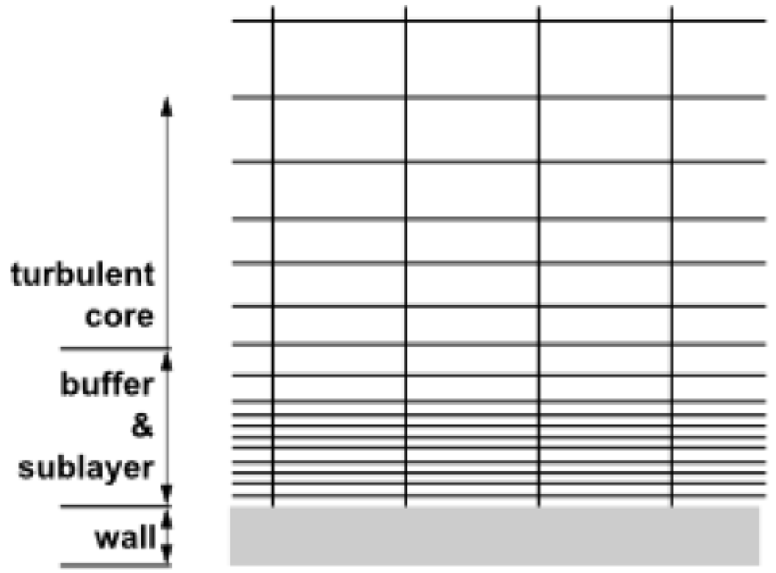
- Near-wall region is resolved all the way down to the wall. Requires a sufficiently fine mesh.
- Turbulence models must be modified to account for the presence of the wall
- More accurate, but more expensive.

Near-wall treatment

- **Wall function** approach:
  - For high-*Re* flows, this approach is economical, robust, reasonably accurate → practical option for industrial flows.
  - Inadequate where low-*Re* effects are important:
    - Hypotheses of wall functions breaks down (actual profile ≠ turbulent law of the wall).
    - Must use near-wall model approach instead.
  - First mesh cell center should be in the log layer $y^{+}>30$ (less true with more recent advanced wall functions; see Fluent Theory Guide).



- **Near-wall model** approach:
  - First mesh cell should be at $y^{+} \sim 1$. Mesh should have at least 10-20 cells in the boundary layer.



Numerical Flow Simulation

Near-wall treatment

- How to determine y^+?
  - Defined as distance to the wall normalized by flow quantities → in principle, need to know flow solution to determine y^+ values!

$$y^+ = \frac{y u_\tau}{\nu} \qquad \text{where} \quad u_\tau = \sqrt{\frac{\tau_w}{\rho}} = \sqrt{\nu \frac{\partial u_x}{\partial y}}$$

  - Estimation of y^+ for the first mesh cell:
    - trial and error (compute, remesh, recompute),
    - from experience,
    - based on simple analytic formula.
  - Online calculators based on formula for flat-plate boundary layer (only useful if flat plate is a good analogy for problem being solved). For example:
    - www.pointwise.com/yplus
    - www.cfd-online.com/Tools/yplus.php

Turbulence models: guidelines

1. Choosing a **turbulence model**
   - Depends on flow physics, accuracy required, computer resources available, etc.
   - Understand capabilities and limitations of turbulence model before using it.

   - For industrial applications: use RANS approach.
     - Aeronautical applications involving wall-bounded flows (boundary-layer flows): consider Spalart-Allmaras or k-ω.
     - “General” flow situations: consider k-ε or k-ω.
     - If low-*Re* effects, compressibility or shear flow spreading are important: consider k-ω.
     - More complex flows (e.g. streamline curvature, swirl, rotation, rapidly changing strain rate): consider RSM.

   - For detailed unsteady turbulent flow studies: use LES approach.

Turbulence models: guidelines

2. Choosing a **model variation**
   - For complex near-wall phenomena (e.g. separation from curved surface), use:
     - RNG or realizable (rather than standard) k-ε model,
     - SST (rather than standard) k-ω model.
   - If solution stability is a problem, use:
     - standard (rather than RNG or realizable) k-ε model,
     - standard (rather than SST) k-ω model.

