1 Partial

Integrale improprii

Th Cauchy (muda): $f : [a,b) \to \mathbb{R}$ integrabla pe $[a,b)$ $\iff$ $\forall\ \varepsilon > 0\ \exists b_\varepsilon$ aî $\forall b', b'' \in [b_\varepsilon, b)$ at $\left|\int_{b'}^{b''} f(x)dx\right| < \varepsilon$

Prop

Ca la Riemann (Aditivitate in rap cu intervalul, orice combinatie liniara e integrabila, pozitivitate)
Th: $\left|\int_a^{b-0} f(x)dx\right| \leq \int_a^{b-0} |f(x)|dx$
Def: absolut convergenta - daca $g(x) = |f(x)|$ este convergenta

Criterii de convergenta - semn pozitiv

Comparatie cu inegalități: Fie $f, g : [a,b) \to \mathbb{R}_+$ aî $f(x) \leq g(x),\ \forall\ x$ at:
a) daca $\int_a^{b-0} g(x)dx \quad (C)$ at $\int_a^{b-0} f(x)dx \quad (C)$
b) daca $\int_a^{b-0} f(x)dx \quad (D)$ at $\int_a^{b-0} g(x)dx \quad (D)$
Comparație cu limită: $f, g : [a,b) \to \mathbb{R}_+$ dacă $\exists \lim\limits_{x \nearrow b} \frac{f(x)}{g(x)} = l$ at:
a) $l \in (0, \infty) \implies \int_a^{b-0} f(x)dx \sim \int_a^{b-0} g(x)dx$
b) $l = 0$ și $\int_a^{b-0} g(x)dx \quad (C) \implies \int_a^{b-0} f(x)dx \quad (C)$
c) $l = \infty$ și $\int_a^{b-0} g(x)dx \quad (D) \implies \int_a^{b-0} f(x)dx \quad (D)$
Crit în α (speța I): $\quad f : [a, \infty) \to \mathbb{R}_+$
a) dacă $\exists\, \alpha > 1$ aî $\exists \lim\limits_{x \nearrow \infty} x^\alpha f(x) < \infty \implies \int_a^\infty f(x)dx \quad (C)$
b) dacă $\exists\, \alpha \leq 1$ aî $\exists \lim\limits_{x \nearrow \infty} x^\alpha f(x) > 0 \implies \int_a^\infty f(x)dx \quad (D)$
Crit în λ (speța II): $\quad f : [a,b) \to \mathbb{R}_+$
a) dacă $\exists\, \lambda < 1$ aî $\exists \lim\limits_{x \nearrow b} (b-x)^\lambda f(x) < \infty \implies \int_a^{b-0} f(x)dx \quad (C)$
b) dacă $\exists\, \alpha \geq 1$ aî $\exists \lim\limits_{x \nearrow b} (b-x)^\lambda f(x) > 0 \implies \int_a^{b-0} f(x)dx \quad (D)$

Criterii de convergenta - semn variabil

Dirichlet Fie $f, g : [a,b) \to \mathbb{R}$. Dacă $\int_a^{b-0} f(x)dx$ are integrale partiale marginite $(\forall A \in [a,b), \left|\int_a^A f(x)dx\right| \leq M)$ și g monotonă cu $\lim\limits_{x \nearrow b} g(x) = 0$ at $\int_a^{b-0} f(x)g(x)dx \quad (C)$

Abel Fie $f, g : [a,b) \to \mathbb{R}$. Dacă $\int_a^{b-0} f(x)dx \quad (C)$ și g monotona și mărginită at $\int_a^{b-0} f(x)g(x)dx \quad (C)$

