

## Gut

$$\int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi} \quad (\text{Gauß})$$

$$\Gamma(p) = \int_0^{\infty} x^{p-1} e^{-x} dx = (p+1)!, \quad p > 0, \quad \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$$

$$\beta(p, q) = \int_0^1 x^{p-1} (1-x)^{q-1} dx = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}, \quad p > 0, q > 0$$

$$\Gamma(p)\Gamma(1-p) = \frac{\pi}{\sin(p\pi)}, \quad p \in (0, 1)$$

Coord polare:  $f(x, y) = \rho f(\rho \cos \theta, \rho \sin \theta)$

Coord polare generalizate:  $f(x, y) = ab\rho f(a\rho \cos \theta, b\rho \sin \theta)$

Coord cilindrice:  $f(x, y, z) = \rho f(\rho \cos \theta, \rho \sin \theta, z)$

Sferice:  $f(x, y, z) = \rho^2 \sin \varphi f(\rho \sin \theta \cos \varphi, \rho \sin \theta \sin \varphi, \rho \cos \theta)$ ,  $\theta \in [0, \pi], \varphi \in [0, 2\pi]$

## Integrale curbilinii

### De speta I

$$\int_{\gamma} f(x, y, z) ds = \int_a^b f(x(t), y(t), z(t)) \sqrt{(x'(t))^2 + (y'(t))^2 + (z'(t))^2} dt$$

### De speta II

$$\int_{\gamma} \bar{F} d\bar{r} = \int_{\gamma} P(x, y) dx + Q(x, y) dy = \int_a^b [P(x(t), y(t))x'(t) + Q(x(t), y(t))y'(t)] dt$$
$$\int_{\gamma} \frac{\partial P}{\partial x} dx + \frac{\partial Q}{\partial y} dy = f(B) - f(A), \quad \text{unde } A, B \text{ extremitatile curbei (sn forma dif exacta)}$$

## Integrale duble

### Masura Jordan în plan

Fie  $E = \bigcup_i D_i$ ,  $D_i$  dreptunghiuri disjuncte 2 cate 2.

$E$  sn multime elementara

$\sigma(E) = \sum_i \sigma(D_i)$ ,  $\sigma$ - aria

$\sigma^{\leq}(M) = \sup\{\sigma(E), E \subset M, E - \text{elementara}\}$

$\sigma^{\geq}(M) = \inf\{\sigma(E), E \supset M, E - \text{elementara}\}$

Daca  $\sigma^{\leq}(M) = \sigma^{\geq}(M)$  at  $M$  este Masurabila Jordan.

Th (caracterizare): O multime  $M \subset \mathbb{R}^2$  este măsurabilă Jordan  $\iff$  frontiera sa este  $J$ -neglijabilă

Prop:  $\sigma(D_1 \cup D_2) = \sigma(D_1) + \sigma(D_2) - \sigma(D_1 \cap D_2)$

### Simplu in raport cu $Oy$

$a \leq x \leq b$ ,  $\alpha(x) \leq y \leq \beta(x)$ ,  $\alpha, \beta$  continue pe  $[a, b]$

Analog cu simplu in raport cu  $Ox$

Daca  $\gamma: [a, b] \rightarrow \mathbb{R}^2$  rectificabil, at  $\gamma([a, b])$   $J$ -neglijabilă

Multimi Jordan nemasurabile - fractals and squiggly stuff

## Integrale duble

Def: luam diviziune, puncte si  $\lim_{\|\Delta \rightarrow 0\|} s(f; \Delta, P) = \sum_{i \geq 0} f(P_i) \sigma(D_i) = \iint_D f(M) d\sigma = \iint_D f(x, y) dx dy$

Proprietati ca la integrale Riemann (Aditivitate cu functia, cu intervalul, monotonia, prop de medie, prop de majorare cu modul)

