

Gut

$$\int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi} \quad (\text{Gauß})$$

$$\Gamma(p) = \int_0^{\infty} x^{p-1} e^{-x} dx = (p+1)!, \quad p > 0, \quad \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$$

$$\beta(p, q) = \int_0^1 x^{p-1} (1-x)^{q-1} dx = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}, \quad p > 0, q > 0$$

$$\Gamma(p)\Gamma(1-p) = \frac{\pi}{\sin(p\pi)}, \quad p \in (0, 1)$$

Coord polare: $f(x, y) = \rho f(\rho \cos \theta, \rho \sin \theta)$

Coord polare generalizate: $f(x, y) = ab\rho f(a\rho \cos \theta, b\rho \sin \theta)$

Coord cilindrice: $f(x, y, z) = \rho f(\rho \cos \theta, \rho \sin \theta, z)$

Sferice: $f(x, y, z) = \rho^2 \sin \varphi f(\rho \sin \theta \cos \varphi, \rho \sin \theta \sin \varphi, \rho \cos \theta)$, $\theta \in [0, \pi], \varphi \in [0, 2\pi]$

Integrale curbilinii

De speta I

$$\int_{\gamma} f(x, y, z) ds = \int_a^b f(x(t), y(t), z(t)) \sqrt{(x'(t))^2 + (y'(t))^2 + (z'(t))^2} dt$$

De speta II

$$\int_{\gamma} \bar{F} d\bar{r} = \int_{\gamma} P(x, y) dx + Q(x, y) dy = \int_a^b [P(x(t), y(t))x'(t) + Q(x(t), y(t))y'(t)] dt$$
$$\int_{\gamma} \frac{\partial P}{\partial x} dx + \frac{\partial Q}{\partial y} dy = f(B) - f(A), \quad \text{unde } A, B \text{ extremitatile curbei (sn forma dif exacta)}$$

Integrale duble

Masura Jordan în plan

Fie $E = \bigcup_i D_i$, D_i dreptunghiuri disjuncte 2 cate 2.

E sn multime elementara

$\sigma(E) = \sum_i \sigma(D_i)$, σ - aria

$\sigma^{\leq}(M) = \sup\{\sigma(E), E \subset M, E - \text{elementara}\}$

$\sigma^{\geq}(M) = \inf\{\sigma(E), E \supset M, E - \text{elementara}\}$

Daca $\sigma^{\leq}(M) = \sigma^{\geq}(M)$ at M este Masurabila Jordan.

Th (caracterizare): O multime $M \subset \mathbb{R}^2$ este măsurabilă Jordan \iff frontiera sa este J -neglijabilă

Prop: $\sigma(D_1 \cup D_2) = \sigma(D_1) + \sigma(D_2) - \sigma(D_1 \cap D_2)$

Simplu in raport cu Oy

$a \leq x \leq b$, $\alpha(x) \leq y \leq \beta(x)$, α, β continue pe $[a, b]$

Analog cu simplu in raport cu Ox

Daca $\gamma: [a, b] \rightarrow \mathbb{R}^2$ rectificabil, at $\gamma([a, b])$ J -neglijabilă

Multimi Jordan nemasurabile - fractals and squiggly stuff

Integrale duble

Def: luam diviziune, puncte si $\lim_{\|\Delta \rightarrow 0\|} s(f; \Delta, P) = \sum_{i \geq 0} f(P_i) \sigma(D_i) = \iint_D f(M) d\sigma = \iint_D f(x, y) dx dy$

Proprietati ca la integrale Riemann

Schimbare de variabila

$$T : \begin{cases} x = \varphi(u, v) \\ y = \psi(u, v), \end{cases} \quad (u, v) \in \Omega$$

Transformarea $T : \Omega' \rightarrow \Omega$ este *regulata* daca $\varphi, \psi \in C^1(\Omega')$, T biunivoca, $J = \frac{D(\varphi, \psi)}{D(u, v)} \neq 0$, $\forall (u, v) \in \Omega$

Def: f *admisibila* daca este marginita si continua cu exceptia unei multimi J -neglijabile

Th: f admisibila, T regulata, $D^* = T^{-1}(D)$ $\iint_D f(x, y) dx dy = \iint_{D^*} f(\varphi(u, v), \psi(u, v)) |J| du dv$

def: D domeniu standard (poate fi descompus in reuniune finita de dom simple in raport cu ambele axe)

Orientare pozitiva: frontiera este lasata in stanga

Formula Riemann-Green: D domeniu standard inchis $P, Q \in C^1(D)$

$$\iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) = \int_{\text{Fr}D} P dx + Q dy, \quad \text{Fr}D \text{ orientata pozitiv}$$

Integrale triple

Masura Jordan in spatiu

Same spiel as in 2d, da cu paralelipede dreptunghice paralele cu axele.

