

might be useful

$$\frac{d}{dx} \left( \int_0^x f(x, y) dy \right) = f(x, x) + \int_0^x \frac{\partial}{\partial x} f(x, y) dy$$

care e obtinuta din formula Leibniz:

$$\frac{d}{dx} \left( \int_{a(x)}^{b(x)} f(x, t) dt \right) = f(x, b(x)) \cdot \frac{d}{dx} b(x) - f(x, a(x)) \cdot \frac{d}{dx} a(x) + \int_{a(x)}^{b(x)} \frac{\partial}{\partial x} f(x, t) dt$$

- laplace operator (laplacian):  $\Delta = \nabla^2 = \nabla \cdot \nabla = \sum_i \frac{\partial^2}{\partial x_i^2}$

- field theory shit: (curl=rotor)  
 $\operatorname{div} \operatorname{grad} f \equiv \nabla \cdot \nabla f \equiv \nabla^2 f$   
 $\operatorname{curl} \operatorname{grad} f \equiv \nabla \times \nabla f = \mathbf{0}$   
 $\operatorname{div} \operatorname{curl} \mathbf{A} \equiv \nabla \cdot (\nabla \times \mathbf{A}) = 0$   
 $\operatorname{curl} \operatorname{curl} \mathbf{A} \equiv \nabla \times (\nabla \times \mathbf{A}) = \nabla(\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A}$   
 $\nabla^2(fg) = f\nabla^2 g + 2\nabla f \cdot \nabla g + g\nabla^2 f$

- Gauß-Ostrogradsky:

$$\int_{\Omega} \frac{\partial u}{\partial x_i} = \int_{\partial\Omega} u \cdot \nu_i d\sigma$$

also:

$$\int_{\Omega} \operatorname{div} F(x) dx = \int_{\partial\Omega} F \cdot \nu d\sigma$$

- Green formula:

$$\int_{\Omega} (\nabla u \cdot \nabla v + v \Delta u) dx = \int_{\partial\Omega} v \sum_i \frac{\partial u}{\partial x_i} \nu_i d\sigma$$

- Green formula 1:

$$-\int_{\Omega} \Delta u v dx = \int_{\Omega} \nabla u \cdot \nabla v dx - \int_{\partial\Omega} \frac{\partial u}{\partial \nu} v d\sigma$$

- Green formula 2:

$$\int_{\Omega} (\Delta u v u \Delta v) dx = \int_{\partial\Omega} \frac{\partial u}{\partial \nu} v - u \frac{\partial v}{\partial \nu} d\sigma$$

- convoluție:

$$f * g(x) = \int_{\mathbb{R}^d} f(x - y) g(y) dy$$

- REMEMBER THE NORM FOR THE FOURIER THING:

$$u(x) = \sum_k \frac{1}{\|\varphi_k\|^2} \int_a^b u(x) \varphi_k(x) dx$$

coś, sin stuff:

- $e^{ix} = \cos x + i \sin x$
- $\sin x = \frac{e^{ix} - e^{-ix}}{2i}$
- $\cos x = \frac{e^{ix} + e^{-ix}}{2}$

### product to sum

- $2 \cos \theta \cos \varphi = \cos(\theta - \varphi) + \cos(\theta + \varphi)$
- $2 \sin \theta \sin \varphi = \cos(\theta - \varphi) - \cos(\theta + \varphi)$
- $2 \sin \theta \cos \varphi = \sin(\theta + \varphi) + \sin(\theta - \varphi)$
- $2 \cos \theta \sin \varphi = \sin(\theta + \varphi) - \sin(\theta - \varphi)$
- $\cos^2 \varphi = \frac{1 + \cos(2\varphi)}{2}$
- $\sin^2 \varphi = \frac{1 - \cos(2\varphi)}{2}$

### sum to product

- $\sin \theta \pm \sin \varphi = 2 \sin \left( \frac{\theta \pm \varphi}{2} \right) \cos \left( \frac{\theta \mp \varphi}{2} \right)$
- $\cos \theta + \cos \varphi = 2 \cos \left( \frac{\theta + \varphi}{2} \right) \cos \left( \frac{\theta - \varphi}{2} \right)$
- $\cos \theta - \cos \varphi = -2 \sin \left( \frac{\theta + \varphi}{2} \right) \sin \left( \frac{\theta - \varphi}{2} \right)$

