1) $\log(A) = U \Lambda U^T$

$\Lambda = \mathrm{diag}(\log(\lambda_i(A))) = \mathrm{diag}\left(\int_0^\infty (1+x)^{-1} - (\lambda_i(A)+x)^{-1}\,dx\right)$.

$$= U \int_0^\infty \left((I + xI)^{-1} - (\mathrm{diag}(\lambda_i(A)) + xI)^{-1}\,dx\right) U^T.$$

$$= \int_0^\infty (I + xI)^{-1} - (A + xI)^{-1}\,dx$$

$$\log(B) - \log(A) = \int_0^\infty -(B + xI)^{-1} + (A + xI)^{-1}\,dx \geq 0.$$

b). Note that $\mathrm{trace}(\exp(A)) = \sum_{i=1}^n \exp(\lambda_i(A))$, $\underbrace{\text{since } B - A > 0 \text{ then } \lambda_i(B) \geq \lambda_i(A)}_{\text{why?}}$

$\Rightarrow$

$$\sum \exp(\lambda_i(B)) > \sum \exp(\lambda_i(A))$$

2). $\mathbb{E}\left(\lambda_{max} \left(\sum_{i=1}^{S} X_i \right) \right) = \frac{1}{\theta} \mathbb{E} \log \left(e^{\lambda_{max} (\theta \sum_{i=1}^{S} X_i)} \right)$

$$\overset{\text{Jensen}}{\leq} \frac{1}{\theta} \log \mathbb{E}\, e^{\lambda_{max} (\theta \sum_{i=1}^{S} X_i)} \quad \text{for } \theta > 0$$

$$= \frac{1}{\theta} \log \mathbb{E}\, \lambda_{max} \left(\exp \left(\theta \sum_{i=1}^{S} X_i \right) \right)$$

$$\leq \frac{1}{\theta} \log \mathbb{E}\, \mathrm{trace} \left(\exp \left(\theta \sum X_i \right) \right)$$

$$\overset{\text{Lemma}}{\leq} \frac{1}{\theta} \log \mathrm{trace} \left(\exp \sum \log \mathbb{E}\, e^{\theta X_i} \right)$$

Then conclude by taking inf of θ.

3a $A = \sum_{i,j} a_{ij} e_i e_j^T$

3b. define $\mathbb{P}\{ X = mn \cdot a_{ij} e_i e_j^T \} = \frac{1}{m \cdot n}$

We use Corollary 6.2.1 in Introduction to Matrix Concentration Inequalities (Tropp 2015)

which can be found in Moodle Books / references.

$$\mathbb{E}[\| \tilde{X}_S - A\|_2] \leq \sqrt{\frac{2 m_2(X) \log(n+m)}{S}} + \frac{2L \log(n+m)}{S}.$$

where $L \geq \|X\|$ $m_2(X) = \max \{ \|\mathbb{E}(XX^*)\|, \|\mathbb{E}(X^*X)\| \}$

We can chose $L = m \cdot n \max_{ij} |a_{ij}|$, and $\mathbb{E}\|XX^*\| = \| \frac{1}{m \cdot n} \sum (mn)^2 a_{ij}^2 e_i e_i^T \|$

$= \| \sum mn\, a_{ij}^2 e_i e_i^T \|$

$\| \mathbb{E}(X^*X) \| = \| \sum mn\, a_{ij}^2 e_j e_j^T \|$

If $\quad s \geq \dfrac{2 m_2(R) \log(n+m)}{\varepsilon^2} + \dfrac{2L \log(n+m)}{3\varepsilon} \quad$ then $\mathbb{E}\|X_s - A\|_2 \leq \varepsilon$.

3c) $- \quad \|X\| \leq \max_{i,j} \left\| \frac{1}{p_{ij}} a_{ij} e_i e_j^T \right\| = \max_{i,j} \frac{1}{p_{ij}} \cdot |a_{ij}| \overset{\text{since } p_{ij} \geq \frac{1}{2} \cdot \frac{|a_{ij}|}{\|A\|_{\ell_1}}}{\leq} 2\|A\|_{\ell_1}$

we take $\quad L = 2\|A\|_{\ell_1}$

$- \quad \mathbb{E}[XX^T] = \sum^n \sum^m \dfrac{|a_{ij}|^2}{p_{ij}} e_i e_i^T \overset{\text{Loewner order and since } p_{ij} \geq \frac{|a_{ij}|}{2\|A\|_F}}{\leq} 2\|A\|_F^2 \sum\sum e_i e_i^T$

$= 2\|A\|_F^2 \, m \cdot I$

similarly

$- \quad \mathbb{E}[X^T X] \leq 2n \|A\|_F^2 \cdot I.$

$m_2(X) = 2 \max\{n, m\} \cdot \|A\|_F^2.$

$\rightsquigarrow$ we can take $\quad s > 4\varepsilon^2 \, \mathrm{srank}(A) \max\{n,m\} \log(n+m)$, (see Tropp 2015 Section 6.32 for more