1.1) $\mathbb{E}[Xf(x)] = \frac{-\sigma^2}{\sqrt{2\pi\sigma^2}}\int_{-\infty}^{\infty} f(x) \left(-\frac{x}{\sigma^2}\right) \exp\left(-\frac{x^2}{2\sigma^2}\right)$

$$= -\sigma^2 \int_{-\infty}^{\infty} f(x) \frac{d}{dx}\left(\frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{x^2}{2\sigma^2}\right)\right) dx$$

$$\overset{\text{integration by parts}}{=} -\sigma^2\left(\left[\left(f(x)\frac{1}{\sqrt{2\pi\sigma^2}}\exp\left(\frac{-x^2}{2\sigma^2}\right)\right)\right]_{-\infty}^{\infty} - \int_{-\infty}^{\infty} f'(x)\frac{1}{\sqrt{2\pi\sigma^2}}\exp\left(\frac{-x^2}{2\sigma^2}\right)dx\right)$$

$$= \sigma^2 \int_{-\infty}^{\infty} f'(x) \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{x^2}{2\sigma^2}\right) dx = \sigma^2 \mathbb{E}[f'(x)]$$

1.2) Let $p = 2n$, $\mathbb{E}\, X X^{2n-1} = \sigma^2 \mathbb{E}\left[\frac{d}{dx} X^{2n-1}\right]$

$$= \sigma^2 (2n-1) \mathbb{E}[X^{2n-2}]$$

$$\vdots$$

$$= \sigma^{2n}(2n-1)!! = \sigma^{2n}\frac{2n!}{2^n \cdot n!}$$

By stirling approximation $(2n)! \sim \sqrt{2\pi(2n)}\left(\frac{(2n)}{e}\right)^{2n}$ and $n! \sim \sqrt{2\pi n}\left(\frac{n}{e}\right)^{2n}$

we have $\left(\mathbb{E}[X^{2n}]\right)^{\frac{1}{2n}} \leq C\sigma\sqrt{p}$

1.3) $\left(\mathbb{E}|X|^p\right)^{1/p} \overset{\text{Montonicity}}{\leq} \left(\mathbb{E}|X|^{p+1}\right)^{1/p+1} \overset{(2)}{\leq} C\cdot\sigma\sqrt{p+1}$

$$\leq C\cdot\sigma\sqrt{2p}$$

$$\leq \sqrt{2}\cdot C\cdot\sigma\sqrt{p}.$$

2.1 $\|\Omega^+\|_F^2 = \sqrt{tr((\Omega^+)^T\Omega^+)}^2$

$$= Tr\left((\Omega^T)^+\Omega^+\right)$$

Note that Ω has full rank with probability 1 and $\Omega\Omega^T$ is invertable. with probablity 1 (both fact can be prove by considering the determinant.) Therefore $Tr((\Omega^T)^+\Omega^+) = Tr((\Omega\Omega^T)^+) = Tr((\Omega\Omega^T)^{-1})$

w.p. 1

2.2. $$\mathbb{E}\|\Omega^{+}\|_F^2 = \mathbb{E}\left(\sum X_i^{-1}\right) = \sum \mathbb{E}(X_i^{-1}) = \frac{m}{n-m-1}$$

2.3. $$\left(\mathbb{E}\|\Omega^{+}\|_F^{2q}\right)^{1/q} \le \sum \left(\mathbb{E}[X_i^{-q}]\right)^{1/q} \le \frac{3m}{n-m+1}$$

2.4 $$\Pr\left[\|\Omega^{+}\|_F^2 \ge t\right] \le \frac{\mathbb{E}(\|\Omega^{+}\|_F^{2q})}{t^q}$$

$$\Rightarrow \Pr\left[\|\Omega^{+}\|_F^2 \ge \frac{3m}{n-m+1}\cdot u\right] \le u^{-q}.$$

3.1 $$\Pr(|e_1^T v_0| \le \varepsilon) = \Pr(-\varepsilon \le e_1^T v_0 \le \varepsilon) = 2\Pr(0 \le e_1^T v_0 \le \varepsilon)$$

$$= 2\int_0^{\varepsilon} C_{1,n} (1-y)^{n-3/2}$$

$$\le 2\varepsilon C_{1,n} \le \sqrt{\frac{2n}{\pi}}\,\varepsilon$$

4. $\Pr(\|A\|_2 > \varepsilon \|Ax\|_2) = \Pr(\|A\|_2^2 > \varepsilon^2 \|Ax\|_2^2)$

$= \Pr(\|A^TA\|_2 > \varepsilon^2 x^TA^TAx)$.

Consider the spectral decomposition $A^TA = U\Lambda U^T = U \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix} U^T$

with $\lambda_1 \geq \lambda_2 \cdots \geq \lambda_n \geq 0$.

$$\varepsilon^2 x^TA^TAx = \varepsilon^2 \sum^n \lambda_j (x^Tu_j)^2 \geq \varepsilon^2 \lambda_1 (x^Tu_1)^2.$$

$$\Rightarrow \Pr\{\|A^TA\|_2 > \varepsilon^2 x^TA^TAx\} \leq \Pr\{\lambda_1 > \varepsilon^2 \lambda_1 (x^Tu_1)^2\}$$

$$= \Pr\{(x^Tu_1)^2 < \frac{1}{\varepsilon^2}\}.$$

$$\overset{\text{by Lecture 1 slide 28}}{\leq} \sqrt{\frac{2}{\pi}} \frac{1}{\varepsilon}.$$

$$\Rightarrow \Pr(\|A\|_2 \leq \varepsilon \|Ax\|_2) \geq 1 - \sqrt{\frac{2}{\pi}} \frac{1}{\varepsilon}$$