3. Choosing a **near-wall treatment**
   - High-*Re* flows and/or limited computer resources: consider wall function.
   - For complex near-wall phenomena (e.g. separation from curved surface), use:
     - near-wall model approach (if practical),
     - non-equilibrium wall functions.

Turbulence models: guidelines

4. Choosing **model parameters**
   - Never change model constants, unless expert user.
   - If solution stability is a problem, consider:
     - decreasing under-relaxation factors for turbulence equations
     - using first-order solution procedure for turbulence equations
   - Include all model options that may be important in the flow (e.g. buoyancy, swirl, compressibility, thermal effects)

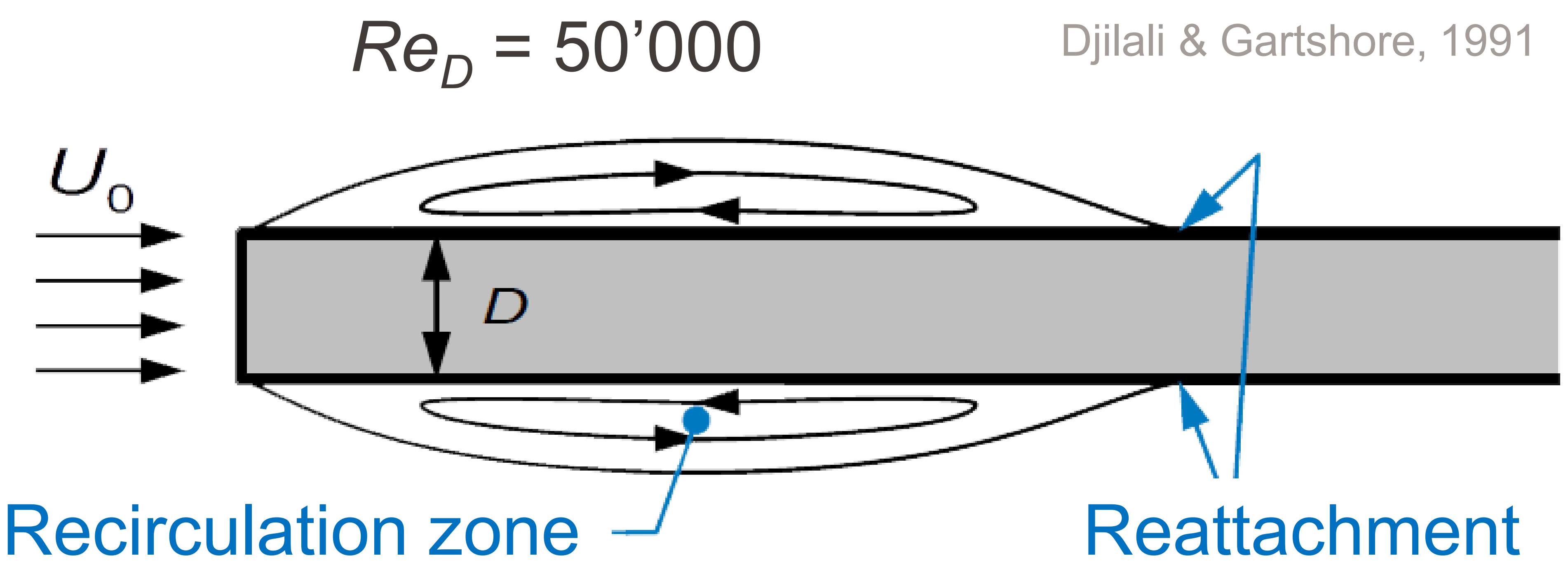
5. Choosing an appropriate **mesh**
   - LES: use fine mesh to resolve all scales down to smallest resolved eddies
   - RANS with near-wall model: set first mesh cell at y+ ~ 1
   - RANS with wall function: set first mesh cell at y^+ ~ 30

