EPFL

EXERCISE 5 – Randomized matrix computations, Fall'24

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1 ► Matrix Chernoff

Consider a finite sequence $\{X_k \in \mathbb{R}^{n\times n}\}$ of independent, random, symmetric matrices. Assume that

$$0 \le \lambda_{\min}(X_k) \quad \text{and} \quad \lambda_{\max}(X_k) \le L \quad \text{for each index } k.$$

Introduce the random matrix

$$Y = \sum_k X_k,$$

and define

$$\mu_{\min} = \lambda_{\min}(\mathbb{E}Y), \text{ and } \mu_{\max} = \lambda_{\max}(\mathbb{E}Y).$$

Prove that for any $\theta > 0$,

$$\mathbb{E}\lambda_{\min}(Y) \ge \frac{1-e^{-\theta}}{\theta}\mu_{\min} - \frac{1}{\theta}L\log n,$$
$$\mathbb{E}\lambda_{\max}(Y) \le \frac{e^{\theta}-1}{\theta}\mu_{\max} + \frac{1}{\theta}L\log n.$$

Hint: Consider using the bound in Exercise 4, question 2.

2 ► Erdős-Rényi graph

In the lecture, we discussed the connectivity of the Erdős-Rényi graph $G(n,p)$. In this exercise, we will study the sharpness of the results given by the matrix Chernoff bound. Let $A \in \mathbb{R}^{n\times n}$ be the adjacency matrix and $\Delta \in \mathbb{R}^{n\times n}$ be the Graph Laplacian matrix associated with the graph $G(n,p)$.

a) Prove that Δ is symmetric positive semidefinite matrix and

$$\|\Delta\|_2 \le \max_{i=1,\dots,n} \deg(i).$$

b) In the lecture, by studying the probability of $\lambda_2(\Delta)$ of the of Erdős-Rényi graph $G(n,p)$ being positive, we conculde that when $p > 2\log(n-1)/n$, with high probability, $\lambda_2(\Delta)$ is positive. We study this result numerically here.

   1) Let $n = 10$, vary $p \in [\frac{0.5\log(n-1)}{n}, \frac{4\log(n-1)}{n}]$, for each fixed (n,p), sample 100 values of $\lambda_2(\Delta)$, plot the emprical probability that $\lambda_2(\Delta)$ is positive.

   2) Repeat 1) with $n = 50, 100, 150$. What do you observe?

      Hint: when increasing p, you may stop the simulation once you detect consecutive empirical probabilities equal to 1.

   3) When we take $n \to \infty$, there exists a threshold $p_0 \in [0,1]$ such that

$$\mathbb{P}\{G(n,p) \text{ is connected.}\} = \begin{cases} 1, & \text{when } p > p_0, \\ 0, & \text{when } p < p_0. \end{cases} \tag{1}$$

      What do you think p_0 is equal to?

3 ▶ Matrix Hoeffding's inequality

Let $\varepsilon_1, \dots, \varepsilon_N$ be independent symmetric Bernoulli random variables, i.e. it takes values -1 and 1 with probabilities $1/2$ and let $A_1, \dots, A_N$ be symmetric $n \times n$ matrices (deterministic). Prove that, for any $t \geq 0$, we have

$$\mathbb{P}\Big(\Big\| \sum_{i=1}^{N} \varepsilon_i A_i \Big\|_2 \geq t \Big) \leq 2n \exp\left(-\frac{t^2}{2\sigma^2} \right),$$

where

$$\sigma^2 = \Big\| \sum_{i=1}^{N} A_i^2 \Big\|_2 .$$

Hint: Prove that $\mathbb{E} \exp(\lambda \varepsilon_i A_i) \leq \exp(\lambda^2 A_i^2/2)$, and proceed as in the proof of the matrix Bernstein inequality.

4 ▶ Tighter bounds on Gaussian embeddings

a) Given a matrix S with d columns and a k-dimensional subspace $\mathcal{U} \subset \mathbb{R}^d$. Let $U \in \mathbb{R}^{d \times k}$ be a matrix whose columns form an orthonormal basis for $\mathcal{U}$. Prove that for $1 > \epsilon > 0$,

$$(1-\epsilon)\|u\|_2^2 \leq \|Su\|_2^2 \leq (1+\epsilon)\|u\|_2^2, \quad \forall u \in \mathcal{U}$$

is equaivent to

$$\sigma_{\min}(SU)^2 \geq 1 - \epsilon \text{ and } \sigma_{\max}(SU)^2 \leq 1 + \epsilon.$$

b) Let $\Omega \in \mathbb{R}^{n \times k}$ be a standard Gaussian random matrix. By [Martinsson/Tropp 2020], we have for $t > 0$,

$$\mathbb{P}\Big\{ \sigma_{\min}\Big(\frac{1}{\sqrt{n}} \Omega \Big) \leq 1 - \frac{\sqrt{k}+1}{\sqrt{n}} - t \Big\} \leq e^{-nt^2/2},$$

$$\mathbb{P}\Big\{ \sigma_{\max}\Big(\frac{1}{\sqrt{n}} \Omega \Big) \geq 1 + \frac{\sqrt{k}}{\sqrt{n}} + t \Big\} \leq e^{-nt^2/2}.$$

Use these results to prove that $\frac{1}{\sqrt{n}}\Omega$ is (k, ϵ, δ)-OSE if

$$n \geq 4\epsilon^{-2}(1 + k + \log(2/\delta)).$$