Introduction to Numerical Analysis
MAA106

Maxime Breden

INSTITUT
POLYTECHNIQUE
DE PARIS

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Thomas Bellotti Yoann Le Calvez Jessie Levillain
Arthur Loison Clément Mantoux Dominik Stantejsky

What is Numerical Analysis?

Why is it useful?

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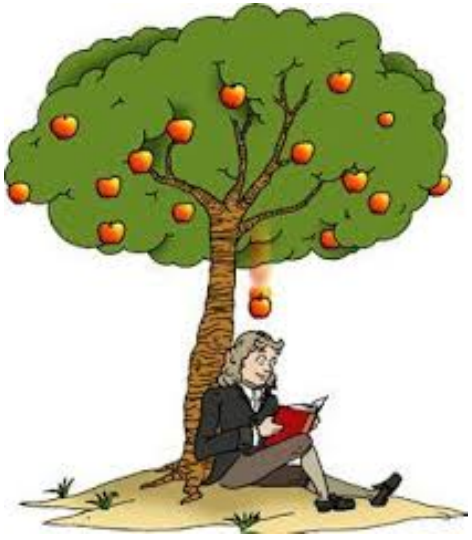
How does one use Maths (and other things) to solve a problem?



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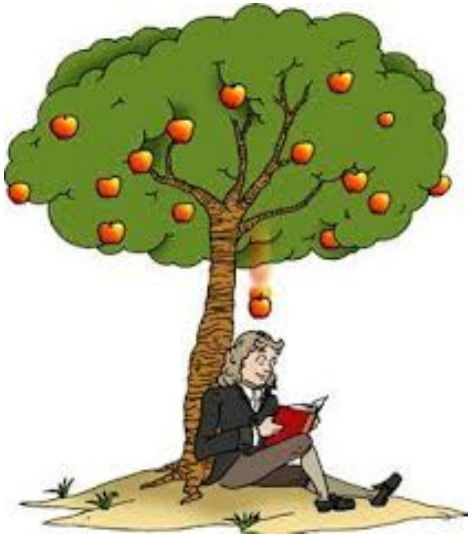

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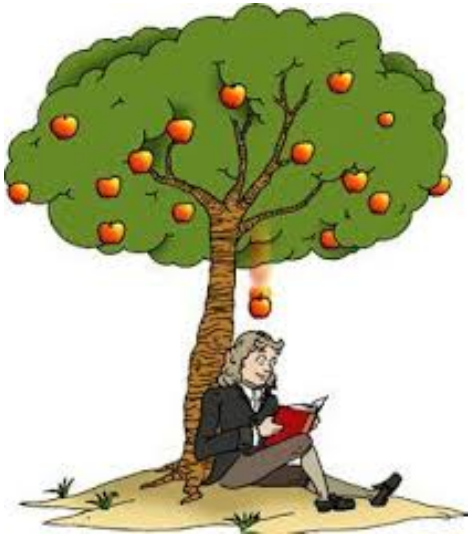

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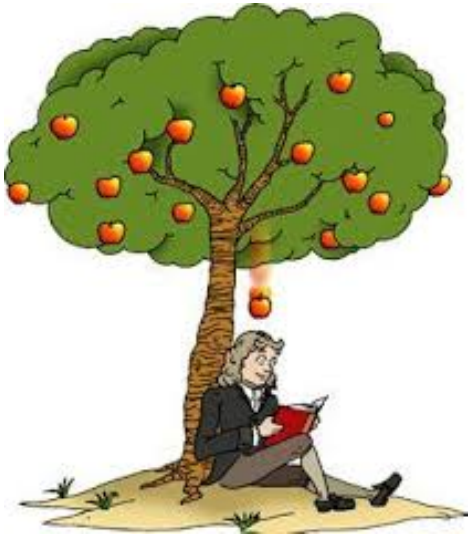

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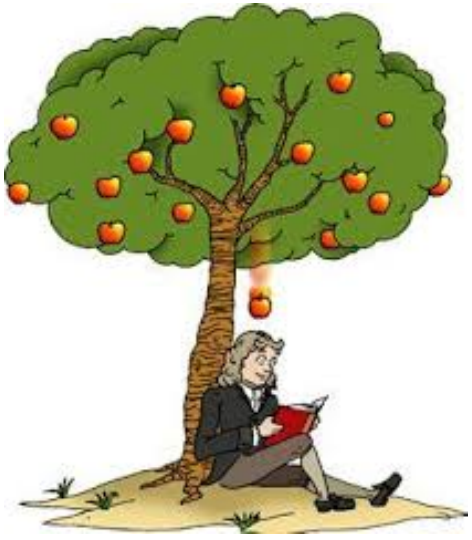