Turbulence models: guidelines

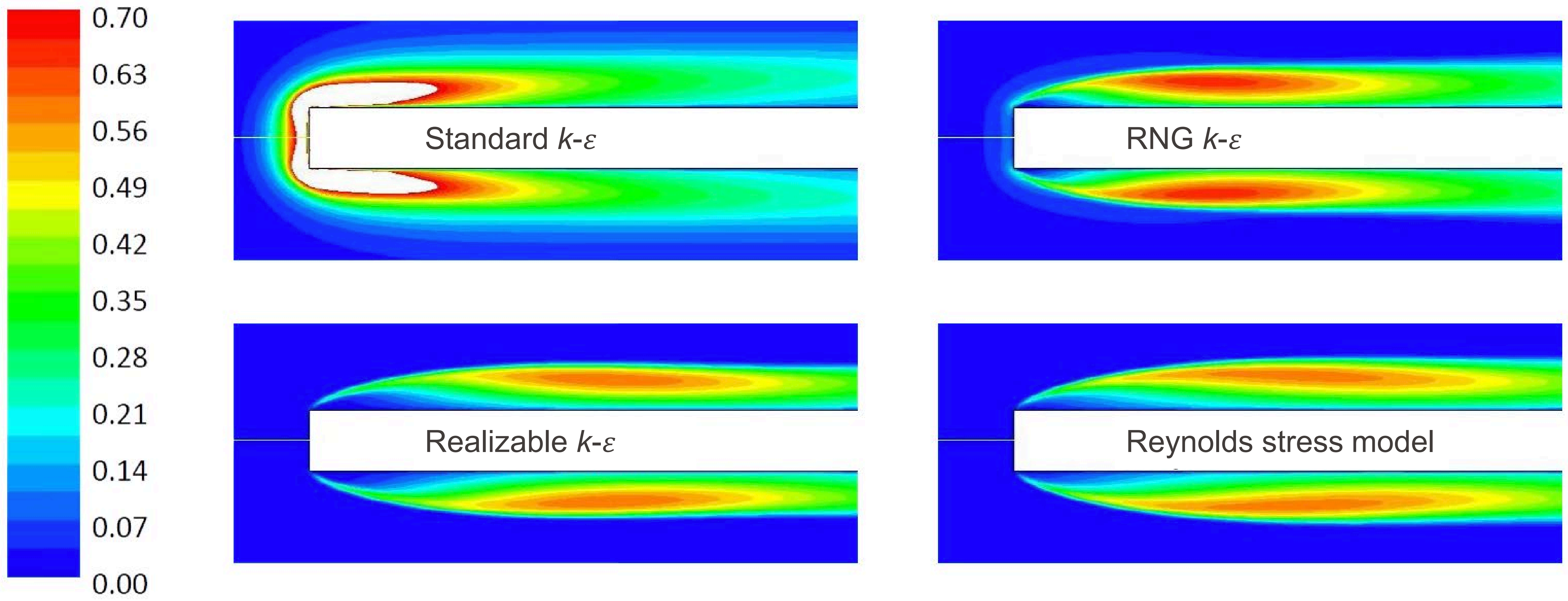
- **General comments**
  - Read Fluent Theory Guide and User’s Guide before applying turbulence models.
  - Study influence of turbulence model on flow solution.
  - Validate flow solution whenever possible (compare with experimental data).

Turbulence models: example

- Turbulent flow past a blunt flat plate.
- Four turbulence models considered.