Domenii simple

Feliute; intre 2 plane: $\iiint_V f(x, y, z) dv = \int_c^d \left(\iint_{D_z} f(x, y, z) dx dy \right) dz$

Bețe; intre 2 suprafete: $\iiint_V f(x, y, z) dv = \iint_D \left(\int_{\varphi_1(x, y)}^{\varphi_2(x, y)} f(x, y, z) dz \right) dx dy$

Schimbarea de variabila

$T : x = \varphi(u, v, w), y = \psi(u, v, w), z = \chi(u, v, w), \varphi, \psi, \chi \in C^1$, biunivoce si cu $J \neq 0$,

$$\iiint_V f(x, y, z) dv = \iiint_{V'} f(\varphi(u, v, w), \psi(u, v, w), \chi(u, v, w)) \left| \frac{D(\varphi, \psi, \chi)}{D(u, v, w)} \right| dv'$$

Integrale de suprafata

ec explicita: $z = f(x, y)$

ec implicita: $F(x, y, z) = 0$

ec param: $x = \varphi(u, v) \dots$

suprafete explicite (aproximam cu diferenciala)

$$d\sigma = \sqrt{p^2 + q^2 + 1} dx dy, \quad p = \frac{\partial f}{\partial x}, q = \frac{\partial f}{\partial y}$$

suprafete param

Functiile de clasa C^1 , Matricea jacobiana are rang maximal 2 in orice pct, reprezentarea e biunivoca

$$d\sigma = \sqrt{A^2 + B^2 + C^2} du dv = \sqrt{EG - F^2} du dv$$

$$A = \frac{D(\psi, \chi)}{D(u, v)}, B = \frac{D(\chi, \varphi)}{D(u, v)}, C = \frac{D(\varphi, \psi)}{D(u, v)}$$

$$a = \left(\frac{\partial \varphi}{\partial u}, \frac{\partial \psi}{\partial u}, \frac{\partial \chi}{\partial u} \right), b = \left(\frac{\partial \varphi}{\partial v}, \frac{\partial \psi}{\partial v}, \frac{\partial \chi}{\partial v} \right) \quad E = \langle a, a \rangle, F = \langle a, b \rangle, G = \langle b, b \rangle$$

speta I

def - same spiel as the previous.

$$\iint_{\Sigma} f(x, y, z) d\sigma = \iint_D f(x(u, v), y(u, v), z(u, v)) \sqrt{EG - F^2} du dv$$

speta II

$$\iint_{\Sigma} \bar{v} \cdot \bar{n} d\sigma, \quad \bar{n} = \text{versorul normalei la suprafață}$$
$$\iint_{\Sigma} P dy dx + Q dz dx + R dx dy = \iint_{\Sigma} P n_x + Q n_y + R n_z d\sigma, \quad \bar{n} = (n_x, n_y, n_z)$$

Stokes, Fr = bord, Σ regulata

$$\iint_{\Sigma} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx dy + \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z} \right) dy dz + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x} \right) dz dx = \int_{\text{Fr}\Sigma} P dx + Q dy + R dz$$

Gauß-Ostrogradsky - domenii simple în raport cu toate axele

$$\iiint_{\text{Fr}\Sigma} P dy dz + Q dz dx + R dx dy = \iiint_V \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z} \right) dx dy dz$$

Teoria Campurilor

$$\nabla = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right)$$
$$\nabla \cdot \nabla = \nabla^2 = \Delta = \text{div}(\nabla) = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$
$$\text{grad} \varphi = \nabla \varphi = \left(\frac{\partial \varphi}{\partial x}, \frac{\partial \varphi}{\partial y}, \frac{\partial \varphi}{\partial z} \right)$$
$$\bar{v}(x, y, z) = (P(x, y, z), Q(x, y, z), R(x, y, z))$$
$$\text{div}_a \bar{v} = \nabla \cdot \bar{v}(a) = \frac{\partial P}{\partial x}(a) + \frac{\partial Q}{\partial y}(a) + \frac{\partial R}{\partial z}(a)$$
$$\text{rot}_a \bar{v} = \nabla \times \bar{v}(a) = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix}$$

Def: \bar{v} sn camp de gradienti în D daca $\exists \varphi \in C^1(D)$ cu $\bar{v} = \text{grad} \varphi$

proprietati

$$\nabla c = 0$$

$$\nabla \cdot \bar{c} = 0$$

$$\nabla \times \bar{c} = \bar{0}$$

Improvisable:

$$\text{div}_a(\alpha \bar{v} + \beta \bar{w}) = \alpha \text{div}_a(\bar{v}) + \beta \text{div}_a(\bar{w})$$

$$\text{rot}_a(\alpha \bar{v} + \beta \bar{w}) = \alpha \text{rot}_a(\bar{v}) + \beta \text{rot}_a(\bar{w})$$

$$\text{div}_a(\varphi \bar{v}) = \varphi(a) \text{div}_a(\bar{v}) + \bar{v}(a) \cdot \text{grad}_a \varphi$$

$$\text{rot}_a(\varphi \bar{v}) = \varphi(a) \text{rot}_a(\bar{v}) - \bar{v}(a) \times \text{grad}_a \varphi$$

$$\text{div}(\bar{v} \times \bar{w}) = \bar{w} \cdot \text{rot} \bar{v} - \bar{v} \cdot \text{rot} \bar{w}$$

$$\text{div}(\bar{c} \times \bar{r}) = \bar{r} \cdot \text{rot} \bar{c} - \bar{c} \cdot \text{rot} \bar{r} = 0, \quad \bar{r} = \text{vector de pozitie}$$

aplicatii

Rieman-Green

$$\int_{\text{Fr}D} \bar{v} \cdot d\bar{r} = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx dy, \quad \bar{v} = (P(x, y), Q(x, y))$$

Stokes

$$\int_{\text{Fr}S} \bar{v} \cdot d\bar{r} = \iint_S \text{rot} \bar{v} \cdot \bar{N} d\sigma$$

Gauß-Ostrogradsky

$$\iiint_V (\text{div} \bar{v}) dx dy dz = \iint_{\text{Fr}V} (\bar{v} \cdot \bar{n}) d\sigma$$