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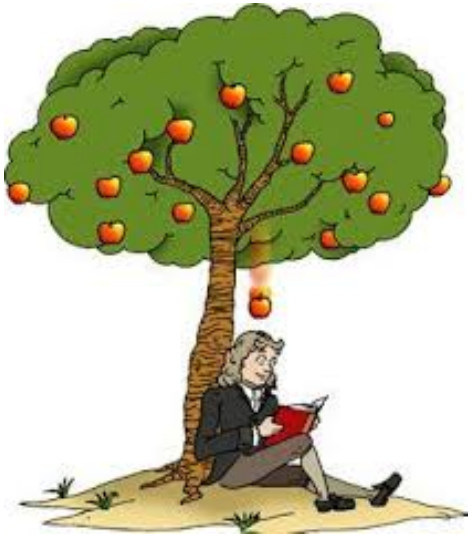

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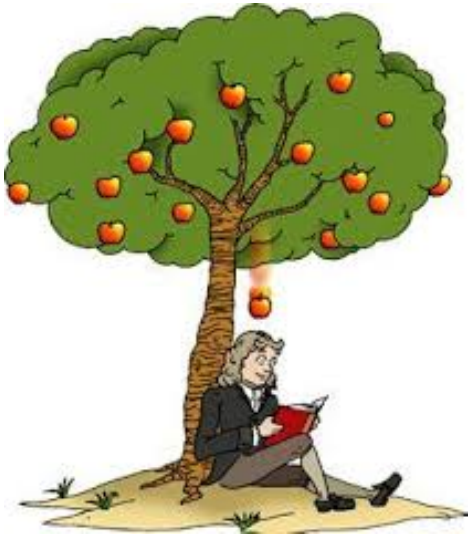

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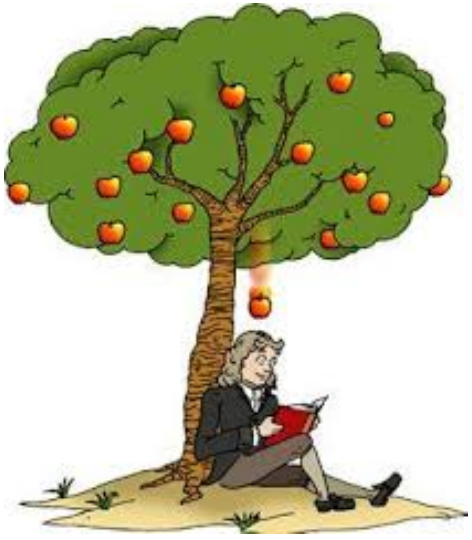

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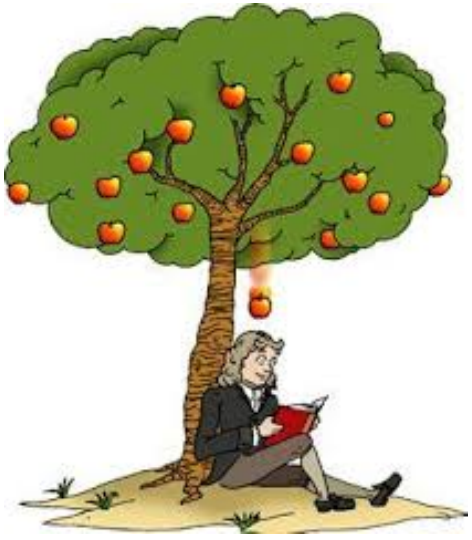

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$$t_1 = \sqrt{\frac{2(h_0 - h_N)}{g}} \simeq \sqrt{\frac{2(3-1)}{9.8}} \simeq 0.64 \text{ sec}$$

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What does it mean to solve mathematical equations?

