might be useful

$$\frac{d}{dx}\left(\int_0^x f(x,y)\,dy\right) = f\big(x,x\big) + \int_0^x \frac{\partial}{\partial x} f(x,y)\,dy$$

care e obtinuta din formula Leibniz:

$$\frac{d}{dx}\left(\int_{a(x)}^{b(x)} f(x,t)\,dt\right) = f\big(x,b(x)\big)\cdot\frac{d}{dx}b(x) - f\big(x,a(x)\big)\cdot\frac{d}{dx}a(x) + \int_{a(x)}^{b(x)} \frac{\partial}{\partial x} f(x,t)\,dt$$

- laplace operator (laplacian): $\Delta = \nabla^2 = \nabla\cdot\nabla = \sum_i \frac{\partial^2}{\partial x_i^2}$
- field therory shit: (curl=rotor)
  div grad $f \equiv \nabla\cdot\nabla f \equiv \nabla^2 f$
  curl grad $f \equiv \nabla\times\nabla f = \mathbf{0}$
  div curl $\mathbf{A} \equiv \nabla\cdot(\nabla\times\mathbf{A}) = 0$
  curl curl $\mathbf{A} \equiv \nabla\times(\nabla\times\mathbf{A}) = \nabla(\nabla\cdot\mathbf{A}) - \nabla^2\mathbf{A}$
  $\nabla^2(fg) = f\nabla^2 g + 2\nabla f\cdot\nabla g + g\nabla^2 f$
- Gauß-Остроградский:

$$\int_\Omega \frac{\partial u}{\partial x_i} = \int_{\partial\Omega} u\cdot\nu_i d\sigma$$

  also:

$$\int_\Omega \operatorname{div} F(x)dx = \int_{\partial\Omega} F\cdot\nu d\sigma$$

- Green formula:

$$\int_\Omega (\nabla u\cdot\nabla v + v\Delta u)dx = \int_{\partial\Omega} v\sum_i \frac{\partial u}{\partial x_i}\nu_i d\sigma$$

- Green formula 1:

$$-\int_\Omega \Delta uvdx = \int_\Omega \nabla u\cdot\nabla vdx - \int_{\partial\Omega}\frac{\partial u}{\partial\nu}vd\sigma$$

- Green formula 2:

$$\int_\Omega (\Delta uvu\Delta v)dx = \int_{\partial\Omega}\frac{\partial u}{\partial\nu}v - u\frac{\partial v}{\partial\nu}d\sigma$$

- convoluție:

$$f * g(x) = \int_{\mathbb{R}^d} f(x-y)g(y)dy$$

- REMEMBER THE NORM FOR THE FOURIER THING:

$$u(x) = \sum_k \frac{1}{\|\varphi_k\|^2}\int_a^b u(x)\varphi_k(x)dx$$

- fourier desmos: `https://www.desmos.com/calculator/xkc2e0emnm`
- Ec dif ordin I

$$x' = a(t)x + b(t) \qquad x = e^{\int a(t)dt}\left(C + \int e^{-\int a(t)dt}b(t)dt\right)$$

coś, sin stuff:

- $e^{ix} = \cos x + i \sin x$
- $\sin(k\pi) = 0$
- $\cos(k\pi) = (-1)^k$
- $\int \sin = -\cos$
- $\int \cos = \sin$
- $\sin' = \cos$
- $\cos' = -\sin$
- $\sin x = \dfrac{e^{ix} - e^{-ix}}{2i}$
- $\cos x = \dfrac{e^{ix} + e^{-ix}}{2}$

product to sum

- $2\cos\theta\cos\varphi = \cos(\theta - \varphi) + \cos(\theta + \varphi)$
- $2\sin\theta\sin\varphi = \cos(\theta - \varphi) - \cos(\theta + \varphi)$
- $2\sin\theta\cos\varphi = \sin(\theta + \varphi) + \sin(\theta - \varphi)$
- $2\cos\theta\sin\varphi = \sin(\theta + \varphi) - \sin(\theta - \varphi)$
- $\cos^2\varphi = \frac{1+\cos(2\varphi)}{2}$
- $\sin^2\varphi = \frac{1-\cos(2\varphi)}{2}$

sum to product

- $\sin\theta \pm \sin\varphi = 2\sin\left(\dfrac{\theta \pm \varphi}{2}\right)\cos\left(\dfrac{\theta \mp \varphi}{2}\right)$
- $\cos\theta + \cos\varphi = 2\cos\left(\dfrac{\theta + \varphi}{2}\right)\cos\left(\dfrac{\theta - \varphi}{2}\right)$
- $\cos\theta - \cos\varphi = -2\sin\left(\dfrac{\theta + \varphi}{2}\right)\sin\left(\dfrac{\theta - \varphi}{2}\right)$