Integrale improprii cu parametru $f : [a,b) \times \Delta \to \mathbb{R}$

Def: $\int_a^b f(x,y)dx$ **uniform convergenta** pe Δ la $I(\cdot)$ dacă $\forall\ \varepsilon > 0, \exists\, \delta \in [a,b)$ aî $\forall\ \beta \in (\delta, b)$
$\left|\int_a^\beta f(x,y)dx - I(y)\right| < \varepsilon, \quad \forall\ y \in \Delta$ **Weierstraß**: Fie $f \ldots$. Daca $\exists\ g : [a,b) \to \mathbb{R}$ aî
a) $|f(x,y)| \leq g(x) \forall y \in \Delta$
b) $\int_a^{b-0} g(x)dx \quad (C)$
at: $\int_a^{b-0} f(x,y)dx$ Uniform și absolut convergentă pe Δ
Dirichlet Fie $f, g : [a,b) \times \Delta \to \mathbb{R}$. Dacă $\int_a^{b-0} f(x,y)dx$ are integrale partiale marginite uniform pe Δ $(\forall A \in [a,b), \forall\, y \in \Delta, \left|\int_a^A f(x,y)dx\right| \leq M)$ și g monotonă în raport cu $x, \forall y \in \Delta$ cu $\lim\limits_{x \nearrow b} g(x) = 0$
at $\int_a^{b-0} f(x,y)g(x,y)dx \quad (UC$ pe $\Delta)$
Abel Fie $f, g : [a,b) \to \mathbb{R}$. Dacă $\int_a^{b-0} f(x,y)dx \quad (UC)$ și g monotona în rap cu $x \forall\ y$ și mărginită
at $\int_a^{b-0} f(x,y)g(x,y)dx \quad (UC)$

Teoreme

Trecerea la limită y_0 pct de acumulare. Dacă
a) $\exists \lim\limits_{y \to y_0} f(x,y) = l$ uniform in rap cu x pe orice compact
b) $\int_a^{b-0} f(x,y)dx \quad (UC)$ pe $\mathcal{V}(y_0)$

at $\int_a^{b-0} l(x)dx(C)$ și $\lim\limits_{y\to y_0}\int_a^{b-0} f(x,y)dx = \int_a^{b-0}\lim\limits_{y\to y_0} f(x,y)dx$
Continuitate $f : [a,b) \times [c,d] \to \mathbb{R}$
f cont
$\int_a^{b-0} f(x,y)dx$ uc pe $[c,d]$
at I cont pe $[c,d]$
Derivabilitate $f : [a,b) \times [c,d] \to \mathbb{R}$
$\exists\ \frac{\partial f}{\partial y}$ cont
$\int_a^{b-0} f(x,y)dx$ uc pe $[c,d]$
at $I'(y) = \int_a^{b-0} \frac{\partial f}{\partial y}(x,y)dx$
Integrabilitate $f : [a,b) \times [c,d] \to \mathbb{R}$
$f(\cdot,\cdot)$ cont
$\int_a^{b-0} f(x,y)dx$ uc pe $[c,d]$
at $\int_d^c I(y)dy = \int_a^b \left(\int_c^d f(x,y)dy\right)dx$

Integrale remarcabile

Dirichlet $\int_0^\infty \frac{\sin tx}{x}dx = \frac{\pi}{2}$
Gauß $\int_{-\infty}^\infty e^{-x^2}dx = \sqrt{\pi}$
Euler
$\Gamma, \ln\circ\Gamma$ - convexe

$$\Gamma(p) = \int_0^\infty x^{p-1}e^{-x}dx = (p+1)!, \quad p>0, \quad \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$$

$$\beta(p,q) = \int_0^1 x^{p-1}(1-x)^{q-1}dx = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}, \quad p>0, q>0$$

$$\Gamma(p)\Gamma(1-p) = \frac{\pi}{\sin(p\pi)}, \quad p\in(0,1)$$

$$\Gamma(p) = \frac{e^{-\gamma p}}{p}\prod_{k\geq 1}\left(1+\frac{p}{k}\right)^{-1}e^{\frac{p}{k}}, \quad \forall\ p>0 \qquad \gamma = \lim_{n\to\infty}(1+\frac{1}{2}+\cdots\frac{1}{n}-\ln n)$$