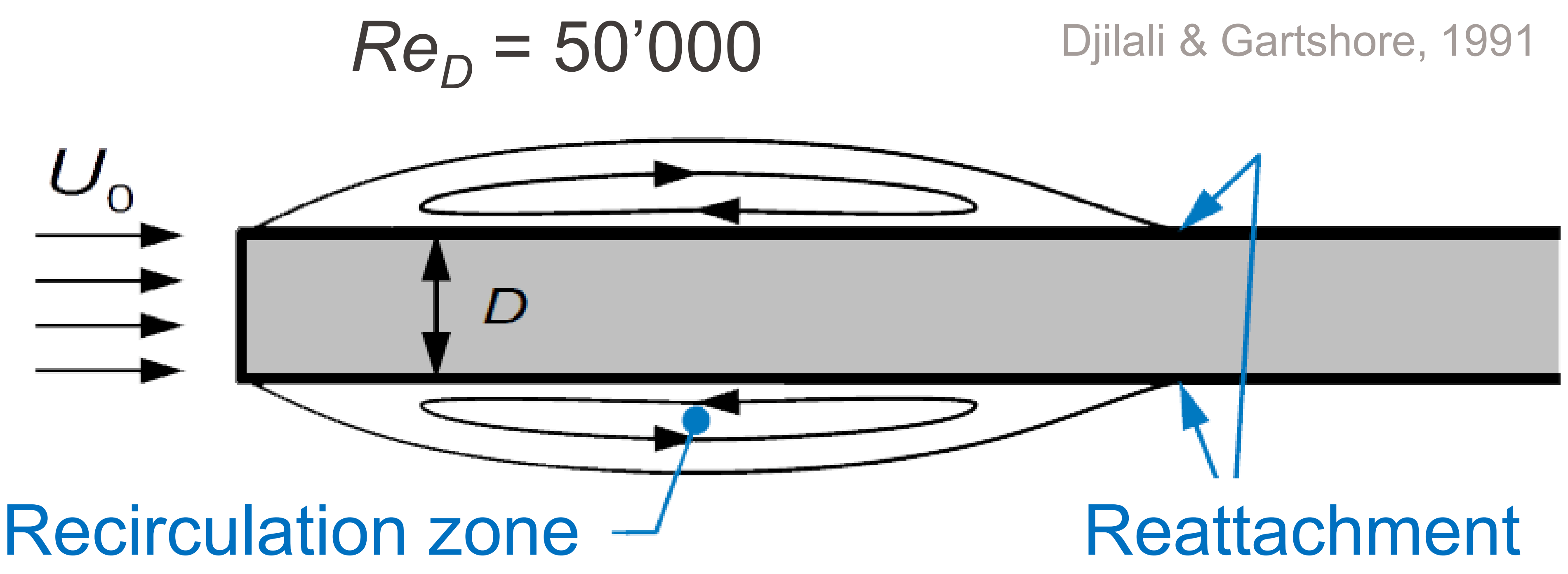
Turbulent kinetic energy:



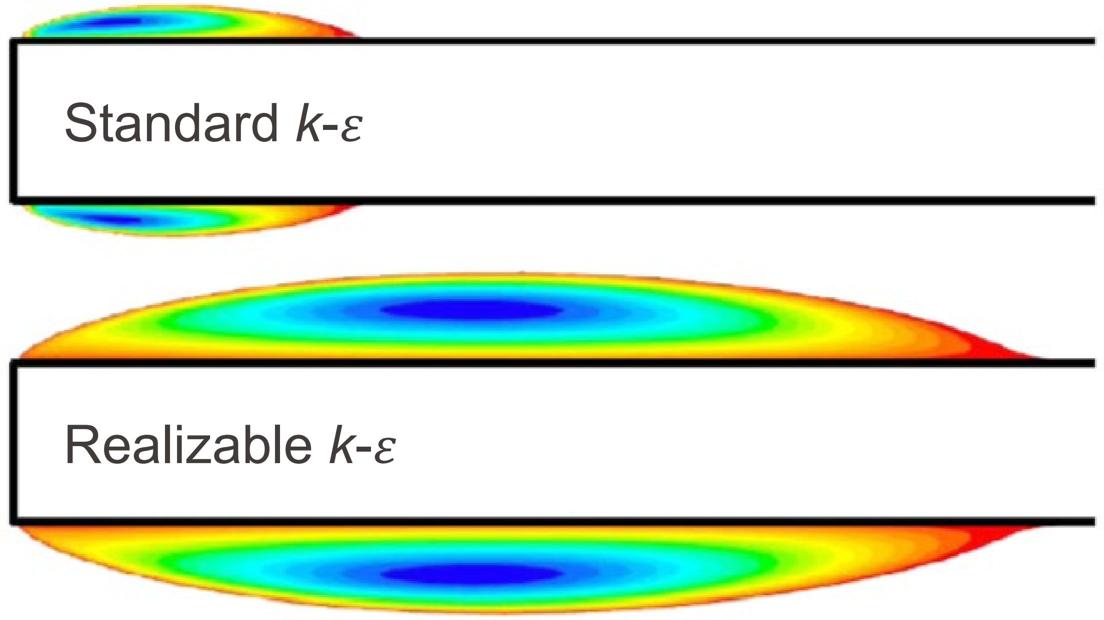
Turbulence models: example

Djilali & Gartshore, 1991

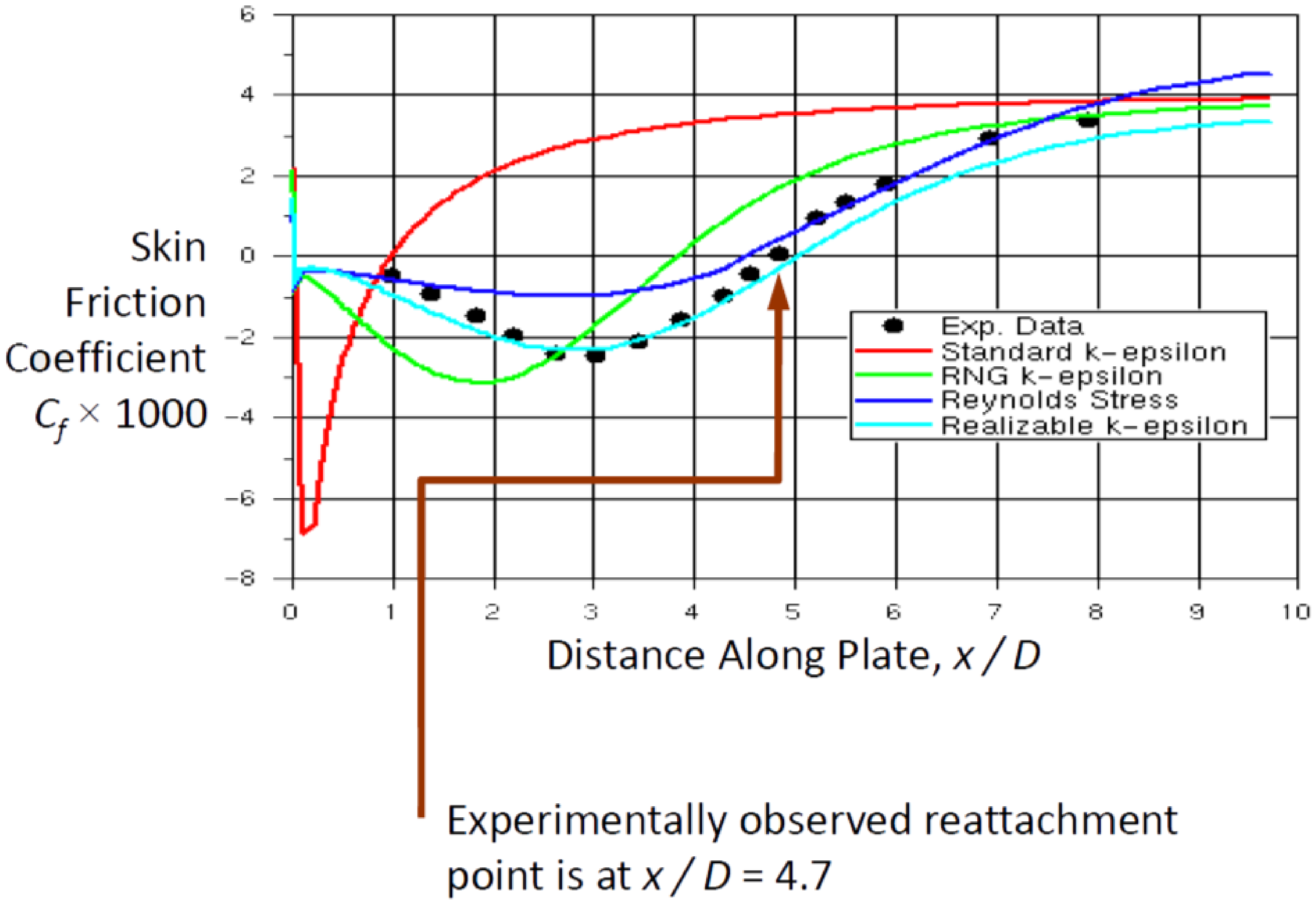
- Turbulent flow past a blunt flat plate.
- Four turbulence models considered.



Predicted recirculation zone:



Standard k-ε severely underpredicts the size of the recirculation zone.
Realizable k-ε is more accurate.



