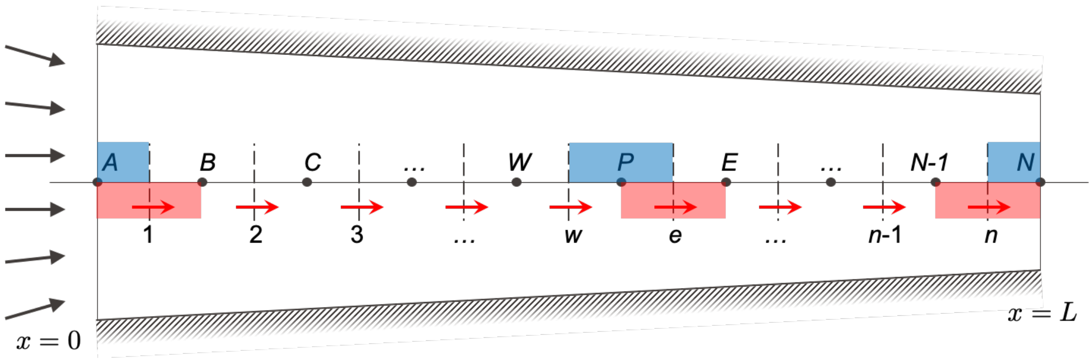
ME-474 Numerical Flow Simulation

Solution: 1D steady incompressible flow; pressure-velocity coupling

Fall 2022



Questions:

1. Discretize the momentum equation on the velocity grid (staggered grid). Deduce the algebraic equation for the guess velocity u^*. For the nonlinear convective term, use Picard's method with the previous velocity u^{old}. For the pressure term, use a guess pressure p^*.

   Governing equations:

$$\text{Continuity eq. (mass conservation):} \quad \frac{\partial(\rho u)}{\partial x} = 0,$$

$$\text{Momentum eq. (momentum conservation):} \quad \frac{\partial(\rho u u)}{\partial x} = -\frac{\partial p}{\partial x}.$$

   Discretization for the guess velocity on the staggered grid:

$$(\rho u^{old} A)_E u_E^* - (\rho u^{old} A)_P u_P^* = (p_P^* - p_E^*) A_e$$
$$F_E u_E^* - F_P u_P^* = (p_P^* - p_E^*) A_e.$$

   The convective flux F expressed at CV centers must be interpolated from face values (because this is where velocity is defined, i.e. where velocity will be computed):

$$F_E = (\rho u^{old} A)_E = \rho \frac{u_e^{old} + u_{ee}^{old}}{2} A_E, \qquad F_P = (\rho u^{old} A)_P = \rho \frac{u_w^{old} + u_e^{old}}{2} A_P.$$

   We use upwind differencing for the unknowns, $u_E^* \approx u_e^*$ and $u_P^* \approx u_w^*$, so we obtain

$$F_E u_e^* - F_P u_w^* = (p_P^* - p_E^*) A_e$$
$$\boxed{a_e u_e^* = a_w u_w^* + S}$$

   where

$$a_e = F_E, \quad a_w = F_P, \quad S = (p_P^* - p_E^*) A_e.$$

2. Express the inlet pressure p_{in} as a function of the upstream chamber stagnation pressure p_0.

Because the flow is steady and inviscid, we can use Bernoulli's principle (first Bernoulli's theorem) on a streamline that goes from far away upstream to the inlet:

$$p_0 + \frac{1}{2}\rho u_0^2 = p_0 + 0 = p_{in} + \frac{1}{2}\rho u_{in}^2 \quad \Rightarrow \quad \boxed{p_{in} = p_0 - \frac{1}{2}\rho u_{in}^2.}$$

3. Derive the equation to be implemented for u_1^* in the first velocity CV.

In the first velocity CV, the discretized momentum equation reads (see question 1):

$$F_B u_B^* - F_A u_A^* = (p_A^* - p_B^*)A_1,$$

which becomes with UD:

$$F_B u_1^* - F_A u_A^* = (p_A^* - p_B^*)A_1,$$

where

$$F_B = \rho\frac{u_1^{old} + u_2^{old}}{2}A_B, \quad F_A = \rho u_A^{old} A_A.$$