### conditions

- dirichlet:  $u(x) = f(x)$  pe  $\partial\Omega$
- newman:  $\frac{\partial u}{\partial \nu}(x) = f(x)$  pe  $\partial\Omega$
- robin:  $\frac{\partial u}{\partial \nu}(x) + u(x) = f(x)$  pe  $\partial\Omega$

### misc

- proiectie pe subspatii inchise:

$$\exists! Pu \in V \text{ a\c{u} } \|Pu - u\| = \inf_{v \in V} \|v - u\|$$

- în plus  $(u - Pu) \perp V$  (ie  $\langle u - Pu, v \rangle = 0, \forall v \in V$ )
- besel inequality:  $V = \text{span}\{v_1, \dots, v_n\}$

$$\|u\|^2 \geq \|P_V u\|^2 = \sum_{j=1}^n \frac{|\langle u, v_j \rangle|^2}{\|v_j\|^2}$$

- daca are loc ineg parseval și  $\{f_j\}$  ortongolală, at e bază Hilbertiană
- scalar product with functions  $f : [a, b] \rightarrow \mathbb{C} \in L^2([a, b])$ :
- weak convergence:

$$\langle f, g \rangle = \int_a^b f(x) \overline{g(x)} dx$$

$$u^n \rightharpoonup u \text{ dac\u{a} } \langle u^n - u, v \rangle \rightarrow 0, \forall v \in H$$

## Sturm-Liouville - S1

### valori proprii

- c3 - pg 5
- Melnig thing pg 7
- do the  $|\cdot\varphi, \int$  to get  $\lambda \geq 0$
- if  $\varphi = 0$  we ignore that one. we dont want null solutions
- we get the characteristic equation<sup>1</sup> (polinom characteristic:

$$\varphi''(x) + \lambda\varphi(x) = 0 \text{ becomes } r^2 + \lambda \cdot 1 = 0$$

- and we get the functions of form (may differ depending on the characteristic equation):

$$\{e^{r_i}, \dots, x^m e^{r_i}\}$$

or, for our example:

$$\sin(\sqrt{\lambda}x), \cos(\sqrt{\lambda}x)$$

so

$$\varphi(x) = \alpha \sin(\sqrt{\lambda}x) + \beta \cos(\sqrt{\lambda}x)$$

- with the initial conditions:  $\varphi(0) = \varphi(l) = 0$  we get some restrictions for  $\alpha$  and  $\beta$

and, tada, ya get some  $\lambda_k, \varphi_k$

### Green's function:

- for n-th order differential equations: see green-kurzgesagt

### separation of variables

- see s6 - pg 2
- we have:

$$\begin{cases} -\Delta u = f, & \text{in } \Omega = (a, b) \times (c, d) \\ \text{some condition like } u = 0, & \text{pe } \partial\Omega \end{cases}$$

- we write stuff with respect to  $x$ :

$$\begin{cases} -\varphi'' = \lambda\varphi, & \text{in } \Omega = (a, b) \\ \text{some condition like } u(a) = u(b) = 0 \end{cases}$$

- and we get some eigen functions and values:  $\{\varphi_k\}, \{\lambda_k\}$
- we write things with the new functions:

$$u(x, y) = \sum_k^{\infty} u_k(y) \varphi_k(x)$$

$$u_{xx} = \dots, u_{yy} = \dots$$

$$f(x, y) = \sum f_k(y) \varphi_k(x) = \sum \frac{1}{\|\varphi_k\|^2} \left( \int_a^b f(t, y) \varphi_k(t) dt \right) \varphi_k(x)$$

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<sup>1</sup>[https://en.wikipedia.org/wiki/Characteristic\\_equation\\_\(calculus\)](https://en.wikipedia.org/wiki/Characteristic_equation_(calculus))

- then we solve it for some  $k$

$$\begin{cases} -\Delta u_k(y) = f_k(y), & \text{in } (c, d) \\ \text{some condition like } u(c) = u(d) = 0 \end{cases}$$

and we get

$$u_k(y) = \int_c^d G_k(y, s) f_k(s) ds$$

- sum things together and we get a  $G((x, y), (t, s))$ :

$$u(x, y) = \int_c^d \sum_k G_k(y, s) \frac{1}{\|\varphi_k\|^2} \left( \int_a^b f(t, y) \varphi_k(t) dt \right) \varphi_k(x) ds$$

aka

$$u(x, y) = \int_c^d \int_a^b \left( \sum_k G_k(y, s) \frac{1}{\|\varphi_k\|^2} \varphi_k(t) \varphi_k(x) \right) f(t, s) dt ds$$

and, tada

$$G((x, y), (t, s)) = \sum_{k=1}^{\infty} G_k(y, s) \frac{1}{\|\varphi_k\|^2} \varphi_k(t) \varphi_k(x)$$

