Exercise 1 – Randomized matrix computations, Fall'24

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1 ▶ Probabilistic method

Given a unit circle in the plane so that a (measurable) subset of 23% of the circle is red and the rest is blue. Show that we can always inscribe a square in the circle so that all four vertices are blue.

Hint: Choose the square at random, and show that there is a positive probability that its vertices are all blue.

2 ▶ χ^2 distribution

Suppose $X \sim N(0,1)$. Prove that the pdf of $Y = X^2$ is given by

$$f_Y(y) = 0 \text{ for } y < 0, \quad f_Y(y) = \frac{1}{\sqrt{2\pi}} y^{-1/2} e^{-y/2}, \text{ for } y \geq 0.$$

Hint: First prove that $F_Y(y) = 2F_X(\sqrt{y}) - 1$, and then take the derivative of $F_Y(y)$.

3 ▶ Marginals of normal random vector

Let $Y \sim N(0, C)$ for a symmetric positive definite matrix $C \in \mathbb{R}^{n\times n}$. Work out the marginal distribution (without consulting the Internet) of Y_1, the first entry of Y_1.

Possible approach: First show that $LY \sim N(0, LCL^T)$ holds for an invertible matrix L. Use the Cholesky decomposition of C to find a suitable L that does not affect Y_1 but decouples the other variables, resulting in a product distribution.

4 ▶ First-moment method

1. Let X be a real random variable, and fix $a \in \mathbb{R}$. Explain why $\mathbb{E}X > a$ implies $X(\omega) > a$ for some $\omega \in \Omega$.
2. Vector balancing: Suppose that $(u_1, \ldots, u_n)$ are unit-norm vectors in $\mathbb{R}^d$. Show that there is a sequence $(\epsilon_1, \ldots, \epsilon_n) \in \{\pm 1\}^n$ of signs for which

$$\Big\| \sum_{i=1}^{n} \epsilon_i u_i \Big\|_2 \leq \sqrt{n}.$$

5 ▶ Randomized rounding of vectors

The randomized rounding procedure involves first choosing a random hyperplane in $\mathbb{R}^n$ that passes through the origin. This hyperplane divides $\mathbb{R}^n$ into two parts. We then assign labels $x_i = 1$ or $x_i = -1$ depending on which part the vector v_i lies. Equivalently, we may choose a standard normal vector $g \sim N(0, I)$ and define

$$x_i := \operatorname{sign}\langle v_i, g\rangle, \quad i = 1, \ldots n. \tag{1}$$

1. Prove the following idenity: Suppose $g \sim N(0, I)$. Then for any fixed unit vectors $u, v \in \mathbb{R}^n$, we have

$$\mathbb{E} \operatorname{sign}\langle g, u\rangle \operatorname{sign}\langle g, v\rangle = \frac{2}{\pi} \arcsin\langle u, v\rangle.$$

   Hint: show the probability that $\langle g, u\rangle$ and $\langle g, v\rangle$ have opposite signs is equal to $\frac{\theta}{\pi}$ where θ is the angle between the vectors u and v.

2. (BONUS: Just for fun) The randomized rounding and the above identity are important insights to approximate the MAX-CUT problem of an undirected graph G. If we partition the set of vertices of G into two disjoint sets, the cut of G is the number of edges connecting a vertex from one set with a vertex from the other set. The maximum cut of G, denoted MAX-CUT(G), is the maximum of all cuts over all partitions. MAX-CUT is an NP complete problem.

   (a) Consider the adjacency matrix A of the graph G. Assume we label the vertices of G by $1, \dots, n$ and denote a partition of vertices by a vector of label

   $$x = (x_i) \in \{-1, 1\}^n,$$

   meaning that the sign of x_i indicates which subset the vertex i belongs to. Explain why the cut of partition x satisfies

   $$\mathrm{CUT}(G, x) = \frac{1}{4} \sum_{i,j=1}^{n} A_{ij}(1 - x_i x_j),$$

   and thus

   $$\text{MAX-CUT}(G) = \frac{1}{4} \max \Big\{ \sum_{i,j=1}^{n} A_{ij}(1 - x_i x_j) : x_i = \pm 1 \text{ for all } i \Big\}.$$

   (b) By performing randomized rounding to the solution of the semidefinite problem:

   $$\mathrm{SDP}(G) := \frac{1}{4} \max \Big\{ \sum_{i,j=1}^{n} A_{ij}(1 - \langle v_i, v_j \rangle) : v_i \in \mathbb{R}^n, \|v_i\|_2 = 1 \text{ for } i = 1, \dots, n \Big\},$$

   one can approximate MAX-CUT(G) fairly well. Prove the following claim:

   Let $x = (x_i)$ be the result of randomized rounding of the solution $v_1, \dots, v_n$ of SDP(G), i.e. x_i defined as in (1). Then

   $$\mathbb{E}\mathrm{CUT}(G, x) \geq 0.878 \ \mathrm{SDP}(G) \geq 0.878 \ \text{MAX-CUT}(G).$$

   In other words, we expect the procedure to yield a partition for which the cut approximates MAX-CUT pretty well.

   Hint: Use $1 - \frac{2}{\pi} \arcsin t = \frac{2}{\pi} \arcsin t \geq 0.878(1 - t), \quad t \in [-1, 1]$.