We need to determine the quantities that are not part of the solution: the velocities u_A^{old} and u_A^*, as well as the pressure p_A^*. We first use continuity to relate u_A^{old} and u_1^{old}:

$$F_A = \rho u_A^{old} A_A = \rho u_1^{old} A_1.$$

Next, we write similarly:

$$\rho u_A^* A_A = \rho u_1^* A_1 \quad \Rightarrow \quad u_A^* = \frac{A_1}{A_A}u_1^*.$$

Finally, we use the inlet pressure found in the previous question:

$$p_A^* = p_0 - \frac{1}{2}\rho {u_A^*}^2 = p_0 - \frac{1}{2}\rho\left(\frac{A_1}{A_A}u_1^*\right)^2.$$

Gathering everything:

$$F_B u_1^* - F_A \frac{A_1}{A_A}u_1^* = \left[p_0 - \frac{1}{2}\rho\left(\frac{A_1}{A_A}u_1^*\right)^2 - p_B^*\right]A_1$$

$$\Leftrightarrow \quad \boxed{\left(F_B - F_A\frac{A_1}{A_A}\right)u_1^* + \frac{1}{2}\rho\left(\frac{A_1}{A_A}\right)^2 A_1 u_1^{*2} = (p_0 - p_B^*)\,A_1.}$$

4. The equation you just found is nonlinear in u_1^*. Linearize the nonlinear term using Picard's method with the previous velocity. Next, use the deferred correction approach if needed.

Rewrite

$$\frac{1}{2}\rho\left(\frac{A_1}{A_A}\right)^2 A_1 u_1^{*2} \approx \frac{1}{2}\rho\left(\frac{A_1}{A_A}\right)^2 A_1 (u_1^{old}u_1^*) = \frac{1}{2}F_A\left(\frac{A_1}{A_A}\right)^2 u_1^*.$$

Therefore the linearized equation for u_1^* is:

$$\left(F_B - F_A\frac{A_1}{A_A} + \frac{1}{2}F_A\left(\frac{A_1}{A_A}\right)^2\right)u_1^* = (p_0 - p_B^*)\,A_1.$$

The coefficient of u_1^* can be negative, so we move the negative term to the RHS (as a known, constant term, which is evaluated with the previous velocity):

$$\left(F_B + \frac{1}{2}F_A\left(\frac{A_1}{A_A}\right)^2\right)u_1^* = (p_0 - p_B^*)\,A_1 + F_A\frac{A_1}{A_A}u_1^{old}$$

i.e.

$$\boxed{a_1 u_1^* = S_u.}$$

5. Derive the equation to be implemented for u_n^* in the last velocity CV.

In the last velocity CV, the discretized momentum equations reads:

$$F_N u_N^* - F_{N-1} u_{N-1}^* = (p_{N-1}^* - p_N^*) A_n,$$

which becomes with UD:

$$F_N u_n^* - F_{N-1} u_{n-1}^* = (p_{N-1}^* - p_N^*) A_n,$$

where

$$F_N = \rho u_N^{old} A_N, \quad F_{N-1} = \rho \frac{u_{n-1}^{old} + u_n^{old}}{2} A_{N-1},$$

We need to determine the velocity u_N^{old}. We use continuity:

$$F_N = \rho u_N^{old} A_N = \rho u_n^{old} A_n,$$

such that

$$\boxed{a_n u_n^* = a_{n-1} u_{n-1}^* + S}$$

where

$$a_n = F_N, \quad a_{n-1} = F_{N-1}, \quad S = (p_{N-1}^* - p_N^*) A_n.$$

6. Discretize the continuity equation on the pressure grid (original grid). Deduce the algebraic equation for the pressure correction p'.