## eigen values for op laplace

- s7 pg 6
- tldr we split it in 2 and get sum the eigenvalues

## max principle and stuff

- $\Delta = \text{op laplace}$
- $\Delta u = 0$  means  $u$  armonică
- $\Delta u \geq 0$  means  $u$  subarmonică
- $\Delta u \leq 0$  means  $u$  super-armonică

## The actual thing

- s9 pg 3

Dacă  $C^2(\Omega) \cap C(\bar{\Omega})$  și  $\Delta \geq 0$  în  $\Omega$  at:

$$\sup_{\bar{\Omega}} u = \sup_{\partial\Omega} u$$

și dacă  $\exists \bar{x} \in \Omega$  aî  $u(\bar{x}) = \sup_{\bar{\Omega}} u$  at  $u \equiv \text{const}$

## unicitatea sol dirichlet

- s9 pg 4
- tl;dr if we have

$$\begin{cases} \Delta u = f, & \text{în } \Omega \subseteq \mathbb{R}^d \\ u = f, & \text{pe } \partial\Omega \end{cases}$$

we give  $v = u_1 - u_2$

$$\begin{cases} \Delta v = 0, & \text{în } \Omega \subseteq \mathbb{R}^d \\ v = 0, & \text{pe } \partial\Omega \end{cases}$$

and by the "max principle" we have:  $\sup_{\Omega} v \leq 0$  we switch  $u_1$  and  $u_2$  and we get  $v = 0$  ie  $u_1 = u_2$

## strong max principle (aka pp Hopf)

Dacă  $\bar{x} \in \partial\Omega$  și  $u(\bar{x}) = M$  at:

$$\frac{\partial u}{\partial \nu}(\bar{x}) > 0$$

sau  $\frac{\partial u}{\partial \nu}(\bar{x}) = 0$  și  $u \equiv M$  în  $\Omega$

## variational principle

- Fundamental sol for laplace:

$$E(x) = \begin{cases} \frac{1}{2\pi} \ln |x|, & d = 2 \\ -\frac{1}{(d-2)\omega_d |x|^{d-2}}, & d > 2 \end{cases}$$

unde (aka aria bilei unitate):

$$\omega_d = \mu_{d-1}(\partial B_1) = \int_{\partial B_1} 1 d\sigma$$

- btw:  $E(x) = E(|x|)$
- Riemann-green: (c8)

$$\int_{\Omega} E(x-y) \Delta u(y) dy - \int_{\Omega} E(x-y) \frac{\partial u}{\partial \nu_y}(y) d\sigma_y + \int_{\partial\Omega} \frac{\partial}{\partial \nu_y} E(x-y) u(y) d\sigma_y = \begin{cases} u(x), & x \in \Omega, \\ \frac{1}{2}u(x), & x \in \partial\Omega, \\ 0, & x \in \mathbb{R}^d \setminus \bar{\Omega} \end{cases}$$

## actual solving - übermelnig - pg 115,117,118, 121,122:

- Sol variatională e sol clasică

având:

$$\begin{cases} -\delta u = f, & \Omega \\ u = 0, & \partial\Omega \end{cases}$$

definim

$$V = \{v \in C_p^1(\Omega) \mid v = 0 \text{ pe } \partial\Omega\}$$

calcul formal (via Green formula 1; it's exactly this),  $u \in C^2(\Omega)$ :

$$-\int_{\Omega} \Delta u v d\mu = \int_{\Omega} \nabla u \nabla v d\mu - \int_{\partial\Omega} \frac{\partial u}{\partial \nu} v d\sigma$$

## fourier transform:

- def:

$$\hat{f}(\lambda) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(x) e^{-i\lambda x} dx$$

- see [https://en.wikipedia.org/wiki/Fourier\\_transform#Functional\\_relationships,\\_one-dimensional](https://en.wikipedia.org/wiki/Fourier_transform#Functional_relationships,_one-dimensional)