## Schimbare de variabila

$$T : \begin{cases} x = \varphi(u, v) \\ y = \psi(u, v), \end{cases} \quad (u, v) \in \Omega$$

Transformarea  $T : \Omega' \rightarrow \Omega$  este *regulata* daca  $\varphi, \psi \in C^1(\Omega')$ ,  $T$  biunivoca,  $J = \frac{D(\varphi, \psi)}{D(u, v)} \neq 0, \quad \forall (u, v) \in \Omega$

Def:  $f$  *admisibila* daca este marginita si continua cu exceptia unei multimi  $J$ -neglijabile

Th:  $f$  admisibila,  $T$  regulata,  $D^* = T^{-1}(D) \quad \iint_D f(x, y) dx dy = \iint_{D^*} f(\varphi(u, v), \psi(u, v)) |J| du dv$

def:  $D$  domeniu standard (poate fi descompus in reuniune finita de dom simple in raport cu ambele axe)

Orientare pozitiva: frontiera este lastata in stanga

Formula Riemann-Green:  $D$  domeniu standard inchis  $P, Q \in C^1(D)$

$$\iint_D \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) = \int_{\text{Fr}D} P dx + Q dy, \quad \text{Fr}D \text{ orientata pozitiv}$$

## Integrale triple

### Masura Jordan in spatiu

Same spiel as in 2d, da cu paralelipede dreptunghice paralele cu axele.

### Domenii simple

Feliute; intre 2 plane:  $\iiint_V f(x, y, z) dv = \int_c^d \left( \iint_{D_z} f(x, y, z) dx dy \right) dz$

Bețe; intre 2 suprafete:  $\iiint_V f(x, y, z) dv = \iint_D \left( \int_{\varphi_1(x, y)}^{\varphi_2(x, y)} f(x, y, z) dz \right) dx dy$

## Schimbarea de variabila

$T : x = \varphi(u, v, w), y = \psi(u, v, w), z = \chi(u, v, w), \varphi, \psi, \chi \in C^1$ , biunivoce si cu  $J \neq 0$ ,

$$\iiint_V f(x, y, z) dv = \iiint_{V'} f(\varphi(u, v, w), \psi(u, v, w), \chi(u, v, w)) \left| \frac{D(\varphi, \psi, \chi)}{D(u, v, w)} \right| dv'$$

## Integrale de suprafata

ec explicita:  $z = f(x, y)$

ec implicita:  $F(x, y, z) = 0$

ec param:  $x = \varphi(u, v) \dots$

### suprafete explicite (aproximam cu diferenciala)

$$d\sigma = \sqrt{p^2 + q^2 + 1} dx dy, \quad p = \frac{\partial f}{\partial x}, q = \frac{\partial f}{\partial y}$$

### suprafete param

Functiile de clasa  $C^1$ , Matricea jacobiana are rang maximal 2 in orice pct, reprezentarea e biunivoca

$$d\sigma = \sqrt{A^2 + B^2 + C^2} du dv = \sqrt{EG - F^2} du dv$$

$$A = \frac{D(\psi, \chi)}{D(u, v)}, B = \frac{D(\chi, \varphi)}{D(u, v)}, C = \frac{D(\varphi, \psi)}{D(u, v)}$$

$$a = \left( \frac{\partial \varphi}{\partial u}, \frac{\partial \psi}{\partial u}, \frac{\partial \chi}{\partial u} \right), b = \left( \frac{\partial \varphi}{\partial v}, \frac{\partial \psi}{\partial v}, \frac{\partial \chi}{\partial v} \right) \quad E = \langle a, a \rangle, F = \langle a, b \rangle, G = \langle b, b \rangle$$

### speta I

def - same spiel as the previous.

$$\iint_{\Sigma} f(x, y, z) d\sigma = \iint_D f(x(u, v), y(u, v), z(u, v)) \sqrt{EG - F^2} du dv$$

