1 By Lemma in slide 22 Lecture 4., we have

$$\log \mathbb{E} e^{\theta X_k} \leq \frac{e^{\theta L}-1}{L} \cdot \mathbb{E} X_k.$$

Plug in to Master bound. (Ex 4, Q2), let $\frac{e^{\theta L}-1}{L} := g(\theta)$,

$$\mathbb{E} \lambda_{max}(Y) \leq \inf_{\theta > 0} \frac{1}{\theta} \log tr \exp \left(g(\theta) \sum_k \mathbb{E} X_k \right).$$

$$\leq \inf_{\theta > 0} \frac{1}{\theta} \log d \lambda_{max} \left(\exp g(\theta) \cdot \mathbb{E} Y \right)$$

$$= \inf_{\theta > 0} \frac{1}{\theta} \log d \exp \lambda_{max} \left(g(\theta) \cdot \mathbb{E} Y \right)$$

$$= \inf_{\theta > 0} \frac{1}{\theta} \log d \exp \left(g(\theta) \lambda_{max} \mathbb{E} Y \right)$$

$$= \inf_{\theta > 0} \frac{1}{\theta} \left[\log d + g(\theta) \mu_{max} \right].$$

we can calculate by variable change $\theta \to \frac{\theta}{L}$. The bound for λ_{min} proceed by similar argument.

2 a). $L = D - A$ $\quad$ L is symmetric because A is symetric and D also symmetric.

Note that $$X^T L X = \sum_{i=1}^{n} deg(i)\, x_i^2 - \sum_{i=1}^{n}\sum_{j=1}^{n} A_{ij} x_i x_j.$$

$$= \sum_i \sum_j A_{ij} x_i^2 - \sum \sum A_{ij} x_i x_j$$

$$= \sum_i \sum_j A_{ij} (x_i^2 - x_i x_j).$$

Note that $A_{ij} \geq 0$ and $A_{ji} = A_{ij}$. we have. by consider the sum in pair $\cancel{A}$ $A_{ij}(x_i^2 - x_i x_j) + A_{ji}(x_j^2 - x_{ij}) = A_{ij}(x_i - x_j)^2 \geq 0$.

With the fact that $A_{ii}(x_i^2 - x_i x_i) = 0$ we conclude $X^T L X \geq 0$. $\forall x \in \mathbb{R}^n$

and we know that $\|\Delta\|_2 = \max_{\|x\|=1} x^T L x \leq \max_{i=1,\dots,n} \deg(i)$

3. We first prove $\mathbb{E}\exp(\lambda \varepsilon_i A_i) \le \exp(\lambda^2 A_i/2)$.

$$\mathbb{E}\exp(\lambda \varepsilon_i A_i) = \frac{1}{2}\exp(\lambda(-A_i)) + \frac{1}{2}\exp(\lambda A_i)$$

$$= \frac{1}{2} U \begin{pmatrix} e^{-\lambda\lambda_i} & & \\ & \ddots & \\ & & e^{-\lambda\lambda_i} \end{pmatrix} U^T + \frac{1}{2} U \begin{pmatrix} e^{\lambda\lambda_i} & & \\ & \ddots & \\ & & e^{\lambda\lambda_i} \end{pmatrix} U^T$$

$$= \frac{1}{2} U \begin{pmatrix} e^{-\lambda\lambda_i}+e^{\lambda\lambda_i} & & \\ & \ddots & \\ & & \end{pmatrix} U^T \le \exp(\lambda^2 A_i^2/2).$$

by $\frac{1}{2}(e^{-\alpha}+e^{\alpha}) \le e^{\alpha^2/2}$ $\forall \alpha \in \mathbb{R}$.

By Lecture 4 slide 6.,

$$\Pr\left(\left\|\sum_{i=1}^{N} \varepsilon_i A_i\right\|_2 \ge t\right) \le \inf_{\theta>0} e^{-\theta t}\operatorname{trace}\left(\exp\left(\sum \log \mathbb{E} e^{\theta \varepsilon_i A_i}\right)\right)$$

$$\le \inf_{\theta>0} e^{-\theta t}\operatorname{trace}\left(\exp\left(\sum \log\left(\exp\left(\theta^2 A_i^2/2\right)\right)\right)\right)$$

$$\le \inf_{\theta>0} e^{-\theta t}\operatorname{trace}\left(\exp\left(\sum \theta^2 A_i^2/2\right)\right)$$

$$\le \inf_{\theta>0} n\, e^{-\theta t} e^{\frac{\theta^2}{2}\sigma^2}$$

We conclude the proof by choosing $\theta = \frac{t}{\sigma^2}$.

4a) $\sigma_{max}(SU)^2 = \lambda_{max}\left((SU)^T SU\right) = \max_{\|x\|_2=1} x^T U^T S^T S U x$

$= \max_{u \in U} \frac{u^T S^T S u}{\|u\|_2^2} \leq (1+\varepsilon)$

4b) take $\varepsilon = \frac{\sqrt{k}+1}{\sqrt{n}} + t$. and $n t^2 = 2 \log (2/\delta)$.

by rearranging the term. we have $2 \log \frac{2}{\delta} = \left(\sqrt{n}\,\varepsilon - (\sqrt{k}+1)\right)^2$

and $n = \varepsilon^{-2}\left(\sqrt{2}\sqrt{\log 2/\delta} + (\sqrt{k}+1)\right)^2 \leq 4\varepsilon^{-2}(k + \log 2/\delta)$