conditions

- dirichlet: $u(x) = f(x)$ pe $\partial\Omega$
- newman: $\frac{\partial u}{\partial \nu}(x) = f(x)$ pe $\partial\Omega$
- robin: $\frac{\partial u}{\partial \nu}(x) + u(x) = f(x)$ pe $\partial\Omega$

misc

- proiectie pe subspatii inchise:

$$\exists! Pu \in V \text{ aî } \|Pu - u\| = \inf_{v \in V} \|v - u\|$$

- în plus $(u - Pu) \perp V$ (ie $\langle u - Pu, v\rangle = 0,\ \forall v \in V$)
- bessel inequality: $V = \text{span}\{v_1, \dots, v_n\}$

$$\|u\|^2 \geq \|P_V u\|^2 = \sum_{j=1}^{n} \frac{|\langle u, v_j\rangle|^2}{\|v_j\|^2}$$

- daca are loc ineg parseval și $\{f_j\}$ ortongolală, at e bază Hilbertiană
- scalar product with functions $f : [a, b] \to \mathbb{C} \in L^2([a, b])$:
- weak convergence:

$$\langle f, g\rangle = \int_a^b f(x)\overline{g(x)}dx$$

$$u^n \rightharpoonup u \text{ dacă} \langle u^n - u, v\rangle \to 0, \forall v \in H$$

Sturm-Liouville - S1

valori proprii

- c3 - pg 5
- Melnig thing pg 7
- do the $| \cdot \varphi, \int$ to get $\lambda \geq 0$
- if $\varphi = 0$ we ignore that one. we dont want null solutions
- we get the characteristic equation[1] (polinom caracteristic:

$$\varphi''(x) + \lambda\varphi(x) = 0 \text{ becomes } r^2 + \lambda \cdot 1 = 0$$

- and we get the functions of form (may differ depending on the characteristic equation):

$$\{e^{r_i}, \dots, x^m e^{r_i}\}$$

  or, for our example:

$$\sin(\sqrt{\lambda}x), \cos(\sqrt{\lambda}x)$$

  so

$$\varphi(x) = \alpha \sin(\sqrt{\lambda}x) + \beta \cos(\sqrt{\lambda}x)$$

- with the initial conditions: $\varphi(0) = \varphi(l) = 0$ we get some restrictions for α and β

and, tada, ya get some λ_k, φ_k

[1] https://en.wikipedia.org/wiki/Characteristic_equation_(calculus)

Green's function:

- for n-th order differential equations: see green-kurzgesagt

separation of variables

- see s6 - pg 2
- we have:

$$\begin{cases} -\Delta u = f, \quad \text{în } \Omega = (a,b) \times (c,d) \\ \text{some condition like } u = 0, \quad \text{pe } \partial\Omega \end{cases}$$

- we write stuff with respect to x:

$$\begin{cases} -\varphi'' = \lambda\varphi, \quad \text{în } \Omega = (a,b) \\ \text{some condition like } u(a) = u(b) = 0 \end{cases}$$

- and we get some eigen functions and values: $\{\varphi_k\}$, $\{\lambda_k\}$
- we write things with the new functions:

$$u(x,y) = \sum_k^{\infty} u_k(y)\varphi_k(x)$$

$$u_{xx} = ..., u_{yy} = ...$$

$$f(x,y) = \sum f_k(y)\varphi_k(x) = \sum \frac{1}{\|\varphi_k\|^2} \left(\int_a^b f(t,y)\varphi_k(t)dt \right) \varphi_k(x)$$

- then we solve it for some k

$$\begin{cases} -\Delta u_k(y) = f_k(y), \quad \text{în } (c,d) \\ \text{some condition like } u(c) = u(d) = 0 \end{cases}$$

and we get

$$u_k(y) = \int_c^d G_k(y,s) f_k(s) ds$$

- sum things together and we get a $G\big((x,y),(t,s)\big)$:

$$u(x,y) = \int_c^d \sum_k G_k(y,s) \frac{1}{\|\varphi_k\|^2} \left(\int_a^b f(t,y)\varphi_k(t)\, dt \right) \varphi_k(x)\, ds$$

aka

$$u(x,y) = \int_c^d \int_a^b \left(\sum_k G_k(y,s) \frac{1}{\|\varphi_k\|^2} \varphi_k(t)\varphi_k(x) \right) f(t,s)\, dt\, ds$$

and, tada

$$G\big((x,y),(t,s)\big) = \sum_{k=1}^{\infty} G_k(y,s) \frac{1}{\|\varphi_k\|^2} \varphi_k(t)\varphi_k(x)$$

eigen values for op laplace

- s7 pg 6
- tl;dr we split it in 2 and get sum the eigenvalues

max principle and stuff

- Δ = op laplace
- $\Delta u = 0$ means u armonică
- $\Delta u \geq 0$ means u subarmonică
- $\Delta u \leq 0$ means u super-armonică