Numerical Flow Simulation

Summary

- Turbulent flows are complex and difficult to simulate numerically.
- Hierarchy of different approaches: DNS, LES, RANS.
- Modeling turbulent flow is difficult. Many different RANS models of varying complexity.
- Choice of wall treatment is important for accuracy and mesh size (y^+).
- Choice of most suitable turbulence model is generally difficult. Should be based on physical guidelines (see manual) and computer resources.
- Validation is particularly important to gauge accuracy of solution.

- For more details on turbulence and its modeling:
  - EPFL SGM Master course “Turbulence” (ME-467, T. Schneider)
  - *Turbulent Flows*, S. Pope

Appendix: derivation of some equations

- Equation for the **fluctuations**:
  - Subtract the RANS eq. (mean flow) from the NS eq. (total flow), and recall the Reynolds decomposition $\mathbf{u}(\mathbf{x},t) = \overline{\mathbf{u}}(\mathbf{x},t) + \mathbf{u}'(\mathbf{x},t)$:

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u}\cdot\nabla)\mathbf{u} = -\frac{1}{\rho}\nabla p + \nu\nabla^2\mathbf{u}$$

$$-\left(\frac{\partial \overline{\mathbf{u}}}{\partial t} + (\overline{\mathbf{u}}\cdot\nabla)\overline{\mathbf{u}} = -\frac{1}{\rho}\nabla \overline{p} + \nu\nabla^2\overline{\mathbf{u}} - \overline{(\mathbf{u}'\cdot\nabla)\mathbf{u}'}\right)$$

$$\frac{\partial \mathbf{u}'}{\partial t} + (\mathbf{u}\cdot\nabla)\mathbf{u} - (\overline{\mathbf{u}}\cdot\nabla)\overline{\mathbf{u}} = -\frac{1}{\rho}\nabla p' + \nu\nabla^2\mathbf{u}' + \overline{(\mathbf{u}'\cdot\nabla)\mathbf{u}'}$$

$$= \frac{\partial \mathbf{u}'}{\partial t} + (\mathbf{u}'\cdot\nabla)\overline{\mathbf{u}} + (\overline{\mathbf{u}}\cdot\nabla)\mathbf{u}' + (\mathbf{u}'\cdot\nabla)\mathbf{u}'$$

$$\Leftrightarrow \quad \frac{\partial \mathbf{u}'}{\partial t} + (\overline{\mathbf{u}}\cdot\nabla)\mathbf{u}' = -\frac{1}{\rho}\nabla p' + \nu\nabla^2\mathbf{u}' + \overline{(\mathbf{u}'\cdot\nabla)\mathbf{u}'} - (\mathbf{u}'\cdot\nabla)\overline{\mathbf{u}} - (\mathbf{u}'\cdot\nabla)\mathbf{u}'$$

Appendix: derivation of some equations

- Equation for the **turbulent kinetic energy**:
  - Start from the eq. for the fluctuations (previous slide):

$$\frac{\partial u'_i}{\partial t} + \overline{u}_j \partial_j u'_i = -\frac{1}{\rho}\partial_i p' + \nu \partial_{jj} u'_i + \overline{u'_j \partial_j u'_i} - u'_j \partial_j \overline{u_i} - u'_j \partial_j u'_i$$

  - Multiply each eq. *i* by *u'ᵢ*, sum the 3 eqs., and take the mean:

$$\overline{u'_i \frac{\partial u'_i}{\partial t}} + \overline{u'_i \overline{u}_j \partial_j u'_i} = -\frac{1}{\rho}\overline{u'_i \partial_i p'} + \nu \overline{u'_i \partial_{jj} u'_i} + \overline{u'_i \overline{u'_j \partial_j u'_i}} - \overline{u'_i u'_j \partial_j \overline{u_i}} - \overline{u'_i u'_j \partial_j u'_i}$$