Discretization of the continuity equation on the original grid:

$$(\rho u A)_e - (\rho u A)_w = 0.$$

In the SIMPLE method, the velocity is expressed as the sum of a guess and a correction,

$$u = u^* + u',$$

where the velocity correction is approximated as

$$u_e' = (p_P' - p_E') d_e \quad \text{with} \quad d_e = \frac{A_e}{a_e}.$$

Substituting in the continuity equation yields

$$\rho \left[u_e^* + (p_P' - p_E') d_e \right] A_e - \rho \left[u_w^* + (p_W' - p_P') d_w \right] A_w = 0,$$

that is:

$$\boxed{a_P p_P' = a_W p_W' + a_E p_E' + b}$$

where

$$a_W = \rho d_w A_w, \quad a_E = \rho d_e A_e, \quad a_P = a_W + a_E, \quad b = \rho u_w^* A_w - \rho u_e^* A_e.$$

Note that the constant term b is a flux balance for u^*, more precisely the discretized version of $-div(\rho u^*)$. It should become close to 0 at convergence.

7. What is the equation to be implemented for p_A' in the first pressure CV?

After solving for the guess velocity u^* and the pressure correction p' (by solving linear systems), the fields are corrected:

$$p_P = p_P^* + p_P', \qquad u_e = u_e^* + u_e' = u_e^* + (p_P' - p_E') d_e.$$

Therefore, in principle we have $p_A = p_A^* + p_A'$ in the first pressure CV, and we need a boundary condition on p_A'. However, we saw that the inlet pressure p_A is known from the upstream stagnation pressure p_0 and the inlet velocity u_A:

$$p_A = p_0 - \frac{1}{2} \rho u_A{}^2 = p_0 - \frac{1}{2} \rho \left(\frac{A_1}{A_A} u_1 \right)^2.$$

Therefore, we can directly compute p_A with this relation using the updated value u_1, and we do not need to use the correction $p_A = p_A^* + p_A'$. So we can set $\boxed{p_A' = 0}$ (i.e. $a_A = 1$, $a_B = 0$, $b = 0$).

8. What is the equation to be implemented for p'_N in the last pressure CV?

   The outlet pressure is given by a Dirichlet boundary condition, $p_N = p_{out} = 0$. If we set $p^*_N = p_{out}$ in the initial guess, we will never need to correct this value, so we can set $\boxed{p'_N = 0}$ (i.e. $a_N = 1$, $a_{N-1} = 0$, $b = 0$).

9. It is actually possible to find an analytical solution to this problem, which will be useful to assess the accuracy of the numerical solution. Find the analytical expressions of $u(x)$ and $p(x)$.

   First, we use continuity (conservation of mass flux):

$$\dot{m} = cst = \rho u_{out} A_{out} = \rho u(x) A(x) \quad \Rightarrow \quad u(x) = \frac{\dot{m}}{\rho A(x)}.$$

   Next, we can again use Bernoulli's principle, this time along a streamline between far away upstream and the outlet:

$$p_0 + \frac{1}{2}\rho u_0^2 = p_0 = p_{out} + \frac{1}{2}\rho u_{out}^2 = \frac{1}{2}\rho u_{out}^2 \quad \Rightarrow \quad u_{out} = \sqrt{\frac{2p_0}{\rho}}.$$

   Therefore the flow rate is

$$\dot{m} = \rho u_{out} A_{out} = A_{out}\sqrt{2\rho p_0},$$

   and the velocity in the channel is

$$\boxed{u(x) = \frac{\dot{m}}{\rho A(x)} = \frac{A_{out}}{A(x)}\sqrt{\frac{2p_0}{\rho}}.}$$

   We use again Bernoulli's principle (between far away upstream and x) to obtain the pressure in the channel:

$$p(x) = p_0 - \frac{1}{2}\rho u(x)^2,$$

   i.e.

$$\boxed{p(x) = p_0 - \frac{\dot{m}^2}{2\rho A(x)^2} = p_0\left[1 - \left(\frac{A_{out}}{A(x)}\right)^2\right].}$$