The actual thing

- s9 pg 3

Dacă $C^2(\Omega) \cap C(\bar{\Omega})$ și $\Delta \geq 0$ în Ω at:
$\sup\limits_{\bar{\Omega}} u = \sup\limits_{\partial\Omega} u$
și dacă $\exists \bar{x} \in \Omega$ aî $u(\bar{x}) = \sup\limits_{\bar{\Omega}} u$ at $u \equiv \text{const}$

unicitatea sol dirichlet

- s9 pg 4
- übermelnig 109
- tl;dr if we have

$$\begin{cases} \Delta u = f, & \text{în } \Omega \subseteq \mathbb{R}^d \\ u = f, & \text{pe } \partial\Omega \end{cases}$$

  we give $v = u_1 - u_2$

$$\begin{cases} \Delta v = 0, & \text{în } \Omega \subseteq \mathbb{R}^d \\ v = 0, & \text{pe } \partial\Omega \end{cases}$$

and by the "max principle" we have: $\sup\limits_{\Omega} v \leq 0$ we switch u_1 and u_2 and we get $v = 0$ ie $u_1 = u_2$

strong max principle (aka pp Hopf)

Dacă $\bar{x} \in \partial\Omega$ și $u(\bar{x}) = M$ at:
$\dfrac{\partial u}{\partial \nu}(\bar{x}) > 0$
sau $\dfrac{\partial u}{\partial \nu}(\bar{x}) = 0$ și $u \equiv M$ în Ω

variational principle

- Fundamental sol for laplace:

$$E(x) = \begin{cases} \dfrac{1}{2\pi} \ln |x|, & d = 2 \\ -\dfrac{1}{(d-2)\omega_d |x|^{d-2}}, & d > 2 \end{cases}$$

  unde (aka aria bilei unitate):

$$\omega_d = \mu_{d-1}(\partial B_1) = \int_{\partial B_1} 1 d\sigma$$

- btw: $E(x) = E(|x|)$
- Riemann-green: (c8)

$$\int_\Omega E(x-y)\Delta u(y)dy - \int_\Omega E(x-y)\frac{\partial u}{\partial \nu_y}(y)d\sigma_y + \int_{\partial\Omega} \frac{\partial}{\partial \nu_y}E(x-y)u(y)d\sigma_y = \begin{cases} u(x), & x \in \Omega, \\ \frac{1}{2}u(x), & x \in \partial\Omega, \\ 0, & x \in \mathbb{R}^d \setminus \bar{\Omega} \end{cases}$$

actual solving - übermelnig - pg 115,117,118, 121,122:

- Sol variationala e sol clasica

având:

$$\begin{cases} -\delta u = f, & \Omega, \\ u = g_1, & \Gamma_1, \\ \frac{\partial u}{\partial \nu} = g_2, & \Gamma_2, \\ \frac{\partial u}{\partial \nu} + u = g_3, & \Gamma_3 \end{cases}$$

definim

$$V = \left\{ v \in C_p^1(\Omega) \mid v = 0 \text{ pe } \partial\Omega \right\}$$

calcul formal (via Green formula 1; it's exacly this), $u \in C^2(\Omega)$:

$$-\int_\Omega \Delta uv \, d\mu = \int_\Omega \nabla u \nabla v \, d\mu - \int_{\partial\Omega} \frac{\partial u}{\partial \nu} v \, d\sigma$$

then we write:

$$\int_{\partial\Omega} \frac{\partial u}{\partial \nu} v \, d\sigma = \int_{\Gamma_1} \frac{\partial u}{\partial \nu} v \, d\sigma + \int_{\Gamma_2} \frac{\partial u}{\partial \nu} v \, d\sigma + \int_{\Gamma_3} \frac{\partial u}{\partial \nu} v \, d\sigma$$

$$\int_{\partial\Omega} \frac{\partial u}{\partial \nu} v \, d\sigma = \int_{\Gamma_1} \frac{\partial u}{\partial \nu} v \, d\sigma + \int_{\Gamma_2} g_1 v \, d\sigma + \int_{\Gamma_3} (g_3 - u) v \, d\sigma$$

split it into
$a(u, v)$ -simetrica, biliniara, and $\ell(v)$, liniara, cont

for unicitate $w = u_1 - u_2$, $v = w$ and we get $a(w, w) = 0$

sigh, see the pages mentioned above

fourier transform:

- def:

$$\hat{f}(\lambda) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(x) \, e^{-i\lambda x} \, dx$$