## speta II

$$\iint_{\Sigma} \bar{v} \cdot \bar{n} d\sigma, \quad \bar{n} = \text{versorul normalei la suprafață}$$
$$\iint_{\Sigma} P dy dx + Q dz dx + R dx dy = \iint_{\Sigma} P n_x + Q n_y + R n_z d\sigma, \quad \bar{n} = (n_x, n_y, n_z)$$

Stokes, Fr = bord,  $\Sigma$  regulata

$$\iint_{\Sigma} \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx dy + \left( \frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z} \right) dy dz + \left( \frac{\partial P}{\partial z} - \frac{\partial R}{\partial x} \right) dz dx = \int_{\text{Fr}\Sigma} P dx + Q dy + R dz$$

Gauß-Ostrogradsky - domenii simple în raport cu toate axele

$$\iiint_{\text{Fr}V} P dy dz + Q dz dx + R dx dy = \iiint_V \left( \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z} \right) dx dy dz$$

## Teoria Campurilor

$$\nabla = \left( \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right)$$
$$\nabla \cdot \nabla = \nabla^2 = \Delta = \text{div}(\nabla) = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$
$$\text{grad} \varphi = \nabla \varphi = \left( \frac{\partial \varphi}{\partial x}, \frac{\partial \varphi}{\partial y}, \frac{\partial \varphi}{\partial z} \right)$$
$$\bar{v}(x, y, z) = (P(x, y, z), Q(x, y, z), R(x, y, z))$$
$$\text{div}_a \bar{v} = \nabla \cdot \bar{v}(a) = \frac{\partial P}{\partial x}(a) + \frac{\partial Q}{\partial y}(a) + \frac{\partial R}{\partial z}(a)$$
$$\text{rot}_a \bar{v} = \nabla \times \bar{v}(a) = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix}$$

Def:  $\bar{v}$  sn camp de gradienti în  $D$  daca  $\exists \varphi \in C^1(D)$  cu  $\bar{v} = \text{grad} \varphi$

## proprietati

$$\nabla c = 0$$

$$\nabla \cdot \bar{c} = 0$$

$$\nabla \times \bar{c} = \bar{0}$$

Improvisable:

$$\text{div}_a(\alpha \bar{v} + \beta \bar{w}) = \alpha \text{div}_a(\bar{v}) + \beta \text{div}_a(\bar{w})$$

$$\text{rot}_a(\alpha \bar{v} + \beta \bar{w}) = \alpha \text{rot}_a(\bar{v}) + \beta \text{rot}_a(\bar{w})$$

$$\text{div}_a(\varphi \bar{v}) = \varphi(a) \text{div}_a(\bar{v}) + \bar{v}(a) \cdot \text{grad}_a \varphi$$

$$\text{rot}_a(\varphi \bar{v}) = \varphi(a) \text{rot}_a(\bar{v}) - \bar{v}(a) \times \text{grad}_a \varphi$$

$$\text{div}(\bar{v} \times \bar{w}) = \bar{w} \cdot \text{rot} \bar{v} - \bar{v} \cdot \text{rot} \bar{w}$$

$$\text{div}(\bar{c} \times \bar{r}) = \bar{r} \cdot \text{rot} \bar{c} - \bar{c} \cdot \text{rot} \bar{r} = 0, \quad \bar{r} = \text{vector de pozitie}$$

## aplicatii

### Rieman-Green

$$\int_{\text{Fr}D} \bar{v} \cdot d\bar{r} = \iint_D \left( \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx dy, \quad \bar{v} = (P(x, y), Q(x, y))$$

### Stokes

$$\int_{\text{Fr}S} \bar{v} \cdot d\bar{r} = \iint_S \text{rot} \bar{v} \cdot \bar{N} d\sigma$$

### Gauß-Ostrogradsky

$$\iiint_V (\text{div} \bar{v}) dx dy dz = \iint_{\text{Fr}V} (\bar{v} \cdot \bar{n}) d\sigma$$