Solving equations

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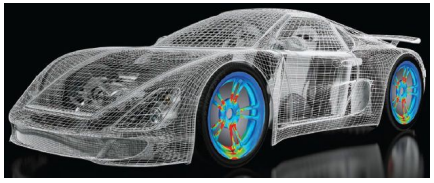
Resistance of materials

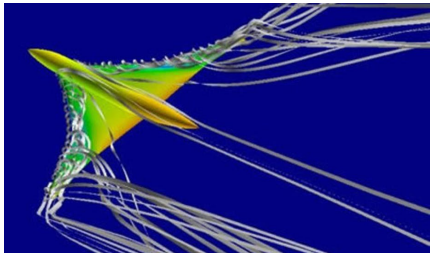
Partial differential equations

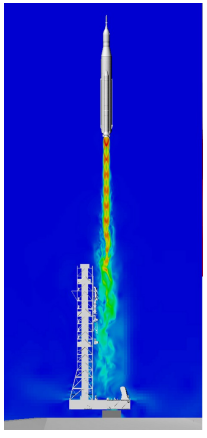
Statistics

Turbulence

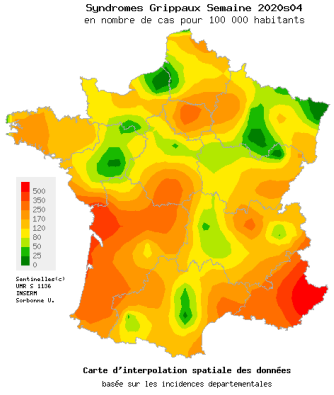
Combustion



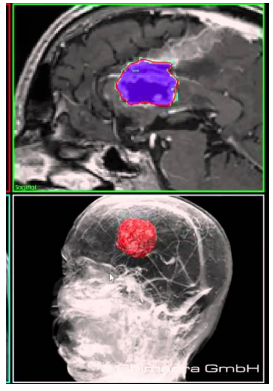


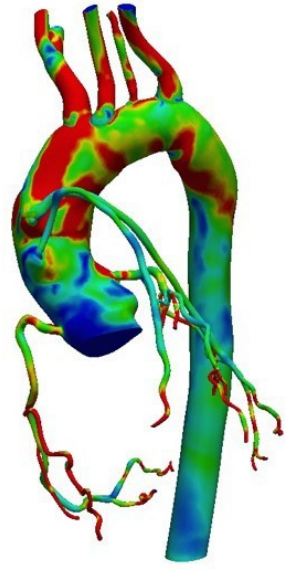


Epidemics propagation



Medical imaging





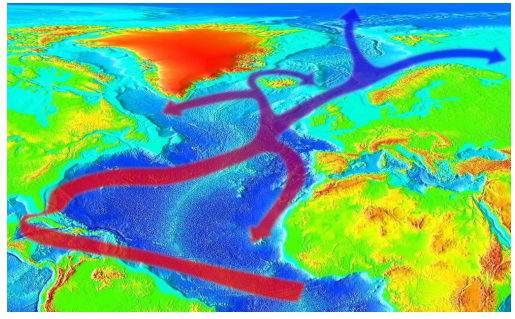
Blood flow

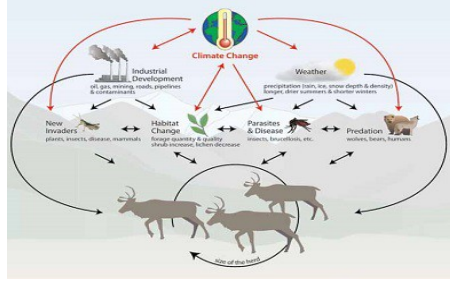
Probabilities

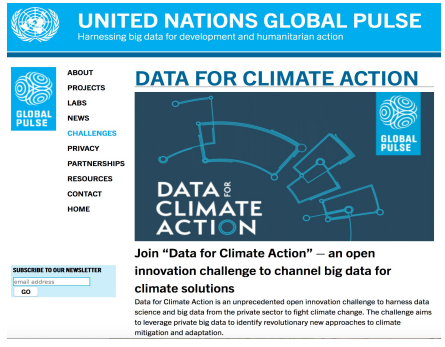
Web and Social networks

Data science

Climate change







Optimization

Crowd motion

Summary – Link with MAA106

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Questions: Which algorithms? How to implement them in practice? How costly are they? How accurate are the approximate solutions provided by these algorithms?

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Objective: Introduce the basic concepts and tools that are at the core of *Computational Mathematics.*

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  - Python and Jupyter Notebooks
  - Experimentation on case studies

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MAA106 – Organization (more details)

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Chapters 1/2/3 (2 weeks each)

- Week 1: Lecture + Lab session
- Week 2: Lecture (starting with a short test about the content of the first week of the cycle) + Lab session
- Before the next chapter: Quiz on Moodle.
- All the material (lectures notes & numerical experiments for the lab sessions) contained in a single notebook per chapter.

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Course Grade: short tests (25%) + quizzes (25%) + exam (50%)