- $\widehat{u^{(k)}}(\lambda) = [(-ix)^k \widehat{u}(x)](\lambda)$

- $\widehat{u^{(k)}}(\lambda) = (2\pi i\lambda)^k \hat{u}(\lambda)$

- $\widehat{u * v}(\lambda) = \hat{u}(\lambda) \hat{v}(\lambda)$

- $\widehat{u \cdot v}(\lambda) = \hat{u}(\lambda) * \hat{v}(\lambda)$

- $\widehat{u_x}(\lambda) = i\lambda \hat{u}(\lambda)$

- $\widehat{u_{xx}}(\lambda) = -\lambda^2 \hat{u}(\lambda)$

- $\widehat{u_t}(\lambda) = \frac{\partial}{\partial t} \hat{u}(\lambda)$

- $\widehat{u_{tt}}(\lambda) = \frac{\partial^2}{\partial t^2} \hat{u}(\lambda)$

- $\widehat{u(x-a)}(\lambda) = e^{-ia\lambda} \hat{u}(\lambda)$

- $\widehat{\hat{u}(x)}(\lambda) = \hat{u}(-\lambda)$

- $\widehat{\hat{u}(ax)}(\lambda) = \frac{1}{|a|} \hat{u}\left(\frac{\lambda}{a}\right)$

## toc

### course

- C1: basic shit
- C2:
  - basic shit (prod scalar and norm)
  - projections
  - besel inequality
- C3:
  - more besel
  - hilbert basis
  - problem with Green's function
  - hilbert spaces examples
- C4:
  - proprietati Green's thing - pg 2
  - Riesz representation theorem - pg 5 (dual stuff)
  - autoadjunct daca  $T = T^*$
- C5:

- weak convergence
- hilbert basis proprieties & stuff
- C6:
  - more weird abstract shit
  - sturm liouville in general form - pg 11
- C7:
  - differential subvariety stuff
  - green's formulas
  - convolutions
  - that weird fundamental thing
- C8:
  - unicity, existence, repr integrals, dependent on data pb, approx numerical
  - fundamental solution for  $\Delta$  - op laplace
  - riemann-green
- C9:
  - riemann green again
  - unicity, existence, repr integrals, dependent on data pb, approx numerical, but actually done
- c10
  - pp maxim general
  - sol variationale, finally pg 8
- s11

## seminaries

### S1

- tldr normal differential equations

$$\begin{cases} u'_k(t) + \lambda_k u_k(t) = f_k(t), t > 0 \\ u_k(0) = u_k^0 \end{cases}$$

$$u_k(t) = e^{-\lambda_k t} u_k^0 + \int_0^t \exp(-\lambda_k(t-s)) f_k(s) ds$$

- sturm-liouville stuff

### S2, s3

- sturm-liouville and fourier exercises

### s4:

- met sep variabilelor pg 4
- fundamental solution pg 10

**s5**

- green shit

**s6, s7**

- separation of variables for Sturm-Liouville problems + green - pg 3

**s7**

- solving eigen-value problems for  $\Delta$

**s8**

- recapitulare

**s9, s10, s11**

- pp de maxim + aplicatii

**s11**

- that weird fundamental thing pg 11

**s12**

- variational thing

**that old book**

- green - pg 39

**melnic thing**

- 7 - val proprii
- 15 - Parseval stuff

**über-melnic thing - maed bai benni**

Most of the stuff are seen in the melnic seminars:

Par example

**Ex 1: Sturm- Liouville: page 6 - 14**

Replace a with smth else ofc.

**Ex2: Ar ca ... ortogonale page 4 - 6**

Also Id Parseval + Dezv in serii Fourier: page 15 - 30



**Ex3: Metoda separarii variabilelor: mostly from page 42 to - 102**

Furthermore, there is the list on which to calculate...

Most seen stuff: metoda separarii, pb parabolica: page 32

An example : page 36

Pb hiperbolica: page 47 and 86

Also check Sem9, page 80

**Ex: 4 problema eliptica la limita: page 91,**

Principiul de maxim: page 103

Formulara variationala pt elipsa: page 115, also s13-14 first pages

**Ex 6: transformata fourier: check s14, page 13**

**things to know**

- sp Hilbert, serii Fourier, pb Sturm-Liouville
- separarea variabilelor (pb val proprii, hip, parab, eliptice - serii fourier)
- fct Green (op laplace+ sturm liouville)
- pp maxim (op eliptici + aplicatii - unicitatea sol si estimari)
- formulara variationala a pb eliptice (si parab si hip)  $\Rightarrow$  sep variabilelor
- transformata fourier - calcul + cateva proprietati
- oral: he asks bout some theory bit