Integrale Curbilinii

Drumuri echivalente: $\gamma_1 \sim \gamma_2 \iff \gamma_1(t) = \gamma_2\varphi(t)$,
Daca φ crescatoare strict, $\implies$ echivalente strict,
Daca φ monotona, $\implies$ echivalente în sens larg

De speta I

$$\int_\gamma f(x,y,z)ds = \int_a^b f(x(t),y(t),z(t))\sqrt{(x'(t))^2+(y'(t))^2+(z'(t))^2}dt$$

De speta II

$$\int_\gamma \overline{F}d\bar{r} = \int_\gamma P(x,y)dx + Q(x,y)dy = \int_a^b \left[P(x(t),y(t))x'(t) + Q(x(t),y(t))y'(t)\right]dt$$

Th

Drum închis Capetele egale
Forma Inchisă $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$ at $\alpha = Pdx + Qdy$
Independența de drum $\int_\gamma df = f(B) - f(A)$
Th caracterizare Independența de drum Sunt echivalente
- α exactă
- $\int_\gamma$ independenta de drum
- $\int_\gamma = 0$ pe orice drum inchis

Poincare $\alpha \in C^1$ închisă pe deschis $\implies \forall x \in D\ df = \alpha$ local
Pe mulțimi stelate, $(\exists x_0$ cu segmentul $[x_0,x] \subseteq D\forall x)$ At α este exacta pe D

2 Sesiune

Gut

Coord polare: $f(x,y) = \rho f(\rho\cos\theta, \rho\sin\theta)$
Coord polare generalizate: $f(x,y) = ab\rho f(a\rho\cos\theta, b\rho\sin\theta)$
Coord cilindrice: $f(x,y,z) = \rho f(\rho\cos\theta, \rho\sin\theta)$,
Sferice: $f(x,y,z) = \rho^2 \sin\theta f(\rho\sin\theta\cos\varphi, \rho\sin\theta\sin\varphi, \rho\cos\theta), \quad \theta \in [0,\pi], \varphi \in [0, 2\pi]$

Integrale duble

Masura Jordan în plan

Fie $E = \bigcup_i D_i, \quad D_i$ dreptunghiuri disjuncte 2 cate 2.
E sn multime elementara
$\sigma(E) = \sum_i \sigma(D_i), \quad \sigma-$ aria
$\sigma^{\leq}(M) = \sup\{\sigma(E), E \subset M, E - \text{elementara}\}$
$\sigma^{\geq}(M) = \inf\{\sigma(E), E \supset M, E - \text{elementara}\}$
Daca $\sigma^{\leq}(M) = \sigma^{\geq}(M)$ at M este Masurabila Jordan.
Th (caracterizare): O multime $M \subset \mathbb{R}^2$ este măsurabilă Jordan $\iff$ frontiera sa este J-neglijabilă
Prop: $\sigma(D_1 \cup D_2) = \sigma(D_1) + \sigma(D_2) - \sigma(D_1 \cap D_2)$

Simplu in raport cu Oy

$a \leq x \leq b, \quad \alpha(x) \leq y \leq \beta(x), \quad \alpha, \beta$ continue pe $[a,b]$
Analog cu simplu in raport cu Oy
Daca $\gamma : [a,b] \to \mathbb{R}^2$ rectificabil, at $\gamma([a,b])$ J-neglijabilă
Multimi Jordan nemasurabile - fractals and squiggly stuff

Integrale duble

Def: luam diviziune, puncte si $\lim\limits_{\|\Delta \to 0\|} s(f;\Delta,P) = \sum\limits_{i\geq 0} f(P_i)\sigma(D_i) = \iint_D f(M)d\sigma = \iint_D f(x,y)dxdy$
Proprietati ca la integrale Riemann (Aditivitate cu functia, cu intervalul, monotonia, prop de medie, prop de majorare cu modul)

