1) Assume we choose the first vertice uniformly at random and denote the angle be θ. We have

$$Pr(\text{1.st vertice is red}) = 0.23.$$

Since we fixed a vertex then the rest of the vertexes would be $90^\circ + \theta$, $180^\circ + \theta$, $270^\circ + \theta$. Denote E_i be the event of the i vertex is red.

$$Pr(\text{All vertices are blue}) = 1 - Pr(E_1 \cup E_2 \cup E_3 \cup E_4).$$

and $$Pr(E_1 \cup E_2 \cup E_3 \cup E_4) \leq \sum_{i=1}^{4} Pr(E_i).$$

Note that the vertices are also random variable. (distribute uniformly at random)

Hence $Pr(E_i) = 0.23$ for $i=1, \cdots, 4$ and $Pr(\text{All vertices are blue}) > 0$

Therefore we must be able to find a square so that all four vertices are blue.

2) For $y<0$, $F_Y(y) = Pr(Y < y) = 0$. and

For $y \geq 0$, $F_Y(y) = Pr(X^2 < y) = Pr(-\sqrt{y} < X < \sqrt{y}) = F_X(\sqrt{y}) - F_X(-\sqrt{y})$

$= 2F_X(\sqrt{y}) - 1$

$$f_Y(y) = \frac{d}{dy} F_Y(y) = 2\frac{d}{dy} F_X(\sqrt{y}) = 2\frac{d}{dy}\int_{-\infty}^{\sqrt{y}} \frac{1}{\sqrt{2\pi}} e^{-t^2/2}\, dt.$$

$$= \frac{1}{\sqrt{2\pi}}\, y^{-1/2} e^{-y/2}$$

3. Let the Cholesky decomposition of $C = BB^T$ and Let $L = B^{-1}$

(check why B is invertable!)

$$LY = B^{-1}Y \sim N(0, B^{-1}BB^TB^{-T}) = N(0, I).$$

$$\Rightarrow [LY]_1 \sim N(0,1).$$

$$[LY]_1 = [B^{-1}Y]_1 = \frac{1}{\sqrt{C_{11}}} Y_1$$

First entry of C, it is positive!

$$\Rightarrow Y_1 = \frac{\sqrt{C_{11}}}{\sqrt{C_{11}}} Y_1 \sim N(0, C_{11})$$

4.1) We proof by contradiction: Suppose $\mathbb{E} X > a$ but $X(\omega) \leq a$ for all $\omega \in \Omega$.

then $\mathbb{E} X \leq \mathbb{E} a = a$. Contradiction arises.

4.2) We take iid $\varepsilon_i = \begin{cases} 1 & \text{with prob } 1/2 \\ -1 & \text{with prob } 1/2. \end{cases}$, denote u_{ij} the j-th entry of vector u_i

$$\mathbb{E} \left\| \sum_{i=1}^{n} \varepsilon_i u_i \right\|_2 = \mathbb{E} \sqrt{\sum_j \left(\sum_i \varepsilon_i u_{ij} \right)^2}$$

$$\overset{\text{Jensen}}{\leq} \sqrt{\mathbb{E} \sum_j \left(\sum_i \varepsilon_i u_{ij} \right)^2}$$

$$= \sqrt{\sum_j \mathbb{E} \left[\sum_i (\varepsilon_i u_{ij}) \right]^2}$$

$$= \sqrt{\sum_j \sum_i u_{ij}^2}$$

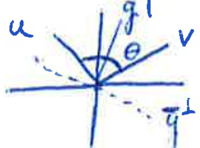
since $\mathbb{E} \varepsilon_i = 0$; $\mathbb{E} \varepsilon_i \varepsilon_j = \mathbb{E} \varepsilon_i \mathbb{E} \varepsilon_j = 0$ and $\mathbb{E} \varepsilon_i^2 = 1$

$$= \sqrt{n}.$$

5 Note that $\operatorname{sign}\langle g,u\rangle \cdot \operatorname{sign}(\langle g,v\rangle) = 1$ iff u,v contains in the same part. By the rotational invariance we can reduce the problem to $\mathbb{R}^2$. * Therefore the probability u,v lies in different parts is $\frac{2\theta}{2\pi}$ factor 2 comes from $g, -g$ span the same plane.

$$
\begin{aligned}
\mathbb{E}\operatorname{sign}\langle g,u\rangle \operatorname{sign}\langle g,v\rangle &= \Pr(u,v \text{ in the same part}) \\
&\quad - \Pr(u,v \text{ in different parts}) \\
&= 1 - \frac{2\theta}{\pi} \\
&= \frac{2}{\pi}\arcsin\left(\sin\left(\frac{\pi}{2}-\theta\right)\right) \\
&= \frac{2}{\pi}\arcsin(\cos\theta) = \frac{2}{\pi}\arcsin(\langle u,v\rangle)
\end{aligned}
$$

(*) By considering the plane span by u,v we have the following Geometric interpretation:



where $\bar{g}$ is orthogonal projection of g to the plane and $\bar{g}$

is gaussian again.

5) 2b) $\mathbb{E}\, CUT(G,x) = \frac{1}{4} \sum_{i,j=1}^{n} A_{ij} \left(1 - \mathbb{E}\, x_i x_j\right).$

$$1 - \mathbb{E}\, x_i x_j = 1 - \mathbb{E}\, \operatorname{sign}\langle x_i, g\rangle \operatorname{sign}(\langle v_j, g\rangle)$$

$$= 1 - \frac{2}{\pi} \arcsin \langle v_i, v_j\rangle$$

$$\geqslant 0.878 (1 - \langle x_i, x_j\rangle$$

$\Rightarrow$ $\mathbb{E}\, CUT(G,x) \geqslant 0.878\, SDP(G)$

and $SDP(G) \geqslant MAX\text{-}CUT(G)$ is trivial.