  - Rearrange each term (use product rule and/or incompressibility where needed):
    - 1st term: $\overline{u'_i \frac{\partial u'_i}{\partial t}} = \frac{1}{2}\frac{\partial \overline{u'^2_i}}{\partial t}$
    - 2nd term: $\overline{u}_j \overline{u'_i \partial_j u'_i} = \frac{1}{2}\overline{u}_j \partial_j \overline{u'^2_i}$
    - 3rd term: $-\frac{1}{\rho}\overline{u'_i \partial_i p'} = -\frac{1}{\rho}\partial_i(\overline{u'_i p'})$
    - 4th term: $\nu \overline{u'_i \partial_{jj} u'_i} = \frac{1}{2}\nu \partial_{jj}\overline{u'^2_i} - \nu \overline{(\partial_j u'_i)^2}$

Appendix: derivation of some equations

- Equation for the **turbulent kinetic energy** (continued):
  - 5th term: $\overline{u_i'\overline{u_j'\partial_j u_i'}} = \overline{u_i'}\;\overline{u_j'\partial_j u_i'} = 0\cdot\overline{u_j'\partial_j u_i'} = 0$
  - 6th term: $-\overline{u_i'u_j'}\partial_j\overline{u_i}$
  - 7th term: $-\overline{u_i'u_j'\partial_j u_i'} = -\frac{1}{2}\overline{u_j'\partial_j({u_i'}^2)} = -\frac{1}{2}\partial_j\overline{u_j'{u_i'}^2}$
- Gather:

$$\frac{1}{2}\frac{\partial\overline{{u_i'}^2}}{\partial t} + \frac{1}{2}\overline{u}_j\partial_j\overline{{u_i'}^2} = -\frac{1}{\rho}\partial_i(\overline{u_i'p'}) + \frac{1}{2}\nu\partial_{jj}\overline{{u_i'}^2} - \nu\overline{(\partial_j u_i')^2} - \overline{u_i'u_j'}\partial_j\overline{u_i} - \frac{1}{2}\partial_j\overline{u_j'{u_i'}^2}$$

$$\frac{\partial k}{\partial t} + \overline{u}_j\partial_j k = -\frac{1}{\rho}\partial_j(\overline{u_j'p'}) + \nu\partial_{jj}k - \nu\overline{(\partial_j u_i')^2} - \overline{u_i'u_j'}\partial_j\overline{u_i} - \frac{1}{2}\partial_j\overline{u_j'{u_i'}^2}$$

$$\frac{\partial k}{\partial t} + \overline{u}_j\partial_j k = \partial_j\left(-\frac{1}{\rho}\overline{u_j'p'} + \nu\partial_j k - \frac{1}{2}\overline{u_j'{u_i'}^2}\right) - \nu\overline{(\partial_j u_i')^2} - \overline{u_i'u_j'}\partial_j\overline{u_i}$$

$$\frac{\partial k}{\partial t} + (\overline{\mathbf{u}}\cdot\nabla)k = \nabla\cdot\left(-\frac{1}{\rho}\overline{\mathbf{u}'p'} + \nu\nabla k - \frac{1}{2}\overline{{\mathbf{u}'}^2\mathbf{u}'}\right) - \nu\overline{||\nabla\mathbf{u}'||^2} + \boldsymbol{\tau}:\nabla\overline{\mathbf{u}}$$

Appendix: derivation of some equations

- Equation for the **Reynolds stresses**:
  - Same idea as for the turbulent kinetic energy (2 previous slides).
  - Start from the eq. for the fluctuations:

$$\frac{\partial u_i'}{\partial t} + \overline{u}_k \partial_k u_i' = \dots$$

  - Multiply by fluctuations and combine to form the quantity

$$u_j' \left(\frac{\partial u_i'}{\partial t} + \overline{u}_k \partial_k u_i' \right) + u_i' \left(\frac{\partial u_j'}{\partial t} + \overline{u}_k \partial_k u_j' \right) = \dots$$

  - Take the mean:

$$\overline{u_j' \left(\frac{\partial u_i'}{\partial t} + \overline{u}_k \partial_k u_i' \right)} + \overline{u_i' \left(\frac{\partial u_j'}{\partial t} + \overline{u}_k \partial_k u_j' \right)} = \dots$$

  - Rearrange:

$$\frac{\partial \overline{u_i' u_j'}}{\partial t} + \overline{u}_k \partial_k \overline{u_i' u_j'} = \dots$$