NEW EQUATIONS FOR KALMAN FILTER

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1. Kalman Filter

We have measurments

$$x_j = \begin{bmatrix} \hat{a}_j^p \\ \hat{a}_j^m \\ r_j \end{bmatrix}$$

where $\hat{a}_j^p$ is the predicted acceleration derived from the thrust data, which we treat as a measurement, $\hat{a}_j^m$ is a noisy measurement of acceleration, and r_j is a noisy measurement of altitude. Let h denote the time interval Δt. We use

$$F = \begin{bmatrix} 1 & 0 & 0 \\ h & 1 & 0 \\ \frac{h^2}{2} & h & 1 \end{bmatrix}.$$

The new H matrix will be

$$H = \begin{bmatrix} \beta & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

where β is a parameter we will have to tune. If $\beta = 1$, we are saying that our prior predictions for acceleration are unbiased. This might not be a case. For example, if our predicted acceleration is always too large (because we ignore air resistance, for example), we could take $\beta < 1$. The predicted acceleration is

$$\hat{a}_j^p = \frac{T(j)}{m(j)} - g$$

where $T(j)$ is the thrust (force in Newtons) at time j and $m(j)$ is the mass at time j, both of which are assumed to be known in advance. Here g is the acceleration due to gravity. Now we must determine the mean of the distribution for z_{j+1} given z_j. We have

$$a_j = \frac{T(j)}{m(j)} - g$$
$$a_{j+1} = \frac{T(j+1)}{m(j+1)} - g$$

which means

$$\begin{aligned} a_{j+1} - \frac{T(j+1)}{m(j+1)} &= a_j - \frac{T(j)}{m(j)} \\ a_{j+1} &= a_j + \frac{T(j+1)}{m(j+1)} - \frac{T(j)}{m(j)}. \end{aligned}$$

Therefore we will have

$$p(z_{j+1}|z_j) = N\left(z_{j+1}|Fz_j + \tau(j), Q\right)$$

with

$$\tau(j) = \begin{bmatrix} \frac{T(j+1)}{m(j+1)} - \frac{T(j)}{m(j)} \\ 0 \\ 0 \end{bmatrix}.$$

The equations that define the HMM are

$$\begin{aligned} p(z_1) &= N(z_1|\mu_0, V_0) \\ p(z_{j+1}|z_j) &= N\left(z_{j+1}|Fz_j + \tau(j), Q\right) \\ p(x_j|z_j) &= N(x_j|Hz_j, R). \end{aligned}$$

The first iteration is

$$p(z_1|x_1) = N(z_1|\mu_1, V_1)$$

with

$$\begin{aligned} \mu_1 &= \mu_0 + K_1(x_1 - H\mu_0) \\ V_1 &= (I - K_1 H)V_0 \\ K_1 &= V_0 H^T (H V_0 H^T + R)^{-1}. \end{aligned}$$

The general recursion is

$$\begin{aligned} p(z_j|x_{1:j}) \underset{z_j}{\propto} p(x_{1:j}, z_j) &= \int p(x_{1:j}, z_{j-1}, z_j)\, dz_{j-1} \\ &= \int p(x_{1:j-1}, z_{j-1}) p(z_j|z_{j-1}) p(x_j|z_j)\, dz_{j-1} \\ &\underset{z_j}{\propto} \int N(z_{j-1}|\mu_{j-1}, V_{j-1}) N(z_j|Fz_{j-1} + \tau(j), Q) N(z_{j-1}|\mu_{j-1}, V_{j-1})\, dz_{j-1} \\ &= N(x_j|Hz_j, R) \int N(z_j|Fz_{j-1} + \tau(j), Q) N(z_{j-1}|\mu_{j-1}, V_{j-1})\, dz_{j-1} \\ &= N(x_j|Hz_j, R) N(z_j|F\mu_j + \tau(j), FV_{j-1}F^T + Q) \\ &\underset{z_j}{\propto} N(z_j|\mu_j, V_j) \end{aligned}$$

where

$$
\begin{aligned}
\mu_j &= F\mu_{j-1} + \tau(j) + K_j(x_j - H(F\mu_{j-1} + \tau(j))),\\
K_j &= P_{j-1}H^T(HP_{j-1}H^T + R)^{-1},\\
V_j &= (I - K_jH)P_{j-1},\\
P_{j-1} &= FV_{j-1}F^T + Q
\end{aligned}
$$

1.1. **Kalman filter algorithm.** Inputs: Observed data $x_1, \ldots, x_n$ and model parameters μ_0, V_0, F, Q, H, R, and τ.