- see `https://en.wikipedia.org/wiki/Fourier_transform#Functional_relationships,_one-dimensional`
- $\hat{u}^{(k)}(\lambda) = [\widehat{(-ix)^k u(x)}](\lambda)$
- $\widehat{u^{(k)}}(\lambda) = (2\pi i\lambda)^k \hat{u}(\lambda)$
- $\widehat{u * v}(\lambda) = \hat{u}(\lambda)\hat{v}(\lambda)$
- $\widehat{u \cdot v}(\lambda) = \hat{u}(\lambda) * \hat{v}(\lambda)$
- $\widehat{u_x}(\lambda) = i\lambda\hat{u}(\lambda)$
- $\widehat{u_{xx}}(\lambda) = -\lambda^2\hat{u}(\lambda)$

- $\widehat{u_t}(\lambda) = \frac{\partial}{\partial t}\hat{u}(\lambda)$
- $\widehat{u_{tt}}(\lambda) = \frac{\partial^2}{\partial t^2}\hat{u}(\lambda)$
- $u\widehat{(x-a)}(\lambda) = e^{-ia\lambda}\hat{u}(\lambda)$
- $\widehat{\hat{u}(x)}(\lambda) = \hat{u}(-\lambda)$
- $\widehat{\hat{u}(ax)}(\lambda) = \frac{1}{|a|}\hat{u}\left(\frac{\lambda}{a}\right)$

toc

course

- C1: basic shit
- C2:
  - basic shit (prod scalar and norm)
  - projections
  - besel inequality
- C3:
  - more besel
  - hilbert basis
  - problem with Green's function
  - hilbert spaces examples
- C4:
  - proprietati Green's thing - pg 2
  - Riesz representation theorem - pg 5 (dual stuff)
  - autoadjunct daca $T = T^*$
- C5:
  - weak convergence
  - hilbert basis proprierties & stuff
- C6:
  - more weird abstract shit
  - sturm liouville in general form - pg 11
- C7:
  - differential subvariety stuff
  - green's formulas
  - convolutions
  - that weird fundam$E$ntal thing
- C8:

- unicitate, existenta, repr integrala, dependenta de datele pb, approx numerica
  - fundamental solution for Δ - op laplace
  - riemann-green
- C9:
  - riemann green again
  - unicitate, existenta, repr integrala, dependenta de datele pb, approx numerica, but actually done
- c10
  - pp maxim general
  - sol variationale, finally pg 8
- s11

seminaries

S1

- tl;dr normal differential equations

$$\begin{cases} u_k'(t) + \lambda_k u_k(t) = f_k(t), t > 0 \\ u_k(0) = u_k^0 \end{cases}$$

$$u_k(t) = e^{-\lambda_k t} u_k^0 + \int_0^t \exp(-\lambda_k(t-s)) f_k(s) ds$$

- sturm-liouville stuff

S2, s3

- sturm-liouville and fourier exercises

s4:

- met sep variabilelor pg 4
- fundamental solution pg 10

s5

- green shit

s6, s7

- separation of variabiles for sturm-liouvile problems + green - pg 3

s7

- solving eigen-value problems for Δ

s8

- recapitulare

s9, s10, s11

- pp de maxim +aplicatii

s11

- that weird fundam$E$ntal thing pg 11

s12

- variational thing

that old book

- green - pg 39

melnig thing

- 7 - val proprii
- 15 - parseval stuff

über-melnig thing - maed bai benni

Most of the stuff are seen in the melnig seminaries:
Par example

Ex 1: Sturm- Liouville: page 6 - 14

Replace a with smth else ofc.

Ex2: Ar ca ... ortogonale page 4 - 6

Also Id Parseval + Dezv in serii Fourier: page 15 - 30

Ex3: Metoda separarii variabilelor: mostly from page 42 to - 102

Furthermore, there is the list on which to calculate...
Most seen stuff: metoda separarii, pb parabolica: page 32
An example : page 36
Pb hiperbolica: page 47 and 86
Also check Sem9, page 80

Ex: 4 problema eliptica la limita: page 91,

Principiul de maxim: page 103
Formularea variationala pt elipsa: page 115, also s13-14 first pages

ex 5: page 115

Ex 6: transformata fourier: check s14, page 13

things to know

- sp Hilbert, serii Fourier, pb Sturm-Liouville

- separarea variabilelor (pb val proprii, hip, parab, eliptice - serii fourier
- fct Green (op laplace+ sturm liouville)
- pp maxim (op eliptici + aplicatii - unicitatea sol si estimari)
- formularea variationala a pb eliptice (si parab si hip) => sep variabilelor
- transformata fourier - calcul + cateva proprietati
- oral: he asks bout some theory bit