Schimbare de variabila

$$T : \begin{cases} x = \varphi(u,v) \\ y = \psi(u,v), \end{cases} \quad (u,v) \in \Omega$$

Transformarea $T : \Omega' \to \Omega$ este *regulata* daca $\varphi, \psi \in C^1(\Omega')$, T biunivoca, $J = \frac{D(\varphi,\psi)}{D(u,v)} \neq 0, \quad \forall (u,v) \in \Omega$
Def: f sn *admisibila* daca este marginita si continua cu exceptia unei multimi J-neglijabile
Th: f admisibila, T regulata, $D^* = T^{-1}(D) \quad \iint_D f(x,y)dxdy = \iint_{D^*} f(\varphi(u,v), \psi(u,v))|J|dudv$
def : D domenu standard (poate fi descompus in reuniune finita de dom simple in raport cu ambele axe)
Orientare pozitiva: frontiera este lastata in stanga
Formula Riemann-Green: D domenu standard inchis $P, Q \in C^1(D)$

$$\iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dxdy = \int_{\mathrm{Fr}D} Pdx + Qdy, \quad \mathrm{Fr}D \text{ orientata pozitiv}$$

Integrale triple

Masura Jordan in spatiu

Same spiel as in 2d, da cu paralelipipede dreptunghice paralele cu axele.

Domenii simple

Feliuțe; intre 2 plane: $\iiint_V f(x,y,z)dv = \int_c^d \left(\iint_{D_z} f(x,y,z)dxdy \right) dz$

Bețe; intre 2 suprafete: $\iiint_V f(x,y,z)dv = \iint_D \left(\int_{\varphi_1(x,y)}^{\varphi_2(x,y)} f(x,y,z)dz \right) dxdy$

Schimbarea de variabila

$T: x = \varphi(u,v,w), y = \psi(u,v,w), z = \chi(u,v,w), cu\ \varphi, \psi, \chi \in C^1$, biunivoce si cu $J \neq 0$,

$$\iiint_V f(x,y,z)dv = \iiint_{V'} f(\varphi(u,v,w), \psi(u,v,w), \chi(u,v,w)) \left| \frac{D(\varphi,\psi,\chi)}{D(u,v,w)} \right| dv'$$

Integrale de suprafață

ec explicita: $z = f(x,y)$
ec implicita: $F(x,y,z) = 0$
ec param: $x = \varphi(u,v)\dots$

suprafete explicite (aproximam cu diferentiala)

$$d\sigma = \sqrt{p^2+q^2+1}dxdy, \quad p = \frac{\partial f}{\partial x}, q = \frac{\partial f}{\partial y}$$

suprafete param

Functiile de clasa C^1, Matricea jacobiana are rang maximal 2 in orice pct, reprezentarea e biunivoca

$$d\sigma = \sqrt{A^2+B^2+C^2}dudv = \sqrt{EG-F^2}dudv$$

$$A = \frac{D(\psi,\chi)}{D(u,v)}, B = \frac{D(\chi,\varphi)}{D(u,v)}, C = \frac{D(\varphi,\psi)}{D(u,v)}$$

$$a = \left(\frac{\partial\varphi}{\partial u}, \frac{\partial\psi}{\partial u}, \frac{\partial\chi}{\partial u} \right), b = \left(\frac{\partial\varphi}{\partial v}, \frac{\partial\psi}{\partial v}, \frac{\partial\chi}{\partial v} \right) \quad E = \langle a,a \rangle, F = \langle a,b \rangle, G = \langle b,b \rangle$$

speta I

def - same spiel as the previous.