(1) Initialize

$$
\begin{aligned}
\mu_1 &= \mu_0 + K_1(x_1 - H\mu_0)\\
V_1 &= (I - K_1H)V_0\\
K_1 &= V_0H^T(HV_0H^T + R)^{-1}.
\end{aligned}
$$

(2) For $j = 2, \ldots, n$:

$$
\begin{aligned}
\mu_j &= F\mu_{j-1} + \tau(j) + K_j(x_j - H(F\mu_{j-1} + \tau(j))),\\
K_j &= P_{j-1}H^T(HP_{j-1}H^T + R)^{-1},\\
V_j &= (I - K_jH)P_{j-1},\\
P_{j-1} &= FV_{j-1}F^T + Q.
\end{aligned}
$$

2. Rauch-Tung-Striebel smoother (backward algorithm)

The recursion is the function $p(z_j|x_{1:n})$ for $j = n, n-1, \ldots, 1$. The first step, $j = n$, is $p(z_n|x_{1:n})$, which happens to be the final step of the Kalman filter. We know $p(z_n|x_{1:n}) = N(z_n|\hat{\mu}_n, \hat{V}_n)$ with $\hat{\mu}_n = \mu_n$ and $\hat{V}_n = V_n$ (where these values come from the forward direction of the Kalman filter).

At the general step, we want to derive $p(z_j|x_{1:n})$ where we know, as a recursion, that $p(z_{j+1}|x_{1:n}) = N(z_{j+1}|\hat{\mu}_{j+1}, \hat{V}_{j+1})$. We have

$$
\begin{aligned}
p(z_j|x_{1:n}) &= \int p(z_j, z_{j+1}|x_{1:n})\, dz_{j+1}\\
&= \int p(z_j|z_{j+1}, x_{1:n})p(z_{j+1}|x_{1:n})\, dz_{j+1}.
\end{aligned}
$$

We know $p(z_{j+1}|x_{1:n}) = N(z_{j+1}|\hat{\mu}_{j+1}, \hat{V}_{j+1})$ and we find

$$\begin{aligned}
p(z_j|z_{j+1}, x_{1:n}) &\underset{z_j}{\propto} p(z_j, z_{j+1}, x_{1:n}) \\
&= p(z_j, x_{1:j})p(z_{j+1}|z_j)p(x_{j+1:n}|z_{j+1}) \\
&\underset{z_j}{\propto} p(z_{j+1}|z_j)p(z_j|x_{1:j}) \\
&= N(z_{j+1}|Fz_j + \tau(j+1), Q)N(z_j|\mu_j, V_j) \\
&\underset{z_j}{\propto} N(z_j|\mu_j + K(z_{j+1} - (F\mu_j + \tau(j+1))), (I - KF)V_j),
\end{aligned}$$

with $K = V_jF^T(FV_jF^T + Q)^{-1} = V_jF^TP_j^{-1}$, where P_j is defined as before. Using this form we find

$$\begin{aligned}
p(z_j|x_{1:n}) &= \int N(z_j|\mu_j + K(z_{j+1} - (F\mu_j + \tau(j+1))), (I - KF)V_j)N(z_{j+1}|\hat{\mu}_{j+1}, \hat{V}_{j+1}) \\
&= \int N(z_j|Kz_{j+1} - K(F\mu_j + \tau(j+1)) + \mu_j, (I - KF)V_j)N(z_{j+1}|\hat{\mu}_{j+1}, \hat{V}_{j+1}) \\
&= N(z_j|K\hat{\mu}_{j+1} - K(F\mu_j + \tau(j+1)) + \mu_j, K\hat{V}_{j+1}K^T + (I - KF)V_j) \\
&= N(z_j|\mu_j + K(\hat{\mu}_{j+1} - F\mu_j - \tau(j+1)), V_j + K(\hat{V}_{j+1} - P_j)K^T).
\end{aligned}$$

If we define $C_j = V_jF_TP_j^{-1}$ then the mean and variance are

$$\begin{aligned}
\hat{\mu}_j &= \mu_j + C_j(\hat{\mu}_{j+1} - F\mu_j - \tau(j+1)) \\
\hat{V}_j &= V_j + C_j(\hat{V}_{j+1} - P_j)C_j^T.
\end{aligned}$$

2.1. **RTS smoother algorithm.** Inputs: μ_j, V_j, and P_j for each $j = 1, \dots, n$ computed by the Kalman filter algorithm.

(1) Initialize: $\hat{\mu}_n = \mu_n$ and $\hat{V}_n = V_n$.
(2) For $j = n-1, \dots, 1$:

$$\begin{aligned}
C_j &= V_jF^TP_j^{-1} \\
\hat{\mu}_j &= \mu_j + C_j(\hat{\mu}_{j+1} - F\mu_j - \tau(j+1)) \\
\hat{V}_j &= V_j + C_j(\hat{V}_{j+1} - P_j)C_j^T.
\end{aligned}$$