$$\iint_\Sigma f(x,y,z)d\sigma = \iint_D f(x(u,v), y(u,v), z(u,v))\sqrt{EG-F^2}dudv$$

speta II

$$\iint_\Sigma \bar{v} \cdot \bar{n} d\sigma, \quad \bar{n} = \text{versorul normalei la suprafață}$$

$$\iint_\Sigma Pdydz + Qdzdx + Rdxdy = \iint_\Sigma Pn_x + Qn_y + Rn_z d\sigma, \quad \bar{n} = (n_x, n_y, n_z)$$

Stokes, Fr = bord, Σ regulata

$$\iint_\Sigma \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dxdy + \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z} \right) dydz + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x} \right) dzdx = \int_{\mathrm{Fr}\Sigma} Pdx + Qdy + Rdz$$

Gauß-Остроградський - domenii simple in raport cu toate axele

$$\iint_{\mathrm{Fr}V} Pdydz + Qdzdx + Rdxdy = \iiint_V \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z} \right) dxdydz$$

Teoria Campurilor

$$\nabla = (\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z})$$

$$\nabla \cdot \nabla = \nabla^2 = \Delta = \mathrm{div}(\nabla) = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

$$\mathrm{grad}\varphi = \nabla\varphi = (\frac{\partial \varphi}{\partial x}, \frac{\partial \varphi}{\partial y}, \frac{\partial \varphi}{\partial z})$$

$$\bar{v}(x,y,z) = (P(x,y,z), Q(x,y,z), R(x,y,z))$$

$$\mathrm{div}_a \bar{v} = \nabla \cdot \bar{v}(a) = \frac{\partial P}{\partial x}(a) + \frac{\partial Q}{\partial y}(a) + \frac{\partial R}{\partial z}(a)$$

$$\mathrm{rot}_a \bar{v} = \nabla \times \bar{v}(a) = \begin{vmatrix} \bar{\imath} & \bar{\jmath} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix}$$

Def: $\bar{v}$ sn *camp de gradienti* în D daca $\exists\, \varphi \in C^1(D)$ cu $\bar{v} = \mathrm{grad}\varphi$

proprietati

$\nabla c = 0$
$\nabla \cdot \bar{c} = 0$
$\nabla \times \bar{c} = \bar{0}$
Improvisable:
$\mathrm{div}_a(\alpha\bar{v} + \beta\bar{w}) = \alpha\mathrm{div}_a(\bar{v}) + \beta\mathrm{div}_a(\bar{w})$
$\mathrm{rot}_a(\alpha\bar{v} + \beta\bar{w}) = \alpha\mathrm{rot}_a(\bar{v}) + \beta\mathrm{rot}_a(\bar{w})$
$\mathrm{div}_a(\varphi\bar{v}) = \varphi(a)\mathrm{div}_a(\bar{v}) + \bar{v}(a) \cdot \mathrm{grad}_a\varphi$
$\mathrm{rot}_a(\varphi\bar{v}) = \varphi(a)\mathrm{rot}_a(\bar{v}) - \bar{v}(a) \times \mathrm{grad}_a\varphi$
$\mathrm{div}(\bar{v} \times \bar{w}) = \bar{w} \cdot \mathrm{rot}\bar{v} - \bar{v} \cdot \mathrm{rot}\bar{w}$
$\mathrm{div}(\bar{c} \times \bar{r}) = \bar{r} \cdot \mathrm{rot}\bar{c} - \bar{c} \cdot \mathrm{rot}\bar{r} = 0, \quad \bar{r} = $ vector de pozitie

aplicatii

Rieman-Green

$$\int_{\mathrm{Fr}D} \bar{v} \cdot d\bar{r} = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dxdy, \quad \bar{v} = (P(x,y), Q(x,y))$$

Stokes

$$\int_{\mathrm{Fr}S} \bar{v} \cdot d\bar{r} = \iint_S \mathrm{rot}\bar{v} \cdot \bar{N} d\sigma$$

Gauß-Остроградський

$$\iiint_V (\mathrm{div}\bar{v}) dxdydz = \iint_{\mathrm{Fr}V} (\bar{v} \cdot \bar{n}) d\sigma$$